A Theory of Non-Coasean Labor Markets*

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Abstract

How does labor market heterogeneity affect the transmission of monetary policy? To answer this question, we develop a theory of non-Coasean labor markets with search frictions, idiosyncratic and aggregate shocks, sticky wages, and two-sided lack of commitment. We formulate the strategic interaction between workers and firms as a nonzero-sum stochastic differential game with stopping times and characterize its equilibrium. We show how to use microdata on wage changes and job transitions to identify the economy's unobserved latent state, namely the distribution of wage-to-productivity ratios. Based on this distribution, we provide sufficient statistics for the aggregate response of employment and real wages to monetary shocks.

Keywords: Inflation; Monetary Policy; Sticky Wages; Wage Inequality; Unemployment; Directed Search; Lack of Commitment; Nonzero-Sum Stochastic Differential Game with Stopping Times

JEL Classification: D20, D31, E12, E32

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1 Introduction

The classical idea that inflation "greases the wheels of the labor market" (Keynes, 1936; Tobin, 1972; Card and Hyslop, 1997) forms the bedrock of many Keynesian theories: Due to nominal wage rigidities, real wages are inefficiently high during recessions (i.e., following the realization of a sequence of negative productivity shocks), which creates a role for monetary policy-induced inflation to bring real wages down toward their efficient level. But Keynesian theories are usually silent on many important labor market phenomena. For example, which jobs are saved, destroyed, or created through inflation? What is the relative importance of worker quits and firm layoffs in the process of inflation-induced job reallocation? And how does the transmission of monetary policy depend on labor market inequality, specifically the distribution of wages and unemployment?

In an attempt to answer these questions, we build a theory of non-Coasean labor markets, which is consistent with mounting empirical evidence of wages being less than fully flexible (Bewley, 2007; Hazell and Taska, 2020; Grigsby *et al.*, 2021; Blanco *et al.*, 2022a). To this end, we depart from the canonical DMP model (Diamond, 1982; Pissarides, 1985; Mortensen and Pissarides, 1994) of search and matching by incorporating two additional frictions into worker-firm relationships. First, wages within job spells are sticky and unresponsive to productivity shocks. Second, neither workers nor firms can commit to their future decisions whether to dissolve or remain in a match. The interaction between productivity shocks, wage rigidity, and two-sided lack of commitment gives a role for monetary policy to affect real labor market outcomes. Our contribution is to provide a theoretical characterization of such a non-Coasean labor market and to link the efficacy of monetary policy to empirically measurable objects in this environment.

Our analysis proceeds in three steps. In the first step, we *characterize the equilibrium of a non-Coasean labor market*. Both a worker's decision to quit and a firm's decision to fire the worker depend not only on the current wage and productivity but also on dynamic strategic considerations, which we formulate in a nonzero-sum stochastic differential game with stopping times. We show that an agent's choice to remain in a presently unprofitable match is relatively less attractive under lack of commitment. In the second step, we show how to *identify the unobserved latent state of the non-Coasean labor market from microdata*. We provide an identification result to infer the unobserved latent state of the economy, namely the steady-state distribution of workers' wage-to-productivity ratios, from microdata on wage changes and worker flows between jobs. Indentification of the latent state relies on a sequence of intuitive links between our theory and the data. For example, if the between-job wage differentials are larger in absolute magnitude, this intuitively indicates greater deviations of wages from productivity arising during job spells. In the third step, we *analyze the macroeconomic consequences of non-Coasean features of the labor market*.

A key insight emerging from this analysis is that labor markets and the macroeconomy are tightly linked to one another. Specifically, we find that the transmission of monetary policy to employment depends on the prevailing inflation regime. We show that in stable economies with low trend inflation monetary shocks do not affect aggregate employment, despite wages being allocative at the micro level. Deviations from low and stable inflation lead to an increased responsiveness of the labor market to aggregate shocks. We theoretically explore the mechanisms behind these effects and provide sufficient statistics for their magnitude. Next, we describe each of the three steps in more detail.

Step 1: Characterizing the Equilibrium of a Non-Coasean Labor Market. Our study remains analytically tractabe by leveraging the powerful tools of optimal control in continuous time. The model labor market is populated by continua of workers and firms. A worker's income depends on their employment state and their idiosyncratic productivity, which follows a Brownian motion with drift. Employed workers receive a wage, while unemployed workers derive consumption from home production. Output in both employment states depends on a worker's productivity. Job search is directed, as in Moen (1997), and segmented across submarkets according to the wage rate and productivity. In each submarket, a set of homogeneous firms post vacancies to recruit workers. Existing matches become obsolete at an exogenous Poisson rate. In addition, worker-firm relationships are characterized by two key frictions. First, contracted wages are fixed within a match. Second, neither workers nor firms can commit to future actions. Together, the two key frictions give rise to the distinguishing feature of our theory: endogenous job separations that can be unilaterally initiated by either the worker or the firm.

The strategic interaction between workers and firms has three features. First, agents in a match play a dynamic nonzero-sum game since one party's payoffs from continuing in the match depend on the other party's future actions, and the joint value of a match is greater than the sum of the two agents' outside options. Second, agents' payoffs are stochastic due to fluctuations in worker productivity. Third, agents' strategies consist of stopping times, which define the stochastic arrival of unilateral job dissolution. In summary, the strategic interaction between workers and firms can be formulated as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman, 1977).

Using the theory of optimal control in continuous time, we prove the existence of a unique block recursive equilibrium. We analytically characterize workers' and firms' decisions to dissolve a match, which we show are functions of only a single state variable, namely the wage-to-productivity ratio. Agents' optimal policy functions reflect both static and dynamic considerations. In terms of static considerations, workers' and firms' respective value functions depend on their flow payoffs and flow opportunity costs from being matched. In the special case when the discount rate tends to infinity, agents behave myopically. In this case, job separations occur either if the wage-to-productivity ratio falls below a threshold that depends on the efficiency of home production, in which case the worker quits, or if the wage-to-productivity ratio rises above unity, in which case the firm fires the worker. More generally, both sides of a match solve a dynamic optimization problem that leads them to optimally delay job separation beyond the stopping time dictated by static considerations. For example, firms continue in a match beyond the time when the wage-to-productivity ratio falls below unity, either because future productivity may increase due to its stochastic component (i.e., the *option value effect*) or because productivity may have a positive drift (i.e., the *anticipatory effect*). Analogously, workers continue in a match either because of the option value of productivity falling stochastically or in anticipation of productivity's negative drift. Surprisingly and unlike in other model contexts, these option value effects are bounded and finite due to the match relationship being characterized by two-sided lack of commitment.

Step 2: Identifying the Unobserved Latent State of the Non-Coasean Labor Market from Microdata. In theory, knowing the distribution of wage-to-productivity ratios is key for measuring the prevalence of inefficient job separations through the lens of our model. In practice, although wages are commonly available in appropriate microdata, individual workers' productivity levels are not directly observed. To get around this challenge, we use our model to derive a mapping between the unobserved prevalence of inefficient job separations on one hand and observed labor market outcomes on the other hand.

To this end, we proceed in four steps. First, we recast agents' state variable—the wage-to-productivity ratio—in terms of the negative of the cumulative productivity shocks since the beginning of the employment spell. We show that this alternative choice of state variable delivers an equivalent representation of both workers' and firms' problems. Second, we identify the parameters governing the stochastic process of idiosyncratic productivity from data on wage changes between employment spells. To achieve this, we exploit properties of continuous-time stochastic processes as summarized in Doob's Optional Stopping Theorem. Third, exploiting the structural features of the model, we recover the distribution of cumulative productivity shocks from observed wage changes between employment spells, given the already-identified stochastic process of idiosyncratic productivity. Fourth and finally, we derive a Kolmogorov forward equation guiding the evolution of the distribution of cumulative productivity shocks, which we show incorporates all the relevant information needed to quantify the prevalence of inefficient job separations in the economy.

Step 3: Analyzing the Macroeconomic Consequences of Non-Coasean Features of the Labor Market. Our model highlights two distinct ways in which aggregate shocks can impact the distribution of employment in the labor market. The first way is by changing the size of the match surplus (i.e., the *surplus channel*). The second way is by changing the way the match surplus is split between workers and firms (i.e., the *redistribution channel*). Because the first channel is the standard one considered in myriad previous studies of labor markets, and also because the allocativeness of wages is our novel focus here, we restrict attention to the effects of the redistribution channel on employment across workers. To this end, we extend our model to a monetary economy, in which wages are nominally sticky while the aggregate price level fluctuates. Since changes in money supply translate one-for-one into inflation and nominal wages are rigid, monetary shocks redistribute the match surplus between workers and firms in existing jobs by moving real wages.

In such a monetary economy, we characterize analytically the transition dynamics of aggregate employment and the average real wage following a one-off monetary shock. On impact, a monetary shock causes real wages of incumbent workers to fall, while the real wages of new hires from unemployment remain constant as their nominal wages adjust one-for-one with inflation. Consequently, the labor market undergoes an adjustment in employment driven entirely by changes in the job separation rate due to increased quits by workers and decreased layoffs by firms. Following such an adjustment, the economy converges back to the previous steady state, with nominal wage growth compensating for the one-off increase in the aggregate price level.

We quantify the effect of a monetary shock on aggregate employment and real wages, relative to their steady-state values, by computing the *cumulative impulse response (CIR)* as the area under the respective impulse response function, building on the seminal work of Álvarez *et al.* (2016) and Alvarez *et al.* (2021) in the product pricing literature. With flexible wages among new hires, the CIR of wages is linked to the variance of cumulative productivity shocks during employment and the covariance of cumulative productivity shocks with the tenure of employed workers. These two moments reflect both the response of employed workers to idiosyncratic productivity shocks in the steady state and also the response of average wages to an aggregate shock. The CIR of employment depends on the steady-state unemployment rate, the average of cumulative productivity shocks in employment, and the average drift of productivity in employment—three objects that with the help of our theory can be inferred from the data. We provide economic intuition behind these results and explore special cases of the model that shed light on the different mechanisms at play. Finally, we highlight the importance of combining theory and microdata to tease out the prevalence of inefficient job separations under non-Coasean labor contracts, which could go undetected in aggregate time series data.

Related Literature. We highlight three contributions. Our first contribution is to embed a richer notion of a labor market into a framework of aggregate fluctuations in the Keynesian tradition. This allows us to study how inflation "greases the wheels of the labor market" (Keynes, 1936; Tobin, 1972). Previous work by Card and Hyslop (1997) has taken a purely reduced-form approach to measuring the effects of inflation on labor market adjustments to negative shocks. In contrast, our framework features wage setting, hiring, quitting, and firing decisions all as equilibrium outcomes. On the other hand, Keynesian models traditionally feature much simpler labor markets and often rely on the assumption of representative workers (Erceg *et al.*, 2000; Blanchard and Galí, 2010; Christiano *et al.*, 2016; Schmitt-Grohé and Uribe, 2016). In comparison, our framework features worker productivity differences, wage inequality, and unemployment risk. Our work complements the analysis in Hall (2005) and Shimer (2005a) by emphasizing endogenous fluctuations in the job separation margin as a consequence of wage rigidity and aggregate productivity shocks. In our framework, inefficient job separations arise as a result of both idiosyncratic and aggregate shocks. In the models of staggered Nash wage bargaining by Gertler and Trigari (2009) and Gertler *et al.* (2020), inefficient job separations may also occur whenever large enough aggregate shocks take a worker-firm match's wage out of the bargaining set. In practice, however, these papers ignore the possibility of inefficient separations as they feature only aggregate (i.e., not idiosyncratic) shocks. In our setting, the interaction between idiosyncratic worker productivity shocks, wage rigidity, and two-sided lack of commitment gives rise to inefficient job separations and thus a role for monetary policy. We show that such a framework has very different implications for the employment and output response to aggregate shocks, compared to alternative frameworks in which all job separations are efficient.

Our second contribution is to analytically characterize the equilibrium of a frictional labor market with aggregate and idiosyncratic shocks subject to wage rigidity. A technical challenge posed by this environment concerns the discontinuities in agents' value functions, and thus the inapplicability of standard dynamic programming results such as the contraction mapping theorem, due to dynamic strategic considerations in discrete time. To overcome this challenge, we leverage the powerful tools of optimal control in continuous time by casting the problem as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman, 1977).[1] Similar continuous-time methods have recently been employed by Bilal *et al.* (2021a,b) to tractably study firm dynamics with random matching and on-the-job search. A distinguishing feature of our analysis is that it allows for the possibility of inefficient job separations, which have been absent in previous work assuming full commitment (Moen, 1997; Acemoglu and Shimer, 1999a,b) or one-sided lack of commitment (Shi, 2009; Menzio and Shi, 2010a,b, 2011; Schaal,

[1]Related models of inaction are studied by Dixit (1991) and Sheshinski and Weiss (1977) in the context of price setting and by Bloom (2009) in the context of investment. See also the overviews contained in Dixit (2001) and Stokey (2008).

2017; Herkenhoff, 2019; Balke and Lamadon, 2020; Fukui, 2020). While Sigouin (2004), Rudanko (2009, 2021), and Bilal *et al.* (2021a,b) also study environments with two-sided lack of commitment, their analysis remains tractable precisely because their assumptions lead to a privately efficient solution (i.e., all agents' decisions maximize the joint value of a match) of the game between workers and firms. In contrast, our focus explicitly lies on the privately inefficient solution (i.e., some of agents' decisions lower the joint value of a match) under sticky wages.

Our third contribution is to import and extend the methods of sufficient statistics from heterogeneous-agent models of inaction to a labor market setting. In the context of the product pricing literature, Álvarez *et al.* (2016) link the CIR of output to monetary shocks to the ratio of the kurtosis and the frequency of price changes. In related work, Baley and Blanco (2021a) characterize the CIR of output in terms of unobserved steady-state objects, which they map to data on price changes. Alvarez *et al.* (2021) and **?** extend this theoretical result to general hazard models and multiple reset points, respectively. Before the current paper, these tools have not been imported to the labor literature. To make this possible, we extend these methods to tractably incorporate workers' endogenous transitions between employment states, which are a central feature of our labor market model.[2]

Outline. The rest of the paper is organized as follows. Section 2 lays out the model environment, defines an equilibrium, and characterizes equilibrium policies. Section 3 establishes the one-to-one mapping between the model's unobservable state variable and data. Section 4 describes the dynamic response of aggregate employment and real wages to a monetary shock. Finally, Section 5 concludes.

2 A Model of Non-Coasean Labor Contracts

We develop a labor market model with search and matching in the spirit of Mortensen and Pissarides (1994) with non-Coasean labor contracts in the form of sticky wages within a job spell and two-sided lack of commitment—i.e., neither workers nor firms can commit to future actions. Our goal is to build a framework of non-Coasen labor markets that can be used to study the two-way interaction between monetary policy and labor market inequality.

[2]While our theory is developed in the context of labor markets, our methods are generalizable to alternative settings with endogenous transitions between a discrete set of states, including firms' entry and exit decisions, traders sorting across segmented asset markets, and individual mobility decisions across geographic locations.

2.1 Environment

The economy is populated by a unit mass of workers and an endogenously determined mass of firms who meet in a frictional labor market. Time is continuous and indexed by t.

Preferences and Technology. Both workers and firms discount the future at a common rate $\rho > 0$. Firms are simply profit maximizers. Workers value an expected discounted consumption stream $\{C_t\}_{t=0}^{\infty}$ with risk-neutral preferences:

$$\mathbb{E}\left[\int_0^{\infty} e^{-\rho t} C_t \, \mathrm{d}t\right].$$

Without loss of generality, we assume that workers consume their flow income Y_t. A worker's flow income depends on her employment state E_t, which can be either employed (h) or unemployed (u), and the worker's productivity level Z_t. While employed, a worker produces an amount of a homogeneous good equal to the worker's productivity and receives flow income equal to a real wage W_t, which we assume is constant within a job spell. While unemployed, a worker receives flow income $B(Z_t)$ from home production.

Henceforth, we use lower-case letters to denote the natural logarithm of variables in upper-case letters. For example, z_t denotes the log of the worker's productivity and w_t denotes the log wage.

Stochastic Process for Worker Productivity. A worker's idiosyncratic productivity follows a Brownian motion in logs and can be written as

$$dz_t = \gamma \, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t^z,$$

where γ is the drift, σ is the volatility, and $\mathcal{W}_t^z$ is a Wiener process. For the time being, we focus on a stationary environment in which the only shocks are to idiosyncratic worker productivity, but we introduce aggregate shocks in Section 4.

Search Frictions. Unemployed workers search for jobs in a frictional labor market. Search is directed, as in Moen (1997) and Menzio and Shi (2010a), and segmented across submarkets according to the log wage w and the worker's log productivity z. In each submarket (w, z), firms post vacancies $\mathcal{V}$ at cost $K(Z_t)$. Given $\mathcal{U}$ unemployed workers and $\mathcal{V}$ vacancies in a submarket, a Cobb-Douglas matching function with constant returns produces $m(\mathcal{U}, \mathcal{V}) = \mathcal{U}^{\alpha}\mathcal{V}^{1-\alpha}$ matches, where α is the elasticity of matches to the unemployment rate. Thus, a worker's job finding rate is $f(w, z) = m(w, z)/\mathcal{U}(w, z) = \theta(w, z)^{1-\alpha}$ and a firm's job filling rate is $q(w,z) = m(w,z)/\mathcal{V}(w,z) = \theta(w,z)^{-\alpha}$, where $\theta(w,z) = \mathcal{V}(w,z)/\mathcal{U}(w,z)$ denotes the market tightness in submarket (w,z). Existing matches can get exogenously dissolved according to a Poisson process with arrival rate δ,[3] or they can be endogenously and unilaterally dissolved by either the worker or the firm.

Wage Determination. We assume that entry wages are competitively set, as in Moen (1997) and Menzio and Shi (2010a), and constant throughout a match.

Agents' Choices. An unemployed worker's choice of a submarket (w,z) induces a stopping time τ^u, which is distributed according to a Poisson process with arrival rate $f(w,z)$. Once matched, the worker chooses the duration of the match before quitting, summarized by the stopping time τ^h, while the firm chooses the duration of the match before firing the worker, summarized by the stopping time τ^j. Given these two choices and the exogenous stopping time τ^δ, the actual duration of a match is determined by the minimum stopping time in the vector $\vec{\tau}^m = (\tau^h, \tau^j, \tau^\delta)$, which we denote by $\tau^m = \min\{\tau^h, \tau^j, \tau^\delta\}$.

Value Functions. In what follows, we describe agents' value functions, which depend on the worker's productivity z and, if matched, the match-specific wage rate w. In theory, value functions may also depend on the aggregate state, which consists of the joint distribution of workers' productivity, wages, and employment states. However, we show below that our model features a unique block recursive equilibrium, as in Shi (2009) and Menzio and Shi (2010a,b, 2011)—i.e., equilibrium objects do not depend on the distribution of workers' idiosyncratic states. Thus, we omit the aggregate state in all notations.

The value of an unemployed worker with productivity z is

$$U(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}_0 \left[\int_0^{\tau^u} e^{-\rho t} B(e^{z_t})\, \mathrm{d}t + e^{-\rho \tau^u} H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u})) \right]. \tag{1}$$

That is, an unemployed worker searches for a job in submarket (w_t, z_t) at time $t \leq \tau^u$, after which she becomes employed at wage w_{τ^u} and receives the value of employment $H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u}))$.

Given a vector of stopping times $\vec{\tau}^m$, the value of a worker employed at wage w with productivity z is

$$H(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} e^{w}\, \mathrm{d}t + e^{-\rho \tau^m} U(z_{\tau^m}) \right]. \tag{2}$$

That is, an employed worker consumes a constant wage w until time τ^m when she either endogenously or

[3] This exogenous separation shock can be interpreted as a permanent shock to the *productivity of the match* that renders the match unproductive forever.

exogenously transitions to unemployment. Similarly, given a vector of stopping times $\vec{\tau}^m$, the value of a firm matched with a worker with wage w and productivity z is

$$J(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} \left[e^{z_t} - e^{w} \right] \mathrm{d}t \right]. \tag{3}$$

That is, the match produces e^{z_t}, of which e^w is paid to the worker until it gets dissolved at time τ^m.

Free Entry. Firms, in choosing the number of vacancies to post in each submarket, trade off the benefit of posting a vacancy—i.e., the product of the vacancy filling rate $q(w, z)$ and the value of a filled vacancy $J(w, z, \vec{\tau}^m(w, z))$—with the vacancy posting cost. In each submarket, firms post vacancies up to the point at which the marginal vacancy posting cost exceeds its expected benefits. Thus, free entry requires that

$$K(e^{z_t}) - q(w, z) J(w, z, \vec{\tau}^m(w, z)) \geq 0 \quad \forall (w, z) \tag{4}$$

and $\theta(w, z) \geq 0$, with complementary slackness, for all (w, z).

Equilibrium Definition. Having described the agents' problems, we are now ready to define an equilibrium. Let $\mathcal{T}$ be the set of stopping times for a match with initial condition (z, w). We say that staying in the match is a weakly dominating strategy for the worker given the state (z, w) if there exists a stopping time $\tau^{h\star}(z, w) \in \mathcal{T}$ such that $\Pr(\tau^{h\star}(z, w) > 0) = 1$ and

$$H(w, z, \tau^{h\star}(w, z), \tau^j, \tau^\delta) \geq H(w, z, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with strict inequality for some τ^j. Similarly, staying in the match is a weakly dominating strategy for the firm given the state (z, w) if there exists a stopping time $\tau^{j\star}(z, w) \in \mathcal{T}$ such that $\Pr(\tau^{j\star}(z, w) > 0) = 1$ and

$$J(w, z, \tau^h, \tau^{j\star}(w, z), \tau^\delta) \geq J(w, z, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with a strict inequality for some τ^h.

Definition 1. *An equilibrium consists of a set of value functions $\{H(w, z, \vec{\tau}^m), J(w, z, \vec{\tau}^m), U(z)\}$, a market tightness function $\theta(w, z)$, and policy functions $\left\{\tau^{h*}(w, z), \tau^{j*}(w, z), w^*(z_t)\right\}$, such that:*

1. *Given $U(z)$, $(\tau^{h*}(w, z), \tau^{j*}(w, z))$ is a non-trivial Nash equilibrium with stopping times (τ^h, τ^j) satisfying*

$$H(w, z, \tau^{h*}(w, z), \tau^{j*}(w, z), \tau^\delta) \geq H(w, z, \tau^h, \tau^{j*}(w, z), \tau^\delta), \quad \forall (w, z) \tag{5}$$

$$J(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}) \geq J(w,z,\tau^{h*}(w,z),\tau^{j},\tau^{\delta}), \quad \forall (w,z) \tag{6}$$

and $Pr(\tau^{h*}(z,w) > 0) = 1$ (resp. $Pr(\tau^{j*}(z,w) > 0) = 1$) whenever staying in the match is a weakly dominating strategy for the worker (resp. firm) given the state (z,w).

2. *Given $H(w,z,\vec{\tau}^{m*}(w,z))$, $U(z)$, and $\theta(w,z)$, $\{w^*(z_t)\}_{t=0}^{\tau^{u*}}$ solves equation (1).*

3. *Given $J(w,z,\vec{\tau}^{m*}(w,z))$, $\theta(w,z)$ solves the free entry condition (4).*

Part 1 of Definition 1 requires that agents' strategies form a Nash equilibrium in weakly dominating strategies (see below the discussion of the equilibrium refinement). That is, the worker's optimal quitting strategy τ^{h*} is the best response to the firm's firing strategy τ^{j*}, and vice versa—see equations (5) and (6). Our equilibrium definition rules out the trivial Nash equilibrium, in which both the worker and the firm choose to dissolve the match immediately. Part 2 requires that unemployed workers' search strategies are optimal. Finally, Part 3 requires that free entry holds.

Allocative Wages and Inefficient Job Separations. Here, we define two key concepts that play an important role in our analysis. We refer to wages as ex-post *allocative* whenever they affect the expected duration of the match, that is, whenever there exist $w, w' \in \mathbb{R}$ such that $\mathbb{E}[\vec{\tau}^m(w,z)] \neq \mathbb{E}[\vec{\tau}^m(w',z)]$.

Relatedly, we refer to a job separation as *inefficient* whenever a match is dissolved in spite of a positive joint match surplus $S(w,z,\vec{\tau}^m) := H(w,z,\vec{\tau}^m) - U(z) + J(w,z,\vec{\tau}^m)$. In our setting, inefficient job separations will be a consequence of non-Coasean labor contracts, which arise due to the interaction between idiosyncratic worker productivity shocks, wage rigidity, and the two-sided lack of commitment. The latter is reflected by the equilibrium definition: the stopping times depend on the history of shocks and (τ^j, τ^h) are optimal for every history.[4] It is important to highlight that the ex-post inefficiencies (i.e., once the match is formed) generated by lack of commitment end up affecting the unemployed worker's policies through $H(w,z,\vec{\tau}^{m*}(w,z))$ as well as the market tightness through $J(w,z,\vec{\tau}^{m*}(w,z))$. Therefore, lack of commitment affects *both* job-finding and job separation rates in the economy.

Homotheticity. Shocks to worker productivity affect agents' choices because they change the relative values of three margins: wages while employed, w, home production while unemployed, $B(Z_t)$, and vacancy posting costs, $K(Z_t)$, all relative to a worker's productivity level Z_t. In order to focus on the margin pertaining to the relative value of wages, which is the main focus of this paper, we assume the

[4] Naturally, we require an agent's stopping times to be measurable with respect to the agent's information set (including the entire history of shocks).

search cost and unemployment income are homothetic in workers' productivity (thereby abstracting from the other two margins). That is, $B(Z_t) = \tilde{B}Z_t$ for $\tilde{B} \in (0,1)$ and $K(Z_t) = \tilde{K}Z_t$ for $\tilde{K} > 0$.

Equilibrium Refinement. To intuitively explain the need for an equilibrium refinement, we temporarily assume that time is discrete and that, with probability one, the match will be dissolved in the following period. In any period of length $\mathrm{d}t$, the game's payoff is described in Table 1. Assume that current productivity z is such that flow payoffs in the match exceed flow payoffs from the outside options—i.e., $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z] > U(z)$. Under these assumptions, there are two Nash equilibria: (i) one in which both agents choose to dissolve the match and (ii) one in which both players decide to stay in the match. The first equilibrium does not survive the iterated elimination of weakly dominated strategies: Independently of what the other agent does, it is *weakly* better to continue in the match.

TABLE 1. PERIOD GAME

	Worker stops	**Worker continues**
Firm stops	$(0, U(z))$	$(0, U(z))$
Firm continues	$(0, U(z))$	$((e^z - e^w)\,\mathrm{d}t, (e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z])$

Notes: Discrete-time approximation of the game played between the worker and the firm under the assumption that in the next period the probability of an exogenous separation is 1.

Finally, observe that, if we take the limit as $\mathrm{d}t \to 0$, we obtain the continuous-time versions of the conditions that make continuing in the match a weakly dominating strategy. That is, if $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z] > U(z)$, then, as $\mathrm{d}t \to 0$, $e^z - e^w > 0$ and $\rho u(z) < e^w + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2}$, respectively.

Discussion of Assumptions. With the goal of giving a clear exposition of the core mechanisms at play in models with non-Coasean labor contracts, we make several simplifying assumptions that are not essential for our undertaking but that would matter in quantitative work: full wage rigidity within the match, homotheticity, and the lack of richer shocks and frictions. Regarding wage rigidity, for ease of exposition, we completely abstract from wage adjustment within the job. An extension with infrequent wage adjustments à la Calvo (1983) would be straight-forward and preserve the main insights from our analysis. Regarding the hometheticity assumption, a worker's wage and productivity could in principle affect equilibrium outcomes in each submarket (e.g., job-finding and separation rates). This assumption implies that equilibrium outcomes depend only on the wage-to-productivity ratio and. Therefore, we abstract away from ex-ante heterogeneity in workers' productivity. Finally, we omit several other relevant features such as match-specific shocks, firm heterogeneity, and on-the-job search, among others. Our goal is to provide the first foundational framework upon which future work can build on and incorporate these important extensions.

2.2 Equilibrium Characterization

Let $u(z)$, $h(z;w)$, $j(z;w)$, and $\theta(z;w)$ denote the values of an unemployed worker, an employed worker, a filled vacancy, and the market tightness function evaluated at equilibrium policies, where the index w references the constant (log) wage. We now derive necessary and sufficient conditions for a block recursive equilibrium. Our equilibrium characterization proceeds in two steps. In the first step, which is standard in search-and-matching models, firms post vacancies to attract workers and workers search for jobs. This problem is characterized by the Hamilton-Jacobi-Bellman (HJB) equation for unemployed workers,

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(w,z)[h(z;w) - u(z)], \tag{7}$$

and the free entry condition for firms, which requires that the equilibrium market tightness satisfies

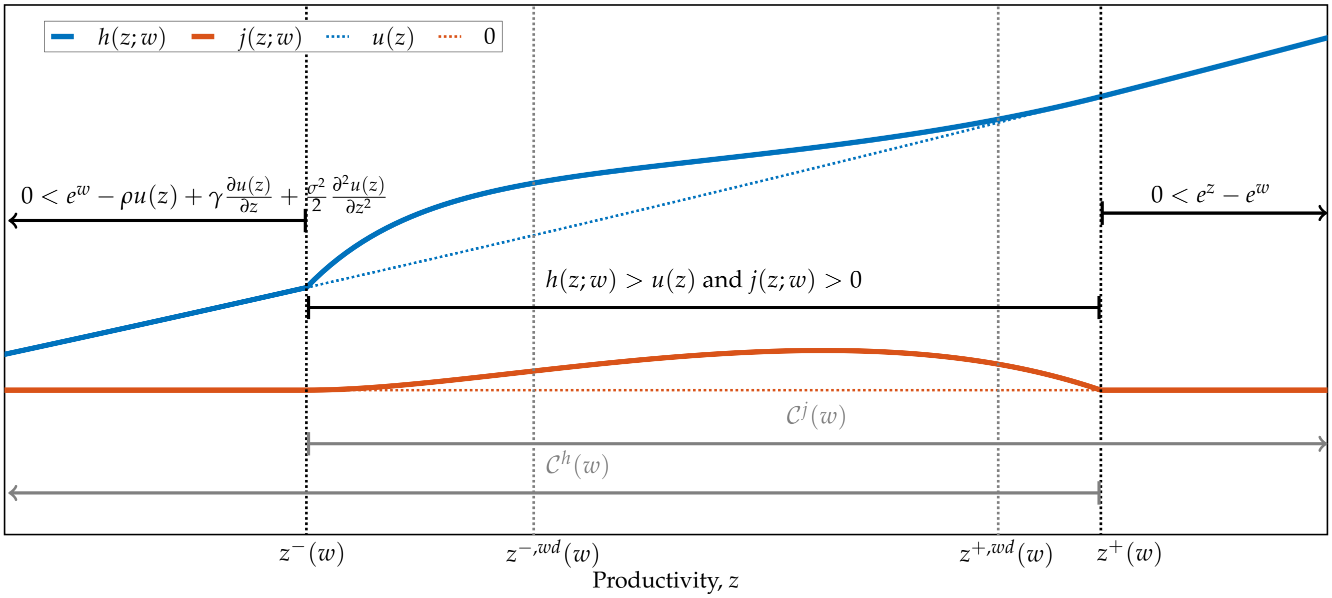
$$K(e^z) - q(w,z)j(z;w) \geq 0 \quad \forall (w,z)$$

and $\theta(w,z) \geq 0$, with complementary slackness, for all (w,z).

In the second step, which is the novel focus of this paper, a matched worker-firm pair plays a game that determines the duration of the match before the worker quits and the firm lays the worker off. The strategic interaction between workers and firms has three features. First, agents play a nonzero-sum game since the value of a match e^z exceeds the value of unemployment $\tilde{B}e^z$ for $\tilde{B} < 1$. Second, agents' payoffs are stochastic and move with productivity, which follows a Wiener process. Third, agents' strategies consist of stopping times. Thus, the strategic interaction between workers and firms can be formulated as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman, 1977). To characterize the equilibrium, we make use of quasi-variational inequalities—a methodological approach that we import from the calculus of variations literature. We highlight that the application of these state-of-the-art tools in the economics literature and the illustration of their broader value is an important contribution of this paper.

Before stating the quasi-variational inequalities that characterize this problem, we describe the equilibrium conditions for a worker-firm match with the aid of Figure 1, which illustrates the equilibrium values, outside options, and optimal policies for both agents.

FIGURE 1. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The figure plots the value functions of workers and firms for a given log wage w as a function of log productivity z. The blue and red solid lines show the value functions for the worker and the firm, respectively. The blue and red dashed lines show the opportunity costs for the worker and the firm, respectively. The firm's optimal job separation trigger based on weakly dominant strategies is $z^{-,wd}(w) := w$. The worker's optimal job separation trigger under weak dominant strategies is $z^{+,wd}(w^*)$ and satisfies $e^{w^*} = \tilde{B}e^{z^{+,wd}(w^*)} + f(z^{+,wd}(w^*); w^*)[h(z^{+,wd}; w^*) - u(z^{+,wd})]$. The optimal job separation triggers for the worker and the firm are $z^+(w) := \sup_z\{z : h(z;w) > u(z)\}$ and $z^-(w) := \inf_z\{z : j(z;w) > 0\}$, respectively. We use the following illustrative parameter values: $(\gamma, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0, 0.005, 0.04, 0.5, 1, 0.021, 0.55)$.

The possibility that both the worker and the firm can unilaterally walk away from a match at any point in time imposes lower bounds on the agents' values $h(z;w)$ and $j(z;w)$. Formally, *individual rationality* of the worker and the firm requires that

$$h(z;w) \geq u(z) \quad \forall z,$$
$$j(z;w) \geq 0 \quad \forall z.$$

Let $\mathcal{C}^h(w)$ denote the interior of the set of productivities for which the worker prefers to stay in the match with wage w under our equilibrium definition. Importantly, this set is made up of two productivity ranges: one in which both the firm and the worker opt to continue the match and one in which only the worker prefers to continue the match (and, thus, the equilibrium refinement based on weak dominance applies). Let $\mathcal{C}^j(w)$ denote the analogous object for the firm.

Agents' Optimality Conditions. We now state the variational inequalities that characterize the worker's and the firm's optimal policies for productivity levels inside and outside the other agent's continuation set. The HJB equation of a worker employed at log wage w with log productivity $z \in \mathcal{C}^j(w)$, for which the firm prefers to continue, is given by

$$\rho h(z;w) = \max\left\{e^w + \gamma \frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta\left[u(z) - h(z;w)\right],\, \rho u(z)\right\}.$$

This value satisfies $h(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^j(w)) \cap \mathbb{C}(\mathbb{R})$; i.e., it needs to be continuously once-differentiable on the continuation set and continuous everywhere. These continuity and differentiability conditions correspond to the *value matching* condition and the *smooth pasting* condition, respectively, in the worker's best response. Importantly, the smooth pasting condition needs to hold whenever the worker has the choice between staying or leaving the match (i.e., in the firm's continuation set). Similarly, the HJB equation of a firm employing a worker at log wage w with log productivity $z \in \mathcal{C}^h(w)$, for which the worker prefers to continue, is given by

$$\rho j(z;w) = \max\left\{e^z - e^w + \gamma \frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta j(z;w),\, 0\right\}.$$

Again, this value must be continuous and differentiable within the worker's continuation set: $j(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^h(w)) \cap \mathbb{C}(\mathbb{R})$.

On the other hand, if any one agent chooses to dissolve the match, then the other agent receives the value of the corresponding outside option. Therefore, the worker's and the firm's values of a match with log productivity z and log wage w satisfy the following conditions:

$$h(z;w) = u(z) \quad \forall z \in (\mathcal{C}^j(w))^c, \tag{8}$$

$$j(z;w) = 0 \quad \forall z \in (\mathcal{C}^h(w))^c, \tag{9}$$

where $X^c := \mathbb{R}\backslash X$. Equations (8)–(9) define the game's *value matching conditions*, which imply the continuity of one agent's value function at the boundary of the other agent's continuation set.

Agents' Continuation Sets. We now characterize the continuation set of each agent. Two conditions characterize these sets. First, the match continues whenever both agents find it strictly optimal to continue in the match; i.e., whenever

$$h(z;w) > u(z) \quad \forall z,,$$
$$j(z;w) > 0 \quad \forall z.$$

Second, each agent prefers to continue whenever staying in the match is a weakly dominant strategy. To explain this last condition, note that, for *any* worker's policy, the firm would weakly prefer to continue the match if flow profits are positive (i.e., $e^z - e^w > 0$). This preference results from the fact that, in this scenario, current profits are positive and the firm's continuation value is non-negative (because the firm always has the option to fire the worker in the future). Therefore, the continuation set for the firm is

$$\mathcal{C}^j(w) := int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}. \tag{10}$$

Similarly, the worker's weakly dominant continuation set includes all productivity levels for which the sum of the current wage and the capital gains from unemployment is positive—i.e.,

$$\mathcal{C}^h(w) := int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^w - \rho u(z) + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2}\right\}. \tag{11}$$

To provide the intuition behind this set, observe that from the HJB equation of the unemployed worker in (7), we have that

$$0 < e^w - \rho u(z) + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} \iff 0 < e^w - \underbrace{\left(\tilde{B}e^z + \max_w f(z;w)[h(z;w) - u(z)]\right)}_{\text{flow opportunity cost}}.$$

That is, if the current wage is larger than the flow value of quitting to look for a new match, then staying in the current match strictly dominates quitting the job.[5]

Figure 1 describes the continuation set of each agent. The firm's continuation set is given by $\mathcal{C}^j(w) = \{z \in (z^-(w), \infty)\}$. This set includes the range of productivities for which the firm makes positive flow profits—$z > z^{-,wd} := w$— and, therefore, it is weakly dominant to choose to retain the worker. Moreover, the set $\mathcal{C}^j(w)$ also includes the range of productivities for which both the firm and the worker find it optimal to remain in the match despite flow profits being negative; i.e., $z \in (z^-, z^{-,wd})$. For $z \in (z^-, z^{-,wd})$, the firm's continuation value is positive and large enough to compensate for current loses. A similar intuition applies to the worker's continuation set $\mathcal{C}^h(w)$. We remark that the existence and uniqueness of

[5]See Chapter 10.1 and 10.3. in **?**.

such a threshold that characterizes the separation policy is a result that we formally show below, and not an assumption.

Equilibrium Policies. The worker quits and the firm fires the worker at times τ^h and τ^j, respectively. These stopping times denote the stochastic time at which productivity falls outside of the worker's and the firm's respective continuation sets. Thus, agents' optimal stopping times are given by

$$\tau^{j*}(w,z) = \inf\left\{t \geq 0 : z_t \in \mathcal{C}^j(w)^c, z_0 = z\right\},$$
$$\tau^{h*}(w,z) = \inf\left\{t \geq 0 : z_t \in \mathcal{C}^h(w)^c, z_0 = z\right\}.$$

The optimality of workers' search decisions implies that the competitive entry wage satisfies

$$w^*(z) = \arg\max_{w} \theta(z;w)^{1-\alpha}\left[h(z;w) - u(z)\right]. \tag{12}$$

The following lemma shows that these conditions are necessary and sufficient to characterize a block recursive equilibrium. We relegate all proofs to the Online Appendix.

Lemma 1. *The policy functions* $\left\{\tau^{h*}, \tau^{j*}, w^*(z)\right\}$ *together with the value functions* $\{U(z), H(w,z,\vec{\tau}^m), J(w,z,\vec{\tau}^m)\}$ *given by* (1), (2) *and* (3)*, and the market tightness function* $\theta(w,z)$ *form a block recursive equilibrium if and only if* $\{u(z), h(z;w), j(z;w)\}$ *satisfy equations* (7)–(12) *and*

$$u(z) = U(z),$$
$$h(z;w) = H(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}),$$
$$j(z;w) = J(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}).$$

Finding the State. To understand the dependence of equilibrium outcomes on the state variables, we first note that we can recast the equilibrium conditions in terms of a reduced state space. Since the flow income of unemployed workers and firms' vacancy costs are both proportional to productivity, Z, it turns out that the relevant state variable for both workers and firms is the log wage-to-productivity ratio, $\hat{w} := w - z$. This result allows us to express agents' values and policies as functions of the scalar $\hat{w}$ instead of the duplet (w,z). To simplify notation, we define the transformed drift $\hat{\gamma} := \gamma + \sigma^2$ and the transformed discount factor $\hat{\rho} := \rho - \gamma - \sigma^2/2$. The equilibrium characterization is summarized in the following Lemma.

Lemma 2. *Suppose that the functions* $(u(z), h(z;w), j(z;w), \theta(w,z))$ *satisfy the equilibrium conditions in* (7)–

(9), *given the continuation sets* $\mathcal{C}^h(w)$ *and* $\mathcal{C}^j(w)$ *defined in* (10)–(11)*. Then, the transformed value and market tightness functions are given by*

$$\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)\right) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w)-u(z)}{e^z}, \theta(w,z)\right)$$

equivalently characterize the equilibrium if the following conditions are satisfied:

1. *The transformed value function of an unemployed worker,* $\hat{U}$*, satisfies*

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}), \tag{13}$$

   where the optimal choice of submarket for an unemployed worker to search in is $\hat{w}^* = w^*(z) - z$.

2. *The lower bounds of the game's values for workers and firms are:* $\hat{W}(\hat{w}) \geq 0$ *and* $\hat{J}(\hat{w}) \geq 0$.

3. *The variational inequalities for workers and firms are satisfied: Given*

$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0 \text{ or } 0 < e^{\hat{w}} - \hat{\rho}\hat{U}\right\} \text{ and } \hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0 \text{ or } 0 < 1 - e^{\hat{w}}\right\},$$

   the transformed value function of an employed worker, $\hat{W}(\hat{w})$*, and that of a filled vacancy,* $\hat{J}(\hat{w})$*, satisfy*

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = \max\left\{e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}), 0\right\}, \ \forall \hat{w} \in \hat{\mathcal{C}}^j, \tag{14}$$

$$(\hat{\rho}+\delta)\hat{J}(\hat{w}) = \max\left\{1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}), 0\right\}, \ \forall \hat{w} \in \hat{\mathcal{C}}^h, \tag{15}$$

   with $\hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j(w)) \cap \mathbb{C}(\mathbb{R})$ *and* $\hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h(w)) \cap \mathbb{C}(\mathbb{R})$, $\tau^{j*} = \inf\{t \geq 0 : \hat{w}_t \notin \hat{\mathcal{C}}^j, w_0 = \hat{w}^*\}$*. The optimal stopping time are given by* $\tau^{h*} = \inf\{t \geq 0 : \hat{w}_t \notin \hat{\mathcal{C}}^h, w_0 = \hat{w}^*\}$.

4. *The value matching conditions are satisfied:* $\hat{W}(\hat{w}) = 0 \ \ \forall \hat{w} \in (\hat{\mathcal{C}}^j)^c$*, and* $\hat{J}(\hat{w}) = 0 \ \ \forall \hat{w} \in (\hat{\mathcal{C}}^h)^c$.

5. *The free entry condition for* $\hat{\theta}(\hat{w})$ *is satisfied:* $\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w}) \geq 0$ *and* $\hat{\theta}(\hat{w}) \geq 0$*, with complementary slackness.*

The equilibrium conditions in Lemma 2 are transformed versions of those stated above and follow similar intuitions. Equation (13) of Part 1 of the lemma encodes the payoff under the optimal log wage-to-productivity ratio of newly employed workers. For an unemployed worker, the optimal wage w^* trades off the job finding rate $\hat{\theta}(\hat{w}^*)^{1-\alpha}$ with the value of employment $\hat{W}(\hat{w}^*)$. Part 2 describes the lower bounds on agents' transformed values. From equations (14)–(15) of Part 3, we can infer the thresholds that render the worker's and the firm's respective transformed flow payoffs negative. If $e^{\hat{w}} < \hat{\rho}\hat{U}$ then the worker's wage is below the flow value of unemployment. Similarly, if $e^{\hat{w}} > 1$, then the firm's flow profits are negative. Part 4 states the transformed value matching conditions. Finally, Part 5 states the transformed free entry condition.

Equilibrium Existence and Uniqueness. Equipped with the equilibrium conditions summarized in Lemma 2, we now demonstrate the existence and uniqueness of a block recursive equilibrium.

Proposition 1. *There exists a unique block recursive equilibrium.*

While the result stated in Proposition 1 is essential for any model with sticky wages and non-Coasean labor contracts, it does not follow from existing results.Theorems for the existence of a block recursive equilibrium with exogenous job separations in discrete time rely on Schauder's fixed point theorem—see, for instance, Menzio and Shi (2010a,b) or Schaal (2017). Two conditions are critical for Schauder's fixed point theorem to apply: continuity in the value functions and continuity in the mapping between value functions characterizing the block recursive equilibrium. In the discrete-time version of our model, idiosyncratic worker productivity shocks, wage rigidity, and two-sided lack of commitment jointly generate endogenous job separations, which break the regularity conditions on which traditional arguments rely. In contrast, our continuous-time setup gets around this technical challenge (e.g., value functions are now continuous). Thus, we proceed with a new approach that leverages techniques from the variational inequalities literature, which allows us to prove the *existence* and *uniqueness* of a block recursive equilibrium. This is where the payoff of our continuous-time methods based on variational inequalities lies.

2.3 Understanding the Mechanisms

Having described the equilibrium conditions, we proceed to characterize the mechanisms that drive workers' and firms' equilibrium behavior.

Equilibrium Policies. Based on the transformed state variable $\hat{w}$ and equilibrium conditions in Lemma 2, we can characterize agents' equilibrium policies. Recalling the definition of the transformed state variable $\hat{w} := w - z$, we postulate that there are exist optimal policies $\hat{w}^- < \hat{w}^* < \hat{w}^+$, where $\hat{w}^-$ is the worker's optimal job separation threshold, $\hat{w}^*$ is the optimal search strategy at match formation, and $\hat{w}^+$ is the firm's optimal job separation threshold. We define the transformed surplus of the match as $\hat{S}(\hat{w}) := \hat{J}(\hat{w}) + \hat{W}(\hat{w})$ and the worker's share of the transformed surplus as $\eta(\hat{w}) := \hat{W}(\hat{w})/\hat{S}(\hat{w})$. The following proposition characterizes the properties of the block recursive equilibrium in its transformed notation.

Proposition 2. *The block recursive equilibrium has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho}), \tag{16}$$

where

$$\mathcal{T}(\hat{w}, \hat{\rho}) := \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\, \mathrm{d}t | \hat{w}_0 = \hat{w}\right] \tag{17}$$

is the expected discounted match duration and $1 > \hat{\rho}\hat{U} > \tilde{B}$.

2. *The competitive entry wage* $\hat{w}^*$ *coincides with the Nash bargaining solution with worker's weight* α*:*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{\hat{W}(\hat{w})^{\alpha}\hat{J}(\hat{w})^{1-\alpha}\right\} = \arg\max_{\hat{w}} \left\{\eta(\hat{w})^{\alpha}(1-\eta(\hat{w}))^{1-\alpha}\mathcal{T}(\hat{w}, \hat{\rho})\right\}, \tag{18}$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right)}_{\textit{Share channel}} = -\underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}}_{\textit{Surplus channel}}. \tag{19}$$

3. *Given* $\eta(\hat{w}^*)$ *and* $\mathcal{T}(\hat{w}^*, \hat{\rho})$*, the equilibrium job finding rate* $f(\hat{w}^*)$ *and the flow opportunity cost of employment* $\hat{\rho}\hat{U}$ *are given by*

$$f(\hat{w}^*) = \left[(1-\eta(\hat{w}^*))(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*, \hat{\rho})/\tilde{K}\right]^{\frac{1-\alpha}{\alpha}}, \tag{20}$$

$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1}\left(1-\eta(\hat{w})\right)^{1-\alpha}\eta(\hat{w})^{\alpha}\left(1-\hat{\rho}\hat{U}\right)\mathcal{T}(\hat{w}^*, \hat{\rho})\right)^{\frac{1}{\alpha}}. \tag{21}$$

4. *Assume* $\gamma \neq 0$ *or* $\sigma \neq 0$*. Given* $\hat{U}$*, the worker's and the firm's continuation sets are connected, and the game's continuation set is bounded; i.e.*

$$\hat{\mathcal{C}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \quad \textit{and } \hat{\mathcal{C}}^j = \{\hat{w} : \hat{w} < \hat{w}^+\}, \tag{22}$$

with $-\infty < \hat{w}^- \leq \log(\hat{\rho}\hat{U}) < 0 \leq \hat{w}^+ < \infty$*. The worker's and firm's value functions satisfy* smooth pasting conditions *at* $\hat{w}^-$ *and* $\hat{w}^+$, *respectively:* $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Starting with Part 1 of Proposition 2, equation (16) states that the surplus of the match is equal to the product between the transformed flow surplus $1 - \hat{\rho}\hat{U}$ and the expected discounted match duration $\mathcal{T}(\hat{w}, \hat{\rho})$ defined in equation (17), which depends on the entry wage $\hat{w}^*$ and the width of the match's continuation set $(\hat{w}^-, \hat{w}^+)$. Additionally, the flow opportunity cost of employment $\hat{\rho}\hat{U}$ is bounded between one (i.e., the transformed value of flow output in the match) and $\tilde{B}$ (i.e., the transformed value of home production). As $1 > \hat{\rho}\hat{U}$, the joint match surplus is always strictly positive—thus, all endogenous job separations are inefficient.

Equations (18)–(19) of Part 2 show that the competitive entry wage $\hat{w}^*$ balances a *share channel* and a *surplus channel*. Unemployed workers search for wages that are competitively set in a way that coincides with the Nash bargaining solution with worker's weight α, thereby satisfying the well-known efficiency condition due to Hosios (1990). This result obtains due to the free entry condition, which implies that a worker's job-finding rate is proportional to the value of a firm. A larger initial wage increases the worker's share by $\eta'(\hat{w}^*)\alpha/\eta(\hat{w}^*)$ but at the same time reduces the job finding probability by $\eta'(\hat{w}^*)(1-\alpha)/(1-\eta(\hat{w}^*))$. This trade-off is reflected in the share channel and is standard in models with directed search (e.g., Moen, 1997; Menzio and Shi, 2010a).

With allocative wages, a novel surplus channel arises. Intuitively, the surplus channel captures the fact that the wage set at match formation affects the expected match duration and therefore the expected surplus. The higher the entry wage, the sooner the firm will dissolve the match in expectation. Conversely, the lower the entry wage, the sooner the worker will dissolve the match in expectation. Only if $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$ will the worker's share of the surplus equal $\eta(\hat{w}^*) = \alpha$, as in efficient models with nonallocative wages. These considerations are unique to our environment with allocative wages.

Part 3 characterizes the unemployed worker's job finding rate (20) and the flow opportunity cost of employment (21) as functions of the worker's surplus share and the expected discounted match duration.

Part 4 shows that the continuation set of the worker and that of the firm (22) follow threshold rules in the log wage-to-productivity ratio $\hat{w}$. Workers refrain from quitting as long as $\hat{w} > \hat{w}^-$, while firms refrain from firing the worker as long as $\hat{w} < \hat{w}^+$. Thus, the continuation set for the match is given by $\hat{\mathcal{C}}^h \cap \hat{\mathcal{C}}^j = (\hat{w}^-, \hat{w}^+)$. These thresholds satisfy $\hat{w}^- \leq \log(\hat{\rho}\hat{U})$ and $\hat{w}^+ \geq 0$, reflecting the fact that both parties are willing to accept flow payoffs below that from their respective outside option. Finally, the smooth pasting conditions apply at the worker's quitting trigger $\hat{w}^-$ and at the firm's firing trigger $\hat{w}^+$, reflecting the optimality of agents' continuation thresholds.

Static Considerations. Before further characterizing the original dynamic problem, it is instructive to consider equilibrium policies when productivity is fixed—i.e., $\gamma = \sigma = 0$.[6] The following proposition characterizes the static considerations in this case.

Proposition 3. *Assume $\gamma = \sigma = 0$. Then, optimal policies are given by*

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log(\hat{\rho}\hat{U}, \alpha + (1-\alpha)\hat{\rho}\hat{U}, 1),$$

with $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}(\hat{w}^*, \hat{\rho}) = 1/(\hat{\rho} + \delta)$.

Note that $\hat{w}^- < \hat{w}^* < \hat{w}^+$ and $\hat{w} = \hat{w}^*$ for the duration of the match, absent productivity fluctuations, so there are no endogenous job separations. From this result, we see that lack of commitment and wage rigidity by themselves do not generate any inefficient job separations. Absent productivity fluctuations, agents' behavior is privately efficient in that it maximizes the joint match surplus.

In addition to the forces outlined in this static example, two important dynamic incentives guide workers' and firms' choices, namely the *option value effect* and the *anticipatory effect*.

Dynamic Considerations I: The Option Value Effect. To understand the role of productivity fluctuations in creating the option value effect, we assume away, for now, the drift of worker productivity—i.e., $\hat{\gamma} = 0$. The following proposition characterizes the option value effect in this case.

Proposition 4. *Assume $\hat{\gamma} = 0$ and $\alpha = 1/2$. Then, to a first-order approximation, the optimal entry wage is given by $\hat{w}^* = \log\left((1+\hat{\rho}\hat{U})/2\right)$ and the job separation triggers satisfy $\hat{w}^{\pm} = \hat{w}^* \pm h(\varphi, \Phi)$ for some function $h(\varphi, \Phi)$ with $\varphi := \sqrt{2(\rho+\delta)}/\sigma$ and $\Phi := (1-\hat{\rho}\hat{U})/(1+\hat{\rho}\hat{U})$. The following properties apply:*

1. *$h(\varphi, \Phi)$ is decreasing in φ and increasing in Φ.*
2. *$\lim_{\varphi \to 0} h(\varphi, \Phi) = 3\Phi$ and $\lim_{\varphi \to \infty} h(\varphi, \Phi) = \Phi$.*
3. *$\varphi h(\varphi, \Phi)$ is increasing in φ.*

Furthermore, the equilibrium surplus share is $\eta(\hat{w}) = \alpha = 1/2$ and the expected discounted match duration

$$\mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1 - 2\left(e^{\varphi h(\varphi,\Phi)} + e^{-\varphi h(\varphi,\Phi)}\right)^{-1}}{\hat{\rho} + \delta}, \tag{23}$$

is increasing in φ and Φ and satisfies $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$.

Proposition 4 demonstrates that idiosyncratic volatility, by itself, does not affect the split of the match surplus between the worker and the firm. Such an economy is symmetric in the sense that $\mathcal{T}_{\hat{w}}(\hat{w}^*, \rho) = 0$

[6] Observe that if $\gamma = \sigma = 0$, then the smooth pasting conditions do not apply.

and $\eta(\hat{w}) = \alpha$. Thus, a larger $\hat{w}^*$ reduces the match duration by increasing the likelihood of a layoff but increases the match duration by reducing the likelihood of a quit. Weighing both forces, $\mathcal{T}(\cdot, \rho)$ is maximized at $\hat{w}^* = (1 + \hat{\rho}\hat{U})/2$ and $\eta(\hat{w}^*) = 1/2$.

This result provides a tight characterization of the worker's and the firm's optimal policy functions, which result in the continuation region of the match $(\hat{w}^-,\ \hat{w}^+)$ being symmetrically centered around the optimal entry wage $\hat{w}^*$. Second, the width of the continuation region is increasing in the volatility σ and decreasing in $\hat{\rho}\hat{U}$ (Part 1). The width of the inaction region increases with σ due to the option value effect: Although the worker's productivity might be low today, the firm is willing to wait before firing the worker in case productivity improves in the future. The width of the inaction region decreases with $\hat{\rho}\hat{U}$ because a higher opportunity cost of employment makes it less attractive to delay job separations.

The option value effect naturally arises in models of inaction. However, our model features a departure from canonical models of inaction (e.g., Dixit, 1991). In those models, the width of the continuation region typically grows unboundedly with the level of volatility σ. Instead, in our model, the width of the continuation region has an upper bound (Part 2). To see the intuition behind this result, consider the problem of a firm that finds itself in a match with negative flow profits. The marginal benefit from remaining in a currently unprofitable match is that, with some probability in the future, productivity increases enough to make the match profitable by rendering the wage-to-productivity ratio less than unity. At the same time, inaction on part of the firm is risky: productivity may increase by a large enough amount for the worker to choose to quit. Given the two job separation triggers, as the volatility goes to infinity, the probability of remaining in the profitable part of the inaction region approaches zero. Thus, the two-sided lack of commitment imposes an upper bound on the option value associated with remaining in a match with negative flow profits.

The inefficiency generated by the lack of commitment also manifests itself in the expected duration of the match given by equation (23). It is easy to see that a bounded option value effect (i.e., bounded separation thresholds), as indexed by $h(\varphi, \Phi)$, implies a lower expected duration as the volatility of productivity shocks increases (Part 3).

Dynamic Considerations II: The Anticipatory Effect. To understand the role of a nonzero productivity drift in generating the anticipatory effect, we assume away, for now, the volatility of worker productivity—i.e., $\sigma = 0$—and focus on the case with weakly positive drift—i.e., $\hat{\gamma} \geq 0$. The following proposition characterizes the anticipatory effect in this case.

Proposition 5. *Assume $\sigma = 0$ and $\hat{\gamma} \geq 0$. Then, $\hat{w}^- = \log\left(\hat{\rho}\hat{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\gamma}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right),$$

where $\tilde{T}(\cdot)$ is increasing in the first argument and decreasing in the second argument—see equation (B.36) in the Online Appendix for its definition. Moreover,

1. *If $\hat{\gamma} = 0$, then $\left(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)\right) \rightarrow \left(\log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right), \frac{1}{\hat{\rho}+\delta}, \alpha\right)$.*
2. *If $\hat{\gamma} \rightarrow \infty$, then $\tilde{T}(\cdot) \rightarrow \tilde{T}^{limit}$, $\mathcal{T}(\hat{w}^*, \hat{\rho}) \rightarrow 0$, and $\eta(\hat{w}^*) \rightarrow \eta^{limit}$, where $\tilde{T}^{limit}$ and η^{limit} are implicitly defined as*

$$\begin{aligned}\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} &= \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}}, \\ \eta^{limit} &= \alpha + \frac{1-\alpha}{\tilde{T}^{limit}} \frac{(1-\hat{\rho}\hat{U})\eta^{limit}}{\eta^{limit} + \hat{\rho}\hat{U}(1-\eta^{limit})}.\end{aligned} \tag{24}$$

When productivity grows at a constant rate, the job separation trigger $\hat{w}^-$ equals the static opportunity cost of employment since there is no value for the worker to further delay the separation. A novel mechanism is embedded in the entry wage $\hat{w}^*$ and, therefore, in the function $\tilde{T}(\cdot)$. From Proposition 5, we can see that $\hat{w}^*$ is increasing in the Nash bargaining target and also in the drift. We refer to the latter as the anticipatory effect: Workers anticipate higher future productivity and modify their search strategy accordingly. Two limiting cases illustrate this point.

As $\gamma \rightarrow 0$ (Part 1), the equilibrium entry wage $\hat{w}^*$ is the same as in the case without drift; thus, $\eta(\hat{w}^*) = \alpha$. As the drift increases, the workers partially compensate for it by searching for a job with a higher entry wage. Therefore, the average wage in the economy increases above the Nash bargaining target—recall that $\hat{w}^-$ remains fixed. This results from the worker internalizing the trade-off that a higher wage implies (i) a reduced job-finding rate and (ii) a lower frequency of inefficient job separations and, thus, a longer expected match duration. As the drift increases unboundedly (Part 2), the entry wage w^* becomes unresponsive to the drift because the job-finding rate becomes so small that it starts to dominate the trade-off. Finally, as we can see in (24), the anticipatory effect makes the worker's share of the surplus increase in the drift.

Relative to the case with no drift, the worker's lack of commitment decreases the value of searching for a job, which is captured by the null response of $\hat{w}^-$ to changes in the drift. To understand this result, assume that the worker commits to any given $\hat{w}^-$ and $\delta \rightarrow 0$. Under these assumptions, the job separation rate is given by $s = \hat{\gamma}/(w^* - \hat{w}^-)$. Thus, the worker minimizes the frequency of inefficient job separations by increasing the size of $w^* - \hat{w}^-$, which is captured by the surplus channel. At the same time, workers choose an entry wage that takes into account the trade-off captured by the share channel. For a given $\hat{w}^-$, the worker has only the choice of $\hat{w}^*$ to achieve two opposing objectives: increase $\hat{w}^*$ to avoid inefficient job separations (i.e., the surplus channel) or keep $\hat{w}^*$ close to the Nash bargaining target (i.e., the share channel). Thus, the lack of commitment distorts both the expected duration of the match and the equilibrium job-finding rates.

3 Identifying the Microeconomic Implications of Allocative Wages

This section proceeds in two steps. First, we show that the prevalence of inefficient job separations in our model critically depends on moments of the distribution of wage-to-productivity ratios $\hat{w}$. Second, we demonstrate how to use microdata on wage changes and worker transitions between jobs to recover the unobserved distribution of wage-to-productivity ratios.

Notation. Our model has a set of testable implications. First, agents' policies imply transitions from employment to unemployment at rate s, from unemployment to employment at rate $f(\hat{w}^*)$, and a level of aggregate employment $\mathcal{E}$. Second, the model predicts a joint distribution over the duration of completed employment spells τ^m, the duration of completed unemployment spells τ^u, and the log wage change between consecutive job spells Δw. We denote the joint distribution of $(\tau^m, \tau^u, \Delta w)$ with $l(\tau^m, \tau^u, \Delta w)$ and the marginal distribution of each variable with $l^m(\tau^m)$, $l^u(\tau^u)$, and $l^w(\Delta w)$. Let $\mathcal{D} := \{\mathcal{E},\ s,\ f(\hat{w}^*),\ l(\tau^m,\ \tau^u,\ \Delta w)\}$ summarize the model's observable implications in the data. Finally, we define $\tau := \tau^m + \tau^u$ as the time elapsed between the starting dates of two consecutive jobs, and we use $\mathbb{E}_{\mathcal{D}}[\cdot]$ to denote the expectation operator under the distribution $l(\tau^m,\ \tau^u,\ \Delta w) \in \mathcal{D}$.

Before proceeding, it will be useful to find the minimum model ingredients needed to characterize $\mathcal{D}$. In principle, we could characterize $\mathcal{D}$ as a function of the joint distribution of workers' employment states, wages, and productivities. In practice, given the parameters guiding the stochastic process of a worker's productivity, all that is needed to characterize $\mathcal{D}$ is the distribution of the negative sum of worker productivity shocks since the beginning of a spell of employment or unemployment. We denote this variable by Δz and refer to it as *cumulative productivity shocks*. [7] Using cumulative productivity shocks as the state variable has three advantages: (i) it is unidimensional; (ii) it is well-defined during spells of

[7] Formally, the negative of cumulative productivity shocks of worker i at time t are $\Delta z_{it} := z_{it_0} - z_{it}$, where t_0 denotes the beginning of the current spell of employment or unemployment. Note that this reflects the negative sum of productivity changes since t_0.

employment and unemployment; (iii) it follows a stationary distribution. By definition of Δz, its law of motion is given by $\mathrm{d}\Delta z = -\gamma \, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t^z$.

Let $g^h(\Delta z)$ and $g^u(\Delta z)$ be the distributions of Δz across employed and unemployed workers, respectively. The support of $g^h(\Delta z)$ is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. We denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under the distributions $g^h(\Delta z)$ and $g^u(\Delta z)$, respectively. Let $\mathcal{M} = \{g^h(\Delta z), g^u(\Delta z), \gamma, \sigma\}$ denote the set of model objects sufficient to characterize $\mathcal{D}$. Online Appendix C provides the analytical mapping from model objects $\mathcal{M}$ to $\mathcal{D}$ in the data. Here, our goal is to link $\mathcal{M}$ to the prevalence of inefficient job separations and to deduce the elements in $\mathcal{M}$ from objects $\mathcal{D}$ that are measurable in labor market microdata.

Characterizing the Equilibrium Distributions of Cumulative Productivity Shocks $g^h(\Delta z)$ and $g^u(\Delta z)$. The equilibrium policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ together with the stochastic process guiding Δz and the exogenous job separation rate, determine the equilibrium distributions of cumulative productivity shocks $g^h(\Delta z)$ and $g^u(\Delta z)$. Due to the law of motion for Δz being independent of the worker's employment state, the Kolmogorov forward equations (KFEs) for employed and unemployed workers are

$$\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+) \backslash \{0\}, \tag{25}$$

$$f(\hat{w}^*) g^u(\Delta z) = \gamma (g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \forall \Delta z \in \mathbb{R} \backslash \{0\}. \tag{26}$$

Here, δ is the exogenous exit rate of employed workers and $f(\hat{w}^*)$ the job finding rate of unemployed workers. Since the entry state for a newly employed or unemployed worker is $\Delta z = 0$, the KFEs (25)–(26) do not hold at this point, but $g^h(\cdot)$ and $g^u(\cdot)$ must be continuous there.

The boundary conditions impose a zero measure of workers at the borders of the support,

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0,$$
$$\lim_{\Delta z \to -\infty} g^u(\Delta z) = \lim_{\Delta z \to \infty} g^u(\Delta z) = 0.$$

These distributions must also be consistent with (i) a unit measure of workers, and (ii) a flow balance equation implying constant steady-state employment:

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z) \, \mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z) \, \mathrm{d}\Delta z, \tag{27}$$

$$\underbrace{f(\hat{w}^*)(1-\mathcal{E})}_{u\text{-to-}h\text{ flows}} = \underbrace{\delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]}_{h\text{-to-}u\text{ flows}}. \tag{28}$$

In equation (27), the unit measure of workers is composed of $\int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z = 1-\mathcal{E}$ unemployed and $\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z = \mathcal{E}$ employed workers. In equation (28), the mass of u-to$h$ flows is $f(\hat{w}^*)(1-\mathcal{E})$, while the mass of h-to-u flows is $\delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]$—i.e., the sum of exogenous and endogenous job separations.

To summarize, equations (25)–(28), together with continuity of $g^u(\Delta z)$ and $g^h(\Delta z)$ at $\Delta z = 0$, constitute the equilibrium conditions for the steady-state distributions of cumulative productivity shocks. Next, we show that $g^h(\Delta z)$ incorporates all the relevant information needed to quantify the prevalence of inefficient job separations in the economy.

The Distribution of Cumulative Productivity Shocks in Employment $g^h(\Delta z)$ is a Sufficient Statistic for the Prevalence of Inefficient Job Separations. In our model, the ratio of the measure of endogenous job separations s^{end} to the measure of all job separations s is given by

$$\frac{s^{end}}{s} = \frac{\frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]}{s}. \tag{29}$$

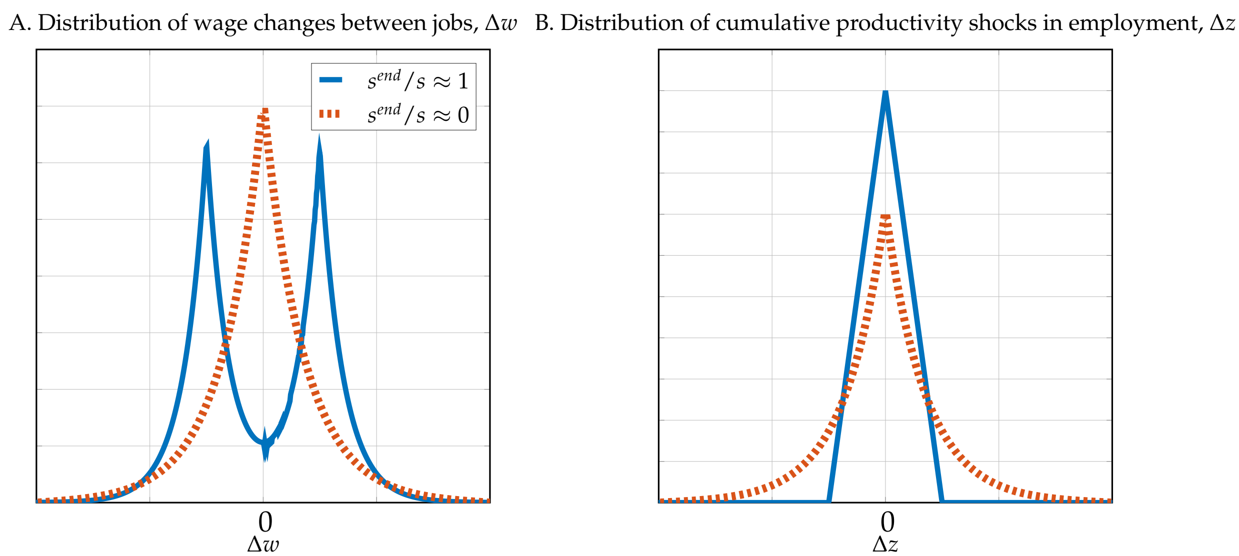
The numerator on the right-hand side of equation (29) is the share of employment resulting in endogenous job separations, which are triggered by cumulative productivity shocks hitting the boundary $-\Delta^-$ from above or the boundary Δ^+ from below. Recall that, by Proposition 2, the match surplus is always strictly positive in equilibrium, which implies that all endogenous job separations are inefficient. Therefore, this ratio summarizes the prevalence of inefficient job separations in the economy. A challenge in operationalizing equation (29) is that the distribution of cumulative productivity shocks in employment $g^h(\Delta z)$ is unobserved. Next, we show how to recover this distribution from labor market microdata.

Inferring the Distribution of Cumulative Productivity Shocks in Employment $g^h(\Delta z)$. A key insight is that, given the parameters of the stochastic process guiding worker productivity, the distribution of wage changes between jobs contains sufficient information to recover $g^h(\Delta z)$ and therefore the prevalence of endogenous job separations. We guide the discussion with the aid of Figure 2, which shows the marginal distribution of wage changes between jobs $l^w(\Delta w)$ (left panel) and the marginal distribution of cumulative productivity changes in employment $g^h(\Delta z)$ (right panel). Each panel plots the respective distribution for two extreme calibrations, one that renders almost all job separations endogenous (blue solid line) and one that renders almost all job separations exogenous (red dashed line).

If, on the one hand, most job separations are endogenous, then most separated workers experienced cumulative productivity shocks during employment of either $-\Delta^-$ or Δ^+. As a result, the probability mass associated with positive wage changes between jobs is concentrated around $-\Delta^-$, and the probability mass associated with negative wage changes is concentrated around Δ^+. This results in a bimodal distribution of wage changes between jobs, with additional dispersion around the two modes caused by cumulative productivity shocks in unemployment.

If, on the other hand, most job separations are exogenous, then most separated workers experienced cumulative productivity shocks in employment close to zero. Because the probability of finding a job is independent of the shocks experienced during unemployment, the shape of the distribution of wage changes between jobs mimics the distribution of cumulative productivity shocks in employment, being symmetric and single peaked at zero.

FIGURE 2. DISTRIBUTIONS OF WAGE CHANGES BETWEEN JOBS AND CUMULATIVE PRODUCTIVITY SHOCKS IN EMPLOYMENT



Notes: The figure plots the distribution of wage changes between jobs $l^w(\Delta w)$ and the distribution of cumulative worker shocks in employment $g^h(\Delta z)$ for two calibrations. In the first calibration, we set $(\Delta^-, \Delta^+, \gamma, \sigma, \delta, p(\hat{w}^*)) = (0.05, 0.05, 0, 0.02, 0, 0.5)$ so that $s^{end}/s \approx 1$ (blue solid line). In the second calibration, we set $(\Delta^-, \Delta^+, \gamma, \sigma, \delta, p(\hat{w}^*)) = (0.2, 0.2, 0, 0.1, 0.04, 0.05)$ so that $s^{end}/s \approx 0$ (red dashed line).

With this intuition in mind, we formalize the argument for the identification of $g^h(\Delta z)$ in three steps. First, we infer the drift γ and volatility σ of worker productivity from microdata on wage changes and worker transitions between jobs. Second, we measure the job finding rate $f(\hat{w}^*)$ and marginal distribution of wage changes between jobs $l^w(\Delta w)$ in order to deduce the CDF of cumulative productivity shocks conditional on a job separation event $\bar{G}^h(\Delta z)$. Third, we recover $g^h(\Delta z)$ along with $g^u(\Delta z)$.

Step 1: Identifying the Parameters of the Stochastic Process Guiding Worker Productivity. A challenge in recovering the drift γ and volatility σ of the stochastic process guiding worker productivity lies in the endogenous job separation of workers into unemployment. The following lemma shows how to recover γ and σ from observables $\mathcal{D}$ by use of Doob's Optional Stopping Theorem.

Lemma 3. *The drift γ and volatility σ of the stochastic process guiding cumulative productivity shocks can be recovered from $\mathcal{D}$ with*

$$\gamma = \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}, \tag{30}$$

$$\sigma^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - \gamma\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]}. \tag{31}$$

Lemma 3 provides a mapping between the drift γ and volatility σ of worker productivity and measurable labor market objects. Equation (30) states that the drift of productivity γ simply equals the mean wage change between jobs $\mathbb{E}_{\mathcal{D}}[\Delta w]$ divided by the mean time elapsed between the starting dates of two consecutive jobs $\mathbb{E}_{\mathcal{D}}[\tau]$. Equation (31) shows that the volatility of productivity σ equals the dispersion of wage changes around the expected wage change between jobs $\mathbb{E}_{\mathcal{D}}[(\Delta w - \gamma\tau)^2]$ divided by the mean time elapsed between the starting dates of two consecutive jobs $\mathbb{E}_{\mathcal{D}}[\tau]$.

Step 2: Identifying the Distribution of Cumulative Productivity Shocks Conditional on Job Transitions. Having identified (γ, σ), we next characterize the distribution of cumulative productivity shocks conditional on a job separation event $\bar{G}^h(\Delta z)$. To understand how to identify this distribution, we first turn to the dynamics of h-to-u and u-to-h worker flows. Consider a worker who at time t_0 starts a job with wage w_{t_0}, at time $t_0 + \tau^m$ separates, and at time $t_0 + \tau^m + \tau^u$ finds a new job with wage $w_{t_0+\tau^m+\tau^u}$. This worker's wage change between jobs is given by

$$\Delta w = w_{t_0+\tau^m+\tau^u} - w_{t_0}, \tag{32}$$

$$= \underbrace{w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u}}_{=\hat{w}^*} - \underbrace{(w_{t_0} - z_{t_0})}_{=\hat{w}^*} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0}}_{=\Delta z \text{ after } h\text{-}u\text{-}h \text{ transition}}, \tag{33}$$

$$= \underbrace{\hat{w}^* - \hat{w}^*}_{=0} + \underbrace{z_{t_0+\tau^m} - z_{t_0}}_{\Delta z | h\text{-}u \text{ transition starting from } z_{t_0}} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m}}_{\Delta z | u\text{-}h \text{ transition starting from } z_{t_0+\tau^m}}. \tag{34}$$

Equation (32) applies the definition of Δw. Equation (33) adds and subtracts $z_{t_0+\tau^h+\tau^u} - z_{t_0}$ before grouping terms into the wage-to-productivity ratio in the old job, the wage-to-productivity ratio in the new job, and the cumulative productivity shocks between the starting dates of the old and new jobs. Finally, equation (34) adds and subtracts $z_{t_0+\tau^m}$ before applying the definition of $\hat{w}^*$ and that of Δz. In summary, equations (32)–(34) show that the wage change across jobs is equal to the sum of three random variables: (i) the difference of entry wage-to-productivity ratios across jobs, which equals zero, (ii) Δz conditional on a job separation starting from productivity z_{t_0}, and (iii) Δz conditional on finding a new job, which is independent of the productivity z_t for $t\ in (t_0+\tau^m, t_0+\tau^m+\tau^u)$. Based on these arguments, we derive the following proposition.

Proposition 6. *The distribution of Δz conditional on a job separation is given by*

$$\bar{G}^h(\Delta z) = \frac{\sigma^2}{2f(\hat{w}^*)}\frac{\mathrm{d}l^w(-\Delta z)}{\mathrm{d}z} - \frac{\gamma}{f(\hat{w}^*)}l^w(-\Delta z) - [1 - L^w(-\Delta z)]\,, \tag{35}$$

where $L^w(\Delta w)$ denotes the cumulative distribution function (CDF) corresponding to the marginal distribution $l^w(\Delta w)$.

Step 3: Identifying the Distribution of Cumulative Productivity Shocks in Employment. Given the distribution of cumulative productivity shocks conditional on a job separation event $\bar{G}^h(\Delta z)$ and $\bar{g}^h(\Delta z)$, we can recover the steady-state cross-sectional distribution of wage-to-productivity ratios in employment.

Proposition 7. *Assume $\gamma \neq 0$. The distribution of cumulative productivity shocks $g^h(\Delta z)$ is given by*

$$g^h(\Delta z) = \frac{s\mathcal{E}}{\gamma}\left[\int_{-\Delta^-}^{\Delta z}\left(1 - e^{\frac{2\gamma}{\sigma^2}(y-\Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z+\Delta^-)}\right]\right]. \tag{36}$$

Proposition 7 provides the functional equation (36) that, when combined with equation (35), maps $l^w(\Delta w)$ into $g^h(\Delta z)$. Depending on the application, one needs to compute specific moments of the distribution $g^h(\Delta z)$. For example, the next section shows that the response of aggregate job separations after a monetary shocks depends only on average tenure—which is directly measurable in the data—and $\mathbb{E}_h[\Delta z]$. Online Appendix D.5 shows how to recover the required moments of the distribution $g^h(\Delta z)$ using moments of the observed distribution of Δw.

We conclude this section with a brief discussion of the assumptions underlying the method described in Propositions 6 and 7. The first assumption is the threshold nature of job separation policies, according to which the job separation rate is equal to δ for $\Delta z \in [-\Delta^-, \Delta^+]$ and infinite for $\Delta z \in \{-\Delta^-, \Delta^+\}$. This assumption is not crucial, and it can be replaced with a general job separation hazard as in Álvarez *et al.* (2020). The second assumption is the lack of other types of wage adjustments, such as those arising from job-to-job transitions or wage adjustments within a job spell. This assumption could be relaxed following the methodology in Baley and Blanco (2021b). Finally, while we assume a particular stochastic process for $\mathrm{d}\Delta z_t$, this assumption can be empirically tested and adjusted if deemed necessary, as in Baley and Blanco (2021a). For example, it would be straightforward to make the parameters of the productivity process depend on the worker's employment state. The critical assumption behind Propositions 6 and 7 is that we have sufficient information about $\bar{g}^u(\Delta z)$, the distribution of productivity changes during unemployment. Given our model assumptions, this is indeed the case, as the lack of selection in job finding and the pre-identified stochastic process for Δz together yield the strong identification result.

4 Analyzing the Macroeconomic Consequences of Allocative Wages

How does the interaction between productivity shocks, wage rigidity, and two-sided lack of commitment—which gives rise to inefficient job separations—matter for the transmission of monetary shocks? To answer this question, we add money as a numeraire to the economic environment.

4.1 A Monetary Economy

We modify the baseline model in four dimensions. First, we introduce preferences over real money holdings:

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(c_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) \mathrm{d}t \right], \tag{37}$$

where $\hat{M}_{it}$ denotes a worker's money holdings, P_t is the relative price of the good in terms of money, and μ is a preference weight on real money holdings.

Second, workers face a budget constraint that reflects access to complete financial markets and ownership of firms' profits. Given a history of labor market decisions regarding job search, acceptance, and dissolution, $lm_i^t := \{lm_{it'}\}_{t'=0}^{t}$, a worker's private income is $y(lm_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and profits of a fully diversified portfolio claims on the individual firms. On the spending side, a worker pays for consumption expenditures $P_t c_{it}$ and the opportunity cost of holding money $i_t \hat{M}_{it}$ at a given interest rate $i_t \geq 0$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t \left(P_t c_{it} + i_t \hat{M}_{it} - y(lm_i^t) - T_{it} \right) \mathrm{d}t \right] \leq M_{i0}. \tag{38}$$

The worker's problem is to choose a consumption stream $\{c_t\}_{t=0}^{\infty}$, labor market decisions $\{lm_t\}_{t=0}^{\infty}$, and money holdings $\{\hat{M}_{it}\}_{t=0}^{\infty}$ to maximize utility (37) subject to the budget constraint (38) at time 0.

Third, the economy is subject to shocks to the aggregate money supply M_t. We assume that the log of the aggregate money supply m_t follows a Brownian motion with drift π and volatility ζ:

$$\mathrm{d}m_t = \pi \, \mathrm{d}t + \zeta \, \mathrm{d}\mathcal{W}_t^m,$$

where $\mathcal{W}_t^m$ is a Wiener process. Because the aggregate money supply moves stochastically over time, fluctuations in m_t constitute aggregate shocks to the economy.

Fourth and finally, we assume that the vacancy posting cost $K(Z_t)$ and the value of home production $B(Z_T)$ are both denominated in real terms.

Given these modifications, the market-clearing conditions for goods and money, respectively, are

$$\int_{i=0}^{1} c_{it} + \theta_{it} \mathbb{1}[E_{it} = u] K(Z_{it}) \, \mathrm{d}i = \int_{i=0}^{1} Z_{it} \mathbb{1}[E_{it} = h] + B(Z_{it}) \mathbb{1}[E_{it} = u] \, \mathrm{d}i, \tag{39}$$

$$\int_{i=0}^{1} \hat{M}_{it} \, \mathrm{d}i = M_t, \tag{40}$$

where $\mathbb{1}[\cdot]$ is an indicator function that takes a logical expression as its argument. Equation (39) states that the sum of real consumption and recruiting expenses must equal the total market and home production of the good. Equation (40) states that the total demand of nominal money holdings across workers equals the aggregate money supply.

The following proposition characterizes the worker's problem in this monetary economy.

Proposition 8. *Let $Q_0 = 1$ be the numéraire and assume $\mu = \rho + \pi - \zeta^2/2$. Then, $P_t = M_t$ and the value of a worker at time 0 is*

$$V_0 = \max_{\{lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{y(lm_i^t)}{P_t} \, \mathrm{d}t \right] + k,$$

where k is a constant independent of the worker's choices and the present discounted value of financial wealth.

Proposition 8 shows that the price level is equal to the aggregate money supply and that maximizing (37) subject to (38) is equivalent to maximizing expected discounted real income. The result relies on the following assumptions: (i) markets are complete, (ii) workers have quasi-linear preferences in consumption, and (iii) the log of aggregate money supply follows a random walk with drift. The first two assumptions imply a constant marginal value of nominal wealth, which combined with the last assumption leads to a constant real interest rate and a one-for-one pass-through of money shocks to inflation.

The introduction of a monetary economy requires minor adjustments to our previous solution approach. Given fluctuations in the log price level p, the relevant state variable becomes the *real wage-to-productivity ratio* $\hat{w} := w - z - p$. Similarly, we keep track of the negative of a worker's cumulative shocks to *revenue productivity* $z + p$ since the beginning of the current employment or unemployment spell, which we denote by Δz.[8] By definition, $\hat{w} = \hat{w}^* + \Delta z$ and the law of motion for Δz is

$$\Delta z = -(\gamma + \pi)\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z + \xi\,\mathrm{d}\mathcal{W}_t^m.$$

All policies $(\hat{w}^+, \hat{w}^*, \hat{w}^-)$ are expressed in real terms. Since productivity growth γ and trend inflation π symmetrically affect revenue productivity, without loss of generality, we set $\pi = 0$. Finally, let $G_h(z, a)$ denote the steady-state joint distribution of cumulative revenue productivity shocks z and tenure a of a job spell. For any integers $k, l \in \mathbb{N}$, we define the moments of this distribution as

$$\mathbb{E}_h(\Delta z^k a^l) \equiv \int_{\Delta z} \int_a \Delta z^k a^l \,\mathrm{d}G^h(\Delta z, a).$$

4.2 Monetary Multipliers for Aggregate Employment and Real Wages

Starting from the steady state without aggregate shocks, so that $\zeta = 0$, we consider a small, unanticipated, one-standard-deviation shock $\zeta > 0$ to aggregate money supply at time $t = 0$—i.e., $\log(M_0) = \lim_{t \uparrow 0} \log(M_t) + \zeta$. The shock leads to a one-for-one increase in the price level. We are interested in the economy's *impulse response function (IRF)* and *cumulative impulse response (CIR)* of aggregate employment and real wages to such a monetary shock.[9]

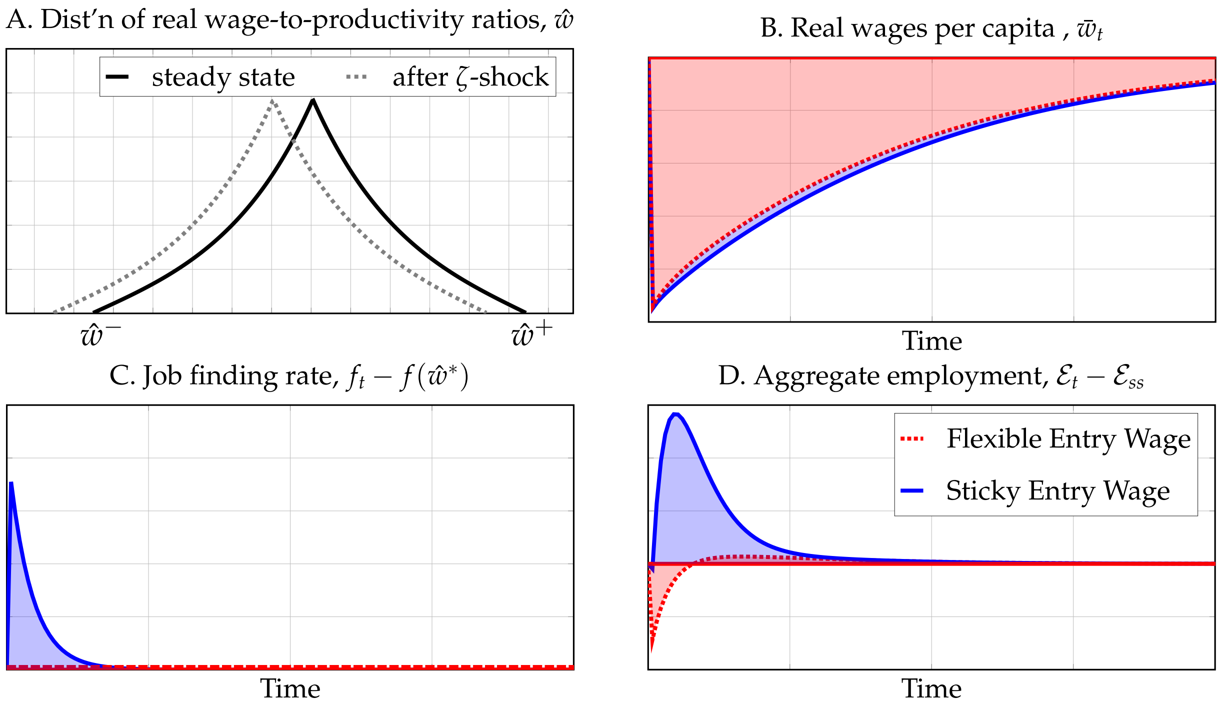
An Illustration. Figure 3 shows the distribution of real wage-to-productivity ratios $\hat{w}$ before and after the monetary shock together with the IRFs of average real per capita wages $\bar{w}_t := \int_0^1 (\mathbb{1}[E_{it} = h] w_{it} - \mathbb{E}_h[\hat{w}^* + \Delta z])\,\mathrm{d}i$, the job-finding rate $f_t - f_{ss}$, and aggregate employment $\mathcal{E}_t - \mathcal{E}_{ss}$.

After the initial increase in the price level, the distribution of real wage-to-productivity ratios shifts to the left (Panel A), leading to a sudden decrease in the aggregate log real wage (Panel B). Consequently, the monetary shock affects both the endogenous job separation rate and aggregate employment (Panel D). Given that the wages of new matches are critical for the job-finding rate (Pissarides, 2009), we study two separate cases: flexible entry wages and sticky entry wages. With flexible entry wages, we assume that unemployed workers fully adjust their search behavior to incorporate the higher price level, so that

[8] We choose this notation to avoid defining new objects. Below, we set $\pi = 0$; thus, steady-state moments of cumulative productivity shocks are equal to the corresponding moments of revenue productivity.

[9] By the certainty equivalence principle, the impulse response function following a money shock departing from the steady-state with steady-state policies is equivalent to the solution based on a first-order perturbation of the model with business cycles fluctuations.

FIGURE 3. IMPULSE RESPONSE FUNCTIONS OF LABOR MARKET VARIABLES



Notes: Panel A shows the distribution of real wage-to-productivity ratios $\hat{w} := \log(W_{it}/(Z_{it}P_t))$ in the steady state and after a monetary shock of size ζ. Panels B, C, and D show the impulse response functions of the average log real per capita wage $\bar{w}_t$, the job-finding rate $f_t - f_{ss}$, and aggregate employment $\mathcal{E}_t - \mathcal{E}_{ss}$, respectively. We use the following illustrative parameter values: $(\gamma, \pi, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0, 0.001, 0.007, 0.03, 0.5, 1, 0.005, 0.4)$.

$\hat{w}^*$ remains at its steady-state level. Thus, the real entry wage and the job-finding probability remain constant (Panel C). The aggregate log real wage is affected by the shock only because the nominal wages of workers already employed at $t = 0$ are rigid. Since entry wages adjust one-for-one with the price level, the firm's real value of a filled vacancy is unaffected, so that both vacancy-filling and job-finding rates remain at their steady-state levels. Therefore, the employment effects are only driven by the effects of the aggregate shock on endogenous job separations.

In the sticky entry wage case, we assume that unemployed workers do not adjust their search behavior to incorporate the higher price level. In this case, the real entry wage reverts to its steady-state level following the worker's first job separation after the shock. Thus, after the shock, the real entry wage also decreases, which induces firms to post more vacancies and the job-finding rate to increase (Panel C). As a consequence, employment dynamics are driven by both the job-separation and job-finding rates. The assumption of sticky entry wages is motivated by the empirical evidence in Grigsby *et al.* (2021), which documents that new hire wages evolve similarly to incumbent workers within a firm at business cycle frequencies, and Hazell and Taska (2020), which shows that wages for new hires rarely change between successive vacancies at the same job. Micro-founding this assumption is outside the scope of this paper. Nevertheless, observe that since the steady-state entry wage is optimal, any perturbation around that level has a second-order welfare effect on the worker. Thus, any first-order cost of wage adjustment would replace this assumption as a result. There is abundant literature that provides a plethora of alternative models to think about imperfect knowledge about aggregate shocks—we chose a simple one to focus on our contribution.[10] For alternative models of rigid entry wages, see Fukui (2020) and Menzio (2022).

Defining IRFs and CIRs. Our goal is to characterize the effects of a monetary shock on aggregate employment $\mathcal{E}$ and aggregate real wages $\bar{w}$. To this end, we denote by $IRF_x(\zeta, t)$ the IRF for variable $x \in \{\mathcal{E}, \bar{w}\}$ at time t relative to its steady-state value, following a monetary shock ζ at time 0. The IRF for aggregate employment is

$$IRF_{\mathcal{E}}(\zeta, t) = \mathcal{E}_t - \mathcal{E}_{ss},$$

where $\mathcal{E}_{ss}$ is the steady-state employment rate. Analogously, the IRF for aggregate real wages is

$$IRF_{\bar{w}}(\zeta, t) = \int_{\hat{w}^-}^{\hat{w}^+} \hat{w}[\mathrm{d}G_t(\hat{w}) - \mathrm{d}G_{ss}(\hat{w})], \tag{41}$$

where $G_t(\hat{w})$ is the CDF of real log wage-to-productivity ratios at time t and $G_{ss}(\hat{w})$ is its steady-state counterpart. It is worth noting that equation (41) implicitly makes use of the fact that the IRF of the mean log real wage $\bar{w}$ is identical to that of the mean log real wage-to-productivity ratio $\hat{w}$, given that the process governing a worker's productivity is unresponsive to the monetary shock.

Following Álvarez *et al.* (2016), we define the CIR of a variable $x \in \{\mathcal{E}, \bar{w}\}$ to a monetary shock ζ as

$$CIR_x(\zeta) = \int_0^{\infty} IRF_x(\zeta, t)\, \mathrm{d}t,$$

which measures the area under the $IRF_x(\zeta, t)$ curve for $t \in [0, \infty)$. The CIR summarizes the response on impact and the persistence of the response of the labor market to the monetary shock in a single scalar. Therefore, the CIR can be interpreted as a *monetary multiplier*. To illustrate the logic behind the CIR, suppose that there are no nominal rigidities so that nominal wages of both newly hired and incumbent workers respond one-for-one to the price level. In this case, $IRF_x(\zeta, t) = 0$ for all t and thus $CIR_x(\zeta) = 0$ for $x \in \{\mathcal{E}, \bar{w}\}$, reflecting the fact that there are no real consequences of inflation. With nominal rigidities,

[10] A few examples are sticky information in Mankiw and Reis (2002), rational inattention in Woodford (2009) and Maćkowiak and Wiederholt (2009), dispersed knowledge in Hellwig *et al.* (2014), level-*k* reasoning in Farhi and Werning (2017), among many others.

an inflationary shock affects both employment and wages, the magnitude of which is reflected in the CIR.

Characterizing the CIR of employment. Now, we characterize the CIR of aggregate employment. The first proposition relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. The idea behind the proof is to exchange the order of integration; we first integrate over time for a given worker and then integrate across workers.

Proposition 9. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\Delta z), g^u(\Delta z))$, the CIR is given by*

$$CIR_{\mathcal{E}}(\zeta) = \underbrace{\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z}_{CIR_{\mathcal{E}} \text{ of initially employed workers}} + \underbrace{\int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z}_{CIR_{\mathcal{E}} \text{ ofinitially unemployed workers}},$$

where the value functions $m_{\mathcal{E},h}(\Delta z)$ and $m_{\mathcal{E},u}(\Delta z, \zeta)$ are defined as:

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m} (1 - \mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},u}(\Delta z, 0) \middle| \Delta z_0 = \Delta z \right], \tag{42}$$

$$m_{\mathcal{E},u}(\Delta z, \zeta) = \mathbb{E}\left[\int_0^{\tau^u(\zeta)} (-\mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},h}(-\zeta) \middle| \Delta z_0 = \Delta z \right]. \tag{43}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z)\, \mathrm{d}\Delta z. \tag{44}$$

with $\tau^u(\zeta)$ being distributed according to a Poisson process with arrival rate $f(\hat{w}^* - \zeta)$.

The proposition shows that to characterize the CIR, we need to keep track of future employment dynamics of initially employed and unemployed workers. When the shock arrives, the real wages of initially employed workers decrease (since $\Delta z_0 = \lim_{t \uparrow 0} \Delta z_t - \zeta$, we have that the starting point is given by the distribution $g^h(\Delta z + \zeta)$) affecting their future employment spells, which is captured by $m_{\mathcal{E},h}(\Delta z)$. During employment, the employed worker's value function accumulates positive deviations from the steady-state level $(1 - \mathcal{E}_{ss})$, and the unemployed worker's value function accumulates negative deviations from the steady-state level $(-\mathcal{E}_{ss})$. Thus, $m_{\mathcal{E},h}(\Delta z)$ measures the cumulative deviations of employment from its steady-state level conditional on being initially employed at revenue productivity Δz. Similarly, $m_{\mathcal{E},u}(\Delta z, \zeta)$ measures the cumulative employment deviations from the steady-state level conditional on being initially unemployed at revenue productivity Δz and search for a job in submarket $\hat{w}^* - \zeta$. Searching in submarket $\hat{w}^* - \zeta$ increases the job-finding probability $f(\hat{w}^* - \zeta)$ and also changes the wages of new hires. Finally, since the Bellman equations (42) and (43) lack discounting (i.e., they simply count non-discounted deviations), it is easy to show that they have infinitely many solutions. The unique relevant solution is pinned down by equation (44), which requires that an economy that departs from the steady-state and experiences no shock must have a null $\text{CIR}_{\mathcal{E}}$—i.e., $\text{CIR}_{\mathcal{E}}(0) = 0$.

Next, we characterize up to first order the CIR of aggregate employment as a set of measurable objects in labor market microdata. Specifically, we argue that certain moments of the joint distribution of tenure and wages in steady-state are informative of the CIR. The key insight below is that the CIR of aggregate employment, which is a summary statistic of its dynamic response, can be characterized only in terms of steady-state cross-sectional moments. The intuition behind this result is that changes in a worker's idiosyncratic productivity and changes in the aggregate price level affect the real wage-to-productivity ratio $W_{it}/(Z_{it}P_t)$ of a match in symmetric ways. Therefore, the response of a match to productivity changes in the steady-state is informative of the aggregate effects of changes in the price level.

CIR of employment with flexible entry wages. To facilitate the exposition of the analysis, we first present the case with flexible entry wages. Proposition 10 characterizes the CIR up to first order.[11]

Proposition 10. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta). \tag{45}$$

The conventional wisdom in macroeconomics is that fluctuations in the job separation rate are not the main driver of aggregate employment dynamics (e.g., Shimer, 2005b). In the context of a monetary shock, equation (45) points to conditions under which aggregate employment fluctuations due to endogenous job separations can be small. More importantly, it also highlights the conditions for these effects to be large. This new result, combined with Corollary 1 below, provides a guide to verify those conditions in the data.

In light of this conventional wisdom, one might also be tempted to conclude that sticky wages cannot lead to inefficiencies at the micro-level. However, equation (45) allows for a small CIR of aggregate employment to a monetary shock *despite* the presence of allocative wages and inefficient job separations. Thus, aggregate time-series data on job flows cannot be used as model discrimination devices between theories of the allocativeness of wages or in assessing the prevalence of inefficient job separations at the micro-level.

To build the intuition behind this result, we first consider the implications of equation (45) under zero productivity drift, $\gamma = 0$. In two cases do the job separation rate and aggregate employment not respond to aggregate shocks. In the first case, all job separations are exogenous and, therefore, the IRF of the job separation rate is zero. In the second case, all job separations are endogenous but the mass of

[11] That is, $CIR_x(\zeta) = CIR_x(0) + (CIR_x)'(0)\zeta + o(\zeta^2)$, where $CIR_x(0) = 0$.

additional worker quits due to lower real wages is exactly compensated by the mass of workers who would have been fired by firms in the absence of the monetary shock. In both cases, the key sufficient statistic referenced by equation (45) is $\mathbb{E}_h[\Delta z] = 0$.

Importantly, the sufficient statistic in equation (45) captures more than the on-impact response of endogenous quits and layoffs. Rather, it measures the response at all times along the IRF. Therefore, the relative mass of workers near the two job separation triggers is not a sufficient statistic for characterizing the CIR of aggregate employment. To illustrate this, consider the following example in which all endogenous separations are quits, but nevertheless, the CIR is zero. Suppose $\gamma > 0$ and $\sigma \downarrow 0$ in this environment. The equilibrium approaches a situation in which workers quit because wages lag behind productivity growth, and firms do not have incentives to fire any worker. Then, $\Delta z_{it} + \gamma a_{it} = \sigma \mathcal{W}^z_t \to 0$ and, therefore, $\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z] = 0$. Intuitively, the increase in worker quits on impact is exactly offset by a reduction in future worker quits, resulting in a null net effect as captured by the CIR.

The sufficient statistic in equation (45) also points to scenarios in which inefficient job separations matter for aggregate employment dynamics. For example, if trend inflation π is large in magnitude, then—all else equal—the rate of inefficient job separations will be more responsive to an inflationary shock. Alternatively, following an unexpected sequence of negative productivity shocks, an inflationary shock reduces the incidence of inefficient job separations due to firings.[12] Furthermore, if it is easy for workers to quit but costly for firms to fire workers or vice versa—for example, due to the presence of mandatory severance pay or unemployment insurance programs—then inflationary shocks can interact with such asymmetries in job separation policies leading to inefficient job separations.

Finally, notice that the CIR is scaled by the steady-state unemployment rate, $1 - \mathcal{E}_{ss}$. This is because the steady-state unemployment rate is informative of workers' steady-state job finding rate $f(\hat{w}^*)$. When this rate is high, relative to the separation rate, then a monetary shock causes temporary unemployment fluctuations but those workers quickly become matched again with new firms. Consequently, aggregate employment remains relatively stable, resulting in a relatively small CIR of aggregate employment to an inflationary shock.

Next, we leverage the mapping from the data to the model provided in Section 3 to express the response of aggregate employment to an inflation shock in terms of observable moments of the distribution of wage changes and tenure. Here, we focus and explain the case with no drift, i.e., $\gamma = 0$. See Online Appendix Section E.4 for the case with non-zero drift, i.e., $\gamma \neq 0$. While the required moments to measure the CIR are different in the case with non-zero drift, they reflect similar economic mechanisms.

[12] See also Blanco *et al.* (2022b) for empirical evidence consistent with this theoretical result.

Corollary 1. *Assume $\gamma = 0$. Up to first order, the $CIR_{\mathcal{E}}(\zeta)$ can be expressed in terms of data moments as follows:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{w}^*)}}_{\textit{avg. u dur.}} \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{\textit{dispersion}} \left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}\right]}_{\textit{asymmetries}} \right] + o(\zeta). \tag{46}$$

Equation (46) shows that, for zero drift, the effect of an inflationary shock on aggregate employment is determined by three statistics: (i) the average duration of unemployment spells, (ii) the inverse of the dispersion of wage changes, and (iii) a measure of the asymmetry of the wage change distribution. Each statistic in turn determines the persistence, initial absolute size, and sign of the effect.

The steady-state average duration of unemployment spells naturally amplifies the CIR as it captures how quickly a separated worker recovers from unemployment. Larger unemployment duration is indicative of larger search frictions, which makes the on-impact effect on employment more persistent. A similar result has been found in price-setting models (Álvarez *et al.*, 2016) and investment models with inaction (Baley and Blanco, 2021a).

In an environment with zero drift, a larger dispersion of wage changes is indicative of a wider inaction region and the presence of more resilient matches to idiosyncratic shocks. A large dispersion arises when the pool of workers experiencing wage changes not only includes previously endogenously separated workers (with large but similar absolute wage changes) but also many exogenously separated ones (with smaller but more dispersed absolute wage changes). Thus, the larger this dispersion, the smaller the share of endogenous separations and the smaller the propagation of shocks.

Finally, the CIR is also affected by the degree of asymmetry of the distribution of wage changes, as captured by the last term in brackets in (46), which is a weighted average of wage changes that puts more weight on larger changes. When the drift is zero, this term captures how asymmetric the policies $\hat{w}^-$ and $\hat{w}^+$ are around the entry wage $\hat{w}^*$. While the previous two statistics capture the degree of amplification of the monetary shock, the asymmetry of the distribution will determine the direction of the effect. A negatively skewed distribution has a longer left tail and the mass concentrated on the right, which reflects a larger fraction of workers quitting to obtain a wage increase relative to the number of workers experiencing a wage cut due to layoffs. Thus, the increase in the price level and the fall in real wages make a large mass of workers quit and the CIR is negative. The opposite holds when the distribution of wage changes is positively skewed.[13] Instead, a symmetric distribution of wages chances is indicative of

[13]When the distribution is positively skewed (i.e., with a longer right tail and the mass concentrated on the left), there is a large mass of workers experiencing wage cuts, which signals a relatively high layoff risk. Thus, higher inflation reduces real wages and increases the firms' incentives to keep their workers; as a result, aggregate employment increases.

symmetric policies when the drift is zero, and monetary shocks do not affect employment.

CIR of employment with sticky entry wages. With the understanding of employment dynamics when entry wages are flexible, we now characterize the case with sticky entry wages.

Proposition 11. *Assume sticky entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\left[-\frac{[\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1}{f(\hat{w}^*)+s}\left[\underbrace{\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}}_{\text{job finding effect on } \mathcal{E}} - \underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}}_{\text{separation effect on } \mathcal{E}}\right]\right] + o(\zeta).$$

Proposition 11 characterizes the new mechanisms affecting employment dynamics when entry wages are sticky. The elasticity of the worker's share of the surplus with respect to the entry wage (i.e., $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$) together with the elasticity of the expected discounted duration of the match (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$) captures the effect of the increase in the job-finding probability following the decrease in the real entry wage. A drop in the real entry wage increases the firm's share and their incentive to post vacancies. On top of this standard mechanism in search and matching models, a drop in the real entry wage could also change the expected duration and, therefore, the total surplus of the match. This new effect also shapes firms' incentives to post vacancies for a given share. These first two mechanisms affect aggregate employment through changes in the job-finding rate. The last term, which is also new, captures the effect of a lower real entry wage on aggregate employment that arises from fluctuations in the job separation rate of initially unemployed workers.

We now characterize the elasticity of the discounted duration of the match to an increase in the entry wage by focusing on two dimensions: (i) how the equilibrium policies determine this elasticity and (ii) how we can discipline this elasticity with data on $g^h(\Delta z)$.

Proposition 12. *The following properties hold for* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$.

1. *Assume that* $\Delta^- = \Delta^+$ *and* $\gamma = 0$. *Then,* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$ *and, up to a 3rd order approximation of* $\mathcal{T}(\hat{w},\hat{\rho})$ *around* $\hat{w} = \hat{w}^*$,

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1}{\hat{\rho}+\delta+(\sigma/\Delta^+)^2}.$$

2. *Up to a 2nd order approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ around $\hat{w}=\hat{w}^*$,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}=\frac{\Delta^+-\Delta^-}{\Delta^+\Delta^-}.$$

3. *If $\hat{\rho}=0$, then*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}=\frac{1}{\sigma^2 g^h(0)}\left[s^{end}\left(\mathcal{E}_{ss}-2G^h(0)\right)+\frac{\sigma^2}{2}\left(\lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z)+\lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z)\right)\right]. \tag{47}$$

4. *If $\hat{\rho}>0$, then*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}=\frac{\mathcal{T}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2}+\frac{\sigma^2}{4}\left[\lim_{\Delta z\downarrow\Delta^-}-\lim_{\Delta z\uparrow\Delta^+}\right]\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}\right]+o(\hat{\rho}^2),$$

with

$$\lim_{\Delta z\downarrow\Delta^-}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}=\lim_{\Delta z\downarrow-\Delta^-}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}H^-(g^h,\hat{\rho},\mathcal{T}(\hat{w}^*,\hat{\rho}),\mathcal{T}(\hat{w}^*,0))$$
$$\lim_{\Delta z\uparrow\Delta^+}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}=\lim_{\Delta z\uparrow\Delta^+}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}H^+(g^h,\hat{\rho},\mathcal{T}(\hat{w}^*,\hat{\rho}),\mathcal{T}(\hat{w}^*,0))$$

where $H^-(\cdot)$ and $H^+(\cdot)$ are described equation (E.41) *of the Online Appendix.*

Items 1 and 2 of Proposition 12 characterize an approximation of the elasticity of the discounted duration with respect to the entry wage as a function of the separation triggers $(-\Delta^-,\Delta^+)$ and model parameters. The proof is based on a Taylor approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ around $\hat{w}^*$ and the HJB equation and border conditions that characterize $\mathcal{T}(\hat{w},\hat{\rho})$. In symmetric economies—i.e., zero drift and symmetric separation triggers—the elasticity of the expected duration with respect to the entry wage is zero. Intuitively, an increase in the entry wage lowers the probability of a quit but increases the probability of a layoff in a similar proportion. Surprisingly, in asymmetric economies, the only mechanism affecting the elasticity of expected duration is the asymmetry of the separation triggers; thus, it is independent of $(\rho,\delta,\gamma,\sigma)$ conditional on Δ^- and Δ^+. For example, a higher discount factor $\hat{\rho}$ decreases the expected discounted duration and, at the same time, it decreases the effect of the initial wage $\hat{w}^*$ on the expected duration. Thus, the ratio $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$ is independent of $\hat{\rho}$.

While items 1 and 2 show the elasticity of the expected duration with respect to the entry wage and the mechanisms that shape it, items 3 and 4 show how to discipline these mechanisms with information about the distribution of Δz for the cases with $\hat{\rho} = 0$ and $\hat{\rho} > 0$. Mechanically, the steady-state distribution of Δz is proportional to the time workers spend at productivity Δz; thus, up to a normalization, $\mathcal{T}(\hat{w}^*, 0)$ could be obtained from $g^h(\Delta z)$. Equation (47) shows the two conditions that generate a positive effect on duration. If the quit rate ($\frac{\sigma^2}{2} \lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z)$) is larger than the firing rate ($-\frac{\sigma^2}{2} \lim_{\Delta z \downarrow \Delta^+} (g^h)'(\Delta z)$), then a higher entry wage increases expected match duration; if these rates are equal, then this term is zero. Similarly, the product between endogenous separations and the CDF of Δz evaluated at 0 also determines the duration elasticity since it measures the asymmetries of the distribution within the separation triggers. While these two statistics measure the effect of asymmetries in the distribution of Δz among employed workers, its effect on expected duration at the entry wage is obtained by re-scaling the statistic by $1/(\sigma^2 g^h(0))$.

Finally, when $\hat{\rho} > 0$, the elasticity of expected duration with respect to the entry wage depends on: (i) the product between $\hat{\rho}$ and the negative of the CIR with flexible entry wages and, (ii) the product between the curvature of the expected discounted duration and the mass of workers at the separation triggers. The main reason the CIR shows up in the elasticity of discounted duration is that a marginal increase in the entry wage has a similar effect on quits and layoffs, and therefore on marginal duration, as a lower price level (once the effect of discounting is properly accounted for).

To finish understanding the role of allocative wages for aggregate fluctuations, we characterize their effect on the elasticity of a worker's share of the surplus. The worker's share is given by the ratio of two Bellman equations—one for the worker's value and the other for the surplus of the match. The first step is to show that their ratio also satisfies a Bellman equation, once properly adjusted due to the endogenous duration of the match. Proposition 13 shows this result.

Proposition 13. *Define*

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $\eta(\hat{w})$ satisfies the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t} - \hat{\rho}\hat{U}}{1 - \hat{\rho}\hat{U}}\, \mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}} \mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w}\right] \tag{48}$$

with

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t, \hat{\rho}) + \sigma^2 \mathcal{T}'_{\hat{w}}(\Gamma_t, \hat{\rho}))\, \mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t, \hat{\rho})(\hat{\rho}+\delta)}\, \mathrm{d}\mathcal{W}^z_t$$

Before discussing this proposition, some properties are important to mention.[14] First, observe that $\lim_{\hat{w}\downarrow\hat{w}^-}\eta(\hat{w}) = 0$ and $\lim_{\hat{w}\uparrow\hat{w}^+}\eta(\hat{w}) = 1$. In other words, if the wage is close to the quitting trigger, then the worker's share of the surplus is zero; if the wage is close to the firing trigger, then the worker's share is one. Second, the flow payoff $(\hat{\rho}+\delta)\frac{e^{\Gamma_t}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}$ is equal to the ratio between the flow value of being employed at the *option value-adjusted wage* Γ_t and the annuity value of the surplus when all separations are exogenous. Therefore, Γ_t and its law of motion encode both the share channel and the surplus channel of a higher $\hat{w}$ on the worker's share $\eta(\hat{w})$.

To understand the new state variable Γ_t, suppose that the inaction region is infinitely large so that all separations are exogenous. Then, $\mathcal{T}(\Gamma_t,\hat{\rho}) = 1/(\hat{\rho}+\delta)$ and $\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}) = 0$. Because the law of motion of Γ_t simplifies to $\mathrm{d}\Gamma_t = \mathrm{d}\hat{w}_t = -\gamma\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, we have that $\Gamma_t = \hat{w}_t$ and $\eta(\hat{w})$ is the average share of the flow surplus. In the other extreme scenario, when the inaction region is bounded and $\Gamma_0 = \hat{w}$ is sufficiently close to a separation trigger, then the law of motion for Γ_t incorporates the net effect of a higher $\hat{w}$: It redistributes part of the surplus to the worker, but also changes the expected duration of the match. To illustrate the relative strength of these forces, suppose that $\Gamma_0 = \hat{w}$ is very close to the upper Ss band $\hat{w}^+$. Then, $\mathcal{T}(\Gamma_t,\hat{\rho}) \approx 0$ and $\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}) < 0$—the expected duration is small and a higher wage further reduces it. Therefore, $\mathrm{d}\Gamma_t < 0$ and the marginal increase in the worker's share is *decreasing* in $\hat{w}$—i.e., the share is concave in $\hat{w}$ because a higher $\hat{w}$ increases the layoff risk. Alternatively, suppose that $\Gamma_0 = \hat{w}$ is very close to the lower Ss band $\hat{w}^-$. Then, $\mathcal{T}(\Gamma_t,\hat{\rho}) \approx 0$, but now $\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}) > 0$ and $\mathrm{d}\Gamma_t > 0$—i.e., the share is convex in $\hat{w}$ because a higher $\hat{w}$ reduces the likelihood of quits.

From Proposition 13, we can characterize the elasticity of the share in symmetric economies.

Proposition 14. *The following properties hold:*

1. *If* $(\hat{w}^-,\hat{w}) \to (-\infty,\infty)$, *then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}. \tag{49}$$

2. *Assume that* $\gamma = 0$, $\Delta^+ = \Delta^-$, *and* Δ^+ *is small enough, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{1}{\alpha(\Delta^+ + \Delta^-)} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}. \tag{50}$$

We explain Proposition 14 with the help of Figure 4, which is computed in two steps. First, we set $\delta = 0$ and calibrate the model to match the average job-finding and separation rates in the US economy, together

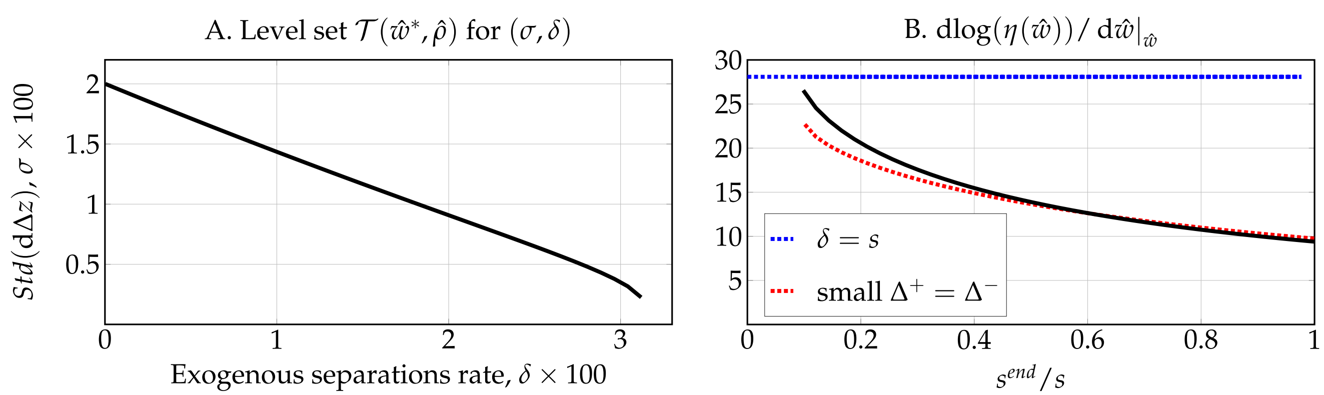
[14] For the Bellman equation (48) to describe the worker's share, $(\hat{w}^-,\hat{w}^+)$ must be a Nash equilibrium of the game between the firm and the worker. This guarantees that (48) properly characterizes a share $\eta(\hat{w}) \in [0,1]$.

with a replacement ratio for new employed workers of 0.46. We choose α such that $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$. Second, we compute the function $\sigma(\delta)$ that keeps $\mathcal{T}(\hat{w}^*, \hat{\rho})$, and therefore the aggregate separation rate, constant. What the function $\sigma(\delta)$ does is to keep, by construction, the opportunity cost $\hat{\rho}\hat{U}$ and the duration of the match constant, but change the share of endogenous separations from 0 to 1. Figure 4-Panels A and B show $\sigma(\delta)$ and $\mathrm{d}\log(\eta(\hat{w}))/\,\mathrm{d}\hat{w}|_{\hat{w}=\hat{w}^*}$, respectively.

As a starting point, assume the case with $\delta = s^{data}$ and $s^{end}/s = 0$. Equation (49) characterize this limit. In this case, all separations are exogenous, and a marginal increase in the entry wage increases the worker's share since wages during the match are higher. Equation (49) also shows the well-known result that, in this limiting case, the elasticity of the share is proportional to the inverse of flow surplus $1 - \hat{\rho}\tilde{U}$ (Shimer, 2005a).

When the share of inefficient separations increases, the elasticity of the worker's share to the entry wage decreases. In this case, a new mechanism that reduces the elasticity arises. With a higher entry wage, the probability that the worker gets fired increases, and the probability that the worker chooses to quit decreases. By construction, the expected duration of the match does not change; thus, the match's joint surplus—i.e., the denominator in the worker's share—does not change. In addition, by the envelope condition, the change in the probability of quitting does not affect the worker's value. Nevertheless, up to a first-order approximation, the increase in the probability of being laid off reduces the worker's value since she did not choose this separation trigger. This mechanism reduces the elasticity of the worker's share to the entry wage whenever the ratio of endogenous to total separation increases. In the limit, equation (50) disciplines this elasticity as a function of observables.

FIGURE 4. $\mathrm{d}\log(\eta(\hat{w}))/\,\mathrm{d}\hat{w}|_{\hat{w}=\hat{w}^*}$ FOR DIFFERENT (δ, σ) AND CONSTANT $\mathcal{T}(\hat{w}^*, \hat{\rho})$



Notes: Panel A shows the level set of $\mathcal{T}(\hat{w}^*, \hat{\rho})$ for different values of (δ, σ). Panels B shows the elasticity of the worker's share with the entry wage and two theoretical limits when $\delta = s$ and $\delta = 0$, respectively. The parameter values for $\delta = 0$ are $(\gamma, \pi, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0, 0, 0.02, 0.0033, 0.45, 2.2, 0, 0.45)$. The steady-state targets for this calibration are: $(f(\hat{w}^*), s) = (0.45, 0.032)$ with a replacement ratio of 0.46 and $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$.

Discussion. We relegate several additional results to the Online Appendix F. There, we present the CIR for large shocks by characterizing the CIR of employment up to a second-order approximation. We also characterize the CIR for average real per capita wages when entry wages are flexible.

5 Conclusion

There is mounting empirical evidence of wages being less than fully flexible. To understand the consequences of wage rigidity at the micro and macro levels, we developed a theory of labor markets with allocative wages. The realistic ingredients of this theory included fluctuations in individual output (i.e., productivity shocks), fixed pay within jobs (i.e., wage rigidity), and the possibility that workers can quit and firms can dissolve jobs at any point in time (i.e., two-sided lack of commitment). Our theory embedded these ingredients in an environment with search frictions, which are central to many macroeconomic analyses of labor markets.

We demonstrated that this theory is useful because it enables us to study the prevalence of inefficient job separations by first identifying the microeconomic implications and then analyzing the macroeconomic consequences of allocative wages. Our study remains analytically tractable by leveraging the powerful tools of optimal control in continuous time. We establish that both a worker's decision to quit and a firm's decision to dissolve a job can be formulated as a nonzero-sum stochastic differential game with stopping times. This formulation allows us to prove the existence of a unique block recursive equilibrium and provide a sharp characterization of agents' equilibrium policies. We show that our theory also has empirical content, as it can be inverted to identify the unobserved distribution of an appropriately defined state variable from microdata on wage changes and worker flows between jobs. The identified model allows us to study the monetary multipliers for aggregate employment and real wages through the use of sufficient statistics, which we show are closely linked to the prevalence of inefficient job separations.

Our work points to several interesting avenues for future research. Possible extensions to our framework include the introduction of on-the-job search, wage renegotiations subject to adjustment frictions à la Rotemberg (1982) or Calvo (1983), a notion of a firm, and alternative types of idiosyncratic and aggregate shocks. While we have abstracted from these features, integrating them into a unified model would make possible a rich, quantitative analysis of labor markets and the macroeconomy.

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A Theory of Non-Coasean Labor Markets

Online Appendix—Not for Publication

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A Auxiliary Theorems

We use the following mathematical notation in this appendix.

1. $H^l(\mathbb{R})$: Sobolev space; i.e., $H^l(\mathbb{R}) \subset L^2(\mathbb{R})$ and its weak derivatives up to order l have a finite L^p norm.
2. Characteristic operator $\mathcal{A}$: Given a function $f : \mathbb{R} \to \mathbb{R}$ and a diffusion process $\{x_t\}$, the characteristic operator of X is given by
$$\mathcal{A}f = \lim_{U \downarrow x} \frac{\mathbb{E}[f(X_{\tau_U}|x_0 = x] - f(x)]}{\mathbb{E}[\tau_U|x_0 = x]}$$
3. Let $u, v : \mathbb{R} \to \mathbb{R}$, $(u, v) = \int_{\mathbb{R}} u(x)v(x)\,\mathrm{d}x$ and $||u|| = \left(\int u(x)^2\,\mathrm{d}x\right)^{1/2}$.
4. $a(u, v)$ is a bilinear continuous form. We say $a(u, v)$ is coercive if $a(u, u) \geq \alpha||u||^2$.

Proposition A.1. *Let $\mathcal{A}$ be the characteristic operator of $\{X_t\}$ with $X_t \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function with compact (i.e., bounded and closed in $\mathbb{R}$) support ($support(f) = \{x : f(x) \neq 0\}$). If τ is a stopping time with $\mathbb{E}_x[\tau] < \infty$, then*

$$\mathbb{E}_x[f(x_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau \mathcal{A}f(X_t)\,\mathrm{d}t\right]. \tag{A.1}$$

Moreover, if τ is the first exit time of a bounded set, then (A.1) holds for any twice differentiable function.

Proof. This is Dynkin's formula, the proof of which can be found in **?**. □

Proposition A.2. *Let x_t be a strong Markov process, τ be a stopping time measurable with the filtration generated by x_t, and τ^δ a exponential random variable independent of τ. Then*

$$\mathbb{E}\left[\int_0^{\tau\wedge\tau^\delta} e^{-\rho t} f(x_t)\,\mathrm{d}t + e^{-\rho(\tau\wedge\tau^\delta)} g(x_{\tau\wedge\tau^\delta}) \middle| x_0 = x\right] = \mathbb{E}\left[\int_0^\tau e^{-(\rho+\delta)t}[f(x_t) + \delta g(x_t)]\,\mathrm{d}t + e^{-(\rho+\delta)\tau} g(x_\tau) \middle| x_0 = x\right].$$

Proposition A.3. *Let V be a Hilbert space and P a closed convex cone of V satisfying*

$$P = \{x \in V : (x, y) \geq 0\ \forall y \in P\}.$$

Let T be an increasing map from V to itself such that there exists a $\underline{x}, \overline{x} \in V$

$$\underline{x} \leq \overline{x},\ \ \underline{x} \leq T(\underline{x}),\ \ T(\overline{x}) \leq \overline{x}.$$

Then, the subset of fixed points x^* of T satisfying $\underline{x} \leq x^* \leq \overline{x}$ is non-empty and has a larger and smallest element.

Proof. See the proof of Proposition 2 of Chapter 15 on page 539 of Aubin (2007). □

Proposition A.4. *Let V be a Hilbert space and P a closed convex set. Assume that $a(u, v)$ with $u, v \in V$ is a coercive bilinear continuous form. Then, there exists a unique solution to*

$$a(u, v - u) \geq (f, v - u), \forall v \in P, u \in P,$$

where f belongs to the dual of V.

Proof. See Lions and Stampacchia (1967). □

B Proofs for Section 2: A Model of Non-Coasean Labor Contracts

B.1 Proof of Lemma 1

To simplify the exposition, we divide the proof into a sequence of lemmas. Define the equilibrium conditions

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(w,z)[h(z;w) - u(z)], \;\; \forall z \in \mathbb{R} \tag{B.1}$$

$$0 = \left[\tilde{K}e^z - q(w,z)j(z;w)\right]^+ \theta(w,z) \quad \forall (w,z) \in \mathbb{R}^2 \tag{B.2}$$

$$h(z;w) \geq u(z), \;\; \forall z \in \mathbb{R}, \tag{B.3}$$

$$j(z;w) \geq 0, \;\; \forall z \in \mathbb{R} \tag{B.4}$$

$$\text{If } z \in (\mathcal{C}^h(w))^c \Rightarrow j(z;w) = 0, \tag{B.5}$$

$$\text{If } z \in (\mathcal{C}^j(w))^c \Rightarrow h(z;w) = u(z), \tag{B.6}$$

$$0 = \max\{u(z) - h(z;w), \mathcal{A}^h h(z;w) + e^w\}, \; \forall z \in \mathcal{C}^j(w), h(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^j(w)) \cap \mathbb{C}(\mathbb{R}), \tag{B.7}$$

$$0 = \max\{-j(z;w), \mathcal{A}^j j(z;w) + e^z - e^w\}, \; \forall z \in \mathcal{C}^h(w), j(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^h(w)) \cap \mathbb{C}(\mathbb{R}), \tag{B.8}$$

$$\mathcal{C}^h(w) := int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } \mathcal{A}^h u(z) + e^w > 0\right\}, \tag{B.9}$$

$$\mathcal{C}^j(w) := int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}, \tag{B.10}$$

$$\mathcal{A}^h(f(z)) := -\rho f + \delta(u(z) - f(z)) + \gamma\frac{\partial f(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z)}{\partial z^2}$$
$$\mathcal{A}^j(f(z)) := -\rho f + \delta(0 - f(z)) + \gamma\frac{\partial f(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z)}{\partial z^2}$$

Proposition B.1. *Let $x := (w,z)$. If there exist two functions $h(z;w)$, $j(z;w)$ satisfying (B.3), (B.4), (B.5), (B.6), (B.7) and (B.8) given the continuation sets (B.9) and (B.10), then*

$$\tau^{h*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^h(w)\right\}$$
$$\tau^{j*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^j(w)\right\}$$

form a non-trivial Nash equilibrium and

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta), j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta).$$

Moreover, if $(\tau^{h*}(x), \tau^{j*}(x))$ is a non-trivial Nash equilibrium, then

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta), j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta).$$

satisfy (B.3) to (B.8).

Proof. **Quasi-variational inequalities as sufficient conditions.** First, we prove that if $h(z;w)$, $j(z;w)$ satisfy (B.3) to (B.8), then

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta}) \geq H(x, \tau^{h}(x), \tau^{j*}(x), \tau^{\delta})$$

for any $\tau^h \in \mathcal{T}$. The proof of the statement

$$j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta}) \geq J(x, \tau^{h*}(x), \tau^{j}(x), \tau^{\delta}),$$

for any $\tau^j \in \mathcal{T}$, follows the same arguments.

Step 1: Here, we show that $h(z;w) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^{\delta})$. Let τ^h be any stopping time (not necessarily the optimal). Without loss of generality, we restrict the attention to $\tau^h \leq \tau_{(-\infty,a)}$, where $\tau_{(-\infty,a)} = \inf\{t > 0 : z_t \notin (-\infty, a)\}$. Intuitively, it is never optimal for the worker to stay in the job at wage w when productivity is sufficiently large. Let $U_k \subset \mathbb{R}$ be an increasing sequence of bounded sets s.t. $\cup_{k=1}^{\infty} U_k = \mathbb{R}$. Let $\tau_k = \inf\{(z_t) : z_t \notin U_k\}$. Since each U_k is bounded, we do not need to assume compact support of the function to apply Proposition A.1. Applying Dynkin's Lemma to the stopping time $\tau_k^h = \tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k$,

$$\mathbb{E}[e^{-\rho\tau_k^h} h(x_{\tau_k^h})|z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t | z_0 = z\right].$$

Using condition (B.3), since $h(z;w) \geq u(z)$ for all z, we have that $\mathbb{E}[e^{-\rho\tau_k^h} h(z_{\tau_k^h};w)|z_0 = z] \geq \mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h})|z_0 = z]$. Thus,

$$\mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h})|z_0 = z] - \mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t | z_0 = z\right] \leq h(z;w).$$

From condition (B.7) , we have $\mathcal{A}^h h(z;w) + e^w \leq 0$ for all z. Thus,

$$\mathbb{E}\left[\int_0^{\tau_k^h} e^{-\rho t} e^w\,\mathrm{d}t | z_0 = z\right] \leq -\mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z;w)\,\mathrm{d}t | z_0 = z\right].$$

Using this result

$$\mathbb{E}\left[e^{-\rho\tau_k^h} u(z_{\tau_k^h}) + \int_0^{\tau_k^h} e^{-\rho t} e^w\,\mathrm{d}t | z_0 = z\right] \leq h(z;w)$$

Now, we take the limit $k \to \infty$. It is easy to see that $\int_0^{\tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k} e^{-\rho t + w}\,\mathrm{d}t \leq \frac{1}{\rho} e^w$ a.e., so using the dominated convergence theorem $\lim_{k\to\infty} \mathbb{E}\left[\int_0^{\tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k} e^{-\rho t + w}\,\mathrm{d}t | z_0 = z\right] = \mathbb{E}\left[\int_0^{\tau^h \wedge \tau^{j*} \wedge \tau^{\delta}} e^{-\rho t + w}\,\mathrm{d}t | z_0 = z\right]$.

As we show below, $u(z) \propto e^z$ and since $e^{z_t} \leq e^a$ for all $t \leq \tau^h \leq \tau_{(-\infty,a)}$, we have that $0 \leq e^{-\rho t} u(z_t) \leq e^a$. Applying the monotone convergence theorem again, we have that

$$\lim_{k\to\infty} \mathbb{E}\left[e^{-\rho(\tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k)} u(z_{\tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k})|z_0 = z\right] = \mathbb{E}\left[e^{-\rho(\tau^h \wedge \tau^{j*} \wedge \tau^{\delta})} u(z_{\tau^h \wedge \tau^{j*} \wedge \tau^{\delta}})|z_0 = z\right].$$

Therefore, taking the limit $k \to \infty$, we finally obtain

$$h(z;w) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^{\delta}).$$

Step 2: Now, we show that $h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta})$. Applying Lemma A.1 to the stopping time $\tau_k^{h*} = \tau^{h*} \wedge \tau^{j*} \wedge$

$\tau_k \wedge \tau^\delta$

$$\mathbb{E}[e^{-\rho\tau_k^{h*}} h(z_{\tau_k^{h*}};w)|z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{\tau_k^{h*}} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t|z_0 = z\right].$$

For all $t < \tau_k^{h*}$, we have that $u(z) < h(z;w)$. Therefore, by (B.7), $\mathcal{A}^h h(z;w) + e^w = 0$ for all z. Thus,

$$\mathbb{E}\left[e^{-\rho\tau_k^{h*}} h(z_{\tau_k^{h*}};w) + \int_0^{\tau_k^{h*}} e^{-\rho t} e^w\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

Taking the limit $k \to \infty$ and following similar arguments as above, we obtain

$$\mathbb{E}\left[e^{-\rho(\tau^{h*}\wedge\tau^{j*}\wedge\tau^\delta)} h(z_{\tau^{h*}\wedge\tau^{j*}\wedge\tau^\delta};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}\wedge\tau^\delta} e^{-\rho t} e^w\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

which, given Proposition A.2, is equivalent to

$$\mathbb{E}\left[e^{-(\rho+\delta)(\tau^{h*}\wedge\tau^{j*})} h(z_{\tau^{h*}\wedge\tau^{j*}};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}} e^{-(\rho+\delta)t}(\delta u(z_t) + e^w)\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

Since $z_{\tau^{h*}\wedge\tau^{j*}} \in \partial(\mathcal{C}^h(w^*(z)) \cap \mathcal{C}^j(w^*(z)))$ and $h(\cdot;w)$ is continuous, we have that

$$\mathbb{E}\left[e^{-(\rho+\delta)(\tau^{h*}\wedge\tau^{j*})} u(z_{\tau^{h*}\wedge\tau^{j*}};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}} e^{-(\rho+\delta)t}(\delta u(z_t) + e^w)\,\mathrm{d}t|z_0 = z\right] = h(z;w).$$

and $h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta)$.

Quasi-variational inequalities as necessary conditions. Now, we prove that if $\tau^{h*}(x)$ and $\tau^{j*}(x)$ is a Non-trivial Nash equilibrium, then $h(z;w)$, $j(z;w)$ satisfy (B.3) to (B.10). It is easy to show that in a Nash equilibrium, the stopping time is Markovian—if any agent chooses to stop, then the game finishes. By definition, we have that

$$h(z;w) = \max_{\tau^h} \mathbb{E}\left[\int_0^{\tau^h\wedge\tau^{j*}\wedge\tau^\delta} e^{-\rho t + w}\,\mathrm{d}t + e^{-\rho(\tau^h\wedge\tau^{j*}\wedge\tau^\delta)} u(z_{\tau^h\wedge\tau^{j*}\wedge\tau^\delta};w)\,\mathrm{d}t|z_0 = z\right]. \tag{B.11}$$

- Condition (B.3): We show it by contradiction. Assume that $h(z;w) < u(z)$. Then $\tau^h(x) = 0$, implies

  $$u(z) > h(z;w) \geq H(w, z, 0, \tau^{j*}(x), \tau^\delta) = \mathbb{E}_0\left[\int_0^0 e^{-\rho t} e^w\,\mathrm{d}t + e^{-\rho\tau^m} u(z_0)|z_0 = z\right] = u(z),$$

  so we have a contradiction.
- Condition (B.6): If $z \in (\mathcal{C}^j(w))^c$, then $\tau^{j*}(x) = 0$ and $Pr[\min\{\tau^{h*}(x), \tau^{j*}(x), \tau^\delta(x)\} \leq \tau^{j*}(x)] = 1$, and $h(z;w) = u(z)$.
- Condition (B.7): Observe that this condition is the best response of the worker, given that the firm continues. See **?** and Brekke and Øksendal (1990) for a discussion of the necessity of the smooth pasting condition.
- Condition (B.9): To show this, we need to characterize the continuation set in the Nash equilibrium that survives the iterated elimination of weakly dominated strategies. First, from the problem (B.11), if $\Pr(\tau^{j*}(x) > 0) = 1$, then $\Pr(\tau^{h*}(x) > 0) = 1$ if and only if

  $$int\,\{z \in \mathbb{R} : h(z;w) > u(z)\}.$$

Second, we show that staying in the match weakly dominates leaving it if

$$0 < e^{-w} + \mathcal{A}^h u(z), \tag{B.12}$$

for all z. Take any stopping time τ such that $\Pr(\tau > 0 | z_0 = z) = 1$. Then, applying Dynkin's Lemma (and using similar arguments as in Step 1), we obtain

$$\mathbb{E}\left[e^{-\rho\tau} u(z_\tau) | z_0 = z\right] = u(z) + \mathbb{E}\left[\int_0^\tau \mathcal{A}u(z_t)\, \mathrm{d}t | z_0 = z\right].$$

Using the inequality in (B.12),

$$u(z) = \mathbb{E}\left[e^{-\rho\tau} u(z_\tau) | z_0 = z\right] - \mathbb{E}\left[\int_0^\tau \mathcal{A}u(z_t)\, \mathrm{d}t | z_0 = z\right] < \mathbb{E}\left[e^{-\rho\tau} u(z_\tau) | z_0 = z\right] + \mathbb{E}\left[\int_0^\tau e^{-\rho t + w}\, \mathrm{d}t | z_0 = z\right].$$

Thus, staying in the match strictly dominates dissolving the match.

□

Proposition B.2. *Define*

$$w^*(z) = \arg\max_w \theta(x)^{1-\alpha}(h(z; w) - u(z)).$$

and $\tau^{u*} = \inf\{t \geq 0 : \Delta N_t^{f(w^*(z_t), z_t)} = 1\}$ *where* $N_t^{f(w^*(z_t), z_t)}$ *is a Poisson counter with arrival rate* $f(w^*(z_t), z_t)$*. The function* $u(z)$ *satisfies* $u(z) \in C^2(\mathbb{R})$ *and* (B.1) *if and only if*

$$u(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}\left[\int_0^{\tau^u} e^{-\rho t} B(z_t)\, \mathrm{d}t + e^{-\rho\tau^u} h(z_{\tau^u}; w)\right].$$

Proof. The proof is the standard optimality conditions in the HJB (see **?**). □

Lemma 1. *Assume* $u(z)$*,* $h(z; w)$*,* $j(z; w)$*,* $\theta(z; w)$ *satisfy* (B.2)—(B.8) *given the continuation sets* (B.9)*. Then* $\left\{\tau^{h*}, \tau^{j*}, \{w_t^*\}_{t=0}^{\tau^u}\right\}$ *constructed with*

$$\tau^{h*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^h(w)\right\}$$
$$\tau^{j*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^j(w)\right\}$$
$$w^*(z) = \arg\max_w \theta(x)^{1-\alpha}(h(z; w) - u(z)).$$

is a block recursive equilibrium with

$$h(z; w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta),$$
$$j(z; w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta),$$
$$u(z) = U(z).$$

If $\{H(w, z, \vec{\tau}^m), J(w, z, \vec{\tau}^m), U(z)\}$*, market tightness* $\theta(w, z)$*, and policy functions* $\left\{\tau^{h*}(w, z), \tau^{j*}(w, z), w^*(z_t)\right\}$ *is a block recursive equilibrium with*

$$h(z; w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta),$$

$$j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta}),$$
$$u(z) = U(z).$$

then $u(z)$, $h(z;w)$, $j(z;w)$, $\theta(z;w)$ *satisfy* (B.2)—(B.8) *given the continuation sets* (B.9).

Proof. The proof is a combination of Results B.1 and B.2. □

B.2 Proof of Lemma 2

For the next proof, it will be useful to define the normalized equilibrium conditions

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*), \tag{B.13}$$
$$0 = \left[\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w})\right]^+ \hat{\theta}(\hat{w}), \tag{B.14}$$
$$\hat{W}(\hat{w}) \geq 0, \tag{B.15}$$
$$\hat{J}(\hat{w}) \geq 0, \tag{B.16}$$
$$if\ \hat{w} \in (\hat{\mathcal{C}}^h)^c \Rightarrow \hat{J}(\hat{w}) = 0, \tag{B.17}$$
$$if\ \hat{w} \in (\hat{\mathcal{C}}^j)^c \Rightarrow \hat{W}(\hat{w}) = 0, \tag{B.18}$$
$$0 = \max\{-\hat{W}(\hat{w}), \hat{\mathcal{A}}\hat{W}(\hat{w}) + e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^j, \hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R}) \tag{B.19}$$
$$0 = \max\{-\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h, \hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R}) \tag{B.20}$$
$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0\ or\ (e^{\hat{w}} - \hat{\rho}\hat{U}) > 0\right\}, \tag{B.21}$$
$$\hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0\ or\ \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{B.22}$$
$$\hat{\mathcal{A}}(f) := -(\hat{\rho} + \delta)f - \hat{\gamma}\frac{\partial f(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 f(\hat{w})}{\partial \hat{w}^2},$$

where $\hat{w} = w - z$, $\hat{\rho} = \rho - \gamma - \sigma^2/2$ and $\hat{\gamma} = \gamma + \sigma^2$.

Lemma 2. *Assume that* $(h(z;w), j(z;w), u(z), \theta(x), w^*(z))$ *satisfy conditions* (B.1) *to* (B.10), *then*

$$\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z), \hat{w}^*\right) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, \theta(w,z), w^*(z) - z\right).$$

satisfy (B.13) *to* (B.22). *Moreover, if* $\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)\right)$ *satisfy* (B.13) *to* (B.22), *then*

$$(u(z), j(z;w), h(z;w), \theta(w,z), w^*(z)) = \left(\hat{U}e^z, \hat{J}(w-z)e^z, (\hat{W}(w-z) + \hat{U})e^z, \hat{\theta}(w-z), \hat{w}^* + z\right)$$

satisfy (B.1) *to* (B.10).

Proof. The general idea for the proof is to use a guess-and-verify strategy for each equilibrium condition.

Condition (B.1) **holds if and only if** (B.13) **is satisfied:** Using $\hat{U} = \frac{u(z)}{e^z}$, we have that

$$\hat{U}e^z = u'(z) \text{ and } \hat{U}e^z = u''(z).$$

Using this result and the fact that $\theta(w,z) = \hat{\theta}(w-z)$, and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$,

$$\begin{aligned}
\rho u(z) &= \tilde{B}e^z + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + \max_{w(z)} \theta(x)^{1-\alpha}[h(z;w) - u(z)] \iff \\
\rho \hat{U}e^z &= \tilde{B}e^z + \gamma \hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + \max_{w(z)} \hat{\theta}(w-z)^{1-\alpha}[\hat{W}(w-z)e^z] \iff \\
\left(\rho - \gamma - \frac{\sigma^2}{2}\right)\hat{U}e^z &= \tilde{B}e^z + e^z \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}) \iff \hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}).
\end{aligned}$$

Condition (B.2) holds if and only if (B.14) is satisfied: Using that $\hat{J}(w-z) = \frac{j(z;w)}{e^z}$, $\theta(w,z) = \hat{\theta}(w-z)$, and the assumption $K(z) = \tilde{K}e^z$, we have that

$$\begin{aligned}
\left[\tilde{K}e^z - q(\theta(x))j(z;w)\right]^+ \theta(x) = 0 &\iff \left[\tilde{K}e^z - q(\theta(x))\hat{J}(w-z)e^z\right]^+ \theta(x) = 0 \\
&\iff \left[\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w})\right]\hat{\theta}(\hat{w}) = 0.
\end{aligned}$$

Since $q(\theta(x)) = q(\hat{\theta}(\hat{w}))$, we have the result.

Condition (B.7) holds if and only if (B.19) is satisfied: Assume $h(z;w)$ satisfies (B.7). Then, for all $z \in \mathcal{C}^j(w)$

$$0 = \max\{u(z) - h(z;w), -\rho h(z;w) + \gamma \frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\}$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, we have that $h(z;w) = \hat{W}(w-z)e^z + \hat{U}e^z$, $\frac{\partial h(z;w)}{\partial z} = \hat{W}(w-z)e^z - \hat{W}'(w-z)e^z + \hat{U}e^z$, and $\frac{\partial^2 h(z;w)}{\partial z^2} = \hat{W}(w-z)e^z - 2\hat{W}'(w-z)e^z + \hat{W}''(w-z)e^z + \hat{U}e^z$. Thus, since $e^z > 0$

$$\begin{aligned}
0 &= \max\left\{u(z) - h(z;w), -\rho h(z;w) + \gamma \frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\right\} \\
&= \max\{-\hat{W}(w-z)e^z, -(\hat{W}(w-z)e^z + \hat{U}e^z) + \gamma(\hat{W}(w-z)e^z - \hat{W}'(w-z)e^z + \hat{U}e^z)\ldots \\
&+ \frac{\sigma^2}{2}(\hat{W}(w-z)e^z - 2\hat{W}'(w-z)e^z + \hat{W}''(w-z)e^z + \hat{U}e^z) - \delta\hat{W}(w-z)e^z + e^w\} \\
&= \max\{-\hat{W}(w-z), -(\rho - \gamma - \sigma^2/2 + \delta)\hat{W}(w-z) - (\gamma + \sigma^2)\hat{W}'(w-z) + \frac{\sigma^2}{2}\hat{W}''(w-z) - \left(\rho - \gamma - \sigma^2/2\right)\hat{U} + e^{w-z}\} \\
&= \max\left\{-\hat{W}(\hat{w}), -(\hat{\rho} + \delta)\hat{W}(\hat{w}) - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) - \hat{\rho}\hat{U} + e^{\hat{w}}\right\}.
\end{aligned}$$

The equivalence between (B.8) and (B.20) can be established following similar steps.

Condition (B.9) holds if and only if (B.21) is satisfied: Assume $z \in \mathcal{C}^h(w)$. Then,

$$h(z;w) > u(z) \text{ or } -\rho u(z) + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + e^w > 0$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, with $e^z > 0$

$$\begin{aligned}
&\hat{W}(w-z) > 0 \text{ or } -\rho\hat{U}e^z + \gamma\hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + e^w > 0 \iff \\
&\hat{W}(w-z) > 0 \text{ or } e^{w-z} - (\rho - \gamma - \sigma^2/2)\hat{U} > 0 \iff \\
&\quad \hat{W}(\hat{w}) > 0 \text{ or } e^{\hat{w}} - \hat{\rho}\hat{U} > 0
\end{aligned}$$

Thus, $z \in \mathcal{C}^h(w)$ if and only if $w - z \in \hat{\mathcal{C}}^h$. The equivalence between (B.10) and (B.22) can be established following similar steps.

Remaining conditions: The equivalence between equations (B.3), (B.4), (B.5), (B.6) and equations (B.15), (B.16), (B.17), and (B.18) is trivially established. □

B.3 Proof of Proposition 1

Proposition 1. *Let $\hat{W}(\hat{w}), \hat{J}(\hat{w}), \hat{\theta}(\hat{w})$ be bounded functions with compact support. Then, there exists a unique solution to*

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*),$$

$$0 = \left[\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w})\right]^+ \hat{\theta}(\hat{w}),$$

$$\hat{W}(\hat{w}) \geq 0, \tag{B.23}$$

$$\hat{J}(\hat{w}) \geq 0, \tag{B.24}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^h)^c \Rightarrow \hat{J}(\hat{w}) = 0, \tag{B.25}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^j)^c \Rightarrow \hat{W}(\hat{w}) = 0, \tag{B.26}$$

$$0 = \max\{-\rho\hat{W}(\hat{w}), \hat{\mathcal{A}}\hat{W}(\hat{w}) + e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^j, \hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R}) \tag{B.27}$$

$$0 = \max\{-\rho\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h, \hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R}) \tag{B.28}$$

$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0\ or\ (e^{\hat{w}} - \hat{\rho}\hat{U}) > 0\right\}, \tag{B.29}$$

$$\hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0\ or\ \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{B.30}$$

$$\hat{\mathcal{A}}(f) := -(\hat{\rho} + \delta)f - \hat{\gamma}\frac{\partial f(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 f(\hat{w})}{\partial \hat{w}^2},$$

The proof uses results from the mathematics literature that, in general, a well-trained economist has not used or seen before. For this reason, before going over the proof, we provide some intuition about the steps we show below. In a nutshell, there are two steps in the proof. First, we need to show that, for a given value of unemployment $\hat{U}$, there is a unique non-trivial Nash equilibrium of the game played by the matched worker-firm pair. To understand the intuition behind this step, define $\hat{w}^+(\hat{w}^-;\rho\hat{U})$ as the best response function of the firm in terms of its layoff threshold, and $\hat{w}^-(\hat{w}^+;\rho\hat{U})$ as the best response function of the worker in terms of her quit threshold. It is easy to show that optimal policies are given by wage-to-productivity thresholds. $\hat{w}^+(\hat{w}^-;\rho\hat{U})$ is the solution to the differential equation

$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}),$$

with border conditions $\hat{J}(\hat{w}^+) = \hat{J}(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$. In the same way, $\hat{w}^-(\hat{w}^+;\rho\hat{U})$ is the solution to the differential equation

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}),$$

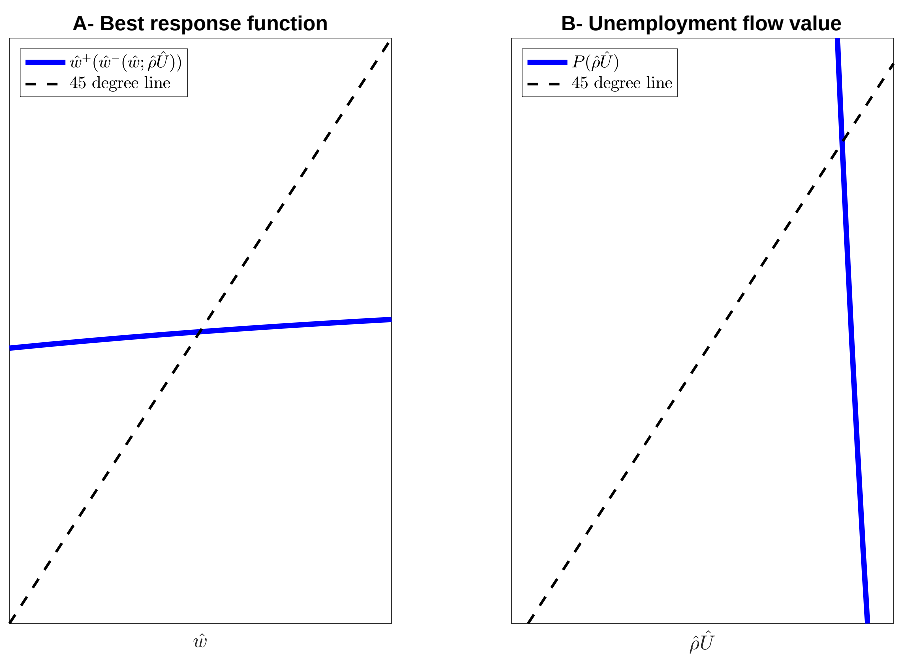
with border conditions $\hat{W}(\hat{w}^+) = \hat{W}(\hat{w}^-) = \hat{W}'(\hat{w}^-) = 0$. Let $\hat{W}(\hat{w};\rho\hat{U})$ and $\hat{J}(\hat{w};\rho\hat{U})$ be the values associated with the non-trivial equilibrium policies.

Second, we need to find a solution to the unemployment value. The equilibrium value of unemployment satisfies

$$\mathbb{P}(\hat{\rho}\hat{U}) = \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}}\hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}}\hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

Figure B1-Panel A shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w};\hat{\rho}\hat{U}))$ and Figure B1-Panel B shows $\mathbb{P}(\hat{\rho}\hat{U})$. As we can see in the figure, the composition of the best responses shows two properties: (i) monotonicity (i.e., $\mathbb{Q}'(\hat{w}) > 0$) and (ii) concavity (i.e., $\mathbb{Q}''(\hat{w}) < 0$). Intuitively, the monotonicity property arises from the fact that if one agent prefers to wait more, then the other agent also prefers to wait more. Concavity arises from the fact that there is a decreasing value of waiting. As the figure clearly shows, a unique non-trivial Nash Equilibrium exists under these two properties. Equipped with the values of the non-trivial Nash Equilibrium as a function of $\hat{U}$, we can then characterize the decision problem of the unemployed worker. The mapping $\mathbb{P}(\hat{\rho}\hat{U})$ satisfies three properties: (i) $\mathbb{P}(\tilde{B}) > \tilde{B}$ with $\mathbb{P}(1) = \tilde{B}$, (ii) it is continuous and (iii) it is decreasing. Intuitively, if the flow value of unemployment is equal to $\tilde{B}$, then the surplus of the match is positive, and the unemployed worker obtains a positive continuation value from searching for a job. If, instead, the flow value of unemployment equals the value of (normalized) output, then the surplus is zero, and the unemployed worker does not benefit from finding a job. Also, the larger the unemployment value, the lower the value of the match, and, therefore, the value of searching for a job. As the figure clearly shows, a unique equilibrium exists under these three properties of $\mathbb{P}(\hat{\rho}\hat{U})$.

FIGURE B1. INTUITION



Notes: The figure illustrates the properties of the policy and value functions. Panel A shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w};\hat{\rho}\hat{U}))$ and the 45 degree line. The non-trivial Nash Equilibrium is given by the intersection between these two lines. Panel B shows the composition of the individual best response and the fixed point in the equilibrium $\mathbb{P}(\hat{\rho}\hat{U})$.

Proof. We divide the proof into four steps.

Step 1 shows the existence of a non-trivial Nash equilibrium for a given $\hat{U}$. In this step, we show the existence of a solution to conditions (B.23) to (B.30). To simplify the exposure, we divide step 1 into three propositions. Proposition B.3 shows the equivalence between the equilibrium conditions and the quasi-variational inequalities. Proposition B.4 shows the existence and uniqueness of the agents' best responses. Proposition B.5 shows the existence of equilibrium by invoking Proposition A.3

(Tartar's fixed point theorem). Observe that we restrict the functions $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ to have bounded support. This property is without loss of generality since it is a result of Proposition 2—i.e., the match's continuation region is bounded.

Step 2 shows the uniqueness of the solution to conditions (B.23) to (B.30). We divide this proof into two propositions. Proposition B.6 shows that the operator defined in step 1 is strong order concave. Using concavity and techniques in the spirit of Marinacci and Montrucchio (2019) applied to our own problem, we show uniqueness in proposition B.7.

Step 3 shows that value functions are continuous and decreasing. We divide this step into two propositions. First, we show in proposition B.8 that the value associated with the worker's "best response" is continuous and decreasing in $\hat{U}$. Proposition B.9 shows these properties for the non-trivial Nash equilibrium. Finally, step 4 proves the uniqueness of the equilibrium by showing the existence of the unique fixed point in the unemployed worker's value $\hat{U}$.

Step 1. We start by defining a continuous bilinear form in a more general space of functions. Let $V = H_0^1(\mathbb{R})$—where $H_0^1(\mathbb{R})$ is a Sobolev space of order 1—be a Hilbert space and define the bilinear continuous form $a : V \times V \to \mathbb{R}$

$$a(v_1, v_2) := \frac{\sigma^2}{2}\int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}}\frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} v_2(\hat{w})\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\mathbb{R}} v_1(\hat{w})v_2(\hat{w})d\hat{w}$$

Proposition B.3. *Define $K^h(\hat{J})$ and $K^j(\hat{W})$ as*

$$K^h(\hat{J}) := \left\{\hat{W} \in V : \hat{W}(\hat{w}) \geq 0 \;\&\; if\; \hat{J}(\hat{w}) = 0 \; and \; \hat{w} \geq 0 \Rightarrow \hat{W}(\hat{w}) = 0\right\},$$
$$K^j(\hat{W}) := \left\{\hat{J} \in V : \hat{J}(\hat{w}) \geq 0 \;\&\; if\; \hat{W}(\hat{w}) = 0 \; and \; \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0\right\},$$

Then (i) $\hat{W}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R})$ and $\hat{J}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R})$ bounded with compact support, where $\hat{\mathcal{C}}^h$ and $\hat{\mathcal{C}}^j$ are constructed with $\hat{W}$ and $\hat{J}$ following (B.29) *and* (B.30)*; (ii) $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ solve*

$$\begin{aligned} \hat{W} \in K^h(\hat{J}),\;\; \hat{J} \in K^j(\hat{W}) \\ a(\hat{J}, v - \hat{J}) \geq \int_{\mathbb{R}} \left(1 - e^{\hat{w}}\right)\left(v - \hat{J}\right)\mathrm{d}\hat{w},\;\; \forall\, v \in K^j(\hat{W}) \\ a(\hat{W}, v - \hat{W}) \geq \int_{\mathbb{R}} (e^{\hat{w}} - \hat{\rho}\hat{U})\left(v - \hat{W}\right)\mathrm{d}\hat{w},\;\; \forall\, v \in K^h(\hat{J}). \end{aligned}$$

if and only if $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ solve (B.23), (B.24), (B.25), (B.26), (B.27), *and* (B.28).

Proof of Step 1 - Proposition B.3. We verify conditions (B.23), (B.24), (B.25), (B.26), (B.27), and (B.28) focusing on the firm (the worker's conditions are verified following similar steps). It is easy to show the converse.

Conditions (B.23) and (B.24) are satisfied. Since $\hat{J} \in K^j(\hat{W})$, we have that $\hat{J}(\hat{w}) \geq 0$.

Conditions (B.25) and (B.26) are satisfied. Define $\hat{\mathcal{C}}^h$ with $\hat{W}$. Then, $(\hat{\mathcal{C}}^h)^c$ is equal to

$$(\hat{\mathcal{C}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) \leq 0 \; and \; (e^{\hat{w}} - \hat{\rho}\hat{U}) \leq 0\}.$$

Since $\hat{W}(\hat{w}) \geq 0$, we have that

$$(\hat{\mathcal{C}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) = 0 \; and \; \hat{w} \leq \log(\hat{\rho}\hat{U})\}.$$

Since $\hat{J} \in K^j(\hat{W})$, if $\hat{w} \in (\hat{\mathcal{C}}^h)^c$, then $\hat{J}(\hat{w}) = 0$.

Conditions (B.27) and (B.28) are satisfied. Take any $v \in K^j(\hat{W})$. Then, if $\hat{w} \in (\hat{\mathcal{C}}^h)^c$, we have that $\hat{J}(\hat{w}) = 0$. Therefore, we have that for every $v \in K^j(\hat{W})$

$$
\begin{aligned}
&a(\hat{J}, v-\hat{J}) \geq \int_{\mathbb{R}} (1-e^{\hat{w}})\left(v-\hat{J}\right) \iff \\
&\frac{\sigma^2}{2}\int_{(\hat{C}^h)^c} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{(\hat{C}^h)^c} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{(\hat{C}^h)^c} \hat{J}(\hat{w})(v(\hat{w})-\hat{J}(\hat{w})))\,\mathrm{d}\hat{w} + \\
&\frac{\sigma^2}{2}\int_{\hat{C}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{\hat{C}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\hat{C}^h} \hat{J}(\hat{w})(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \geq \\
&\int_{\hat{C}^h}(1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w} + \int_{(\hat{C}^h)^c}(1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w} \iff \\
&\frac{\sigma^2}{2}\int_{\hat{C}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{\hat{C}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\hat{C}^h} \hat{J}(\hat{w})(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \geq \\
&\int_{\hat{C}^h}(1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w}.
\end{aligned}
$$

Using integration by parts, we obtain

$$
\begin{aligned}
&\frac{\sigma^2}{2}\int_{\hat{C}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} \\
&=^{(1)} \underbrace{\frac{\sigma^2}{2}\frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\Big|_{\hat{w}\in\partial\in\hat{C}^h}}_{=0} - \frac{\sigma^2}{2}\int_{\hat{C}^h}\frac{d^2\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}.
\end{aligned}
$$

In (1), there could be two cases. The first case is a finite limit of integration. In this case, we use continuity and the fact that if $\hat{w} \to \partial\hat{C}^h$ ($\hat{C}^h$ is open) , then $\hat{w} \to (\hat{C}^h)^c$ and, therefore, $\hat{J}(\hat{w}) = 0$. The second case is an infinite limit of integration. In this case, the assumption of bounded support implies $\hat{J}(\hat{w}) = 0$ for sufficiently large or small $\hat{w}$, thus $\hat{J}'(\hat{w}) = 0$.

In conclusion,

$$
\int_{\hat{C}^h}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \leq 0
$$

Before continue, we remark that the previous equality holds for all $v(\hat{w}) \in K^j(\hat{W})$. Let $\mathcal{O}$ be an open ball in $\hat{C}^h$ that cover an arbitrary point $\hat{w} \in \hat{C}^h$. Then, we can find a family of smooth functions index by n with $o_{\hat{w}}(n) \in [0,1]$, s.t. $o_{\hat{w}}(n) = 0$ outside $\hat{C}^h$, $o_{\hat{w}}(n) \to 1$ in $\mathcal{O}$, and $o_{\hat{w}}(n) \to 0$ outside $\mathcal{O}$. Since $\hat{J}(\hat{w}) + \theta_{\hat{w}}(n) \geq 0$, $\hat{J}(\hat{w}) + o_{\hat{w}}(n) \in K^j(\hat{W})$ and

$$
\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w} + \int_{\hat{C}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w} \leq 0.
$$

Taking the limit $n \to \infty$, we have that

$$
\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\mathrm{d}\hat{w} \leq 0.
$$

Since $\mathcal{O}$ is arbitrary, $\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}} \leq 0$ a.e. in $\hat{C}^h$. Since $\hat{J}(\hat{w}) \in \mathbb{C}^1(\hat{C}^h)$, then $\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}} \leq 0$ for all $\hat{w}$ whenever the second derivative is defined. To obtain the other inequality, take $\hat{J}(\hat{w})(1-o_{\hat{w}}(n)) + 0o_{\hat{w}}(n) \in K^j(\hat{W})$ and we have that

$$
-\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1-e^{\hat{w}}\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w} - \int_{\hat{C}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w} \leq 0
$$

Taking the limit $n \to \infty$, we have that $\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \leq 0$ a.e.. Since $\hat{J}(\hat{w}) \in C^1(\hat{C}^h)$, we have that for all $\hat{w} \in \hat{C}^h$

$$
\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w})) \leq 0.
$$

Since $\hat{J}(\hat{w}) \geq 0$ and $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right) \leq 0$, we have that $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w})) \geq 0$. Thus, $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\right)(-\hat{J}(\hat{w})) =$

0 or written more compactly

$$0 = \max\{-\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h,$$

with $\hat{J}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R})$. □

Proposition B.4. *Define the value functions that are obtained from the best responses as $BR^h : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ and $BR^j : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ such that*

$$BR^h(\hat{J}) = \{\hat{W} \in H^1(\mathbb{R}) : a(\hat{W}, v - \hat{W}) \geq (e^{\hat{w}} - \hat{\rho}\hat{U}, v - \hat{W}),\ \forall\, v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})\},$$
$$BR^j(\hat{W}) = \{\hat{J} \in H^1(\mathbb{R}) : a(\hat{J}, v - \hat{J}) \geq (1 - e^{\hat{w}}, v - \hat{J}),\ \forall\, v \in K^j(\hat{W}),\ \hat{J} \in K^j(\hat{W})\}.$$

Then, $BR^h(\hat{J})$ and $BR^j(\hat{W})$ exist and unique.

Proof of Step 1 - Proposition B.4. Here, we show that the value functions that are obtained from the best responses are well-defined. For this, we need to verify the conditions in Proposition A.4. Basically, we need to show that $K^j(\hat{W})$ is closed and convex, and that $a(\cdot,\cdot)$ is coercive.

$\mathbf{K^j(\hat{W})}$ is closed and convex. First, we show that $K^j(\hat{W})$ is closed. Take a sequence $\hat{J}^n \in K^j(\hat{W})$ s.t. $\hat{J}^n$ converges to some $\hat{J}^*$. Since $\hat{J}^n \in K^j(\hat{W})$,

$$\hat{J}^n \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^n = 0$$

for all n. Taking the limit,

$$\hat{J}^* \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^* = 0$$

where we use the fixed domain in the second limit. Thus, $K^j(\hat{W})$ is closed.

To show that $K^j(\hat{W})$ is convex, take $\hat{J}^1, \hat{J}^2 \in K^j(\hat{W})$, then

$$\hat{J}^1 \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^1 = 0,$$
$$\hat{J}^2 \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^2 = 0.$$

Taking the convex combination with $\lambda \in [0,1]$

$$\lambda\hat{J}^1 + (1-\lambda)\hat{J}^2 \geq 0, \text{if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \geq 0, \text{ then } \lambda\hat{J}^1 + (1-\lambda)\hat{J}^2 = 0.$$

Thus, $K^j(\hat{W})$ is convex.

$\mathbf{a(u,v)}$ is coercive. Operating over the bilinear operator

$$\begin{aligned}
a(v,v) &= \frac{\sigma^2}{2}\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}} v(\hat{w})\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&=^{(1)} \underbrace{\frac{\sigma^2}{2}\int_{\mathbb{R}} \left(\frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w}}_{\geq 0} + \hat{\gamma}\,\underbrace{v(\hat{w})^2\Big|_{-\infty}^{\infty}}_{=0} + (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&\geq^{(2)} (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&= (\hat{\rho}+\delta)||v||^2
\end{aligned}$$

Step (1) integrates $\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}} v(\hat{w})\,\mathrm{d}\hat{w} = v(\hat{w})^2|_{-\infty}^{\infty}$ and uses compact support. Step (2) uses the non-negativity of the squared derivative term.

With the properties verified, we can apply Proposition A.4. Thus, the best response exists, and it is unique. □

Proposition B.5. *Define* $Q(\hat{W}) = (BR^h \circ BR^j)(\hat{W})$*, then there exists a fixed point* $Q(\hat{W}^*) = \hat{W}^*$ *and* $\hat{J}^* = BR^j(\hat{W}^*)$*. The set of fixed points is bounded above and below by*

$$0 \leq \underline{\hat{W}} \leq \hat{W}^* \leq \overline{\hat{W}},$$
$$0 \leq \underline{\hat{J}} \leq \hat{J}^* \leq \overline{\hat{J}},$$

where

$$a(\underline{\hat{W}}, v - \underline{\hat{W}}) \geq (e^{\hat{w}} - \hat{\rho}\hat{U}, \underline{\hat{W}}),\ \forall\, v \in K^{small}, \underline{\hat{W}} \in K^{small},$$
$$a(\underline{\hat{J}}, v - \underline{\hat{J}}) \geq (1 - e^{\hat{w}}, \underline{\hat{J}}),\ \forall\, v \in K^{small}, \underline{\hat{J}} \in K^{small},$$
$$a(\overline{\hat{W}}, v - \overline{\hat{W}}) \geq (e^{\hat{w}} - \hat{\rho}\hat{U}, \overline{\hat{W}}),\ \forall\, v \in K^{big}, \overline{\hat{W}} \in K^{big},$$
$$a(\overline{\hat{J}}, v - \overline{\hat{J}}) \geq (1 - e^{\hat{w}}, \overline{\hat{J}}),\ \forall\, v \in K^{big}, \overline{\hat{J}} \in K^{big},$$

with

$$K^{small} := \{v \in V : v(\hat{w}) \geq 0 \,\&\, if\ \ \hat{w} \notin (-\log(\hat{\rho}\hat{U}), 0) \Rightarrow v(\hat{w}) = 0\},$$
$$K^{big} := \{v \in V : v(\hat{w}) \geq 0\},$$

with a maximum and minimum element.

Proof of Step 1 - Proposition B.5. The first step consists in showing that the function $Q(W)$ is monotonically increasing—i.e., if $\hat{W}_1 \geq \hat{W}_2$, then $Q(\hat{W}_1) \geq Q(\hat{W}_2)$. To show this result, first, we need to prove that $K^j(\hat{W})$ is increasing—i.e., if $\hat{W}_1 \geq \hat{W}_2$, then $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$. Take $\hat{J}_2 \in K^j(\hat{W}_2)$, then

$$\hat{J}_2 \geq 0,\ \&\, if\ \hat{W}_2(\hat{w}) = 0\ and\ \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}_2(\hat{w}) = 0.$$

Since $\hat{W}_2(\hat{w}) \geq 0$, we have that

$$\hat{J}_2 \geq 0,\ \&\ \hat{J}_2(\hat{w}) = 0\ \forall \hat{w} \in \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\}.$$

Now, we show that $\{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\}$. Take $\hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\}$. Then $\hat{W}_1(\hat{w}) \leq 0$ and since $\hat{W}_1(\hat{w}) \geq \hat{W}_2(\hat{w})$, we have that $\hat{W}_2(\hat{w}) \leq 0$. Since $\{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\}$, the previous condition holds for the larger set, so it will also hold for the smaller set

$$\hat{J}_2 \geq 0,\ \&\ \hat{J}_2(\hat{w}) = 0, \forall \hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \leq 0 \,\&\, \hat{w} \leq \log(\hat{\rho}\hat{U})\}.$$

Thus, $\hat{J}_2 \in K^j(W_1)$ and $K^j(\hat{W}_2) \subset K^j(W_1)$.

Now, let $\hat{W}_1 \geq \hat{W}_2$. We need to show that $\hat{J}_1 = BR^j(\hat{W}_1) \geq BR^j(\hat{W}_2) = \hat{J}_2$. Since $K^j(\hat{W})$ is increasing—i.e., $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$—$\hat{J}_1, \hat{J}_2 \in K^j(\hat{W}_1)$ and the envelope $\max\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_1)$. Now, we show that $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. Since $\hat{J}_1, \hat{J}_2 \geq 0$, we have that $\min\{\hat{J}_1, \hat{J}_2\} \geq 0$. Moreover, take a $\hat{w}$ s.t. $\hat{W}_2(\hat{w}) \leq 0$ and $\hat{w} \leq \log(\hat{\rho}\hat{U})$, then $0 = \hat{J}_2 = \min\{\hat{J}_2, \hat{J}_1\}$. Thus, $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. In conclusion, we can use $\max\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_1)$ and $\min\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_2)$:

$$\begin{aligned}\min\{\hat{J}_1, \hat{J}_2\} &= \hat{J}_2 - \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_2)\\ \max\{\hat{J}_1, \hat{J}_2\} &= \hat{J}_1 + \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_1)\end{aligned}$$

Using the quasi-variational inequality

$$\begin{aligned}a(\hat{J}_2, -\max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, -\max\{\hat{J}_2 - \hat{J}_1, 0\})\\ a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).\end{aligned}$$

Thus,

$$\begin{aligned}-a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq -(1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\})\\ a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).\end{aligned}$$

Summing these two equalities, we obtain

$$a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq 0$$

or equivalently,

$$a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0.$$

Next, we show that the previous inequality implies $a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0$. Define the set $\mathbb{X} = \{x : \hat{J}_2 > \hat{J}_1\}$. Then,

$$\begin{aligned}&a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\})\\ &= \frac{\sigma^2}{2}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0 \,\mathrm{d}x\right)\\ &\cdots + \hat{\gamma}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}} (\hat{J}_2 - \hat{J}_1) \,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}} (\hat{J}_2 - \hat{J}_1) \,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0 \,\mathrm{d}x\right)\\ &\cdots + (\hat{\rho} + \delta)\left(\int_{\mathbb{X}} \hat{J}_2(\hat{J}_2 - \hat{J}_1) \,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \hat{J}_1(\hat{J}_2 - \hat{J}_1) \,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0 \,\mathrm{d}\hat{w}\right)\\ &= \frac{\sigma^2}{2}\int_{\mathbb{X}} \left(\frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w} + \hat{\gamma}\int_{\mathbb{X}} \frac{\mathrm{d}(\hat{J}_2(\hat{w}) - \hat{J}_1)}{\mathrm{d}\hat{w}} (\hat{J}_2 - \hat{J}_1) \,\mathrm{d}\hat{w} + (\hat{\rho} + \delta)\left(\int_{\mathbb{X}} (\hat{J}_2 - \hat{J}_1)^2 \,\mathrm{d}\hat{w}\right)\\ &= a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).\end{aligned}$$

In conclusion, since $a(\cdot, \cdot)$ is a coercive bilinear form, $0 \geq a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq K||\max\{\hat{J}_2 - \hat{J}_1, 0\}||^2$. Thus, $\hat{J}_1 \geq \hat{J}_2$ a.e., and by continuity $\hat{J}_1 \geq \hat{J}_2$ for all $\hat{w}$. Applying similar arguments to $BR^h(\hat{J})$, we have that if $\hat{W}_1 \geq \hat{W}_2$, then $Q(\hat{W}_1) \geq Q(\hat{W}_2)$, so by Proposition A.3, there exists a fixed point. Moreover, the set of fixed points has a maximum and a minimum, i.e.,

$$\{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$$

has a $\hat{W}^{\min}$ and $\hat{W}^{\max}$ s.t. $\hat{W}^{\min} \leq \hat{W}^{*} \leq \hat{W}^{\max}$ for all $\hat{W}^{*} \in \{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$. To find the upper and lower bound, observe that we can write the non-trivial Nash equilibrium policies as

$$\hat{J}^{*}(w) = \max_{\{\tau^j \in \mathcal{T} : \tau^j \leq \tau^{h*}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right].$$

Since $\infty > \tau^{h*} \geq \tau_{(\log(\hat{\rho}\hat{U},0))}$,[15] we have that

$$\begin{aligned}
0 \leq \underline{\hat{J}} &= \max_{\{\tau^j \in \mathcal{T} : \tau^j \leq \tau_{(\log(\hat{\rho}\hat{U},0))}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \\
&\leq \max_{\{\tau^j \in \mathcal{T} : \tau^j \leq \tau^{h*}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \\
&= \hat{J}^{*}(w) \\
&\leq \max_{\{\tau^j \in \mathcal{T}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \\
&= \overline{\hat{J}}.
\end{aligned}$$

□

Step 2. This step proves the uniqueness of the fixed point. The first proposition shows that $Q : H_0^1(\mathbb{R}) \to H_0^1(\mathbb{R})$ is concave. Since the Q operator is only defined for non-negative functions, we assume that the domain is restricted to non-negative functions without loss of generality. Since the game's continuation region is bounded, flow payoffs are bounded. Therefore, the equilibrium value functions are also bounded. For these reasons, without loss of generality, we restrict the $Q : \mathcal{A} \to A$ operator in

$$\mathcal{A} = \{v \in H_0^1(\mathbb{R}) : v(\hat{w}) \in [0, \overline{v}],\ \forall \hat{w}\}$$

Observe that $\mathcal{A}$ order convex; i.e., if $a, b \in \mathcal{A}$ with $a \leq c \leq b$, then $c \in \mathcal{A}$.

Define the operator $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where

$$\alpha(\hat{W}', \hat{W}'') = \alpha(\hat{w})\hat{W}'(\hat{w}) + (1 - \alpha(\hat{w}))\hat{W}''(\hat{w}),$$

with $\alpha(\hat{w}) \in [0, 1]$.

Proposition B.6. *$Q : \mathcal{A} \to A$ is strongly order concave; i.e.,*

$$Q(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(Q(\hat{W}'), Q(\hat{W}''))$$

for all $\hat{W}' \leq \hat{W}''$.

Proof of Step 2 - Proposition B.6. Take $\hat{W}' \leq \hat{W}''$. The proof has three arguments. First, we show that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$. Then, with this result in hand, we show that the $BR^j(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(BR^j(\hat{W}'), BR^j(\hat{W}''))$. Finally, we show that $Q(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(Q(\hat{W}'), Q(\hat{W}''))$.

To see that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$, observe that since $\alpha(\hat{W}', \hat{W}'') \leq \hat{W}''$ and since $K(\hat{W}')$ is increasing, we have that

[15] $\tau_{(\log(\hat{\rho}\hat{U},0))} := \inf\{t \geq 0 : \hat{w}_t \notin (\log(\hat{\rho}\hat{U}, 0))\}$.

$K^j(\alpha\hat{W}' + (1-\alpha)\hat{W}'') \subset K^j(\hat{W}'')$. Now, we show that $K^j(\hat{W}'') \subset K^j(\alpha(\hat{W}', \hat{W}''))$. Take any $\hat{J} \in K^j(\hat{W}'')$. Then,

$$\hat{J} \geq 0, \ \& \ if \ \hat{W}''(\hat{w}) = 0 \ and \ \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0.$$

If $\hat{W}''(\hat{w}) = 0$, then $\hat{W}''(\hat{w}) \geq \hat{W}'(\hat{w}) = 0$, which is then also true for any convex combination. Thus, $\alpha(\hat{W}', \hat{W}'') \leq \hat{W}'' = 0$ and

$$\hat{J} \geq 0, \ \& \ if \ \alpha(\hat{W}', \hat{W}'') = 0 \ and \ \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0.$$

In conclusion, $\hat{J} \in K^j(\alpha(\hat{W}', \hat{W}''))$ and $K^j(\hat{W}'') \subset K^j(\alpha(\hat{W}', \hat{W}''))$. Therefore, we have that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$.

Since the constraint set—i.e., $\hat{W}$ and any test function v in $K^j(\cdot)$—is the same for $\alpha(\hat{W}', \hat{W}'')$ and $\hat{W}''$, we have that

$$\begin{aligned}
BR^j(\alpha(\hat{W}', \hat{W}'')) &= BR^j(\hat{W}''), \\
&= \alpha(BR^j(\hat{W}''), BR^j(\hat{W}'')), \\
&\geq \alpha(BR^j(\hat{W}'), BR^j(\hat{W}'')),
\end{aligned}$$

where we used the monotonicity of $BR^j(\hat{W})$ in the last inequality. A similar property holds for $BR^h(\hat{J})$. In conclusion, $BR^j(\hat{W})$ and $BR^h(\hat{J})$ are increasing and strongly order concave. Using this result, for $\hat{W}' \leq \hat{W}''$, we have

$$\begin{aligned}
Q(\alpha(\hat{W}', \hat{W}'')) &= BR^h(BR^j(\alpha(\hat{W}', \hat{W}''))) \\
&\geq^{(1)} BR^h(\alpha(BR^j(\hat{W}'), BR^j(\hat{W}''))) \\
&\geq^{(2)} \alpha(BR^h(BR^j(\hat{W}')), BR^h(BR^j(\hat{W}''))) \\
&= \alpha(Q(\hat{W}'), Q(\hat{W}'')).
\end{aligned}$$

Step (1) uses the monotonicity of $BR^h(\hat{J})$ and the strongly order concavity of $BR^j(\hat{W})$. Step (2) uses the strongly order concavity of $BR^h(\hat{J})$. □

Proposition B.7. *$Q : \mathcal{A} \to A$ has a unique fixed point.*

Proof of Step 2 - Proposition B.7. We have shown that $Q(\hat{W})$ is monotone and order concave defined in an order convex set. Now, we prove the result by contradiction. Let $\underline{\hat{W}}$ be the minimum fixed point and let $\hat{W}^*$ be another fixed point with $\hat{W}^* > \underline{\hat{W}}$. Then, we can write $\underline{\hat{W}} = \alpha^*(0, \hat{W}^*)$ for some $\alpha^*(\hat{w})$ function, where zero is the lower bound in the domain. Importantly, it is easy to see that $\alpha^*(\hat{w}) \in (0,1)$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}, 0))$. Thus,

$$\begin{aligned}
\underline{\hat{W}} &=^{(1)} Q(\underline{\hat{W}}) \\
&=^{(2)} Q(\alpha^*(0, \hat{W}^*)) \\
&\geq^{(3)} \alpha^*(Q(0), Q(\hat{W}^*)) \\
&=^{(4)} \alpha^*(Q(0), \hat{W}^*) \\
&>^{(5)} \alpha^*(0, \hat{W}^*) \\
&=^{(6)} \underline{\hat{W}}
\end{aligned}$$

Step (1) uses the fact that $\underline{\hat{W}}$ is a fixed point and step (2) uses the fact that $\underline{\hat{W}} = \alpha^*(0, \hat{W}^*)$. Step (3) uses the strongly order concavity of Q. Step (4) uses the fact that $\hat{W}^*$ is a fixed point. Step (5) uses that $Q(0) > 0$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}), 0)$. Since it cannot be that $\underline{\hat{W}} > \underline{\hat{W}}$, we have a contradiction. □

Step 3. Let $\hat{W}^*(\hat{w};\hat{\rho}\hat{U})$ and $\hat{J}^*(\hat{w};\hat{\rho}\hat{U})$ be the value functions from the unique non-trivial Nash equilibrium. We now show that they are continuous and decreasing in $\hat{U}$.

Proposition B.8. *Fix $\hat{J}$. Let $\hat{W}(\hat{w};\hat{\rho}\hat{U}) = BR^h(\hat{J};\hat{\rho}\hat{U})$ be the solution of*

$$a(\hat{W}, v-\hat{W}) \geq (1-\hat{\rho}\hat{U}, v-\hat{W}),\ \forall v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})$$

Then, $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is continuous and decreasing in $\hat{\rho}\hat{U}$.

Proof of Step 3 - Proposition B.8. First, we prove continuity. Take $\hat{U}_1$ and $\hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J};\hat{\rho}\hat{U}_1)$ and $\hat{W}_2 = BR^h(\hat{J};\hat{\rho}\hat{U}_2)$. Then,

$$a(\hat{W}_1, v-\hat{W}_1) \geq (1-\hat{\rho}\hat{U}_1, v-\hat{W}_1), \tag{B.31}$$

$$a(\hat{W}_2, v-\hat{W}_2) \geq (1-\hat{\rho}\hat{U}_2, v-\hat{W}_2). \tag{B.32}$$

Let $\hat{W}_2$ be the test function for (B.31) and let $\hat{W}_1$ be the test function for (B.32). Summing both equations

$$a(\hat{W}_1, \hat{W}_2-\hat{W}_1) + a(\hat{W}_2, \hat{W}_1-\hat{W}_2) \geq (1-\hat{\rho}\hat{U}_1, \hat{W}_2-\hat{W}_1) + (1-\hat{\rho}\hat{U}_2, \hat{W}_1-\hat{W}_2)$$

or equivalently

$$a(\hat{W}_1-\hat{W}_2, \hat{W}_2-\hat{W}_1) \geq (\hat{\rho}(\hat{U}_2-\hat{U}_1), \hat{W}_2-\hat{W}_1).$$

Multiplying by -1 on both sides and under the observation that $(\hat{\rho}(\hat{U}_2-\hat{U}_1), \hat{W}_2-\hat{W}_1) = \hat{\rho}(\hat{U}_2-\hat{U}_1)(1, \hat{W}_2-\hat{W}_1)$, we obtain

$$a(\hat{W}_2-\hat{W}_1, \hat{W}_2-\hat{W}_1) \leq \hat{\rho}(\hat{U}_1-\hat{U}_2)(1, \hat{W}_2-\hat{W}_1).$$

Given that the operator is coercive and that

$$(1, \hat{W}_2-\hat{W}_1) = \int_{\mathbb{R}} \left(\hat{W}(\hat{w};\hat{\rho}\hat{U}_2) - \hat{W}(\hat{w};\hat{\rho}\hat{U}_1)\right) \mathrm{d}\hat{w} \leq \left(\int_{\mathbb{R}} \left(\hat{W}(\hat{w};\hat{\rho}\hat{U}_2) - \hat{W}(\hat{w};\hat{\rho}\hat{U}_1)\right)^2 \mathrm{d}\hat{w}\right)^{1/2},$$

we have that

$$\beta||\hat{W}_2-\hat{W}_1||^2 \leq a(\hat{W}_2-\hat{W}_1, \hat{W}_2-\hat{W}_1) \leq \hat{\rho}(\hat{U}_1-\hat{U}_2)(1, \hat{W}_2-\hat{W}_1) \leq \hat{\rho}|\hat{U}_1-\hat{U}_2|||\hat{W}_2-\hat{W}_1||$$

for some $\beta > 0$. Thus,

$$||\hat{W}_2-\hat{W}_1|| \leq \frac{\hat{\rho}}{\beta}|\hat{U}_1-\hat{U}_2|$$

With this inequality, we can verify the continuity of $\hat{W}(\hat{w};\hat{\rho}\hat{U})$. Let $\epsilon > 0$ and choose $|\hat{U}_1-\hat{U}_2| < \epsilon\frac{\beta}{\hat{\rho}}$. Then

$$||\hat{W}_2-\hat{W}_1|| < \epsilon.$$

Thus, $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is continuous.

Now, we prove that $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is decreasing in the second argument. Let $\hat{U}_1 > \hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J};\hat{\rho}\hat{U}_1)$ and

$\hat{W}_2 = BR^h(\hat{J};\hat{\rho}\hat{U}_2)$. Observe that $\hat{W}_1, \hat{W}_2 \in K^h(\hat{J})$. Thus, $\min\{\hat{W}_1, \hat{W}_2\}$ and $\max\{\hat{W}_1, \hat{W}_2\} \in K^h(\hat{J})$. Therefore, we can use $\min\{\hat{W}_1, \hat{W}_2\} = \hat{W}_1 - \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_1$ and $\max\{\hat{W}_1, \hat{W}_2\} = \hat{W}_2 + \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_1$. Therefore,

$$-a(\hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq -(1 - \hat{\rho}\hat{U}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}),$$
$$a(\hat{W}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq (1 - \hat{\rho}\hat{U}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Adding both inequalities, we obtain

$$a(\hat{W}_2 - \hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq \hat{\rho}\left(\hat{U}_1 - \hat{U}_2\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Multiplying by -1 and under the observation that $a(\hat{W}^1 - \hat{W}^2, \max\{\hat{W}^1 - \hat{W}^2, 0\}) = a(\max\{\hat{W}^1 - \hat{W}^2, 0\}, \max\{\hat{W}^1 - \hat{W}^2, 0\}) \geq \beta|| \max\{\hat{W}^1 - \hat{W}^2, 0\}||^2$ for some $\beta > 0$, we have that

$$|| \max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{\rho}}{\beta}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Since $\hat{U}_1 > \hat{U}_2$, we have that $\hat{U}_2 - \hat{U}_1 < 0$. Assume, by contradiction, that $\hat{W}_1 > \hat{W}_2$, then$(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) > 0$. Operating,

$$0 < || \max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{\rho}}{\beta}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) < 0.$$

Thus, we have a contradiction. In conclusion, $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is decreasing in the second argument. Observe that $\hat{J}(\hat{w}) = BR^j(\hat{W})$ is independent of $\hat{\rho}\hat{U}$. □

Proposition B.9. *Let $\hat{W}^*(\hat{w};\hat{\rho}\hat{U})$ be the non-trivial Nash Equilibrium, then it is continuous and decreasing in the second argument.*

Proof of Step 3 - Proposition B.9. First, we show that the value function from the non-trivial Nash equilibrium is decreasing in $\hat{U}$. If $\hat{U}_1 > \hat{U}_2$, we have, by the previous step, that

$$Q(\hat{W}, \hat{\rho}\hat{U}_1) \leq Q(\hat{W}, \hat{\rho}\hat{U}_2).$$

Define recursively $Q^n(\hat{W}, \hat{\rho}\hat{U}_1) = Q \circ Q^{n-1}(\hat{W}, \hat{\rho}\hat{U}_1)$. By monotonicity,

$$Q^n(\hat{W}, \hat{\rho}\hat{U}_1) \leq Q^n(\hat{W}, \hat{\rho}\hat{U}_2)$$

also holds for all n. By Theorem 18 of Marinacci and Montrucchio (2019)

$$Q^n(\hat{W}, \hat{\rho}\hat{U}_1) \to \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_1) \text{ and } Q^n(\hat{W}, \hat{\rho}\hat{U}_2) \to \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_2).$$

Thus,

$$\hat{W}^*(\hat{w};\hat{\rho}\hat{U}_1) \leq \hat{W}^*(\hat{w};\hat{\rho}\hat{U}_2).$$

In conclusion, the non-trivial Nash equilibrium is decreasing in $\hat{U}$.

Now, we show continuity. Take $\hat{U}_n \uparrow \hat{U}^*$ (resp. $\hat{U}_n \downarrow \hat{U}^*$). Then, it is easy to see that $\hat{W}^*(\hat{w};\hat{\rho}\hat{U}^n)$ is monotonic, and by completeness, it is easy to see that $\hat{W}^*(\hat{w};\hat{\rho}\hat{U}^n)$ is a convergent series. Thus, $\hat{W}^*(\hat{w};\hat{\rho}\hat{U})$ is continuous in the second element. □

Step 4. We now show the existence of the unique fixed point in $\hat{\rho}\hat{U}$. Using the free entry condition, we can define the value of the unemployed worker as

$$P(\hat{\rho}\hat{U}) := \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

We now show two propositions: (i) we show some properties of $P(\hat{\rho}\hat{U})$, (ii) we use these properties to show the existence of a unique fixed point $P(\hat{\rho}\hat{U}^*) = \hat{\rho}\hat{U}^*$.

Proposition B.10. *The following properties hold for $P(\hat{\rho}\hat{U})$:*

- *$P(\hat{\rho}\hat{U})$ exists and is unique.*
- *$P(\hat{\rho}\hat{U})$ is continuous.*
- *$P : [\tilde{B}, \bar{P}] \to [\tilde{B}, \bar{P}]$ and it is decreasing.*

Proof of Step 4 - Proposition B.10. From Proposition 2, we have that $\hat{\mathcal{C}}^h \cap \hat{\mathcal{C}}^j$ is bounded, thus

$$\max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}) = \max_{\hat{w}\in cl\{\mathcal{C}^j\cap\mathcal{C}^h\}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

Since $\hat{J}(\cdot;\hat{\rho}\hat{U})$ and $\hat{W}(\cdot;\hat{\rho}\hat{U})$ are continuous and the optimization is conducted over a compact support, by the extreme value theorem there exists a maximum and, clearly, is unique. By Proposition 2, we have that $\bar{P} =: P(\tilde{B}) > \tilde{B}$. Since $\hat{J}(\hat{w};\hat{\rho}\hat{U})$ and $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ are continuous in both arguments, by the maximum theorem, the maximal value is continuous. Let $\hat{w}^*(\hat{\rho}\hat{U})$ be the solution to the optimization problem. Then, if $\hat{U} < \hat{U}'$,

$$\begin{aligned}
\frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U});\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U});\hat{\rho}\hat{U}) &\geq^{(1)} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U}');\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}) \\
&\geq^{(2)} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}')^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}').
\end{aligned}$$

Step (1) uses the optimality of $\hat{w}^*(\hat{\rho}\hat{U})$ and step (2) uses the fact that $\hat{J}$ and $\hat{W}$ are decreasing in the second argument. Thus, $P(\hat{\rho}\hat{U})$ is decreasing. Since $P(\hat{\rho}\hat{U}) \geq \tilde{B}$ ($\hat{J}(\cdot)$ and $\hat{W}(\cdot)$ are non-negative), we have that $P : [\tilde{B}, \bar{P}] \to [\tilde{B}, \bar{P}]$. □

Proposition B.11. *$P(\hat{\rho}\hat{U})$ has a unique fixed point.*

Proof of Step 4 - Proposition B.11. The existence of the fixed point follows directly from Brouwer's fixed point theorem. To show uniqueness, observe that if there were two fixed points $\hat{U}_1 < \hat{U}_2$, by definition, we would have that $P(\hat{\rho}\hat{U}_1) = \hat{\rho}\hat{U}_1 < \hat{\rho}\hat{U}_2 = P(\hat{\rho}\hat{U}_2)$ and $P(\hat{\rho}\hat{U})$ would be strictly increasing. By Step 4-Proposition B.10, this is a contradiction. □

B.4 Proof of Proposition 2

Proposition 2. *The recursive block equilibrium has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho}),$$

where

$$\mathcal{T}(\hat{w},\hat{\rho}) := \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\, \mathrm{d}t | \hat{w}_0 = \hat{w}\right]$$

is the expected discounted match duration and $1 > \hat{\rho}\hat{U} > \tilde{B}$.

2. *The competitive entry wage* $\hat{w}^*$ *coincides with the Nash bargaining solution with worker's weight* α*:*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{ \hat{W}(\hat{w})^{\alpha} \hat{J}(\hat{w})^{1-\alpha} \right\} = \arg\max_{\hat{w}} \left\{ \eta(\hat{w})^{\alpha} (1-\eta(\hat{w}))^{1-\alpha} \mathcal{T}(\hat{w}, \hat{\rho}) \right\},$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*) \left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)} \right)}_{\textit{Share channel}} = - \underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}}_{\textit{Surplus channel}}.$$

3. *Given* $\eta(\hat{w}^*)$ *and* $\mathcal{T}(\hat{w}^*, \hat{\rho})$*, the equilibrium job finding rate* $f(\hat{w}^*)$ *and the flow opportunity cost of employment* $\hat{\rho}\hat{U}$ *are given by*

$$f(\hat{w}^*) = \left[(1-\eta(\hat{w}^*))(1-\hat{\rho}\hat{U}) \mathcal{T}(\hat{w}^*, \hat{\rho}) / \tilde{K} \right]^{\frac{1-\alpha}{\alpha}},$$
$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1} (1-\eta(\hat{w}))^{1-\alpha} \eta(\hat{w})^{\alpha} \left(1-\hat{\rho}\hat{U}\right) \mathcal{T}(\hat{w}^*, \hat{\rho}) \right)^{\frac{1}{\alpha}}.$$

4. *Given* $\hat{U}$*, the worker's and the firm's continuation sets are connected, and the game's continuation set is bounded, i.e.*

$$\{\hat{w} : \hat{w} > \hat{w}^-\} = \hat{\mathcal{C}}^h,$$
$$\{\hat{w} : \hat{w} < \hat{w}^+\} = \hat{\mathcal{C}}^j.$$

with $-\infty < \hat{w}^- \leq \log(\hat{\rho}\hat{U}) < 0 \leq < \hat{w}^+ < \infty$ *if* $\hat{\rho} + \delta + \hat{\gamma} - \sigma^2/2 > 0$.

Proof. Now, we prove each equilibrium property.

1. Using free entry condition and By the equilibrium conditions, for all $\hat{w}$ we have that $\hat{\theta}(\hat{w}) \geq 0$ and $\hat{W}(\hat{w}) \geq 0$; thus, the product is also non-negative in $\hat{w}^*$

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha} \hat{W}(\hat{w}^*) \geq \hat{B} \iff \hat{\rho}U \geq \tilde{B}$$

Using the recursive definition of the value function, we have that

$$\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (e^{\hat{w}_t} - \hat{\rho}\hat{U}) \, \mathrm{d}t \right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (1 - e^{\hat{w}_t}) \, \mathrm{d}t \right]$$

where τ^{m*} is the Nash equilibrium in the math with the exogenous separations. Summing up the previous two equations

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (1 - \hat{\rho}\hat{U}) \, \mathrm{d}t \right] = (1-\hat{\rho}\hat{U}) \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} \, \mathrm{d}t \right].$$

Since $\hat{W}(\hat{w}), \hat{J}(\hat{w}) \geq 0$, $\hat{S}(\hat{w}) \geq 0$ and

$$0 \leq \hat{S}(\hat{w}^*) = (1-\hat{\rho}\hat{U}) \underbrace{\mathcal{T}(\hat{w}^*, \hat{\rho})}_{>0} \iff 0 \leq 1 - \hat{\rho}\hat{U} \iff 1 \geq \hat{\rho}\hat{U}.$$

So, $1 \geq \hat{\rho}\hat{U} \geq \tilde{B}$.

Now, we show the strict inequality by contradiction. Assume that $\hat{\rho}\hat{U} = \tilde{B}$. Then $\forall \hat{w}$, we have that $0 = \hat{\theta}(\hat{w}) = \hat{W}(\hat{w}) = \hat{S}(\hat{w}) = (1-\tilde{B})\mathcal{T}(\hat{w},\hat{\rho})$. Thus, $\forall \hat{w}$, $\mathcal{T}(\hat{w},\hat{\rho}) = 0$ which breaks the non-trivial Nash equilibrium. Assume that $\hat{\rho}\hat{U} = 1$. Then $\tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*) = \hat{B} < 1$, and we have the contradiction.

2. To show this property, first we show that $\hat{J}(\hat{w}) > 0$ for all $\hat{w} \in (\hat{\rho}\hat{U}, 0)$. Define

$$\tau_{(w^-,0)} = \inf_t \{t : \hat{w}_t \notin (\tilde{\rho}\tilde{U}, 0)\}.$$

By optimality of the firm,

$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta)t}(1-e^{\hat{w}_t})\,\mathrm{d}t\right] \geq \mathbb{E}_{\hat{w}}\left[\int_0^{\min\{\tau_{(\hat{\rho}\hat{U},0)},\tau^{m*}\}} e^{-(\tilde{\rho}+\delta)t}(1-e^{\hat{w}_t})\,\mathrm{d}t\right] > 0.$$

Thus, there is an open set s.t. $\hat{J}(\hat{w}) > 0$, $\hat{\theta}(\hat{w}) > 0$, and $\hat{J}(\hat{w}) - \hat{K}\hat{\theta}(\hat{w})^{\alpha} = 0$. Therefore,

$$\arg\max_{\hat{w}} \left\{p(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})\right\} = \arg\max_{\hat{w}} \left\{\left(\frac{\hat{J}(\hat{w})}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w})\right\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-\alpha}\hat{W}(\hat{w})^{\alpha}\right\}.$$

Since $\hat{W}(\hat{w}) = \eta(\hat{w})\hat{S}(\hat{w})$ and $\hat{J}(\hat{w}) = (1-\eta(\hat{w}))\hat{S}(\hat{w})$ and $\hat{S}(\hat{w}) = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})$,

$$\arg\max_{\hat{w}} \left\{p(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})\right\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-\alpha}\hat{W}(\hat{w})^{\alpha}\right\} = \arg\max_{\hat{w}} \left\{(1-\eta(\hat{w}))^{1-\alpha}\eta(\hat{w})^{\alpha}\mathcal{T}(\hat{w},\hat{\rho})\right\}.$$

Taking first order conditions

$$\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}.$$

3. This step comes directly from the optimality conditions.

4. Now, we show the property. To show that $\hat{C}^h$ and $\hat{C}^j$ are connected, assume they are not. Without loss of generality, assume that $(\hat{C}^h)^c = \{\hat{w} : \hat{w} > \hat{w}^-\} \cup (a,b)$ with $a < b < w^-$. Then, since $\hat{w}^- \leq \tilde{\rho}\tilde{U}$, it must be the case that for all $\hat{w} \in (a,b)$, we have $(e^{\hat{w}} - \tilde{\rho}\tilde{U}) < 0$ for all $\hat{w} \in (a,b)$, and $\tilde{W}(x) = \mathbb{E}_x\left[\int_0^{\tau_{(S^h)^c\cap(S^j)^c}} e^{-(\tilde{\rho}+\delta)t}(e^{w_t} - \tilde{\rho}\tilde{U})\,\mathrm{d}t\right] < 0$ for all $x \in (a,b)$ due to continuity of Brownian motions. Since $\hat{W}(x) \geq 0$, we have a contradiction. A similar argument holds for the firm's continuation set.

We prove that $-\infty < \hat{w}^-$ by contradiction. Assume that $-\infty = \hat{w}^-$, then

$$\hat{W}(\hat{w},\hat{w}^+) = \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)}} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t\right].$$

Then, since $\hat{\rho}\hat{U} < e^{\hat{w}^+}$, it is easy to show s

$$\begin{aligned}
\hat{W}(\hat{w}) &= \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)}\wedge\tau^{\delta}} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t|\hat{w}_0 = \hat{w}\right] \\
&\leq \mathbb{E}\left[\int_0^{\infty} e^{-(\hat{\rho}+\delta)t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t|\hat{w}_0 = \hat{w}\right] \\
&= \frac{e^{w_0}}{\hat{\rho}+\delta+\hat{\gamma}-\sigma^2/2} - \frac{\hat{\rho}\hat{U}}{\hat{\rho}+\delta}.
\end{aligned}$$

Thus, there exists a small enough $\hat{w}_0$ s.t. $\hat{W}(\hat{w}) < 0$ and we have a contradiction. A similar argument holds for the firm's separation threshold.

$\square$

B.5 Proof of Propositions 3, 4, and 5

Define $\hat{C} = (\hat{w}^-, \hat{w}^+)$. From proposition 2 and , we can work with the following HJB conditions

$$(\tilde{\rho}+\delta)\tilde{W}(\hat{w}) = e^{\hat{w}} - \tilde{\rho}\tilde{U} - \hat{\gamma}\tilde{W}'(\hat{w}) + \frac{\sigma^2}{2}\tilde{W}''(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{B.33}$$

$$(\tilde{\rho}+\delta)\tilde{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\tilde{J}'(\hat{w}) + \frac{\sigma^2}{2}\tilde{J}''(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{B.34}$$

$$\tilde{\rho}\tilde{U} = \tilde{B} + \kappa^{1-\alpha^{-1}}\tilde{J}(\hat{w}^*)^{\frac{1-\alpha}{\alpha}}\tilde{W}(\hat{w}^*)$$

$$(1-\alpha)\frac{\mathrm{d}\log\tilde{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{d}\log\tilde{W}(\hat{w}^*)}{\mathrm{d}\hat{w}},$$

with the value matching

$$\tilde{W}(\hat{w}^-) = \tilde{J}(\hat{w}^-) = \tilde{W}(\hat{w}^+) = \tilde{J}(\hat{w}^+) = 0;$$

and smooth pasting conditions

$$\tilde{W}'(\hat{w}^-) = \tilde{J}'(\hat{w}^+) = 0.$$

Proposition 3. *Assume $\hat{\gamma} = \sigma = 0$. Then, the optimal policies are given by*

$$(\hat{w}^-,\ \hat{w}^*,\ \hat{w}^+) = \log(\hat{\rho}\hat{U}, \alpha + (1-\alpha)\tilde{\rho}\tilde{U}, 1),$$

with $\eta(\hat{w}) = \alpha$ and $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = (\hat{\rho}+\delta)^{-1}$.

Proof. If $\hat{\gamma} = \sigma = 0$, conditions (B.33) and (B.34) imply

$$\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - e^{\hat{w}^-}}{\hat{\rho}+\delta} \quad ; \quad \hat{J}(\hat{w}) = \frac{e^{\hat{w}^+} - e^{\hat{w}}}{\hat{\rho}+\delta}.$$

The variation inequalities imply

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = \max\{0, e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \tilde{\mathcal{C}}^j,$$

$$(\hat{\rho}+\delta)\tilde{J}(\hat{w}) = \max\{0, 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \tilde{\mathcal{C}}^h,.$$

Thus, $\tilde{W}(\hat{w}^-) = \tilde{J}(\hat{w}^+) = 0$ and

$$\hat{w}^+ = 0 \quad ; \quad \hat{w}^- = \log\left(\tilde{\rho}\tilde{U}\right).$$

Since

$$\mathcal{T}(\hat{w}, \hat{\rho}) = \begin{cases} (\hat{\rho}+\delta)^{-1} & \text{if } \hat{w} \in [\hat{w}^-, \hat{w}^+] \\ 0 & \text{Otherwise} \end{cases}$$

Since $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$, we have that $\eta(\hat{w}) = \alpha$. $\square$

Proposition 4. *Assume $\hat{\gamma} = 0$, $\alpha = 1/2$ and a first-order approximation of the flow payoffs around $\hat{w}^*$. Then $\hat{w}^{\pm} = \hat{w}^* \pm h(\varphi, \Phi)$*

$$e^{\hat{w}^*} = \frac{1+\tilde{\rho}\tilde{U}}{2} \text{ and } k(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$$

with $\varphi = \sqrt{2(\rho+\delta)}/\sigma$, $\Phi = \frac{1-\tilde{\rho}\tilde{U}}{1+\tilde{\rho}\tilde{U}}$. The following properties hold for ω: (i) $\omega(z)$ decreases for all $z \in (0,\infty)$, (ii) $\lim_{z\to 0}\omega(z) = 3$, (iii) $\lim_{z\to\infty}\omega(z) = 1$, (iv) $\omega(2\varphi\Phi)\Phi$ is increasing in Φ, and (v) $\varphi\omega(2\varphi\Phi)$ increasing. $\eta(\hat{w}) = \alpha$ and

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1 - sech(\varphi k(\varphi,\Phi))}{\tilde{\rho}+\delta}$$

increasing in φ and Φ.

Proof. Let us guess and verify the following solution $w^* = \log\left(\frac{1+\hat{\rho}\hat{U}}{2}\right)$ and $\hat{w}^- = \hat{w}^* - k$ and $\hat{w}^+ = \hat{w}^* + k$ for a given k. Using a Taylor approximation over the flow profits around w^*

$$e^{\hat{w}} - \hat{\rho}\hat{U} \approx e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) - \tilde{\rho}\tilde{U} = \frac{1-\tilde{\rho}\tilde{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*),$$
$$1 - e^{\hat{w}} \approx 1 - e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) = \frac{1-\tilde{\rho}\tilde{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*).$$

We can write the optimality conditions as

$$(\tilde{\rho}+\delta)\tilde{W}(\hat{w}) = \frac{1-\tilde{\rho}\tilde{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\tilde{W}''(\hat{w}), \quad \forall \hat{w} \in (w^* - k, w^* + k)$$
$$(\tilde{\rho}+\delta)\tilde{J}(\hat{w}) = \frac{1-\tilde{\rho}\tilde{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\tilde{J}''(\hat{w}), \quad \forall \hat{w} \in (w^* - k, w^* + k)$$

with the border conditions

$$\hat{W}(\hat{w}^* - k) = \hat{J}(\hat{w}^* - k) = \hat{W}(\hat{w}^* + k) = \hat{J}(\hat{w}^* + k) = 0,$$
$$\hat{W}'(\hat{w}^* - k) = \hat{J}'(\hat{w}^* + k) = 0.$$

Now, we show that we can transform $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\tilde{\rho}\tilde{U}}{2(\tilde{\rho}+\delta)}}{e^{w^*}}$. A similar argument applies to the value function of the worker. Making the following transformation $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\tilde{\rho}\tilde{U}}{2(\tilde{\rho}+\delta)}}{e^{w^*}}$, and using (B.34)

$$\begin{aligned}(\tilde{\rho}+\delta)J(x) &= (\tilde{\rho}+\delta)\left(\frac{\hat{J}(x+\hat{w}^*) - \frac{1-\tilde{\rho}\tilde{U}}{2(\tilde{\rho}+\delta)}}{e^{w^*}}\right),\\ &= -x + \frac{\sigma^2}{2}\frac{1}{e^{w^*}}\hat{J}''(x+\hat{w}^*),\\ &= -x + \frac{\sigma^2}{2}J''(x).\end{aligned}$$

Thus,

$$(\tilde{\rho}+\delta)W(x) = x + \frac{\sigma^2}{2}W''(x) \quad \forall x \in (-k,k)$$
$$(\tilde{\rho}+\delta)J(x) = -x + \frac{\sigma^2}{2}J''(x) \quad \forall x \in (-k,k)$$

Defining

$$\Phi = \frac{\frac{1-\tilde{\rho}\tilde{U}}{2}}{e^{w^*}} = \frac{1-\tilde{\rho}\tilde{U}}{1+\tilde{\rho}\tilde{U}} > 0,$$

it is easy to show

$$W(k) = J(k) = W(-k) = J(-k) = -\frac{\Phi}{\tilde{\rho}+\delta} \quad ; \quad W'(-k) = J'(k) = 0.$$

Thus, $W(x) = J(-x)$. Given that this problem symmetric, we verify the guess of symmetry of the Ss bands and $\frac{1}{2}W'(0) = -\frac{1}{2}J(-0)$. The latter property, implies that w^* satisfies the Nash bargaining solution.

Now, we show that $k = \omega(\varphi)\Phi$ with $\varphi = \sqrt{2\rho+\delta}/\sigma$. Note that $W(x) = J(-x)$. Thus, we can only focus on $W(x)$ using the smooth pasting condition over $-k$. The solution to this system of differential equations is given by

$$\begin{aligned} W(x) &= Ae^{\varphi x} + Be^{-\varphi x} + \frac{x}{\tilde{\rho}+\delta} \\ W(k) &= W(-k) = -\frac{\Phi}{\tilde{\rho}+\delta} \;\; and \;\; W'(-h) = 0 \end{aligned}$$

with $\varphi = \sqrt{2(\tilde{\rho}+\delta)/\sigma^2}$. Writing the value matching conditions

$$\begin{aligned} Ae^{\varphi k} + Be^{-\varphi k} + \frac{k}{\tilde{\rho}+\delta} &= -\frac{\Phi}{\tilde{\rho}+\delta} \\ Ae^{-\varphi k} + Be^{\varphi k} - \frac{k}{\tilde{\rho}+\delta} &= -\frac{\Phi}{\tilde{\rho}+\delta} \end{aligned}$$

Taking the difference and the sum

$$\begin{aligned} A\left(e^{\varphi k} + e^{-\varphi k}\right) + B\left(e^{-\varphi k} + e^{\varphi k}\right) &= -2\frac{\Phi}{\tilde{\rho}+\delta} \\ A\left(e^{\varphi k} - e^{-\varphi k}\right) + B\left(e^{-\varphi k} - e^{\varphi k h}\right) &= -2\frac{k}{\tilde{\rho}+\delta} \end{aligned}$$

Therefore

$$\begin{aligned} A &= \frac{-2\frac{\Phi}{\tilde{\rho}+\delta}\left(e^{-\varphi k} - e^{\varphi k}\right) + 2\frac{h}{\tilde{\rho}+\delta}\left(e^{-\varphi k} + e^{\varphi k}\right)}{\left(e^{\varphi k} + e^{-\varphi k}\right)\left(e^{-\varphi k} - e^{\varphi k}\right) - \left(e^{-\varphi k} + e^{\varphi k}\right)\left(e^{\varphi k} - e^{-\varphi k}\right)} \\ &= \frac{e^{-\varphi k}\left(-\frac{\Phi}{\tilde{\rho}+\delta} + \frac{h}{\tilde{\rho}+\delta}\right) + e^{\varphi k}\left(\frac{h}{\tilde{\rho}+\delta} + \frac{\Phi}{\tilde{\rho}+\delta}\right)}{\left(e^{\varphi k} + e^{-\varphi k}\right)\left(e^{-\varphi k} - e^{\varphi k}\right)} \\ &= -\frac{1}{\tilde{\rho}+\delta}\frac{e^{-\varphi k}\left(-\Phi + k\right) + e^{\varphi k}\left(k + \Phi\right)}{e^{2\varphi k} - e^{-2\varphi k}} \\ B &= \frac{-2\frac{h}{\tilde{\rho}+\delta}\left(e^{\varphi k} + e^{-\varphi k}\right) + 2\frac{\Phi}{\tilde{\rho}+\delta}\left(e^{\varphi k} - e^{-\varphi k}\right)}{\left(e^{\varphi k} + e^{-\lambda k}\right)\left(e^{-\varphi k} - e^{\varphi k}\right) - \left(e^{-\varphi k} + e^{\varphi k}\right)\left(e^{\varphi k} - e^{-\varphi k}\right)} \\ &= -\frac{e^{\varphi k}\left(-\frac{\Phi}{\tilde{\rho}+\delta} + \frac{h}{\tilde{\rho}+\delta}\right) + e^{-\varphi k}\left(\frac{h}{\tilde{\rho}+\delta} + \frac{\Phi}{\tilde{\rho}+\delta}\right)}{\left(e^{\varphi k} + e^{-\varphi k}\right)\left(e^{-\lambda k} - e^{\lambda k}\right)} \\ &= \frac{1}{\tilde{\rho}+\delta}\frac{e^{\varphi k}\left(-\Phi + k\right) + e^{-\varphi k}\left(k + \Phi\right)}{e^{2\varphi k} - e^{-2\varphi k}} \end{aligned}$$

Therefore

$$W(x) = -\frac{1}{\tilde{\rho}+\delta}\frac{e^{-\varphi k}\left(-\Phi + k\right) + e^{\varphi k}\left(k + \Phi\right)}{e^{2\varphi k} - e^{-2\varphi k}}e^{\lambda x} + \frac{1}{\tilde{\rho}+\delta}\frac{e^{\varphi k}\left(\Phi + k\right) + e^{-\varphi k}\left(k - \Phi\right)}{e^{2\varphi k} - e^{-2\varphi k}}e^{-\lambda x} + \frac{x}{\tilde{\rho}+\delta}$$

Taking the derivative and evaluating in $x = -k$

$$W'(-h) = -\frac{e^{-\varphi k}\left(-\Phi+k\right)+e^{\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}\varphi e^{-\varphi k} - \frac{e^{\varphi k}\left(-\Phi+k\right)+e^{-\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}\varphi e^{\varphi k}+1=0$$

or equivalently

$$-\Phi(e^{-2\varphi k}+e^{2\varphi k}-2) = \frac{1}{\varphi}(e^{2\varphi k}-e^{-2\varphi k}) - \frac{1}{2\varphi}2\varphi k\left(e^{2\varphi k}+e^{-2\varphi k}+2\right) \tag{B.35}$$

It would be useful to express equation (B.35) using $\sinh(x) = \frac{e^x-e^{-x}}{2}$ and $\cosh(x) = \frac{e^x+e^{-x}}{2}$. Using the hyperbolic functions

$$-\Phi 2\left(\cosh(2\varphi k)-1\right) = \frac{2\sinh(2\varphi k)}{\varphi} - 2k\left(\cosh(2\varphi k)+1\right)$$

Multiplying both sides by φ

$$-\Phi 2\varphi\left(\cosh(2\varphi k)-1\right) = 2\sinh(2\varphi k) - \varphi 2k\left(\cosh(2\varphi k)+1\right)$$

While in principle $k(\varphi,\Phi)$, we can change variables with $x \equiv 2\varphi k$ and x as the implicit solution of

$$-\Phi 2\varphi\left(\cosh(x)-1\right) + x\left(\cosh(x)+1\right) = 2\sinh(x).$$

Thus, $k = \frac{x(\Phi 2\varphi)}{2\varphi}$. Let $b = \Phi 2\varphi > 0$, then we can express the function $x(\cdot)$ as the solution of

$$b = -\frac{2\sinh(x(b)) - x(b)\left(\cosh(x(b))+1\right)}{\left(\cosh(x(b))-1\right)}$$

Notice that if we define

$$f(x) = -\frac{2\sinh(x) - x\left(\cosh(x)+1\right)}{\left(\cosh(x)-1\right)}$$

The following properties hold over $f(x)$

1. $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$.
2. $f(x)$ is increasing and convex, with $\lim_{x\to 0} f'(x) = 1/3$ and $\lim_{x\to\infty} f(x) = 1$.
3. $\frac{\mathrm{d}\log(f(x))}{\mathrm{d}\log(x)} > 1$.

Given these properties, we can write $k(\varphi,\Phi) = \frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ and show the following properties over $k(\varphi,\Phi)$

1. $k(\varphi,\Phi)$ is increasing in Φ: Since $f^{-1}(\cdot)$ is increasing ,we have the result.
2. $k(\varphi,\Phi)$ is decreasing in φ: Taking derivative of $k(\varphi,\Phi) = \frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ with φ and operating

$$\begin{aligned}
\frac{\partial k(\varphi,\Phi)}{\partial\varphi} &= \left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\Phi}{2\varphi} - \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2} \\
&= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)} - 1\right] \\
&= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}\log(x)}{\mathrm{d}\log(f(x))}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)} - 1\right] \\
&< 0.
\end{aligned}$$

3. $\lim_{\varphi\downarrow 0} k(\varphi,\Phi) = 3\Phi$ and $\lim_{\varphi\to\infty} k(\varphi,\Phi) = \Phi$: Applying L'hospital and using the derivative of the inverse property

$$\lim_{\varphi\to\infty} k(\varphi,\Phi) = \lim_{\varphi\to\infty} \frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\to\infty} \frac{1}{f'(2\varphi\Phi)}\Phi = \Phi$$
$$\lim_{\varphi\downarrow 0} k(\varphi,\Phi) = \lim_{\varphi\downarrow 0} \frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\downarrow 0} \frac{1}{f'(2\varphi\Phi)}\Phi = 3\Phi$$

4. $k(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$: Define $\omega(z) = \frac{f^{-1}(z)}{z}$, then it is easy to see that $h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$. Moreover, from property 2 and 3, $\omega(z)$ is decreasing with $\lim_{z\downarrow 0}\omega(z) = 3$ and $\lim_{z\to\infty}\omega(z) = 1$. Moreover, with similar argument, it is easy to show that $\omega(2\varphi\Phi)\Phi$ is increasing in Φ and $\omega(2\varphi\Phi)\varphi$ increasing in φ.

Now, we can compute $\eta(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*,\hat{\rho})$. Note that we can define $T(x) = \mathcal{T}(\hat{w} - \hat{w}^*,\hat{\rho})$, which solves

$$(\hat{\rho}+\delta)T(x) = 1 + \frac{\sigma^2}{2}T''(x), \text{ with } T(\pm k(\varphi,\Phi)) = 0.$$

The solution to this differential equation is given by

$$T(x) = \frac{-\frac{e^{\varphi x}+e^{-\varphi x}}{e^{\varphi k}+e^{-\varphi k}} + 1}{\tilde{\rho}+\delta}.$$

Thus, $T'(0) = 0$ and $\eta(\hat{w}^*) = \alpha$

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{-\frac{e^{\varphi x}-e^{-\varphi x}}{e^{\varphi k}-e^{-\varphi k}} + 1}{\tilde{\rho}+\delta}$$

Using the property that $sech(x) = \frac{2}{e^x+e^{-x}}$, we have

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1 - sech(\varphi\omega(2\varphi\Phi)\Phi)}{\tilde{\rho}+\delta}.$$

□

Proposition 5. *Assume $\sigma = 0$ and $\hat{\gamma} \geq 0$. Then $\hat{w}^- = \log\left(\tilde{\rho}\tilde{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right).$$

$\tilde{T}(\cdot)$ is defined as

$$a = e^{\tilde{T}(a,b,c)}\frac{1-e^{-(1+b)\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\frac{b}{b+1} - cb\frac{e^{-b\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\left[1 - \frac{b+1}{b}\frac{1-a^{-b\tilde{T}(a,b,c)}}{e^{\tilde{T}(a,b,c)} - e^{-b\tilde{T}(a,b,c)}}\right] \tag{B.36}$$

where $\tilde{T}(\cdot)$ is increasing in the first argument and decreasing in the second argument. The expected discounted duration and worker's share satisfies:

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}{\hat{\rho}+\delta}$$
$$\eta(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)}\frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}\frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U}$$

Moreover,

1. *If $\hat{\gamma} = 0$, then*

$$\left(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)\right) \to \left(\log\left(\frac{\alpha + (1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}\right), \frac{1}{\hat{\rho}+\delta}, \alpha\right).$$

2. *If $\hat{\gamma} \to \infty$, then $\tilde{T}(\cdot) \to \tilde{T}^{limit}$ where*

$$\frac{\alpha + (1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}} = \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\rho}\tilde{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}},$$

$\mathcal{T}(\hat{w}^*, \hat{\rho}) \to 0$ and $\eta(\hat{w}^*) \to \eta^{limit}$

$$\eta^{limit} = \alpha + \frac{1-\alpha}{\tilde{T}^{limit}} \frac{(1-\hat{\rho}\tilde{U})\eta^{limit}}{\eta^{limit} + \hat{\rho}\tilde{U}(1-\eta^{limit})}$$

Proof. Now, we take the $\sigma \downarrow 0$. The equilibrium conditions in this case are

$$\begin{aligned}
(\tilde{\rho}+\delta)\tilde{W}(\hat{w}) &= e^{\hat{w}} - \tilde{\rho}\tilde{U} - \hat{\gamma}\tilde{W}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{C}} \\
(\tilde{\rho}+\delta)\tilde{J}(\hat{w}) &= 1 - e^{\hat{w}} - \hat{\gamma}\tilde{J}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{C}} \\
(1-\alpha)\frac{\mathrm{d}\log \tilde{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} &= -\alpha\frac{\mathrm{d}\log \tilde{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}
\end{aligned}$$

with VM and SP

$$\tilde{W}(\hat{w}^-) = \tilde{J}(\hat{w}^-) = \tilde{W}(\hat{w}^+) = \tilde{J}(\hat{w}^+) = 0 \quad ; \quad \tilde{W}'(\hat{w}^-) = \tilde{J}'(\hat{w}^+) = 0.$$

Without idiosyncratic shocks and $\gamma > 0$ the upper Ss band is not active. Thus, we discard the optimality condition for this $\hat{w}^+$. The stopping time is a deterministic function in this case, hence, it is easier to work in the sequential formulation.

$$\begin{aligned}
\tilde{W}(\hat{w}) &= \max_T \int_0^T e^{-(\tilde{\rho}+\delta)s}\left(e^{\hat{w}-\hat{\gamma}s} - \hat{\rho}\tilde{U}\right)\mathrm{d}s \\
\tilde{J}(\hat{w}) &= \int_0^{T(w)} e^{-(\hat{\rho}+\delta)s}\left(1 - e^{\hat{w}-\hat{\gamma}s}\right)\mathrm{d}s.
\end{aligned} \tag{B.37}$$

In equation (B.37), $T(\hat{w})$ is the optimal policy of the worker. Taking the first order conditions with $T(\hat{w})$

$$e^{\hat{w}-\hat{\gamma}T(\hat{w})} = \hat{\rho}\hat{U}.$$

Solving the previous equation

$$T(\hat{w}) = \frac{\hat{w} - \log\left(\hat{\rho}\hat{U}\right)}{\hat{\gamma}}.$$

Thus, if $\hat{w} = \hat{w}^*$ we have that $\hat{w}^* - \hat{\gamma}T(\hat{w}^*) = \hat{w}^-$ satisfies

$$\hat{w}^- = \log(\hat{\rho}\tilde{U}).$$

Taking the derivatives of $\tilde{W}(\hat{w})$ and $\tilde{J}(\hat{w})$, and using the envelope condition for $\tilde{W}'(\hat{w})$, we have that

$$\tilde{W}'(\hat{w}) = \int_0^{T(w)} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}-\hat{\gamma}s}\right)\mathrm{d}s, \tag{B.38}$$

$$\tilde{J}'(\hat{w}) = -\int_0^{T(w)} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}-\hat{\gamma}s}\right)\mathrm{d}s + e^{-(\hat{\rho}+\delta)T(\hat{w})}\left(1 - e^{\hat{w}-\hat{\gamma}T(\hat{w})}\right)\underbrace{T'(\hat{w})}_{=1/\hat{\gamma}}. \tag{B.39}$$

From equations (B.38) and (B.39), we get the Nash bargaining solution

$$-\alpha \frac{\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(e^{\hat w^*-\hat\gamma s}\right)\mathrm{d}s}{\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(e^{\hat w^*-\hat\gamma s}-\hat\rho\tilde U\right)\mathrm{d}s} = (1-\alpha)\frac{\left[-\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(e^{\hat w^*-\hat\gamma s}\right)\mathrm{d}s + e^{-(\hat\rho+\delta)T^*}\frac{(1-\hat\rho\tilde U)}{\hat\gamma}\right]}{\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(1-e^{\hat w^*-\hat\gamma s}\right)\mathrm{d}s} \tag{B.40}$$

Define

$$\Omega(a,T^*) := \frac{1-e^{-aT^*}}{a}$$
$$\mathcal{Z} := \frac{e^{-(\hat\rho+\delta)T^*}(1-\hat\rho\tilde U)}{\hat\gamma\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(e^{\hat w^*-\hat\gamma s}\right)\mathrm{d}s}$$

Operating

$$\alpha\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(1-e^{\hat w^*-\hat\gamma s}\right)\mathrm{d}s = (1-\alpha)\int_0^{T^*} e^{-(\hat\rho+\delta)s}\left(e^{\hat w^*-\hat\gamma s}-\hat\rho\tilde U\right)\mathrm{d}s\,[1-\mathcal{Z}] \iff$$

$$\alpha\left[\Omega(\hat\rho+\delta,T^*) - e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)\right] = (1-\alpha)\left[e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*) - \hat\rho\tilde U\Omega(\hat\rho+\delta,T^*)\right]\times$$
$$\ldots\left[1-\frac{e^{-(\hat\rho+\delta)T^*}(1-\hat\rho\tilde U)}{\hat\gamma e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)}\right] \iff$$

$$\left(\alpha+(1-\alpha)\hat\rho\tilde U\right)\Omega(\hat\rho+\delta,T^*) = e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)+\ldots$$
$$\ldots(1-\alpha)\left[e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*) - \hat\rho\tilde U\Omega(\hat\rho+\delta,T^*)\right]\frac{e^{-(\hat\rho+\delta)T^*}(1-\hat\rho\tilde U)}{\hat\gamma e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)} \iff$$

$$\left(\alpha+(1-\alpha)\hat\rho\tilde U\right)\Omega(\hat\rho+\delta,T^*) = e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*) - \frac{(1-\alpha)e^{-(\hat\rho+\delta)T^*}(1-\hat\rho\tilde U)}{\hat\gamma}\left[1-\hat\rho\tilde U\frac{\Omega(\hat\rho+\delta,T^*)}{e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)}\right]$$

The policy $(T^*,\hat w^*)$ solves

$$e^{\hat w^*-\hat\gamma T^*} = \hat\rho\tilde U$$
$$\left(\alpha+(1-\alpha)\hat\rho\tilde U\right)\Omega(\hat\rho+\delta,T^*) = e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*) - \frac{(1-\alpha)e^{-(\hat\rho+\delta)T^*}(1-\hat\rho\tilde U)}{\hat\gamma}\left[1-\hat\rho\tilde U\frac{\Omega(\hat\rho+\delta,T^*)}{e^{\hat w^*}\Omega(\hat\rho+\delta+\hat\gamma,T^*)}\right]$$

Define $\tilde T = \hat\gamma T^*$ and $\Omega(a,T^*) := \frac{1-e^{-aT^*}}{a} = \hat\gamma^{-1}\Omega(\frac{a}{\hat\gamma},\tilde T)$. Then

$$e^{\hat w^*-\tilde T} = \hat\rho\tilde U$$
$$\left(\alpha+(1-\alpha)\hat\rho\tilde U\right)\hat\gamma^{-1}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right) = e^{\hat w^*}\hat\gamma^{-1}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right) - \frac{(1-\alpha)e^{-\frac{\hat\rho+\delta}{\hat\gamma}\tilde T}(1-\hat\rho\tilde U)}{\hat\gamma}\left[1-\hat\rho\tilde U\frac{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)}{e^{\hat w^*}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)}\right]$$

Therefore, the optimal stopping is given by

$$\frac{\alpha+(1-\alpha)\hat\rho\tilde U}{\hat\rho\tilde U}\frac{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)}{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)} = e^{\tilde T} - \frac{(1-\alpha)(1-\hat\rho\tilde U)\left[1-\frac{\hat\rho+\delta}{\hat\gamma}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)\right]}{\hat\rho\tilde U\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)}\left[1-\frac{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)}{e^{\tilde T}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)}\right]$$

or

$$\frac{\alpha+(1-\alpha)\hat\rho\tilde U}{\hat\rho\tilde U} = e^{\tilde T}\frac{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)}{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)} - \frac{(1-\alpha)(1-\hat\rho\tilde U)\left[1-\frac{\hat\rho+\delta}{\hat\gamma}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)\right]}{\hat\rho\tilde U\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)}\left[1-\frac{\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma},\tilde T\right)}{e^{\tilde T}\Omega\left(\frac{\hat\rho+\delta}{\hat\gamma}+1,\tilde T\right)}\right]$$

Now, we show the property over the $\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\rho}\tilde{U}}\right)$. Let us define the following function:

$$\begin{aligned}
f(a,b,c) &:= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - c\left[\frac{1-b\frac{1-e^{-ba}}{b}}{\frac{1-e^{-ba}}{b}}\right]\left[1-\frac{1-e^{-ba}}{1-e^{-(1+b)a}}\frac{b+1}{be^a}\right] \\
&= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - c\left[\frac{b}{1-e^{-ba}} - b\right]\left[\frac{b(e^a-e^{-ba})-(1-e^{-ba})(b+1)}{b(e^a-e^{-ba})}\right] \\
&= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[\frac{be^a-(b+1)+e^{-ba}\pm be^{-ba}}{b(e^a-e^{-ba})}\right] \\
&= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[\frac{b(e^a-e^{-ba})-(b+1)+(1+b)e^{-ba}}{b(e^a-e^{-ba})}\right] \\
&= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-a^{-ba}}{e^a-e^{-ba}}\right].
\end{aligned}$$

Observe that with this function:

$$\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}} = f\left(\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\rho}\tilde{U}}\right), \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\rho}\tilde{U}}\right).$$

The following properties are easy to show:

1. $f(a,b,c)$ is increasing in a.
2. If $a, c > 0$, $b \to \infty$, then $f(a,b,c) \to e^a$: To see this property, taking the limit

$$\begin{aligned}
&= \lim_{a>0, b\to\infty, c\propto b}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^a-e^{-ba}}\right]\right] \\
&= e^a \underbrace{\lim_{a>0,b\to\infty} \frac{1-e^{-(1+b)a}}{1-e^{-ba}}}_{=1} \underbrace{\lim_{a>0,b\to\infty} \frac{b}{b+1}}_{=1} - \underbrace{\lim_{a>0,b\to\infty} cb\frac{e^{-ba}}{1-e^{-ba}}}_{=0}\left[1-\underbrace{\lim_{b\to\infty}\frac{b+1}{b}}_{=1}\underbrace{\lim_{a>0,b\to\infty}\frac{1-e^{-ba}}{e^a-e^{-ba}}}_{=e^{-a}}\right] \\
&= e^a.
\end{aligned}$$

3. If $a, c > 0$ and $b \to 0$ then $f(a,b,c) \to \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$: To see this property, taking the limit

$$\begin{aligned}
&= \lim_{a>0, b\to 0}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^a-e^{-ba}}\right]\right] \\
&= e^a(1-e^{-a}) \underbrace{\lim_{a>0,b\to 0} \frac{b}{1-e^{-ba}}}_{=1/a} - c\underbrace{\lim_{a>0,b\to 0} \frac{b}{1-e^{-ba}}}_{=1/a}\left[1-\frac{1}{e^a-1}\underbrace{\lim_{b\to\infty}\frac{1-e^{-ba}}{b}}_{=a}\right] \\
&= \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}.
\end{aligned}$$

4. $e^a \geq f(a,b,c) \geq \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$ where the upper bound is reach when $b \to \infty$ and the lower bound when $b \downarrow 0$.
5. Duration of the match: It is simple to show that

$$\mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}{\hat{\rho}+\delta}$$

6. Worker's share:

$$\eta(\hat{w}^*) = \frac{e^{\hat{\gamma}T^*(\cdot)+\log(\hat{\rho}\hat{U})}\int_0^{T^*} e^{-(\hat{\rho}+\delta+\hat{\gamma})t}\,\mathrm{d}t - \hat{\rho}\hat{U}\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t}{(1-\hat{\rho}\hat{U})\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t}$$
$$= \frac{e^{\tilde{T}(\cdot)}\frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}\frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U}$$

With these properties, we can proof the equilibrium policies:

1. $\tilde{T}(\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\gamma}\hat{\rho}\tilde{U}})$ is increasing in the first argument.
2. If $\hat{\gamma} \to 0$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to \infty$

$$\lim_{(\hat{\rho}+\delta)/\hat{\gamma}\to\infty} \tilde{T}(\cdot) = \log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}\right)$$

The expected discounted duration in the limit is equal to

$$\lim_{\hat{\gamma}\to 0} \mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1}{\hat{\rho}+\delta}$$

The worker's share in the limit is equal to

$$\eta(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)}\frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}\frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \frac{e^{\tilde{T}(\cdot)}-1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \frac{\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}}-1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \alpha$$

3. If $\hat{\gamma} \to \infty$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to 0$, which provides the same $\tilde{T}(\cdot)$ as $\hat{\rho}+\delta \to 0$. As we have shown before, under this limit, $\tilde{T}(\cdot)$ converges to the implicit solution given by

$$\frac{\alpha+(1-\alpha)\hat{\rho}\tilde{U}}{\hat{\rho}\tilde{U}} = \frac{e^{\tilde{T}(\cdot)}-1-\frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\rho}\tilde{U}}\left(1-\frac{\tilde{T}(\cdot)}{e^{\tilde{T}(\cdot)}-1}\right)}{\tilde{T}(\cdot)}.$$

Given the converge, we now show the limit for $\eta(\hat{w}^*)$ since clearly $\mathcal{T}(\hat{w}^*, \rho) \to 0$. Let us depart from equation (B.40)

$$-\alpha\frac{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}-\hat{\rho}\tilde{U}\right)\mathrm{d}s} = (1-\alpha)\frac{\left[-\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s + e^{-(\hat{\rho}+\delta)T^*}\frac{(1-\hat{\rho}\tilde{U})}{\hat{\gamma}}\right]}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(1-e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}$$

Taking the limit $\hat{\rho}+\delta \to 0$

$$\alpha\int_0^{T^*}(1-e^{w_t})\,\mathrm{d}t = (1-\alpha)\int_0^{T^*}\left(e^{w_t}-\hat{\rho}\tilde{U}\right)\mathrm{d}t - \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\gamma}}\frac{\int_0^{T^*}\left(e^{w_t}-\hat{\rho}\tilde{U}\right)\mathrm{d}t}{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}.$$

Operating and using the occupancy measure

$$\alpha+(1-\alpha)\hat{\rho}\tilde{U} + \frac{(1-\alpha)(1-\hat{\rho}\tilde{U})}{\hat{\gamma}T^*}\frac{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}-\hat{\rho}\tilde{U}}{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}} = \frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}$$

It is easy to check that

$$\alpha+(1-\alpha)\hat{\rho}\tilde{U} + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{\mathbb{E}[e^{\hat{w}}]-\hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\tilde{U}) = \mathbb{E}[e^{\hat{w}}].$$

Since whenever $\hat{\rho} + \delta \to 0$, we have that $\eta(\hat{w}^*) = \frac{\mathbb{E}[e^{\hat{w}}] - \hat{\rho}\tilde{U}}{1 - \hat{\rho}\tilde{U}}$. Using this result

$$\alpha + (1-\alpha)\hat{\rho}\tilde{U} + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{\mathbb{E}[e^{\hat{w}}] - \hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\tilde{U}) = \mathbb{E}[e^{\hat{w}}] \iff$$

$$\alpha + (1-\alpha)\hat{\rho}\tilde{U} + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{(1-\hat{\rho}\tilde{U})\eta(\hat{w}^*)}{(1-\hat{\rho}\tilde{U})\eta(\hat{w}^*) + \hat{\rho}\tilde{U}}(1-\hat{\rho}\tilde{U}) = (1-\hat{\rho}\tilde{U})\eta(\hat{w}^*) + \hat{\rho}\tilde{U} \iff$$

$$\alpha(1-\hat{\rho}\tilde{U}) + \frac{1-\alpha}{\hat{\gamma}T^*}\frac{(1-\hat{\rho}\tilde{U})\eta(\hat{w}^*)}{(1-\hat{\rho}\tilde{U})\eta(\hat{w}^*) + \hat{\rho}\tilde{U}}(1-\hat{\rho}\tilde{U}) = (1-\hat{\rho}\tilde{U})\eta(\hat{w}^*)$$

$$\eta(\hat{w}^*) = \alpha + \frac{1-\alpha}{\tilde{T}}\frac{(1-\hat{\rho}\tilde{U})\eta(\hat{w}^*)}{\eta(\hat{w}^*) + \hat{\rho}\tilde{U}(1-\eta(\hat{w}^*))}$$

□

C Proofs for Section 3: Identifying the Microeconomic Implications of Allocative Wages

C.1 Proof of Lemma 3

Lemma 3. *The drift γ and volatility σ of the stochastic process guiding cumulative productivity shocks can be recovered from $\mathcal{D}$ with*

$$\gamma = \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]},$$
$$\sigma^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - \gamma\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]}.$$

Proof. From the law of motion $\mathrm{d}z_t = \gamma\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z$ and the fact that at the beginning of a new job spell $w_{t_0} - z_{t_0} = \hat{w}^*$, we have that

$$\Delta w = -\Delta z_\tau = \gamma\tau + \sigma\mathcal{W}_\tau^z. \tag{C.1}$$

Drift: Taking expectation on both sides conditional on a h-to-u-to-h transition, we have that $\sigma\mathbb{E}[\mathcal{W}_\tau^z] = \mathbb{E}_{\mathcal{D}}[\Delta w] - \gamma\mathbb{E}_{\mathcal{D}}[\tau]$. Since $\mathcal{W}_t^z$ is a martingale, by Doob's Optional Stopping Theorem (OST), $\mathcal{W}_\tau^z$ is a martingale, and $E[\mathcal{W}_\tau^z] = \mathbb{E}[\mathcal{W}_0^z] = 0$. Thus,

$$\gamma = \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}.$$

Idiosyncratic volatility: Let us define $Y_t = (\Delta z_t + \gamma t)^2$. We apply Itô's Lemma to Y_t and obtain

$$\mathrm{d}Y_t = 2\,(\Delta z_t + \gamma t)\,(\mathrm{d}\Delta z_t + \gamma\,\mathrm{d}t) + \frac{1}{2}2(\mathrm{d}\Delta z_t)^2 = 2\sigma\,(\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z + \sigma^2\,\mathrm{d}t$$

Integrating the previous equation between 0 and τ and using condition (C.1), we obtain

$$(\Delta w - \gamma\tau)^2 = 2\sigma\int_0^\tau (\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z + \sigma^2\tau.$$

Since $\int_0^t (\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z$ is a martingale, by the OST, $\int_0^\tau (\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z$ is a martingale and $\mathbb{E}[\int_0^\tau (\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z] = 0$. Thus,

$$\mathbb{E}_{\mathcal{D}}[(\Delta w - \gamma\tau)^2] = 2\sigma\mathbb{E}\left[\int_0^\tau (\Delta z_t + \gamma t)\,\mathrm{d}\mathcal{W}_t^z,\right] + \sigma^2\mathbb{E}_{\mathcal{D}}[\tau] = \sigma^2\mathbb{E}_{\mathcal{D}}[\tau]$$

which completes the proof of Lemma 3. □

C.2 Proof of Proposition 6

Proposition 6. *The distribution of Δz conditional on a job separation is given by*

$$\bar{G}^h(\Delta z) = \frac{\sigma^2}{2f(\hat{w}^*)}\frac{\mathrm{d}l^w(-\Delta z)}{\mathrm{d}z} - \frac{\gamma}{f(\hat{w}^*)}l^w(-\Delta z) - [1 - L^w(-\Delta z)]\,. \tag{C.2}$$

where $L^w(\Delta w)$ denotes the cumulative distribution function (CDF) corresponding to the marginal distribution $l^w(\Delta w)$.

Proof. The objective in this proof is to use the non-differentiability of the distribution of $\bar{g}_s(\Delta z)$ for $s = \{h, u\}$ at $\Delta z = 0$ to express the distribution of Δz conditional on a separation. Observe that

$$\begin{aligned}
L^w(a) &= Pr^{I^w}(\Delta w \leq a) \\
&=^{(1)} Pr^{\bar{G}^h,\bar{G}^u}(-(\Delta z^h + \Delta z^u) \leq a) \\
&=^{(2)} Pr^{\bar{G}^h,\bar{G}^u}(\Delta z^h + \Delta z^u \geq -a) \\
&=^{(3)} 1 - Pr(\Delta z^h + \Delta z^u \leq -a) \\
&=^{(4)} 1 - \int_{-\infty}^{\infty} \bar{G}^h(-a-y)\bar{g}^u(y)\,\mathrm{d}y.
\end{aligned}$$

Here, in step (1) we use the definition of Δw, and steps (2) to (4) operate and use the independence of $\bar{G}^h(\cdot)$ and $\bar{g}^u(\cdot)$. From Proposition D.1, we have

$$\bar{g}^u(\Delta z) = \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

and

$$L^w(\Delta w) = 1 - C_1(\Delta w) - C_2(\Delta w), \tag{C.3}$$

where

$$\begin{aligned}
C_1(\Delta w) &= \mathcal{G}_u \int_0^{\infty} \bar{G}^h(-\Delta w - u)e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u, \\
C_2(\Delta w) &= \mathcal{G}_u \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)e^{\beta_2(f(\hat{w}^*))u}\,\mathrm{d}u.
\end{aligned}$$

Departing from $L^w(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}^h(-\Delta w - y)\bar{g}^u(y)\,\mathrm{d}y$ and doing the change of variable $x = -\Delta w - y$ with $dx = -dy$, we obtain

$$L^w(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}^h(x)\bar{g}^u(-\Delta w - x)\,\mathrm{d}x.$$

Taking the derivative on both sides with respect to Δw we obtain

$$l^w(\Delta w) = \int_{-\infty}^{\infty} \bar{G}^h(x)\,(\bar{g}^u)'\,(-\Delta w - x)\,\mathrm{d}x.$$

Reverting the change of variables and using the fact that $\bar{g}^u(-\Delta w - x)$ is non-differentiable at 0, we obtain

$$\begin{aligned}
l^w(\Delta w) &= \int_{-\infty}^{\infty} \bar{G}^h(-\Delta w - u)(\bar{g}^u)'(u)\,\mathrm{d}u \\
&= \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)\mathcal{G}_u\beta_2(f(\hat{w}^*))e^{\beta_2(f(\hat{w}^*))u}\,\mathrm{d}u + \int_0^{\infty} \bar{G}^h(-\Delta w - u)\mathcal{G}_u\beta_1(f(\hat{w}^*))e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u \\
&= \beta_1(f(\hat{w}^*))C_1(\Delta w) + \beta_2(f(\hat{w}^*))C_2(\Delta w).
\end{aligned}$$

Thus,

$$l^w(\Delta w) = \beta_1(f(\hat{w}^*))C_1(\Delta w) + \beta_2(f(\hat{w}^*))C_2(\Delta w). \tag{C.4}$$

To obtain the last condition, observe that

$$C_1(\Delta w) = \int_0^{\infty} \bar{G}^h(-\Delta w - u)\mathcal{G}_u e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u,$$

$$
\begin{aligned}
&= -\mathcal{G}_u \int_{-\Delta w}^{-\infty} \bar{G}^h(y) e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)}\, \mathrm{d}y, \\
&= \mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}^h(y) e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)}\, \mathrm{d}y.
\end{aligned}
$$

and

$$
\begin{aligned}
C_2(\Delta w) &= \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)\mathcal{G}_u e^{\beta_2(f(\hat{w}^*))u}\, \mathrm{d}u. \\
&= -\mathcal{G}_u \int_{\infty}^{-\Delta w} \bar{G}^h(y) e^{\beta_2(f(\hat{w}^*))(-\Delta w - y)}\, \mathrm{d}y, \\
&= \mathcal{G}_u \int_{-\Delta w}^{\infty} \bar{G}^h(y) e^{\beta_2(f(\hat{w}^*))(-\Delta w - y)}\, \mathrm{d}y.
\end{aligned}
$$

Taking the derivative with respect to Δw and using the Leibniz rule, we obtain

$$
\begin{aligned}
C_1'(\Delta w) &= -\mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_1(f(\hat{w}^*))\mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}^h(y) e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)}\, \mathrm{d}y, \\
&= -\mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_1(f(\hat{w}^*))C_1(\Delta w) \qquad \text{(C.5)} \\
C_2'(\Delta w) &= \mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_2(f(\hat{w}^*))C_2(\Delta w). \qquad \text{(C.6)}
\end{aligned}
$$

Taking derivative of (C.4),

$$
(l^w)'(\Delta w) = \beta_1(f(\hat{w}^*))C_1'(\Delta w) + \beta_2(f(\hat{w}^*))C_2'(\Delta w)
$$

and using conditions (C.5) and (C.6),

$$
(l^w)'(\Delta w) = \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] - \beta_1(f(\hat{w}^*))^2 C_1(\Delta w) - \beta_2(f(\hat{w}^*))^2 C_2(\Delta w). \qquad \text{(C.7)}
$$

Equations (C.3), (C.4), and (C.7) provide the following system of three functional equations with three unknowns

$$
\begin{aligned}
1 - L^w(\Delta w) &= C_1(\Delta w) + C_2(\Delta w), \\
l^w(\Delta w) &= \beta_1(f(\hat{w}^*))C_1(\Delta w) + \beta_2(f(\hat{w}^*))C_2(\Delta w), \\
(l^w)'(\Delta w) &= \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] - \beta_1(f(\hat{w}^*))^2 C_1(\Delta w) - \beta_2(f(\hat{w}^*))^2 C_2(\Delta w).
\end{aligned}
$$

Operating over the system of functional equations

$$
(l^w)'(\Delta w) + [\beta_2(f(\hat{w}^*)) + \beta_1(f(\hat{w}^*))]\, l^w(\Delta w) + \beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))\,[1 - L^w(\Delta w)] = \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))],
$$

with

$$
\begin{aligned}
\mathcal{G}_u &= \left(\beta_2(f(\hat{w}^*))^{-1} - \beta_1(f(\hat{w}^*))^{-1}\right)^{-1} \\
\beta_1(f(\hat{w}^*)) &= \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}, \\
\beta_2(f(\hat{w}^*)) &= \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] &= \frac{\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))}{\beta_2(f(\hat{w}^*))^{-1} - \beta_1(f(\hat{w}^*))^{-1}}, \\
&= -\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*)), \\
&= -\left(\frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right)\left(\frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right) \\
&= \frac{2f(\hat{w}^*)}{\sigma^2} \\
\frac{\beta_2(f(\hat{w}^*)) + \beta_1(f(\hat{w}^*))}{\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))]} &= \frac{\beta_2(f(\hat{w}^*)) + \beta_1(f(\hat{w}^*))}{\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))}, \\
&= \frac{\left(\frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)} - \gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right)}{\frac{2f(\hat{w}^*)}{\sigma^2}} \\
&= -\frac{\gamma}{f(\hat{w}^*)}. \\
\frac{\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))}{\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))]} &= -1.
\end{aligned}
$$

Therefore the differential equation is given by (C.2).

□

C.3 Proof of Proposition 7

Proposition 7. *Assume $\gamma \neq 0$. The distribution of cumulative productivity shocks $g^h(\Delta z)$ is given by*

$$
g^h(\Delta z) = \frac{s\mathcal{E}}{\gamma}\left[\int_{-\Delta^-}^{\Delta z}\left(1 - e^{\frac{2\gamma}{\sigma^2}(y - \Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z + \Delta^-)}\right]\right].
$$

Proof. During employment, the distribution of cumulative productivity shocks satisfies the following KFE and the boundary conditions

$$
\begin{aligned}
&\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \\
&g^h(-\Delta^-) = g^h(\Delta^+) = 0, G^h(\Delta^+) = \mathcal{E}, \\
&g^h(\Delta z) \in \mathbb{C}.
\end{aligned}
$$

The distribution of cumulative productivity shocks conditional on a job separation satisfies

$$
\bar{G}^h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta_+, \infty) \\ \frac{1}{s\mathcal{E}}\left[\frac{\sigma^2}{2}\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \delta\int_{-\Delta^-}^{\Delta z} g^h(x)\,\mathrm{d}x\right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-). \end{cases}
$$

Combining these two conditions, we obtain

$$
\begin{aligned}
&s\mathcal{E}\bar{g}^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \\
&g^h(-\Delta^-) = g^h(\Delta^+) = 0, G^h(\Delta^+) = \mathcal{E}.
\end{aligned}
$$

Multiplying both sides of the first equation by $e^{\frac{2\gamma}{\sigma^2}\Delta z}$ we get

$$\begin{aligned} s\mathcal{E}e^{\frac{2\gamma}{\sigma^2}\Delta z}\bar{g}^h(\Delta z) &= \gamma e^{\frac{2\gamma}{\sigma^2}\Delta z}(g^h)'(\Delta z) + \frac{\sigma^2}{2}e^{\frac{2\gamma}{\sigma^2}\Delta z}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \\ &= \frac{\sigma^2}{2}\frac{\mathrm{d}(e^{\frac{2\gamma}{\sigma^2}\Delta z}(g^h)'(\Delta z))}{\mathrm{d}\Delta z}. \end{aligned}$$

Integrating both sides from $-\Delta^-$ to Δz, we obtain

$$\begin{aligned} s\mathcal{E}\int_{-\Delta^-}^{\Delta z} e^{\frac{2\gamma}{\sigma^2}x}\bar{g}^h(x)\,\mathrm{d}x &= \frac{\sigma^2}{2}\left[e^{\frac{2\gamma}{\sigma^2}\Delta z}(g^h)'(\Delta z) - \lim_{x\downarrow -\Delta^-} e^{\frac{2\gamma}{\sigma^2}x}(g^h)'(x)\right], \\ &= \frac{\sigma^2}{2}e^{\frac{2\gamma}{\sigma^2}\Delta z}(g^h)'(\Delta z) - s\mathcal{E}e^{-\frac{2\gamma}{\sigma^2}\Delta^-}\bar{G}^h(-\Delta^-), \end{aligned}$$

where the last equation uses the value of $\bar{G}^h(\Delta z)$ evaluated at $\Delta z = -\Delta^-$. Solving for $(g^h)'(\Delta z)$,

$$\frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z} e^{\frac{2\gamma}{\sigma^2}(x-\Delta z)}\bar{g}^h(x)\,\mathrm{d}x + e^{-\frac{2\gamma}{\sigma^2}(\Delta^- + \Delta z)}\bar{G}^h(-\Delta^-)\right] = (g^h)'(\Delta z).$$

Integrating this equation from $-\Delta^-$ to Δz, we obtain

$$\begin{aligned} \int_{-\Delta^-}^{\Delta z}(g^h)'(x)\,\mathrm{d}x &= g^h(\Delta z) - \underbrace{g^h(-\Delta^-)}_{=0} \\ &= \frac{2s\mathcal{E}}{\sigma^2}\int_{-\Delta^-}^{\Delta z}\left[\int_{-\Delta^-}^{x} e^{\frac{2\gamma}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}y + e^{-\frac{2\gamma}{\sigma^2}(\Delta^-+x)}\bar{G}^h(-\Delta^-)\right]\mathrm{d}x \\ &= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{-\Delta^-}^{x} e^{\frac{2\gamma}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{\sigma^2}{2\gamma}\bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z + \Delta^-)}\right]\right] \\ &= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{y}^{\Delta z} e^{\frac{2\gamma}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{\sigma^2}{2\gamma}\bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z + \Delta^-)}\right]\right] \\ &= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\underbrace{\left[\int_{y}^{\Delta z} e^{\frac{2\gamma}{\sigma^2}(y-x)}\,\mathrm{d}x\right]}_{=\frac{\sigma^2}{2\gamma}\left(1-e^{\frac{2\gamma}{\sigma^2}(y-\Delta z)}\right)}\bar{g}^h(y)\,\mathrm{d}y + \frac{\sigma^2}{2\gamma}\bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z + \Delta^-)}\right]\right] \\ &= \frac{s\mathcal{E}}{\gamma}\left[\int_{-\Delta^-}^{\Delta z}\left(1 - e^{\frac{2\gamma}{\sigma^2}(y-\Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1 - e^{-\frac{2\gamma}{\sigma^2}(\Delta z + \Delta^-)}\right]\right]. \end{aligned}$$

□

D Additional Results for Section 3: Identifying the Microeconomic Implications of Allocative Wages

D.1 Characterization of $g^h(\Delta z)$ and $g^u(\Delta z)$

Proposition D.1. *Assume $\delta > 0$. Then, $g^h(\Delta z)$ and $g^u(\Delta z)$ are given by*

$$g^h(\Delta z) = \mathcal{E}\mathcal{G}_h \begin{cases} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} & \text{if } \Delta z \in (-\Delta^-, 0] \\ \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} & \text{if } \Delta z \in [0, \Delta^+) \end{cases}$$

$$g^u(\Delta z) = (1-\mathcal{E})\mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where

$$\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2},$$

$$\mathcal{E} = \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2}\mathcal{G}_h \left[\frac{\beta_1(\delta)-\beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta)-\beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]},$$

$$\mathcal{G}_h = \left[\frac{\frac{e^{\beta_1(\delta)\Delta^-}-1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-}-1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \frac{\frac{1-e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1-e^{-\beta_2\Delta^+}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]^{-1},$$

$$\mathcal{G}_u = \left[-\beta_1(f(\hat{w}^*))^{-1} + \beta_2(f(\hat{w}^*))^{-1} \right]^{-1}.$$

Proof. Let us write the KFE and border conditions:

$$\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \tag{D.1}$$

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0, \tag{D.2}$$

$$f(\hat{w}^*) g^u(\Delta z) = \gamma (g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \forall \Delta z \in (-\infty, \infty)/\{0\}, \tag{D.3}$$

$$\lim_{\Delta z \to -\infty} g^u(\Delta z) = \lim_{\Delta z \to \infty} g^u(\Delta z) = 0, \tag{D.4}$$

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\Delta z, \tag{D.5}$$

$$f(\hat{w}^*)(1-\mathcal{E}) = \delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z) \right], \tag{D.6}$$

$$g^h(\Delta z), g^u(\Delta z) \in \mathbb{C}.$$

We guess and verify the proposed solution. Substituting the guess for $g^h(\Delta z)$ in (D.1) for $\Delta z < 0$, we have

$$0 = -\delta\mathcal{E}\mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \gamma\beta_1(\delta)\mathcal{E}\mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \mathcal{E}\mathcal{G}_h \frac{\sigma^2}{2}\beta_1(\delta)^2 \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} \iff$$

$$0 = -\delta + \gamma\beta_1(\delta) + \frac{\sigma^2}{2}\beta_1(\delta)^2,$$

mutatis mutandis for the terms that include $\beta_2(\delta)$. Given the definition of $\beta_1(\delta)$, the guess satisfies (D.1). A similar argument applies when (D.1) is evaluated at $\Delta z > 0$. It is easy to verify that the boundary conditions (D.2) are satisfied and that $g^h(\Delta z)$ is continuous at $\Delta z = 0$. Following the same steps for $g^u(\Delta z)$, we verify conditions (D.3) and (D.4). Next, we verify condition (D.5):

$$
\begin{aligned}
&\int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\int_{-\infty}^{0} e^{\beta_2(f(\hat{w}^*))\Delta z}\,\mathrm{d}\Delta z + \int_{0}^{\infty} e^{\beta_1(f(\hat{w}^*))\Delta z}\,\mathrm{d}\Delta z \right] + \dots \\
&\dots \mathcal{E}\mathcal{G}_h \left[\int_{-\Delta^-}^{0} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}}\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}}\,\mathrm{d}\Delta z \right] \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\frac{1-\lim_{\Delta z\to-\infty} e^{\beta_2(f(\hat{w}^*))\Delta z}}{\beta_2(f(\hat{w}^*))} + \frac{\lim_{\Delta z\to\infty} e^{\beta_1(f(\hat{w}^*))\Delta z} - 1}{\beta_1(f(\hat{w}^*))} \right] + \dots \\
&\dots \mathcal{E}\mathcal{G}_h \left[\frac{\frac{e^{\beta_1(\delta)\Delta^-}-1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-}-1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \frac{\frac{1-e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1-e^{-\beta_2\Delta^+}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right] \\
&= (1-\mathcal{E}) + \mathcal{E} = 1.
\end{aligned}
$$

Finally, combining condition (D.6) with the definition of $g^h(\Delta z)$, the employment rate is

$$
\begin{aligned}
\mathcal{E} &= \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z)\right]}, \\
&= \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2}\mathcal{G}_h\left[\frac{\beta_1(\delta)-\beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-}-e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta)-\beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+}-e^{-\beta_2(\delta)\Delta^+}}\right]}.
\end{aligned}
$$

□

D.2 Characterization of the job finding rate $f(\hat{w}^*)$ and job separation rate s

Proposition D.2. *The job finding rate $f(\hat{w}^*)$ and the job separating rate s are given by*

$$
\begin{aligned}
f(\hat{w}^*) &= \frac{\sigma^2}{2(1-\mathcal{E})}\left[\lim_{\Delta z\uparrow 0}(g^h)'(\Delta z) - \lim_{\Delta z\downarrow 0}(g^h)'(\Delta z)\right], \\
s &= \frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z\uparrow 0}(g^u)'(\Delta z) - \lim_{\Delta z\downarrow 0}(g^u)'(\Delta z)\right].
\end{aligned}
$$

The ratio of endogenous s^{end} to total job separations s is given by

$$
\frac{s^{end}}{s} = \frac{\frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z)\right]}{s}
$$

and $\delta = s - s^{end}$.

Proof. The proof comes directly from the equilibrium conditions described in Section 3.

□

D.3 Characterization of $l^u(\tau^u)$ and $l^m(\tau^m)$

Proposition D.3. *Let $\Delta_{+-} := \Delta^+ + \Delta^-$. The distributions of τ^u and τ^m are given by*

$$
\begin{aligned}
l^u(\tau^u) &= f(\hat{w}^*)e^{-f(\hat{w}^*)\tau^u}, \\
l^m(\tau^m) &= e^{-\delta\tau^m}\left[\delta + \sum_{n=1}^{\infty}\mathcal{A}(n)\left(e^{-\mathcal{B}(n)\tau^m} - \frac{\delta}{\mathcal{B}(n)}\left(1-e^{-\mathcal{B}(n)\tau^m}\right)\right)\right].
\end{aligned}
\tag{D.7}
$$

with

$$
\begin{aligned}
\mathcal{A}(n) &= \frac{\pi\sigma^2 n(-1)^{n-1}}{\Delta_{+-}^2}\left[\sin\left(\pi n\frac{\Delta^-}{\Delta_{+-}}\right)e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n\Delta^+}{\Delta_{+-}}\right)e^{\frac{\gamma}{\sigma^2}\Delta^-}\right], \\
\mathcal{B}(n) &= \frac{1}{2}\left(\frac{n^2\pi^2\sigma^2}{\Delta_{+-}^2} + \frac{\gamma^2}{\sigma^2}\right).
\end{aligned}
$$

Proof. The distribution $l^u(\tau^u)$ is the exponential distribution that arises from a Poisson process with arrival rate $f(\hat{w}^*)$. Next, we derive the formula for $l^m(\tau^m)$. Since $\tau^m = \min\{\tau^\delta, \tau^{h*}, \tau^{j*}\}$, we have that

$$
\begin{aligned}
\mathbb{P}r(\tau^m \le T) &= \mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T|\tau^\delta \ge T)\mathbb{P}r(\tau^\delta \ge T) + \mathbb{P}r(\tau^\delta \le T) \\
&= \mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T|\tau^\delta \ge T)[1-\mathbb{P}r(\tau^\delta \le T)] + \mathbb{P}r(\tau^\delta \le T) \\
&= \mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T)[e^{-\delta T}] + [1-e^{-\delta T}],
\end{aligned}
$$

where the last equation uses the fact that τ^δ is distributed according to an exponential distribution with rate δ. Taking the derivative with respect to T, we obtain

$$
l^m(T) = e^{-\delta T}\left[\frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T)}{\mathrm{d}T} - \delta\mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T) + \delta\right].
\tag{D.8}
$$

From Kolkiewicz (2002), we have that

$$
\begin{aligned}
&\frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*},\tau^{j*}\} \le T)}{\mathrm{d}T} \\
&= \frac{\pi\sigma^2}{(\hat{w}^+ - \hat{w}^-)^2}\sum_{n=1}^{\infty} n(-1)^{n-1}e^{-\frac{n^2\pi^2\sigma^2}{2(\hat{w}^+-\hat{w}^-)^2}T}\left[\sin\left(\pi n\frac{\hat{w}^*-\hat{w}^-}{\hat{w}^+-\hat{w}^-}\right)e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^+-\hat{w}^*)+\gamma T)} + \sin\left(\pi n\frac{\hat{w}^+-\hat{w}^*}{\hat{w}^+-\hat{w}^-}\right)e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^--\hat{w}^*)+\gamma T)}\right] \\
&= \frac{\pi\sigma^2}{\Delta_{+-}^2}\sum_{n=1}^{\infty} n(-1)^{n-1}e^{-\left(\frac{n^2\pi^2\sigma^2}{2\Delta_{+-}^2}+\frac{\gamma^2}{2\sigma^2}\right)T}\left[\sin\left(\frac{\pi n\Delta^-}{\Delta_{+-}}\right)e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n\Delta^+}{\Delta_{+-}}\right)e^{\frac{\gamma}{\sigma^2}\Delta^-}\right] \\
&= \sum_{n=1}^{\infty}\mathcal{A}(n)e^{-\mathcal{B}(n)T}.
\end{aligned}
\tag{D.9}
$$

Combining (D.8) and (D.9), we obtain (D.7). □

D.4 Characterization of $l^w(\Delta w)$

Proposition D.4. *The distribution of log nominal wage changes satisfies*

$$
l^w(\Delta w) = \mathcal{G}_u\left[\beta_2(f(\hat{w}^*))e^{-\beta_2(f(\hat{w}^*))\Delta w}\Gamma_2(\Delta w) + \beta_1(f(\hat{w}^*))e^{-\beta_1(f(\hat{w}^*))\Delta w}\Gamma_1(\Delta w)\right]
$$

with

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*)))x} \bar{G}^h(x)\, \mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x} \bar{G}^h(x)\, \mathrm{d}x \right).$$

Proof. Fix a date t_0 and focus on a newly hired worker. Then, the distribution of wage changes between two new jobs is given by

$$\begin{aligned}
Pr(\Delta w \leq c) &= Pr(w_{t_0+\tau^m+\tau^u} - w_{t_0} \leq c) \\
&=^{(1)} Pr(w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u} - (w_{t_0} - z_{t_0}) + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(2)} Pr(\hat{w}^* - \hat{w}^* + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(3)} Pr(-(\Delta z^h + \Delta z^u) \leq c),
\end{aligned}$$

where Δz^h and Δz^u denote cumulative productivity shocks during completed employment and unemployment spells, respectively. Here, step (1) adds and subtracts productivity at the beggining of both job spells. In step (2), we use the result that $\hat{w}^*$ is constant across jobs. Steps 3 uses the facts that τ^u and the Brownian motion increments are independent of the filtration $\mathcal{F}_{\tau_u}$. Therefore, the distributions of cumulative productivity shocks for completed employment and unemployment spells are given by

$$\begin{aligned}
\bar{G}^h(\Delta z) &= \begin{cases} 1 & \text{if } \Delta z \in [\Delta^+, \infty) \\ \frac{1}{s\mathcal{E}} \left[\frac{\sigma^2}{2} \lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) + \delta \int_{-\Delta^-}^{\Delta z} g^h(x)\, \mathrm{d}x \right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases} \\
\bar{g}^u(\Delta z) &= \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Pr(\Delta w \leq c) \\
&= \Pr(-(\Delta z^u + \Delta z^h) \leq c) \\
&= 1 - \Pr(\Delta z^u + \Delta z^h \leq -c) \\
&=^{(1)} 1 - \int_{-\infty}^{\infty} \bar{G}^h(-(c + \Delta z)) \bar{g}^u(\Delta z)\, \mathrm{d}\Delta z \\
&=^{(2)} 1 - \mathcal{G}_u \left[\int_{-\infty}^{0} e^{\beta_2(f(\hat{w}^*))\Delta z} \bar{G}^h(-(c + \Delta z))\, \mathrm{d}\Delta z + \int_{0}^{\infty} e^{\beta_1(f(\hat{w}^*))\Delta z} \bar{G}^h(-(c + \Delta z))\, \mathrm{d}\Delta z \right] \\
&=^{(3)} 1 + \mathcal{G}_u \left[\int_{\infty}^{-c} e^{-\beta_2(f(\hat{w}^*))(c+x)} \bar{G}^h(x)\, \mathrm{d}x + \int_{-c}^{-\infty} e^{-\beta_1(f(\hat{w}^*))(c+x)} \bar{G}^h(x)\, \mathrm{d}x \right] \\
&=^{(4)} 1 - \mathcal{G}_u \left[e^{-\beta_2(f(\hat{w}^*))c} \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x} \bar{G}^h(x)\, \mathrm{d}x + e^{-\beta_1(f(\hat{w}^*))c} \int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*))x} \bar{G}^h(x)\, \mathrm{d}x \right].
\end{aligned}$$

In step (1), we use the independence of Δz^u and Δz^h. In step (2), we use the definition of $\bar{g}^u(\Delta z)$. In step (3), we integrate by substituting $x = -c - \Delta z$ and in step (4), we use the properties of an integral. The last step involves defining

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*))x} \bar{G}^h(x)\, \mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x} \bar{G}^h(x)\, \mathrm{d}x \right).$$

$\square$

D.5 Characterization of $\mathbb{E}_h[\Delta z^n]$

We denote by $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively.

Proposition D.5. *Define the weights* $\omega^{hn}(\Delta z) = \frac{\Delta z^n}{\bar{\mathbb{E}}_h[\Delta z^n]}$ *with the property that*

$$\bar{\mathbb{E}}_h\left[\omega^{hn}(\Delta z)\right] = 1.$$

If $\gamma = 0$, *then* $\mathbb{E}_h[(\Delta z)^n]$ *can be recovered from*

$$\mathbb{E}_h[(\Delta z)^n] = \frac{2\mathcal{E}}{(n+1)(n+2)}\bar{\mathbb{E}}_h\left[(\Delta z)^n\omega^{h2}(\Delta z)\right]. \tag{D.10}$$

If $\gamma \neq 0$, *then* $\mathbb{E}_h[(\Delta z)^n]$ *can be recovered recursively from*

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1}\bar{\mathbb{E}}_h[(\Delta z)^n\omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2\gamma}\mathbb{E}_h[(\Delta z)^{n-1}]. \tag{D.11}$$

The moments $\bar{\mathbb{E}}_h\left[(\Delta z)^n\omega^{hk}(\Delta z)\right] = \frac{\bar{\mathbb{E}}_h\left[(\Delta z)^{n+k}\right]}{\bar{\mathbb{E}}_h\left[(\Delta z)^k\right]}$ *can be recovered from the following linear system of equations:*

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\Delta w^n] &= (-1)^n\sum_{i=0}^{n}\begin{pmatrix} n \\ i\end{pmatrix}\bar{\mathbb{E}}_h[\Delta z^i]\bar{\mathbb{E}}_u[\Delta z^{n-i}],\\
\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] &= \frac{(n-i)!}{\mathcal{L}_1^{n-i}\left(\mathcal{L}_2+\mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)}\right),
\end{aligned}$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}}.$$

Proof. We divide the proof into 3 steps.

Step 1. We first show that

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1}\bar{\mathbb{E}}_h[(\Delta z)^n\omega^{h1}(\Delta z)] - \frac{\sigma^2 n}{2\gamma}\mathbb{E}_h[(\Delta z)^{n-1}].$$

when $\gamma \neq 0$. For the case with $\gamma = 0$, see Baley and Blanco (2021a).

Let us define $Y_t = (\Delta z_t)^n$. The law of motion for Δz_t is given by $\mathrm{d}\Delta z_t = -\gamma\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z$. Applying Itô's Lemma, we obtain

$$\begin{aligned}
\mathrm{d}Y_t &= n(\Delta z_t)^{n-1}\,\mathrm{d}\Delta z_t + \frac{1}{2}n(n-1)(\Delta z_t)^{n-2}(\mathrm{d}\Delta z_t)^2\\
&= \left[-\gamma n(\Delta z_t)^{n-1} + \frac{\sigma^2}{2}n(n-1)(\Delta z_t)^{n-2}\right]\mathrm{d}t + n\sigma(\Delta z_t)^{n-1}\,\mathrm{d}\mathcal{W}_t^z
\end{aligned}$$

Thus,

$$(\Delta z_{\tau^m})^n = -\gamma n\int_0^{\tau^m}(\Delta z_t)^{n-1}\,\mathrm{d}t + \frac{\sigma^2}{2}n(n-1)\int_0^{\tau^m}(\Delta z_t)^{n-2}\,\mathrm{d}t + n\int_0^{\tau^m}(\Delta z_t)^{n-1}\sigma\,\mathrm{d}\mathcal{W}_t^z.$$

Following the same arguments as in the proof of Lemma 3 and using the Renewal Principle to have $\mathbb{E}_{\mathcal{D}}[\tau^m] = 1/s$, we obtain

$$\bar{\mathbb{E}}_h[(\Delta z)^n] = -\gamma n\mathbb{E}_{\mathcal{D}}[\tau^m]\frac{\mathbb{E}_h[(\Delta z)^{n-1}]}{\mathcal{E}} + \frac{\sigma^2 n(n-1)}{2s}\frac{\mathbb{E}_h[(\Delta z)^{n-2}]}{\mathcal{E}}$$

or equivalently

$$\mathbb{E}_h[(\Delta z)^n] = -\frac{\mathcal{E}}{\gamma \mathbb{E}_{\mathcal{D}}[\tau^m]} \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{n+1} + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

From Lemma 3, we have that $\gamma \mathbb{E}_{\mathcal{D}}[\tau^m] = -\bar{\mathbb{E}}_h[(\Delta z)]$ and $\frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{\bar{\mathbb{E}}_h[(\Delta z)]} = \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)]$. Thus,

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

Step 2. Here we show that

$$\mathbb{E}_{\mathcal{D}}[\Delta w^n] = (-1)^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\Delta z^n] \bar{\mathbb{E}}_u[\Delta z^{n-i}].$$

Using the independence of cumulative productivity shocks during employment and unemployment, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\Delta w^n] &= \bar{\mathbb{E}}[(-\Delta z^h - \Delta z^u)^n], \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}[(-\Delta z^h)^i (-\Delta u^u)^{n-i}], \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[(-\Delta z)^i] \bar{\mathbb{E}}_u[(-\Delta z)^{n-i}], \\
&= (-1)^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}],
\end{aligned}$$

Step 3. Here we show that

$$\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i} \left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)} \left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)}\right).$$

Let us depart from the definition of $\bar{g}^u(\Delta z)$, which is given by

$$\bar{g}^u(\Delta z) = \left[-\beta_1(f(\hat{w}^*))^{-1} + \beta_2(f(\hat{w}^*))^{-1}\right]^{-1} \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*)\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where $\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}$ and $\beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}$. This step consist of showing that $\bar{g}^u(\Delta z)$ is an asymmetric Laplace distribution with parameters

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}}$$

The ratio between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\begin{aligned}
\frac{\mathcal{L}_1}{\mathcal{L}_2} &= \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2} \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}} \\
&= \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2} (-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}) \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\left(\sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}\right)^2 - \gamma^2}} \\
&= (-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}) \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2} \frac{1}{2\sigma^2 f(\hat{w}^*)}}
\end{aligned}$$

$$= \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2} = \beta_2(f(\hat{w}^*)).$$

The product between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\begin{aligned}
-\mathcal{L}_1\mathcal{L}_2 &= -\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}} \\
&= -\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}(\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)})\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma^2 + \left(\sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}\right)^2}} \\
&= -(\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)})\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{1}{2\sigma^2 f(\hat{w}^*)}} \\
&= -\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2} = \beta_1(f(\hat{w}^*)).
\end{aligned}$$

Therefore, we can write $\bar{g}^u(\Delta z)$

$$\bar{g}^u(\Delta z) = \frac{\mathcal{L}_1}{\mathcal{L}_2 + \mathcal{L}_2^{-1}} \begin{cases} e^{\frac{\mathcal{L}_1}{\mathcal{L}_2}\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{-\mathcal{L}_1\mathcal{L}_2\Delta z} & \text{if } \Delta z \in [0, \infty), \end{cases}$$

which is the probability distribution function of an asymmetric Laplace distribution. It is a standard result that the n-th moment for an asymmetric Laplace distribution is given by

$$\bar{\mathbb{E}}_u[(\Delta z)^n] = \frac{n!}{\mathcal{L}_1^n\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n+1)} - (-\mathcal{L}_2)^{(n+1)}\right).$$

□

E Proofs for Section 4: Analyzing the Macroeconomic Consequences of Allocative Wages

E.1 Proof of Proposition 8

Proposition 8. *Let $Q_0 = 1$ be the numéraire and assume $\mu = \rho + \pi - \frac{\zeta^2}{2}$. Then, $P_t = M_t$ and the worker's optimal value solves*

$$V_0 = \max_{\{lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{y(lm_i^t)}{P_t} \, \mathrm{d}t \right] + k,$$

where k is a constant independent of a worker's policy.

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{c_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(c_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) \mathrm{d}t \right],$$

subject to

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t \left(P_t c_{it} + i_t \hat{M}_{it} - y(lm_i^t) - T_{it} \right) \mathrm{d}t \right] \leq M_{i0}. \tag{E.1}$$

The first-order conditions for consumption and money holdings, combined with the definition of the nominal interest rate, are given by

$$e^{-\rho t} = \Lambda_i Q_t P_t, \tag{E.2}$$

$$\mu \frac{e^{-\rho t}}{\hat{M}_{it}} = \Lambda_i Q_t i_t, \tag{E.3}$$

$$\mathbb{E}[\mathrm{d}Q_t] = -i_t Q_t \, \mathrm{d}t. \tag{E.4}$$

Here, Λ_i is the Lagrange multiplier of (E.1) for each worker. Equation (E.2) shows that $\Lambda_i = \Lambda$ for all i. Taking integrals over (E.3), we can replace $\hat{M}_{ti} = M_t$. With these results, we guess and verify the following equilibrium outcomes

$$\begin{aligned} P_t &= A^p M_t, \\ i_t &= A^i, \\ Q_t &= \frac{A^Q e^{-\rho t}}{M_t}. \end{aligned} \tag{E.5}$$

given a set of constants A^p, A^i, and A^Q. Using the guess in (E.2) and (E.3)

$$1 = \Lambda A^Q A^P, \tag{E.6}$$

$$\mu = \Lambda A^Q A^i. \tag{E.7}$$

Equations (E.6) and (E.7) provide the equilibrium values for A^Q and A^P given A^i. Applying Ito's lemma and using the guess over (E.4)

$$\mathrm{d}Q_t = A^Q \, \mathrm{d} \left(\frac{e^{-\rho t}}{e^{\log(M_t)}} \right),$$

$$
\begin{aligned}
&= -\rho A^Q \left(\frac{e^{-\rho t}}{e^{\log(M_t)}}\right) \mathrm{d}t - A^Q \frac{e^{-\rho t}}{e^{\log(M_t)}} \mathrm{d}\log(M_t) + A^Q \frac{e^{-\rho t}}{2e^{\log(M_t)}} (\mathrm{d}\log(M_t))^2, \\
&= -\rho Q_t \,\mathrm{d}t - \pi Q_t \,\mathrm{d}t - \zeta Q_t \,\mathrm{d}\mathcal{W}_t^m + \frac{\zeta^2}{2} Q_t \,\mathrm{d}t.
\end{aligned}
$$

Thus, using the guess (E.5) and $\mathbb{E}[\mathrm{d}\mathcal{W}_t^m] = 0$

$$
\mathbb{E}[\mathrm{d}Q_t] = -\underbrace{\left(\rho + \pi - \frac{\zeta^2}{2}\right)}_{=A^i} Q_t \,\mathrm{d}t.
$$

If we take as numéraire $Q_0 = 1$, then we verify the guess with $\mu = \rho + \pi - \frac{\zeta^2}{2}$:

$$
\begin{aligned}
A^Q &= M_0, \\
A^i &= \rho + \pi - \frac{\zeta^2}{2} = \mu, \\
\Lambda &= \frac{\mu}{M_0(\rho + \pi - \zeta^2/2)} = \frac{1}{M_0}, \\
A^P &= \frac{\rho + \pi - \zeta^2/2}{\mu} = 1.
\end{aligned}
$$

Using the budget constraint (E.1)

$$
\begin{aligned}
\mathbb{E}_0 \left[\int_0^\infty Q_t \left(P_t c_{it} + i_t \hat{M}_{it} - y(lm_i^t) - T_{it}\right) \mathrm{d}t\right] &= M_{i0} \iff \\
\mathbb{E}_0 \left[\int_0^\infty \frac{M_0 e^{-\rho t}}{M_t} \left(M_t c_{it} + \mu M_t - y(lm_i^t) - T_{it}\right) \mathrm{d}t\right] &= M_{i0} \iff \\
M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} c_{it} \,\mathrm{d}t\right] = M_{i0} + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{y(lm_i^t)}{M_t}\right] + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{T_{it}}{M_t} \,\mathrm{d}t\right] &- \frac{M_0}{\rho} \mu \iff \\
\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} c_{it} \,\mathrm{d}t\right] = \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{y(lm_i^t)}{M_t}\right] &+ k_i
\end{aligned}
$$

where k_i is a constant independent of the worker's policies. Thus,

$$
\begin{aligned}
V_0 &= \max_{\{c_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(c_{it} + \mu \log\left(\frac{\hat{M}_{it}}{P_t}\right)\right) \mathrm{d}t\right], \\
&= \max_{\{c_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(c_{it} + \mu \log\left(\frac{\mu}{\rho + \pi - \zeta^2/2}\right)\right) \mathrm{d}t\right], \\
&= \max_{\{c_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} c_{it} \,\mathrm{d}t\right], \\
&= \max_{\{lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{y(lm_i^t)}{M_t}\right] + k_i.
\end{aligned}
$$

□

E.2 Proof of Proposition 9: CIR of employment

Proposition 9. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\Delta z), g^u(\Delta z))$, the CIR is given by*

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z,$$

where the value functions $m_{\mathcal{E},h}(\Delta z)$ and $m_{\mathcal{E},u}(\Delta z, \zeta)$ are defined as:

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m} (1 - \mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},u}(0,0) \middle| \Delta z_0 = \Delta z\right], \tag{E.8}$$

$$m_{\mathcal{E},u}(\Delta z, \zeta) = \mathbb{E}\left[\int_0^{\tau^u(\zeta)} (-\mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},h}(-\zeta) \middle| \Delta z_0 = \Delta z\right]. \tag{E.9}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z)\, \mathrm{d}\Delta z.$$

with $\tau^u(\zeta)$ being distributed according to a Poisson process with arrival rate $f(\hat{w}^* - \zeta)$.

Proof. We define the cumulative impulse response of aggregate employment to a monetary shock as

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_0^{\infty} \int_{-\infty}^{\infty} \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z)\right) \mathrm{d}\Delta z\, \mathrm{d}t$$

Note that $\mathcal{E}_t = \int_{-\infty}^{\infty} g^h(\Delta z, \zeta, t)\, \mathrm{d}\Delta z$ is a function of ζ since aggregate shocks affect the entry rate to employment. The proof proceeds in three steps. Step 1 rewrites the CIR as the integral over time of two value functions, one for employed and unemployed workers, up to a finite time $\mathcal{T}$. Step 2 expresses the CIR as $\mathcal{T} \to \infty$. Step 3 uses the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the sequential problem and the corresponding HJB equations and boundary conditions).

Step 1. Here, we follow a recursive representation for the CIR. The CIR satisfies

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \lim_{\mathcal{T} \to \infty} \left[m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta) + m_{\mathcal{E},u}(\Delta z, \mathcal{T}) g^u(\Delta z + \zeta)\right] \mathrm{d}\Delta z$$

where we defined

$$m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, t|\Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t|\Delta z_0, h))\right] \mathrm{d}\Delta z\, \mathrm{d}t\right] \tag{E.10}$$

$$m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, \zeta, t|\Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t|\Delta z_0, u)\right] \mathrm{d}\Delta z\, \mathrm{d}t\right] \tag{E.11}$$

Proof of Step 1. Starting from the definition of the CIR, (1) adds and subtracts employment in t; (2) operates over the integral; (3) and (4) use the fact that the integral operator is a linear operator; (5) applies the definition of a conditional expectation, where $g^h(\Delta z, t|\Delta z_0, h)\, \mathrm{d}\Delta z$ is the probability of a worker being in the state Δz at time t when the initial productivity is Δz_0 and the initial employment state is h (mutatis mutandis if the initial employment state is u); (5) uses the fact that conditional on being initially employed, the transition probabilities are independent of ζ; (6), (7) and (8) apply Fubini's theorem and the definition of the limit of an integral, (8) relabels the resulting terms.

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_0^{\infty} \int_{-\infty}^{\infty} \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z)\right) \mathrm{d}\Delta z\, \mathrm{d}t$$

$$
\begin{aligned}
&=^{(1)} \int_0^\infty \int_{-\infty}^\infty \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z)(\mathcal{E}_t + 1 - \mathcal{E}_t) \right) \mathrm{d}\Delta z \, \mathrm{d}t \\
&=^{(2)} \int_0^\infty \left[\int_{-\infty}^\infty g^h(\Delta z, \zeta, t) \, \mathrm{d}\Delta z - \mathcal{E}_{ss} \left(\int_{-\infty}^\infty g^h(\Delta z, \zeta, t) \, \mathrm{d}\Delta z + \int_{-\infty}^\infty g^u(\Delta z, \zeta, t) \, \mathrm{d}\Delta z \right) \right] \mathrm{d}t \\
&=^{(3)} \int_0^\infty \int_{-\infty}^\infty (1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t) \, \mathrm{d}\Delta z \, \mathrm{d}t + \int_0^\infty \int_{-\infty}^\infty (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t) \, \mathrm{d}\Delta z \, \mathrm{d}t \\
&=^{(4)} \int_0^\infty \int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t) \right] \mathrm{d}\Delta z \, \mathrm{d}t \\
&=^{(5)} \int_0^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h)) \right] g^h(\Delta z_0, 0) \, \mathrm{d}\Delta z_0 \, \mathrm{d}\Delta z \, \mathrm{d}t \right] \dots \\
&\dots + \int_0^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] g^u(\Delta z_0, 0) \, \mathrm{d}\Delta z_0 \, \mathrm{d}\Delta z \, \mathrm{d}t \right] \\
&=^{(6)} \int_{-\infty}^\infty \int_0^\infty \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h)) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right] g^h(\Delta z_0 + \zeta) \, \mathrm{d}\Delta z_0 \dots \\
&\dots + \int_{-\infty}^\infty \int_0^\infty \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right] g^u(\Delta z_0 + \zeta) \, \mathrm{d}\Delta z_0 \\
&=^{(7)} \int_{-\infty}^\infty \lim_{\mathcal{T} \to \infty} \underbrace{\int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h)) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right]}_{m_{\mathcal{E},h}(\Delta z_0, \mathcal{T})} g^h(\Delta z_0 + \zeta) \, \mathrm{d}\Delta z_0 \dots \\
&\dots + \int_{-\infty}^\infty \lim_{\mathcal{T} \to \infty} \underbrace{\int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right]}_{m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T})} g^u(\Delta z_0 + \zeta) \, \mathrm{d}\Delta z_0 \\
&=^{(8)} \int_{-\infty}^\infty \lim_{\mathcal{T} \to \infty} m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta) \, \mathrm{d}\Delta z + \int_{-\infty}^\infty \lim_{\mathcal{T} \to \infty} m_{\mathcal{E},u}(\Delta z, \zeta, \mathcal{T}) g^u(\Delta z + \zeta) \, \mathrm{d}\Delta z
\end{aligned} \tag{E.12}
$$

where we define

$$
\begin{aligned}
m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right] \\
m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \left[(1 - \mathcal{E}_{ss}) \, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z \, \mathrm{d}t \right].
\end{aligned}
$$

Step 2. The CIR satisfies

$$
\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^\infty m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta) \, \mathrm{d}\Delta z + \int_{-\infty}^\infty m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta) \, \mathrm{d}\Delta z
$$

and the value functions $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0, \zeta)$ satisfy the following HJB and border conditions:

$$
0 = 1 - \mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)), \tag{E.13}
$$

$$
0 = -\mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z^2} + f(\hat{w}^* - \zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z, \zeta)) \tag{E.14}
$$

$$
m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+)
$$

$$
0 = \lim_{\Delta z \to -\infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} = \lim_{\Delta z \to \infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} \tag{E.15}
$$

$$
0 = \int_{-\infty}^\infty m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \, \mathrm{d}\Delta z + \int_{-\infty}^\infty m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z) \, \mathrm{d}\Delta z. \tag{E.16}
$$

Proof of Step 2. We divide this proof in steps *a* to *d*.

a. We show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},h}(\Delta z,\mathcal{T}) = m_{\mathcal{E},h}(\Delta z)$ and $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T}) = m_{\mathcal{E},u}(\Delta z,\zeta)$: This property holds due to the convergence of the distribution of Δz over time to its ergodic distribution for any initial condition (Stokey, 1989).

b. We show that $0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z,\mathcal{T})g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z,0,\mathcal{T})g^u(\Delta z)\,\mathrm{d}\Delta z$: The logic of the proof is to repeat the steps behind (E.12) in the reverse order. Departing from the definition,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})g^h(\Delta z_0)\,\mathrm{d}\Delta z_0 + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z_0,0,\mathcal{T})g^u(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&=^{(1)} \int_{-\infty}^{\infty}\int_0^{\mathcal{T}}\left[\int_{-\infty}^{\infty}\Big[(1-\mathcal{E}_{ss})\,g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss})g^u(\Delta z,t|\Delta z_0,h))\Big]\,\mathrm{d}\Delta z\,\mathrm{d}t\right] g^h(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&\cdots + \int_{-\infty}^{\infty}\int_0^{\mathcal{T}}\left[\int_{-\infty}^{\infty}\Big[(1-\mathcal{E}_{ss})\,g^h(\Delta z,0,t|\Delta z_0,u) + (-\mathcal{E}_{ss})g^u(\Delta z,0,t|\Delta z_0,u)\Big]\,\mathrm{d}\Delta z\,\mathrm{d}t\right] g^u(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&=^{(2)} \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Big[(1-\mathcal{E}_{ss})\,g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss})g^u(\Delta z,t|\Delta z_0,h))\Big]\, g^h(\Delta z_0)\,\mathrm{d}z_0\,\mathrm{d}\Delta z\right]\mathrm{d}t\\
&\cdots + \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Big[(1-\mathcal{E}_{ss})\,g^h(\Delta z,0,t|\Delta z_0,u) + (-\mathcal{E}_{ss})g^u(\Delta z,0,t|\Delta z_0,u)\Big]\, g^u(\Delta z_0)\;\,\mathrm{d}\Delta z_0\,\mathrm{d}\Delta z\right]\mathrm{d}t\\
&=^{(3)} \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\underbrace{\int_{-\infty}^{\infty} g^h(\Delta z,0,t|\Delta z_0)g(\Delta z_0)\,\mathrm{d}\Delta z_0}_{=g^h(\Delta z)}\mathrm{d}\Delta z + (-\mathcal{E}_{ss})\underbrace{\int_{-\infty}^{\infty} g^u(\Delta z,0,t|\Delta z_0)g(\Delta z_0)\,\mathrm{d}\Delta z_0}_{=g^u(\Delta z)}\mathrm{d}\Delta z\right]\mathrm{d}t\\
&=^{(4)} \int_0^{\mathcal{T}}(1-\mathcal{E}_{ss})\,\mathcal{E}_{ss}\,\mathrm{d}t + \int_0^{\mathcal{T}}(-\mathcal{E}_{ss})\,(1-\mathcal{E}_{ss})\,\mathrm{d}t\\
&= 0
\end{aligned}
$$

In (1), we apply the definitions (E.10) and (E.11); (2) applies Fubini's theorem; (3) uses the steady-state conditions for $g^h(\cdot)$ and $g^u(\cdot)$, and the definition $g(\Delta z) = g^h(\Delta z) + g^u(\Delta z)$; and (4) computes the integral using the definitions of aggregate employment and unemployment.

c. We show that $0 = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z,0)g^u(\Delta z)\,\mathrm{d}\Delta z$: See **?**.

d. We show that the CIR satisfies (E.12) with $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta)$ satisfying (E.13)–(E.16): Writing the HJB for $m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})$, we have that

$$
\begin{aligned}
0 &= 1-\mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\mathcal{T}} - \gamma\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0,\mathcal{T}) - m_{\mathcal{E},h}(\Delta z,\mathcal{T})),\\
0 &= -\mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\mathcal{T}} - \gamma\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z^2} + f(\hat{w}^*-\zeta)(m_{\mathcal{E},h}(-\zeta,\mathcal{T}) - m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T}))
\end{aligned}
$$

$m_{\mathcal{E},u}(0,0,\mathcal{T}) = m_{\mathcal{E},h}(\Delta z,\mathcal{T}),$ for all $\Delta z \notin (-\Delta^-,\Delta^+)$

$$
\begin{aligned}
0 &= \lim_{\Delta z\to-\infty}\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} = \lim_{\Delta z\to\infty}\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z}\\
0 &= \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z,\mathcal{T})g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z,0,\mathcal{T})g^u(\Delta z)\,\mathrm{d}\Delta z.
\end{aligned}
$$

The border condition for $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is implied from the fact that the job finding rate $f(\hat{w}^*)$ is independent of Δz, so the function $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is constant in the entire domain. Taking the limit $\mathcal{T}\to\infty$ and using point-wise convergence of $m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})$, we have the desired result.

Step 3. The solution of the differential equations (E.13) to (E.15) satisfy (E.8) and (E.9).

Proof of Step 3. This is just an application of **?**, Chapter 9.

□

E.3 Proof of Proposition 10: Flexible Entry Wages

Proposition 10 relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. Before starting with the proof, we summarize the conditions that characterize this distribution.

Steady-State Cross-Sectional Distribution Δz

$$\begin{aligned}
\delta g^h(\Delta z) &= \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-, \Delta^+)/\{0\}, \\
f(\hat{w}^*) g^u(\Delta z) &= \gamma (g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \text{for all } \Delta z \in (-\infty, \infty)/\{0\}. \\
g^h(\Delta z) &= 0, \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \\
\lim_{\Delta z \to -\infty} g^u(\Delta z) &= \lim_{\Delta z \to \infty} g^u(\Delta z) = 0. \\
1 &= \int_{-\infty}^{\infty} g^u(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\Delta z, \\
f(\hat{w}^*)(1-\mathcal{E}) &= \delta \mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z)\right], \\
g^h(\Delta z), g^u(\Delta z) &\in \mathbb{C}, \mathbb{C}^1(\{0\}), \mathbb{C}^2(\{0\})
\end{aligned} \tag{E.17}$$

Proposition 10. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta).$$

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\Delta z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock ζ, so we omit this argument). Step 2 computes the value for the employed worker at $\Delta z = 0$—i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}),$$

with

$$\begin{aligned}
0 &= 1 - \mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z)\right), \\
-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0) &= m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \\
0 &= \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).
\end{aligned} \tag{E.18}$$

Proof of Step 1. To show this result, observe that the solution to (E.14) and (E.15) is

$$m_{\mathcal{E},u}(\Delta z) = m_{\mathcal{E},u}(0), \text{ for all } \Delta z$$

Thus,

$$0 = -\mathcal{E}_{ss} + f(\hat{w}^*)(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0). \tag{E.19}$$

Replacing (E.19) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\Delta z)$ satisfies the following recursive representation

$$m_{\mathcal{E},h}(\Delta z) \;=\; \mathbb{E}\left[\int_0^{\tau^m} (1-\mathcal{E}_{ss})\,\mathrm{d}t + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\middle|\,\Delta z_0 = \Delta z\right].$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) \;=\; \mathbb{E}\left[\int_0^{\tau^m} e^{\varphi t}(1-\mathcal{E}_{ss})\,\mathrm{d}t + e^{\varphi\tau^m}\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\middle|\,\Delta z_0 = \Delta z\right]. \tag{E.20}$$

and note that $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$. The auxiliary function $\Psi(\Delta z|\varphi)$ satisfies the following HJB and border conditions:

$$-\varphi\Psi(\Delta z|\varphi) + \delta\left(\Psi(\Delta z|\varphi) - \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\right) = (1-\mathcal{E}_{ss}) - \gamma\frac{\partial\Psi(\Delta z|\varphi)}{\partial\Delta z} + \frac{\sigma^2}{2}\frac{\partial^2\Psi(\Delta z|\varphi)}{\partial\Delta z^2}, \tag{E.21}$$
$$\Psi(\Delta z,\varphi) \;=\; \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right) \text{ for all } \Delta z \notin (-\Delta^-,\Delta^+).$$

Taking the derivative with respect to φ in (E.21), we have that

$$\begin{aligned}(\delta-\varphi)\frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} - \Psi(\Delta z|\varphi) \;&=\; -\gamma\frac{\partial^2\Psi(\Delta z,\varphi)}{\partial\Delta z\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3\Psi(\Delta z|\varphi)}{\partial\Delta z^2\partial\varphi},\\ \frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} \;&=\; 0 \text{ for all } \Delta z \notin (-\Delta^-,\Delta^+).\end{aligned}$$

Using the Schwarz's theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$, we obtain

$$\delta\frac{\partial\Psi(\Delta z|0)}{\partial\varphi} - m_{\mathcal{E},h}(\Delta z) = -\gamma\frac{\partial}{\partial\Delta z}\left(\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\Delta z^2}\left(\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}\right), \tag{E.22}$$

$$\frac{\partial\Psi(-\Delta^-|0)}{\partial\varphi} \;=\; \frac{\partial\Psi(\Delta^+|0)}{\partial\varphi} = 0. \tag{E.23}$$

Equations (E.22) and (E.23) correspond to the HJB and border conditions of the function $\frac{\partial\Psi(\Delta z|0)}{\partial\varphi} = \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t\middle|\,\Delta z_0 = \Delta z\right]$. Evaluating $\frac{\partial\Psi(\Delta z|0)}{\partial\varphi}$ at $\Delta z = 0$, using the occupancy measure and result (E.18), we write the previous equation as:

$$\begin{aligned}\frac{\partial\Psi(0|0)}{\partial\varphi} \;&=\; \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t\middle|\,\Delta z_0 = 0\right]\\ &=\; \mathbb{E}_{\mathcal{D}}[\tau^m]\,\frac{\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z}{\mathcal{E}_{ss}}\end{aligned}$$

$$= \mathbb{E}_{\mathcal{D}}[\tau^m]\left(\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - m_{\mathcal{E},h}(0)\right)\frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}}, \tag{E.24}$$

where $\mathbb{E}_{\mathcal{D}}[\cdot]$ is the mean duration of completed employment spells (the subscript highlights that the moment can be easily computed from the data). From (E.20), we also have that

$$\begin{aligned}\frac{\partial\Psi(0|0)}{\partial\varphi} &= \mathbb{E}\left[\int_0^{\tau^m} s\,(1-\mathcal{E}_{ss})\,\mathrm{d}s + \tau^m\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\middle|\,\Delta z_0 = 0\right] \\ &= \mathbb{E}_{\mathcal{D}}\left[\tau^m\right]\left[(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\right],\end{aligned} \tag{E.25}$$

Combining (E.24) and (E.25), and solving for $m_{\mathcal{E},h}(0)$ we obtain:

$$m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$$

Step 3. Up to a first-order approximation, the CIR is given by:

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = -(1-\mathcal{E}_{ss})\frac{\gamma\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}\zeta + o(\zeta^2).$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) \tag{E.26}$$

with

$$\delta m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - \gamma\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta m_{\mathcal{E},u}(0), \tag{E.27}$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\Delta z) \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \tag{E.28}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}). \tag{E.29}$$

1. **Zero-order:** If $\zeta = 0$, condition (E.29) implies

$$\mathrm{CIR}_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) = 0.$$

2. **First-order:** Taking the derivative of (E.26) we obtain

$$\mathrm{CIR}'_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta)\,\mathrm{d}\Delta z,$$

which evaluated at $\zeta = 0$ becomes

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z.$$

Using condition (E.17) to replace $\delta = \frac{\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into equation (E.27), we obtain

$$\frac{\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - \gamma m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2}m''_{\mathcal{E},h}(\Delta z) + \frac{\gamma g'(\Delta z) + \frac{\sigma^2}{2}g''(\Delta z)}{g(\Delta z)}m_{\mathcal{E},u}(0).$$

Multiplying both sides by $g^h(\Delta z)\Delta z$ and integrating between $-\Delta^-$ and Δ^+,

$$
\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 \\
T_1 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z \\
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\Delta z \\
T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right) \mathrm{d}\Delta z.
\end{aligned}
\tag{E.30}
$$

Next, we operate on the terms T_1, T_2, and T_3. The term T_1 is equal to

$$
\begin{aligned}
T_1 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z \\
&=^{(2)} \int_{-\Delta^-}^{0} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\right)}{\mathrm{d}\Delta z} \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\right)}{\mathrm{d}\Delta z} \mathrm{d}\Delta z \\
&=^{(3)} \underbrace{\Delta z m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} \\
&\quad \cdots - \left[\int_{-\Delta^-}^{0} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z\right] \\
&=^{(4)} -\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z \\
&=^{(5)} m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}).
\end{aligned}
\tag{E.31}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the property of the derivative of a product of functions; (3) integrates and uses the border conditions for $g^h(\Delta z)$; (4) uses the continuity of $m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)$; and (5) uses (E.29).

The term T_2 satisfies

$$
\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\Delta z \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)g(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\Delta z \\
&=^{(2)} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{-\Delta^-}^{0} + \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{0}^{\Delta^+} \\
&\quad \cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z\right] \\
&=^{(3)} \underbrace{\Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{\Delta^-}^{\Delta^+}}_{= -m_{\mathcal{E},u}(0)\,\Delta z(g^h)'(\Delta z)|_{\Delta^-}^{\Delta^+}} \\
&\quad \cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\Delta z\right]
\end{aligned}
\tag{E.32}
$$

$$
\begin{aligned}
&=^{(4)} -m_{\mathcal{E},u}(0)\ \Delta z(g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z \\
&=^{(5)} -m_{\mathcal{E},u}(0)\ \Delta z(g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \left[\underbrace{m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} - \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z\right] + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z \\
&= -m_{\mathcal{E},u}(0)\ \Delta z(g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} + 2\int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z)\,\mathrm{d}\Delta z.
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the equality $m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)$ $= \frac{\mathrm{d}\left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\Delta z}$ and integrates by parts; (3) uses conditions (E.28) and the border conditions of $g^h(\Delta z)$; and (4)-(5) integrate by parts and operate.

Finally, the term T_3 is equal to

$$
\begin{aligned}
T_3 &= \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right) \mathrm{d}\Delta z \qquad \text{(E.33)} \\
&=^{(1)} \gamma \left[\int_{-\Delta^-}^{0} \Delta z (g^h)'(\Delta z)\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\,\mathrm{d}\Delta z\right] + \frac{\sigma^2}{2}\left[\int_{\Delta^-}^{0} \Delta z (g^h)''(\Delta z)\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)''(\Delta z)\,\mathrm{d}\Delta z\right] \\
&=^{(2)} \gamma \left[\underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z}_{=\mathcal{E}_{ss}}\right] \\
&\cdots + \frac{\sigma^2}{2}\left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z (g^h)'(\Delta z)\Big|_{0}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} (g^h)'(\Delta z)\,\mathrm{d}\Delta z\right] \\
&=^{(3)} -\gamma\mathcal{E}_{ss} + \frac{\sigma^2}{2}\left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \underbrace{g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0}\right] \\
&=^{(4)} -\gamma\mathcal{E}_{ss} + \frac{\sigma^2}{2}\left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+}\right]
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) integrates by parts; and (3)-(4) use the border conditions of $g^h(\Delta z)$.

Combining results (E.30), (E.31), (E.32), (E.33) and those in Step 2, we obtain

$$
\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2} T_2 + m_{\mathcal{E},u}(0) T_3 \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - \gamma m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}) + \frac{\sigma^2}{2}\left[-m_{\mathcal{E},u}(0)\ \Delta z(g^h)'(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} + 2\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z\right] \\
&\cdots + m_{\mathcal{E},u}(0)\left[-\gamma\mathcal{E}_{ss} + \frac{\sigma^2}{2}\left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{\Delta^+}\right]\right] \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - \gamma m_{\mathcal{E},u}(0) + \sigma^2 \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z,
\end{aligned}
$$

which implies

$$\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z = \frac{\gamma\left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)}+\frac{\mathcal{E}_{ss}}{f(\theta^*)}-(1-\mathcal{E}_{ss})\mathbb{E}_h[a]\right)-(1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z]}{\sigma^2},$$
$$= -(1-\mathcal{E}_{ss})\frac{[\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2}.$$

□

E.4 Proof of Corollary 1: Flexible entry wages

Corollary 1. *Up to first order, the $CIR_{\mathcal{E}}(\zeta)$ can be expressed in terms of data moments as follows:*

(i) If $\gamma = 0$,

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{w}^*)}}_{avg.\ u\ dur.}\underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{dispersion}\left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w\frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]}_{asymmetries}\right]+o(\zeta).$$

(ii) If $\gamma \neq 0$,

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{w}^*)}}_{avg.\ u\ dur.}\underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w-\frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}}_{dispersion}$$
$$\times\left[\underbrace{\mathbb{E}_{\mathcal{D}}[\Delta w]\left(\mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w},\widetilde{\Delta w}-\tilde{\tau}\right]+\frac{\mathbb{V}ar_{\mathcal{D}}[\tilde{\tau}]-\mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\tau^m}\right]}{2}\right)+(1-\mathcal{E}_{ss})\left(\frac{\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1}{2}\right)\right)}_{asymmetries}\right]+o(\zeta).$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\Delta z]$, in terms of moments of the distribution of Δw and (τ^u,τ^m). We do so separately for the case with $\gamma = 0$ and $\gamma \neq 0$. Let $\tilde{x} \equiv x/\mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Proposition D.5 expresses moments of the wage distribution as a linear combination of moments of the distribution of productivity changes among completed employment an unemployment spells:

$$\mathbb{E}_{\mathcal{D}}[\Delta w] = -\left[\bar{\mathbb{E}}_u[\Delta z]+\bar{\mathbb{E}}_h[\Delta z]\right]$$
$$\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] = \left[\bar{\mathbb{E}}_u\left[\Delta z^2\right]+2\bar{\mathbb{E}}_h[\Delta z]\,\bar{\mathbb{E}}_u[\Delta z]+\bar{\mathbb{E}}_h\left[\Delta z^2\right]\right]$$
$$\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] = -\left[\bar{\mathbb{E}}_u\left[\Delta z^3\right]+3\bar{\mathbb{E}}_h[\Delta z]\,\bar{\mathbb{E}}_u\left[\Delta z^2\right]+3\bar{\mathbb{E}}_h\left[\Delta z^2\right]\bar{\mathbb{E}}_u[\Delta z]+\bar{\mathbb{E}}_h\left[\Delta z^3\right]\right],$$

where $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ denote the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively (see Proposition 6). Using results from the same Proposition, we can express the moments of productivity changes for completed unemployment spells in terms of model parameters:

$$\bar{\mathbb{E}}_u[\Delta z] = \frac{\left(\mathcal{L}_2^{-1}-\mathcal{L}_2\right)}{\mathcal{L}_1}$$

$$\bar{\mathbb{E}}_u\left[\Delta z^2\right] = \frac{2\left(\mathcal{L}_2^{-2} + \mathcal{L}_2^2 - 1\right)}{\mathcal{L}_1^2}$$

$$\bar{\mathbb{E}}_u\left[\Delta z^3\right] = \frac{6\left(-\mathcal{L}_2^3 + \mathcal{L}_2 - \mathcal{L}_2^{-1} + \mathcal{L}_2^{-3}\right)}{\mathcal{L}_1^3},$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}}.$$

From these two sets of equations, we solve for the moments of productivity changes for completed employment spells and obtain

$$\bar{\mathbb{E}}_h\left[\Delta z\right] = -\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \mathbb{E}_{\mathcal{D}}\left[\Delta w\right]$$

$$\bar{\mathbb{E}}_h\left[\Delta z^2\right] = \mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] + 2\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \frac{2}{\mathcal{L}_1^2}$$

$$\bar{\mathbb{E}}_h\left[\Delta z^3\right] = -\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] - 3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) + \frac{6}{\mathcal{L}_1^2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right].$$

The remaining steps are case-specific.

Case I: $\gamma = 0$. To obtain $\mathbb{E}_h[\Delta z]$, evaluate (D.10) at $m = 1$, use the fact that $\mathcal{L}_2 = 1$, $\mathbb{E}_{\mathcal{D}}\left[\Delta w\right] = 0$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute σ^2 from Lemma 3:

$$\begin{aligned}
\mathbb{E}_h\left[\Delta z\right] &= \frac{\mathcal{E}_{ss}}{3}\frac{\bar{\mathbb{E}}_h\left[\Delta z^3\right]}{\bar{\mathbb{E}}_h\left[\Delta z^2\right]} \\
&= \frac{\mathcal{E}_{ss}}{3}\left(\frac{-\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] + \mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\frac{6}{\mathcal{L}_1^2}}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - \frac{2}{\mathcal{L}_1^2}}\right) \\
&= -\frac{\mathcal{E}_{ss}}{3}\left(\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - \sigma^2\mathbb{E}_{\mathcal{D}}[\tau^u]}\right) \\
&= -\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}.
\end{aligned}$$

Finally, replace this expression into (45):

$$\begin{aligned}
\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} &= -\left(1 - \mathcal{E}_{ss}\right)\frac{\mathbb{E}_h\left[\Delta z\right]}{\sigma^2} \\
&= \left(1 - \mathcal{E}_{ss}\right)\frac{\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}}{\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}{\mathbb{E}_{\mathcal{D}}[\tau]}} \\
&= \frac{1}{f\left(\hat{w}^*\right)}\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]^2} \\
&= \frac{1}{f\left(\hat{w}^*\right)}\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\Delta w^2\right]}\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w\frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}\right].
\end{aligned}$$

Case II: $\gamma \neq 0$. To obtain $\mathbb{E}_h[\Delta z]$, evaluate (D.11) at $m = 1$, use the result that $\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)/\mathcal{L}_1 = -\gamma/f(\hat{w}^*)$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute γ and σ^2 from Lemma 3:

$$
\begin{aligned}
\mathbb{E}_h\left[\Delta z\right] &= \frac{\mathcal{E}_{ss}}{2}\frac{\bar{\mathbb{E}}_h\left[\Delta z^2\right]}{\bar{\mathbb{E}}_h\left[\Delta z\right]} - \frac{\sigma^2}{2\gamma} \\
&= \frac{\mathcal{E}_{ss}}{2}\left(\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\Delta w]^2\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} - \sigma^2\mathbb{E}_{\mathcal{D}}[\tau^u]}{-\mathbb{E}_{\mathcal{D}}[\Delta w]\left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right)}\right) - \frac{\sigma^2}{2\gamma} \\
&= -\frac{1}{2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) + \frac{1}{2}\frac{\sigma^2}{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}\left(\mathbb{E}_{\mathcal{D}}[\tau^u] - \mathbb{E}_{\mathcal{D}}[\tau]\right) \\
&= -\frac{1}{2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) + \frac{1}{2}\frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - \gamma\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}\left(\mathbb{E}_{\mathcal{D}}[\tau^u] - \mathbb{E}_{\mathcal{D}}[\tau]\right) \\
&= -\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) + \frac{1}{2}\mathbb{E}_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})^2]\left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right)\right) \\
&= -\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(var_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}var_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})]\right)\right).
\end{aligned}
$$

The average cross-sectional age of a job spell is obtained from from the occupancy measure:

$$
\mathbb{E}_h[a] = \mathcal{E}_{ss}\frac{\mathbb{E}[\int_0^{\tau^m} t\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*]}{\mathbb{E}[\tau^m|\hat{w}_0 = \hat{w}^*]} = \frac{\mathcal{E}_{ss}}{2}\frac{\mathbb{E}_{\mathcal{D}}[\tau^{m2}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]},
$$

where we solve the Reimann integral.

Finally, we replace these expressions into (45):

$$
\begin{aligned}
&\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -\left(1 - \mathcal{E}_{ss}\right)\frac{\mathbb{E}_h\left[\Delta z\right] + \gamma\mathbb{E}[a]}{\sigma^2} \\
&= -\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\frac{\mathbb{E}_{\mathcal{D}}[\tau]}{\mathbb{E}_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)^2\right]}\left(-\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right)\right) + \gamma\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right) - \frac{1}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)\left(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss}\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right) - \frac{1}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{2f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)\left(1 - \mathcal{E}_{ss}\right) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]\right) - \frac{2}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}[a]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{2f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)\left(1 - \mathcal{E}_{ss}\right) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]\right) - \mathcal{E}_{ss}\frac{\mathbb{E}_{\mathcal{D}}[\tau^m]}{\mathbb{E}_{\mathcal{D}}\left[\tau\right]}\mathbb{E}_{\mathcal{D}}[\widetilde{\tau^m}^2]\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{2f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)\left(1 - \mathcal{E}_{ss}\right) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - \mathcal{E}_{ss}\mathbb{E}_{\mathcal{D}}[\widetilde{\tau^m}^2]\right)\right) \\
&= \frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]}{f\left(\hat{w}^*\right)\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}\left(\frac{\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)}{2}\left(1 - \mathcal{E}_{ss}\right) + \mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w}, \widetilde{\Delta w} - \tilde{\tau}\right] + \frac{\mathbb{V}ar_{\mathcal{D}}\left[\tilde{\tau}\right] - \mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}[\widetilde{\tau^m}^2]}{2}\right)\right).
\end{aligned}
$$

□

E.5 Proof of Proposition 11

Proposition 11. *Assume sticky entry wages. Up to first order, the CIR of employment*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1 - \mathcal{E}_{ss}) \left[-\frac{[\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1}{f(\hat{w}^*) + s} \left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)} \right] \right] + o(\zeta). \tag{E.34}$$

Proof. We divide the proof in two steps. Step 1 characterizes $m_{\mathcal{E},u}(\Delta z, \zeta)$. Steps 2 uses the equilibrium conditions to show (E.34). Let us depart from the CIR for employment given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z$$

with

$$0 = 1 - \mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)), \text{ for all } \Delta z \in (-\Delta^-, \Delta^+) \tag{E.35}$$

$$0 = -\mathcal{E}_{ss} - \gamma \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2} \frac{d^2 m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z^2} + f(\hat{w}^* - \zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z, \zeta)) \tag{E.36}$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+)$$

$$0 = \lim_{\Delta z \to -\infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} = \lim_{\Delta z \to \infty} \frac{\mathrm{d} m_{\mathcal{E},u}(\Delta z, \zeta)}{\mathrm{d}\Delta z} \tag{E.37}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z) g^u(\Delta z)\, \mathrm{d}\Delta z$$

The key differences between the CIR with flexible wages and the CIR with sticky wages are found in the HJB equation at the moment of the shock. With sticky entry wages, the job-finding probability is given by $f(\hat{w}^* - \zeta)$, since now the real entry wage is lower. As a consequence, we need to evaluate $m_{\mathcal{E},h}(\Delta z)$ at $\Delta z = -\zeta$ because conditional on finding a job, the real entry wage is lower. Observe that following the first job separation, the monetary shock is fully absorbed (see the term $m_{\mathcal{E},u}(0,0)$ in equation (E.35)).

Step 1. The value function $m_{\mathcal{E},u}(\Delta z, \zeta)$ is independent of Δz and satisfies

$$m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^* - \zeta)} + m_{\mathcal{E},h}(-\zeta).$$

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\Delta z, \zeta) = m_{\mathcal{E},u}(0, \zeta)$ for all Δz. From the equilibrium conditions (E.36) and (E.37),

$$0 = -\mathcal{E}_{ss} + f(\hat{w}^* - \zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(0, \zeta)).$$

Thus,

$$m_{\mathcal{E},u}(0, \zeta) = m_{\mathcal{E},u}(\Delta z, \zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^* - \zeta)} + m_{\mathcal{E},h}(-\zeta).$$

Step 2. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_{\mathcal{E}}(\zeta) = -(1 - \mathcal{E}_{ss}) \frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} \zeta + \frac{(1 - \mathcal{E}_{ss})}{f(\hat{w}^*) + s} \left(\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})} - \frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)} \right) \zeta + o(\zeta^2).$$

Proof of Step 2. From Step 1, we have that

$$\mathrm{CIR}'_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss} f'(\hat{w}^*)}{f(\hat{w}^*)^2} - m'_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).$$

Since $\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z$ satisfies the same system of functional equations as the CIR with flexible entry wages characterized in Online Appendix E.3,

$$\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z = -(1-\mathcal{E}_{ss})\frac{\gamma\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}. \tag{E.38}$$

Observe that we can write

$$\begin{aligned} m_{\mathcal{E},h}(\Delta z) &= \mathbb{E}\left[\int_0^{\tau^m} (1-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},u}(\Delta z, 0) \middle| \Delta z_0 = \Delta z\right], \\ &= (1-\mathcal{E}_{ss})\mathcal{T}(\hat{w}^* + \Delta z, 0) - \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0). \end{aligned}$$

Taking the derivative with respect to Δz, evaluating it at $\Delta z = 0$, and using $s = 1/\mathcal{T}(\hat{w}^*, 0)$ from the Renewal Principle, we have that

$$m'_{\mathcal{E},h}(0) = (1-\mathcal{E}_{ss})\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{s}{f(\hat{w}^*) + s}\mathcal{T}'_{\hat{w}}(\hat{w}^*, 0) = \frac{1}{f(\hat{w}^*) + s}\frac{\mathcal{T}'(\hat{w}^*, 0)}{\mathcal{T}(\hat{w}^*, 0)}. \tag{E.39}$$

From the free entry condition

$$f(\hat{w}^*) = \left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}},$$

and the definition $(1-\eta(\hat{w}^*)) = \hat{J}(\hat{w}^*)/\hat{S}(\hat{w}^*)$, we can compute the elasticity of the job finding rate with respect to the entry wage:

$$\begin{aligned} \frac{f'(\hat{w}^*)}{f(\hat{w}^*)} &= \frac{\frac{1-\alpha}{\alpha}\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}-1}\frac{\hat{J}'(\hat{w}^*)}{\tilde{K}}}{\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}}}, \\ &= \frac{1-\alpha}{\alpha}\frac{\hat{J}'(\hat{w}^*)}{\hat{J}(\hat{w}^*)}, \\ &= \frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]. \end{aligned}$$

Finally, combining this result with the fact that $\mathcal{E}_{ss} = \frac{f(\hat{w}^*)}{f(\hat{w}^*)+s}$, $s = \frac{1}{\mathcal{T}(\hat{w}^*,0)}$, $\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) = -\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}$, and operating, we obtain

$$\begin{aligned} -\frac{\mathcal{E}_{ss} f'(\hat{w}^*)}{f(\hat{w}^*)^2} &= -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)}\frac{f'(\hat{w}^*)}{f(\hat{w}^*)} \\ &= \frac{1}{f(\hat{w}^*) + s}\left[-\frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]\right] \\ &= \frac{1}{f(\hat{w}^*) + s}\left[-\frac{1}{\alpha}\left[-\frac{\eta'(\hat{w}^*)(1-\alpha)}{(1-\eta(\hat{w}^*))} + (1-\alpha)\frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]\right] \\ &= \frac{1}{f(\hat{w}^*) + s}\left[-\frac{1}{\alpha}\left[-\frac{\eta'(\hat{w}^*)\alpha}{\eta(\hat{w}^*)} - \alpha\frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]\right] \\ &= \frac{1}{f(\hat{w}^*) + s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}\right]. \end{aligned} \tag{E.40}$$

Combining results in equations (E.38), (E.39), and (E.40), we obtain the desired result:

$$\mathrm{CIR}'_{\mathcal{E}}(0) = -(1-\mathcal{E}_{ss})\frac{[\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1-\mathcal{E}_{ss}}{f(\hat{w}^*)+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right].$$

□

E.6 Proof of Proposition 12

Proposition 12. *The following properties hold for* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$.

a. Assume that $\Delta^- = \Delta^+$ *and* $\gamma = 0$*. Then,* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$ *and, up to a 3rd order approximation of* $\mathcal{T}(\hat{w},\hat{\rho})$ *around* $\hat{w} = \hat{w}^*$*,*

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1}{\hat{\rho}+\delta+(\sigma/\Delta^+)^2}.$$

b. Up to a 2nd order approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ *around* $\hat{w} = \hat{w}^*$*,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+ - \Delta^-}{\Delta^+\Delta^-}.$$

c. If $\hat{\rho} = 0$*, then*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)} = \frac{1}{\sigma^2 g^h(0)}\left[s^{end}\left(\mathcal{E}_{ss} - 2G^h(0)\right) + \frac{\sigma^2}{2}\left(\lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z) + \lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z)\right)\right].$$

d. If $\hat{\rho} > 0$*, then*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\mathcal{T}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2} + \frac{\sigma^2}{4}\left[\lim_{\Delta z\downarrow\Delta^-} - \lim_{\Delta z\uparrow\Delta^+}\right]\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}\right] + o(\hat{\rho}^2),$$

where $\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})$ *solves, up to a first-order approximation around* $\hat{\rho}=0$*, the 2-step procedure given by*

$$\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho}) = \frac{g^h(\Delta z)}{g^h(0)}\left[\mathcal{T}(\hat{w}^*,\hat{\rho})e^{\frac{\gamma^2}{\sigma^2}\Delta z} + \frac{2g^h(0)}{\sigma^2}\begin{cases}\int_{\Delta z}^0 e^{\frac{\gamma^2}{\sigma^2}(\Delta z-s)}\int_{-\Delta^-}^s \frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s & \text{if } \Delta z<0\\ \int_0^{\Delta z} e^{\frac{\gamma^2}{\sigma^2}(\Delta z-s)}\int_s^{\Delta^+}\frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s & \text{if } \Delta z>0\end{cases}\right] \tag{E.41}$$

with

$$\lim_{\Delta z\downarrow\Delta^-}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\downarrow-\Delta^-}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{-\frac{\gamma^2}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{\gamma^2}{\sigma^2}(-\Delta^--s)}\int_{-\Delta^-}^s\frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s\right]$$

$$\lim_{\Delta z\uparrow\Delta^+}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\uparrow\Delta^+}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{\frac{\gamma^2}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{\gamma^2}{\sigma^2}(\Delta^+-s)}\int_s^{\Delta^+}\frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s\right]$$

Proof. We proceed to prove items *a-d* of the Proposition. To show these properties, it would be useful to change the state variable of $\mathcal{T}(\hat{w},\hat{\rho})$ from $\hat{w}$ to Δz. Define $\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) := \mathcal{T}(\hat{w}^* + \Delta z,\hat{\rho})$. Then, applying Itô's Lemma, we obtain

$$\delta\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 1 - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) - \gamma\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho}) + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho}) \;\forall\; \Delta z \in (-\Delta^-,\Delta^+), \tag{E.42}$$

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 0 \;\forall\; \Delta z \notin (-\Delta^-,\Delta^+). \tag{E.43}$$

a. Assume that $\Delta^- = \Delta^+$ and $\gamma = 0$. Then, it is easy to show that $\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(-\Delta z,\hat{\rho})$, and by definition of a derivative,

$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \lim_{\epsilon\downarrow 0}\underbrace{\frac{\tilde{\mathcal{T}}(\epsilon,\hat{\rho}) - \tilde{\mathcal{T}}(-\epsilon,\hat{\rho})}{2\epsilon}}_{=0} = 0.$$

A similar argument applies to $\tilde{\mathcal{T}}'''_{\Delta z^3}(0,\hat{\rho})$. Thus, $\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \tilde{\mathcal{T}}'''_{\Delta z^3}(0,\hat{\rho}) = 0$. Applying a third-order Taylor approximation to $\tilde{\mathcal{T}}(\Delta z,\hat{\rho})$ around $\Delta z = 0$,

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(0,\hat{\rho}) + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\Delta z^2 + O(\Delta z^4).$$

From the HJB equation in (E.42),

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1 + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho} + \delta}.$$

Combining the Taylor approximation with the border conditions in (E.43), we obtain (we omit the term $O(\Delta z^4)$ to save on notation)

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2.$$

Using these results, we have that

$$-\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2 = \frac{1 + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho} + \delta} \iff \tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho}) = -\frac{1}{\hat{\rho}+\delta}\left(\frac{(\Delta^+)^2}{2} + \frac{\sigma^2}{2(\hat{\rho}+\delta)}\right)^{-1}$$

and

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1}{\hat{\rho} + \delta + \left(\frac{\sigma}{\Delta^+}\right)^2} \qquad ; \qquad \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho}) = 0.$$

b. Now, we let $\gamma \neq 0$ and $\Delta^+ \neq \Delta^-$. In this case, we proceed with a second-order Taylor approximation of $\tilde{\mathcal{T}}(\Delta z,\hat{\rho})$ around $\Delta z = 0$,

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta z + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\Delta z^2 + O(\Delta z^3).$$

From the border conditions in (E.43), we obtain (we omit the term $O(\Delta z^3)$ to save on notation)

$$\begin{aligned}\tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta^+ + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2 &= 0,\\ \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(-\Delta^-) + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^-)^2 &= 0.\end{aligned} \tag{E.44}$$

Taking the difference

$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(\Delta^+ + \Delta^-) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2 - (\Delta^-)^2\right) \iff \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left(\Delta^+ - \Delta^-\right).$$

Replacing this last equation into the HJB equation in (E.42) evaluated at $\Delta z = 0$ and into (E.44), we obtain

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1 + \left(\frac{\sigma^2 + \gamma(\Delta^+ - \Delta^-)}{2}\right)\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho} + \delta}$$

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2 - \Delta^+\left(\Delta^+ - \Delta^-\right)\right).$$

Combining these equations and solving for $\tilde{\mathcal{T}}(0,\hat{\rho})$, we have

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1}{\hat{\rho} + \delta + \frac{\sigma^2 + \gamma(\Delta^+ - \Delta^-)}{(\Delta^+)^2 - \Delta^+(\Delta^+ - \Delta^-)}},$$
$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \frac{(\Delta^+ - \Delta^-)}{(\hat{\rho}+\delta)\left((\Delta^+)^2 - \Delta^+(\Delta^+ - \Delta^-)\right) + \sigma^2 + \gamma(\Delta^+ - \Delta^-)},$$
$$\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho}) = -\frac{2}{(\hat{\rho}+\delta)\left((\Delta^+)^2 - \Delta^+(\Delta^+ - \Delta^-)\right) + \sigma^2 + \gamma(\Delta^+ - \Delta^-)}.$$

and

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+ - \Delta^-}{\Delta^+\Delta^-}.$$

c. Set $\hat{\rho} = 0$. Using (E.17), (E.42), and (E.43)

$$0 = g^h(\Delta z) - \gamma\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)\right) + \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)''(\Delta z)\right)$$

Integrating between $-\Delta^-$ and $\Delta z < 0$,

$$\begin{aligned}
0 &= G^h(\Delta z) - \gamma\int_{-\Delta^-}^{\Delta z}\left((g^h)'(x)\tilde{\mathcal{T}}(x,0) + g^h(x)\tilde{\mathcal{T}}'_{\Delta z}(x,0)\right)\mathrm{d}x + \frac{\sigma^2}{2}\int_{-\Delta^-}^{\Delta z}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(x,0)g^h(x) - \tilde{\mathcal{T}}(x,0)(g^h)''(x)\right)\mathrm{d}x \\
&= G^h(\Delta z) - \gamma\int_{-\Delta^-}^{\Delta z}\frac{\mathrm{d}\left(g^h(x)\tilde{\mathcal{T}}(x,0)\right)}{\mathrm{d}x}\mathrm{d}x + \frac{\sigma^2}{2}\int_{-\Delta^-}^{\Delta z}\frac{\mathrm{d}\left(\tilde{\mathcal{T}}'_{\Delta z}(x,0)g^h(x) - \tilde{\mathcal{T}}(x,0)(g^h)'(x)\right)}{\mathrm{d}x}\mathrm{d}x, \\
&= G^h(\Delta z) - \gamma g^h(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) + \frac{\sigma^2}{2}(\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)'(\Delta z)).
\end{aligned}$$

In the last step, we use the fact that $\lim_{\Delta z\downarrow -\Delta^-} g^h(\Delta z) = \lim_{\Delta z\downarrow -\Delta^-}\tilde{\mathcal{T}}(\Delta z,0) = 0$. Applying similar steps to integrate from $\Delta z > 0$ to Δ^+, we have that

$$0 = \mathcal{E}_{ss} - G^h(\Delta z) + \gamma g^h(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) - \frac{\sigma^2}{2}(\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)'(\Delta z)).$$

Thus, $\tilde{\mathcal{T}}(\Delta z,0)$ satisfies the following first order differential equation, once we write it as a function of $g^h(\Delta z)$:

$$\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0) = \begin{cases} -\frac{2}{\sigma^2}\frac{G^h(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2\gamma}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z,0) & \text{if } \Delta z \in (-\Delta^-,0) \\ \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - G^h(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2\gamma}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z,0) & \text{if } \Delta z \in (0,\Delta^+) \end{cases} \tag{E.45}$$

Integrating the Kolmogorov forward equation (E.17) from 0 to Δ^+ and from $-\Delta^-$ to 0, we obtain

$$-\frac{2}{\sigma^2}\left[\delta\left(\mathcal{E}_{ss} - G^h(0)\right) - \frac{\sigma^2}{2}\lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z) + \gamma g^h(0)\right] = \lim_{\Delta z\downarrow 0}(g^h)'(\Delta z), \tag{E.46}$$
$$\frac{2}{\sigma^2}\left[\delta G^h(0) + \frac{\sigma^2}{2}\lim_{\Delta z\downarrow -\Delta^-}(g^h)'(\Delta z) - \gamma g^h(0)\right] = \lim_{\Delta z\uparrow 0}(g^h)'(\Delta z), \tag{E.47}$$

respectively. Next, we sum the limits of (E.45) as $\Delta z \to 0$ from the left and right, use the continuity of $G^h(\Delta)$, $g^h(\Delta)$, $\tilde{\mathcal{T}}'(\Delta z,0)$

together with (E.46) and (E.47) to obtain

$$2\tilde{\mathcal{T}}'_{\Delta z}(0,0) = \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)} + \left(\frac{\lim_{\Delta z \downarrow 0}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow 0}(g^h)'(\Delta z)}{g^h(0)} + \frac{4\gamma}{\sigma^2}\right)\tilde{\mathcal{T}}(0,0),$$

$$= \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)}\left[1 - \delta\tilde{\mathcal{T}}(0,0)\right] + \frac{\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)}{g^h(0)}\tilde{\mathcal{T}}(0,0),$$

$$= \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)}\underbrace{\left[1 - \frac{s^{exo}}{s}\right]}_{= s^{end}/s} + \frac{\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)}{g^h(0)}\tilde{\mathcal{T}}(0,0),$$

where the last equation uses $s = 1/\tilde{\mathcal{T}}(0,0)$. Operating the last expression, we obtain

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,0)}{\tilde{\mathcal{T}}(0,0)} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)} = \frac{1}{\sigma^2}\left[s^{end}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)} + \frac{\sigma^2}{2}\frac{\lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z) + \lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z)}{g^h(0)}\right].$$

d. Now, we study the case with $\hat{\rho} > 0$. Let $\Psi(\Delta z, \hat{\rho}) := \tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})$. Differentiating the HJB in (E.42), we obtain a new HJB

$$\delta\Psi(\Delta z, \hat{\rho}) = -\hat{\rho}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) - \gamma\Psi'_{\Delta z}(\Delta z, \hat{\rho}) + \frac{\sigma^2}{2}\Psi''_{\Delta z^2}(\Delta z, \hat{\rho}),\ \forall\ \Delta z \in (-\Delta^-, \Delta^+)$$

with new border conditions for $\Psi(\Delta z, \hat{\rho})$

$$\tilde{\mathcal{T}}'_{\Delta z}(-\Delta^-, \hat{\rho}) = \Psi(-\Delta^-, \hat{\rho})\ ;\tilde{\mathcal{T}}'_{\Delta z}(\Delta^+, \hat{\rho}) = \Psi(\Delta^+, \hat{\rho}).$$

Thus,

$$\Psi(\Delta z, \hat{\rho}) = \mathbb{E}\left[\int_0^{\tau^m} -\hat{\rho}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z_t, \hat{\rho})\,dt + \tilde{\mathcal{T}}'_{\Delta z}(\Delta z_{\tau^m}, \hat{\rho})\mathbb{1}[\Delta z_{\tau^m} = \Delta^+ \textit{ or } \Delta z_{\tau^m} = -\Delta^-] | \Delta z_0 = \Delta z\right].$$

Evaluating at zero and using the occupancy measure,

$$\Psi(0,\hat{\rho}) = -\hat{\rho}\frac{\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})]}{\mathcal{E}_{ss}}\tilde{\mathcal{T}}(0,0) + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})\frac{(g^h)'(\Delta z)}{s\mathcal{E}_{ss}} - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})\frac{(g^h)'(\Delta z)}{s\mathcal{E}_{ss}}\right]$$

Using the fact that $s = \frac{1}{\tilde{\mathcal{T}}(0,0)}$, we have that

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})] + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z)\right]\right]$$

Notice that for small $\hat{\rho}$, we can apply a 1st-order approximation to $\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})]$ around $\hat{\rho} = 0$:

$$\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})] = \left(\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] + \underbrace{\hat{\rho}\mathbb{E}_h\left[\frac{\partial\tilde{\mathcal{T}}'_{\Delta z}}{\partial\hat{\rho}}(\Delta z, 0)\right]}_{=0}\right)\hat{\rho} + o(\hat{\rho}^2) = \mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)]\hat{\rho} + O(\hat{\rho}^2).$$

Thus,

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z)\right]\right] + O(\hat{\rho}^2).$$

Next, we show that $\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] = \frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}$:

$$\begin{aligned}
\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] &= \int_{-\Delta^-}^{\Delta^+} \tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0) g^h(\Delta z)\,\mathrm{d}\Delta z, \\
&=^{(1)} \mathcal{T}(\Delta z, 0) g^h(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} \mathcal{T}(\Delta z, 0)(g^h)'(\Delta z)\,\mathrm{d}\Delta z, \\
&=^{(2)} -\int_{-\Delta^-}^{\Delta^+} \left[\frac{m_{\mathcal{E},h}(\Delta z)}{1-\mathcal{E}_{ss}} - \frac{m_{\mathcal{E},u}(0,0)}{1-\mathcal{E}_{ss}}\right](g^h)'(\Delta z)\,\mathrm{d}\Delta z, \\
&=^{(3)} \frac{m_{\mathcal{E},u}(0)}{1-\mathcal{E}_{ss}}\, g^h(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} \frac{m_{\mathcal{E},h}(\Delta z)}{1-\mathcal{E}_{ss}}(g^h)'(\Delta z)\,\mathrm{d}\Delta z, \\
&=^{(4)} \frac{[\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2}.
\end{aligned}$$

Step (1) applies integration by parts; step (2) uses the border conditions for $g^h(\Delta z)$; step (3) uses the recursive definition of $m_{\mathcal{E},h}(\Delta z) = (1-\mathcal{E}_{ss})\tilde{\mathcal{T}}(\Delta z) + m_{\mathcal{E},u}(0,0)$; and step (4) uses the results in Subsection E.3. Thus,

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{[\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-} \tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} \tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z)\right]\right] + o(\hat{\rho}^2).$$

To further operate on the limits, observe that

$$\frac{\mathrm{d}^2(\tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)(\Delta z))}{\mathrm{d}\Delta z^2} = \tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) + 2\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z) + \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z)$$

The limits of this expression as $\Delta z \downarrow -\Delta^-$ and $\Delta z \downarrow \Delta^+$ are given by

$$\left[\lim_{\Delta z \downarrow -\Delta^-} - \lim_{\Delta z \uparrow \Delta^+}\right]\frac{\mathrm{d}^2(\tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)(\Delta z))}{\mathrm{d}\Delta z^2} = \left[\lim_{\Delta z \downarrow -\Delta^-} - \lim_{\Delta z \uparrow \Delta^+}\right] 2\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z). \tag{E.48}$$

Replacing this expression into (E.48), we obtain

$$\frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\mathcal{T}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{\gamma \mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} + \frac{\sigma^2}{4}\left[\lim_{\Delta z \downarrow -\Delta^-} - \lim_{\Delta z \uparrow \Delta^+}\right]\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^* + \Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}\right] + O(\hat{\rho}^2).$$

Finally, we characterize the marginal duration at the separation triggers as a function of $g^h(\Delta z)$. Using (E.17), (E.42), and (E.43)

$$0 = g^h(\Delta z) - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho})g^h(\Delta z) - \gamma\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})\right) + \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z)\right).$$

After applying a similar first-order approximation to $\hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho})g^h(\Delta z)$ around $\hat{\rho} = 0$, we obtain

$$0 = g^h(\Delta z) - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,0)g^h(\Delta z) - \gamma\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})\right) + \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z)\right) + O(\hat{\rho}^2).$$

To save on notation, we omit the term $O(\hat{\rho}^2)$. Define

$$\begin{aligned}
\Phi^-(\Delta z) &= \int_{-\Delta^-}^{\Delta z} (1 - \hat{\rho}\tilde{\mathcal{T}}(x,0))g^h(x)\,\mathrm{d}x \\
\Phi^+(\Delta z) &= \int_{\Delta z}^{\Delta^+} (1 - \hat{\rho}\tilde{\mathcal{T}}(x,0))g^h(x)\,\mathrm{d}x.
\end{aligned}$$

Then, after applying similar steps as in item c, $\tilde{\mathcal{T}}(\Delta z, \hat{\rho})$ satisfies a first order differential equation, once we write it as a function of $g^h(\Delta z)$:

$$\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0) = \begin{cases} -\frac{2}{\sigma^2}\frac{\Phi^-(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2\gamma}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z, 0) & \text{if } \Delta z \in (-\Delta^-, 0) \\ \frac{2}{\sigma^2}\frac{\Phi^+(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2\gamma}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z, 0) & \text{if } \Delta z \in (0, \Delta^+) \end{cases}$$

Guessing and verifying the solution, it is easy to see that the solution is given by

$$\begin{aligned} \tilde{\mathcal{T}}(\Delta z, \hat{\rho}) &= \tilde{\mathcal{T}}(0, \hat{\rho})\frac{g^h(\Delta z)}{g^h(0)}e^{\frac{2\gamma}{\sigma^2}\Delta z} + \begin{cases} \frac{2g(\Delta z)}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z < 0 \\ \frac{2g(\Delta z)}{\sigma^2}\int_0^{\Delta z} e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^+(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z > 0 \end{cases} \\ &= \frac{g^h(\Delta z)}{g^h(0)}\left[\tilde{\mathcal{T}}(0, \hat{\rho})e^{\frac{2\gamma}{\sigma^2}\Delta z} + \frac{2g(0)}{\sigma^2}\begin{cases} \int_{\Delta z}^0 e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z < 0 \\ \int_0^{\Delta z} e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^+(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z > 0 \end{cases}\right] \end{aligned}$$

Next, we characterize the marginal duration. Taking the derivative of the solution, and using L'Hopital's rule, $\lim_{\Delta \downarrow -\Delta^-}\left(\frac{G^h(\Delta z)}{g^h(\Delta z)}, \frac{G^h(\Delta z)}{g^h(\Delta z)^2}\right) = (0, 1)$, when $\Delta z \downarrow -\Delta^-$

$$\frac{\tilde{\mathcal{T}}(0, \hat{\rho})}{g(0)}e^{\frac{2\gamma}{\sigma^2}\Delta z}\left[g'(\Delta z) + \frac{2\gamma}{\sigma^2}g(\Delta z)\right] \to \frac{\tilde{\mathcal{T}}(0, \hat{\rho})}{g(0)}e^{-\frac{2\gamma}{\sigma^2}\Delta^-}g'(-\Delta^-),$$

$$\frac{2g'(\Delta z)}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s \to \frac{2g'(-\Delta^-)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s,$$

$$\frac{2g(\Delta z)}{\sigma^2}\frac{2\gamma}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2\gamma}{\sigma^2}(\Delta z - s)}\frac{g^h(s)}{g^h(s)^2}\,\mathrm{d}s + \frac{2g(\Delta z)}{\sigma^2}\frac{2\gamma}{\sigma^2}\frac{G^h(\Delta z)}{g^h(\Delta z)^2}\,\mathrm{d}s \to 0.$$

Combining these results, we obtain

$$\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) = \lim_{\Delta z \downarrow -\Delta^-}\frac{(g^h)'(\Delta z)}{g^h(0)}\left[\tilde{\mathcal{T}}(0, \hat{\rho})e^{-\frac{2\gamma}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2\gamma}{\sigma^2}(-\Delta^- - s)}\frac{\Phi^-(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right]$$

and

$$\lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) = \lim_{\Delta z \uparrow \Delta^+}\frac{(g^h)'(\Delta z)}{g(0)}\left[\tilde{\mathcal{T}}(0, \hat{\rho})e^{\frac{2\gamma}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{2\gamma}{\sigma^2}(\Delta^+ - s)}\frac{\Phi^+(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right].$$

Therefore,

$$\begin{aligned} \lim_{\Delta z \downarrow \Delta^-}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^* + \Delta z, \hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} &= \lim_{\Delta z \downarrow -\Delta^-} 2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0, \hat{\rho})e^{-\frac{2\gamma}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2\gamma}{\sigma^2}(-\Delta^- - s)}\frac{\Phi^-(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right] \\ \lim_{\Delta z \uparrow \Delta^+}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^* + \Delta z, \hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} &= \lim_{\Delta z \uparrow \Delta^+} 2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0, \hat{\rho})e^{\frac{2\gamma}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{2\gamma}{\sigma^2}(\Delta^+ - s)}\frac{\Phi^+(\Delta z)}{g^h(s)^2}\,\mathrm{d}s,\right] \end{aligned}$$

which proves the result. □

E.7 Proof of Proposition 13

Proposition 13. *Define*

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $\eta(\hat{w})$ satisfies the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\,\mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+]|\Gamma_0 = \hat{w}\right]$$

with

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t,\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}))\,\mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t,\hat{\rho})(\hat{\rho}+\delta)}\,\mathrm{d}\mathcal{W}^z_t.$$

Proof. The HJB equations for the worker's value and the surplus of the match are

$$\begin{aligned}(\hat{\rho}+\delta)\hat{W}(\hat{w}) &= e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w}\in(\hat{w}^-,\hat{w}^+)\\ (\hat{\rho}+\delta)\hat{S}(\hat{w}) &= 1 - \hat{\rho}\hat{U} - \hat{\gamma}\hat{S}'(\hat{w}) + \frac{\sigma^2}{2}\hat{S}''(\hat{w}) \quad \forall \hat{w}\in(\hat{w}^-,\hat{w}^+),\end{aligned}$$

respectively. Replacing the definition of the worker's share $\eta(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ into the worker's value function, we obtain

$$(\hat{\rho}+\delta)(\eta(\hat{w})\hat{S}(\hat{w})) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\left(\eta(\hat{w})\hat{S}'(\hat{w}) + \eta'(\hat{w})\hat{S}(\hat{w})\right) + \frac{\sigma^2}{2}\left(\eta(\hat{w})\hat{S}''(\hat{w}) + 2\eta'(\hat{w})\hat{S}'(\hat{w}) + \eta''(\hat{w})\hat{S}(\hat{w})\right)\forall \hat{w}\in(\hat{w}^-,\hat{w}^+).$$

Using the HJB equation of the surplus to replace $(\hat{\rho}+\delta)\hat{S}(\hat{w})$ on the left hand side,

$$(1-\hat{\rho}\hat{U})\eta(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} + \eta'(\hat{w})(-\hat{\gamma}\hat{S}(\hat{w}) + \sigma^2\hat{S}'(\hat{w})) + \eta''(\hat{w})\frac{\sigma^2}{2}\hat{S}(\hat{w}) \quad \forall \hat{w}\in(\hat{w}^-,\hat{w}^+).$$

Since $\hat{S}(\hat{w}) = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})$,

$$\eta(\hat{w}) = \frac{e^{\hat{w}}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} + \eta'(\hat{w})(-\hat{\gamma}\mathcal{T}(\hat{w},\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\hat{w},\hat{\rho})) + \eta''(\hat{w})\frac{\sigma^2}{2}\mathcal{T}(\hat{w},\hat{\rho}) \quad \forall \hat{w}\in(\hat{w}^-,\hat{w}^+).$$

Multiplying by $(\hat{\rho}+\delta)$, we arrive at

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} + \eta'(\hat{w})(\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\hat{w},\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\hat{w},\hat{\rho})) + \eta''(\hat{w})\frac{\sigma^2}{2}(\hat{\rho}+\delta)\mathcal{T}(\hat{w},\hat{\rho}) \quad \forall \hat{w}\in(\hat{w}^-,\hat{w}^+).$$

Finally, recall the value matching conditions

$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0,$$

and the smooth pasting conditions

$$\hat{W}'(-\Delta^-) = \hat{J}'(\Delta^+) = 0.$$

The L'Hôpital's rule implies

$$\begin{aligned}\lim_{\hat{w}\downarrow\hat{w}^-}\eta(\hat{w}) &= \lim_{\hat{w}\downarrow\hat{w}^-}\frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-}\frac{\hat{W}'(\hat{w})}{\hat{J}'(\hat{w})} = 0\\ \lim_{\hat{w}\uparrow\hat{w}^+}\eta(\hat{w}) &= \lim_{\hat{w}\uparrow\hat{w}^+}\frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\uparrow\hat{w}^+}\frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1,\end{aligned}$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $\eta(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\,\mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+]|\Gamma_0 = \hat{w}\right],$$

where

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

and

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t,\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}))\,\mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t,\hat{\rho})(\hat{\rho}+\delta)}\,\mathrm{d}\mathcal{W}^z_t.$$

□

E.8 Proof of Proposition 14

Proposition 14. *The following properties hold:*

1. *If $(\hat{w}^-, \hat{w}) \to \infty$, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\left[\alpha + (1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}.$$

2. *Assume $\gamma = 0$, $\Delta^+ = \Delta^-$, and Δ^+ small enough, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{1}{\alpha(\Delta^+ + \Delta^-)} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

Proof. Below, we prove each property.

1. If $(\hat{w}^-, \hat{w}^+) \to (-\infty, \infty)$, then $\mathcal{T}(\hat{w},\hat{\rho}) = \int_0^\infty e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t = \frac{1}{\hat{\rho}+\delta}$. The optimality condition for $\hat{w}^*$ implies

$$0 = -\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) \iff \alpha = \eta(\hat{w}^*).$$

Therefore, by definition of $\eta(\hat{w})$

$$\alpha = \eta(\hat{w}^*) = \frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right] - \hat{\rho}\hat{U}\mathcal{T}(\hat{w},\hat{\rho})}{(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})} \iff \left[\alpha + (1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho}) = \mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right].$$

Since, $\mathcal{T}(\hat{w},\hat{\rho})$ is constant, the HJB equation of the worker's share $\eta(\hat{w})$ is given by

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}-\hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta'(\hat{w}) + \eta''(\hat{w})\frac{\sigma^2}{2} \;\; \forall\hat{w} \in (-\infty,\infty). \tag{E.49}$$

Taking the derivative of (E.49) with respect to $\hat{w}$ yields

$$(\hat{\rho}+\delta)\eta'(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta''(\hat{w}) + \eta'''(\hat{w})\frac{\sigma^2}{2} \;\; \forall\hat{w} \in (-\infty,\infty).$$

This expression corresponds to the HJB of the function $\eta'(\hat{w})$, which can be expressed as

$$\eta'(\hat{w}^*) = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0=\hat{w}^*\right]}{1-\hat{\rho}\hat{U}}$$

Combining all these results, we finally obtain

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\eta'(\hat{w}^*)}{\alpha} = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0=\hat{w}^*\right]}{\alpha(1-\hat{\rho}\hat{U})} = (\hat{\rho}+\delta)\frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho})}{\alpha(1-\hat{\rho}\hat{U})} = \frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}.$$

2. If $\gamma=0$ and $\Delta^+=\Delta^-$, then $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})=0$ and $\eta(\hat{w}^*)=\alpha$ (see the proof of Proposition 12, item *a*). If $(\Delta^+ + \Delta^-)$ is small enough, then we can use a second order approximation of $\eta'(\hat{w})$ around $\hat{w}=\hat{w}^*$ to characterize $\eta'(\hat{w}^*)$ *only* using the border conditions. The approximation is given by

$$\eta(\hat{w}) = \eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}-\hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}-\hat{w}^*)^2 + O((\hat{w}-\hat{w}^*)^3).$$

Evaluating this expression at $\hat{w}^-$ and $\hat{w}^+$, and omiting any terms of the order $O((\hat{w}-\hat{w}^*)^3)$, we obtain

$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^- -\hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^- -\hat{w}^*)^2 = 0,$$
$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^+ -\hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^+ -\hat{w}^*)^2 = 1,$$

respectively. The difference between both equations is given by

$$\eta'(\hat{w}^*) = \frac{1}{\Delta^+ + \Delta^-}.$$

From the proof of Proposition 12 item *b*, we know that $\tilde{\mathcal{T}}(0,0) = 1/s = 1/(\delta + (\sigma/\Delta^+)^2) \Rightarrow s^{end} = (\sigma/\Delta^+)^2$. Replacing this result in the previous equation, we obtain,

$$\eta'(\hat{w}^*) = \frac{1}{\Delta^+ + \Delta^-} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

□

F Additional Results for Section 4: Analyzing the Macroeconomic Consequences of Allocative Wages

F.1 Characterization of CIR employment as a function of CIR job-separations and job-finding

Proposition F.1. *Define*

$$IRF^{\mathcal{E}}(\zeta,t) = \mathcal{E}_t - \mathcal{E}_{ss}, \;\; IRF^{s}(\zeta,t) = s_t - s_{ss}, \;\; IRF^{p}(\zeta,t) = f_t - f_{ss}$$

and

$$CIR^{x}(\zeta) = \int_0^\infty IRF^{x}(\zeta,t)\,\mathrm{d}t, \; \textit{with } x \in \{\mathcal{E}, s, f\},$$

Assume that $\frac{\mathcal{E}_0(0)}{\mathrm{d}\zeta} = 0$ *and* $\lim_{t\to\infty}\mathcal{E}_t(\zeta) = \mathcal{E}_{ss}$. *Then*

$$\frac{CIR^{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}\frac{CIR_f(0)}{\mathrm{d}\zeta} - s_{ss}\frac{CIR_s(0)}{\mathrm{d}\zeta}\right) + o(\zeta)$$

Proof. Since $(\mathcal{E}_t(0), p_t(0), s_t(0)) = (\mathcal{E}_{ss}, p_{ss}, s_{ss})$ a first order Taylor approximation over ζ

$$\begin{aligned}
\mathcal{E}_t(\zeta) &= \mathcal{E}_{ss} + \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2),\\
f_t(\zeta) &= f_{ss} + \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2),\\
s_t(\zeta) &= s_{ss} + \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2).
\end{aligned}$$

and the law of motion

$$\mathrm{d}\mathcal{E}_t = (-s_t\mathcal{E}_t + f_t(1-\mathcal{E}_t))\,\mathrm{d}t \quad , \quad \mathcal{E}_0(\zeta) = \mathcal{E}_{ss} + o(\zeta^2).$$

Using the first order Taylor approximation over ζ in the law of motion of employment

$$\begin{aligned}
\mathrm{d}\mathcal{E}_t &\approx \mathrm{d}(\mathcal{E}_{ss} + \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\zeta) + o_t(\zeta^2),\\
&= \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\zeta\\
&= (-s_t\mathcal{E}_t + f_t(1-\mathcal{E}_t))\,\mathrm{d}t,\\
&\approx \left(\underbrace{-s_{ss}\mathcal{E}_{ss} + (1-\mathcal{E}_{ss})}_{=0} + (1-\mathcal{E}_{ss})\frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta} - \mathcal{E}_{ss}\frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta} - (f_{ss}+s_{ss})\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\mathrm{d}t\zeta.
\end{aligned}$$

Canceling ζ from both sizes

$$\mathrm{d}\left(\frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta}\right) = \left((1-\mathcal{E}_{ss})\frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta} - \mathcal{E}_{ss}\frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta} - (f_{ss}+s_{ss})\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\mathrm{d}t.$$

Taking the integral between 0 and T

$$\int_0^T \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right) = (1-\mathcal{E}_{ss})\int_0^T \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \mathcal{E}_{ss}\int_0^T \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - (f_{ss}+s_{ss})\int_0^T \frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Since $\int_0^T \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right) = \frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta} - \frac{\mathrm{d}\hat{\mathcal{E}}_0(0)}{\mathrm{d}\zeta} = \frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta}$, we have that

$$\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta} = (1-\mathcal{E}_{ss})\int_0^T \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \mathcal{E}_{ss}\int_0^T \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - (f_{ss}+s_{ss})\int_0^T \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Taking the limit, since $\lim_{t\to\infty}\mathcal{E}_t(\zeta) = \mathcal{E}_{ss}$, we have that $\lim_{T\to\infty}\frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta} = 0$ and

$$\int_0^\infty \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t = \frac{(1-\mathcal{E}_{ss})}{(f_{ss}+s_{ss})}\int_0^\infty \frac{\mathrm{d}\hat{f}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \frac{\mathcal{E}_{ss}}{(f_{ss}+s_{ss})}\int_0^\infty \frac{\mathrm{d}\hat{s}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Since

$$\begin{aligned}
\frac{\mathrm{dCIR}^{\mathcal{E}}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}\hat{\mathcal{E}}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t,\\
\frac{\mathrm{dCIR}^{f}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}\hat{f}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t,\\
\frac{\mathrm{dCIR}^{s}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}\hat{s}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.
\end{aligned}$$

Using the previous result and $\mathcal{E}_{ss} = \frac{p_{ss}}{s_{ss}+f_{ss}}$,

$$\frac{\mathrm{dCIR}^{\mathcal{E}}(0)}{\mathrm{d}\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}\frac{\mathrm{dCIR}_f(0)}{\mathrm{d}\zeta} - s_{ss}\frac{\mathrm{dCIR}_s(0)}{\mathrm{d}\zeta}\right).$$

Since $\mathrm{CIR}^{\mathcal{E}}(\zeta) = \mathrm{CIR}_{\mathcal{E}}(0) + \frac{\mathrm{dCIR}_{\mathcal{E}}(0)}{\mathrm{d}\zeta}\zeta + o(\zeta^2)$, we have

$$\frac{\mathrm{CIR}^{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}\frac{\mathrm{CIR}_f(0)}{\mathrm{d}\zeta} - s_{ss}\frac{\mathrm{CIR}_s(0)}{\mathrm{d}\zeta}\right) + o(\zeta)$$

□

F.2 Second-Order Approximation of the CIR of Employment with Flexible Wages

Proposition F.2. *Up-to-second order appromation*

$$\begin{aligned}
(CIR^{\mathcal{E}})''(0) = {} & \frac{(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma(CIR^{\mathcal{E}})'(0)}{\sigma^2} - \frac{1}{2}\left(\Delta z m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}\\
& + \frac{1}{2}\left((g^h)'(0_-) - (g^h)'(0_+)\right)\left(m_{\mathcal{E},u}(0) - m_{\mathcal{E},h}(0)\right).
\end{aligned}$$

Proof. Taking the second derivative of (E.26) we obtain

$$\begin{aligned}
\mathrm{CIR}''_{\mathcal{E}}(\zeta) = & -\lim_{\Delta z\to-\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \lim_{\Delta z\to-\Delta^- -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \int_{-\Delta^- -\zeta}^{-\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z+\zeta)\,\mathrm{d}\Delta z\\
& -\lim_{\Delta z\to\Delta^+ -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \lim_{\Delta z\to-\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \int_{-\zeta}^{\Delta^+ -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z+\zeta)\,\mathrm{d}\Delta z,
\end{aligned}$$

which evaluated at $\zeta = 0$ becomes

$$\mathrm{CIR}''_{\mathcal{E}}(0) = -\, m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\ |_{-\Delta^-}^{0} + \int_{-\Delta^-}^{0} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) d\Delta z.$$

$$- m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \mid_0^{\Delta^+} + \int_0^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) d\Delta z.$$

Differentiating condition (D.1) to replace $\delta = \frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)}$ into equation (E.27) we obtain

$$\frac{\gamma(g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)} m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - \gamma m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2} m''_{\mathcal{E},h}(\Delta z) + \frac{\gamma g''(\Delta z) + \frac{\sigma^2}{2} g'''(\Delta z)}{(g^h)'(\Delta z)} m_{\mathcal{E},u}(0).$$

Multiplying by $(g^h)'(\Delta z)\Delta z$ and taking the integral between $-\Delta^-$ and Δ^+

$$\begin{aligned}
0 &= (1 - \mathcal{E}_{ss})\, T_1 - \gamma T_2 + \frac{\sigma^2}{2} T_3 + m_{\mathcal{E},u}(0) T_4 \qquad \text{(F.1)}\\
T_1 &= \int_{-\Delta^-}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\, \Delta z \\
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) + m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z \\
T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z) \right] \mathrm{d}\, \Delta z \\
T_4 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2} (g^h)'''(\Delta z) \right) \mathrm{d}\, \Delta z.
\end{aligned}$$

T_1 is equal to

$$\begin{aligned}
T_1 &= \int_{-\Delta -}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\, \Delta z \qquad \text{(F.2)}\\
&= \int_{-\Delta -}^{0} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\, \Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\, \Delta z \\
&= \underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\, \Delta z}_{=\mathcal{E}_{ss}} \\
&= -\mathcal{E}_{ss}.
\end{aligned}$$

T_2 satisfies

$$\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\, \Delta z \qquad \text{(F.3)}\\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z + \int_{0}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z \\
&=^{(2)} \Delta z m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{-\Delta^-}^{0} + \Delta z m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{0}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z + \int_{0}^{\Delta^+} \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z \right] \\
&=^{(3)} m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z)\, \mathrm{d}\, \Delta z \\
&=^{(4)} m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - CIR'_{\mathcal{E}}(0).
\end{aligned}$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) + m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) =$

$\frac{\mathrm{d}\left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; (3) uses conditions (E.28) and (F.7); and (4) uses the definition of $CIR'_{\mathcal{E}}(0)$.

T_3 satisfies

$$\begin{aligned}
T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\,\Delta z \qquad \text{(F.4)}\\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\,\Delta z\\
&=^{(2)} \Delta z \left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}\\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z\right]\\
&=^{(3)} \Delta z \left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},u}(0)(g^h)''(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}\\
&\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{-\Delta^-}^{0} - 2\int_{-\Delta^-}^{0} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z\right]\\
&\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{0}^{\Delta^+} - 2\int_{0}^{\Delta^+} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z\right].
\end{aligned}$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z) = \frac{\mathrm{d}\left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; and (3) uses conditions (E.28) and integrates by parts

Finally, for T_4

$$\begin{aligned}
T_4 &= \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma(g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\,\Delta z \qquad \text{(F.5)}\\
&=^{(1)} \int_{-\Delta-}^{0} \Delta z \left(\gamma(g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left(\gamma(g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\,\Delta z\\
&=^{(2)} \Delta z \left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{0} + \Delta z \left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{0}^{\Delta^+}\\
&\cdots - \int_{-\Delta^-}^{\Delta^+} \gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\,\mathrm{d}\,\Delta z\\
&=^{(3)} \Delta z \left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{\Delta^+} - \gamma \underbrace{\left[g^h(\Delta z)\Big|_{-\Delta^-}^{0} + g^h(\Delta z)\Big|_{0}^{\Delta^+}\right]}_{=0}\\
&\cdots - \frac{\sigma^2}{2}\left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+}\right]\\
&= \Delta z \left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2}\left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+}\right].
\end{aligned}$$

Here, (1) divides the integral at the discontinuity point; (2) integrates by parts; (3) operates the integral and uses the border conditions.

Combining results (F.1), (F.2), (F.3), (F.4), (F.5), we obtain

$$\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\,T_1 - \gamma T_2 + \frac{\sigma^2}{2}T_3 + m_{\mathcal{E},u}(0)T_4\\
0 &= -(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma\left(m_{\mathcal{E},u}(0)\ \Delta z(g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - CIR'_{\mathcal{E}}(0)\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdots + \frac{\sigma^2}{2}\Bigg[\ \Delta z\left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},u}(0)(g^h)''(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+} \\
&\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{-\Delta^-}^{0} - 2\int_{-\Delta^-}^{0}\left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right]\mathrm{d}\,\Delta z\right] \\
&\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta)\Big|_{0}^{\Delta^+} - 2\int_{0}^{\Delta^+}\left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right]\mathrm{d}\,\Delta z\right]\Bigg] \\
&\cdots + m_{\mathcal{E},u}(0)\left(\Delta z\left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Bigg|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2}\left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+}\right]\right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
(CIR^{\mathcal{E}})''(0) = &\ \frac{(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma(CIR^{\mathcal{E}})'(0)}{\sigma^2} - \frac{1}{2}\left(\Delta z m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+} \\
&+ \frac{1}{2}\left((g^h)'(0_-) - (g^h)'(0_+)\right)\left(m_{\mathcal{E},u}(0) - m_{\mathcal{E},h}(0)\right).
\end{aligned}
$$

□

F.3 Characterizing the CIR for real wages

We define the CIR of the average wage to a monetary shock as

$$
\mathrm{CIR}^w(\zeta) = \int_0^\infty \int_{\hat{w}^-}^{\hat{w}^+} \hat{w}\left(g^h(\hat{w},t) - g(\hat{w})\right)\mathrm{d}\hat{w}\,\mathrm{d}t
$$

The strategy to find the sufficient statistic is similar to the strategy used for $\mathrm{CIR}_{\mathcal{E}}(\zeta)$, with few differences in the implementation of the steps. For this reason, we skip the proof of some of the similar steps. The main difference is the associated Bellman equation that describes the total sum of the differences between average wage in period t and the steady-state average wage.

Step 1. The CIR satisfies

$$
\mathrm{CIR}^w(\zeta) = \int_{-\infty}^{\infty} m^{w,h}(\Delta z)g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m^{w,u}(\Delta z)g^u(\Delta z+\zeta)\,\mathrm{d}\Delta z
$$

with

$$
\begin{aligned}
m^{w,h}(\Delta z) &\equiv \mathbb{E}\left[\int_0^{\tau^m}[\Delta z_t - \mathbb{E}_h[\Delta z]]\,\mathrm{d}t + m^{w,u}(0)|\Delta z_0 = \Delta z\right] \\
m^{w,u}(\Delta z) &\equiv \mathbb{E}\left[m^{w,h}(0)|\Delta z_0 = \Delta z\right] \\
0 &= \int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m^{w,u}(\Delta z)g^u(\Delta z)\,\mathrm{d}\Delta z,
\end{aligned}
$$

Step 2. Up to first order, the $CIR^w(\zeta)$ is the solution of

$$
\mathrm{CIR}_w(\zeta) = \int_{-\infty}^{\infty} m^{w,h}(\Delta z)g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + m^{w,h}(0)(1-\mathcal{E}_{ss}),
$$

where

$$0 = \Delta z - \mathbb{E}_h[\Delta z] - \gamma (m^{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m^{w,h})''(\Delta z) + \delta(m^{w,h}(0) - (m^{w,h})'(\Delta z)) \tag{F.6}$$

$$m^{w,h}(0) = m^{w,h}(-\Delta^-) = m^{w,h}(\Delta^+),$$

$$0 = \int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + m^{w,h}(0)(1 - \mathcal{E}_{ss}),$$

$$\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-, \Delta^+)/\{0\},$$

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0, \tag{F.7}$$

$$\mathcal{E}_{ss} = \int_{-\Delta^-}^{\Delta_+} g^h(\Delta z)\, \mathrm{d}\Delta z,$$

$$g^h(\Delta z) \in \mathbb{C}, \mathbb{C}^1(\{0\}), \mathbb{C}^2(\{0\}).$$

Step 3. We show that $m^{w,h}(0) = \frac{\mathbb{C}ov_h[a, \Delta z]}{1 - \mathcal{E}_{ss}}$.

Proof of Step 3. Define $f(\Delta z_t) = \Delta z_t - \mathbb{E}_h[\Delta z]$. Observe that $m^{w,h}(\Delta z)$ satisfies the following recursive representation

$$m^{w,h}(\Delta z) \;=\; \mathbb{E}\left[\int_0^{\tau^m} f(\Delta z_t)\, \mathrm{d}t + m^{w,h}(0) \middle| \Delta z_0 = \Delta z\right].$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) \;=\; \mathbb{E}\left[\int_0^{\tau^m} e^{\varphi t} f(\Delta z_t)\, \mathrm{d}t + e^{\varphi \tau^m} m^{w,h}(0) \middle| \Delta z_0 = \Delta z\right]. \tag{F.8}$$

Then, following similar steps, we obtain

$$\begin{aligned} \frac{\partial \Psi(0|0)}{\partial \varphi} \;&=\; \mathbb{E}\left[\int_0^{\tau^m} m^{w,h}(\Delta z_t)\, \mathrm{d}t \middle| \Delta z_0 = 0\right] \\ &=\; \mathbb{E}[\tau^m] \frac{\int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z}{\mathcal{E}_{ss}} \\ &=\; -\mathbb{E}[\tau^m] m^{w,h}(0) \frac{(1 - \mathcal{E}_{ss})}{\mathcal{E}_{ss}} \end{aligned} \tag{F.9}$$

From (F.8), we have that

$$\begin{aligned} \frac{\partial \Psi(0|0)}{\partial \varphi} \;&=\; \mathbb{E}\left[\int_0^{\tau^m} s f(\Delta z_s)\, \mathrm{d}s + \tau^m m^{w,h}(0) \middle| \Delta z_0 = 0\right] \\ &= \overline{\mathbb{E}}\left[\tau^m\right] \left[\frac{\mathbb{E}_h[a f(\Delta z_s)]}{\mathcal{E}_{ss}} + m^{w,h}(0)\right], \end{aligned} \tag{F.10}$$

Combining (F.9) and (F.10), and solving for $m^{w,h}(0)$ we obtain:

$$m^{w,h}(0) = \frac{\mathbb{C}ov_h[a, \Delta z]}{1 - \mathcal{E}_{ss}}$$

Step 4. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_w(\zeta) = -\frac{\mathbb{C}ov_h[\Delta z + \gamma a, \Delta z]}{\sigma^2} + o(\zeta^2).$$

Proof of Step 4. To help the reader, we summarize below the conditions we use in this proof.

$$\text{CIR}_w(\zeta) = \int_{-\infty}^{\infty} m^{w,h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + m^{w,h}(0)(1 - \mathcal{E}_{ss}), \tag{F.11}$$

where

$$0 = f(\Delta z) - \gamma (m^{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m^{w,h})''(\Delta z) + \delta(m^{w,h}(0) - m^{w,h}(\Delta z))$$

$$m^{w,h}(0) = m^{w,h}(-\Delta^-) = m^{w,h}(\Delta^+), \tag{F.12}$$

$$0 = \int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + m^{w,h}(0)(1 - \mathcal{E}_{ss}). \tag{F.13}$$

1. **Zero-order:** If $\zeta = 0$, condition (F.13) implies

$$\text{CIR}_w(0) = \int_{-\infty}^{\infty} m^{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + m^{w,h}(0)(1 - \mathcal{E}_{ss}) = 0.$$

2. **First-order:** Taking the derivative of (F.11) we obtain

$$\text{CIR}_w'(\zeta) = \int_{-\infty}^{\infty} m^{w,h}(\Delta z)(g^h)'(\Delta z + \zeta)\, \mathrm{d}\Delta z,$$

which evaluated at $\zeta = 0$ becomes

$$\text{CIR}_w'(0) = \int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z.$$

Using condition (D.1) to replace $\delta = \frac{\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into the HJB equation (F.6), we obtain

$$\frac{\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m^{w,h}(\Delta z) = f(\Delta z) - \gamma(m^{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m^{w,h})''(\Delta z) + \frac{\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m^{w,h}(0).$$

Multiplying by $g^h(\Delta z)\Delta z$ and taking the integral between $-\Delta^-$ and Δ^+

$$\begin{aligned}
0 &= \mathbb{V}ar_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2} T_2 + m^{w,h}(0) T_3 \\
T_1 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta z) + m^{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z \\
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\, \Delta z \\
T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) \mathrm{d}\, \Delta z.
\end{aligned}$$

T_1 is equal to

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta z) + m^{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\, \Delta z$$

$$
\begin{aligned}
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta z) + m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta z) + m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z \\
&=^{(2)} \int_{-\Delta^-}^{0} \Delta z \frac{d\left(m^{w,h}(\Delta z) g^h(\Delta z)\right)}{\mathrm{d}\,\Delta z} \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \frac{d\left(m^{w,h}(\Delta z) g^h(\Delta z)\right)}{\mathrm{d}\,\Delta z} \mathrm{d}\,\Delta z \\
&=^{(3)} \underbrace{\Delta z m^{w,h}(\Delta z) g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z m^{w,h}(\Delta z) g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} \\
&\cdots - \left[\int_{-\Delta^-}^{0} m^{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} m^{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z\right] \\
&=^{(4)} -\int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z \\
&=^{(5)} m^{w,h}(0)(1-\mathcal{E}_{ss})
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the property of the derivative of a product of functions; (3) integrates and uses the border conditions (F.7); (4) uses continuity of $m^{w,h}(\Delta z) g^h(\Delta z)$; and (6) uses (F.13).

T_2 satisfies

$$
\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[(m^{w,h})''(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z \\
&=^{(2)} \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{-\Delta^-}^{0} + \Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{0}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[(m^{w,h})'(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[(m^{w,h})'(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z\right] \\
&=^{(3)} \underbrace{\Delta z \left[(m^{w,h})'(\Delta z) g^h(\Delta) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right]\Big|_{\Delta^-}^{\Delta^+}}_{=-m^{w,h}(0)\,\Delta z (g^h)'(\Delta z)|_{\Delta^-}^{\Delta^+}} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[(m^{w,h})'(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[(m^{w,h})'(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z\right] \\
&=^{(4)} -m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} (m^{w,h})'(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z + \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \\
&=^{(5)} -m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \left[\underbrace{m^{w,h}(\Delta z) g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} - \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z\right] + \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \\
&= -m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} + 2\int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z.
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $(m^{w,h})''(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) = \frac{\mathrm{d}\left[(m^{w,h})'(\Delta z) g^h(\Delta z) - m^{w,h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; (3) uses conditions (F.12) and (F.7); and (4)-(5) integrate by parts and operate.

Finally, for T_3

$$
T_3 = \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right) \mathrm{d}\,\Delta z
$$

$$
\begin{aligned}
&= \gamma \left[\int_{-\Delta^-}^{0} \Delta z (g^h)'(\Delta z)\,\mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \right] + \frac{\sigma^2}{2} \left[\int_{\Delta^-}^{0} \Delta z (g^h)''(\Delta z)\,\mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)''(\Delta z)\,\mathrm{d}\,\Delta z \right] \\
&= \gamma \left[\underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\,\Delta z}_{=\mathcal{E}_{ss}} \right] \\
&\cdots + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z (g^h)'(\Delta z)\Big|_{0}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} (g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \right] \\
&= -\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \underbrace{g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} \right] \\
&= -\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} \right].
\end{aligned}
$$

Combining all these results

$$
\begin{aligned}
0 &= \mathbb{V}ar_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2} T_2 + m^{w,h}(0) T_3 \\
0 &= \mathbb{V}ar_h[\Delta z] - \gamma m^{w,h}(0)(1 - \mathcal{E}_{ss}) \\
&\quad + \frac{\sigma^2}{2} \left(-m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} + 2 \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \right) \\
&\quad + m^{w,h}(0) \left(-\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} \right] \right) \\
0 &= \mathbb{V}ar_h[\Delta z] - \gamma m^{w,h}(0) + \sigma^2 \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_{-\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z) &= \frac{\gamma m^{w,h}(0) - \mathbb{V}ar_h[\Delta z]}{\sigma^2} \\
&= -\frac{\mathbb{C}ov_h[\Delta z + \gamma a, \Delta z]}{\sigma^2}.
\end{aligned}
$$

3. **Second-order:** Taking the second derivative of (E.26) we obtain

$$
\begin{aligned}
\mathrm{CIR}_w''(\zeta) = &- \lim_{\Delta z \to -\zeta} m^{w,h}(\Delta z)(g^h)'(\Delta z + \zeta) + \lim_{\Delta z \to -\Delta^- - \zeta} m^{w,h}(\Delta z)(g^h)'(\Delta z + \zeta) + \int_{-\Delta^- - \zeta}^{-\zeta} m^{w,h}(\Delta z)(g^h)''(\Delta z + \zeta)\,\mathrm{d}\Delta z \\
&- \lim_{\Delta z \to \Delta^+ - \zeta} m^{w,h}(\Delta z)(g^h)'(\Delta z + \zeta) + \lim_{\Delta z \to -\zeta} m^{w,h}(\Delta z)(g^h)'(\Delta z + \zeta) + \int_{-\zeta}^{\Delta^+ - \zeta} m^{w,h}(\Delta z)(g^h)''(\Delta z + \zeta)\,\mathrm{d}\Delta z,
\end{aligned}
$$

which evaluated at $\zeta = 0$ becomes

$$
\begin{aligned}
\mathrm{CIR}_w''(0) = &- m^{w,h}(\Delta z)(g^h)'(\Delta z)\ |_{-\Delta^-}^{0} + \int_{-\Delta^-}^{0} m^{w,h}(\Delta z)(g^h)''(\Delta z) d\Delta z. \\
&- m^{w,h}(\Delta z)(g^h)'(\Delta z)\ |_{0}^{\Delta^+} + \int_{0}^{\Delta^+} m^{w,h}(\Delta z)(g^h)''(\Delta z) d\Delta z.
\end{aligned}
$$

Differentiating condition (D.1) to replace $\delta = \frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)}$ into equation (F.6), we obtain

$$\frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)} m^{w,h}(\Delta z) = f(\Delta z) - \gamma (m^{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m^{w,h})''(\Delta z) + \frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)} m^{w,h}(0).$$

Multiplying by $(g^h)'(\Delta z)\Delta z$ and taking the integral between $-\Delta^-$ and Δ^+

$$\begin{aligned} 0 &= T_1 - \gamma T_2 + \frac{\sigma^2}{2} T_3 + m^{w,h}(0) T_4 \\ T_1 &= \int_{-\Delta^-}^{\Delta^+} f(\Delta z)\Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \\ T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) + (m^{w,h})'(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z \\ T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\,\Delta z \\ T_4 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\,\Delta z. \end{aligned} \tag{F.14}$$

T_1 is equal to

$$\begin{aligned} T_1 &= \int_{-\Delta-}^{\Delta^+} f(\Delta z)\Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \\ &= \int_{-\Delta-}^{0} f(\Delta z)\Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} f(\Delta z)\Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \\ &= \underbrace{\Delta z \left(\frac{\Delta z^2}{2} - \mathbb{E}_h[\Delta z]\Delta z\right) g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z \left(\frac{\Delta z^2}{2} - \mathbb{E}_h[\Delta z]\Delta z\right) g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} \left(\frac{\Delta z^2}{2} - \mathbb{E}_h[\Delta z]\Delta z\right) g^h(\Delta z)\, \mathrm{d}\,\Delta z}_{=\frac{\mathbb{E}_h[\Delta z^2]}{2} - \mathbb{E}_h[\Delta z]^2} \\ &= -\left(\frac{\mathbb{E}_h[\Delta z^2]}{2} - \mathbb{E}_h[\Delta z]^2\right). \end{aligned} \tag{F.15}$$

T_2 satisfies

$$\begin{aligned} T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) - (m^{w,h})'(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\,\Delta z \\ &=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) - (m^{w,h})'(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) - (m^{w,h})'(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z \\ &=^{(2)} \Delta z m^{w,h}(\Delta z)(g^h)'(\Delta)\Big|_{-\Delta^-}^{0} + \Delta z m^{w,h}(\Delta z)(g^h)'(\Delta)\Big|_{0}^{\Delta^+} \\ &\cdots - \left[\int_{-\Delta^-}^{0} \left[m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[m^{w,h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\,\Delta z \right] \\ &=^{(3)} m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m^{w,h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \\ &=^{(4)} m^{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - CIR'_w(0). \end{aligned} \tag{F.16}$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $m^{w,h}(\Delta z)(g^h)''(\Delta z) + (m^{w,h})'(\Delta z)(g^h)'(\Delta z) = \frac{\mathrm{d}\left[m^{w,h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; (3) uses the border conditions; and (4) uses the definition of $CIR'_w(0)$.

T_3 satisfies

$$
\begin{aligned}
T_3 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)'''(\Delta z) \right] \mathrm{d}\,\Delta z \qquad \text{(F.17)} \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[(m^{w,h})''(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)'''(\Delta z) \right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left[(m^{w,h})''(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)'''(\Delta z) \right] \mathrm{d}\,\Delta z \\
&=^{(2)} \Delta z \left((m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) \right) \Big|_{-\Delta^-}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[(m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[(m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \right] \\
&=^{(3)} \Delta z \left((m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(0)(g^h)''(\Delta z) \right) \Big|_{-\Delta^-}^{\Delta^+} \\
&\cdots - \left[m^{w,h}(\Delta z)(g^h)'(\Delta) \Big|_{-\Delta^-}^{0} - 2 \int_{-\Delta^-}^{0} \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \right] \\
&\cdots - \left[m^{w,h}(\Delta z)(g^h)'(\Delta) \Big|_{0}^{\Delta^+} - 2 \int_{0}^{\Delta^+} \left[m^{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \right].
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $(m^{w,h})''(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)'''(\Delta z) = \frac{\mathrm{d}\left[(m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(\Delta z)(g^h)''(\Delta z) \right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; and (3) uses the border conditions and integrates by parts

Finally, for T_4

$$
\begin{aligned}
T_4 &= \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2} (g^h)'''(\Delta z) \right) \mathrm{d}\,\Delta z \qquad \text{(F.18)} \\
&=^{(1)} \int_{-\Delta-}^{0} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2} (g^h)'''(\Delta z) \right) \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2} (g^h)'''(\Delta z) \right) \mathrm{d}\,\Delta z \\
&=^{(2)} \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \right] \Bigg|_{-\Delta^-}^{0} + \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \right] \Bigg|_{0}^{\Delta^+} \\
&\cdots - \int_{-\Delta^-}^{\Delta^+} \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z)\, \mathrm{d}\,\Delta z \\
&=^{(3)} \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \right] \Bigg|_{-\Delta^-}^{\Delta^+} - \gamma \underbrace{\left[g^h(\Delta z) \Big|_{-\Delta^-}^{0} + g^h(\Delta z) \Big|_{0}^{\Delta^+} \right]}_{=0} \\
&\cdots - \frac{\sigma^2}{2} \left[(g^h)'(\Delta z) \Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z) \Big|_{0}^{\Delta^+} \right] \\
&= \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \right] \Bigg|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2} \left[(g^h)'(\Delta z) \Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z) \Big|_{0}^{\Delta^+} \right].
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point; (2) integrates by parts; (3) operates the integral and uses the border conditions.

Combining results (F.14), (F.15), (F.16), (F.17), (F.18), we obtain

$$
\begin{aligned}
0 &= T_1 - \gamma T_2 + \frac{\sigma^2}{2} T_3 + m^{w,h}(0) T_4 \\
0 &= -\left(\frac{\mathbb{E}_h[\Delta z^2]}{2} - \mathbb{E}_h[\Delta z]^2 \right) - \gamma \left(m^{w,h}(0)\, \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - CIR_w'(0) \right) \\
&\cdots + \frac{\sigma^2}{2} \Bigg[\Delta z \left((m^{w,h})'(\Delta z)(g^h)'(\Delta z) - m^{w,h}(0)(g^h)''(\Delta z) \right) \Big|_{-\Delta^-}^{\Delta^+}
\end{aligned}
$$

$$\cdots - \left[m^{w,h}(\Delta z)(g^h)'(\Delta)\Big|_{-\Delta^-}^{0} - 2\int_{-\Delta^-}^{0} \left[m^{w,h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\, \Delta z \right]$$

$$\cdots - \left[m^{w,h}(\Delta z)(g^h)'(\Delta)\Big|_{0}^{\Delta^+} - 2\int_{0}^{\Delta^+} \left[m^{w,h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\, \Delta z \right] \Bigg]$$

$$\cdots + m^{w,h}(0) \left(\Delta z \left[\gamma(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right] \Bigg|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2} \left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+} \right] \right),$$

which implies

$$(CIR^w)''(0) = \frac{\left(\frac{\mathbb{E}_h[\Delta z^2]}{2} - \mathbb{E}_h[\Delta z]^2\right) - \gamma(CIR^w)'(0)}{\sigma^2} - \frac{1}{2}\left(\Delta z (m^{w,h})'(\Delta z)(g^h)'(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}.$$

Online Appendix References

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