Bad Trade: The Loss of Domestic Varieties[*][†]

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Abstract

In the context of Krugman's (1979) canonical New Trade model, we show that a country is better off in autarky than in free and costless trade if its trading partner's marginal cost sufficiently exceeds its own. The initiation of trade hurts *both* countries. This also holds on a sector by sector basis, irrespective of the overall distance to autarky. We derive the lowest import tariffs that guarantee that countries gain from trade. As an aside, we prove existence of equilibrium in the (generalized) Krugman model—a result missing from the literature.

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Globalization, or worldwide McDonaldization, destroys diversity. A tidal wave of the worst Western culture is creeping across the globe like a giant strawberry milkshake oozing over the planet, with a flavor that is distinctly sweet, sickly and manifestly homogenous.

Wole Akande

1 Introduction

Awarding the Nobel Prize to Paul Krugman, the Swedish Academy observed that "trade [...] enables specialization and large-scale production, which result in lower prices and a greater diversity of commodities" (Nobel Committee, 2008). This insight relied, at least in part, on Krugman's seminal paper *Increasing Returns, Monopolistic Competition, and International Trade* that kicked off the New Trade literature (Krugman, 1979).[1] Krugman showed that, when production exhibits increasing returns and consumers have a taste for variety, Pareto improving international trade takes place even in the absence of comparative advantage. Specifically, he established that symmetric countries are better off under free and costless trade than under autarky.[2]

The topic of the current paper is the contingency of Krugman's findings as to the gains from trade. Specifically, we relax three of Krugman's simplifying assumptions and show that, in each case, his result partially reverses. (1) Relaxing symmetry, we show that a country is strictly better off in autarky than in free and costless trade, if its trading partner's marginal production cost sufficiently exceeds its own. The reason is that, as costs rise, the low-cost country's gains from trade become second-order, while the welfare losses from the associated drop in domestic varieties remain first-order. (2) Examining what happens in between the two extremes of autarky and free and costless trade, we show that in the absence of tariffs, a bit of trade is worse than no trade at all.[3] That is, when trade costs dip below their prohibitive level, the initiation of trade reduces welfare in both countries. Again, the reason is that the gains are second-order, while the losses due to a fall in domestic varieties are first-order—but now this holds for both trading partners. In a multi-sector economy, 'Bad Trade' holds on a sector by sector basis, irrespective of the overall distance to autarky. Furthermore, variety shrinks in all sectors—not only in the newly tradable one—further reducing welfare. (3) Distinguishing between tariffs (policy instruments) and trade costs (less so), we show that countries can easily

[1] For a historical overview, see Neary (2009).

[2] Here, 'free trade' refers to the absence of import quotas or revenue-generating tariffs. 'Costless trade' refers to the absence of (non-revenue-generating) iceberg trade costs.

[3] Kokovin *et al.* (2022) have independently examined this particular generalization. We compare our findings in Section 7.

be worse off in a fully liberalized (i.e., zero-tariff) economy than in autarky.[4] Rather than full liberalization, we show that judiciously chosen import tariffs prevent these welfare losses. These 'critical' tariffs prove to be particularly simple and independent of the tariff, or any other characteristic, of the trading partner. In the symmetric setting, we also derive a first-order condition for socially optimal tariffs as a function of trade costs, both at and away from the initiation of trade. Finally, as an aside, we prove equilibrium existence in the (generalized) Krugman model, a result missing from the literature.

To develop an intuition for our findings, first consider the original single-sector Krugman model, adding iceberg trade costs but no tariffs. When trade costs dip below the prohibitive level, foreign firms (varieties) enter the domestic market, and domestic firms (varieties) enter the foreign market. While this results in a first-order increase in firm size, the envelope theorem implies that the associated rise in profits is only second-order. In general equilibrium, increased supply pushes down the overall price level, leading firms to make losses. To re-establish equilibrium some firms cease operating, resulting, on balance, in fewer but larger firms.[5] Variety-loving households may now consume small quantities of many new foreign varieties, but they must do without some domestic varieties that they used to enjoy in large quantities.

Consumers enjoy surplus only on *infra-marginal* units, while the utility from a marginal unit equals its price. At the initiation of trade, there are no infra-marginal units of foreign goods. Therefore, the surplus gained from a new foreign variety is second-order, while the surplus lost from the disappearance of a 'beloved' domestic variety is first-order. Even though more foreign varieties enter than domestic varieties are pushed out, the sum of many second-order gains is strictly smaller than the sum of a few first-order losses. Hence, some trade is worse than no trade at all.

A similar intuition explains why a country is strictly better off in autarky than in free and costless trade when facing a trading partner with marginal costs sufficiently greater than its own. High marginal costs imply small-scale production. This reduces the number of infra-marginal units of high-cost-country varieties and, thus, consumer surplus. For sufficiently high costs, the surplus that low-cost-country households derive from high-cost-country varieties is so low relative to that from domestic varieties, that it would benefit from a discrete switch from free and costless trade to autarky. Even though this move pushes out more foreign varieties than that it creates new domestic ones, the many small losses are smaller in total than the few large gains. Hence, for the low-cost country, autarky trumps free and costless trade.

While the intuition for 'Bad Trade' in multi-sector economies is essentially the same as for

[4] Solely comparing autarky with free and costless trade, Krugman's (1979) model has neither trade costs nor tariffs. Krugman (1980) does have trade costs, but no tariffs. The same is true for Kokovin *et al.* (2022).

[5] With CES preferences, firm size is famously *invariant* to iceberg trade costs (see, e.g., Krugman, 1980). Recall, however, that Krugman (1979) excludes CES from consideration by assuming finite marginal utility at zero and, hence, a finite choke price. Under these conditions, firm size increases at the initiation of trade.

a single sector, there is one additional effect to consider. When sectoral trade costs dip below their prohibitive level, the initiation of trade leads to fewer but larger firms in that industry, and the surplus gained from many new foreign varieties is smaller than the surplus lost from the disappearance of some domestic varieties. However, the drop of import prices now also leads to a reallocation of resources toward the newly tradable sector—obviously, an impossibility in the single-sector model. Firm size in other sectors remains unchanged, because sectoral price levels have not moved. Fewer resources and unchanged firm size imply that, also there, the number of firms/varieties fall. This inflicts an additional first-order utility loss on households in both countries.

Finally, suppose countries levy import tariffs that are rebated lump sum to domestic households. This drives a wedge between the private and the social cost of foreign goods consumption, making the social cost smaller than the private one by an amount equal to the tariff levied. A country can avoid 'Bad Trade' by setting its tariff such that, at the initiation of trade, the social cost per util is the same for foreign as for domestic goods. This makes the displacement of domestic by foreign varieties welfare neutral. Indeed, any lower tariff yields welfare losses, while any larger tariff entails foregoing welfare gains from an 'earlier' initiation of trade—i.e., trade costs need not fall as far. As we show, the critical tariff takes a particularly simple form, which is independent of the trade policy, or any other characteristic, of the trading partner.

While Bad Trade critically depends on varieties having finite choke prices, it is robust in other dimensions.[6] In particular, it survives multiple sectors, fixed costs of exporting, and a finite type-space of heterogeneous firms.[7] Under CES, which is excluded in the baseline model, marginal utility at zero is unbounded, and choke prices are infinite. Without fixed costs of exporting, this means that all varieties are traded, irrespective of production and trade costs—hence, there is no *initiation* of trade. With fixed cost of exporting, the initiation of trade has no effect on CES utility, because the marginal social surplus from imports is exactly equal to that from domestic goods. This is a singularity, and it can be viewed as cautioning against an over-reliance on CES.

The paper proceeds as follows. Section 2.1 lays out the single-sector baseline model, which generalizes Krugman (1979) allowing for trade costs, import tariffs, and general asymmetries between countries. In Section 2.2 we prove existence of equilibrium. Section 3 studies the welfare effects of the initiation of trade. We derive the 'Bad Trade' result for the baseline model and calculate the critical tariff that protects against it. In Section 4 we show that a country is better off in autarky than in free and costless trade, when its trading partner's marginal cost is sufficiently greater than its own. Section 5 deals with socially optimal import tariffs, both at

[6] The choke price refers to the lowest price (if any) at which the demanded quantity equals zero.

[7] A finite type-space ensures that a marginal reduction of trade costs below the prohibitive level has a first-order effect on exports. With atomless types (and no fixed cost of exporting) all first-order effects are zero.

and away from the initiation of trade. Section 6 extends the country-symmetric version of the baseline model to allow for, in turn, multiple sectors, fixed costs of exporting, and heterogeneous firms. Section 7 contains a review of the related literature. Section 8 concludes. Throughout the paper, examples illustrate our findings. Formal proofs have been relegated to Appendix A.

2 Model

Our baseline model is a generalization of Krugman (1979), allowing for iceberg trade costs, import tariffs, and general asymmetries between countries. Because the model is more or less standard, we content ourselves with a sketch.[8]

2.1 Setup

There are two countries, A and B. For concreteness, we take the perspective of country A. The situation for country B is the mirror image. Country A has a fixed mass $L_A > 0$ of households and a variable mass $n_A > 0$ of active firms. The mass of potentially active firms is unbounded, and market entry occurs until the marginal firm just breaks even. Rivalry between firms is monopolistically competitive. Each household inelastically supplies one unit of labor and, using labor as the only scarce input, each active firm $i_A \in [0, n_A]$ produces a differentiated good, also denoted by i_A. We refer to these differentiated goods as varieties. While labor is domestically supplied, firms can sell theirs goods both domestically and abroad. When exporting from A to B, a firm incurs an iceberg trade cost $t_B \geq 1$ and pays an *ad valorem* import tariff $0 \leq r_B \leq 1$. Iceberg costs mean that t_B units of the good must be shipped from A for one unit to arrive in B. The remainder 'melts' in transit and is wasted. Tariffs mean that an exporting firm only receives a fraction $1 - r_B$ of the sale price of a unit sold abroad. The remainder goes to country B's customs authority and is remitted as a lump sum to households in B.

For future reference, we define $\varphi_B \equiv t_B/(1 - r_B) \geq 1$ to be the 'friction' in exporting to B. We call $\Phi \equiv \varphi_A \varphi_B$ the trade friction index, which proves the essential measure for whether trade takes place.

Let $z_{i_A}, z_{i_B} \geq 0$ denote the quantities consumed of varieties i_A and i_B, while p_{i_A} and s_{i_B} denote their prices. Household income is I_A. The utility maximization problem of a household in A is given by

$$\begin{array}{ll} \max_{z_{i_A}, z_{i_B}} & U_A = \int_{i_A=0}^{n_A} v_A [z_{i_A}] \, di_A + \int_{i_B=0}^{n_B} v_A [z_{i_B}] \, di_B \\ \text{st.} & \int_{i_A=0}^{n_A} p_{i_A} z_{i_A} di_A + \int_{i_B=0}^{n_B} s_{i_B} z_{i_B} di_B = I_A \ , \end{array} \qquad (1)$$

The sub-utility function $v_A [\cdot]$ is twice differentiable with $v_A [0] = 0$. Furthermore, $0 < v'_A [\cdot] < \infty$

[8] A more detailed exposition can be found at https://tinyurl.com/7hwty48z.

and $-\infty < v''_A[\cdot] < 0$. Finite marginal utility at zero implies that each variety, i_k, $k \in \{A, B\}$, has a 'choke price,' i.e., a finite price above which households stop consuming that variety.[9] Households have a taste for variety. To see this, notice that, jointly, $v_A[0] = 0$ and concavity of v_A imply that $nv_A[z/n]$ is strictly increasing in n for all $n, z > 0$.

Let $\varepsilon_{v'_A}$ denote *minus* the elasticity of marginal utility v'_A with respect to z, i.e., $\varepsilon_{v'_A}[z] \equiv -zv''_A[z]/v'_A[z]$.[10] Observe that $\varepsilon_{v'_A}[0] = 0$, whereas for $z > 0$, $\varepsilon_{v'_A}$ is strictly positive. Hence, at $z = 0$, $\varepsilon_{v'_A}$ must be locally strictly increasing. Slightly relaxing Krugman (1979), who assumes that $\varepsilon_{v'_A}$ is strictly increasing everywhere, we assume that $\varepsilon_{v'_A}$ is non-decreasing for $z > 0$.

The utility maximization problem in (1) yields the following first-order conditions (FOC):

$$v'_A[z_{i_A}] = \lambda_A p_{i_A}, \quad v'_B[z_{i_B}] = \lambda_A s_{i_B}, \quad (i_A, i_B) \in [0, n_A] \times [0, n_B] \ . \tag{2}$$

Here, $\lambda_A \in (0, \infty)$ denotes the Lagrangian multiplier on the budget constraint, i.e., the shadow price of income I_A. Since the Lagrangian is equal to the marginal utility of income, the marginal price index (that is, the cost of an additional util) is $1/\lambda_A$. We denote it by P_A, $0 < P_A < \infty$. Notice that demand for each good is only a function of its own price p_{i_A} (or s_{i_B}) and the marginal price index P_A. In other words, P_A is a 'sufficient statistic' that encodes not only for the effect on demand of the prices of all other goods—domestic as well as imported—but also of income, I_A.

Other than producing different varieties, firms are identical within each country. (For heterogeneous firms, see Section 6.3.) Let $y_{i_A} \geq 0$ denote the quantity of variety i_A that firm i_A sells in the domestic market (i.e., in A), and let $x_{i_A} \geq 0$ denote the quantity of i_A that it sells abroad (i.e., in B). We say that a firm is active if $y_{i_A} + x_{i_A} > 0$.

Expressed in domestic labor units, firm i_A's cost function is

$$C_A[y_{i_A} + t_B x_{i_A}] = F_A + c_A(y_{i_A} + t_B x_{i_A}) \ .$$

Here, $F_A > 0$ denotes the fixed cost of operating, which is sunk, while $c_A > 0$ denotes the constant marginal cost of production. To transform the labor cost $C_A[y_{i_A} + t_B x_{i_A}]$ into monetary units, it must be multiplied by the domestic wage rate, $w_A > 0$.

The inverse-domestic-demand curve, $p_{i_A}[y_{i_A}, P_A]$, is found by aggregating the FOCs $v'_A[z_{i_A}] = \lambda_A p_{i_A}$ in (2) over all households in A and using that demand must equal supply y_{i_A}. Similarly, the inverse-foreign-demand curve $s_{i_A}[x_{i_A}, P_B]$ is found by aggregating the FOCs $v'_B[z_{i_A}] =$

[9] Notice, however, that a variety's choke price is not a constant; it is increasing in the prices of other varieties.

[10] Notice that $\varepsilon_{v'_A}$ is, in fact, equal to the conventional measure of relative risk aversion.

$\lambda_B s_{i_A}$ of households in country B. This yields

$$p_{i_A}[y_{i_A}, P_A] = P_A v'_A\left[\frac{y_{i_A}}{L_A}\right] \text{ and } s_{i_A}[x_{i_A}, P_B] = P_B v'_B\left[\frac{x_{i_A}}{L_B}\right] \ . \tag{3}$$

Firm i_A's profit, π_{i_A}, is

$$\pi_{i_A} = p_{i_A} y_{i_A} + (1 - r_B)\, s_{i_A} x_{i_A} - w_A C_A [y_{i_A} + t_B x_{i_A}] \ .$$

Since firms are atomistic, individually, they do not influence wages w_A or price levels P_A, P_B. Then the FOCs for optimal interior y_{i_A} and x_{i_A} are

$$\begin{aligned} FOC_{y_{i_A}}: &\quad p_{i_A} + \tfrac{\partial p_{i_A}}{\partial y_{i_A}} y_{i_A} - w_A c_A = 0 \\ FOC_{x_{i_A}}: &\quad s_{i_A} + (1 - r_B)\tfrac{\partial s_{i_A}}{\partial x_{i_A}} x_{i_A} - t_B w_A c_A = 0 \ . \end{aligned} \tag{4}$$

Using (3) we can rewrite (4) as

$$\begin{aligned} FOC_{y_{i_A}}: &\quad v'_A\left[\tfrac{y_{i_A}}{L_A}\right]\left(1 - \varepsilon_{v'_A}\left[\tfrac{y_{i_A}}{L_A}\right]\right) = \tfrac{c_A}{P_A/w_A} \\ FOC_{x_{i_A}}: &\quad v'_B\left[\tfrac{x_{i_A}}{L_B}\right]\left(1 - \varepsilon_{v'_B}\left[\tfrac{x_{i_A}}{L_B}\right]\right) = \varphi_B \tfrac{c_A}{P_B/w_A} \ . \end{aligned} \tag{5}$$

The LHS of $FOC_{y_{i_A}}$ in (5) represents marginal revenue of home-bound production y_{i_A}, normalized by the price level P_A. The RHS represents normalized marginal cost. Hence, $FOC_{y_{i_A}}$ simply states that, at an interior optimum, marginal revenue equals marginal cost. The interpretation of $FOC_{x_{i_A}}$ is the same, except that here we have also divided by $1 - r_B$, giving rise on the RHS to trade friction φ_B.

For future reference, we denote normalized marginal revenue by $m_k[z] \equiv v'_k[z]\left(1 - \varepsilon_{v'_k}[z]\right)$, $z \geq 0$, $k \in \{A, B\}$. Notice that $m_k[\cdot]$ takes the same functional form for domestic production y as for export x.

Monotonicity in y_{i_A} and x_{i_A} implies that each of the FOCs in (5) has at most one solution. Assuming strictly decreasing marginal revenues ensures that, at a solution, the SOCs for a maximum are satisfied.[11] We denote firm i_A's optimal quantities by $\hat{y}_{i_A}[P_A/w_A]$ and $\hat{x}_{i_A}[P_B/w_A]$. Depending on wage-normalized price levels in A and B, the firm either enters and produces the unique interior maximizer for that market, or it stays out and produces zero. Since optimal quantities are uniquely determined, all active firms in country A behave identically, such that we only need to keep track of the number of active firms, n_A, and not of their identities, $i_A \in [0, n_A]$. With slight abuse of notation, we write y_A, p_A, x_A, s_A for y_{i_A}, p_{i_A}, x_{i_A}, s_{i_A}.

Beyond paying fixed cost $F_A > 0$, there are no barriers to becoming active, nor to ceasing

[11] In terms of primitives, decreasing marginal revenues require that $\varepsilon_{v''_A} < 2$. Here, $\varepsilon_{v''_A}$ denotes *minus* the elasticity of v''_A with respect to z.

activity. Hence, in equilibrium, the number of active firms, n_A, is such that the marginal firm makes zero profit. Because firms are symmetric within a country, this means that *all* firms make zero profits:

$$\pi_A = p_A y_A + (1 - r_B) s_A x_A - w_A C_A [y_A + t_B x_A] = 0 \ . \tag{6}$$

Notice that n_A and n_B do not directly enter into the zero-profit condition (6)—that is, the number of firms only affects π_A indirectly, via price levels and wages.

While households are the ultimate owners of firms, firms make no profits in equilibrium. Hence, household income only consists of wages and lump sum tariff revenues, i.e., $I_A = w_A + n_B r_A s_B x_B / L_A$. Substituting this expression back into the budget constraint and using market clearing yields the budget balance equation

$$n_A p_A y_A + n_B (1 - r_A) s_B x_B = w_A L_A \ . \tag{7}$$

Labor market clearing requires that

$$n_A C_A [y_A + t_B x_A] = L_A \ . \tag{8}$$

Finally, to close the model, we impose balance of payments, i.e.

$$n_A (1 - r_B) s_A x_A = n_B (1 - r_A) s_B x_B \ . \tag{9}$$

This means that, net of tariffs, the two countries spend the same amount on imports.

2.2 Equilibrium

Equilibrium consists of a tuple $(P_k, w_k, n_k)_{k \in \{A,B\}}$ of price indices P_k, wages w_k, and numbers of active firms n_k, inducing optimal quantities $\hat{y}_k [P_k, w_k]$, $\hat{x}_k [P_l, w_k]$ and prices $p_k [\hat{y}_k, P_k]$, $s_k [\hat{x}_k, P_l]$, $l \neq k$, such that zero profits (ZP, (6)), budget balance (BB, (7)), labor market clearing (LM, (8)), and balance of payments (BP, (9)) hold. However, in line with Walras' Law, one of these (pairs of) equations is redundant. To see this, substitute LM and BP into ZP to find BB. Equilibrium is thus characterized by the following system:

For $k, l \in \{A, B\}$, $l \neq k$,

$$\begin{aligned} ZP^k: &\quad p_k \hat{y}_k + (1 - r_l) s_k \hat{x}_k - w_k C_k [\hat{y}_k + t_l \hat{x}_k] = 0 \\ LM^k: &\quad n_k C_k [\hat{y}_k + t_l \hat{x}_k] = L_k \\ BP: &\quad n_k (1 - r_l) s_k \hat{x}_k = n_l (1 - r_k) s_l \hat{x}_l \ . \end{aligned} \tag{10}$$

This system contains five equations—BP and two each of ZP and LM—and six unknowns—

P_k, w_k, n_k, $k \in \{A, B\}$. To solve the system, we express all solutions in terms of the wage ratio $w_B/w_A \in (0, \infty)$.

We denote quantities on the threshold between autarky and trade by a tittle '$\dot{}$'. (The dot is a mnemonic for the *point* where trade is initiated.) The threshold index $\dot{\Phi}$ is defined as

$$\dot{\Phi} \equiv \frac{v_A' [0]}{m_A [\dot{y}_A/L_A]} \frac{v_B' [0]}{m_B [\dot{y}_B/L_B]} > 1 \ . \tag{11}$$

Recall that $m_k [z] \equiv v_k' [z] \left(1 - \varepsilon_{v_k'} [z]\right)$ denotes normalized marginal revenue, while $\dot{y}_k$ denotes the threshold value of y_k, which is also its autarky value. The inequality $\dot{\Phi} > 1$ now follows from $\dot{y}_k/L_k > 0$ and strict decreasingness of $m_k [\cdot]$. The autarky value $\dot{y}_k > 0$ is in fact the unique solution to

$$\frac{\varepsilon_{v_k'} [\dot{y}_k/L_k]}{1 - \varepsilon_{v_k'} [\dot{y}_k/L_k]} \dot{y}_k = \frac{F_k}{c_k} \ . \tag{12}$$

(See Lemma A.4 in the Appendix for a derivation.) At the initiation of trade, $x = 0$ and $y = \dot{y}$. Since $m_k [0] = v_k' [0]$, index $\dot{\Phi}$ in (11) is the product of the marginal revenue ratios of exporting at the initiation of trade versus producing for the home market. Hence, the larger is $\dot{\Phi}$, the greater is the incentive to export.

In the next proposition we prove existence of equilibrium, a result that is missing from the literature. We show that whether trade occurs solely depends on the value of trade-friction index $\Phi \equiv \varphi_A \varphi_B$ relative to the threshold $\dot{\Phi}$. This is surprising, since the four sources of trade frictions—tariffs r_k and trade costs t_k, $k \in \{A, B\}$—enter the equilibrium system irreducibly, i.e., not only in terms of Φ.

Proposition 1 *Equilibrium exists. Countries trade iff* $\Phi < \dot{\Phi}$.

Threshold index $\dot{\Phi}$ *is strictly decreasing in* c_k, L_k, *and strictly increasing in* F_k, $k \in \{A, B\}$.

Uniqueness of equilibrium remains an open question; we have neither a proof nor a counter example.

When trade-friction index Φ is equal to its threshold value $\dot{\Phi}$, the unique solution to the first-order conditions FOC_{x_A}, FOC_{x_B} is $(x_A, x_B) = (0, 0)$ (see (5)). Here, trade is *just* precluded. Let $\tau \equiv ((t_A, r_A), (t_B, r_B))$ denote a tuple of trade costs and tariffs, and let $\dot{\tau} \equiv \left(\left(\dot{t}_A, \dot{r}_A\right), \left(\dot{t}_B, \dot{r}_B\right)\right)$ denote a tuple τ such that $\Phi = \dot{\Phi}$. The set of all threshold tuples $\dot{\tau}$ is $\dot{T}$, i.e.,

$$\dot{T} \equiv \left\{\tau \in ([1, \infty) \times [0, 1))^2 : \Phi = \dot{\Phi}\right\} \ .$$

Finally, the upper and lower contour sets of $\dot{T}$ are denoted by $\bar{T}$ and $\underline{T}$, respectively. In region $\bar{T}$, trade frictions are too high for trade to occur. In region $\underline{T}$, frictions are sufficiently low that trade does take place. Locus $\dot{T}$ constitutes the boundary between the two. Since $\dot{\Phi} > 1$, $\bar{T}$, $\underline{T}$, and $\dot{T}$ are all non-empty. The upper right panel of Figure 2 depicts these regions for symmetric tariffs and trade costs $r_A = r_B \equiv r$ and $t_A = t_B \equiv t$. In that case, $\dot{T}$ describes a straight line in (t, r)-space, given by $\dot{t}[r] = (1 - r)\sqrt{\dot{\Phi}}$ or, equivalently, by $\dot{r}[t] = 1 - \left(1/\sqrt{\dot{\Phi}}\right) t$.

Threshold $\dot{\Phi}$ is strictly decreasing in L_k (see Lemma A.4 in the Appendix). Intuitively, the larger is the domestic market, the smaller are the additional scale advantages afforded by exporting. In an informal sense, this makes larger countries less likely to trade with each other. Scale advantages are also smaller when production is less fixed-cost intensive. Hence, as F_k/c_k decreases, $\dot{\Phi}$ falls and trade becomes less likely. In today's digital economy, the opposite happens: With marginal costs close to zero, production is highly fixed-cost intensive. All else equal, this encourages trade.

3 Welfare Effects of the Initiation of Trade

In this section, we show that the initiation of trade in the absence of tariffs leaves both countries strictly worse off. We also determine what import tariff a country should charge to ensure that it gains from trade. This 'critical' tariff turns out to depend solely on the country's domestic economic structure. As such, it is independent of the tariff, or any other characteristic, of the trading partner.

From Proposition 2 we know that whether trade occurs solely depends on whether the trade-friction index Φ is smaller than the threshold value $\dot{\Phi}$. Thus, trade can be initiated either by a fall in frictions or by a rise in the threshold. Our main focus is on the former. In Section 3.2 we briefly consider the latter.

3.1 A Fall in Trade Frictions

To study the welfare effects of the initiation of trade, we fix all model parameters other than the trade frictions $\tau = ((t_A, r_A), (t_B, r_B))$. Then we calculate the directional derivative of utility with respect to (the elements of) τ, departing from a point on the trade threshold $\dot{T}$ and moving into the trade region $\underline{T}$.

Even though the occurrence of trade only depends on whether $\Phi \lesseqgtr \dot{\Phi}$, we cannot simply differentiate utility with respect to Φ. The reason is that tariffs r_A, r_B and iceberg costs t_A, t_B enter the equilibrium system not only in terms of the index Φ, but also separately. Put differently, while Φ is a sufficient statistic for *whether* trade occurs, it is not sufficient for knowing *how* trade unfolds. When differentiating utility, we therefore need to keep track of our starting point on $\dot{T}$, as well as of the particulars of the changes in iceberg costs and tariffs that lead us into $\underline{T}$.

Fix a point $\dot{\tau}$ in $\dot{T}$. At this point, the trade friction index satisfies $\Phi = \dot{\Phi}$. Let the direction of change starting from $\dot{\tau}$ be given by $\Delta\tau \equiv ((\Delta t_A, \Delta r_A), (\Delta t_B, \Delta r_B)) \in \mathbb{R}^4$. Scaling by $\sigma \in [-1, 1]$, the new value of τ is then $\tau = \dot{\tau} + \sigma\Delta\tau$. Direction $\Delta\tau$ is *permissible* iff it is such that when σ is small and negative, we are in the trade region $\underline{T}$, and when σ is small and positive, we are in the no-trade region $\bar{T}$. This means that $\Phi = \frac{t_A}{1-r_A}\frac{t_B}{1-r_B} \overset{(<)}{\underset{(>)}{=}} \dot{\Phi}$ iff $\sigma \overset{(<)}{\underset{(>)}{=}} 0$. It is now easy to show that

Lemma 1 *At $\dot{\tau} \in \dot{T}$, the set $D[\dot{\tau}]$ of 'permissible' directions is given by*

$$D[\dot{\tau}] = \left\{ \Delta\tau \in \mathbb{R}^4 \middle| \frac{1}{t_A}\Delta t_A + \frac{1}{1-r_A}\Delta r_A + \frac{1}{t_B}\Delta t_B + \frac{1}{1-r_B}\Delta r_B > 0 \right\} .$$

Observe that, for country-symmetric trade costs and tariffs (t, r), the set of permissible directions simplifies to $D[\dot{\tau}] = \left\{ \Delta\tau \in \mathbb{R}^2 \middle| \Delta t / \Delta r > -\dot{\Phi} \right\}$, which is independent of starting point $\dot{\tau}$. This is illustrated in the upper right panel of Figure 2.

We now present our first 'Bad Trade' result. We show that, unless country k's tariff is above some critical value r_k^*, $0 < r_k^* < 1$, the initiation of trade leaves country $k \in \{A, B\}$ strictly worse off. An important implication is that, in a tariff-free world, the initiation of trade necessarily produces welfare losses for *both* trading partners. The result holds regardless of the starting point $\dot{\tau}$ or the direction of change $\Delta\tau \in D[\dot{\tau}]$.

To prove our claim, we fix $(\dot{\tau}, \Delta\tau) \in \dot{T} \times D[\dot{\tau}]$ and calculate the *left*-directional derivative $\overleftarrow{\nabla}_{\Delta\tau} U_k[\dot{\tau}]$ of household utility U_k at $\dot{\tau}$ in the direction of $\Delta\tau$. By construction, it is equal to the left-derivative of utility with respect to scalar σ, evaluated at $\sigma = 0$.[12]

Proposition 2 (Bad Trade) *The initiation of trade due to a fall in trade frictions leaves country $k \in \{A, B\}$ strictly worse (better) off, if and only if its import tariff r_k is strictly smaller (greater) than the critical value $r_k^* \equiv 1 - \varepsilon_{v_k}[\dot{y}_k / L_k] > 0$.*

Formally, for all $\dot{\tau} \in \dot{T}$ and $\Delta\tau \in D[\dot{\tau}]$,

$$\overleftarrow{\nabla}_{\Delta\tau} U_k \Big|_{\tau=\dot{\tau}} \overset{(\geq)}{\underset{(<)}{=}} 0 \iff r_k \overset{(\leq)}{\underset{(>)}{=}} r_k^* \; . \tag{13}$$

Notice that the critical tariff $r_k^* = 1 - \varepsilon_{v_k}[\dot{y}_k / L_k]$ has a remarkably simple form: it depends only on the elasticity of domestic autarky consumption and, thus, it is *independent* of the characteristics and tariffs of the trading partner.

In the remainder of this section, we provide a calculus-based, a verbal, and a graphical intuition for Proposition 2. In calculus terms, the forces driving the proposition can be seen most easily if we restrict attention to symmetric countries and trade that is initiated either by a drop in tariffs or by a drop in trade costs. Here, we sketch the argument for the case of a change in trade costs. (The argument for a change in tariffs is analogous.)

Under symmetry, $t_A = t_B = t$ and $r_A = r_B = r$. Focusing on changes in trade costs, we can dispense with directional derivatives. That is, to find the welfare effect of the initiation of trade,

[12] Notice that the *right*-derivative is necessarily zero, because $\sigma > 0$ pushes the countries deeper into autarky.

we simply calculate the (left-)derivative of utility with respect to t and evaluate at $t = \dot{t}[r]$.

Differentiating U yields

$$\frac{dU}{dt} = n\left(v'\left[\frac{\hat{y}}{L}\right]\frac{d\hat{y}/L}{dt} + v'\left[\frac{\hat{x}}{L}\right]\frac{d\hat{x}/L}{dt}\right) + \frac{dn}{dt}\left(v\left[\frac{\hat{y}}{L}\right] + v\left[\frac{\hat{x}}{L}\right]\right) \ . \tag{14}$$

Eq. (14) says that the change in utility solely depends on the (change in) per-household consumption of each domestic and foreign variety, and the (change in the) number of varieties. At the initiation of trade, $\hat{x} = 0$ and $\hat{y} = \dot{y}$. Thus, it remains to use the equilibrium conditions and the FOCs to find $d\hat{y}/dt$, $d\hat{x}/dt$, and dn/dt, evaluate at $t = \dot{t}[r]$, and substitute the results back into (14). Here and in the remainder of this section, derivatives with respect to t refer to left-derivatives. All right-derivatives are zero.

Under symmetry, the balance of payments condition, BP, is automatically satisfied. Normalizing the wages in both countries to 1, two equations suffice to characterize a symmetric equilibrium (P, n), namely,

$$\begin{aligned} ZP: &\quad \pi[\hat{y}, \hat{x}, P, t] = 0 \\ LM: &\quad nC[\hat{y} + t\hat{x}] = L \ . \end{aligned} \tag{15}$$

Implicitly differentiating ZP with respect to t yields

$$\frac{d\pi[\hat{y}, \hat{x}, P, t]}{dt} = \frac{\partial\pi}{\partial\hat{y}}\frac{d\hat{y}}{dt} + \frac{\partial\pi}{\partial\hat{x}}\frac{d\hat{x}}{dt} + \frac{\partial\pi}{\partial P}\frac{dP}{dt} + \frac{\partial\pi}{\partial t} = 0 \ . \tag{16}$$

At the initiation of trade, the FOCs hold with equality; i.e., $\partial\pi/\partial\hat{y}|_{\tau=\dot{\tau}} = \partial\pi/\partial\hat{x}|_{\tau=\dot{\tau}} = 0$. Furthermore, exports are zero. So, a drop in t has no direct, first-order impact on profits: $\partial\pi/\partial t|_{\tau=\dot{\tau}} = 0$. Jointly, these observations imply that (16) simplifies to

$$\left.\frac{d\pi[\hat{y}, \hat{x}, P, t]}{dt}\right|_{\tau=\dot{\tau}} = \left.\frac{\partial\pi}{\partial P}\frac{dP}{dt}\right|_{\tau=\dot{\tau}} = 0 \ . \tag{17}$$

For (17) to hold, either $\partial\pi/\partial P|_{\tau=\dot{\tau}} = 0$ or $dP/dt|_{\tau=\dot{\tau}} = 0$. Since $\partial p[y, P]/\partial P > 0$ and $\partial s[x, P]/\partial P \geq 0$, we have $\partial\pi/\partial P > 0$. Thus, we may conclude that $dP/dt|_{\tau=\dot{\tau}} = 0$; i.e., at the initiation of trade, the price level remains constant.

Because the price level remains constant, so too does home-bound production y of each (surviving) firm. To see this, consider the optimality condition for y:

$$FOC_y: \quad p[y, P] + \tfrac{\partial p[y,P]}{\partial y}y = c \ .$$

It immediately follows that

$$\left.\frac{d\hat{y}}{dt}\right|_{\tau=\dot{\tau}} = 0 \ . \tag{18}$$

These results are intuitive. Since firms were optimizing to begin with, the envelope theorem implies that the indirect effect of a drop in trade costs on profits is second order. At the initiation of trade, the direct effect is zero as well, because there is no existing stock of exports. Finally, as the price level remains constant, each firm's home-bound production remains unchanged.

Having dealt with ZP, we now implicitly differentiate LM in (15). This yields

$$nc\left(\frac{d\hat{y}}{dt}+t\frac{d\hat{x}}{dt}+\hat{x}\right)+\frac{dn}{dt}C\left[\hat{y}+t\hat{x}\right]=0\ . \tag{19}$$

Since $d\hat{y}/dt|_{\tau=\dot{\tau}}=\hat{x}|_{\tau=\dot{\tau}}=0$ and $\hat{y}|_{\tau=\dot{\tau}}=\dot{y}$, (19) implies that

$$\left.\frac{dn}{dt}\right|_{\tau=\dot{\tau}}=-\frac{nc\dot{t}\left[r\right]}{C\left[\dot{y}\right]}\left.\frac{d\hat{x}}{dt}\right|_{\tau=\dot{\tau}}\ . \tag{20}$$

At the initiation of trade, the optimality condition for x reduces to

$$FOC_x:\quad s=\tfrac{\dot{t}[r]}{1-r}c\ .$$

Using FOC_x to substitute for $c\dot{t}\left[r\right]$ and noting that $C\left[\dot{y}\right]=p\dot{y}$ (from ZP), we may rewrite (20) as

$$\left.\frac{dn}{dt}\right|_{\tau=\dot{\tau}}=-\frac{n\left(1-r\right)s}{p\dot{y}}\left.\frac{d\hat{x}}{dt}\right|_{\tau=\dot{\tau}}\ . \tag{21}$$

Differentiating FOC_x with respect to t, it can be easily verified—and it is intuitive—that $d\hat{x}/dt|_{\tau=\dot{\tau}}<0$; i.e., exports rise as trade costs fall. Thus, equation (21) says that a drop in trade costs reduces the number n of firms. Intuitively: as firms' exports rise and their home-bound production remains unchanged, firm size increases. The overall resource constraint then implies that there must be fewer firms in equilibrium.

Substituting (21), $\hat{y}=\dot{y}|_{\tau=\dot{\tau}}$, and $\hat{x}=0=d\hat{y}/dt|_{\tau=\dot{\tau}}$ back into the the derivative of the utility function in (14), we find that

$$\left.\frac{dU}{dt}\right|_{\tau=\dot{\tau}}=\left(v\left[\frac{\dot{y}}{L}\right]-\frac{\dot{y}}{L}\frac{p}{\left(1-r\right)s}v'\left[0\right]\right)\left.\frac{dn}{dt}\right|_{\tau=\dot{\tau}}\ .$$

Finally, substituting inverse-demand functions $p=Pv'\left[y/L\right]$ and $s=Pv'\left[x/L\right]$ evaluated at $y=\dot{y}$ and $x=0$, respectively, yields

$$\left.\frac{dU}{dt}\right|_{\tau=\dot{\tau}}=\left(v\left[\frac{\dot{y}}{L}\right]-\frac{1}{1-r}\frac{\dot{y}}{L}v'\left[\frac{\dot{y}}{L}\right]\right)\left.\frac{dn}{dt}\right|_{\tau=\dot{\tau}}\underset{(<)}{\overset{(>)}{=}}0\iff r\underset{(>)}{\overset{(<)}{=}}1-\varepsilon_v\left[\frac{\dot{y}}{L}\right]\ ,$$

where the (in)equalities follow from: 1) strict concavity of v together with $v\left[0\right]=0$; and 2) $dn/dt|_{\tau=\dot{\tau}}>0$.

This proves Proposition 2 for the special case of trade between symmetric countries initiated by a drop in iceberg costs. ■

The upshot of the foregoing calculations is that, while surviving firms scale up to satisfy export demand, some domestic firms are displaced by the entry of many foreign varieties/firms.

Whether this is beneficial in welfare terms turns on whether tariff r is greater or smaller than r^*.

We now complement the calculus-based argument above with verbal and graphical intuitions for Proposition 2. These intuitions clarify that, in a tariff-free world, the disappearance of a few domestic varieties constitutes a first-order welfare loss, while the appearance of many foreign varieties constitutes a second-order gain. Import tariffs ensure that the *social* gains from the latter become first-order. At the critical tariff r^*, marginal social gains at the initiation of trade are equal to the losses.

When trade costs dip below the prohibitive level, foreign firms enter the domestic market and domestic firms enter the foreign market. While exporting results in a first-order increase in firm size, the envelope theorem implies that the increase in profits is only second-order. In general equilibrium, increased supply pushes down the overall price level, leading firms to make losses. To re-establish equilibrium some firms fold, resulting, on balance, in fewer but larger firms.[13] Variety-loving households now get to consume small quantities of many new foreign varieties, but they must do without some domestic varieties that they used to enjoy in relatively large quantities.

Strictly positive consumer surplus is enjoyed only on *infra-marginal* units, while the utility from a marginal unit is equal to its price. At the initiation of trade, there are no infra-marginal units of foreign goods. Therefore, the (private) surplus gained from a new foreign variety is only second order, while the surplus lost from the disappearance of a domestic variety is first order. Even though new foreign varieties outnumber displaced domestic varieties, the sum of second-order gains is dwarfed by the sum of first-order losses. This explains why the initiation of trade without tariffs necessarily produces welfare losses for both trading partners.[14]

Now suppose countries levy import tariffs. Such tariffs drive a wedge between the private and the social cost of imports, keeping benefits unchanged. The private cost of imports is equal to their retail price, s_l. By contrast, the social cost is strictly less, because a fraction r_k of price s_l consists of tariffs, which are rebated lump sum to domestic households. Hence, even though infra-marginal units are still zero, the marginal *social* surplus gained from foreign varieties at the initiation of trade is now $r_k s_l > 0$.

Tariff $r_k^* = 1 - \varepsilon_{v_k} [\dot{y}_k / L]$ is such that, at the initiation of trade, the displacement of domestic

[13] Increased firm size depends on increasing $\varepsilon_{v'}$. (See Krugman, 1980, footnote 3, keeping in mind that the elasticity of demand is equal to $1/\varepsilon_{v'}$.) At the initiation of trade, $\varepsilon_{v'}$ must be increasing, because $\varepsilon_{v'} = 0$ at a choke price. By contrast, under CES, $\varepsilon_{v'}$ is constant everywhere. In that case, choke prices are infinite, trade takes place for all levels of iceberg costs, and firm size is constant (ibid.). To study the initiation trade with CES, one needs to introduce a fixed cost of exporting. We consider this case in Appendix 6.3.

[14] Dixit and Stiglitz (1977) have shown that, when $\varepsilon_{v'_k}$ is increasing, monopolistic competition leads to excess variety relative to the constrained optimum without lump sum subsidies. Since the number of domestic varieties falls at the initiation of trade, bringing us 'closer' to the constrained optimum, one might expect welfare to rise. The fact that this is not so is reminiscent of the Theory of the Second-Best (Lipsey and Lancaster, 1956): in the presence of multiple distortions, reducing one of them may make things worse. In this case, the loss of domestic varieties is harmful, because foreign goods crowd out domestic varieties with higher consumer surplus.

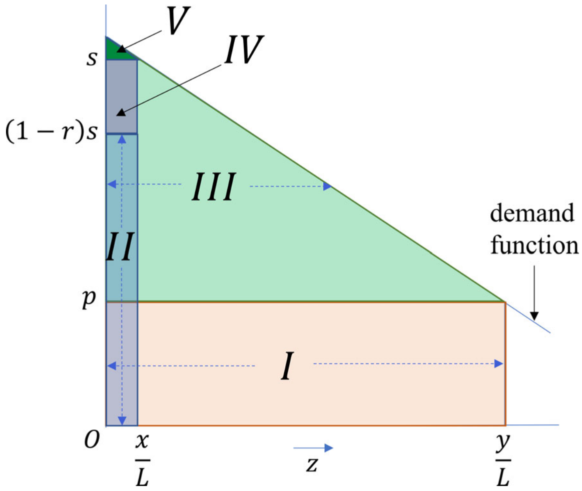

Figure 1: First-order losses in consumer surplus dominate second-order gains.

by foreign varieties is welfare neutral. To see this, consider Figure 1. Here, area I represents household expenditure $p\dot{y}_k/L$ on each domestic variety, while $I + III$, the area under the demand curve, is equal to the utility derived therefrom. The social as well as private 'cost per util' (CPU) of domestic goods consumption is therefore equal to the ratio $I/(I + III)$. For foreign goods, the social CPU is $II/(II + IV + V)$, which is strictly smaller than the private CPU $(II + IV)/(II + IV + V)$. The welfare effect of the displacement of domestic by foreign varieties depends on whether it makes the social CPU of households in country k go up or down—that is, on whether

$$\frac{II}{II + IV + V} \underset{(>)}{\overset{(<)}{=}} \frac{I}{I + III} . \tag{22}$$

The inverse-demand curve (i.e., price) is given by $Pv_k'[z]$. Thus, suppressing subscripts, we have $II = (1 - r)\, Pv' \left[\frac{x}{L}\right] \frac{x}{L}$ and

$$II + IV + V = \int_0^{\frac{x}{L}} Pv'[z]\, dz = Pv \left[\frac{x}{L}\right] .$$

The expressions for I and $I + III$ are analogous. The condition in (22) is then equivalent to

$$\frac{(1 - r)\, v' \left[\frac{x}{L}\right] \frac{x}{L}}{v \left[\frac{x}{L}\right]} \underset{(>)}{\overset{(<)}{=}} \frac{v' \left[\frac{\dot{y}}{L}\right] \frac{\dot{y}}{L}}{v \left[\frac{\dot{y}}{L}\right]} \iff (1 - r)\, \varepsilon_v \left[\frac{x}{L}\right] \underset{(>)}{\overset{(<)}{=}} \varepsilon_v \left[\frac{\dot{y}}{L}\right] . \tag{23}$$

At the initiation of trade, exports are infinitesimal. Taking the limit for $x \to 0$ and solving for r, we find that welfare turns on

$$r \underset{(<)}{\overset{(>)}{=}} \lim_{x \to 0} 1 - \frac{\varepsilon_v \left[\frac{\dot{y}}{L}\right]}{\varepsilon_v \left[\frac{x}{L}\right]} = 1 - \varepsilon_v \left[\frac{\dot{y}}{L}\right] = r^* ,$$

where have used that $\lim_{z\to 0} \varepsilon_v[z] = 1$ (see Lemma A.2 in the Appendix). Notice that the argument relies solely on the demand curve and tariffs of the importing country and does not depend on the characteristics and tariffs of its trading partner.

3.2 A Rise in $\dot{\Phi}$

Other than by a fall in trade frictions Φ, trade can be initiated by a rise in the value of the threshold, $\dot{\Phi}$. Recall from Proposition 1 that $\dot{\Phi}$ is strictly decreasing in c_k and L_k, and strictly increasing in F_k. Thus, starting from $\Phi = \dot{\Phi}$, trade can be initiated by a drop in marginal cost c_k, a rise in fixed cost F_k, or a shrinking of the population L_k, $k \in \{A, B\}$. Suppose that it is country B's parameters that change in this way, while country A's remain unchanged. Our next proposition shows that the Bad Trade result of Proposition 2 carries over unchanged for country A. By contrast, the (overall) effect on country B's welfare is ambiguous. Here, $\overleftarrow{d}$ and $\overrightarrow{d}$ refer to left and right-hand derivatives, respectively.

Proposition 3 *Trade initiated by 1) a drop in trading partner B's marginal cost c_B; 2) a rise in fixed costs F_B; or 3) a fall in population L_B leaves country A strictly worse (better) off if and only if $r_A \overset{(>)}{<} r_A^*$.*

Formally, for all $\dot{\tau} \in \dot{T}$,

$$\left.\begin{array}{l} \left.\frac{dU_A}{\overleftarrow{d}\, c_B}\right|_{\tau=\dot{\tau}} \overset{(>)}{\underset{(<)}{=}} 0 \iff \\ \left.\frac{dU_A}{\overrightarrow{d}\, F_B}\right|_{\tau=\dot{\tau}} \overset{(<)}{\underset{(>)}{=}} 0 \iff \\ \left.\frac{dU_A}{\overleftarrow{d}\, L_B}\right|_{\tau=\dot{\tau}} \overset{(>)}{\underset{(<)}{=}} 0 \iff \end{array}\right\} r_A \overset{(<)}{\underset{(>)}{=}} r_A^* \ .$$

The intuition for the fall in country A's utility is as before: without tariffs, the first-order loss from the fall in domestic varieties dominates the second-order gain from new foreign varieties. Since changes in c_B, F_B, and L_B affect B's utility also directly—i.e., not just via the trade channel—the net effect on welfare in B is more difficult to ascertain. For example, *ceteris paribus*, a fall in c_B clearly benefits country B, while the initiation of (tariff-free) trade hurts it. The net effect of these countervailing forces is ambiguous. On the other hand, a rise in F_B or a fall in L_B lowers country B's utility. The welfare loss from the initiation of tariff-free trade merely reinforces the negative direct impact.

Using directional derivatives, Proposition 3 can be easily extended to allow for simultaneous changes in c_B, F_B, and L_B, as well as in r_k and t_k, $k \in \{A, B\}$. What matters is that, jointly, the marginal changes take us from a point on the boundary $\dot{T}$ into the trading region $\underline{T}$

3.3 Example

Example 1 *The following example illustrates our findings. Suppose households have Pollak preferences, i.e.,* $v_k[z] = (\gamma + z)^\beta - \gamma^\beta$, *where* $0 < \beta < 1$ *and* $\gamma > 0$. *(Here,* $\gamma > 0$ *ensures finite choke prices, while* $\gamma = 0$ *would correspond to CES.) Consider the country-symmetric case with* $\beta = 1/2$; $\gamma = 100$; $F = 5 \times 10^5$; $c = 10$; *and* $L = 1000$. *The symmetric trade costs and tariffs* (t, r) *are variable. Let* $r_1 = 0.3 > r^* = 0.17$ *denote a tariff higher than the critical one.*

The panels in Figure 2 depict the relationship between trade costs t, *tariffs* r *and HH utility* U. *Clockwise from the upper left, the first panel plots* U *as a function of* t *and* r, *followed by two-dimensional projections onto the* t-*by*-r *plane, the* r-*by*-U *plane, and the* t-*by*-U *plane.*

The upper-right panel depicts the partitioning of the (t, r)-*space into the trade area* $\underline{T}$, *the no-trade area* $\bar{T}$, *and the threshold between the two,* $\dot{T}$. *Threshold* $\dot{T}$ *corresponds to the straight line* $\dot{r}[t] \equiv 1 - t/\sqrt{\dot{\Phi}}$. *The horizontal line at the critical tariff* r^* *divides* $\dot{T}$ *in two parts: Trade initiated from a point on* $\dot{T}$ *above* r^**—such as* $\left(\dot{t}[r_1], r_1\right)$*—raises welfare (green). Trade initiated from a point on* $\dot{T}$ *below* r^**—such as* $\left(\dot{t}[0], 0\right)$*— reduces welfare (red).*

The lower-left panel depicts utility U *as a function of trade cost* t, *for tariffs fixed at* $r = 0$, r^*, *and* r_1. *(The optimal tariff,* $r^{**}[t]$, *is also shown, but we defer discussion thereof to the Section 5). At the initiation of trade, utility drops (rises) iff* $r \overset{(>)}{<} r^*$. *Hence, at that point, utility falls for* $r = 0$, *rises for* $r = r_1$, *and exhibits no first-order change for* $r = r^*$. *Since* $0 < r^* < r_1$, *we have* $\dot{t}[r_1] < \dot{t}[r^*] < \dot{t}[0]$. *That is, the larger the tariff* r, *the more iceberg costs must fall before trade begins. As trade costs vanish, trade-cost savings on the existing stock of exports come to dominate initial losses (if any) due to the fall in domestic varieties. In particular, utility under free and costless trade exceeds utility under autarky.*[15]

The lower right panel depicts U *as a function of tariff* r, *for trade costs fixed at* $t = \dot{t}[r^*]$, $\dot{t}[r_1]$, *and* 1 *(as well as for the locus of trade costs* $t^{**}[r]$ *that makes* r *optimal).*[16,17] *At the initiation of trade, once again, welfare falls (rises) iff at that point* $r \overset{(>)}{<} r^*$*—or, equivalently, iff* $t \overset{(<)}{>} \dot{t}[r^*]$. *Thus, locally, utility falls for* $t < \dot{t}[r^*]$, *rises for* $t = \dot{t}[r_1]$ *and* $t = 1$, *and exhibits no first-order change for* $t = \dot{t}[r^*]$.

Finally, the upper-left panel provides a 3D-visualization of U *as a function of both* t *and* r.

4 When Autarky Trumps Free and Costless Trade

Krugman (1979) shows that, for symmetric countries, free and costless trade Pareto-dominates autarky. As we have seen, focusing on this dichotomy overlooks important non-monotonicities

[15] *As we shall see in Section 4, this is an artifact of the symmetric model. When countries are asymmetric, a country can be strictly better off in autarky than in free-and-costless trade.*

[16] *Notice that* $1 = \dot{t}[\dot{r}[1]]$, *where* $\dot{r}[1] = 0.58 > r_1$.

[17] *Notice that for* $t = \dot{t}[0]$, *utility is constant at the autarky level for all* r, *since tariffs have to fall all the way to zero before trade is iniated.*

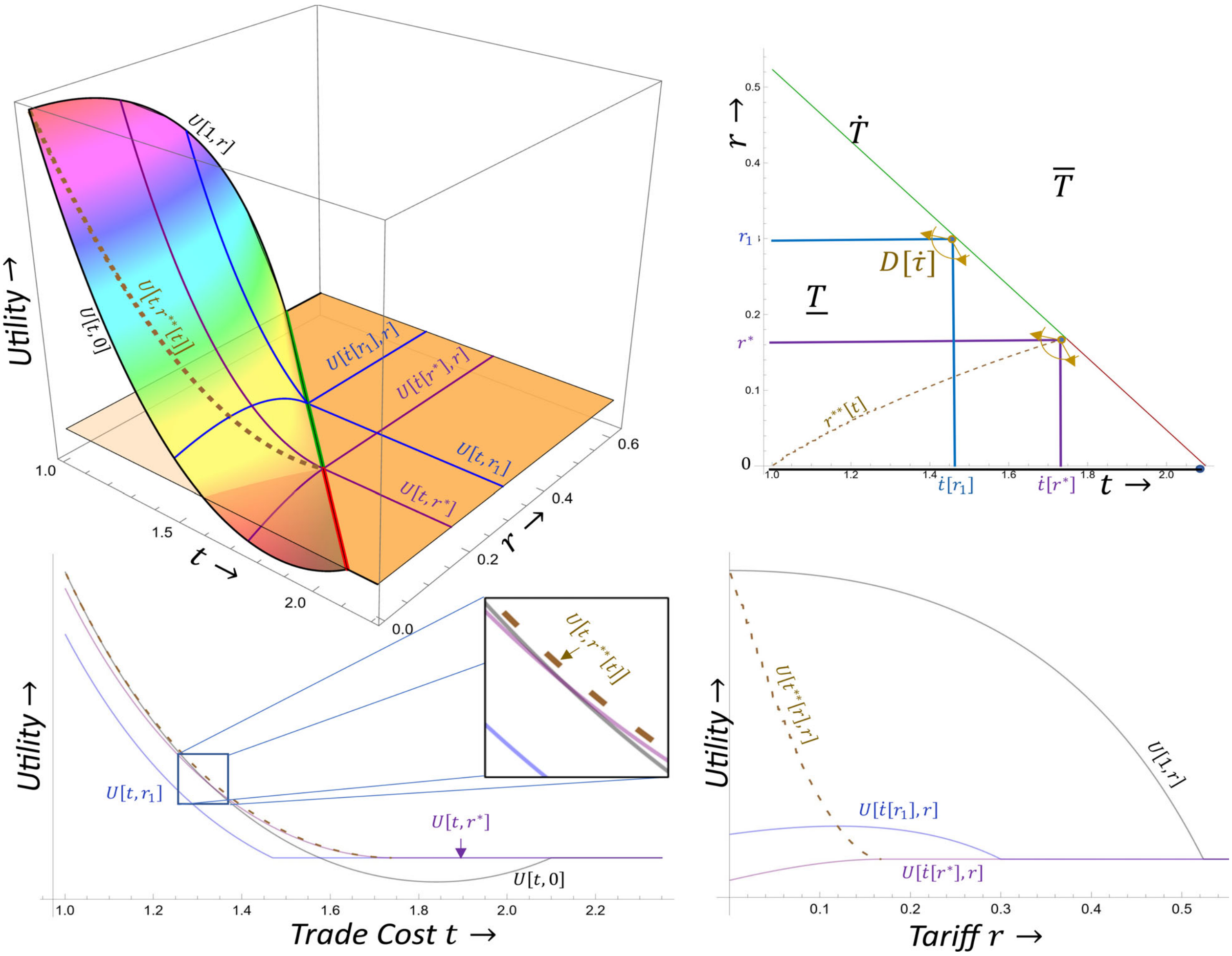

Figure 2: The upper right panel depicts the partitioning of the (t, r)-space into the trade area $\underline{T}$, the no-trade area $\bar{T}$, and the threshold $\dot{T}$ between the two. The remaining panels depict utility as a function of both t and r (upper left); as a function of trade costs t for different levels of tariffs r (lower left); and as a function of tariff r for different levels of trade costs t (lower right). See Example 1 for details.

between the two extremes. We now show that focusing on symmetric countries is similarly restrictive. Allowing countries to be asymmetric reveals that a country may in fact be better off in autarky than in free and costless trade.

For concreteness, we take again the perspective of country A. Let $\dot{U}_A$ denote household utility in A under autarky. Fixing all model parameters other than $c_B \in (0, \infty)$, let $U_A [c_B]$ denotes utility in A as a function of the marginal cost in B. Since we have not proved uniqueness of equilibrium with trade, $U_A [c_B]$ might be multi-valued for some c_B. By contrast, autarky utility $\dot{U}_A$ is unique and, obviously, independent of c_B.

Proposition 4 (Autarky Trumps Free Trade) *Country A is strictly better off in autarky than in free and costless trade, if its trading partner, B, has sufficiently greater marginal costs.*

Formally: Let $t_A = t_B = 1$. There exists a $\underline{c}_B > 0$ such that $\dot{U}_A > U_A [c_B]$ for all $c_B > \underline{c}_B$, iff $r_A < r_A^*$.

To develop an intuition for Proposition 4, suppose there are neither trade costs nor tariffs—i.e., we are (and remain) in free and costless trade. When c_B rises, export quantities per firm, x_B, and the number of firms, n_B, fall, while exports $n_B x_B$ vanish entirely when $c_B \to \infty$. This means that in the limit, country A simultaneously lives in autarky *and* in free and costless trade(!). Starting from the limit and reversing course, a 'small reduction' in c_B initiates trade. As in the Bad Trade result of Proposition 3, this leaves country A strictly worse off since, in the absence of tariffs, the gains from trade are second-order, while the losses from the fall in domestic varieties are first order. Thus, moving away from autarky—but remaining in free and costless trade—*reduces* utility in A for sufficiently large c_B. This intuition also explains the recurring role of the critical tariff r_A^*, which equalizes the gains and losses from the initiation of trade for country A.

Figure 1 may also help in developing an intuition for why autarky can dominate free and costless trade for country A. When c_B becomes large, x_B and n_B, become small. At the same time, in country A, per-firm home-bound production y_A converges to its autarky level $\dot{y}_A > 0$. In terms of Figure 1, consumer surplus in A from each foreign variety (region V) vanishes, while the surplus from each domestic variety (region I) remains large. At some point—i.e., for c_B sufficiently large—a discrete switch from free and costless trade to autarky is beneficial for households in A, because it entails giving up ever-lower-surplus units x_B in return for high-surplus units y_A.

While households in A benefit, a switch to autarky leaves households in B strictly worse off. The reason is that, for large c_B, country B's home-bound production per-firm, y_B, becomes *equal* to zero, rather than converging to zero as do x_A and x_B.[18] In the absence of infra-marginal units for y_B, the switch to autarky entails country B giving up higher-surplus units

[18] Country B turns into a 'tourism economy,' not consuming any of its own products.

x_A for, on the margin, zero-surplus units y_B. Thus, while country A gains from a switch to autarky, country B loses.

One might expect country A to be also better off in autarky than in free and costless trade, if its trading partner, B, has sufficiently higher fixed costs F_B or a sufficiently smaller population L_B. After all, when $F_B \to \infty$ or $L_B \to 0$, country A again ends up living both in autarky and in free and costless trade. Furthermore, starting from the limit and reversing course, a 'small reduction' in F_B or a rise in L_B initiates trading. Despite the analogy, the conjecture is false. That is, in these cases, country A is not be better off in autarky. To see why, recall that A benefitted from autarky for large c_B, because trading entailed giving up ever-lower-surplus units x_B in return for high-surplus units y_A. The low surplus that A derived from x_B was due to the disappearance of infra-marginal units as $x_B \to 0$. However, unlike for $c_B \to \infty$, x_B does not go to zero when $F_B \to \infty$ or $L_B \to 0$. In fact, while total exports, $n_B x_B$, do vanish since $n_B \to 0$, per-firm exports x_B *rise*. The continued presence of infra-marginal units of x_B makes that country A benefits from the initiation of trade. As a result, Proposition 4 fails to carry over for large F_B or small L_B.

Now consider utility in A as a function of trade cost t_A when $r_A = 0$. Proposition 4 implies that, for low-cost country A, the Bad Trade result in Proposition 2 is not merely a local phenomenon but a *global* one—i.e., it extends all the way from the initiation of trade to free and costless trade. To see this, observe that A's utility at the initiation of trade is equal to its utility in autarky, which for large c_B is strictly greater than its utility in free and costless trade. Thus, as trade cost t_A falls below the initiation-of-trade threshold $\dot{t}_A$ and then vanishes altogether (i.e., $t_A = 1$), country A's utility drops below autarky and never recovers. Finally, notice that levying the critical tariff r_A^* remedies the problem. In fact, it leaves country A strictly better off than in either autarky or free and costless trade, whenever trade takes place.

Example 2 *Utility is as in Example 1. Country A is parametrized by* $F_A = 5 \times 10^5$; $c_A = 10$; $L_A = 10000$, *country B by* $F_B = 1 \times 10^5$; $L_B = 1000$, *while* c_B *is variable. We set tariffs to zero* ($r_A = r_B = 0$) *and let* $t_1 \equiv 1.13 > 1$, $c_{B1} \equiv 20$, $c_{B2} \equiv 31$, *and* $c_{B3} \equiv 71$.

Figure 3 depicts the relationship between marginal cost c_B, *(symmetric) trade costs* t, *and utility* U_A *in A. The middle panel plots these quantities together in three dimensions, while the left and right panels are two-dimensional projections onto the* t*-by-*U_A *and* c_B*-by-*U_A *planes, respectively.*

The left panel depicts U_A *as a function of* t, *for* $c_B \in \{c_{B1}, c_{B2}, c_{B3}, \infty\}$. *When trade is initiated,* U_A *decreases locally, for all finite* c_B. *Provided* c_B *is sufficiently low (e.g.,* $c_B = c_{B1}$*) then, as* t *continues to fall, gains from trade eventually offset the welfare losses due to the loss of domestic variety, and* U_A *climbs back above its autarky level. For* $c_B = c_{B2}$, *though,* U_A *does not return to its autarky level until trade is completely costless* ($t = 1$), *while for all higher* c_B

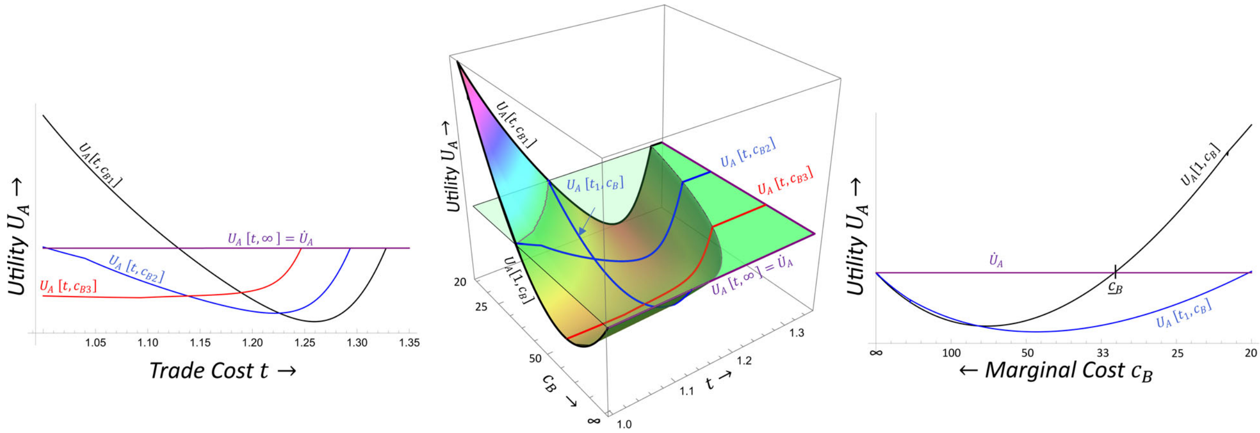

Figure 3: The figure depicts utility in country A as a function of (symmetric) trade costs t for various marginal cost levels c_B (left panel); as a function of marginal cost c_B for various trade cost levels t (right panel); and as a function of both t and c_B (center panel). See Example 2 for details.

(e.g., $c_B = c_{B3}$) it never does. As $c_B \to \infty$, production in country B and trade vanish, and hence, U_A approaches its autarky level $\dot{U}_A$.

The right panel depicts U_A as a function of c_B, for $t \in \{1, t_1\}$. At $c_B = \infty$, $U_A = \dot{U}_A$, since country B neither produces nor trades. For all $c_B \in (\underline{c}_B, \infty)$, country A strictly prefers autarky to free-and-costless trade (i.e., $U_A < \dot{U}_A$). Only for $c_B < \underline{c}_B$ does A prefer free and costless trade to autarky ($U_A > \dot{U}_A$).

5 Optimal Symmetric Tariffs

We now turn to the question of socially optimal symmetric tariffs, both at and away from the initiation of trade. In the absence of side-payments, notice that countries must be symmetric for such tariffs to exist. Otherwise, tariffs that are optimal for one country generally fail to be so for the other country.[19] Asymmetric tariffs exhibit this problem in spades—i.e., even when countries are symmetric. That is why we set $r_A = r_B = r$.

Remark 1 *Suppose countries are symmetric. In the interior, socially optimal tariff $r^{**}[t]$ solves*

$$r = 1 - \frac{v'\left[\frac{x}{L}\right] - tcU}{v'\left[\frac{y}{L}\right] - cU} \frac{v'[y/L]\,\varepsilon_m[y/L]}{v'[x/L]\,\varepsilon_m[x/L]} \,. \tag{24}$$

At costless trade, $r^{**} = 0$. At the initiation of trade, $r^{**} = r^*$.

[19] Even when they exist, socially optimal symmetric tariffs are generally not self-enforcing. That is, countries have an incentive to unilaterally deviate from them. The exceptions are costless trade with zero tariffs and the initiation of trade with critical tariffs. It can be verified that these (optimal) tariffs are mutual best responses—at least *locally*. Analyzing tariff setting as a non-cooperative game is complicated and beyond the scope of this paper.

The proof of Remark 1 is similar to (the sketch of) the proof of Proposition 2 presented earlier. However, now we differentiate utility with respect to r rather than t. Also, we no longer evaluate at $\tau = \dot{\tau}$, since extrema away from the initiation of trade also interest.

To better understand the first-order condition in (24), we proceed in three steps: First, we differentiate U with respect to r. Then we rewrite the FOC such that it is solely a function of per-household consumption of domestic and imported varieties, y/L and x/L, and the ratio of their derivatives with respect to r. Finally, we derive an expression for this ratio from the zero-profit condition and use it to eliminate the ratio of derivatives from the FOC.

Step 1. Differentiating utility $U = n\left(v\left[\frac{y}{L}\right] + v\left[\frac{x}{L}\right]\right)$ with respect to r we find that, at an interior extremum,

$$\frac{dU}{dr} = n\left(v'\left[\frac{y}{L}\right]\frac{dy/L}{dr} + v'\left[\frac{x}{L}\right]\frac{dx/L}{dr}\right) + \frac{dn}{dr}\left(v\left[\frac{y}{L}\right] + v\left[\frac{x}{L}\right]\right) = 0\ . \tag{25}$$

(Cf., the analogous expression in (14).)

Step 2. Labor market clearing $LM : n = L/C\,[y + tx]$ implies that n is solely a function of y and x. Hence, the total derivative dn/dr can be written

$$\frac{dn}{dr} = \frac{\partial n}{\partial y/L}\frac{dy/L}{dr} + \frac{\partial n}{\partial x/L}\frac{dx/L}{dr}\ . \tag{26}$$

Substituting (26) into (25) and rearranging yields

$$\left(nv'\left[\frac{y}{L}\right] + \frac{U}{n}\frac{\partial n}{\partial y/L}\right)\frac{dy/L}{dr} + \left(nv'\left[\frac{x}{L}\right] + \frac{U}{n}\frac{\partial n}{\partial x/L}\right)\frac{dx/L}{dr} = 0\ . \tag{27}$$

The expression in (27) is intuitive. It says that the effect of a marginal change in r on utility can be decomposed into the effects of marginal changes in per-household consumption of domestic and foreign varieties y/L and x/L, respectively. In turn, changes in y/L and x/L have both direct and indirect effects. The direct effect on utility of a marginal change in y/L is simply $nv'[y/L]$, i.e., marginal utility multiplied by the number of existing domestic varieties. Similarly, the direct effect of a change in x/L is $nv'[x/L]$. Changes in y and x also have *indirect* effects, because they modify the scale of production. This changes the number of firms/varieties, via the labor market clearing condition LM—see (26) above. The indirect effect on utility of a change in y/L equals the change in the number of varieties it engenders, $\frac{\partial n}{\partial y/L}$, multiplied by the utility per variety, U/n. Analogously, the indirect effect of a change in x/L is $\frac{U}{n}\frac{\partial n}{\partial x/L}$. Notice that $\frac{\partial n}{\partial y/L}, \frac{\partial n}{\partial x/L} < 0$; i.e., an increase in the scale of production reduces the number of varieties. Since a drop in tariff r leads to the replacement of domestic varieties with foreign ones, x/L goes up and y/L goes down. At an interior extremum, the effects on utility cancel each other.

To eliminate dn/dr in (25), we compute the derivative *explicitly*, rather than use the implicit form in (26). Implicitly differentiating labour market clearing $LM : n = L/C\,[y + tx]$ with respect to r and reusing LM yields

$$\frac{dn}{dr} = -n^2 c \left(\frac{dy/L}{dr} + t \frac{dx/L}{dr} \right) \ . \tag{28}$$

Substituting (28) into the first-order condition (25) for optimal r and rearranging yields

$$\left(v' \left[\frac{y}{L} \right] - cU \right) \frac{dy/L}{dr} = - \left(v' \left[\frac{x}{L} \right] - tcU \right) \frac{dx/L}{dr} \ . \tag{29}$$

Obviously, the condition in (29) is equivalent to the somewhat more intuitive one in (27). A final rearrangement (29) implies an interior optimal r must satisfy

$$\frac{d\,(y/L)\,/dr}{d\,(x/L)\,/dr} = - \frac{v' \left[\frac{x}{L} \right] - tcU}{v' \left[\frac{y}{L} \right] - cU} \ . \tag{30}$$

Step 3. Straightforward differentiation of the zero-profit condition and rearranging—steps relegated to Lemma A.19 in the Appendix—yields

$$\frac{d\,(y/L)\,/dr}{d\,(x/L)\,/dr} = -\,(1-r)\, \frac{v' \left[\frac{x}{L} \right] \varepsilon_m \left[x/L \right]}{v' \left[\frac{y}{L} \right] \varepsilon_m \left[y/L \right]} \ . \tag{31}$$

Equating (31) with (30) yields (24).

The two expressions for $\frac{d(y/L)/dr}{d(x/L)/dr}$ in (30) and (31) differ in nature. The former follows from a smooth optimization problem that trades off the utility-costs and benefits of a marginal change in the import tariff. As such, it yields an economic intuition based on standard marginal reasoning. By contrast, the latter follows from the assumption of infinitely elastic and, therefore, discontinuous, entry and exit in response to arbitrarily small profits and losses. This makes (31) wholly 'mechanical.' As r changes, the equation simply dictates what must happen to y and x to maintain zero profits in general equilibrium, without providing much economic intuition.

Example 3 *(Continuation of Example 1) The optimal tariff, $r^{**}[t]$, is depicted in each panel of Figure 2. Notice that zero tariffs are optimal if and only if trade is costless ($t = 1$); the critical tariff r^* is optimal at the initiation of trade ($t = \dot{t}\,[r^*]$); while for trade costs greater than $\dot{t}\,[r^*]$ it is optimal to impose tariffs that block trade entirely.*

As a function of t, utility under $r = 0$ crosses utility under $r = r^*$ once. This crossing can be seen more clearly in the magnification lens. For trade costs between 1 and $\dot{t}\,[r^*]$, the optimal tariff r^{**} lies between 0 and r^*. Formally, for all $t \in \left(1, \dot{t}\,[r^*]\right)$, $0 < r^{**} < r^*$. Thus, in the face of 'exogenous' trade frictions $t > 1$, it is beneficial to impose additional 'endogenous' trade frictions, in the form of tariffs $r > 0$. This is in line with the theory of the Second-Best (Lipsey and Lancaster, 1956).

6 Generalizations

We now generalize the model in various directions. These generalizations serve two purposes. First, we want to get a feel for the robustness of 'Bad Trade.' Second, we want to make the model more amenable to empirical analysis, by accommodating certain real-world aspects of the data, such as multiple sectors, fixed costs of exporting, and heterogeneous firms. Extending Bad Trade to more than two (*symmetric*) countries is straightforward and therefore omitted.

Despite relentless globalization, there are still important sectors of the economy with little or no international trade, especially in services. At the same time, some of these sectors are opening up, due to technological advances especially in the field of ICT. This motivates the multi-sector extension we undertake in Section 6.1. There we show that Bad Trade holds sector-by-sector, irrespective of the aggregate distance from autarky. Furthermore, we find that the initiation of trade in one sector has negative spill-overs in terms of a loss of variety in all other sectors. These findings make Bad Trade relevant for today's world, which is of course far from autarky in the aggregate.

Like producing for the domestic market, exporting may have both a fixed and a variable cost component. When firms face a fixed cost of exporting, they never export marginal quantities: if they export at all, they sell a discrete quantity large enough to make up for the additional fixed cost incurred. Since Bad Trade relies on a marginal analysis, it may seem doubtful that the result survives in such an environment. Perhaps surprisingly, in Section 6.2 we show that Bad Trade not only survives but is strengthened: the drop in utility becomes *discontinuous* and, thus, is no longer 'small.'

Finally, in Section 6.3 we extend the model to allow for heterogeneous firms. Firm heterogeneity has played an important role in trade theory and empirics since at least Melitz (2003). In the absence of a fixed cost of exporting, we show that Bad Trade carries over to any finite type space of heterogenous firms. Arguably, this is the empirically more natural environment. A finite type-space ensures that a small reduction of trade costs below the prohibitive level has a first-order effect on exports. With atomless types and no fixed cost of exporting, all first-order effects are zero.

There is one respect in which the extensions presented here are *less* general than the baseline model: we assume that countries are symmetric. For the extension to heterogeneous firms, this is solely a matter of convenience. For the extensions to multiple sectors and fixed costs of exporting, the restriction to symmetry is not as innocuous: we need it to ensure existence of equilibrium.

6.1 Multiple Sectors

We now extend the country-symmetric version of the baseline model to allow for a finite number of asymmetric sectors with sector-specific trade costs. We show that 'Bad Trade' holds sector-by-sector, and that the intuition carries over almost unchanged from the single-sector model. The multi-sector model also reveals a cross-sectoral spill-over effect from the initiation of trade: variety drops in *all* sectors—not only in the newly tradable one.

The model is essentially the same as in Section 2.1, except that the economies of countries A and B now consist of N different sectors or industries, which we denote by $\xi \in \{1, ..., N\}$. The endogenous mass of firms in sector ξ is $n_{\xi k} \geq 0$, $k \in \{A, B\}$, while $\mathbf{n}_k \equiv [n_{1k}, ..., n_{Nk}] \in [0, \infty)^N$ denotes a vector of firm masses, one for each sector. Allowing for arbitrary asymmetries between sectors, we do assume that the countries are symmetric. This guarantees existence of equilibrium. It also simplifies the algebra.

A sector-ξ variety produced in country k is denoted by $i_{\xi k}$, where $i_{\xi k} \in [0, n_{\xi k}]$, $\xi \in \{1, ..., N\}$, and $k \in \{A, B\}$. A household consumption bundle of sector-ξ varieties produced in country k is a measurable function $\mathbf{z}_{\xi k} : [0, n_k] \to [0, \infty)$, assigning a quantity $z_{i_{\xi k}} = \mathbf{z}_{\xi k}[i_{\xi k}]$ to each variety $i_{\xi k} \in [0, n_{\xi k}]$. Vector $\mathbf{z}_k \equiv [z_{1k}, ...z_{Nk}]$ denotes an array of consumption bundles of sector-ξ varieties produced in country k. Hence, $\mathbf{z}_k : [0, n_{\xi k}]^N \to [0, \infty)^N$.

Taking the perspective of a household in country A, utility $U_A : [0, \infty)^{\mathcal{N}_A} \times [0, \infty)^{\mathcal{N}_B} \to [0, \infty)$, $\mathcal{N}_k \equiv \sum_{\xi=1}^{N} n_{\xi k}$, is given by

$$U_A[\mathbf{z}_A, \mathbf{z}_B] = \sum_{\xi=1}^{N} u_\xi \left[\int_{i_{\xi A}=0}^{n_{\xi A}} v_\xi \left[z_{i_{\xi A}} \right] di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} v_\xi \left[z_{i_{\xi B}} \right] di_{\xi B} \right] . \qquad (32)$$

Notice that the utility function in (32) has a nested structure. That is, sub-utility function $u_\xi : [0, \infty) \to [0, \infty)$ imposes on a sectoral level the same additive structure that sub-sub-utility function $v_\xi : [0, \infty) \to [0, \infty)$ imposes on the varietal level. Sectors and varieties do differ in the sense that the number of sectors, N, is discrete and exogenous, while the number of varieties within a sector, $n_{\xi k}$, is continuous and endogenous.

The (sub-)sub-utility functions u_ξ, v_ξ have the same properties as v_k in the baseline model, except that now $\lim_{\boldsymbol{v} \to 0} u'_\xi[\boldsymbol{v}] = \infty$—i.e., on the sectoral level, marginal utility at zero is unbounded. This is an assumption of convenience that precludes corner solutions for $n_{\xi k}$, which guarantees that production takes place in every sector.

Prices of domestically produced goods in sector ξ of country A are denoted by $p_{i_{\xi A}}$, and prices of imported goods are denoted by $s_{i_{\xi B}}$. Households solve:

$$\begin{array}{ll} \max_{\{\mathbf{z}_A, \mathbf{z}_B\}} & U_A = \sum_{\xi=1}^{N} u_{\xi A} \\ \text{st.} & \sum_{\xi=1}^{N} \int_{i_{\xi A}=0}^{n_{\xi A}} p_{i_{\xi A}} z_{i_{\xi A}} di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} s_{i_{\xi B}} z_{i_{\xi B}} di_{\xi B} = I_A \ . \end{array} \qquad (33)$$

Here, $u_{\xi A} \equiv u_\xi \left[\int_{i_{\xi A}=0}^{n_{\xi A}} v_\xi \left[z_{i_{\xi A}} \right] di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} v_\xi \left[z_{i_{\xi B}} \right] di_{\xi B} \right]$. The country subscript in $u_{\xi A}$ reflects that, even though the function $u_\xi [\cdot]$ is the same across countries, the arguments are country-specific and, thus, they may differ.

The FOCs for utility maximization are

$$u'_{\xi A} v'_\zeta \left[z_{i_{\zeta A}} \right] = \lambda_A p_{i_{\zeta A}} \text{ and } u'_{\xi A} v'_\zeta \left[z_{i_{\zeta B}} \right] = \lambda_A s_{i_{\xi B}} \text{ , for } \xi = \{1, ...N\} \ . \tag{34}$$

As in the single sector case, $\lambda_A \in (0, \infty)$ denotes the Lagrangian multiplier on the budget constraint, which is equal to the marginal utility of income. The next lemma shows that the FOCs give rise to a well-defined set of demand functions (or rather, vectors of operators). Here, $\bar{v}_\xi'^{-1} [\cdot]$ denotes the extended inverse of $v_\xi [\cdot]$. That is,

$$\bar{v}_\xi'^{-1} [\omega] = \begin{cases} 0 & \text{if } \omega \geq v'_\xi [0] \\ v_\xi'^{-1} [\omega] & \text{if } v'_\xi [\infty] \leq \omega < v'_\xi [0] \\ \infty & \text{if } 0 \leq \omega < v'_\xi [\infty] \ . \end{cases}$$

Lemma 2 *For every price profile* $(\mathbf{p}_A, \mathbf{s}_B) \in (0, \infty)^{\mathcal{N}_A} \times (0, \infty)^{\mathcal{N}_B}$*, the utility maximization problem in* (33) *has a unique solution,* $(\hat{\mathbf{z}}_A, \hat{\mathbf{z}}_B)$*. The solution is implicitly given by*

$$\hat{\mathbf{z}}_A [i_{\xi A}] = \bar{v}_\xi'^{-1} \left[\frac{p_{i_{1A}}}{u'_{\xi A} / \lambda_A} \right] \text{ and } \hat{\mathbf{z}}_B [i_{\xi B}] = \bar{v}_\xi'^{-1} \left[\frac{s_{i_{1B}}}{u'_{\xi A} / \lambda_A} \right] \ , \tag{35}$$

for $(i_{\xi A}, i_{\xi B}) \in [0, n_{\xi A}] \times [0, n_{\xi B}]$ *and* $\xi \in \{1, ..., N\}$.

Notice that $u'_{\xi k} / \lambda_k$ plays the role of a sector-specific price level in country k. Therefore, we write $P_{\xi k} \equiv u'_{\xi k} / \lambda_k$, and $\mathbf{P}_k \equiv \{P_{1k}, ..., P_{Nk}\} \in (0, \infty)^N$, $k \in \{A, B\}$.

Firm $i_{\xi A}$'s profit, $\pi_{i_{\xi A}}$, is given by

$$\pi_{i_{\xi A}} = p_{i_{\xi A}} y_{i_{\xi A}} + (1 - r_\xi) s_{i_{\xi A}} x_{i_{\xi A}} - w_A \left(F_\xi + c_\xi \left(y_{i_{\xi A}} + t_\xi x_{i_{\xi A}} \right) \right) \ .$$

Here, $F_\xi > 0$, $c_\xi > 0$, $0 \leq r_\xi \leq 1$, $t_\xi \geq 1$ denote industry-specific fixed costs, marginal costs, *ad valorem* import tariffs, and iceberg trade costs, respectively. The firm's FOCs for optimal interior $y_{i_{\xi A}}$ and $x_{i_{\xi A}}$ can be written as (Cf. (5)):

$$\begin{aligned} FOC_{y_{i_{\xi A}}} : \quad & v'_\xi \left[\tfrac{y_{i_{\xi A}}}{L_A} \right] \left(1 - \varepsilon_{v'_\xi} \left[\tfrac{y_{i_{\xi A}}}{L_A} \right] \right) = \tfrac{c_\xi}{P_{\xi A} / w_A} \\ FOC_{x_{i_{\xi A}}} : \quad & v'_\xi \left[\tfrac{x_{i_{\xi A}}}{L_B} \right] \left(1 - \varepsilon_{v'_\xi} \left[\tfrac{x_{i_{\xi A}}}{L_B} \right] \right) = \varphi_\xi \tfrac{c_\xi}{P_{\xi B} / w_A} \ , \end{aligned} \tag{36}$$

where $\varphi_\xi \equiv t_\xi / (1 - r_\xi)$ denotes the sector-specific trade friction. As in the single-sector model, monotonicity in $y_{i_{\xi A}}$ and $x_{i_{\xi A}}$ implies that the FOCs in (36) each admit at most one solution. The SOCs also carry over and, other than being sector-specific, firms' optimal quantities

$\hat{y}_{i_{\xi A}}[P_{\xi A}/w_A]$ and $\hat{x}_{i_{\xi A}}[P_{\xi B}/w_A]$ are unchanged as well. Since all active firms in sector ξ behave identically, we write $y_{\xi A}$, $p_{\xi A}$, $x_{\xi A}$, $s_{\xi A}$ for $y_{i_{\xi A}}$, $p_{i_{\xi A}}$, $x_{i_{\xi A}}$, $s_{i_{\xi A}}$, respectively.

In this country-symmetric multi-sector model, a country-symmetric equilibrium $(\mathbf{P}, \mathbf{n})$ is characterized by the system

$$\left.\begin{array}{ll} FOC_{y_\xi}: & v'_\xi\left[\frac{y_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]\right) = \frac{c_\xi}{P_\xi} \\ FOC_{x_\xi}: & v'_\xi\left[\frac{x_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{x_\xi}{L}\right]\right) \overset{(\leq)}{=} \varphi_\xi \frac{c_\xi}{P_\xi} \quad \text{if} \quad x_\xi \overset{(=)}{>} 0 \\ ZP_\xi: & v'_\xi\left[\frac{y_\xi}{L}\right] y_\xi + (1-r_\xi)\, v'_\xi\left[\frac{x_\xi}{L}\right] x_\xi = \frac{C_\xi[y_\xi + t_\xi x_\xi]}{P_\xi} \\ LM: & \sum_{\xi=1}^N n_\xi C_\xi [y_\xi + t_\xi x_\xi] = L\ . \end{array}\right. , \quad \xi \in \{1, ...N\} \tag{37}$$

Notice that we have normalized $w = 1$, and that BP has disappeared because it is automatically satisfied under country-symmetry. BB is again redundant. Since our focus is on country-symmetric equilibria, variables in (37) have lost their country subscripts $k \in \{A, B\}$.

For $\xi \in \{1, ...N\}$, define

$$\dot{\varphi}_\xi \equiv \frac{v'_\xi[0]}{m_\xi[\dot{y}_\xi/L]} > 1\ . \tag{38}$$

Here, $m_\xi[z] \equiv v'_\xi[z]\left(1-\varepsilon_{v'_\xi}[z]\right)$, while $\dot{y}_\xi > 0$ denotes the 'sector-ξ-autarky' level of y_ξ, which is in fact equal to the autarky value of y in the symmetric version of the one-sector baseline model of Section 2.1. That is, $\dot{y}_\xi$ is the unique solution to

$$\frac{\varepsilon_{v'_\xi}[y_\xi/L]}{1-\varepsilon_{v'_\xi}[y_\xi/L]} y_\xi = \frac{F_\xi}{c_\xi}\ .$$

The autarky value $\dot{y}_\xi$ in sector ξ is independent of the characteristics of all other sectors $\psi \neq \xi$. This is, of course, an artifact of the additive utility structure.

Proposition 5 *A country-symmetric equilibrium exists. Trade occurs in sector $\xi \in \{1, ...N\}$, iff $\varphi_\xi < \dot{\varphi}_\xi$.*

Observe that the threshold level $\dot{\varphi}_\xi$ of trade frictions in sector ξ is independent of all other sectors $\psi \neq \xi$ and only depends on autarky production $\dot{y}_\xi$ in sector ξ. When $\varphi_\xi = \dot{\varphi}_\xi$, the unique solution to the first-order condition FOC_{x_ξ} is $x_\xi = 0$. In that case, trade in sector ξ is *just* precluded. Denote by $\boldsymbol{\tau} \in \mathbf{T} \equiv ([1,\infty) \times [0,1))^N$ a profile $((t_1, r_1), ..., (t_N, r_N))$ of trade cost and tariff pairs, one for each sector. Denote by $\dot{\boldsymbol{\tau}}^\xi \in \mathbf{T}$ a cost and tariff profile such that sector ξ is on the verge of trading. That is, its ξ-th component $\dot{\boldsymbol{\tau}}^\xi_\xi = (t_\xi, r_\xi)$ is such that $\varphi_\xi \equiv t_\xi/(1-r_\xi) = \dot{\varphi}_\xi$, while all other components of $\dot{\boldsymbol{\tau}}^\xi$ are unrestricted. Let $\dot{\mathbf{T}}^\xi$ denote the locus of all such $\dot{\boldsymbol{\tau}}^\xi$, i.e., $\dot{\mathbf{T}}^\xi \equiv \{\boldsymbol{\tau} \in \mathbf{T} : \varphi_\xi = \dot{\varphi}_\xi\}$. The upper and lower contour sets of $\dot{\mathbf{T}}^\xi$ are given by $\bar{\mathbf{T}}^\xi \equiv \{\boldsymbol{\tau} \in T : \varphi_\xi > \dot{\varphi}_\xi\}$ and $\underline{\mathbf{T}}^\xi \equiv \{\boldsymbol{\tau} \in T : \varphi_\xi < \dot{\varphi}_\xi\}$, respectively. In region $\bar{\mathbf{T}}^\xi$, trade frictions in ξ are too high for trade to occur in that sector, while in $\underline{\mathbf{T}}^{\xi}$ they are low enough for trade to take place.

To study the effects of the initiation of trade in sector ξ, we pick a threshold point $\dot{\boldsymbol{\tau}}^{\xi} \in \dot{\mathbf{T}}^{\xi}$, where $\varphi_{\xi} = \dot{\varphi}_{\xi}$. Fixing all components of $\dot{\boldsymbol{\tau}}^{\xi}$ other than $\dot{\boldsymbol{\tau}}^{\xi}_{\xi} = (t_{\xi}, r_{\xi})$, we then calculate the directional derivative of utility, departing from $\dot{\boldsymbol{\tau}}^{\xi}$ and moving into the trade region $\underline{T}$ by solely changing $\dot{\boldsymbol{\tau}}^{\xi}_{\xi}$.

Denote changes in $\boldsymbol{\tau}$ by $\Delta\boldsymbol{\tau} \equiv ((\Delta t_1, \Delta r_1), ..., (\Delta t_N, \Delta r_N)) \in \mathbb{R}^{2N}$, and let changes solely in the ξ-th component of $\boldsymbol{\tau}$ be denoted by $\Delta\boldsymbol{\tau}^{\xi} \equiv \left(\Delta\boldsymbol{\tau}^{\xi}_{\xi}, \Delta\boldsymbol{\tau}^{\xi}_{-\xi}\right) \equiv \left((\Delta t_{\xi}, \Delta r_{\xi}), (0,0)^{N-1}\right)$. Starting from $\dot{\boldsymbol{\tau}}^{\xi}$ and moving in the direction of $\Delta\tau^{\xi}$, the new value of τ is then given by $\boldsymbol{\tau} = \dot{\boldsymbol{\tau}}^{\xi} + \sigma\Delta\tau^{\xi}$, with scaling factor $\sigma \in [-1, 1]$. Direction $\Delta\boldsymbol{\tau}^{\xi}$ is 'permissible' iff it is such that, when σ is small and negative, we are in the sector-ξ trade region $\underline{\mathbf{T}}^{\xi}$, and when σ is small and positive, we are in its no-trade region $\bar{\mathbf{T}}^{\xi}$. Since this means that $\varphi_{\xi} \equiv \frac{t_{\xi}}{1-r_{\xi}} \underset{(>)}{\overset{(<)}{=}} \dot{\varphi}_{\xi}$ iff $\sigma \underset{(>)}{\overset{(<)}{=}} 0$, it immediately follows that

Lemma 3 *At threshold point $\dot{\boldsymbol{\tau}}^{\xi} \in \dot{\mathbf{T}}^{\xi}$, the set of 'permissible' directions $\Delta\tau^{\xi} = ((\Delta t_{\xi}, \Delta r_{\xi}),$ $(0,0)^{N-1})$ is*

$$D_{\xi}\left[\dot{\boldsymbol{\tau}}^{\xi}\right] = \left\{ \Delta\boldsymbol{\tau}^{\xi} \middle| \frac{1}{\dot{t}_{\xi}[r_{\xi}]}\Delta t_{\xi} + \frac{1}{1-r_{\xi}}\Delta r_{\xi} > 0 \right\} .$$

We now extend Proposition 2 to this multi-sector environment.

Proposition 6 (Bad Trade—Sector-by-Sector) *The initiation of trade in sector ξ due to a fall in frictions φ_{ξ} leaves both countries strictly worse (better) off iff tariff r_{ξ} is strictly smaller (greater) than its critical value $r^*_{\xi} \equiv 1 - \varepsilon_{v_{\xi}}[\dot{y}_{\xi}/L]$. Formally, for all threshold points $\dot{\boldsymbol{\tau}}^{\xi} \in \dot{\mathbf{T}}^{\xi}$, $\xi \in \{1, ...N\}$, and for all permissible directions $\Delta\boldsymbol{\tau}^{\xi} \in D_{\xi}\left[\dot{\boldsymbol{\tau}}^{\xi}\right]$,*

$$\overleftarrow{\nabla}_{\Delta\boldsymbol{\tau}^{\xi}} U_k \Big|_{\boldsymbol{\tau}=\dot{\boldsymbol{\tau}}^{\xi}} \underset{(<)}{\overset{(>)}{=}} 0 \iff r_{\xi} \underset{(>)}{\overset{(<)}{=}} r^*_{\xi} .$$

At the initiation of trade in sector ξ, variety in that sector falls. Variety in all other sectors $\psi \neq \xi$ falls (rises) iff $\dot{r}_{\xi}$ is smaller (greater) than r^*_{ξ}. Formally,

$$\begin{aligned} \forall r_{\xi} &\in [0,1) : \overleftarrow{\nabla}_{\Delta\tau^{\xi}} \mathbf{n}_{\xi}\Big|_{\boldsymbol{\tau}=\dot{\boldsymbol{\tau}}^{\xi}} > 0 \text{ and} \\ \forall \psi &\neq \xi : \overleftarrow{\nabla}_{\Delta\boldsymbol{\tau}^{\xi}} \mathbf{n}_{\psi}\Big|_{\boldsymbol{\tau}=\dot{\boldsymbol{\tau}}^{\xi}} \underset{(<)}{\overset{(>)}{=}} 0 \iff r_{\xi} \underset{(>)}{\overset{(<)}{=}} r^*_{\xi} . \end{aligned}$$

Proposition 6 shows that 'Bad Trade' holds sector-by-sector.[20] The intuition is the same as for the single-sector model. There is, however, one additional effect to consider. The initiation of trade in sector ξ leads to fewer but larger firms in ξ. If tariff r_{ξ} is below r^*_{ξ}, total production

[20] Lemma (A.31), in the Appendix, shows that the critical tariff is never so high as to prevent trade for sufficiently low trade costs.

in sector ξ increases, which entails a reallocation of resources (i.e., labor) from sectors $\psi \neq \xi$ to ξ—something that did not and could not occur in the single-sector model of Section 2.1. Firm sizes in sectors other than ξ remain unchanged because, there, price levels have not moved. Less labor and unchanged firm size mean that the number of firms in these sectors must fall. This inflicts an *additional* first-order utility loss on households. By contrast, when $r_\xi > r_\xi^*$, the number of firms in sectors $\psi \neq \xi$ increases, while $r_\xi = r_\xi^*$ leaves them unchanged.

Notice that the Bad Trade result in Proposition 6 concerns trade that is initiated by a drop in *sector-specific* trade frictions φ_ξ. If the initiation of trade in sector ξ were due to an economy-wide fall in frictions, then the welfare effect might be beneficial even for $r_\xi < r_\xi^*$. To see this, notice that an across-the-board reduction in t creates first-order iceberg cost savings on the existing stock of exports in all sectors. Depending on the size of that stock, the first-order gain from these cost savings may outweigh the first-order loss from the initiation of trade in sector ξ.

Example 4 *Reconsider Example 1, except that now each (symmetric) economy consists of two sectors,* 1 *and* 2*, that are identical in all respects other than trade costs. Let utility be as in eqn. (32), where sub-utility functions* u_ξ *are given by* $u_1[\boldsymbol{v}] = u_2[\boldsymbol{v}] = \sqrt{\boldsymbol{v}}$*, and* $v_\xi[z]$ *are of the form and parametrization specified in Example 1. We set tariffs* r_ξ *to zero (*$r_1 = r_2 = 0$*). Let* $t_{\xi i}$ *denote a value of sector-*ξ *trade costs, indexed by* $i \in \{1, 2, 3\}$*. In particular,* $t_{21} \equiv 1.25$*,* $t_{22} \equiv 1.85$*, and* $t_{23} \equiv 2.60$*.*

The left panel of Figure 4 plots utility as a function of sectoral trade costs t_1*,* t_2*. The right panel depicts the two-dimensional projection onto the* t_1*-by-*U *plane. (The essentially identical projection onto the* t_2*-by-*U *plane has been omitted.)*

In the left panel, consider the curves depicting U *as a function* t_1*, for trade costs* $t_2 \in \{1, t_{21}, t_{22}, t_{23}\}$ *in sector* 2*. If* t_2 *is so high that no trade occurs in sector* 2 *(e.g.,* $t_2 = t_{23}$*), then utility traces the usual pattern as* t_1 *falls—it decreases from the autarky level at the initiation of trade but eventually recovers and surpasses the autarky level. If* t_2 *is such that utility is below the autarky level before trade in sector* 1 *begins (e.g.,* $t_2 = t_{22}$*), welfare falls even further as* t_1 *decreases to allow trade in sector* 1*. This is due both to the crowding out of domestic varieties in sector* 1 *by foreign substitutes, as well as to a further loss of domestic varieties in sector* 2 *as labor shifts from sector* 2 *to sector* 1*. Even when sector-*2 *trade costs are low enough that, without trade in sector* 1*, welfare has recovered to levels above autarky (e.g.,* $t_2 = 1$ *or* $t_2 = t_{21}$*), there is still a welfare drop when trade in sector* 1 *begins.*

6.2 Fixed Cost of Exporting

When entering a foreign market entails paying a fixed cost, exporting small quantities is never profitable. Since we have relied on a marginal analysis and the envelope theorem, it would seem

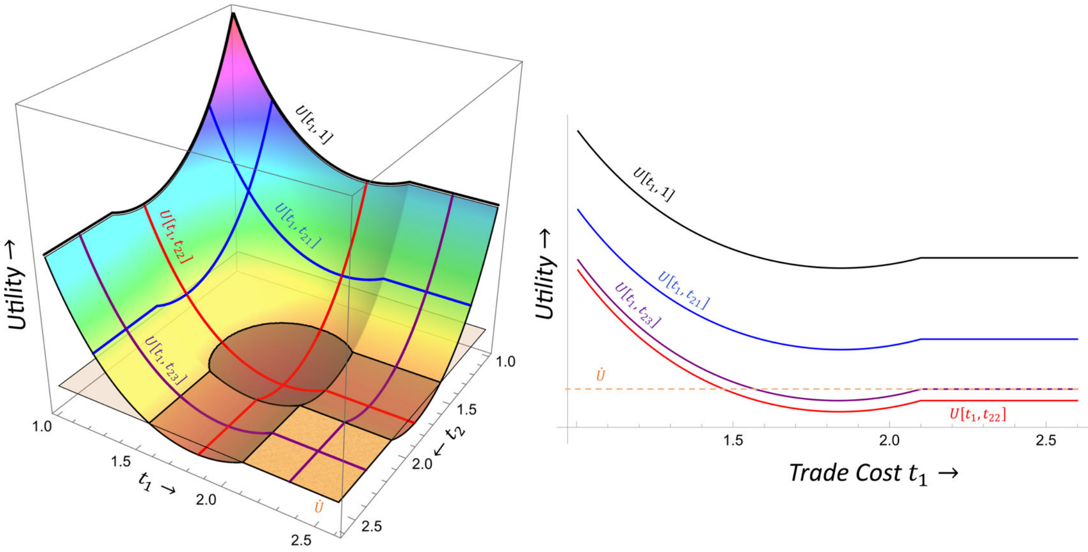

Figure 4: The figure depicts utility as a function of sector-specific trade costs t_1 and t_2 (left panel) and as a function of sector 1 *trade costs t_1 for various levels of sector* 2 *trade costs t_2 (right panel). See Example 4 for details.*

doubtful that Bade Trade survives in such an environment. Perhaps surprisingly, we now show that Bad Trade does survive. In fact, the drop in utility becomes *discontinuous* and, as such, is no longer 'small.'

Suppose the model is as in Section 2.1 but that, in addition to fixed cost F of operating, firms face a fixed cost f of exporting, $0 < f < F$. To ensure existence of equilibrium, we restrict attention to the symmetric model.[21]

As we shall see, the (t, r)-space $[1, \infty) \times [0, 1)$ of trade costs and tariffs can again be partitioned into contiguous trade, no-trade, and threshold areas. We denote these areas by $\underline{T}_F^f$, $\bar{T}_F^f$, and $\dot{T}_F^f$, respectively, with the fixed cost of exporting as superscript and the fixed cost of operating as subscript. However, trade friction index $\varphi = t/(1-t)$ is no longer a sufficient statistic for whether trade occurs, and the threshold locus $\dot{T}_F^f$ is no longer a straight line in (t, r)-space. To see this, divide (the symmetric versions of) FOC_x and FOC_y in (5) to get

$$\frac{m\,[x/L]}{m\,[y/L]} = \frac{t}{1-r} \; . \tag{39}$$

At the initiation of trade, home-bound production y is again equal to its autarky value $\dot{y}$. *Without* fixed cost of exporting, we have $x = 0$ at that point, and equation (39) then implies that the associated locus $\dot{T}_F^0$ describes a downward-sloping straight line in (t, r)-space given by $t = (1-r)\, v'\,[0]\, / m\,[\dot{y}/L]$. *With* fixed cost of exporting we have $x > 0$, because the net revenue

[21] With asymmetric countries and fixed cost of exporting, equilibrium may not exist in a neighborhood of the initiation of trade.

from exporting must make up for the additional fixed cost, f, incurred. Hence, at the initiation of trade, x solves

$$(1-r)\,sx = f + tcx \ .$$

Substituting using $s = Pv'[x/L]$ and FOC_x and simplifying yields

$$\frac{\varepsilon_{v'}[x/L]}{1-\varepsilon_{v'}[x/L]}x = \frac{f}{tc} \ . \tag{40}$$

Notice that r has dropped out. Denote by $\tilde{x}[t]$ a solution to (40), parameterized by t and independent of r. The LHS of (40) is strictly increasing and differentiable in x. Furthermore, it runs from zero at $x = 0$ to infinity as $x \to \infty$. Hence, $\tilde{x}[t]$ is a well-defined, strictly decreasing and differentiable function, that assigns a single value $x \in \mathbb{R}$ to every $t \in [1, \infty)$. Threshold $\dot{T}_F^f$ is now found by substituting $x = \tilde{x}[t]$ and $y = \dot{y}$ into (39). Formally,

$$\dot{T}_F^f \equiv \left\{ (t,r) \in [1,\infty) \times [0,1) \left| \frac{t}{1-r} = \frac{m[\tilde{x}[t]/L]}{m[\dot{y}/L]} \right. \right\} \ . \tag{41}$$

Since the RHS of the equation in (41) depends on t, the threshold locus $\dot{T}_F^f$ is no longer linear in t and r. However, it is still a smooth, single-valued, and downward-sloping curve. To see this, solve (41) for r and notice that the solution, $\dot{r}_F^f[t]$, is a strictly decreasing function of t, namely,

$$\dot{r}_F^f[t] = 1 - \frac{m[\dot{y}/L]}{m[\tilde{x}[t]/L]}t \ . \tag{42}$$

The inverse of $\dot{r}_F^f[t]$, i.e. the solution of (41) in t, is denoted by $\dot{t}_F^f[r]$.

The following proposition establishes existence of equilibrium and describes its structure.

Proposition 7 *In the symmetric model with fixed cost of exporting, there exists a symmetric equilibrium.*

1. *For $(t,r) \in \bar{T}_F^f$, countries live in autarky.*

2. *For $\left(\dot{t}[r], r\right) \in \dot{T}_F^f$, firms mix between exporting $\tilde{x}\left[\dot{t}\right] > 0$ and not exporting. The probability of exporting, α, can take on any value in $[0,1]$. For all $\alpha \in [0,1]$, P and y are constant and equal to their autarky levels $\dot{P}$ and $\dot{y}$. As α rises, n falls while the number of consumed varieties, $(1+\alpha)\,n[\alpha]$, strictly increases.*

3. *For $(t,r) \in \underline{T}_F^f$, all firms export. Equilibrium values are the same as in the model with fixed cost $F+f$ of operating and no fixed cost of exporting.*

Proposition 7 shows that, at the initiation of trade, there exists a continuum of equilibria. Along this continuum, domestic varieties are replaced by foreign varieties, while firms increase in size and decrease in number. Nevertheless, the total number of varieties consumed increases.

Recall that $\dot{t}_F^f[r]$ denotes the threshold level of trade costs under tariff r. That is, $\dot{t}_F^f[r]$ is such that $\left(\dot{t}_F^f[r], r\right) \in \dot{T}_F^f$. By analogy, $\dot{t}_F^0[r]$ denotes the threshold level of trade costs under tariff r in the (symmetric) baseline model—i.e., with fixed cost F of operating but *without* fixed costs of exporting. Finally, $\dot{t}_{F+f}^0[r]$ refers to the threshold trade cost under r in a model with fixed cost of operating $F+f$ but no fixed cost of exporting.

In the following Remark, we rank these thresholds that measure how far trade costs must fall under a given tariff for trade to be initiated. The remark implies that the locus $\dot{T}_F^f$ in (t, r)-space lies strictly to the 'left' of $\dot{T}_{F+f}^0$ which, in turn, lies strictly to the left of $\dot{T}_F^0$. For ease of exposition, here, we allow t to take on values in $(0, \infty)$.[22] Allowing for iceberg costs smaller than 1 ensures existence of each of the thresholds $\dot{t}$ for all $r \in (0,1)$, but otherwise it does not play a role.

Remark 2 *Let $t \in (0, \infty)$. For every $r \in (0,1)$,*

$$\dot{t}_F^f[r] < \dot{t}_{F+f}^0[r] < \dot{t}_F^0[r] \quad . \tag{43}$$

The second inequality in (43) follows immediately from the observation that, without fixed cost of exporting, the threshold index $\dot{\Phi}$ is strictly decreasing in the fixed costs of production (see the last paragraph of Section 2.1). The first inequality follows from the easily verified fact that, at $\left(\dot{t}_F^f[r], r\right) \in \dot{T}_F^f$, the equilibrium with $\alpha = 1$ is also the unique equilibrium of the standard model with fixed cost of operating $F+f$ and no fixed cost of exporting. To see that this observation implies $\dot{t}_F^f[r] < \dot{t}_{F+f}^0[r]$, recall that when t drops marginally below $\dot{t}_{F+f}^0[r]$ in the model with fixed cost of operating $F+f$ and no fixed cost of exporting, exports (which are uniquely determined) increase continuously from zero to some small, infinitesimal amount. Since at $t = \dot{t}_F^f[r]$, exports in both models are 'large,' i.e., non-infinitesimal, while exports are monotone in trade costs, it must be that $\dot{t}_F^f[r] < \dot{t}_{F+f}^0[r]$.

Turning to Bad Trade, we first introduce critical tariffs in the context of fixed costs of exporting. Recall that the critical tariff in the symmetric model *without* fixed cost of exporting is $r^* = 1 - \varepsilon_v[\dot{y}/L]$. Under this tariff, trade is initiated at iceberg cost $\dot{t}_F^0[r^*]$, which solves $t/(1-r^*) = v'[0]/m[\dot{y}/L]$. In the model with $f > 0$ and $t \in [1, \infty)$, define the 't-critical tariff' as

$$r_F^{f\,*}[t] \equiv 1 - \frac{\varepsilon_v[\dot{y}/L]}{\varepsilon_v[\tilde{x}[t]/L]} \,,^{23} \tag{44}$$

[22] Technically, iceberg costs smaller than 1 pose no particular problem. Economically, they mean that shipping abroad is cheaper than shipping domestically.

Under tariff 'strategy' ${r_F^f}^* [t]$, which depends on t, trade is initiated at an iceberg cost solving

$$\frac{t}{1 - {r_F^f}^* [t]} = \frac{m [\tilde{x} [t] / L]}{m [\dot{y} / L]} \ . \tag{45}$$

This follows from Proposition 7 and the definition of $\dot{T}_F^f$ in (41). Denote a solution to (45) by ${\dot{t}_F^f}{}^*$, while the associated t-critical tariff is ${\dot{r}_F^f}{}^*$; i.e., ${\dot{r}_F^f}{}^* \equiv {r_F^f}^* \left[{\dot{t}_F^f}{}^*\right]$. We refer to this fixed point ${\dot{r}_F^f}{}^*$ as *the* critical tariff with fixed cost f of exporting. Finally, let ${\dot{\tau}_F^f}{}^* \equiv \left({\dot{t}_F^f}{}^*, {\dot{r}_F^f}{}^*\right)$.

Lemma 4 *${\dot{\tau}_F^f}{}^*$ exists and is unique. Furthermore, ${\dot{t}_F^f}{}^* > 1$ and $0 < {\dot{r}_F^f}{}^* < 1$.*

Having established existence and uniqueness of the critical tariff ${\dot{r}_F^f}{}^*$, we now show that Bad Trade is robust to the introduction of a fixed cost of exporting. In fact, the drop in utility at the initiation of trade becomes *discontinuous* and, thus, it is no longer 'small.'

Proposition 8 (Bad Trade—Fixed Cost of Exporting) *In the model with fixed costs of exporting, the initiation of trade due to a fall in φ leaves both countries strictly worse (better) off if and only if tariffs are strictly smaller (greater) than ${\dot{r}_F^f}{}^*$.*

Formally,

$$\left.\frac{dU}{d\alpha}\right|_{\tau = \dot{\tau}} \overset{(<)}{\underset{(>)}{=}} 0 \iff r \overset{(<)}{\underset{(>)}{=}} {r_F^f}^* \ .$$

Recall the definition of ${r_F^f}^*$ in (44). Proposition 8 says that the welfare effect of the initiation of trade hinges on whether

$$\varepsilon_v [\dot{y}/L] \overset{(<)}{\underset{(>)}{=}} (1 - r) \, \varepsilon_v \left[\tilde{x}\left[\dot{t}_F^f [r]\right] / L\right] \ . \tag{46}$$

Condition (46) is essentially the same as (23) in the main text, except that now $x \nrightarrow 0$, because $f > 0$. This clarifies that, at the initiation of trade, $x = 0$ is a sufficient but not a necessary condition for Bad Trade. It suffices that the number of infra-marginal units of x—and hence consumer surplus—are smaller than $\dot{y}$.

Proposition 8 can be viewed as a general equilibrium version of Venables (1982), with two-way trade in differentiated goods. Inequality (46) is, in essence, the condition found by Venables (1982) for when displacement of domestic varieties by foreign varieties is welfare decreasing. Both in Venables (1982) and along the continuum of equilibria at $\dot{\tau}[r] = \left(\dot{t}_F^f [r], r\right)$, conditional on being active in a market, firms' prices and quantities in that market remain unchanged. The same holds for the price level P—a role taken on in Venables' model by the marginal rate of substitution between the differentiated goods and the numeraire. Our general equilibrium analysis reveals that the initiation of trade is *universally* deleterious, because domestic variety drops in both countries.

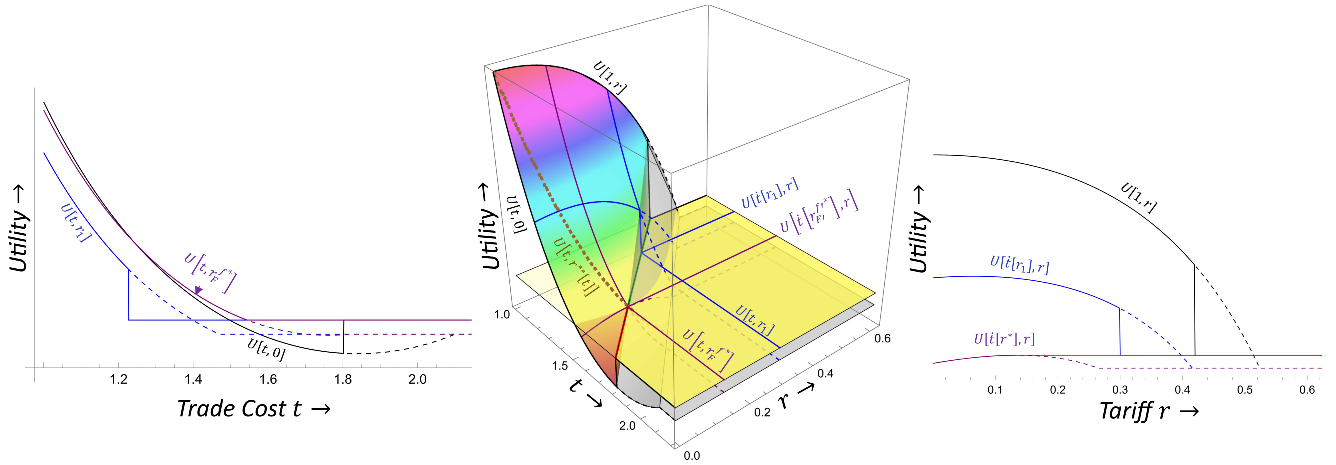

Figure 5: The figure depicts utility as a function of trade costs t for various tariff levels r (left panel); as a function of tariff r for various levels of trade costs t (right panel); and as a function of both t and r (center panel). See Example 5 for details.

Example 5 *Reconsider Example 1. We reduce the fixed cost of operating from $F = 5 \times 10^5$ to $F' = 4.8 \times 10^5$ and introduce a fixed cost of exporting $f' = 20 \times 10^4$, such that $F' + f' = F$. The new critical tariff is $r_{F'}^{f'*} = 0.13$, which is strictly smaller than $r_{F'+f'}^{0*} = 0.17$. Let $r_1 \equiv 0.3 > r_{F'}^{f'*}$ denote a tariff higher than the critical one.*

Figure 5 depicts utility as a function of (symmetric) trade costs and tariffs. Dashed lines and, in the middle panel, the gray surface indicate utility levels in Example 1, for reference.

The shift in fixed costs from operating to exporting implies that, for any level of tariffs, trade costs need to fall farther for trading to begin; i.e., $\dot{t}_{F'}^{f'}[r] < \dot{t}_{F'+f'}^{0}[r]$, for all r. However, once trade is initiated, the equilibrium values are exactly the same as those in Example 1—and so is the optimal tariff $r^{**}[t]$ denoted by the brown dotted line in the middle panel. When countries levy the critical tariff $r_{F'}^{f'*}$, there is no first-order change in utility at the initiation of trade. For all higher (lower) tariff levels, utility jumps up (down) discretely.

CES utility

Fixed costs of exporting also allow us to study CES utility. So far we have assumed that individual varieties have finite choke prices. CES violates this assumption, because the marginal utility at zero is infinite. Infinite choke prices imply that households want to consume every variety no matter the price. Without fixed cost of exporting, this means that trade takes place for all iceberg costs $t \in [1, \infty)$ and tariffs $r \in [0, 1)$, precluding studying its *initiation*. In that case, utility is monotonically decreasing in t and r.

With fixed costs of exporting, the welfare effect of the initiation of trade hinges on whether $\varepsilon_v \left[\frac{\dot{y}}{L}\right] \lesseqgtr (1-r)\, \varepsilon_v \left[\frac{\tilde{x}[t]}{L}\right]$. For CES preferences, $\varepsilon_v[\cdot]$ is constant. Hence,

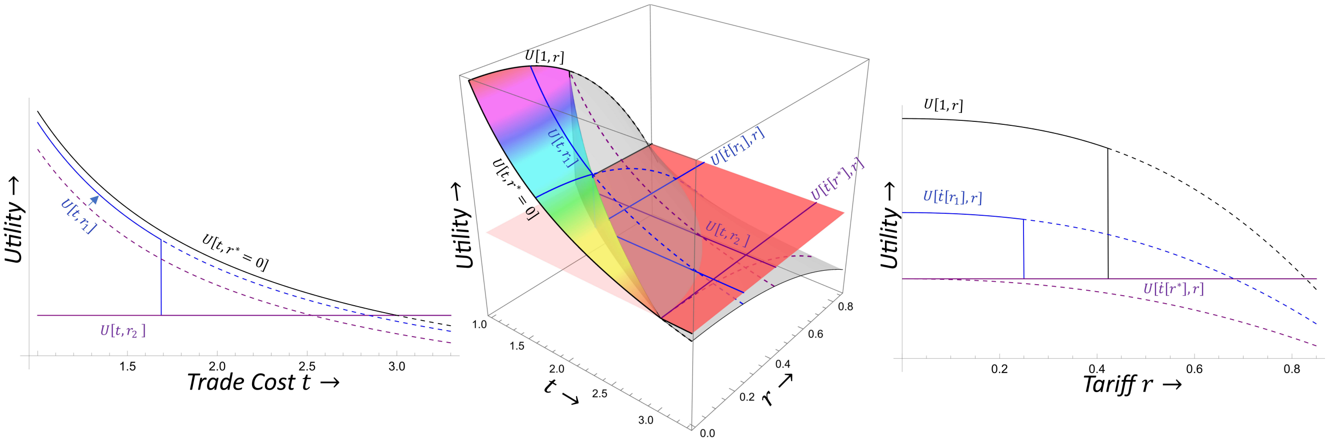

Figure 6: The figure depicts utility as a function of trade cost t for various tariff levels r (left panel); as a function of tariff r for various levels of trade costs t (right panel); and as a function of both t and r (center panel).See Example 6 for details.

Remark 3 *With CES preferences and fixed cost of exporting,*

$$\left.\frac{dU}{d\alpha}\right|_{\tau=\dot{\tau}} \gtreqless 0 \iff r \gtreqless 0 \; .$$

Remark 3 implies that, with CES preferences, the critical tariff is zero. This is a singularity, and it can be viewed as cautioning against an over-reliance on CES.

Example 6 *Reconsider Example 5 with CES preferences (i.e., $\gamma = 0$) and fixed costs of operating and exporting equal to $F'' = 7.5{\times}10^5$ and $f'' = 2.5{\times}10^5$, respectively. With CES preferences, the critical tariff is $r^* = 0$, for all $F'', f'' > 0$. Define $r_1 \equiv 0.25$ and $r_2 \equiv \dot{r}^{f''}_{F''}[1] = 0.42$.*

Figure 6 depicts utility as a function of trade costs and tariffs. The panels are analogous to those of Figure 5. For reference, dashed lines and, in the middle panel, the gray surface indicate utility in a model with $F = F'' + f''$ and $f = 0$.

Two singular properties of CES utility are apparent. First, without fixed cost of exporting, trade occurs for all $t \in [1, \infty)$ and $r \in [0, 1)$. This is due to infinite marginal utility at zero. Second, with fixed cost of exporting, trade is initiated at finite trade costs, but the critical tariff is zero ($r^* = 0$).

Decreasing elasticity

Fixed cost of exporting and globally *decreasing* $\varepsilon_{v'}$ (and, hence, increasing ε_v) imply strictly increasing utility at the initiation of trade. However, as Krugman (1980) observes, "increasing elasticity of demand when the variety of products grows [which is equivalent to *increasing* $\varepsilon_{v'}$] seems plausible, since the more finely differentiated are the products, the better substitutes they are likely to be for one another. Thus an increase in scale as well as diversity is probably the 'normal' case. The constant elasticity case, however, is much easier to work with."

6.3 Heterogeneous firms

Finally, we generalize the baseline model to allow for heterogeneous firms. For ease of exposition, we focus on symmetric countries and one-dimensional heterogeneity. Neither is essential.[24] We show that the Bad Trade result of Proposition 2 holds for any finite type space of marginal costs. A finite type space ensures that a marginal reduction of trade frictions φ below the prohibitive level $\dot{\varphi}$ has a first-order effect on exports. By contrast, with a continuum of zero-measure types (and no fixed cost of exporting), all marginal effects are zero.

The model is as follows. Potential producers in country $k \in \{A, B\}$ must 'pre-pay' their cost F of operating, to find out their marginal cost $c \in \mathrm{C} \equiv \{c_1, ..., c_h\}$. Here, $h \in \mathbb{N}$ and $c_1 < c_2 < ... < c_h$. Let $\mathbf{g} \equiv [g_{c_1}, ..., g_{c_h}]$ denote the vector of probabilities that a randomly drawn marginal cost is equal to $c \in \mathrm{C}$. Hence, $\sum_{c \in \mathrm{C}} g_c = 1$. The (endogenously determined) highest marginal cost type that produces for the domestic market is denoted by $\bar{c}_y$, i.e., $\bar{c}_y \equiv \max\{c \in \mathrm{C} | \, y > 0\}$. Similarly, the highest cost type that produces for the export market is denoted by $\bar{c}_x$. Expected production costs are

$$C[\mathbf{y}, t\mathbf{x}] \equiv F + \sum_{c=c_1}^{\bar{c}_y} c y_c g_c + t \sum_{c=c_1}^{\bar{c}_x} c x_c g_c \ .$$

where $\mathbf{y} \equiv \left[y_{c_1}, ..., y_{\bar{c}_y}\right]$, $\mathbf{x} \equiv [x_{c_1}, ..., x_{\bar{c}_x}]$ denote production strategies, one for each possible marginal cost realization.

A symmetric equilibrium (P, n) solves, for $c \in \mathrm{C}$,

$$\begin{array}{lll}
FOC_{y_c}: & p[y_c, P] + \frac{\partial p[y_c, P]}{\partial y_c} y_c \overset{(\leq)}{=} c & \text{if} \quad y_c \overset{(=)}{>} 0 \\
FOC_{x_c}: & s[x_c, P] + \frac{\partial s[x_c, P]}{\partial x_c} x_c \overset{(\leq)}{=} \frac{t}{1-r} c & \text{if} \quad x_c \overset{(=)}{>} 0 \\
ZP: & \sum_{c=c_1}^{\bar{c}_y} p[y_c, P] y_c g_c + (1-r) \sum_{c=c_1}^{\bar{c}_x} s[x_c, P] x_c g_c = C[\mathbf{y}, t\mathbf{x}] & \\
LM: & nC[\mathbf{y}, t\mathbf{x}] = L \ . &
\end{array}$$

Wages have been normalized to 1, and BP has disappeared because it is automatically satisfied under symmetry. As usual, BB is redundant.

While we have a proof of existence, including for the asymmetric model, it has been omitted from the paper. At the initiation of trade,

$$\varphi = \frac{t}{1-r} = \frac{v'[0]}{m[\dot{y}_{c_1}/L]} \equiv \dot{\varphi}_{c_1} \ , \tag{47}$$

while trade takes place iff $\varphi < \dot{\varphi}_{c_1}$. The locus of $\tau = (\tau, r)$ for which (47) holds with equality is denoted by $\dot{T}_1$. An element on that locus is $\dot{\tau}_1 = \left(\dot{t}[r], r\right)$. The upper and lower contour sets are $\bar{T}_1$ and $\underline{T}_1$, respectively. Fix a starting point $\dot{\tau}_1 \equiv \left(\dot{t}[r], r\right) \in \dot{T}_1$ and move in the direction

[24] Extending the model to allow for asymmetric countries and heterogeneity in fixed as well as marginal costs is straightforward but tedious.

of $\Delta\tau = (\Delta t, \Delta r)$. Scaling $\Delta\tau$ by $\sigma \in [-1, 1]$, the new value of τ is $\tau = \dot{\tau}_1 + \sigma\Delta\tau$. Direction $\Delta\tau$ is 'permissible' iff it is such that, when σ is small and negative, we are in the trade region $\underline{T}_1$, and when σ is small and positive, we are in the no-trade region $\bar{T}_1$. Since this means that $\varphi \equiv \frac{t}{1-r} \overset{(<)}{\underset{(>)}{=}} \varphi$ iff $\sigma \overset{(<)}{\underset{(>)}{=}} 0$, the set of permissible directions $\Delta\tau$ at $\dot{\tau}_1$ is given by

$$D[\dot{\tau}_1] = \left\{ \Delta\tau \in \mathbb{R}^2 \middle| \frac{1}{\dot{t}[r]}\Delta t + \frac{1}{1-r}\Delta r > 0 \right\} .$$

Define

$$r^*_{\mathbf{g}} \equiv 1 - \varepsilon_v\left[\frac{\dot{\mathbf{y}}}{L}\right] \text{ and } \varepsilon_v\left[\frac{\dot{\mathbf{y}}}{L}\right] \equiv \frac{\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{\dot{y}_c}{L}\right]\frac{\dot{y}_c}{L} g_c}{\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{\dot{y}_c}{L}\right] g_c} , \tag{48}$$

where $\dot{\mathbf{y}}$ denotes the vector of autarky quantities $\left[\dot{y}_{c_1}, ..., \dot{y}_{\bar{c}_y}\right]$ and $r^*_{\mathbf{g}}$ is the critical tariff.

Proposition 9 (Bad Trade—Heterogeneous Firms) *In the model with heterogeneous firms, the initiation of trade due to a fall in frictions φ leaves both countries strictly worse (better) off if and only if the tariff is strictly smaller (greater) than the critical value $r^*_{\mathbf{g}}$.*

Formally, for all $\dot{\tau}_1 \in \dot{T}_1$ and $\Delta\tau \in D[\dot{\tau}_1]$,

$$\overleftarrow{\nabla}_{\Delta\tau} U\Big|_{\tau=\dot{\tau}_1} \overset{(>)}{\underset{(<)}{=}} 0 \iff r \overset{(<)}{\underset{(>)}{=}} r^*_{\mathbf{g}} .$$

The proof of Proposition 9 is analogous to the proof of Proposition 2. The intuition is also the same.

Melitz

With heterogeneous firms, fixed costs of exporting, and CES utility, the model closely resembles a standard Melitz model. Other than the finite type space, the only difference is that Melitz has an additional fixed cost for servicing the domestic market. CES utility implies that the initiation of trade without tariffs has no effect on household utility. To see this, let $v[z] = z^\rho$, $0 < \rho < 1$, and notice that $\varepsilon_v[\dot{\mathbf{y}}/L]$ in (48) reduces to ρ, which is in turn equal to $\varepsilon_v[\tilde{x}_{c_1}/L]$. The claim then follows from (46).

7 Related Literature

We have uncovered three contingencies in Krugman's (1979) seminal argument for gains from trade in the absence of comparative advantage: (1) Generalizing his symmetric model to asymmetric countries, we found that, with large differences in marginal costs, the more productive country is better off in autarky than in free and costless trade. (2) Examining what happens in between the two extremes of autarky and free and costless trade, we showed that the welfare of both countries falls at the initiation of trade. (3) By disentangling (policy-driven) tariffs from (technologically-driven) trade costs, we saw that both countries may be better off in autarky than under full liberalization (i.e., zero tariffs). We also provided a solution for 'Bad Trade' by deriving the lowest unilateral import tariffs ensuring that trade, when it occurs, is always 'good.' Finally, we proved equilibrium existence for (a generalized version of) Krugman's model, a result missing from the literature. Uniqueness of equilibrium remains an open question; we have neither a proof nor a counter example.

While we know of no other work showing harmful free and costless trade in general equilibrium, others have observed the harmfulness of the *initiation* of trade between symmetric. Chen and Zeng (2014) numerically show but do not explain the phenomenon (see their Figure 7). In a symmetric, single-sector model without tariffs, Bykadorov *et al.* (2016) and Kokovin *et al.* (2022) derive the result. In addition to generalizing to asymmetric countries and multiple sectors, we provide a straightforward intuition that resonates politically: little-consumer-surplus-generating imports crowd out beloved (i.e., large-surplus-generating) domestic varieties.[25]

Our analysis also establishes the robustness of 'Bad Trade,' by showing that even full liberalization may fail to solve the problem: Not only can both countries be worse off in a zero-tariff world than in autarky, one country may continue to lose out relative to autarky, even if international trade costs, including for transport, were to magically disappear. Furthermore, when a good or service becomes newly tradable due to a fall in sector-specific trade costs, then both countries' welfare drops regardless of trade volumes or trade costs in all other sectors of the economy.

As to disentangling tariffs from trade costs, since Samuelson (1952), the theoretical literature has tended to bundle the two into a single 'iceberg cost.' However, two recent exceptions are notable. Contrasting the effect of cost-shifting iceberg costs to demand-shifting and revenue-generating tariffs, Felbermayr *et al.* (2015) show that Arkolakis *et al.*'s (2012) formula to compute the welfare gains of trade underestimates them: gains from trade are greater with tariffs than with similar-sized iceberg costs. Demidova's (2017) model with heterogenous firms also distinguishes between revenue-generating tariffs and cost-shifting trade costs. Her quadratic utility function implies that the aggregate quantity of goods consumed is a sufficient statistic for per capita utility—i.e., the effect of changes in the number of varieties, which drives our results, is absent. She derives the optimal unilateral tariff in this setting and shows that a drop in cost-shifting trade costs always increases welfare when tariffs are absent.

To put our existence proof in perspective, recall that existence of equilibrium for the monopolistic competition model of Dixit and Stiglitz (1977) was only formally proved by Zhelobodko *et al.* (2012).[26] Our paper fills an analogous hole for the New Trade model of Krugman (1979).

[25] The record shows that Bykadorov et al. (2016) were the first to post the mathematical result. However, in Morgan et al. (2020), we were the first to give an economic intuition.

[26] Zhelobodko *et al.* also show that increasing elasticity of marginal utility (in absolute value) gives rise to

In addition to the papers already mentioned, our paper is perhaps most closely related to Venables (1982), Brander and Krugman (1983), Venables (1985), Bulow *et al.* (1985), Melitz (2003), Hsieh *et al.* (2016), Dhingra and Morrow (2019), and Arkolakis *et al.* (2019).

Venables (1982) studies a small price-taking economy. It exports a homogeneous commodity produced competitively under non-increasing returns to scale. It imports differentiated goods that compete with a local monopolistically competitive industry. Firms in this industry cannot export, by assumption. Therefore, trade forces local varieties out of the market. Venables studies the welfare implications of this displacement. Foreclosing general equilibrium effects, he shows that trade increases welfare if and only if the elasticity of utility of the foreign variety is smaller than that of the domestic variety it displaces.[27]

Our paper can be interpreted as a general-equilibrium version of Venables (1982), with two-way trade in differentiated goods. We show that, at the initiation of trade, firms increase in size while total profits remain unchanged. This implies that the number of firms must go down. Even though there are many foreign varieties that enter for each domestic variety that exits, we establish that the trade-off is always welfare-decreasing. Thus, compared to Venables (1982), the contributions of the current paper are as follows. First, we identify a situation (i.e., the initiation of trade) where displacement of domestic varieties is both 'unforced' and occurs in general equilibrium. Second, we show that domestic welfare effects of displacement are always negative. Third, we establish that trade is *universally* deleterious, because displacement takes place not only at home, but also in the rest of the world. Fourth, we extend the analysis to an arbitrary number of monopolistically competitive sectors.

With the number of firms fixed exogenously, the initiation of trade is also universally harmful in the 'reciprocal dumping' models of Brander and Krugman (BK, 1983) and Venables (1985). (The intuition is essentially the same as in Bulow, Geanakoplos, and Klemperer, 1985, discussed below.) However, recall that free entry undoes this finding: when the number of firms is endogenously determined, trade is unambiguously welfare improving.

Not withstanding the reverse chronology, Krugman (1979) can be viewed as taking BK's imperfect-competition framework and adding heterogeneous goods and a taste for variety. These additional ingredients do not change the *gains* from trade which, around the prohibitive level, are second-order in both models. They do change the *losses*, however. The reason gains are second order is that households only enjoy consumer surplus on infra-marginal units, while, at the initiation of trade, no such infra-marginal units exist for foreign varieties. In both models, the initiation of trade also leads to fewer and larger firms. While fewer firms has no utility

intuitive, pro-competitive effects of market entry—i.e., lower prices and mark-ups—while decreasing elasticity does the opposite.

[27] With symmetric CES preferences, notice that the elasticity is the same at all consumption levels and across all varieties. Hence, the replacement has no net effect on utility. In fact, this is in line with the findings of Hsieh *et al.* (2018), which we discuss in more detail below.

implications *per se* in the homogeneous-goods world of BK, it strictly harms households in the heterogeneous-goods world of Krugman (1979). The reason is that, in the latter, fewer firms means less domestic variety. Since each domestic variety is consumed in non-infinitesimal quantities, their disappearance does entail the loss of infra-marginal units and, thus, a first-order loss in utility.

Melitz (2003) extends Krugman (1980) to allow for heterogeneous marginal costs across firms. In the standard Melitz model with CES preferences, fixed cost of exporting, and a continuum of marginal cost types, utility is invariant to the initiation of trade (see Section 6.3). With increasing elasticity of marginal utility, 'Bad Trade' also arises in Melitz. If we eliminate the fixed cost of exporting, both the set of firms and the quantity exported have zero measure at the initiation of trade. Hence, there are no first-order effects at all. Assuming a finite type space brings back 'Bad Trade.'

Hsieh *et al.* (2018) decompose the gains from trade in the Melitz model into 'new' trade gains, which arise from changes in the set of firms serving a country, and 'traditional' trade gains resulting from price reductions on existing varieties. They find that the 'new' gains exactly offset the utility losses from domestic varieties being pushed out of the market. (This is consistent with Venables (1982)—see footnote 27.) Applying their model to the Canada-US Free Trade Agreement, they estimate that 'new' gains were in fact negative for Canada, while 'traditional' gains more than made up for these losses. This aligns with our findings, at least for symmetric countries: even though welfare losses dominate gains at the initiation of trade, trade cost savings on infra-marginal units eventually overcome the welfare losses if iceberg costs fall far enough. Hence, free and costless trade dominates autarky for both countries. For asymmetric countries, this may no longer be true. As we have shown, the lower-cost country can be better off in autarky than in free and costless trade.

While CES utility has long been the norm in the Trade literature, there now exist a fair number of papers that emphasize its limitations.[28] These include Dhingra and Morrow (2019) and Arkolakis *et al.* (2019), among others. Dhingra and Morrow (2019) compare and contrast CES with utility functions that give rise to variable mark-ups over marginal cost. Essentially, they extend the original Dixit and Stiglitz (1977) analysis of product variety under monopolistic competition to heterogeneous firms and show that the market generates optimal variety iff utility is CES. The intuition is that CES generates constant markups, which makes prices proportional to marginal costs. Since utility maximization implies that prices are also proportional to marginal utility, marginal utility is proportional to marginal costs. Notice, however, that proportionality of marginal utility and cost is precisely what social optimality requires. Finally, observe that this argument does not depend on marginal costs being identical across firms.

[28] See Krugman (1979, 1980) and Zhelobodko *et al.*, 2012, for arguments why increasing elasticity is, in fact, the 'normal' case.

Outside of CES, Dhingra and Morrow show that the market generates multiple distortions: in the 'normal' case of increasing elasticity of marginal utility, high-cost firms produce too much, while low-cost firms produce too little. At the same time, there is too much entry in the sense that some high-cost firms that should stay out choose to enter the market.[29]

Arkolakis *et al.* (2019) study the effect of variable markups on the gains from trade liberalization, relative to the gains in the constant markup world of CES. In the 'normal,' increasing elasticity case, trade liberalization reduces markups on domestic goods, because competition is increased (see Zhelobodko *et al.*, 2012, above). At the same time, markups on imported goods go up, because foreign firms pass on only part of the savings in trade costs. The authors show that the latter effect dominates the former such that, with variable markups and increasing elasticity, gains from trade liberalization are smaller than with CES. Notice that this is consistent with our finding. Nevertheless, while gains remain strictly positive in Arkolakis *et al*, we find that they are strictly negative at the initiation of trade. Gains remain positive in their model, because they abstract from the change in varieties. By contrast, changing varieties are a critical element in our analysis.[30]

Our finding that some trade is worse than no trade is reminiscent of Bulow, Geanakoplos, and Klemperer (BGK, 1985), who show that a bit of competition can reduce welfare. BGK study quantity competition in homogeneous products and ask what happens when a firm with marginal costs just below the prevailing price enters a market dominated by an incumbent monopolist. They find that entry, and thus competition, reduces welfare when quantities are strategic substitutes—as they are, for example, with linear demand. To see why, notice that the entrant produces a small amount at a cost almost equal to the social value of his output. Hence, the (positive) direct effect of entry on surplus is second-order. With strategic substitutes, the incumbent responds by decreasing output. This has a first-order negative effect on consumer surplus, equal to the reduction in the incumbent's output times the price minus his marginal cost. Since the negative first-order effect dominates the positive second-order effect, competition reduces welfare.

The literature has identified other negative effects of trade. For example, trade may damage global working conditions (Davies and Vadlamannati, 2013) and the natural environment (Esty, 2001); it can create economic dependencies that reduce a small country's bargaining power (McLaren, 1997)—an argument national security hawks parlay into protectionist measures; and

[29] Dhingra and Morrow distinguish between utility functions that give rise to "aligned incentives" versus "misaligned incentives," depending on whether the derivatives of the elasticities of utility and marginal utility take on the same sign. Subsequently, they focus on the case of aligned incentives. In Lemma A.2 in the Appendix, we show that incentives are in fact always aligned, provided elasticities are monotone. Hence, their restricting attention to this case is actually without loss of generality.

[30] They write: "Our baseline analysis (...) abstracts from welfare gains from new varieties, because of our focus on small changes in variable trade costs, and from changes in the distribution of markups, because of our focus on Pareto distributions." Cf., our heterogenous firms model in Section 6.3: unlike Arkolakis *et al.*, we focus on a finite type space, such that a small change in variable trade costs does materially change the number of varieties.

if trade drives an infant industry offshore, then intellectual property, learning-by-doing, and other positive spillovers may go with it (Melitz, 2005). In each case, an externality grafted onto the standard model is to blame. By contrast, in this paper we identify an externality in the standard model itself, namely, the trade-induced fall in domestic variety.

In other cases, apparent trade-induced welfare losses stem from a partial equilibrium perspective. For example, the popular notion that trade destroys domestic jobs ignores the fact that new, more productive employment is created in export sectors, eventually more than offsetting initial job losses. Relatedly, as highlighted above, in BK's and Venables' 'reciprocal dumping' models, trade can reduce welfare *unless* firms can enter and leave the economy freely. In these cases, general equilibrium or a total accounting of winners and losers restores the net benefits of trade.

Our Bad Trade results are different: First, they survive general equilibrium dynamics. Second, the underlying externality has been lurking in the seminal New Trade model for forty years.

8 Conclusion

We have studied Krugman's (1979) seminal model away from symmetry and the extremes of autarky or free and costless trade. Relaxing symmetry, we have derived a sufficient condition for a country to prefer autarky over free and costless trade. In the absence of tariffs, we showed that the initiation of trade leaves both countries strictly worse off, even in the symmetric model. With multiple sectors, 'Bad Trade' holds on a sector-by-sector basis and irrespective of the economies' aggregate distance from autarky. Furthermore, variety drops in all sectors, not just the newly tradable one. Simple import tariffs solve these problems. They ensure that trade is always 'good' for both countries.

To some extent, our results rationalize anti-globalists' complaints about the homogenizing effects of economic integration. However, our findings challenge only one of many justifications for free trade. For example, none of the models we have studied allow countries to exploit comparative advantage. Also, the magnitude of the concerns raised by 'Bad Trade' remains an open empirical question.

References

[1] Akande, W. (2002). The drawbacks of cultural globalization. Available at: https://www.globalpolicy.org/component/content/article/162/27594.html.

[2] Arkolakis, C., Costinot, A., and Rodriguez-Clare, A. (2012). New trade models, same old gains? *American Economic Review* 102(1), 94-130.

[3] Arkolakis, C., Costinot, A., Donaldson, D., and Rodriguez-Clare, A. (2019). The elusive pro-competitive effects of trade. *The Review of Economic Studies* 86(1), 46-80.

[4] Bulow, J. I., Geanakoplos, J. D. and Klemperer, P.D. (1985). Multimarket oligopoly: strategic substitutes and complements. *Journal of Political Economy* 93(3), 488-511.

[5] Brander, J., and Krugman, P. (1983). A 'reciprocal dumping' model of international trade. *Journal of International Economics* 15(3), 313-321.

[6] Bykadorov, I., Ellero, A., Funari, S., Kokovin, S. and Molchanov, P. (2016). Painful birth of trade under classical monopolistic competition. Available at: https://dx.doi.org/10.2139/ssrn.2759872.

[7] Chamberlain, E. (1933). *The theory of monopolistic competition.* Cambridge, MA: Harvard University Press.

[8] Chen, C. and Zeng, D. (2014). Home market effects:Beyond the constant elasticity of substitution. Available at SSRN: https://ssrn.com/abstract=2538066.

[9] Davies, R.B. and Vadlamannati, K.C. (2013). A race to the bottom in labor standards? An empirical investigation. *Journal of Development Economics* 103, 1-14.

[10] Demidova, S. (2017). Trade policies, firm heterogeneity, and variable markups. *Journal of International Economics* 108, 260-273.

[11] Dhingra, S. and Morrow, J. (2019). Monopolistic competition and optimum product diversity under firm heterogeneity. *Journal of Political Economy* 127(1), 196-232.

[12] Dixit, A.K., and Stiglitz, J.E. (1977). Monopolistic competition and optimum product diversity. *The American Economic Review* 67(3), 297-308.

[13] Esty, D. C. (2001). Bridging the trade-environment divide. *Journal of Economic Perspectives* 15(3), 113-130.

[14] Felbermayr, G., Jung, B., and Larch, M. (2015). The welfare consequences of import tariffs: A quantitative perspective. *Journal of International Economics* 97(2), 295-309.

[15] Hsieh, C.T., Li, N., Ossa, R. and Yang, M. (2016). Accounting for the new gains from trade liberalization. National Bureau of Economic Research No. w22069.

[16] Kokovin, S., Molchanov, P., and Bykadorov, I. (2022). Increasing returns, monopolistic competition, and international trade: Revisiting gains from trade. *Journal of International Economics* 137.

[17] Krugman, P. R. (1979). Increasing returns, monopolistic competition, and international trade. *Journal of international Economics* 9(4), 469-479.

[18] Krugman, P. R. (1980). Scale economies, product differentiation, and the pattern of trade. *The American Economic Review* 70(5), 950-959.

[19] Lipsey, R. and Lancaster, K. (1956), The general theory of second best. *The Review of Economic Studies* 24(1), 11-32.

[20] McLaren, J. (1997). Size, sunk costs, and Judge Bowker's objection to free trade. *The American Economic Review* 87(3), 400-420.

[21] Melitz, M. J. (2003). The impact of trade on intra-industry reallocations and aggregate industry productivity. *Econometrica* 71(6), 1695-1725.

[22] Melitz, M. J. (2005). When and how should infant industries be protected? *Journal of International Economics* 66(1), 177-196.

[23] Morgan, J. and Tumlinson, J., and Várdy, F. (2020), Bad trade: The loss of domestic variety. Available at: http://dx.doi.org/10.2139/ssrn.3529246.

[24] Nobel Committee. Press release on the Sveriges Riksbank Prize in Economic Sciences in memory of Alfred Nobel 2008. Royal Swedish Academy of Sciences. 13 October 2008. Available at: www.nobelprize.org/prizes/economic-sciences/2008/press-release.

[25] Samuelson, P.A. (1952). The transfer problem and transport costs: The terms of trade when impediments are absent. *The Economic Journal* 62(246), 278-304.

[26] Venables, A.J. (1982). Optimal tariffs for trade in monopolistically competitive commodities. *Journal of International Economics* 12(3-4), 225-241.

[27] Venables, A. J. (1985). Trade and trade policy with imperfect competition: The case of identical products and free entry. *Journal of International Economics* 19(1), 1-19.

[28] Zhelobodko, E., Kokovin, S., Parenti, M., and Thisse, J. (2012). Monopolistic competition: Beyond the constant elasticity of substitution. *Econometrica* 80(6), 2765-2784.

A Online Appendix: Proofs

A.1 Preliminaries

The following results will be used repeatedly.

Lemma A.1 *At interior extrema y_{i_k}, x_{i_k},*

$$\begin{array}{l} 0 \leq \varepsilon_{v'_k}\left[\frac{y_{i_k}}{L_k}\right] < 1 \\ 0 \leq \varepsilon_{v'_l}\left[\frac{x_{i_k}}{L_l}\right] < 1 \ , \end{array} \tag{49}$$

for $k, l \in \{A, B\}$ and $k \neq l$.

Proof Since $\varepsilon_{v'_k} \equiv -zv''_k[z]/v'_k[z]$, while $0 < v'_k < \infty$ and $-\infty < v''_k < 0$, the non-negativity of the elasticities is immediate.

Next, observe that the FOCs in (5) can be written as

$$\begin{array}{l} FOC_{y_{i_k}} : \varepsilon_{v'_k}\left[\frac{y_{i_k}}{L_k}\right] = \frac{P_k v'_k[y_{i_k}/L_k] - w_k c_k}{P_k v'_k[y_{i_k}/L_k]} < 1 \\ FOC_{x_{i_k}} : \varepsilon_{v'_l}\left[\frac{x_{i_k}}{L_l}\right] = \frac{P_l v'_l[x_{i_k}/L_l] - \varphi_l w_k c_k}{P_l v'_l[x_{i_k}/L_l]} < 1 \ , \end{array}$$

where the inequalities follow from $w_k, c_k > 0$, and $\varphi_l \geq 1$. ■

Lemma A.2 $\varepsilon_{v_k}[\cdot]$ *is non-negative and strictly decreasing, while* $\lim_{z \to 0} \varepsilon_{v_k}[z] = 1$.

Proof Non-negativity of ε_{v_k} follows immediately from its definition and the properties of $v[\cdot]$.

Recall that $\varepsilon_{v'}[\cdot]$ is strictly increasing in a neighborhood of $z = 0$ and non-decreasing everywhere else. Therefore,

$$\begin{aligned} \varepsilon_{v'}[z]\, v[z] &= \frac{-zv''[z]}{v'[z]} v[z] = \frac{-zv''[z]}{v'[z]} \int_0^z v'[\zeta]\, d\zeta > \int_0^z \frac{-\zeta v''[\zeta]}{v'[\zeta]} v'[\zeta]\, d\zeta \\ &= -\int_0^z \zeta v''[\zeta]\, d\zeta = -\left(\zeta v'[\zeta]\big|_0^z - \int_0^z v'[\zeta]\, d\zeta \right) = v[z] - zv'[z] \,, \end{aligned}$$

where we have used that $v[0] = 0$. Rearranging, we find

$$\frac{-zv''[z]}{v'[z]} v[z] > v[z] - zv'[z] \iff \left(v'[z] + zv''[z]\right) v[z] - zv'[z]^2 < 0 \ . \tag{50}$$

Differentiating $\varepsilon_v[z]$ yields

$$\frac{d}{dz} \frac{zv'[z]}{v[z]} = \frac{(v'[z] + zv''[z])\, v[z] - zv'[z]^2}{v[z]^2} < 0 \ ,$$

where the inequality follows from (50). This proves that $\varepsilon_v[\cdot]$ is decreasing.

Finally, using Hopital's rule,

$$\lim_{z\to 0}\varepsilon_v[z]=\lim_{z\to 0}\frac{v'[z]\,z}{v[z]}=\lim_{z\to 0}\frac{v'[z]+v''[z]\,z}{v'[z]}=1\ .$$

■

Define $\varepsilon_{m_k}[z]\equiv z\frac{m_k'[z]}{m_k[z]}$.

Lemma A.3 *1. For $k,l\in\{A,B\}$ and $k\neq l$,*

$$\frac{d}{d\sigma}\frac{m_k[z_k[\sigma]]}{m_l[z_l[\sigma]]}=\frac{m_k[z_k[\sigma]]}{m_l[z_l[\sigma]]}\left(\varepsilon_{m_k}[z_k[\sigma]]\frac{1}{z_k[\sigma]}\frac{dz_k[\sigma]}{d\sigma}-\varepsilon_{m_l}[z_l[\sigma]]\frac{1}{z_l[\sigma]}\frac{dz_l[\sigma]}{d\sigma}\frac{\sigma}{\sigma}\right)$$

Here, $z_k,z_l:[-1,1]\to[0,\infty)$ are differentiable functions, and $\sigma\in[-1,1]$.

2. For $z\geq 0$,

$$\frac{d}{dz}\left[\frac{\varepsilon_{v_k'}[z]}{1-\varepsilon_{v_k'}[z]}z\right]=-\frac{\varepsilon_{m_k}[z]}{1-\varepsilon_{v_k'}[z]}\ .$$

3. For $z\geq 0$,

$$z\varepsilon_{v_A'}'[z]=\varepsilon_{v_A'}[z]\left(1+\varepsilon_{v_A''}[z]+\varepsilon_{v_A'}[z]\right)$$

Proof Tedious but trivial. ■

A.2 Equilibrium

In this section we prove existence of equilibrium. The proof of Proposition 1 proceeds in two steps: we separately consider the case where countries live in autarky and where they trade.

A.2.1 Autarky

A country k, $k\in\{A,B\}$, lives in autarky if, in equilibrium, it does not import—i.e., $x_l=0$, $k\neq l$. Clearly, $x_k=0\iff x_l=0$. Otherwise, the balance of payments BP is violated. Hence, if one country lives in autarky, so does the other. Furthermore, if $x_k=0$, then $y_k>0$, since, otherwise, the labor market does not clear (i.e. LM^k cannot be satisfied). An autarky equilibrium is thus characterized by the following system:

For, $k,l\in\{A,B\}$ and $l\neq k$,

$$\begin{aligned}
FOC_y^k:&\quad P_k m_k[y_k/L_k]=w_k c_k\\
FOC_x^k:&\quad P_l v_l'[0]\leq\varphi_l w_k c_k\\
ZP^k:&\quad P_k v_k'[y_k/L_k]\,y_k=w_k C_k[y_k]\\
LM^k:&\quad n_k C_k[y_k]=L_k\ .
\end{aligned}\tag{51}$$

Comparing (51) with (10), we see that: 1) BP has disappeared, as it is automatically satisfied when $x_A = x_B = 0$; 2) the firms' FOCs have been added, making explicit—and replacing—the generic optimal quantities $\hat{y}_k [P_k, w_k]$ and $\hat{x}_k [P_l, w_k]$; 3) using (3), prices p_k,s_k have been replaced by $P_k v'_k \left[\frac{y_k}{L_k}\right]$ and $P_l v'_l [0]$, respectively.

Below, we use '$\dot{}$' to denote autarky values; e.g., $\dot{y}_k > 0$ denotes the autarky output per firm in country k, while $\dot{x}_k = 0$. In the next lemma, we show that the autarky equilibrium system in (51) allows for a solution iff $\Phi \geq \dot{\Phi}$.

Lemma A.4 *Countries live in autarky iff $\Phi \geq \dot{\Phi}$. The boundary, $\dot{\Phi}$, is strictly decreasing in L_k and strictly increasing in F_k/c_k, $k \in \{A, B\}$.*

For $k \in \{A, B\}$, autarky output per firm, $\dot{y}_k$, is the unique solution to

$$\frac{\varepsilon_{v'_k} [y_k/L_k]}{1 - \varepsilon_{v'_k} [y_k/L_k]} y_k = \frac{F_k}{c_k} \ .$$

Furthermore,

$$\begin{aligned} \dot{\left(\frac{P_k}{w_k}\right)} &= \frac{C_k [\dot{y}_k]}{\dot{y}_k} \frac{1}{v'_k [\dot{y}_k/L_k]}, \ \dot{n}_k = \frac{L_k}{C_k [\dot{y}_k]} \ , \\ \textit{and} \qquad \frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{v'_k [0]}{m_k [\dot{y}_k/L_k]} &\leq \dot{\left(\frac{w_l}{w_k}\right)} \leq \varphi_l \frac{c_k}{c_l} \frac{m_l [\dot{y}_l/L_l]}{v'_l [0]} \ . \end{aligned}$$

Only at $\Phi = \dot{\Phi}$ is the wage ratio—and, hence, equilibrium—uniquely determined.

Proof Let $k, l \in \{A, B\}$, $l \neq k$. We begin by deriving autarky production, $\dot{y}_k$.

Dividing ZP^k in (51) by FOC^k_y yields

$$\frac{y_k}{1 - \varepsilon_{v'_k} [y_k/L_k]} = \frac{C_k [y_k]}{c_k} \ . \tag{52}$$

Rearranging,

$$\frac{\varepsilon_{v'_k} [y_k/L_k]}{1 - \varepsilon_{v'_k} [y_k/L_k]} y_k = \frac{F_k}{c_k} \ . \tag{53}$$

Notice that the LHS is strictly increasing in y_k, running from zero at $y_k = 0$ to infinity. Hence, equation (53) has a unique solution, $\dot{y}_k$.

The number of firms in autarky now follows from LM^k :

$$\dot{n}_k = \frac{L_k}{C_k [\dot{y}_k]} \ ,$$

while ZP^k yields for the normalized price level

$$\dot{\left(\frac{P_k}{w_k}\right)} = \frac{C_k[\dot{y}_k]}{\dot{y}_k v_k'[\dot{y}_k/L_k]} \ . \tag{54}$$

It remains to determine the autarky wage ratio. Dividing FOC_x^l by w_k yields

$$\frac{P_k}{w_k} v_k'[0] \le \varphi_k \frac{w_l c_l}{w_k} \ .$$

Substituting (54) and rewriting, we find

$$\frac{1}{\varphi_k} \frac{C_k[\dot{y}_k]}{c_k \dot{y}_k} \frac{c_k}{c_l} \frac{v_k'[0]}{v_k'[\dot{y}_k/L_k]} \le \frac{w_l}{w_k} \ .$$

Applying (52) gives

$$\frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} \le \frac{w_l}{w_k} \ . \tag{55}$$

Repeating these steps with FOC_x^k yields the analogous condition

$$\frac{1}{\varphi_l} \frac{c_l}{c_k} \frac{v_l'[0]}{m_l[\dot{y}_l/L_l]} \le \frac{w_k}{w_l} \ . \tag{56}$$

Combining (55) and (56), we find that the autarky wage ratio must satisfy

$$\frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} \le \dot{\left(\frac{w_l}{w_k}\right)} \le \varphi_l \frac{c_k}{c_l} \frac{m_l[\dot{y}_l/L_l]}{v_l'[0]} \ . \tag{57}$$

Such a wage ratio exists iff

$$\dot{\Phi} = \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} \frac{v_l'[0]}{m_l[\dot{y}_l/L_l]} \le \Phi \ .$$

In Lemma A.5, below, it is shown that $\dot{y}_k/L_k$ is strictly decreasing in L_k and strictly increasing in F_k/c_k. Since $m_k[\cdot]$ is strictly decreasing in its argument, $\dot{\Phi}$ is strictly decreasing in L_k and strictly increasing in F_k/c_k.

For $\Phi > \dot{\Phi}$, the FOC_x^ks are slack. In that case, the autarky wage ratio is not uniquely determined and can take on any value in the interval given by (57). For $\Phi = \dot{\Phi}$, the FOC_x^ks bind and the interval in (57) reduces to a single point

$$\dot{\left(\frac{w_l}{w_k}\right)} = \frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} = \varphi_l \frac{c_k}{c_l} \frac{m_l[\dot{y}_l/L_l]}{v_l'[0]} \ . \tag{58}$$

Hence, only at $\Phi = \dot{\Phi}$ is equilibrium uniquely determined. ■

Lemma A.5 *$\dot{y}_k/L_k$ is strictly decreasing in L_k and strictly increasing in F_k/c_k.*

Proof Multiplying the LHS and RHS of (53) by $1/L_k$ and implicitly differentiating with respect to L_k using the chain rule yields on the LHS

$$\frac{d}{d\left[y_k/L_k\right]}\left[\frac{\varepsilon_{v_k'}\left[y_k/L_k\right]}{1-\varepsilon_{v_k'}\left[y_k/L_k\right]}\frac{y_k}{L_k}\right]\frac{d\left[y_k/L_k\right]}{dL_k}=\frac{d}{dL_k}\left[\frac{F_k}{c_k}\frac{1}{L_K}\right] .$$

Now use $\frac{d}{dz}\left[\frac{\varepsilon_{v_k'}[z]}{1-\varepsilon_{v_k'}[z]}z\right]=-\frac{\varepsilon_{m_k}[z]}{1-\varepsilon_{v_k'}[z]}$ (see Lemma A.3.2) to find

$$-\frac{\varepsilon_{m_k}\left[y_k/L_k\right]}{1-\varepsilon_{v_k'}\left[\dot{y}_k/L_k\right]}\frac{d\left[y_k/L_k\right]}{dL_k}=-\frac{F_k}{c_k}\frac{1}{L_k^2} .$$

Hence,

$$\frac{d\left[y_k/L_k\right]}{dL_k}=\frac{F_k}{c_k}\frac{1}{L_k^2}\frac{1-\varepsilon_{v_k'}\left[y_k/L_k\right]}{\varepsilon_{m_k}\left[y_k/L_k\right]}<0 ,$$

where the inequality follows from $\varepsilon_{m_k}<0$.

Since the LHS of (53) is strictly increasing in y_k, the solution $\dot{y}_k$ is strictly increasing in F_k/c_k. ■

A.2.2 Trade

In an equilibrium with trade, $x_A, x_B > 0$, which means that the FOC_x^ks must hold with equality. However, FOC_y^k may be slack. In that case, $y_k = 0$, and firms in country k produce only for the export market. A trade equilibrium is thus characterized by the system:

For $k, l \in \{A, B\}$, $l \neq k$,

$$\begin{aligned}
FOC_y^k: &\quad P_k m_k\left[y_k/L_k\right] \overset{(\leqq)}{=} w_k c_k \quad \text{if} \quad y_k \overset{(=)}{>} 0\\
FOC_x^k: &\quad P_l m_l\left[x_k/L_l\right]=\varphi_l w_k c_k\\
ZP^k: &\quad P_k v_k'\left[y_k/L_k\right]y_k+\left(1-r_l\right)P_l v_l'\left[x_k/L_l\right]x_k=w_k C_k\left[y_k+t_l x_k\right]\\
LM^k: &\quad n_k C_k\left[y_k+t_l x_k\right]=L_k\\
BP: &\quad n_k\left(1-r_l\right)P_l v_l'\left[x_k/L_l\right]x_k=n_l\left(1-r_k\right)P_k v_k'\left[x_l/L_k\right]x_l .
\end{aligned} \tag{59}$$

The next lemma permits focusing on a self-contained sub-system of equations that is only a function of quantities and the wage ratio.

Lemma A.6 *The system in* (59) *gives rise to the following sub-system of five equations and five unknowns: for,* $k, l \in \{A, B\}$, $l \neq k$,

$$\begin{aligned}
FOC^k: &\quad \varphi_k\frac{c_l}{c_k}\frac{m_k[y_k/L_k]}{m_k[x_l/L_k]} \overset{(\leqq)}{=} \frac{w_k}{w_l} \quad \textit{if } y_k \overset{(=)}{>} 0\\
ZP^k: &\quad \frac{\varepsilon_{v_k'}[y_k/L_k]}{1-\varepsilon_{v_k'}[y_k/L_k]}y_k+\frac{\varepsilon_{v_l'}[x_k/L_l]}{1-\varepsilon_{v_l'}[x_k/L_l]}t_l x_k=\frac{F_k}{c_k}\\
BP: &\quad \frac{L_l}{L_k}\frac{1-\varepsilon_{v_l'}[x_k/L_l]}{1-\varepsilon_{v_k'}[x_l/L_k]}\frac{C_k[y_k+t_l x_k]}{C_l[y_l+t_k x_l]}\frac{t_k c_l x_l}{t_l c_k x_k}=\frac{w_k}{w_l} .
\end{aligned} \tag{60}$$

The five unknowns are: $y_A, x_A, y_B, x_B,$ *and* w_B/w_A.

Proof We eliminate P_k, P_l, and n_k in (59) through a series of substitutions.

Substituting FOC_y^k and FOC_x^k into ZP^k yields

$$ZP^k: \quad \frac{w_k c_k}{1-\varepsilon_{v_k'}[y_k/L_k]} y_k + \frac{t_l w_k c_k}{1-\varepsilon_{v_l'}[x_k/L_l]} x_k = w_k \{F_k + c_k (y_k + t_l x_k)\} \quad .$$

Rewriting yields the form of ZP^k in the lemma. Notice that this substitution holds even when FOC_y^k is slack, since then $y_k = 0$, and the first term in ZP^k disappears.

Similarly, substituting FOC_x^k and LM^k into BP yields

$$BP: \quad \frac{L_k}{C_k[y_k+t_l x_k]} \frac{t_l w_k c_k}{1-\varepsilon_{v_l'}[x_k/L_l]} x_k = \frac{L_l}{C_l[y_l+t_k x_l]} \frac{t_k w_l c_l}{1-\varepsilon_{v_k'}[x_l/L_k]} x_l \quad ,$$

which we then solve for w_k/w_l to get the form of BP in the lemma.

Finally, substituting FOC_x^k into FOC_y^l yields the form of FOC^k in the lemma. ■

We now show for which trade costs the reduced system in (60) has a solution.

Lemma A.7 *The system in (60) has a solution iff* $\Phi \leq \dot{\Phi}$*. Trade takes place (i.e.,* $x_A, x_B > 0$*) iff* $\Phi < \dot{\Phi}$*.*

Proof : " $\Longrightarrow$ ": Here we show that, if $\Phi < \dot{\Phi}$, then the system in (60) has a solution with $x_A, x_B > 0$. To do so, we proceed through the following steps. First, using ZP^k, we show that y_k can be written as a decreasing function of x_k. Next, using FOC^A and FOC^B, we show that when $\Phi \leq \dot{\Phi}$, then x_B can be written as a weakly decreasing and bounded function of x_A. Finally, we show that when $\Phi \leq \dot{\Phi}$, then there exists a wage ratio that satisfies the balance of payments condition.

Consider ZP^A in (60) and notice that its LHS is strictly increasing in both y_A and in x_A. Since its RHS is a constant, this implies that y_A is a strictly decreasing function of x_A, which we denote by $y_A[x_A]$.

Let $\bar{x}_A > 0$ denote the unique solution to $y_A[x_A] = 0$; i.e., $\bar{x}_A$ is the largest value of x_A consistent with ZP^A, and it uniquely solves

$$\frac{\varepsilon_{v_B'}[x_A/L_B]}{1-\varepsilon_{v_B'}[x_A/L_B]} t_B x_A = \frac{F_A}{c_A} \quad .$$

Similarly, the largest value of y_A consistent with ZP^A uniquely solves

$$\frac{\varepsilon_{v_A'}[y_A/L_A]}{1-\varepsilon_{v_A'}[y_A/L_A]} y_A = \frac{F_A}{c_A} \quad . \tag{61}$$

Notice that this is simply $\dot{y}_A$, the autarky level of production.

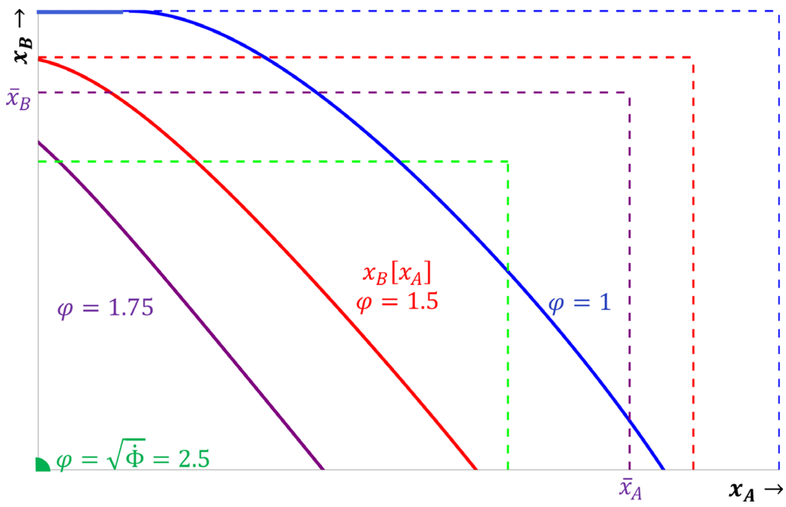

Figure A.1: The figure depicts $x_B[x_A]$ at different levels of tariff-augmented trade costs, $\varphi_A = \varphi_B = \varphi$. The following parameterization was used: $f_A = f_B = 5 \times 10^5; c_A = 4, c_B = 20$; $L_A = L_B = 1000$, where HHs have Pollak preferences with $\rho = 1/2$ and $\gamma = 100$.

These definitions and observations imply that the function $y_A[x_A] : [0, \bar{x}_A] \to [0, \dot{y}_A]$ decreases monotonically from $y_A[0] = \dot{y}_A > 0$ to $y_A[\bar{x}_A] = 0$ as x_A runs from 0 to $\bar{x}_A$. The function $y_B[x_B] : [0, \bar{x}_B] \to [0, \dot{y}_B]$ is analogously defined and has similar properties.

Substituting $y_A[x_A]$ and $y_B[x_B]$ into FOC^A, FOC^B in (60)—and taking the reciprocal of the LHS and RHS of FOC^A—yields

$$\begin{array}{llll} FOC^A: & \frac{1}{\varphi_A}\frac{c_A}{c_B}\frac{m_A[x_B/L_A]}{m_A[y_A[x_A]/L_A]} \overset{(\geq)}{=} \frac{w_B}{w_A} & \text{if} & x_A \overset{(=)}{<} \bar{x}_A \\ FOC^B: & \varphi_B\frac{c_A}{c_B}\frac{m_B[y_B[x_B]/L_B]}{m_B[x_A/L_B]} \overset{(\leq)}{=} \frac{w_B}{w_A} & \text{if} & x_B \overset{(=)}{<} \bar{x}_B \ . \end{array} \tag{62}$$

Let X denote the set of points $(x_A, x_B) \in [0, \bar{x}_A] \times [0, \bar{x}_B]$ that satisfy the (in)equalities in (62) for some $w_B/w_A \in (0, \infty)$, and let $x_B[x_A] : D \rightrightarrows [0, \bar{x}_B]$, $D \subset [0, \bar{x}_A]$, denote the correspondence that describes the set X. (Figure A.1 depicts some of the shapes that X—or, equivalently, $x_B[x_A]$—may take on, as formally described in Lemmas A.8 and A.9 below.)

Lemma A.8 *The domain D of $x_B[\cdot]$ is*

$$D = \begin{cases} [0, x'_A] & \text{if } \Phi < \dot{\Phi} \\ \{0\}, \text{ while } x_B[0] = 0 & \text{if } \Phi = \dot{\Phi} \\ \emptyset & \text{if } \Phi > \dot{\Phi} \end{cases}, \text{ for some } x'_A \in (0, \bar{x}_A] \ .$$

Let $1 \leq \Phi < \dot{\Phi}$. If $\bar{x}_A \notin D$, then $x_B[\cdot]$ is a continuous and weakly decreasing function, running from $(x_A, x_B) = (0, x'_B)$ to $(x'_A, 0)$, for some $x'_B \in (0, \bar{x}_B]$ and $x'_A \in (0, \bar{x}_A)$.

If $\bar{x}_A \in D$, then on $D \setminus \{\bar{x}_A\}$, $x_B[\cdot]$ is a continuous and weakly decreasing function, starting at $(0, x'_B)$, $x'_B \in (0, \bar{x}_B]$, and converging to a point $(\bar{x}_A, x''_B)$, $x''_B \in (0, x'_B)$. Furthermore, at

$x_A = \bar{x}_A$, $x_B[\bar{x}_A] = [0, x''_B]$.

Proof Within the half-open rectangle $[0, \bar{x}_A) \times [0, \bar{x}_B)$, solutions (x_A, x_B) to (62) must satisfy

$$\frac{1}{\varphi_A} \frac{m_A[x_B/L_A]}{m_A[y_A[x_A]/L_A]} = \varphi_B \frac{m_B[y_B[x_B]/L_B]}{m_B[x_A/L_B]} .$$

Rearranging yields

$$\Phi \frac{m_A[y_A[x_A]/L_A]}{m_B[x_A/L_B]} = \frac{m_A[x_B/L_A]}{m_B[y_B[x_B]/L_B]} . \tag{63}$$

The LHS of (63) is strictly increasing in x_A, because $m_k[\cdot]$ and $y_k[\cdot]$ are strictly decreasing, while the RHS is strictly decreasing in x_B. Hence, provided there are any solutions $(x_A, x_B) \in [0, \bar{x}_A) \times [0, \bar{x}_B)$, x_B is a strictly decreasing, continuous function of x_A in this rectangle.

A necessary condition for the domain of $x_B[\cdot]$ to be non-empty on $[0, \bar{x}_A)$ is that, at $x_A = x_B = 0$, LHS $\leq$ RHS. Otherwise, the smallest value that the LHS of (63) takes on is greater than the largest value that the RHS takes on. Hence, we need

$$\Phi \frac{m_A[\dot{y}_A/L_A]}{v'_B[0]} \leq \frac{v'_A[0]}{m_B[\dot{y}_B/L_B]} , \tag{64}$$

where we have used that $m_k[\cdot] = v'_k[\cdot]\left(1 - \varepsilon_{v'_k}[\cdot]\right)$ and $\varepsilon_{v'_k}[0] = 0$. This is equivalent to

$$\Phi \leq \frac{v'_A[0]}{m_A[\dot{y}_A/L_A]} \frac{v'_B[0]}{m_B[\dot{y}_B/L_B]} = \dot{\Phi} .$$

At the other extreme, for the domain of $x_B[\cdot]$ to be non-empty on $[0, \bar{x}_A)$, in (63) we need that LHS $>$ RHS when $(x_A, x_B) \to (\bar{x}_A, \bar{x}_B)$. Otherwise, the largest value that the LHS of (63) takes on is smaller than the smallest value that the RHS takes on. Hence, we need that

$$\Phi \frac{v'_A[0]}{m_B[\bar{x}_A/L_B]} > \frac{m_A[\bar{x}_B/L_A]}{v'_B[0]} . \tag{65}$$

This inequality is trivially satisfied, however, since $\Phi \geq 1$, $m_k[\cdot] = v'_k[\cdot]\left(1 - \varepsilon_{v'_k}[\cdot]\right)$, $v'_k[\cdot]$ is strictly decreasing, and $\varepsilon_{v'_k}[\cdot]$ is weakly increasing.

It remains to show what happens to the graph of (x_A, x_B) if the function $x_B[x_A]$ converges to one of the outer boundaries of the rectangle $[0, \bar{x}_A) \times [0, \bar{x}_B)$. Specifically, suppose that $(\bar{x}_A, x''_B)$ satisfies FOC^A and FOC^B in (62) for some $x''_B \in [0, \bar{x}_B)$. We claim that, in that case, all $(\bar{x}_A, x_B)$ with $x_B \in [0, x''_B]$ also satisfy these (in)equalities. To see this, notice that when $x_A = \bar{x}_A$ (and, hence, $y_A = 0$), the inequality FOC^A in (62) becomes slack, making x_B a variable that can be freely reduced without violating any of the conditions. Similarly, when $(x''_A, \bar{x}_B)$ satisfies FOC^A and FOC^B for some $x''_A \in [0, \bar{x}_A)$, then all $(x_A, \bar{x}_B)$ for $x_A \in [0, x'_A]$ also satisfy the same (in)equalities. ■

Lemma A.9 *Suppose* $1 \leq \Phi \leq \dot{\Phi}$. *At all points* $(x_A, x_B[x_A]) \in X$, *at least one of the conditions* FOC^A, FOC^B *in* (62) *holds with equality. There exists a point in* X *where both hold with equality.*

Proof The inequality in (65) implies that $(\bar{x}_A, \bar{x}_B) \notin X$. Therefore, for all elements in X, either FOC^A or FOC^B in (62), or both, hold with equality. Together with Lemma A.8, $(\bar{x}_A, \bar{x}_B) \notin X$ also implies that, unless X is empty, it contains an element in $[0, \bar{x}_A) \times [0, \bar{x}_B)$. It remains to observe that, at any such point, both FOC^A and FOC^B hold with equality. ■

For $1 \leq \Phi \leq \dot{\Phi}$, it remains to find a point $(x_A, x_B[x_A])$ in X that is consistent with BP—or, rather, we must show that such a point exists. This is equivalent to showing that there exists a pair $\left(x_A, \frac{w_B}{w_A}\right) \in D \times (0, \infty)$ such that:

$$\begin{array}{lllll} FOC^A: & \frac{1}{\varphi_A}\frac{c_A}{c_B}\frac{m_A[x_B[x_A]/L_A]}{m_A[y_A[x_A]/L_A]} \overset{(\geq)}{=} \frac{w_B}{w_A} & \text{if} & x_A \overset{(=)}{<} \bar{x}_A \\ FOC^B: & \varphi_B\frac{c_A}{c_B}\frac{m_B[y_B[x_A]/L_B]}{m_B[x_A/L_B]} \overset{(\leq)}{=} \frac{w_B}{w_A} & \text{if} & x_B[x_A] \overset{(=)}{<} \bar{x}_B \\ BP: & \frac{L_A}{L_B}\frac{1-\varepsilon_{v'_A}[x_B[x_A]/L_A]}{1-\varepsilon_{v'_B}[x_A/L_B]}\frac{C_B[y_B[x_B[x_A]]+t_A x_B[x_A]]}{C_A[y_A[x_A]+t_B x_A]}\frac{t_B c_A x_A}{t_A c_B x_B[x_A]} = \frac{w_B}{w_A} \; . \end{array} \tag{66}$$

Here we have expressed all variables, save w_B/w_A, in terms of x_A.

Denote the LHS of FOC^A in (66) by $FOC^A[x_A]$, and let $FOC^B[x_A]$ and $BP[x_A]$ be similarly defined. From Lemma A.9 we know that at every $x_A \in D$, either $FOC^A[x_A]$, $FOC^B[x_A]$ or both are binding—i.e., they are equal to $w_B/w_A \in (0, \infty)$. Let $w[x_A] : D \to R \subset [0, \infty)$ denote this binding value.

The following lemma guarantees that every point in the range of $w[x_A]$ is also in the range of $BP[x_A]$.

Lemma A.10 *Suppose* $1 \leq \Phi \leq \dot{\Phi}$. *Wage* $w[x_A]$ *is a continuous function. Its range* R *is positive, bounded away from zero and bounded from above.*

$BP[x_A]$ *is continuous. Its range is* $(0, \infty)$.

Proof Continuity of $w[x_A]$ follows from the continuity of $FOC^A[x_A]$ and $FOC^B[x_A]$, and from the fact that at least one of the two is binding for every $x_A \in D$.

Next, notice that $y_k \in [0, \dot{y}_k]$, $x_k \in [0, \bar{x}_k]$, $k \in \{A, B\}$ are all bounded. This implies that in $FOC^A[x_A]$ and $FOC^B[x_A]$, the values of $m_k[\cdot] \equiv v'_k[\cdot]\left(1 - \varepsilon_{v'_k}[\cdot]\right)$ are positive, bounded away from zero and bounded from above. Hence, the same must hold for the range R of $w[x_A]$.

Recall that

$$BP[x_A] = \frac{L_A}{L_B}\frac{1-\varepsilon_{v'_A}[x_B[x_A]/L_A]}{1-\varepsilon_{v'_B}[x_A/L_B]}\frac{c_A}{c_B}\frac{C_B[y_B[x_B[x_A]]+t_A x_B[x_A]]}{C_A[y_A[x_A]+t_B x_A]}\frac{t_B}{t_A}\frac{x_A}{x_B[x_A]} \; .$$

Clearly, $BP[x_A]$ is continuous. To see that it spans $(0,\infty)$, notice that all factors except the last are positive, bounded away from zero and bounded from above, while the last factor runs from 0 to ∞, as x_A runs across its domain, starting at zero. ■

Jointly, Lemma A.10 and the intermediate value theorem imply that, for all $1 \leq \Phi \leq \dot{\Phi}$, the equation

$$BP[x_A] - w[x_A] = 0$$

has a solution $x_A^* \in D$. By construction, this solution gives rise to an equilibrium with values $x_B^* = x_B[x_A^*]$, $y_k^* = y_k[x_k^*]$, $(w_B/w_A)^* = w[x_A^*]$, $n_k^* = L_k/C[y_k^* + t_l x_k^*]$, while price levels P_k^* are determined by FOC_k^x in (59). Finally, notice that in this solution, $x_A, x_B \overset{(=)}{>} 0$ if $\Phi \overset{(=)}{<} \dot{\Phi}$.

" $\Longleftarrow$ ": Next, we show that $\Phi \geq \dot{\Phi}$ is inconsistent with trade in equilibrium. To see this, notice that FOC^k and FOC^l in (60) imply that

$$\varphi_l \frac{c_k}{c_l} \frac{m_l[y_l/L_l]}{m_l[x_k/L_l]} \overset{(\leq)}{=} \frac{w_l}{w_k} \overset{(\leq)}{=} \frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{m_k[x_l/L_k]}{m_k[y_k/L_k]} \ .$$

Recall from ZP^k that y_k can be written as a decreasing function of x_k. Hence, if $x_k, x_l > 0$, then $y_k \leq \dot{y}_k$, $y_l \leq \dot{y}_l$ and

$$\varphi_l \frac{c_k}{c_l} \frac{m_l[\dot{y}_l/L_l]}{v_l'[0]} < \varphi_l \frac{c_k}{c_l} \frac{m_l[y_l/L_l]}{m_l[x_k/L_l]} \overset{(\leq)}{=} \frac{w_l}{w_k} \overset{(\leq)}{=} \frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{m_k[x_l/L_k]}{m_k[y_k/L_k]} < \frac{1}{\varphi_k} \frac{c_k}{c_l} \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} \ .$$

Therefore, rearranging the outer inequalities we find

$$\Phi = \varphi_k \varphi_l < \frac{v_k'[0]}{m_k[\dot{y}_k/L_k]} \frac{v_l'[0]}{m_l[\dot{y}_l/L_l]} = \dot{\Phi} \ ,$$

which is a contradiction.

This completes the proof of Lemma A.7. ■

Jointly, Lemmas A.4 and A.7 imply Proposition 1.

A.3 Welfare Effects of the Initiation of Trade

Define a function $\tau[\cdot] : [-1,1] \to [1,\infty)^2 \times [0,1)^2$, $\sigma \mapsto \tau[\sigma] = (t_A[\sigma], t_B[\sigma], r_A[\sigma], r_B[\sigma])$, where for some $0 < \delta < 1$,

$$\tau[\sigma] = \dot{\tau} + \sigma\Delta\tau \in \begin{cases} \bar{T} & \text{if } \sigma \in (0,\delta) \\ \underline{T} & \text{if } \sigma \in (-\delta,0) \end{cases} \ .$$

Also, let $\varphi_k[\sigma] \equiv t_k[\sigma] / (1 - r_k[\sigma])$.

Proof of Lemma 1:

Observe that

$$0 < \frac{d\Phi[\sigma]}{d\sigma} = \frac{d\varphi_A[\sigma]\varphi_B[\sigma]}{d\sigma} = \Phi[\sigma]\left(\frac{1}{\varphi_A[\sigma]}\frac{d\varphi_A[\sigma]}{d\sigma} + \frac{1}{\varphi_B[\sigma]}\frac{d\varphi_B[\sigma]}{d\sigma}\right) . \tag{67}$$

Evaluated at some $\dot{\tau} \in \dot{T}$,

$$\begin{aligned} \left.\frac{d\Phi[\sigma]}{d\sigma}\right|_{\sigma=0} &= \dot{\Phi}\left.\left\{ \begin{array}{c} \frac{1-\dot{r}_A}{\dot{t}_A}\left(\frac{\frac{dt_A[\sigma]}{d\sigma}}{1-\dot{r}_A} - \frac{\dot{t}_A}{1-\dot{r}_A}\frac{-\frac{dr_A[\sigma]}{d\sigma}}{1-r_A[\sigma]}\right) \\ +\frac{1-\dot{r}_B}{\dot{t}_B}\left(\frac{\frac{dt_B[\sigma]}{d\sigma}}{1-\dot{r}_B} - \frac{\dot{t}_B}{1-\dot{r}_B}\frac{-\frac{dr_B[\sigma]}{d\sigma}}{1-\dot{r}_B}\right) \end{array} \right\}\right|_{\sigma=0} \\ &= \dot{\Phi}\left(\frac{\Delta t_A}{\dot{t}_A} + \frac{\Delta r_A}{1-\dot{r}_A} + \frac{\Delta t_B}{\dot{t}_B} + \frac{\Delta r_B}{1-\dot{r}_B}\right) . \end{aligned}$$

Since threshold $\dot{T}$ is characterized by $\Phi = \dot{\Phi}$, while $\bar{T}$ is characterized by $\Phi > \dot{\Phi}$, it must be that

$$\left.\frac{d\Phi[\sigma]}{d\sigma}\right|_{\sigma=0} > 0 \iff \frac{\Delta t_A}{\dot{t}_A} + \frac{\Delta r_A}{1-\dot{r}_A} + \frac{\Delta t_B}{\dot{t}_B} + \frac{\Delta r_B}{1-\dot{r}_B} > 0 .$$

Hence, the set of all permissible vectors $\Delta\tau$ at $\dot{\tau}$ is given by

$$D[\dot{\tau}] = \left\{\Delta\tau \in \mathbb{R}^4 \middle| \frac{\Delta t_A}{\dot{t}_A} + \frac{\Delta r_A}{1-\dot{r}_A} + \frac{\Delta t_B}{\dot{t}_B} + \frac{\Delta r_B}{1-\dot{r}_B} > 0\right\} .$$

■

Proof of Proposition 2:

We first show that, in the relevant region, all FOCs must be binding.

Lemma A.11 *Fix a $\dot{\tau} \in \dot{T}$. There exists a neighborhood Q of $\dot{\tau}$ such that both FOC_y^k and FOC_x^k, $k \in \{A, B\}$, are binding in $Q \cap \left(\underline{T} \cup \dot{T}\right)$.*

Proof From Lemma A.4 we know that, at $\dot{\tau}$, $y_k = \dot{y}_k > 0$. The implicit function theorem then implies that the FOC_y^ks must be binding in a neighborhood Q of $\dot{\tau}$. From (the proof of) Lemma A.7 we know that the FOC_x^ks are binding for all (φ_A, φ_B) such that $\Phi \leq \dot{\Phi}$—i.e., for all $(t_A, t_B) \in \underline{T} \cup \dot{T}$. Jointly, these observations imply that both the FOC_y^ks and the FOC_y^ks are binding in $Q \cap \left(\underline{T} \cup \dot{T}\right)$. ■

Lemma A.11 implies that, in S, equilibrium is characterized by the following system: for

$k, l \in \{A, B\}$, $l \neq k$,

$$\begin{array}{ll} FOC_y^k: & P_k m_k [y_k/L_k] = w_k c_k \\ FOC_x^k: & P_l m_l [x_k/L_l] = \varphi_l w_k c_k \\ ZP^k: & P_k v_k' [y_k/L_k] y_k + (1 - r_l) P_l v_l' [x_k/L_l] x_k = w_k C_k [y_k + t_l x_k] \\ LM^k: & n_k C_k [y_k + t_l x_k] = L_k \\ BP: & n_k P_l v_l' [x_k/L_l] x_k = n_l P_k v_k' [x_l/L_k] x_l \ . \end{array} \tag{68}$$

The next lemma, which is almost identical to Lemma A.6, reduces the system in (68).

Lemma A.12 *The system in* (68) *gives rise to the following sub-system of five equations and five unknowns: for* $k, l \in \{A, B\}$, $l \neq k$,

$$\begin{array}{ll} FOC^k: & \varphi_k \frac{c_l}{c_k} \frac{m_k[y_k/L_k]}{m_k[x_l/L_k]} = \frac{w_k}{w_l} \\ ZP^k: & \frac{\varepsilon_{v_k'}[y_k/L_k]}{1-\varepsilon_{v_k'}[y_k/L_k]} y_k + \frac{\varepsilon_{v_l'}[x_k/L_l]}{1-\varepsilon_{v_l'}[x_k/L_l]} t_l x_k = \frac{F_k}{c_k} \\ BP: & \frac{L_l}{L_k} \frac{1-\varepsilon_{v_l'}[x_k/L_l]}{1-\varepsilon_{v_k'}[x_l/L_k]} \frac{c_l}{c_k} \frac{C_k[y_k+t_l x_k]}{C_l[y_l+t_k x_l]} \frac{t_k x_l}{t_l x_k} = \frac{w_k}{w_l} \end{array} \ . \tag{69}$$

The five unknowns are y_A, x_A, y_B, x_B, *and* w_B/w_A.

Proof The proof is identical to the proof of Lemma A.6. ■

In essence, Proposition 2 now follows from replacing $\tau = (t_A, t_B, r_A, r_B)$ in (69) with the function $\tau[\sigma] = (t_A[\sigma], t_B[\sigma], r_A[\sigma], r_B[\sigma])$, implicitly differentiating the system with respect to σ, and evaluating at $\sigma = 0$.

Proof of Proposition 2:

First we replace $\tau = (t_A, t_B, r_A, r_B)$ in (69) with $\tau[\sigma] = (t_A[\sigma], t_B[\sigma], r_A[\sigma], r_B[\sigma])$. Next, through a series of lemmas, we implicitly differentiate the system with respect to σ. Finally, we use the derivatives to sign $\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0}$. *All derivatives below refer to left-derivatives.*

Lemma A.13 *At the initiation of trade, a marginal drop in trade frictions has no effect on home-bound output per firm. Formally, for* $k \in \{A, B\}$,

$$\left.\frac{dy_k}{d\sigma}\right|_{\sigma=0} = 0 \ .$$

Proof Having replaced t_A and t_B with $t_A[\sigma]$ and $t_B[\sigma]$, we differentiate ZP^k in (69) with respect to σ and use Lemma A.3.2 to find

$$\begin{array}{ll} & \left(-\frac{\varepsilon_{m_k}[y_k/L_k](y_k/L_k)}{1-\varepsilon_{v_k'}[y_k/L_k]} + \frac{\varepsilon_{v_k'}[y_k/L_k]}{1-\varepsilon_{v_k'}[y_k/L_k]}\right) \frac{dy_k}{d\sigma} + \frac{\varepsilon_{v_l'}[x_k/L_l] x_k}{1-\varepsilon_{v_l'}[x_k/L_l]} \frac{dt_l[\sigma]}{d\sigma} \\ + & \left(-\frac{\varepsilon_{m_k}[x_k/L_l](x_k/L_l)}{1-\varepsilon_{v_k'}[x_k/L_l]} + \frac{\varepsilon_{v_l'}[x_k/L_l]}{1-\varepsilon_{v_l'}[x_k/L_l]}\right) t_l[\sigma] \frac{dx_k}{d\sigma} = 0 \ . \end{array} \tag{70}$$

At $\sigma = 0$, we have $y_k = \dot{y}_k$ and $x_k = \varepsilon_{v'_l}[x_k/L_l] = 0$. Then (70) reduces to

$$\left(-\frac{\varepsilon_{m_k}[\dot{y}_k/L_k]}{1-\varepsilon_{v'_k}[\dot{y}_k/L_k]}\frac{\dot{y}_k}{L_k} + \frac{\varepsilon_{v'_k}[\dot{y}_k/L_k]}{1-\varepsilon_{v'_k}[\dot{y}_k/L_k]}\right)\left.\frac{dy_k}{d\sigma}\right|_{\sigma=0} = 0 \ .$$

Since $\dot{y}_k$ is an interior extremum, $\varepsilon_{m_k}[\dot{y}_k/L_k] < 0$, and $0 < \varepsilon_{v'_k}[\dot{y}_k/L_k] < 1$ (Lemma A.1), it follows that $\left. dy_k/d\sigma\right|_{\sigma=0} = 0$. ■

Lemma A.14

$$\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0} = \left(\dot{\overline{\frac{w_l}{w_k}}}\right)\frac{\dot{n}_l}{\dot{n}_k}\frac{\dot{t}_k c_l}{\dot{t}_l c_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} \ . \qquad (71)$$

Proof Having replaced t_A and t_B with $t_A[\sigma]$ and $t_B[\sigma]$, we differentiate BP in (69) with respect to σ to find

$$\left\{\begin{array}{l}\frac{d}{d\sigma}\left\{\left(1-\varepsilon_{v'_l}\left[\frac{x_k}{L_l}\right]\right)\frac{C_k[y_k+t_l x_k]}{c_k}w_l t_k[\sigma]\right\}\frac{x_l}{L_k} \\ +\left(1-\varepsilon_{v'_l}\left[\frac{x_k}{L_l}\right]\right)\frac{C_k[y_k+t_l x_k]}{c_k}w_l t_k[\sigma]\frac{1}{L_k}\frac{dx_l}{d\sigma}\end{array}\right\} = \left\{\begin{array}{l}\frac{d}{d\sigma}\left\{\left(1-\varepsilon_{v'_k}\left[\frac{x_l}{L_k}\right]\right)\frac{C_l[y_l+t_k x_l]}{c_l}w_k t_l[\sigma]\right\}\frac{x_k}{L_l} \\ +\left(1-\varepsilon_{v'_k}\left[\frac{x_l}{L_k}\right]\right)\frac{C_l[y_l+t_k x_l]}{c_l}w_k t_l[\sigma]\frac{1}{L_l}\frac{dx_k}{d\sigma}\end{array}\right. .$$

Evaluating at $\sigma = 0$, where $y_k = \dot{y}_k$, $x_k = x_l = 0$ and $(t_k[0], t_l[0]) = (\dot{t}_k, \dot{t}_l)$, we get

$$\frac{C_k[\dot{y}_k]}{c_k}\dot{w}_l\dot{t}_k\frac{1}{L_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} = \frac{C_l[\dot{y}_l]}{c_l}\dot{w}_k\dot{t}_l\frac{1}{L_l}\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0} \ .$$

Rewriting yields

$$\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0} = \left(\dot{\overline{\frac{w_l}{w_k}}}\right)\frac{C_k[\dot{y}_k]/L_k}{C_l[\dot{y}_l]/L_l}\frac{\dot{t}_k c_l}{\dot{t}_l c_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} = \left(\dot{\overline{\frac{w_l}{w_k}}}\right)\frac{\dot{n}_l}{\dot{n}_k}\frac{\dot{t}_k c_l}{\dot{t}_l c_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} ,$$

where we have applied LM^k. ■

Lemma A.15 *At the initiation of trade, a marginal drop in trade frictions increases per-firm exports in both countries. Formally, for* $k \in \{A, B\}$,

$$\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0} < 0 \ .$$

Proof Differentiate FOC^k in (69) with respect to σ :

$$\frac{d\varphi_k}{d\sigma}\frac{c_l}{c_k}\frac{m_k[y_k/L_k]}{m_k[x_l/L_k]} + \varphi_k\frac{c_l}{c_k}\frac{d}{d\sigma}\left[\frac{m_k[y_k/L_k]}{m_k[x_l/L_k]}\right] = \frac{d}{d\sigma}\left(\frac{w_k}{w_l}\right) \ .$$

Using Lemma A.3.1 then yields

$$\varphi_k\frac{c_l}{c_k}\frac{m_k\left[\frac{y_k}{L_k}\right]}{m_l\left[\frac{x_l}{L_k}\right]}\left(\frac{1}{\varphi_k}\frac{d\varphi_k}{d\sigma} + \frac{\varepsilon_{m_k}\left[\frac{y_k}{L_k}\right]}{y_k/L_k}\frac{d\left[\frac{y_k}{L_k}\right]}{d\sigma} - \frac{\varepsilon_{m_l}\left[\frac{x_l}{L_k}\right]}{x_l/L_k}\frac{d\left[\frac{x_l}{L_k}\right]}{d\sigma}\right) = \frac{1}{w_l}\left(\frac{dw_k}{d\sigma} - \frac{w_k}{w_l}\frac{dw_l}{d\sigma}\right) \ .$$

Substituting using FOC^k,

$$\frac{w_k}{w_l}\left(\frac{1}{\varphi_k}\frac{d\varphi_k}{d\sigma}+\frac{\varepsilon_{m_k}[y_k/L_k]}{y_k/L_k}\frac{d[y_k/L_k]}{d\sigma}-\frac{\varepsilon_{m_l}[x_l/L_k]}{x_l/L_k}\frac{d[x_l/L_k]}{d\sigma}\right)=\frac{1}{w_l}\left(\frac{dw_k}{d\sigma}-\frac{w_k}{w_l}\frac{dw_l}{d\sigma}\right) .$$

Recall from Lemma A.13 that $dy_k/d\sigma|_{\sigma=0}=0$. Hence, at $\sigma=0$,

$$\frac{1}{\varphi_k}\frac{d\varphi_k}{d\sigma}-\frac{\varepsilon_{m_l}[x_l/L_k]}{x_l/L_k}\frac{d[x_l/L_k]}{d\sigma}=\frac{1}{w_k}\frac{dw_k}{d\sigma}-\frac{1}{w_l}\frac{dw_l}{d\sigma} .$$

for $k,l\in\{A,B\}$, $k\neq l$. Adding up the equations,

$$\frac{\Phi[\sigma]}{\Phi[\sigma]}\left(\frac{1}{\varphi_A}\frac{d\varphi_A}{d\sigma}+\frac{1}{\varphi_B}\frac{d\varphi_B}{d\sigma}\right)-\frac{\varepsilon_{m_B}[x_B/L_A]}{x_B/L_A}\frac{d[x_B/L_A]}{d\sigma}-\frac{\varepsilon_{m_A}[x_A/L_B]}{x_A/L_B}\frac{d[x_A/L_B]}{d\sigma}=0 .$$

Since $\frac{1}{\varphi_A}\frac{d\varphi_A}{d\sigma}+\frac{1}{\varphi_B}\frac{d\varphi_B}{d\sigma}>0$ (see (67)), it follows that

$$-\frac{\varepsilon_{m_B}[x_B/L_A]}{x_B/L_A}\frac{d[x_B/L_A]}{d\sigma}-\frac{\varepsilon_{m_A}[x_A/L_B]}{x_A/L_B}\frac{d[x_A/L_B]}{d\sigma}<0 .$$

From (71) we know that $dx_l/d\sigma|_{\sigma=0}$ and $dx_k/d\sigma|_{\sigma=0}$ take on the same sign. Furthermore, $\varepsilon_{m_k}[0]<0$. Hence,

$$\left.\frac{dx_A}{d\sigma},\frac{dx_B}{d\sigma}\right|_{\sigma=0}<0 .$$

■

Lemma A.16 *At the initiation of trade, a marginal drop in trade frictions reduces the number of firms in both countries. Formally, for $k\in\{A,B\}$,*

$$\left.\frac{dn_k}{d\sigma}\right|_{\sigma=0}=-\dot{n}_k\frac{\dot{t}_l c_k}{C_k[\dot{y}_k]}\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0}>0 .$$

Proof Differentiating LM^k in (68) with respect to σ yields

$$\frac{dn_k}{d\sigma}C_k[y_k+t_l[\sigma]x_k]+n_k c_k\left(\frac{dy_k}{d\sigma}+\frac{dt_l}{d\sigma}x_k+t_l[\sigma]\frac{dx_k}{d\sigma}\right)=0 .$$

Evaluating at $\sigma=0$, where $x_k=dy_k/d\sigma|_{\sigma=0}=0$, we find

$$\left.\frac{dn_k}{d\sigma}\right|_{\sigma=0}=-\dot{n}_k\frac{\dot{t}_l c_k}{C_k[\dot{y}_k]}\left.\frac{dx_k}{d\sigma}\right|_{\sigma=0}>0 ,$$

where the inequality follows from $dx_k/d\sigma|_{\sigma=0}<0$ (Lemma A.15). ■

Lemma A.17 *For $k\in\{A,B\}$,*

$$\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0}\overset{(>)}{\underset{(<)}{=}}0 \iff r_k\overset{(<)}{\underset{(>)}{=}}1-\varepsilon_{v_k}\left[\frac{\dot{y}_k}{L_k}\right] .$$

Proof Utility in country k is

$$U_k = n_k v_k \left[\frac{y_k}{L_k}\right] + n_l v_k \left[\frac{x_l}{L_k}\right] .$$

Differentiating with respect to σ yields

$$\frac{dU_k}{d\sigma} = \frac{dn_k}{d\sigma} v_k \left[\frac{y_k}{L_k}\right] + n_k v_k' \left[\frac{y_k}{L_k}\right] \frac{1}{L_k}\frac{dy_k}{d\sigma} + \frac{dn_l}{d\sigma} v_k \left[\frac{x_l}{L_k}\right] + n_l v_k' \left[\frac{x_l}{L_k}\right] \frac{1}{L_k}\frac{dx_l}{d\sigma} .$$

At $\sigma = 0$, we have $x_l = dy_k/d\sigma|_{\sigma=0} = 0$, while $v[0] = 0$. Hence,

$$\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0} = \frac{dn_k}{d\sigma} v_k \left[\frac{\dot{y}_k}{L_k}\right] + \dot{n}_l v_k' [0] \frac{1}{L_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} .$$

Substitute $dn_k/d\sigma|_{\sigma=0}$ using Lemma A.16 to find

$$\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0} = -\dot{n}_k \frac{\dot{t}_l c_k}{C_k[\dot{y}_k]}\frac{dx_k}{d\sigma} v_k \left[\frac{\dot{y}_k}{L_k}\right] + \dot{n}_l v_k' [0] \frac{1}{L_k}\left.\frac{dx_l}{d\sigma}\right|_{\sigma=0} .$$

Substitute $dx_k/d\sigma|_{\sigma=0}$ using (71) to find

$$\begin{aligned}
\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0} &= \left. -\dot{n}_k \frac{\dot{t}_l c_k}{C_k[\dot{y}_k]}\dot{\left(\frac{w_l}{w_k}\right)}\frac{\dot{n}_l}{\dot{n}_k}\frac{\dot{t}_k c_l}{\dot{t}_l c_k}\frac{dx_l}{d\sigma} v_k \left[\frac{\dot{y}_k}{L_k}\right] + n_l v_k' [0] \frac{1}{L_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} \\
&= \left.\left(-\frac{\dot{y}_k}{C_k[\dot{y}_k]}\dot{\left(\frac{w_l}{w_k}\right)}\dot{t}_k c_l v_k\left[\frac{\dot{y}_k}{L_k}\right] + v_k'[0]\frac{\dot{y}_k}{L_k}\right)\frac{\dot{n}_l}{\dot{y}_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} \\
&= \left.\left(-\frac{\dot{y}_k}{w_k C_k[\dot{y}_k]}\dot{t}_k\frac{\dot{\varphi}_k}{\dot{\varphi}_k} w_l c_l v_k\left[\frac{\dot{y}_k}{L_k}\right] + v_k'[0]\frac{\dot{y}_k}{L_k}\right)\frac{\dot{n}_l}{\dot{y}_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} .
\end{aligned}$$

Recall from ZP^k that, at the initiation of trade, $P_k v_k'[y_k/L_k] = w_k C_k[y_k]/y_k$, and from FOC_x^k that $P_k v_k'[0] = \dot{\varphi}_k w_l c_l$. So, we may write

$$\begin{aligned}
\left.\frac{dU_k}{d\sigma}\right|_{\sigma=0} &= \left.\left(-\frac{1}{P_k v_k'[y_k/L_k]}(1-\dot{r}_k) P_k v_k'[0] v_k\left[\frac{\dot{y}_k}{L_k}\right] + v_k'[0]\frac{\dot{y}_k}{L_k}\right)\frac{\dot{n}_l}{\dot{y}_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} \\
&= \left.\left(\frac{\frac{\dot{y}_k}{L_k} v_k'\left[\frac{\dot{y}_k}{L_k}\right]}{v_k\left[\frac{\dot{y}_k}{L_k}\right]} - (1-\dot{r}_k)\right) v_k\left[\frac{\dot{y}_k}{L_k}\right]\frac{v_k'[0]}{v_k'[\dot{y}_k/L_k]}\frac{\dot{n}_l}{\dot{y}_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} \\
&= \left.\left(\dot{r}_k - \left(1-\varepsilon_v\left[\frac{\dot{y}_k}{L_k}\right]\right)\right) v_k\left[\frac{\dot{y}_k}{L_k}\right]\frac{v_k'[0]}{v_k'[\dot{y}_k/L_k]}\frac{\dot{n}_l}{\dot{y}_k}\frac{dx_l}{d\sigma}\right|_{\sigma=0} .
\end{aligned}$$

Clearly, the second, third and fourth factors on the RHS are strictly positive, while (from Lemma A.15) $dx_l/d\sigma|_{\sigma=0} < 0$. Hence, the expression turns on the first factor. ■

Lemma A.18 *Critical tariff $r_k^* \equiv 1 - \varepsilon_{v_k}[\dot{y}_k/L_k]$ is strictly decreasing in L_k and strictly increasing in F_k/c_k. Furthermore, $0 < r_k^* < 1$, and $(1-r_A^*)(1-r_B^*)\dot{\Phi} > 1$. Hence, under the* *critical tariffs, trade takes place for sufficiently low trade costs $t_A, t_B \geq 1$.*

Proof Recall from Lemma A.5 that $\dot{y}_k/L_k$ is strictly decreasing in L_k and strictly increasing in F_k/c_k. Since $\varepsilon_{v_k}[\cdot]$ is a strictly decreasing function (by Lemma A.2), r_k^* is also strictly decreasing in L_k. Together with $0 < \dot{y}_k < \infty$, Lemma A.2 also implies that $0 < r_k^* < 1$.

It remains to show that $(1 - r_A^*)(1 - r_B^*)\dot{\Phi} > 1$. From the proof of Lemma A.2 we know that $\varepsilon_{v_k'}[z]\, v_k[z] > v_k[z] - z v_k'[z]$, which implies

$$\varepsilon_{v_k}[z] > 1 - \varepsilon_{v_k'}[z] \; .$$

Therefore,

$$\begin{aligned}
(1 - r_A^*)(1 - r_B^*)\dot{\Phi} &= \frac{\varepsilon_{v_A}[\dot{y}_A/L_A]\, v_A'[0]}{m_A[\dot{y}_A/L_A]} \frac{\varepsilon_{v_B}[\dot{y}_B/L_B]\, v_B'[0]}{m_B[\dot{y}_B/L_B]} \\
&> \frac{\left(1 - \varepsilon_{v_A'}[\dot{y}_A/L_A]\right) v_A'[0]}{m_A[\dot{y}_A/L_A]} \frac{(1 - \varepsilon_{v_B}[\dot{y}_B/L_B])\, v_B'[0]}{m_B[\dot{y}_B/L_B]} \\
&= \frac{v_A'[0]}{v_A'[\dot{y}_A/L]} \frac{v_B'[0]}{v_B'[\dot{y}_B/L]} > 1 \; .
\end{aligned}$$

■

Jointly, Lemmas A.16, A.17 and A.18 imply Proposition 2. ■

Proof of Remark 1:

Remark 1 was partially proved in the main text. The following lemma constitutes the missing part.

Lemma A.19 *Suppose countries are symmetric. In equilibrium with trade,*

$$\frac{d[y/L]/dr}{d[x/L]/dr} = -(1 - r)\frac{v'\left[\frac{x}{L}\right] \varepsilon_m[x/L]}{v'\left[\frac{y}{L}\right] \varepsilon_m[y/L]} \; .$$

Proof Under symmetry, ZP^k in (69) reduces to

$$\frac{\varepsilon_{v'}[y/L]}{1 - \varepsilon_{v'}[y/L]} y + \frac{\varepsilon_{v'}[x/L]}{1 - \varepsilon_{v'}[x/L]} tx = \frac{F}{c} \; . \tag{72}$$

Implicitly differentiating (72) with respect to r using Lemma A.3.2 yields

$$\frac{dy}{dr} = -t \frac{\frac{\varepsilon_m[x/L]}{1-\varepsilon_{v'}[x/L]}}{\frac{\varepsilon_m[y/L]}{1-\varepsilon_{v'}[y/L]}} \frac{dx}{dr} \; .$$

Using FOC^k in (69) which, under symmetry reduces to $\frac{m[x/L]}{m[y[x]/L]} = \frac{t}{1-r}$, we find

$$\frac{dy}{dr} = -(1-r)\frac{v'\left[\frac{x}{L}\right]}{v'\left[\frac{y}{L}\right]}\frac{\varepsilon_m[x/L]}{\varepsilon_m[y/L]}\frac{dx}{dr} \ .$$

■

This completes the proof of Remark 1. ■

A.4 When Autarky Trumps Free and Costless Trade

A.4.1 Equilibrium as $c_B \to \infty$

Consider the equilibrium system in (60). Throughout, we set $t_A = t_B = 1$. Define $\check{c}_B \equiv 1/c_B$, and observe that $c_B \to \infty$ is equivalent to $\check{c}_B \to 0$. To focus on the area of interest, we assume that tariffs r_A, r_B are 'sufficiently small,' specifically,

$$(1-r_A)(1-r_B) > m_A[\dot{y}_A/L_A]/v_A'[0] \ . \tag{73}$$

This guarantees that trade takes place for all c_B. To see this, recall the trade condition in the absence of trade costs:

$$(1-r_A)(1-r_B) > \frac{m_A[\dot{y}_A/L_A]}{v_A'[0]}\frac{m_B[\dot{y}_B/L_B]}{v_B'[0]}$$

where $m_B[\dot{y}_B/L_B]/v_B'[0] < 1$.

Lemma A.20 *In equilibrium,*

$$\lim_{\check{c}_B\to 0} y_B = \lim_{\check{c}_B\to 0} x_B = 0 \ .$$

Proof Consider ZP^B in (60). Taking the limit as $\check{c}_B \to 0$ yields

$$\lim_{\check{c}_B\to 0} \frac{\varepsilon_{v_B'}[y_B/L_B]}{1-\varepsilon_{v_B'}[y_B/L_B]} y_B + \frac{\varepsilon_{v_A'}[x_B/L_A]}{1-\varepsilon_{v_A'}[x_B/L_A]} x_B = 0 \ .$$

Since $\frac{\varepsilon_{v'}[z]}{1-\varepsilon_{v'}[z]} > 0$ for $z > 0$, both y_B and x_B must go to zero as $\check{c}_B \to 0$. ■

Lemma A.21 *In equilibrium,*

$$\lim_{\check{c}_B\to 0} y_A = \dot{y}_A; \ \lim_{\check{c}_B\to 0} x_A = 0 \ .$$

Proof Equating FOC^A and BP in (60),

$$\frac{1}{1-r_A}\frac{v'_A\left[y_A/L_A\right]}{v'_A\left[x_B/L_A\right]}\left(1-\varepsilon_{v'_A}\left[y_A/L_A\right]\right)=\frac{L_B}{L_A}\left(1-\varepsilon_{v'_B}\left[x_A/L_B\right]\right)\frac{C_A\left[y_A+x_A\right]}{C_B\left[y_B+x_B\right]}\frac{x_B}{x_A}\ .$$

Rewriting,

$$x_A=(1-r_A)\frac{L_B}{L_A}\frac{v'_A\left[x_B/L_A\right]}{v'_A\left[y_A/L_A\right]}\frac{1-\varepsilon_{v'_B}\left[x_A/L_B\right]}{1-\varepsilon_{v'_A}\left[y_A/L_A\right]}\frac{C_A\left[y_A+x_A\right]}{C_B\left[y_B+x_B\right]}x_B\ . \tag{74}$$

Taking the limit as $\check{c}_B\to 0$,

$$\lim_{\check{c}_B\to 0}x_A=(1-r_A)\frac{L_B}{L_A}v'_A\left[0\right]\times\lim_{\check{c}_B\to 0}\left\{\frac{1-\varepsilon_{v'_B}\left[x_A/L_B\right]}{1-\varepsilon_{v'_A}\left[y_A/L_A\right]}\frac{C_A\left[y_A+x_A\right]}{v'_A\left[y_A/L_A\right]}\right\}\times\lim_{\check{c}_B\to 0}\frac{x_B}{F_B+\frac{y_B+x_B}{\check{c}_B}}=0\ ,$$

where we have used the fact that the braced factor is finite because ZP^A in (60) guarantees that $0\le y_A,x_A<\infty$, while $\lim_{\check{c}_B\to 0}x_B=0$. Finally, using $\lim_{\check{c}_B\to 0}x_A=0$, ZP^A implies that $\lim_{\check{c}_B\to 0}y_A=\dot{y}_A$. ■

Lemma A.22 *In equilibrium, $y_B=0$ for $\check{c}_B$ sufficiently small.*

Proof Observe that optimality implies that, if the inequality in FOC^k of (60) is strict, then y_k must be cornered at 0. Hence, combining FOC^A and FOC^B in (60) we have

$$\frac{1}{R}\frac{v'_A\left[y_A/L_A\right]\left(1-\varepsilon_{v'_A}\left[y_A/L_A\right]\right)}{v'_A\left[x_B/L_A\right]\left(1-\varepsilon_{v'_A}\left[x_B/L_A\right]\right)}<\frac{v'_B\left[x_A/L_B\right]\left(1-\varepsilon_{v'_B}\left[x_A/L_B\right]\right)}{v'_B\left[y_B/L_B\right]\left(1-\varepsilon_{v'_B}\left[y_B/L_B\right]\right)}\implies y_B=0\ , \tag{75}$$

where $R=(1-r_A)(1-r_B)$. For the RHS of the inequality in (75), observe that

$$\lim_{\check{c}_B\to 0}\frac{v'_B\left[x_A/L_B\right]\left(1-\varepsilon_{v'_B}\left[x_A/L_B\right]\right)}{v'_B\left[y_B/L_B\right]\left(1-\varepsilon_{v'_B}\left[y_B/L_B\right]\right)}=\frac{v'_B\left[0\right]\left(1-\varepsilon_{v'_B}\left[0\right]\right)}{v'_B\left[0\right]\left(1-\varepsilon_{v'_B}\left[0\right]\right)}=1\ ,$$

while for the LHS,

$$\lim_{\check{c}_B\to 0}\frac{1}{R}\frac{v'_A\left[y_A/L_A\right]\left(1-\varepsilon_{v'_A}\left[y_A/L_A\right]\right)}{v'_A\left[x_B/L_A\right]\left(1-\varepsilon_{v'_A}\left[x_B/L_A\right]\right)}=\frac{1}{R}\frac{v'_A\left[\dot{y}_A/L_A\right]\left(1-\varepsilon_{v'_A}\left[\dot{y}_A/L_A\right]\right)}{v'_A\left[0\right]\left(1-\varepsilon_{v'_A}\left[0\right]\right)}\ .$$

This last expression is strictly smaller than 1, such that the strict inequality in (75) holds in the limit, if

$$(1-r_A)(1-r_B)>\frac{m_A\left[\dot{y}_A/L_A\right]}{v'_A\left[0\right]}\ . \tag{76}$$

Finally, notice that the inequality in (76) indeed holds, by the assumption in (73). ■

Lemma A.23 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} \frac{x_A}{\check{c}_B} = \frac{(1-r_A)\, v_A' [0]\, C_A [\dot{y}_A]}{v_A' [\dot{y}_A/L_A] \left(1-\varepsilon_{v_A'} [\dot{y}_A/L_A]\right)} \frac{L_B}{L_A} < \infty \ .$$

Proof When $\check{c}_B$ is sufficiently small, so that $y_B = 0$ (see Lemma A.22), then ZP^B in (60) reduces to

$$\frac{\varepsilon_{v_A'} [x_B/L_A]}{1-\varepsilon_{v_A'} [x_B/L_A]} x_B = F_B \check{c}_B \ . \tag{77}$$

Once more using $y_B = 0$, this implies that

$$C_B [y_B + x_B]\, \check{c}_B = F_B \check{c}_B + x_B = \frac{\varepsilon_{v_A'} [x_B/L_A]}{1-\varepsilon_{v_A'} [x_B/L_A]} x_B + x_B \ .$$

Hence, for sufficiently small $\check{c}_B$, (74) can be rewritten as

$$x_A = (1-r_A) \frac{L_B}{L_A} \frac{v_A' [x_B/L_A]}{v_A' [y_A/L_A]} \frac{1-\varepsilon_{v_B'} [x_A/L_B]}{1-\varepsilon_{v_A'} [y_A/L_A]} \frac{C_A [y_A + x_A]}{C_B [y_B + x_B]} x_B \ .$$

Taking the limit as $\check{c}_B \to 0$ and using Lemmas A.20 and A.21 as well as the fact that $\varepsilon_{v_k'} [0] = 0$, yields the lemma. ■

A.4.2 Bad Trade for c_B Sufficiently Large

Proof of Proposition 4:

Recall that $\check{c}_B \equiv 1/c_B$, and observe that $c_B \to \infty$ is equivalent to $\check{c}_B \to 0$.

Lemma A.24 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} \frac{dx_A}{d\check{c}_B} = \lim_{\check{c}_B \to 0} \frac{x_A}{\check{c}_B} < \infty \ .$$

Proof Consider x_A as a function of $\check{c}_B$. From the definition of a derivative,

$$\left.\frac{dx_A}{d\check{c}_B}\right|_{\check{c}_B = k} = \lim_{h \to 0} \frac{x_A [k+h] - x_A [k]}{(k+h)-k} \ .$$

Taking the limit as $k \to 0$,

$$\begin{aligned} \lim_{k \to 0} \left(\left.\frac{dx_A}{d\check{c}_B}\right|_{\check{c}_B = k} \right) &= \lim_{k \to 0} \left(\lim_{h \to 0} \frac{x_A [k+h] - x_A [k]}{(k+h)-k} \right) = \lim_{h \to 0} \left(\lim_{k \to 0} \frac{x_A [k+h] - x_A [k]}{(k+h)-k} \right) \\ &= \lim_{h \to 0} \frac{x_A [h] - x_A [0]}{h} = \lim_{h \to 0} \frac{x_A [h]}{h} \ . \end{aligned}$$

Here we have used the Moore-Osgood Theorem to interchange limits. Finiteness now follows from Lemma A.23. ■

Lemma A.25 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} \frac{dy_A}{d\check{c}_B} = 0 \ .$$

Proof Differentiating ZP^A in (60) with respect to $\check{c}_B$,

$$\left\{ \frac{\left(\frac{y_A}{L_A}\right) \varepsilon'_{v'_A} \left[\frac{y_A}{L_A}\right]}{\left(1 - \varepsilon_{v'_A} \left[\frac{y_A}{L_A}\right]\right)^2} + \frac{\varepsilon_{v'_A} \left[\frac{y_A}{L_A}\right]}{1 - \varepsilon_{v'_A} \left[\frac{y_A}{L_A}\right]} \right\} \frac{dy_A}{d\check{c}_B} + \left\{ \frac{\left(\frac{x_A}{L_B}\right) \varepsilon'_{v'_B} \left[\frac{x_A}{L_B}\right]}{\left(1 - \varepsilon_{v'_B} \left[\frac{x_A}{L_B}\right]\right)^2} + \frac{\varepsilon_{v'_B} \left[\frac{x_A}{L_B}\right]}{1 - \varepsilon_{v'_B} \left[\frac{x_A}{L_B}\right]} \right\} t_B \frac{dx_A}{d\check{c}_B} = 0 \ .$$

Taking the limit as $\check{c}_B \to 0$, using that $\lim_{\check{c}_B \to 0} y_A = \dot{y}_A$, $\lim_{\check{c}_B \to 0} x_A = x_B = \varepsilon_{v'_B} [x_A/L_B] = 0$, and $0 < \lim_{\check{c}_B \to 0} \frac{dx_A}{d\check{c}_B} < \infty$, we find that

$$\left\{ \frac{(\dot{y}_A/L_A)\, \varepsilon'_{v'_A} [\dot{y}_A/L_A]}{\left(1 - \varepsilon_{v'_A} [\dot{y}_A/L_A]\right)^2} + \frac{\varepsilon_{v'_A} [\dot{y}_A/L_A]}{1 - \varepsilon_{v'_A} [\dot{y}_A/L_A]} \right\} \lim_{\check{c}_B \to 0} \frac{dy_A}{d\check{c}_B} = 0 \ .$$

Since $\dot{y}_A$ is an interior extremum, $\varepsilon_{v'_A} [\dot{y}_A/L_A] > 0$ and $\varepsilon'_{v'_A} [\dot{y}_A/L_A] > 0$, implying that the braced factor is strictly positive. Hence, the lemma follows. ■

Lemma A.26 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} n_B \frac{dx_B}{d\check{c}_B} = \frac{L_B}{2} \ .$$

Proof Differentiating (77) in (60) with respect to $\check{c}_B$,

$$\frac{\varepsilon'_{v'_A} [x_B/L_A]}{\left(1 - \varepsilon_{v'_A} [x_B/L_A]\right)^2} \frac{dx_B}{d\check{c}_B} \frac{x_B}{L_A} + \frac{\varepsilon_{v'_A} [x_B/L_A]}{1 - \varepsilon_{v'_A} [x_B/L_A]} \frac{dx_B}{d\check{c}_B} = F_B \ .$$

Isolating $\frac{dx_B}{d\check{c}_B}$,

$$\frac{dx_B}{d\check{c}_B} = F_B \frac{1 - \varepsilon_{v'_A} [x_B/L_A]}{\frac{(x_B/L_A)\varepsilon'_{v'_A} [x_B/L_A]}{1 - \varepsilon_{v'_A} [x_B/L_A]} + \varepsilon_{v'_A} [x_B/L_A]} \ . \tag{78}$$

Using that $n_B = \frac{L_B}{F_B + x_B/\check{c}_B}$, which follows from LM^B in (60),

$$n_B \frac{dx_B}{d\check{c}_B} = \frac{\check{c}_B F_B L_B}{\check{c}_B F_B + x_B} \frac{1 - \varepsilon_{v'_A} [x_B/L_A]}{\frac{(x_B/L_A)\varepsilon'_{v'_A} [x_B/L_A]}{1 - \varepsilon_{v'_A} [x_B/L_A]} + \varepsilon_{v'_A} [x_B/L_A]} \ .$$

Reusing (77) and simplifying,

$$n_B \frac{dx_B}{d\check{c}_B} = \frac{L_B \left(1 - \varepsilon_{v'_A} [x_B/L_A]\right)}{\frac{(x_B/L_A)\varepsilon'_{v'_A} [x_B/L_A]}{\varepsilon_{v'_A} [x_B/L_A]} \frac{1}{1 - \varepsilon_{v'_A} [x_B/L_A]} + 1} \ . \tag{79}$$

From Lemma A.3.3, and because $\lim_{\check{c}_B \to 0} x_B = 0$, and $\varepsilon_{v''_A}[0] = \varepsilon_{v'_A}[0] = 0$,

$$\lim_{\check{c}_B \to 0} n_B \frac{dx_B}{d\check{c}_B} = \frac{L_B}{1+1} = \frac{L_B}{2} \ .$$

■

Lemma A.27 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} \frac{\check{c}_B}{x_B} \frac{dx_B}{d\check{c}_B} = \frac{1}{2} \ .$$

Proof From (78),

$$\frac{\check{c}_B}{x_B} \frac{dx_B}{d\check{c}_B} = \frac{\check{c}_B}{x_B} F_B \frac{1 - \varepsilon_{v'_A}[x_B/L_A]}{\frac{(x_B/L_A)\varepsilon'_{v'_A}[x_B/L_A]}{1-\varepsilon_{v'_A}[x_B/L_A]} + \varepsilon_{v'_A}[x_B/L_A]} \ .$$

Using (77) and simplifying,

$$\frac{\check{c}_B}{x_B} \frac{dx_B}{d\check{c}_B} = \frac{1}{\frac{(x_B/L_A)\varepsilon'_{v'_A}[x_B/L_A]}{\varepsilon_{v'_A}[x_B/L_A]} \frac{1}{1-\varepsilon_{v'_A}[x_B/L_A]} + 1} \ .$$

From Lemma A.3.3, $\lim_{\check{c}_B \to 0} x_B = 0$, and $\varepsilon_{v''_A}[0] = \varepsilon_{v'_A}[0] = 0$, it now follows that

$$\lim_{\check{c}_B \to 0} \frac{\check{c}_B}{x_B} \frac{dx_B}{d\check{c}_B} = \frac{1}{1+1} = \frac{1}{2} \ .$$

■

Lemma A.28 *In equilibrium,*

$$\lim_{\check{c}_B \to 0} \frac{dn_A}{d\check{c}_B} = -\frac{L_B}{\dot{y}_A} \frac{(1-r_A)\, v'_A[0]}{v'_A[\dot{y}_A/L_A]} < 0 \ .$$

Proof Recall from LM^A in (60) that

$$n_A = \frac{L_A}{F_A + c_A (y_A + x_A)} \ .$$

Differentiating with respect to $\check{c}_B$ yields

$$\frac{dn_A}{d\check{c}_B} = -\frac{L_A}{(F_A + c_A (y_A + x_A))^2} c_A \left(\frac{dy_A}{d\check{c}_B} + \frac{dx_A}{d\check{c}_B} \right) \ .$$

Recall from Lemma A.25 that $\lim_{\check{c}_B \to 0} \frac{dy_A}{d\check{c}_B} = 0$, and from Lemmas A.24 and A.26 that $\lim_{\check{c}_B \to 0} \frac{dx_A}{d\check{c}_B} =$

$\frac{(1-r_A)v'_A[0]C_A[\dot{y}_A]}{v'_A[\dot{y}_A/L_A]\left(1-\varepsilon_{v'_A}[\dot{y}_A/L_A]\right)}\frac{L_B}{L_A}$. Therefore,

$$\lim_{\check{c}_B\to 0}\frac{dn_A}{d\check{c}_B}=-\frac{L_A}{C_A\left[\dot{y}_A\right]^2}c_A\frac{(1-r_A)\,v'_A\left[0\right]C_A\left[\dot{y}_A\right]}{v'_A\left[\dot{y}_A/L_A\right]\left(1-\varepsilon_{v'_A}\left[\dot{y}_A/L_A\right]\right)}\frac{L_B}{L_A}\ .$$

Recall from (52) that in autarky, $\frac{c_A}{C_A[y_A]}=\frac{1}{y_A}\left(1-\varepsilon_{v'_A}\left[y_A/L_A\right]\right)$. Hence, the claim follows. ■

Lemma A.29 *In equilibrium,*

$$\lim_{\check{c}_B\to 0}\frac{dn_B}{d\check{c}_B}x_B=\frac{L_B}{2}\ .$$

Proof For $\check{c}_B$ sufficiently small, $y_B=0$. Hence, from LM^B in (60),

$$n_B=\frac{\check{c}_B L_B}{\check{c}_B F_B+x_B}\ .$$

Differentiating with respect to $\check{c}_B$,

$$\frac{dn_B}{d\check{c}_B}=\frac{L_B}{\check{c}_B F_B+x_B}-\frac{\check{c}_B L_B}{\check{c}_B F_B+x_B}\frac{F_B+\frac{dx_B}{d\check{c}_B}}{\check{c}_B F_B+x_B}=x_B L_B\frac{1-\frac{\check{c}_B}{x_B}\frac{dx_B}{d\check{c}_B}}{\left(\check{c}_B F_B+x_B\right)^2}\ .$$

Using (77),

$$\frac{dn_B}{d\check{c}_B}=x_B L_B\frac{1-\frac{\check{c}_B}{x_B}\frac{dx_B}{d\check{c}_B}}{\left(\frac{\varepsilon_{v'_A}[x_B/L_A]}{1-\varepsilon_{v'_A}[x_B/L_A]}x_B+x_B\right)^2}=\frac{L_B}{x_B}\left(1-\frac{\check{c}_B}{x_B}\frac{dx_B}{d\check{c}_B}\right)\left(1-\varepsilon_{v'_A}\left[x_B/L_A\right]\right)^2\ .$$

Hence,

$$\frac{dn_B}{d\check{c}_B}x_B=L_B\left(1-\frac{\check{c}_B}{x_B}\frac{dx_B}{d\check{c}_B}\right)\left(1-\varepsilon_{v'_A}\left[x_B/L_A\right]\right)^2\ .$$

Since $\lim_{\check{c}_B\to 0}\frac{\check{c}_B}{x_B}\frac{dx_B}{d\check{c}_B}=\frac{1}{2}$ and $\lim_{\check{c}_B\to 0}x_B=\varepsilon_{v'_A}\left[0\right]=0$,

$$\frac{dn_B}{d\check{c}_B}x_B=L_B\left(1-\frac{1}{2}\right)(1-0)^2=\frac{L_B}{2}\ .$$

■

Lemma A.30 *In equilibrium,*

$$\lim_{\check{c}_B\to 0}\frac{dU_A}{d\check{c}_B}=\left(1-\frac{1-r_A}{\varepsilon_{v_A}\left[\dot{y}_A/L_A\right]}\right)\frac{L_B}{L_A}v'_A\left[0\right]\underset{(<)}{\overset{(>)}{=}}0\iff r_A\underset{(<)}{\overset{(>)}{=}}1-\varepsilon_{v_A}\left[\frac{\dot{y}_A}{L_A}\right]\ .$$

Proof Utility in country A is

$$U_A=n_A v_A\left[\frac{y_A}{L_A}\right]+n_B v_A\left[\frac{x_B}{L_A}\right]\ .$$

Differentiating with respect to $\check{c}_B$ yields

$$\frac{dU_A}{d\check{c}_B} = \frac{dn_A}{d\check{c}_B} v_A \left[\frac{y_A}{L_A}\right] + n_A v_A' \left[\frac{y_A}{L_A}\right] \frac{1}{L_A} \frac{dy_A}{d\check{c}_B} + \frac{dn_B}{d\check{c}_B} v_A \left[\frac{x_B}{L_A}\right] + n_B v_A' \left[\frac{x_B}{L_A}\right] \frac{1}{L_A} \frac{dx_B}{d\check{c}_B} \ .$$

Using $\lim_{\check{c}_B \to 0} x_B = \lim_{\check{c}_B \to 0} dy_A/d\check{c}_B = v_A[0] = 0$, $\lim_{\check{c}_B \to 0} \frac{dn_B}{d\check{c}_B} x_B = \lim_{\check{c}_B \to 0} n_B \frac{dx_B}{d\check{c}_B} = \frac{L_B}{2}$, and $\lim_{\check{c}_B \to 0} \frac{dn_A}{d\check{c}_B} = -\frac{L_A}{\dot{y}_A} \frac{v_A'[0]}{v_A'[\dot{y}_A/L_A]} \frac{L_B}{L_A}$ yields

$$\begin{aligned} \lim_{\check{c}_B \to 0} \frac{dU_A}{d\check{c}_B} &= -\frac{L_A}{\dot{y}_A} \frac{(1-r_A)\, v_A'[0]}{v_A'[\dot{y}_A/L_A]} \frac{L_B}{L_A} v_A \left[\frac{\dot{y}_A}{L_A}\right] + \frac{L_B}{2} \frac{v_A'[0]}{L_A} + \frac{v_A'[0]}{L_A} \frac{L_B}{2} \\ &= \left(1 - \frac{1-r_A}{\varepsilon_{v_A}[\dot{y}_A/L_A]}\right) \frac{L_B}{L_A} v_A'[0] \ . \end{aligned}$$

The result follows because the parenthesized factor is strictly positive iff $r_A > 1 - \varepsilon_{v_A}[\dot{y}_A/L_A]$.
■

Lemma A.30 implies Proposition 4 ■

A.5 Multiple Sectors

A.5.1 Existence

Proof of Lemma 2: Fix some price profiles $(\mathbf{p}_A, \mathbf{s}_B) \in (0,\infty)^{\mathcal{N}_A} \times (0,\infty)^{\mathcal{N}_B}$. The FOCs for an interior extremum are given by (34). Notice that the SOCs for a maximum are automatically satisfied due to strict concavity of u_ξ and v_ξ for all $\xi \in \{1, ...N\}$.

First we show that, for given Lagrangian $\lambda_A \in (0,\infty)$ but ignoring the budget constraint, the maximization problem in (33) permits a unique solution $(\hat{\mathbf{z}}_A, \hat{\mathbf{z}}_B)$. Then we show that there exists a unique λ_A such that $(\hat{\mathbf{z}}_A, \hat{\mathbf{z}}_B)$ also satisfies the budget constraint. Together, these two steps prove the Lemma.

Step 1. Fix $\lambda_A \in (0,\infty)$, and define $\hat{\mathbf{z}}_{\xi A} : [0, n_{\xi A}] \times (0,\infty) \to [0,\infty)$, $\left(i_{\xi A}, u_{\xi A}'\right) \longmapsto \bar{v}_\xi'^{-1}\left[\frac{p_{i_{\xi A}}}{u_{\xi A}'/\lambda_A}\right]$. Observe that $\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u_{\xi A}'\right]$ is the uniquely determined optimal consumption quantity of variety $i_{\xi A}$, given price $p_{i_{\xi A}}$ and marginal utility $u_{\xi A}'$ in sector ξ. Let $\hat{\mathbf{z}}_{\xi B}\left[i_{\xi B}, u_{\xi A}'\right]$ be analogously defined. Finally, let $\hat{\mathbf{z}}_k \equiv [\hat{\mathbf{z}}_{1k}, ...\hat{\mathbf{z}}_{Nk}]^T$, $k \in \{A, B\}$.

Next, we construct a self-map of values $\mathbf{u}_A' \equiv [u_{1A}', ...u_{NA}']^T \in (0,\infty)^N$. For given $\lambda_A \in (0,\infty)$, notice that every $\mathbf{u}_A' \in (0,\infty)^N$ yields a uniquely determined pair $(\hat{\mathbf{z}}_A, \hat{\mathbf{z}}_B)$, by construction. Conversely, every $(\hat{\mathbf{z}}_A, \hat{\mathbf{z}}_B)$ yields a uniquely determined vector $\mathbf{u}_A'$ according to

$$u_{\xi A}' = u_\xi' \left[\int_{i_{\xi A}=0}^{n_{\xi A}} v_\xi \left[\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u_{\xi A}'\right]\right] di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} v_\xi \left[\hat{\mathbf{z}}_{\xi B}\left[i_{\xi B}, u_{\xi A}'\right]\right] di_{\xi B}\right], \ \xi \in \{1, ..., N\} \ . \tag{80}$$

Notice that the system in (80) is fully separable and that each equation gives rise to a unique solution, $u_{\xi A}'$, $\xi \in \{1, ..., N\}$. To see this, notice that $\hat{\mathbf{z}}_{\xi A}$ and $\hat{\mathbf{z}}_{\xi B}$ are strictly increasing in $u_{\xi A}'$

for $u'_{\xi A} > \lambda_A p_{i_{\xi A}} / v'_\xi[0]$ and $u'_{\xi A} > \lambda_A s_{i_{\xi B}} / v'_\xi[0]$, respectively, and zero at or below those values. Hence, the RHS of (80) is weakly decreasing in $u'_{\xi A}$. Since the LHS is strictly increasing in $u'_{\xi A}$ and spans $(0, \infty)$, for every $\lambda_A \in (0, \infty)$, there exists a unique value for $u'_{\xi A}$ that solves (80), for each $\xi \in \{1, ..., N\}$. We denote this value by $u'^*_{\xi A}$.

Step 2. It remains to solve for $\lambda_A \in (0, \infty)$ such that the household budget constraint is satisfied. Subbing $\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u'^*_{\xi A}\right] = \bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]$ into the budget constraint yields

$$\sum_{\xi=1}^{N} \left\{ \int_{i_{\xi A}=0}^{n_{\xi A}} p_{i_{\xi A}} \bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right] di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} s_{i_{\xi B}} \bar{v}'^{-1}_\xi \left[\frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A}\right] di_{\xi B} \right\} = I_A \; . \tag{81}$$

We now prove a series of three claims. First, in Claim 1, we show that the "sectoral price index" $u'^*_{\xi A}/\lambda_A$ decreases monotonically in λ_A, implying that there exists at most one λ_A that satisfies (81). Then, in Claim 2, we show that as $\lambda_A \to \infty$, the argument of $\bar{v}'^{-1}_\xi[\cdot]$ in (81) converges to $v'_\xi[0]$ for almost all varieties $i_{\xi k}$, implying that the LHS of (81) converges to 0. Finally, in Claim 3, we show that as $\lambda_A \to 0$, the argument $\bar{v}'^{-1}_\xi[\cdot]$ in (81) converges to $v'[\infty] \equiv \lim_{z\to\infty} v'[z]$ for a positive measure of varieties, implying that the LHS of (81) converges to ∞. Together, these claims imply that there exists a unique $\lambda_A \in (0, \infty)$ that solves (81).

Claim 1: $\frac{d}{d\lambda_A}\left[u'^*_{\xi A}/\lambda_A\right] < 0$.

Proof: Implicitly differentiating (80) with respect to λ_A yields

$$\frac{du'^*_{\xi A}}{d\lambda_A} = u''_\xi \cdot \left(\int_{i_{\xi A}=0}^{n_{\xi A}} \frac{d}{d\lambda_A} v_\xi\left[\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u'^*_{\xi A}\right]\right] di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} \frac{d}{d\lambda_A} v_\xi\left[\hat{\mathbf{z}}_{\xi B}\left[i_{\xi B}, u'^*_{\xi A}\right]\right] di_{\xi B} \right) \; . \tag{82}$$

Recall that $\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u'^*_{\xi A}\right] = \bar{v}'^{-1}_\xi\left[\frac{p_{i_{\xi A}}}{u'_{\xi A}/\lambda_A}\right]$. Hence,

$$\begin{aligned} \frac{d}{d\lambda_A} v_\xi\left[\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u'^*_{\xi A}\right]\right] &= v'_\xi\left[\hat{\mathbf{z}}_{\xi A}\left[i_{\xi A}, u'^*_{\xi A}\right]\right] \frac{d}{d\lambda_A} \bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right] \\ &= \frac{v'_{\xi A}}{v''_{\xi A}} p_{i_{\xi A}} \frac{1}{u'^*_{\xi A}} \left(1 - \frac{\lambda_A}{u'^*_{\xi A}} \frac{du'^*_{\xi A}}{d\lambda_A}\right) \cdot 1_{\left\{\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A} \leq v'_\xi[0]\right\}} \; , \end{aligned}$$

and similarly,

$$\frac{d}{d\lambda_A} v_\xi\left[\hat{\mathbf{z}}_{\xi B}\left[i_{\xi B}, u'^*_{\xi A}\right]\right] = \frac{v'_{\xi B}}{v''_{\xi B}} s_{i_{\xi B}} \frac{1}{u'^*_{\xi A}} \left(1 - \frac{\lambda_A}{u'^*_{\xi A}} \frac{du'^*_{\xi A}}{d\lambda_A}\right) \cdot 1_{\left\{\frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A} \leq v'_\xi[0]\right\}} \; .$$

Therefore, we may write (82) as

$$\frac{du'^*_{\xi A}}{d\lambda_A} = \left(1 - \frac{\lambda_A}{u'^*_{\xi A}} \frac{du'^*_{\xi A}}{d\lambda_A}\right) Z \; , \tag{83}$$

where

$$Z \equiv \frac{u''_\xi}{u'^*_{\xi A}} \cdot \left(\int_{i_{\xi A}=0}^{n_{\xi A}} \frac{v'_{\xi A}}{v''_{\xi A}} p_{i_{\xi A}} \cdot 1_{\left\{ \frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A} \le v'_\xi[0] \right\}} di_{\xi A} + \int_{i_{\xi B}=0}^{n_{\xi B}} \frac{v'_{\xi B}}{v''_{\xi B}} s_{i_{\xi B}} \cdot 1_{\left\{ \frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A} \le v'_\xi[0] \right\}} di_{\xi B} \right) \ge 0 \ .$$

Solving (83) for $\frac{du'^*_{\xi A}}{d\lambda_A}$,

$$\frac{du'^*_{\xi A}}{d\lambda_A} = \frac{Z}{1 + \frac{\lambda_A}{u'^*_{\xi A}} Z} \ . \tag{84}$$

Finally, we calculate $\frac{d}{d\lambda_A}\left[u'^*_{\xi A}/\lambda_A\right]$ as

$$\frac{d}{d\lambda_A}\left[\frac{u'^*_{\xi A}}{\lambda_A}\right] = \frac{1}{\lambda_A}\left(\frac{du'^*_{\xi A}}{d\lambda_A} - \frac{u'^*_{\xi A}}{\lambda_A}\right) = \frac{1}{\lambda_A}\left(\frac{Z}{1 + \frac{\lambda_A}{u'^*_{\xi A}} Z} - \frac{u'^*_{\xi A}}{\lambda_A}\right) = \frac{1}{\lambda_A}\frac{-u'^*_{\xi A}}{\lambda_A\left(1 + \frac{\lambda_A}{u'^*_{\xi A}} Z\right)} < 0 \ ,$$

where in the first step we simply apply the quotient rule, and in the second step we substitute (84).

This proves the claim.

Our second claim implies that when $\lambda_A \to \infty$, the argument of $\bar{v}'^{-1}_\xi[\cdot]$ on the LHS of (81) is greater than or equal to $v'[0]$. Hence, the LHS of (81) goes to zero for $\lambda_A \to \infty$.

Claim 2:

a) For almost all $i_{\xi A} \in [0, n_{\xi A}]$, $\lim_{\lambda_A \to \infty} u'^*_{\xi A}/\lambda_A \le \frac{p_{i_{\xi A}}}{v'[0]}$.

b) For almost all $i_{\xi B} \in [0, n_{\xi B}]$, $\lim_{\lambda_A \to \infty} u'^*_{\xi A}/\lambda_A \le \frac{s_{i_{\xi B}}}{v'[0]}$.

Proof: Notice that $\lim_{\lambda_A \to \infty} u'^*_{\xi A}/\lambda_A$ exists, since $\frac{d}{d\lambda_A}\left[u'^*_{\xi A}/\lambda_A\right] < 0$ (Claim 1).

a) Suppose not. Then there exists a strictly positive measure of $i_{\xi A} \in [0, n_{\xi A}]$, such that $\lim_{\lambda_A \to \infty} u'^*_{\xi A}/\lambda_A > \frac{p_{i_{\xi A}}}{v'[0]}$. Hence, $\lim_{\lambda_A \to \infty} u'^*_{\xi A} = \infty$. Due to strict concavity of u_ξ, this can only happen if the argument of $u'_\xi[\cdot]$ in (80) goes to zero when $\lambda_A \to \infty$. Therefore, for almost all $i_{\xi A} \in [0, n_{\xi A}]$,

$$\lim_{\lambda_A \to \infty} v_\xi\left[\bar{v}'^{-1}_\xi\left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = 0 \iff \lim_{\lambda_A \to \infty} \bar{v}'^{-1}_\xi\left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right] = 0$$

$$\iff \lim_{\lambda_A \to \infty} \frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A} \ge v'_\xi[0] \iff \lim_{\lambda_A \to \infty} u'^*_{\xi A}/\lambda_A \le \frac{p_{i_{\xi A}}}{v'_\xi[0]} \ .$$

Contradiction.

b) The proof is analogous to that of *a)*.

This proves the claim.

Recall that $v'[\infty] \equiv \lim_{z \to \infty} v'[z]$. Our third claim implies that when $\lambda_A \to 0$, the argument of $\bar{v}'^{-1}_\xi[\cdot]$ on the LHS of (81) is goes $v'[\infty]$. Hence, the LHS of (81) goes to infinity for $\lambda_A \to 0$.

Claim 3:

a) If $v'[\infty] = 0$, then $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A = \infty$.

b) If $0 < v'[\infty] < v'[0]$, then for a strictly positive measure of $i_{\xi A} \in [0, n_{\xi A}]$, $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A \geq \frac{p_{i_{\xi A}}}{v'[\infty]}$, or for a strictly positive measure of $i_{\xi B} \in [0, n_{\xi B}]$, $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A \geq \frac{s_{i_{\xi B}}}{v'[\infty]}$.

Proof: Notice that $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A$ exists, since $\frac{d}{d\lambda_A}\left[u'^*_{\xi A}/\lambda_A\right] < 0$ (claim 1).

a) Suppose not. Then $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A < \infty$. This implies that $\lim_{\lambda_A \to \infty} u'^*_{\xi A} = 0$. Due to strict concavity of u_ξ, this can only happen if the argument of $u'_\xi[\cdot]$ in (80) goes to ∞ when $\lambda_A \to 0$. Hence, for a positive measure of $i_{\xi A} \in [0, n_{\xi A}]$ or $i_{\xi B} \in [0, n_{\xi B}]$,

$$\lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \text{ or } \lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{s_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \ .$$

Now notice that

$$\lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \iff \lim_{\lambda_A \to 0} \bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right] = \infty \iff \lim_{\lambda_A \to 0} \frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A} = 0 \ .$$

and similarly

$$\lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \iff \lim_{\lambda_A \to 0} \frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A} = 0 \ .$$

Since $p_{i_{\xi A}}, s_{i_{\xi B}} > 0$, this implies that $\lim_{\lambda_A \to 0} u'^*_{\xi k}/\lambda_k = \infty$. Contradiction.

b) Suppose not. Then for almost all $i_{\xi A} \in [0, n_{\xi A}]$, $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A < \frac{p_{i_{\xi A}}}{v'[\infty]}$, and for almost all $i_{\xi B} \in [0, n_{\xi B}]$, $\lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A < \frac{s_{i_{\xi B}}}{v'[\infty]}$. For these $i_{\xi k}$, the argument of $u'_\xi[\cdot]$ in (80) must go to ∞ when $\lambda_A \to 0$. Hence, for a positive measure of $i_{\xi A} \in [0, n_{\xi A}]$ or $i_{\xi B} \in [0, n_{\xi B}]$,

$$\lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \text{ or } \lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{s_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \ .$$

Now notice that

$$\begin{aligned} \lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \quad &\iff \quad \lim_{\lambda_A \to 0} \bar{v}'^{-1}_\xi \left[\frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A}\right] = \infty \\ &\iff \quad \lim_{\lambda_A \to 0} \frac{p_{i_{\xi A}}}{u'^*_{\xi A}/\lambda_A} \leq v'[\infty] \iff \lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A \geq \frac{p_{i_{\xi A}}}{v'[\infty]} \ , \end{aligned}$$

and similarly

$$\lim_{\lambda_A \to 0} v_\xi \left[\bar{v}'^{-1}_\xi \left[\frac{s_{i_{\xi B}}}{u'^*_{\xi A}/\lambda_A}\right]\right] = \infty \iff \lim_{\lambda_A \to 0} u'^*_{\xi A}/\lambda_A \geq \frac{s_{i_{\xi B}}}{v'[\infty]} \ .$$

Contradiction.

This proves the claim.

Jointly, Claims 1, 2, and 3 complete Step 2 and, hence, the proof of Lemma 2. ■

Proof of Proposition 5: Fix some $\xi \in \{1, ..., N\}$ and a profile $\boldsymbol{\tau}_{-\xi} \in ([1, \infty) \times [0, 1))^{N-1}$ of trade costs and tariffs for all sectors other than ξ. First suppose FOC_{x_ξ} is slack and, thus, $x_\xi = 0$. (As we shall see, this holds for $\varphi_\xi = t_\xi / (1 - r_\xi)$ sufficiently large.) Substituting FOC_{y_ξ} and $x_\xi = 0$ into ZP_ξ, we find that sector-ξ-autarky level $\dot{y}_\xi$ is the unique solution to

$$\frac{\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]}{1 - \varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]} y_\xi = \frac{F_\xi}{c_\xi} . \tag{85}$$

Next, suppose FOC_{x_ξ} binds. (As we shall see, this holds for $\varphi_\xi = t_\xi / (1 - r_\xi)$ sufficiently small, i.e., close to 1). Dividing FOC_{y_ξ} by FOC_{x_ξ}, as well as substituting FOC_{y_ξ} and FOC_{x_ξ} into ZP_ξ, yields the following system of two equations and two unknowns y_ξ, x_ξ:

$$\begin{aligned} FOC_\xi : \quad & \frac{v'_\xi\left[\frac{y_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]\right)}{v'_\xi\left[\frac{x_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{x_\xi}{L}\right]\right)} \equiv \frac{m_\xi\left[\frac{y_\xi}{L}\right]}{m_\xi\left[\frac{x_\xi}{L}\right]} = \frac{1}{\varphi_\xi} \\ ZP_\xi : \quad & \frac{\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]}{1-\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]} y_\xi + \frac{\varepsilon_{v'_\xi}\left[\frac{x_\xi}{L}\right]}{1-\varepsilon_{v'_\xi}\left[\frac{x_\xi}{L}\right]} t_\xi x_\xi = \frac{F_\xi}{c_\xi} . \end{aligned}$$

Notice that ZP_ξ implies that y_ξ is a strictly decreasing function of x_ξ, which we denote by $y_\xi[x_\xi]$. Substituting $y_\xi[x_\xi]$ into FOC_ξ yields a single equation with one unknown, x_ξ. Since $m_\xi\left[\frac{y_\xi[x_\xi]}{L}\right] / m_\xi\left[\frac{x_\xi}{L}\right]$ is strictly increasing in x_ζ, there exists at most one solution for $x_\xi \geq 0$. Furthermore, for $\varphi_\xi \geq 1$ sufficiently small, notice that such a solution must indeed exist. Home-bound production $y_\xi = y_\xi[x_\xi]$ then follows immediately.

Dividing FOC_{y_ξ} by FOC_{y_1}, $\xi \neq 1$, and using the fact that $P_\xi \equiv u'_\xi / \lambda$ gives

$$FOC_{y_\xi / y_1} : \quad \frac{u'_\xi\left[n_\xi\left(v_\xi\left[\frac{y_\xi}{L}\right]+v_\xi\left[\frac{x_\xi}{L}\right]\right)\right]}{u'_1\left[n_1\left(v_1\left[\frac{y_1}{L}\right]+v_1\left[\frac{x_1}{L}\right]\right)\right]} \frac{v'_\xi\left[\frac{y_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{y_\xi}{L}\right]\right)}{v'_1\left[\frac{y_1}{L}\right]\left(1-\varepsilon_{v'_1}\left[\frac{y_1}{L}\right]\right)} = \frac{c_\xi}{c_1} . \tag{86}$$

This yields n_ξ as a function of n_1, which we denote by $n_\xi[n_1]$. Notice that $n_\xi[n_1]$ is strictly increasing. Furthermore, using our assumption of convenience that $\lim_{\boldsymbol{v}\to 0} u'_\xi[\boldsymbol{v}] = \infty$, it follows that $\lim_{n_1 \to 0} n_\xi[n_1] = 0$. Finally, subbing $n_\xi[n_1]$ into LM yields

$$LM : \quad \textstyle\sum_{\xi=1}^N n_\xi[n_1] \, C[y_\xi + t_\xi x_\xi] = L .$$

Since $n_\xi[n_1]$ is strictly increasing and $\lim_{n_1 \to 0} n_\xi[n_1] = 0$, the LHS of LM spans $[0, \infty)$ as n_1 runs from zero to infinity. Hence, the entire profile of masses of firms, $\mathbf{n}$, is uniquely determined.

It remains to the find the boundary between 'large' and 'small' φ_ξ. Let $\dot{\varphi}_\xi[\boldsymbol{\tau}_{-\xi}]$ denote the threshold level of trade frictions in sector ξ, conditional on $\boldsymbol{\tau}_{-\xi}$. That is, at $\varphi_\xi = \dot{\varphi}_\xi[\mathbf{t}_{-\xi}]$, $x_\xi = 0$ solves FOC_{x_ξ} in (37) with equality. Dividing FOC_{x_ξ} by FOC_{y_ξ} and substituting $y_\xi = \dot{y}_\xi$

and $x_\xi = 0$ yields

$$FOC_\xi : \quad \frac{v'_\xi[0]}{v'_\xi\left[\frac{\dot{y}_\xi}{L}\right]\left(1-\varepsilon_{v'_\xi}\left[\frac{\dot{y}_\xi}{L}\right]\right)} = \varphi_\xi \ .$$

Hence, $\dot{\varphi}_\xi\left[\boldsymbol{\tau}_{-\xi}\right]$ does not depend on $\boldsymbol{\tau}_{-\xi}$ and is indeed equal to $\dot{\varphi}_\xi$ as defined in (38). ■

A.5.2 Bad Trade

Recall that $\dot{\boldsymbol{\tau}}^\xi \in \dot{T}^\xi$ is a profile of trade costs and tariffs such that sector ξ is on the threshold of trading; i.e., its ξ-th component, $\dot{\boldsymbol{\tau}}^\xi_\xi = (t_\xi, r_\xi)$, is such that $\varphi_\xi \equiv t_\xi / (1 - r_\xi) = \dot{\Phi}$. All other components of $\dot{\boldsymbol{\tau}}^\xi$ are unrestricted. Also, recall that $\Delta\mathbf{t}^\xi = \left((\Delta t_\xi, \Delta r_\xi), (0,0)^{N-1}\right)$ is a profile of 'component-ξ-only' changes: that is, $\Delta\mathbf{t}^\xi_\xi = (\Delta t_\xi, \Delta r_\xi) \in \mathbb{R}^2$, while all other components $\psi \neq \xi$ are $\Delta\mathbf{t}^\xi_\psi = (0,0)$.

Fixing $\xi \in \{1, ..., N\}$, below, we define a function $\boldsymbol{\tau}[\sigma]$ that associates a profile $\boldsymbol{\tau} \in T$ of trade costs and tariffs to each $\sigma \in [-1, 1]$. Only varying its ξ-th component $\boldsymbol{\tau}_\xi[\sigma] = (t_\xi[\sigma], r_\xi[\sigma])$, function $\boldsymbol{\tau}[\cdot]$ takes us from a point $\dot{\boldsymbol{\tau}}^\xi$ on the sector-ξ trade threshold $\dot{T}^\xi$ for $\sigma = 0$, to a point in the sector-ξ no-trade region $\bar{T}^\xi$ for $\sigma > 0$. For $\sigma < 0$, $\boldsymbol{\tau}[\cdot]$ takes us to a point in the sector-ξ trade region $\underline{T}^\xi$. All components of $\boldsymbol{\tau}[\cdot]$ other than ξ are constant in σ.

Formally, the function $\boldsymbol{\tau}[\cdot]$ is defined as $\boldsymbol{\tau} : [-1, 1] \to [1, \infty)^2 \times [0, 1)^2$, $\sigma \mapsto \boldsymbol{\tau}[\sigma] = ((t_1, r_1), ..., (t_\xi[\sigma], r_\xi[\sigma]), ..., (t_N, r_N))$, where for some $0 < \delta < 1$,

$$\boldsymbol{\tau}[\sigma] = \dot{\boldsymbol{\tau}}^\xi + \sigma\Delta\boldsymbol{\tau}^\xi \in \begin{cases} \bar{T}^\xi & \text{if } \sigma \in (0, \delta) \\ \underline{T}^\xi & \text{if } \sigma \in (-\delta, 0) \end{cases} .$$

Also, let $\varphi_\xi[\sigma] \equiv t_\xi[\sigma] / (1 - r_\xi[\sigma])$.

The welfare effect of the initiation of trade in sector ξ is found by calculating the left directional derivative $\overleftarrow{\nabla}_{\Delta\boldsymbol{\tau}^\xi} U\Big|_{\boldsymbol{\tau}=\dot{\boldsymbol{\tau}}^\xi}$ of utility U at the point $\dot{\boldsymbol{\tau}}^\xi$ in the permissible direction $\Delta\boldsymbol{\tau}^\xi$. Notice that is equivalent to differentiating $\boldsymbol{\tau}[\sigma]$ with respect to σ and evaluating at $\sigma = 0$.

Proof of Lemma 3:

Direction $\Delta\tau^\xi$ is permissible, iff $\boldsymbol{\tau} = \dot{\boldsymbol{\tau}}^\xi + \sigma\Delta\boldsymbol{\tau}^\xi$ lies in $\bar{T}_\xi$ for sufficiently small $\sigma > 0$ and in $\underline{T}^\xi$ for sufficiently small $\sigma < 0$. Recall that threshold point $\dot{\boldsymbol{\tau}}^\xi \in \dot{T}_\xi$ is characterized by $\varphi_\xi = \dot{\varphi}_\xi$, while points in $\bar{T}_\xi$ and $\underline{T}^\xi$ are characterized by $\varphi_\xi > \dot{\varphi}_\xi$ and $\varphi_\xi < \dot{\varphi}_\xi$, respectively. Hence, permissibility of $\Delta\tau^\xi$ is equivalent to

$$\frac{d\varphi_\xi[\sigma]}{d\sigma}\bigg|_{\sigma=0} = \frac{d}{d\sigma}\left[\frac{t_\xi[\sigma]}{1 - r_\xi[\sigma]}\right]\bigg|_{\sigma=0} > 0 \iff \frac{1}{\dot{t}_\xi}\Delta t_\xi + \frac{1}{1 - \dot{r}_\xi}\Delta r_\xi > 0 \ .$$

This proves the lemma. ■

Proof of Proposition 6:

The proof is analogous to the one for Bad Trade in the single-sector model. First we replace (t_ξ, r_ξ) in the system of equilibrium equations (37) with the ξ-th component of $\boldsymbol{\tau}[\sigma]$, i.e., $(t_\xi[\sigma], r_\xi[\sigma])$. Then we implicitly differentiate the system with respect to σ and evaluate the change in utility at $\sigma = 0$.

Focusing on functional dependencies, we rewrite ZP in (37) as

$$\begin{aligned} ZP_\xi: &\quad \pi_\xi[y_\xi, x_\xi, P_\xi, t_\xi[\sigma]] = P_\xi\left(v_\xi'\left[\tfrac{y_\xi}{L}\right] y_\xi + (1-r_\xi)\, v_\xi'\left[\tfrac{x_\xi}{L}\right] x_\xi\right) - C_\xi[y_\xi + t_\xi[\sigma]\, x_\xi] = 0 \\ ZP_\psi: &\quad \pi_\psi[y_\psi, x_\psi, P_\psi] = P_\psi\left(v_\psi'\left[\tfrac{y_\psi}{L}\right] y_\psi + (1-r_\psi)\, v_\psi'\left[\tfrac{x\psi}{L}\right] x_\psi\right) - C[y_\psi + t_\psi x_\psi] = 0\ . \end{aligned} \tag{87}$$

where $\xi, \psi \in \{1, ..., N\}$ and $\xi \neq \psi$. Implicitly differentiating ZP with respect to σ yields

$$\begin{aligned} \frac{d\pi_\xi[y_\xi, x_\xi, P_\xi, t_\xi[\sigma]]}{d\sigma} &= \frac{\partial \pi_\xi}{\partial y_\xi}\frac{dy_\xi}{d\sigma} + \frac{\partial \pi_\xi}{\partial x_\xi}\frac{dx_\xi}{d\sigma} + \frac{\partial \pi}{\partial P_\xi}\frac{dP_\xi}{d\sigma} + \frac{\partial \pi_\xi}{\partial t_\xi}\frac{dt_\xi}{d\sigma} + \frac{\partial \pi_\xi}{\partial r_\xi}\frac{dr_\xi}{d\sigma} = 0 \\ \frac{d\pi_\psi[y_\psi, x_\psi, P_\psi]}{d\sigma} &= \frac{\partial \pi_\psi}{\partial y_\psi}\frac{dy_\psi}{d\sigma} + \frac{\partial \pi_\psi}{\partial x_\psi}\frac{dx_\psi}{d\sigma} + \frac{\partial \pi_\psi}{\partial P_\psi}\frac{dP_\psi}{d\sigma} = 0\ . \end{aligned}$$

Since $y_\xi, y_\psi > 0$, the envelope theorem implies that $\frac{\partial \pi_\xi}{\partial y_\xi} = \frac{\partial \pi_\psi}{\partial y_\psi} = 0$. At $\sigma = 0$, sector ξ is on the verge of trading. Again applying the envelope theorem yields $\frac{\partial \pi_\xi}{\partial x_\xi} = 0$. The zero stock of exports, $x_\xi = 0$, implies that $\frac{\partial \pi_\xi}{\partial t_\xi} = \frac{\partial \pi_\xi}{\partial r_\xi} = 0$. Sector $\psi \neq \xi$ can be in any state; namely, autarky, trading, or on the boundary between the two. If ψ is trading or on the boundary, $\frac{\partial \pi_\psi}{\partial x_\psi} = 0$. If ψ is strictly in autarky, then $\frac{dx_\psi}{d\sigma} = 0$. Together, these observations imply that

$$\begin{aligned} \left.\frac{d\pi_\xi}{d\sigma}\right|_{\sigma=0} &= \left.\frac{\partial \pi}{\partial P_\xi}\frac{dP_\xi}{d\sigma}\right|_{\sigma=0} = 0 \\ \left.\frac{d\pi_\psi}{d\sigma}\right|_{\sigma=0} &= \left.\frac{\partial \pi}{\partial P_\psi}\frac{dP_\psi}{d\sigma}\right|_{\sigma=0} = 0\ . \end{aligned} \tag{88}$$

From $y_\xi > 0$ it immediately follows that $\partial\pi_\xi/\partial P_\xi > 0$. Using (88) we then find that $dP_\xi/d\sigma|_{\sigma=0} = 0$. Similarly, $dP_\psi/d\sigma|_{\sigma=0} = 0$. In turn, the FOC_{y_ξ}s (see eqns. (37)) imply that home-bound production per firm in all sectors, y_ξ and y_ψ, does not change, while FOC_{x_ψ} implies that exports per firm in sectors ψ, $\psi \neq \xi$, do not change. That is,

$$\left.\frac{dy_\xi}{d\sigma}\right|_{\sigma=0} = \left.\frac{dy_\psi}{d\sigma}\right|_{\sigma=0} = \left.\frac{dx_\psi}{d\sigma}\right|_{\sigma=0} = 0\ . \tag{89}$$

Exports per firm do rise in sector ξ. To see this, implicitly differentiate FOC_{x_ξ} to find that

$$\begin{aligned} &\frac{dP_\xi}{d\sigma} v_\xi'\left[\frac{x_\xi}{L}\right]\left(1 - \varepsilon_{v_\xi'}\left[\frac{x_\xi}{L}\right]\right) + \frac{P_\xi}{L}\frac{dx_\xi}{d\sigma} v_\xi''\left[\frac{x_\xi}{L}\right]\left(1 - \varepsilon_{v_\xi'}\left[\frac{x_\xi}{L}\right]\right) \\ - &\frac{P_\xi}{L} v_\xi'\left[\frac{x_\xi}{L}\right]\frac{dx_\xi}{d\sigma}\varepsilon_{v_\xi'}'\left[\frac{x_\xi}{L}\right] = c_\xi \frac{d\varphi_\xi[\sigma]}{d\sigma}\ . \end{aligned}$$

Evaluate at $\sigma = 0$, recalling that $dP_\xi/d\sigma|_{\sigma=0} = 0$:

$$\left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} = \frac{c_\xi}{\frac{P_\xi}{L}\left(v''_\xi[0] - v'_\xi[0]\,\varepsilon'_{v'_\xi}[0]\right)} \left.\frac{d\varphi_\xi[\sigma]}{d\sigma}\right|_{\sigma=0} < 0 \ , \tag{90}$$

where the inequality follows because $d\varphi_\xi[\sigma]/d\sigma|_{\sigma=0} > 0$, $v''_\xi[0] < 0$, $v'_\xi[0] > 0$, and $\varepsilon'_{v'_\xi}[0] > 0$.

Implicitly differentiating LM with respect to σ (see eqns. (37)) yields

$$n_\xi c_\xi \left(\frac{dy_\xi}{d\sigma} + t_\xi \frac{dx_\xi}{d\sigma} + x_\xi \frac{dt_\xi[\sigma]}{d\sigma}\right) + \frac{dn_\xi}{d\sigma} C[y_\xi + t_\xi x_\xi] \tag{91}$$

$$+ \quad \sum_{\psi \neq \xi}^{N} \left\{ n_\psi c \left(\frac{dy_\psi}{d\sigma} + x_\psi \frac{dt_\psi}{d\sigma} + t_\psi \frac{dx_\psi}{d\sigma}\right) + \frac{dn_\psi}{d\sigma} C[y_\psi + t_\psi x_\psi] \right\} = 0 \ .$$

At $\sigma = 0$, it follows from (89) and $x_\xi = 0$ that

$$\dot{n}_\xi c_\xi \dot{t}_\xi \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} + C[\dot{y}_\xi] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} + \sum_{\psi \neq \xi}^{N} C[y_\psi + t_\psi x_\psi] \left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} = 0 \ .$$

Using that, at the initiation of trade in sector ξ, $\dot{t}_\xi c_\xi = (1 - r_\xi) P_\xi v'_\xi[0]$ (from FOC_{x_ξ}), while $C[\dot{y}_\xi] = P_\xi v'_\xi[\dot{y}_\xi/L]\,\dot{y}_\xi$ and $C[y_\psi + t_\psi x_\psi] = P_\psi\left(v'_\psi\left[\frac{y_\psi}{L}\right] y_\psi + v'_\psi\left[\frac{x_\psi}{L}\right] x_\psi\right)$ (from ZP), we find

$$\dot{n}_\xi (1 - r_\xi) P_\xi v'_\xi[0] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} + P_\xi v'_\xi\left[\frac{\dot{y}_\xi}{L}\right] y_\xi \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} \tag{92}$$

$$+ \quad \sum_{\psi \neq \xi}^{N} P_\psi \left(v'_\psi\left[\frac{y_\psi}{L}\right] y_\psi + v'_\psi\left[\frac{x_\psi}{L}\right] x_\psi\right) \left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} = 0 \ . \tag{93}$$

Next, observe from (86) that

$$m_\xi\left[\frac{y_\xi}{L}\right] u'_\xi\left[n_\xi\left(v_\xi\left[\frac{y_\xi}{L}\right] + v_\xi\left[\frac{x_\xi}{L}\right]\right)\right] c_\psi = c_\xi m_\psi\left[\frac{y_\psi}{L}\right] u'_\psi\left[n_\psi\left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right)\right] \ .$$

Implicitly differentiating FOC_{y_ξ}/FOC_{y_ψ} wrt. σ, and using (89), $x_\xi = v_\xi[0] = 0$, and $m_\xi/m_\psi = u'_\psi/u'_\xi$ yields

$$v_\xi\left[\frac{\dot{y}_\xi}{L}\right] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} + \frac{\dot{n}_\xi}{L} v'_\xi[0] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} = \frac{c_\xi}{c_\psi} \frac{u''_\psi/u'_\psi}{u''_\xi/u'_\xi} \left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right) \left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} \ . \tag{94}$$

Define the shorthand aggregator variable

$$\Theta \equiv \sum_{\psi \neq \xi}^{N} P_\psi \frac{c_\xi}{c_\psi} \frac{u''_\xi/u'_\xi}{u''_\psi/u'_\psi} \frac{v'_\psi[y_\psi/L]\,y_\psi + v'_\psi[x_\psi/L]\,x_\psi}{v_\psi[y_\psi/L] + v_\psi[x_\psi/L]} > 0 \ ,$$

and sub (94) into (92) to get

$$\begin{aligned}
& \dot{n}_\xi \left(1-r_\xi\right) P_\xi v_\xi' \left[0\right] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} + P_\xi v_\xi' \left[\frac{\dot{y}_\xi}{L}\right] y_\xi \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} \\
+ & \left(v_\xi \left[\frac{\dot{y}_\xi}{L}\right] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} + \frac{\dot{n}_\xi}{L} v_\xi' \left[0\right] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} \right) \Theta = 0 \ .
\end{aligned}$$

Rearranging,

$$\frac{\dot{n}_\xi}{L} v_\xi' \left[0\right] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} = -v_\xi \left[\frac{\dot{y}_\xi}{L}\right] \frac{\frac{\Theta}{L} + P_\xi \varepsilon_{v_\xi} \left[\dot{y}_\xi/L\right]}{\frac{\Theta}{L} + \left(1-r_\xi\right) P_\xi} \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} \ . \tag{95}$$

Hence, varieties in sector ξ fall, since

$$sign\left[\left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0}\right] = -sign\left[\left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0}\right] > 0 \ .$$

Subbing (95) back into (94) and simplifying yields

$$\left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right) \left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} = \frac{c_\xi}{c_\psi} \frac{u_\xi''/u_\xi'}{u_\psi''/u_\psi'} P_\xi v_\xi \left[\frac{\dot{y}_\xi}{L}\right] \frac{\left(1-r_\xi\right) - \varepsilon_{v_\xi}\left[\dot{y}_\xi/L\right]}{\frac{\Theta}{L} + \left(1-r_\xi\right) P_\xi} \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} \ . \tag{96}$$

Hence, for $\psi \neq \xi$ we find that

$$\left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} \overset{(>)}{\underset{(<)}{=}} 0 \iff \left(1-r_\xi\right) - \varepsilon_{v_\xi}\left[\frac{\dot{y}_\xi}{L}\right] \overset{(>)}{\underset{(<)}{=}} 0 \iff r_\xi \overset{(<)}{\underset{(>)}{=}} 1 - \varepsilon_{v_\xi}\left[\frac{\dot{y}_\xi}{L}\right] \ .$$

That is, variety in all sectors $\psi \neq \xi$ falls (rises) iff $r_\xi \overset{(>)}{<} 1 - \varepsilon_{v_\xi}\left[\dot{y}_\xi/L\right]$.

Recall that household utility is

$$U = u_\xi\left[n_\xi\left(v_\xi\left[\frac{y_\xi}{L}\right] + v_\xi\left[\frac{x_\xi}{L}\right]\right)\right] + \sum_{\psi\neq\xi}^{N} u_\psi\left[n_\psi\left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right)\right] \ .$$

Differentiating with respect to σ yields

$$\begin{aligned}
\frac{dU}{d\sigma} &= u_\xi' \cdot \left\{ \frac{n_\xi}{L}\left(v_\xi'\left[\frac{y_\xi}{L}\right]\frac{dy_\xi}{d\sigma} + v_\xi'\left[\frac{x_\xi}{L}\right]\frac{dx_\xi}{d\sigma}\right) + \frac{dn_\xi}{d\sigma}\left(v_\xi\left[\frac{y_\xi}{L}\right] + v_\xi\left[\frac{x_\xi}{L}\right]\right)\right\} \\
&\quad + \sum_{\psi\neq\xi}^{N} u_\psi' \cdot \left\{ \frac{n_\psi}{L}\left(v_\psi'\left[\frac{y_\psi}{L}\right]\frac{dy_\psi}{d\sigma} + v_\psi'\left[\frac{x_\psi}{L}\right]\frac{dx_\psi}{d\sigma}\right) + \frac{dn_\psi}{d\sigma}\left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right)\right\} \ .
\end{aligned}$$

Evaluating at $\sigma = 0$,

$$\left.\frac{dU}{d\sigma}\right|_{\sigma=0} = u_\xi' \cdot \left(\frac{\dot{n}_\xi}{L} v_\xi'\left[0\right] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} + v_\xi\left[\frac{\dot{y}_\xi}{L}\right] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0}\right) + \sum_{\psi\neq\xi}^{N} u_\psi' \cdot \left(v_\psi\left[\frac{y_\psi}{L}\right] + v_\psi\left[\frac{x_\psi}{L}\right]\right) \left.\frac{dn_\psi}{d\sigma}\right|_{\sigma=0} \ .$$

Substituting (96) and using that $u'_\psi/\lambda = P_\psi$, we can write

$$\left.\frac{dU}{d\sigma}\right|_{\sigma=0} = \lambda P_\xi \left\{ \left(\frac{(1-r_\xi) - \varepsilon_{v_\xi}[\dot{y}_\xi/L]}{\frac{\Theta}{L} + (1-r_\xi) P_\xi} \theta + 1 \right) v_\xi \left[\frac{\dot{y}_\xi}{L}\right] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} + \frac{\dot{n}_\xi}{L} v'_\xi[0] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} \right\} , \quad (97)$$

where $\theta \equiv \sum_{\psi\neq\xi}^N P_\psi \frac{u''_\xi/u'_\xi}{u''_\psi/u'_\psi} > 0$.

From (95) it follows that

$$v_\xi \left[\frac{\dot{y}_\xi}{L}\right] \left.\frac{dn_\xi}{d\sigma}\right|_{\sigma=0} = -\frac{\frac{\Theta}{L} + (1-r_\xi) P_\xi}{\frac{\Theta}{L} + P_\xi \varepsilon_{v_\xi}[\dot{y}_\xi/L]} \frac{\dot{n}_\xi}{L} v'_\xi[0] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} .$$

Substituting this expression into (97) and rearranging, we get

$$\begin{aligned} \left.\frac{dU}{d\sigma}\right|_{\sigma=0} &= -\lambda P_\xi \left(1 - \varepsilon_{v_\xi}[\dot{y}_\xi/L] - r_\xi\right) \frac{\theta + P_\xi}{\frac{\Theta}{L} + P_\xi \varepsilon_{v_\xi}[\dot{y}_\xi/L]} \frac{\dot{n}_\xi}{L} v'_\xi[0] \left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} \overset{(>)}{\underset{(<)}{=}} 0 \\ \iff & \quad r_\xi \overset{(<)}{\underset{(>)}{=}} 1 - \varepsilon_{v_\xi}\left[\frac{\dot{y}_\xi}{L}\right] , \end{aligned}$$

where the inequalities follow from $\left.\frac{dx_\xi}{d\sigma}\right|_{\sigma=0} < 0$ (see eqn. (90)).

This completes the proof of Proposition 6. ■

Lemma A.31 $0 < r^*_\xi < 1$, *and* r^*_ξ *is strictly decreasing in* L *and strictly increasing in* F_ξ/c_ξ. *Furthermore,* $\left(1 - r^*_\xi\right)\dot{\varphi}_\xi > 1$. *Hence, under the critical tariff, trade takes place in sector* ξ *for sufficiently low trade costs* $t_\xi \geq 1$.

Proof To prove that $\left(1 - r^*_\xi\right)\dot{\varphi}_\xi > 1$, recall from the proof of Lemma A.2 that $\varepsilon_{v'}[z]\, v[z] > v[z] - zv'[z]$, which implies that

$$\varepsilon_v[z] > 1 - \varepsilon_{v'}[z] .$$

Therefore,

$$\begin{aligned} \left(1 - r^*_\xi\right)\dot{\varphi}_\xi &= \frac{\left(1 - r^*_\xi\right) v'_\xi[0]}{m_\xi[\dot{y}_\xi/L]} = \frac{\varepsilon_{v_\xi}[\dot{y}_\xi/L]\, v'_\xi[0]}{v'_\xi[\dot{y}_\xi/L]\left(1 - \varepsilon_{v'_\xi}[\dot{y}_\xi/L]\right)} \\ &> \frac{\left(1 - \varepsilon_{v'_\xi}[\dot{y}_\xi/L]\right) v'_\xi[0]}{v'_\xi[\dot{y}_\xi/L]\left(1 - \varepsilon_{v'_\xi}[\dot{y}_\xi/L]\right)} = \frac{v'_\xi[0]}{v'_\xi[\dot{y}_\xi/L]} > 1 . \end{aligned}$$

The proof of the other claims is essentially the same as in Lemma A.18 in the baseline model.

■

A.6 Fixed Cost of Exporting

A.6.1 Existence

Here, we prove existence of equilibrium in the symmetric model with fixed costs of exporting.

Proof of Proposition 7:

Clearly, for all $t \geq \dot{t}_F^0$, the autarky equilibrium $\left(\dot{P}, \dot{n}\right)$ of the (symmetric) standard model with fixed cost of operating $F > 0$ and no fixed cost of exporting is also an equilibrium of the model with fixed cost of operating F and fixed cost of exporting $f > 0$. The reason is that exporting cannot be profitable.

For $t < \dot{t}_F^0$, $\left(\dot{P}, \dot{n}\right)$ continues to be an equilibrium as long as (hypothetical) net export revenue fails to exceed the fixed cost f of exporting—i.e. , as long as

$$\left((1-r)\,\dot{P}v'\left[\tilde{x}/L\right] - tc\right)\tilde{x} - f \leq 0\ . \tag{98}$$

Let $\left(\dot{t}_F^f, \dot{r}\left[\dot{t}_F^f\right]\right)$ denote an element of $\dot{T}_F^f$, where the associated $\tilde{x}$ is denoted $\tilde{x}\left[\dot{P}, \dot{t}_F^f\right]$ (see (40) and (41)). Clearly $\dot{t}_F^f < \dot{t}_F^0$.

At $\dot{t}_F^f$, firms are indifferent between not exporting and exporting $x = \tilde{x}\left[\dot{P}, \dot{t}_F^f\right]$. Therefore, provided it exists, $\dot{t}_F^f$ gives rise to two symmetric pure-strategy equilibria, both with price level $\dot{P}$: one is the autarky equilibrium, the other a trade equilibrium with $x = \tilde{x}\left[\dot{P}, \dot{t}_F^f\right]$, $y = \dot{y}$, $P = \dot{P}$, and

$$n = L\Big/\left\{F + f + c\left(\dot{y} + \dot{t}_F^f\tilde{x}\right)\right\} < \dot{n}\ .$$

Allowing for mixing, any convex combination of the autarky and the trade equilibrium constitutes another symmetric equilibrium. Here, firms export with probability $\alpha \in [0, 1]$ and do not export with probability $1 - \alpha$. (Alternatively, a fraction α of firms export, while the remainder do not. Such an equilibrium is in pure strategies, but not symmetric.) Hence, at $t = \dot{t}_F^f$, there exists a continuum of symmetric mixed-strategy equilibria.

It is easily verified that the equilibrium with $\alpha = 1$ is also an equilibrium of the standard model with fixed cost of operating $F + f$ and no fixed cost of exporting. Furthermore, this implies that $\dot{t}_F^f < \dot{t}_{F+f}^0$. To see this, recall that when t drops marginally below $\dot{t}_{F+f}^0$ in the model with fixed cost of operating equal to $F + f$ and no fixed cost of exporting, exports increase continuously from zero to some small, infinitesimal amount. Since at $t = \dot{t}_F^f$, exports in the model with fixed cost of exporting are 'large,' i.e., non-infinitesimal, while exports are monotone in trade costs, it must be that $\dot{t}_F^f < \dot{t}_{F+f}^0$.

In the next lemma, we show that net export revenues are strictly decreasing in t. This implies single crossing; i.e., for all $t < \dot{t}_F^f$, an equilibrium of the standard model with fixed cost of operating $F + f$ and no fixed of exporting is also an equilibrium of the model with fixed cost of operating F and fixed cost of exporting f.

Lemma A.32 *Net export revenues are strictly decreasing in trade costs. Formally, in a symmetric equilibrium of the standard model without fixed cost of exporting,*

$$\frac{d}{dt}\left[\left((1-r)\,Pv'\left[\frac{x}{L}\right]-tc\right)x\right]<0\ ,$$

for all $t<\dot{t}^{0}_{F+f}$.

Proof In a trade equilibrium,

$$\begin{array}{ll} FOC: & \frac{m[x/L]}{m[y/L]}=\frac{t}{1-r} \\ ZP: & \frac{\varepsilon_{v'}[y/L]}{1-\varepsilon_{v'}[y/L]}y+\frac{\varepsilon_{v'}[x/L]}{1-\varepsilon_{v'}[x/L]}tx=\frac{F+f}{c}\ . \end{array} \tag{99}$$

Differentiating FOC with respect to t, using Lemma A.3.1,

$$\frac{m\,[x/L]}{m\,[y/L]}\left(\varepsilon_{m_k}\,[x/L]\,\frac{1}{x/L}\frac{d\,(x/L)}{dt}-\varepsilon_{m_l}\left[\frac{y}{L}\right]\frac{1}{y/L}\frac{d\,(y/L)}{dt}\right)=\frac{1}{1-r}\ .$$

Re-substituting the FOC back into the result,

$$\frac{t}{1-r}\left(\varepsilon_{m_k}\,[x/L]\,\frac{1}{x}\frac{dx}{dt}-\varepsilon_{m_l}\left[\frac{y}{L}\right]\frac{1}{y}\frac{dy}{dt}\right)=\frac{1}{1-r}\ .$$

Solving for dx/dt,

$$\frac{dx}{dt}=\left(\frac{1}{t}+\varepsilon_m\left[\frac{y}{L}\right]\frac{1}{y}\frac{dy}{dt}\right)\Big/\left(\varepsilon_m\left[\frac{x}{L}\right]\frac{1}{x}\right)\ . \tag{100}$$

Differentiating ZP with respect to t yields

$$-\frac{\varepsilon_m\,[y/L]}{1-\varepsilon_{v'}\,[y/L]}L\frac{dy/L}{dt}+\frac{d}{dt}\left(\frac{\varepsilon_{v'}\,[x/L]}{1-\varepsilon_{v'}\,[x/L]}tx\right)=0\ , \tag{101}$$

where we have used Lemma A.3.2. Completing the differentiation of ZP in (101),

$$-\frac{\varepsilon_m\,[y/L]}{1-\varepsilon_{v'}\,[y/L]}\frac{dy}{dt}-\frac{\varepsilon_m\,[x/L]}{1-\varepsilon_{v'}\,[x/L]}\frac{dx}{dt}t+\frac{\varepsilon_{v'}\,[x/L]}{1-\varepsilon_{v'}\,[x/L]}x=0\ .$$

Substituting dx/dt from (100) and solving for dy/dt yields

$$\frac{dy}{dt}=-\frac{x}{\left(\frac{1}{1-\varepsilon_{v'}[y/L]}+\frac{\frac{tx}{y}}{1-\varepsilon_{v'}[x/L]}\right)\varepsilon_m\left[\frac{y}{L}\right]}=-\frac{xy}{\left(\frac{y}{1-\varepsilon_{v'}[y/L]}+\frac{tx}{1-\varepsilon_{v'}[x/L]}\right)\varepsilon_m\left[\frac{y}{L}\right]}>0\ ,$$

where the inequality follows from $y>0,x>0,0\leq\varepsilon_{v'}<1$, and $\varepsilon_m<0$.

Equation (101) then implies that

$$\frac{d}{dt}\left(\frac{\varepsilon_{v'}\,[x/L]}{1-\varepsilon_{v'}\,[x/L]}tx\right)<0\ . \tag{102}$$

Notice, however, that $\frac{\varepsilon_{v'}[x/L]}{1-\varepsilon_{v'}[x/L]}tx$ is, essentially, another way to write net export revenue using FOC_x. To see this, observe that

$$\left((1-r)\,Pv'\left[\frac{x}{L}\right]-tc\right)x=\left(\frac{tc}{1-\varepsilon_{v'}\,[x/L]}-tc\right)x=\frac{\varepsilon_{v'}\,[x/L]}{1-\varepsilon_{v'}\,[x/L]}tcx\;. \tag{103}$$

Since c is a constant, the result now follows from combining (103) and (102). ■

Finally, we show that, at the initiation of trade, the number of firms falls, while the number of consumed varieties increases.

Lemma A.33 *At $t=\dot{t}_F^f$, the number of firms is strictly decreasing in the probability of exporting, $\alpha\in[0,1]$. Specifically,*

$$\left.\frac{dn}{d\alpha}\right|_{t=\dot{t}_F^f}=-\frac{(1-r)\,v'\,[\tilde{x}/L]\,\tilde{x}}{v'\,[\dot{y}/L]\,\dot{y}+\alpha\,(1-r)\,v'\,[\tilde{x}/L]\,\tilde{x}}n<0\;, \tag{104}$$

where $\tilde{x}\equiv\tilde{x}\left[\dot{t}_F^f\right]<\dot{y}$. By contrast, the number of consumed varieties, $(1+\alpha)\,n$, is strictly increasing in α.

Proof Let $\tilde{x}\equiv\tilde{x}\left[\dot{P},\dot{t}_F^f\right]$. From LM we know that $n|_{t=\dot{t}_F^f}=L\Big/\left(F+c\dot{y}+\alpha\left(f+c\dot{t}_F^f\tilde{x}\right)\right)$. Differentiating $n|_{t=\dot{t}_F^f}$ with respect to α yields

$$\left.\frac{dn}{d\alpha}\right|_{t=\dot{t}_F^f}=-\frac{L\left(f+c\dot{t}_F^f\tilde{x}\right)}{\left(F+c\dot{y}+\alpha\left(f+c\dot{t}_F^f\tilde{x}\right)\right)^2}=-\left.\frac{n\,(1-r)\,v'\left[\frac{x}{L}\right]\tilde{x}}{v'\left[\frac{\dot{y}}{L}\right]\dot{y}+\alpha\,(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\tilde{x}}\right|_{t=\dot{t}_F^f}<0\;,$$

where we have used the zero-profit conditions pinning down $\dot{y}$ and $\tilde{x}=\tilde{x}\left[\dot{P},\dot{t}_F^f\right]$, namely,

$$\begin{array}{ll} ZP_{\dot{y}}: & Pv'\left[\frac{y}{L}\right]y=F+cy \\ ZP_{\tilde{x}}: & (1-r)\,Pv'\left[\frac{x}{L}\right]x=f+c\dot{t}_F^f x\;. \end{array} \tag{105}$$

Next,

$$\begin{aligned}\left.\frac{d\,(1+\alpha)\,n}{d\alpha}\right|_{t=\dot{t}_F^f} &= \left.(1+\alpha)\frac{dn}{d\alpha}+n\right|_{t=\dot{t}_F^f}=-\left.\frac{(1+\alpha)\,n\,(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\tilde{x}}{v'\left[\frac{\dot{y}}{L}\right]\dot{y}+\alpha\,(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\tilde{x}}+n\right|_{t=\dot{t}_F^f} \\ &= \left.\frac{v'\left[\frac{\dot{y}}{L}\right]\dot{y}-(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\tilde{x}}{v'\left[\frac{\dot{y}}{L}\right]\dot{y}+\alpha\,(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\tilde{x}}n\right|_{t=\dot{t}_F^f}\;,\end{aligned}$$

which takes the sign of

$$v'\left[\frac{\dot{y}}{L}\right]\frac{\dot{y}}{L}-(1-r)\,v'\left[\frac{\tilde{x}}{L}\right]\frac{\tilde{x}}{L}\;. \tag{106}$$

Now notice that $v'[z]z$ is strictly increasing for $z > 0$, since

$$\left[v'[z]z\right]' = v'[z] + zv''[z] = v'[z](1 - \varepsilon_{v'}[z]) > 0 \; . \tag{107}$$

Because $\dot{y} > \tilde{x}$, this implies that (106) > 0 and, therefore,

$$\left.\frac{d(1+\alpha)n}{d\alpha}\right|_{t=\dot{t}_F^f} > 0 \; .$$

To see that $\dot{y} > \tilde{x}$, observe that the zero-profit conditions (105) that pin down $\dot{y}$ and $\tilde{x}$ can be rewritten as

$$\frac{\varepsilon_{v'}[y/L]}{1-\varepsilon_{v'}[y/L]}y = \frac{F}{c} \text{ and } \frac{\varepsilon_{v'}[x/L]}{1-\varepsilon_{v'}[x/L]}x = \frac{f}{tc} \; ,$$

where we have used FOC_y and FOC_x. The ranking of $\dot{y}$ and $\tilde{x}$ now follows from $F > f$, $t \geq 1$ and the fact that $\varepsilon_{v'}[\cdot]/(1-\varepsilon_{v'}[\cdot])$ is strictly increasing. ■

This completes the proof of Proposition 7. ■

A.6.2 Bad Trade

Proof of Lemma 4: We prove the lemma through a sequence of lemmas.

Lemma A.34 *Then $r_F^{f\,*}$ exists, is unique, and satisfies $0 < r_F^{f\,*} < 1$. Furthermore, $t_F^{f\,*} \equiv \dot{t}_F^f\left[r_F^{f\,*}\right] > 1$.*

Proof Recall from (42) that the fixed point $r_F^{f\,*}$ is defined as a solution to

$$r = 1 - \frac{\varepsilon_v[\dot{y}/L]}{\varepsilon_v\left[\tilde{x}\left[\dot{t}_F^f[r]\right]/L\right]} \; . \tag{108}$$

Rather than proving that (108) allows for a unique solution $r_F^{f\,*} \in [0,1]$ with an associated $\dot{t}_F^f\left[r_F^{f\,*}\right] \equiv t_F^{f\,*} \in (1,\infty)$, we show that

$$\dot{r}_F^f[t] = 1 - \frac{\varepsilon_v[\dot{y}/L]}{\varepsilon_v[\tilde{x}[t]/L]} \tag{109}$$

allows for a unique solution $t_F^{f\,*} \in (1,\infty)$, with an associated $\dot{r}_F^f\left[t_F^{f\,*}\right] \equiv r_F^{f\,*} \in (0,1)$.

Consider (109) and notice that for $t = \dot{t}_F^f[0] > 1$,

$$\text{LHS} = 0 < 1 - \frac{\varepsilon_v[\dot{y}/L]}{\varepsilon_v\left[\tilde{x}\left[\dot{t}_F^f[0]\right]/L\right]} = \text{RHS} \; ,$$

where the inequality follows from $\varepsilon_v[\cdot]$ being strictly decreasing (Lemma A.2) and $x\left[\dot{t}_F^f\right] < \dot{y}$ (Lemma A.33). Next, we show that the inequality reverses for $t = 1$. Recall from (42) that for

$t \in [1, \infty)$,

$$\dot{r}_F^f [t] = 1 - \frac{m [\dot{y}/L]}{m [\tilde{x} [t] /L]} t \ .$$

Hence, in (109), LHS > RHS at $t = 1$ iff

$$\frac{m [\dot{y}/L]}{m [\tilde{x} [1] /L]} < \frac{\varepsilon_v [\dot{y}/L]}{\varepsilon_v [\tilde{x} [1] /L]} \ ,$$

which is equivalent to

$$\frac{1 - \varepsilon_{v'} [\dot{y}/L]}{1 - \varepsilon_{v'} [\tilde{x} [1] /L]} < \frac{\frac{\dot{y}/L}{v[\dot{y}/L]}}{\frac{\tilde{x}[1]/L}{v[\tilde{x}[1]]}} \ . \tag{110}$$

Since $\varepsilon_{v'} [\cdot]$ is weakly increasing by assumption, and $\tilde{x} [1] < \dot{y}$ (Lemma A.33), the LHS of (110) is ≤ 1. Since $\varepsilon_v [\cdot]$ is strictly decreasing (Lemma A.2), and $\tilde{x} [1] < \dot{y}$, the RHS of (110) is > 1. Hence, the inequality in (110) holds, which implies, in turn, that in (109), LHS > RHS at $t = 1$. The intermediate value theorem now yields existence of a solution $t = t_F^{f\,*} \in \left(1, \dot{t}_F^f [0]\right)$ to (109). Since $t_F^{f\,*} = \dot{t}_F^f \left[r_F^{f\,*}\right]$, this proves the second part of the lemma.

Uniqueness of $t_F^{f\,*}$ follows from the fact that the LHS of (109) is (obviously) strictly decreasing in $t \in [1, \infty)$, while the RHS is increasing in t. To see that the RHS is indeed increasing in t, recall that $\varepsilon_v [\cdot]$ is strictly decreasing in its argument, while $\tilde{x} [t]$ is strictly decreasing in t (see (40)). Hence, we may conclude that $\dot{t}_F^{f\,*} \in \left(1, \dot{t}_F^f [0]\right)$ is unique. Finally, since $r_F^{f\,*} = \dot{r}_F^f \left[t_F^{f\,*}\right]$, it follows from (42) that $r_F^{f\,*} \in (0, 1)$ and unique. ■

Lemma A.35 *For* $\alpha \in [0, 1]$,

$$\left.\frac{dU}{d\alpha}\right|_{t = \dot{t}_F^f} \overset{(<)}{\underset{(>)}{=}} 0 \iff r \overset{(<)}{\underset{(>)}{=}} 1 - \frac{\varepsilon_v [\dot{y}/L]}{\varepsilon_v \left[\tilde{x} \left[\dot{t}_F^f\right] /L\right]} \ .$$

Proof At $t = \dot{t}_F^f$, utility is

$$U = n \left(v [\dot{y}/L] + \alpha v \left[\tilde{x} \left[\dot{t}_F^f\right] /L\right]\right) \ .$$

Differentiating with respect to α and using eqn. (104) to substitute for $\left.\frac{dn}{d\alpha}\right|_{t = \dot{t}_F^f}$ yields

$$\begin{aligned}
\left.\frac{dU}{d\alpha}\right|_{t = \dot{t}_F^f} &= nv \left[\frac{\tilde{x}}{L}\right] + \frac{dn}{d\alpha} \left(v \left[\frac{\dot{y}}{L}\right] + \alpha v \left[\frac{\tilde{x}}{L}\right]\right) \\
&= nv \left[\frac{\tilde{x}}{L}\right] - \frac{(1-r) v' [\tilde{x}/L] \tilde{x}}{v' [\dot{y}/L] \dot{y} + \alpha (1-r) v' [\tilde{x}/L] \tilde{x}} n \left(v \left[\frac{\dot{y}}{L}\right] + \alpha v \left[\frac{\tilde{x}}{L}\right]\right) \\
&= n \frac{v [\tilde{x}/L] v' [\dot{y}/L] \dot{y} - (1-r) v [\dot{y}/L] v' [\tilde{x}/L] \tilde{x}}{v' [\dot{y}/L] \dot{y} + \alpha (1-r) v' [\tilde{x}/L] \tilde{x}} \ .
\end{aligned}$$

Hence,

$$\left.\frac{dU}{d\alpha}\right|_{t=\dot{t}_F^f} \underset{(>)}{\overset{(<)}{=}} 0 \iff \frac{v'[\dot{y}/L]\,\dot{y}/L}{v[\dot{y}/L]} \underset{(>)}{\overset{(<)}{=}} (1-r)\frac{v'[\tilde{x}/L]\,\tilde{x}/L}{v[\tilde{x}/L]} ,$$

which proves the claim. ■

Finally, $\dot{y} > \tilde{x}\left[\dot{t}_F^f\right]$ by Lemma A.33, while $\varepsilon_v[\cdot] \geq 0$ is strictly decreasing in its argument by Lemma A.2. Hence, we may conclude that

$$0 < r^* = 1 - \frac{\varepsilon_v[\dot{y}/L]}{\varepsilon_v\left[\tilde{x}\left[\dot{t}_F^f\right]/L\right]} < 1 .$$

This completes the proof of Lemma 4. ■

A.7 Heterogeneous Firms

Proof of Proposition 9: Define

$$\pi \equiv P\left(\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{y_c}{L}\right] y_c g_c + (1-r)\sum_{c=c_1}^{\bar{c}_x} v'\left[\frac{x_c}{L}\right] x_c g_c\right) - C[\mathbf{y},\mathbf{x}] .$$

Focusing on functional dependencies, we write ZP as

$$ZP: \quad \pi[\mathbf{y},\mathbf{x},P,t] = 0 , \tag{111}$$

where $\mathbf{y}$ and $\mathbf{x}$ denote the vectors of production schedules.

Implicitly differentiating ZP in (111) with respect to t yields

$$\frac{d\pi[\mathbf{y},\mathbf{x},P,t]}{dt} = \left(\frac{\partial\pi}{\partial\mathbf{y}}\right)^T \cdot \frac{d\mathbf{y}}{dt} + \left(\frac{\partial\pi}{\partial\mathbf{x}}\right)^T \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial\pi}{\partial P}\frac{dP}{dt} + \frac{\partial\pi}{\partial t} = 0 .$$

Here, '$\cdot$' denotes the inner-product, and ^{T} the transpose. Optimality of $\mathbf{y}$ implies that $\partial\pi/\partial\mathbf{y}|_{t=\dot{t}} \leq 0$ and $dy_c/dt = 0$ if $\partial\pi/\partial y_c|_{t=\dot{t}} < 0$. Similarly, for $\mathbf{x}$, $\partial\pi/\partial\mathbf{x}|_{t=\dot{t}} \leq 0$ and $dx_c/dt = 0$ if $\partial\pi/\partial x_c|_{t=\dot{t}} < 0$. At the initiation of trade, $\bar{c}_x = c_1$ and $x_c = 0$. Because the stock of exports is zero, a drop in t has no first-order direct impact on profits either: $\partial\pi/\partial t|_{t=\dot{t}} = 0$. Therefore,

$$\left.\frac{d\pi}{dt}\right|_{t=\dot{t}} = \left.\frac{\partial\pi}{\partial P}\frac{dP}{dt}\right|_{t=\dot{t}} = 0 .$$

Since $\partial p[y_c,P]/\partial P > 0$ and $\partial s[x_c,P]/\partial P > 0$, we have $\partial\pi/\partial P > 0$. Hence, it must be that $dP/dt|_{t=\dot{t}} = 0$. With the price level unchanged, FOC_{y_c} implies that home-bound production levels, $\mathbf{y}$, do not change either. That is, for all $c \in C$,

$$\left.\frac{dy_c}{dt}\right|_{t=\dot{t}} = 0 .$$

Finally, notice that the boundary type $\bar{c}_y$ also remains unchanged: with a discrete type space, boundary types are not differentiable and only change in jumps.

Implicitly differentiating $FOC_{x_c} : Pm[x_c/L] = \frac{t}{1-r}c$ with respect to t, we find

$$m[x_c/L]\frac{dP}{dt} + Pm'[x_c/L]\frac{d(x_c/L)}{dt} = \frac{c}{1-r} \ .$$

At the initiation of trade, $x_c = 0 = dP/dt|_{t=\dot{t}}$. Re-using FOC_{x_c} and solving for $dx_c/dt|_{t=\dot{t}}$ then yields

$$\left.\frac{dx_c}{dt}\right|_{t=\dot{t}} = L \Big/ \left(t\frac{m'[0]}{m[0]}\right) < 0 \ , \qquad (112)$$

where the inequality follows from

$$\frac{m'[0]}{m[0]} = \frac{v''[0]}{v'[0]} - \varepsilon'_{v'_k}[0] < 0 \ .$$

Implicitly differentiating LM with respect to t yields

$$n\left(\sum_{c=c_1}^{\bar{c}_y} c\frac{dy_c}{dt}g_c + t\sum_{c=c_1}^{\bar{c}_x} c\frac{dx_c}{dt}g_c\right) + \frac{dn}{dt}C[\mathbf{y}, t\mathbf{x}] = 0 \ .$$

At the initiation of trade, $x_c = 0 = dy_c/dt|_{t=\dot{t}}$ and $\bar{c}_x = c_1$. Substituting these values and solving for $dn/dt|_{t=\dot{t}}$, we find

$$\left.\frac{dn}{dt}\right|_{t=\dot{t}} = -\frac{n\dot{t}c_1 g_{c_1}}{C[\mathbf{y}, \mathbf{0}]}\left.\frac{dx_{c_1}}{dt}\right|_{t=\dot{t}} = -\frac{n(1-r)v'[0]g_{c_1}}{\sum_{c=c_1}^{\bar{c}_y} v'[y_c/L]y_c g_c}\left.\frac{dx_{c_1}}{dt}\right|_{t=\dot{t}} > 0 \ . \qquad (113)$$

Here, we have used ZP and $p[y_c, P] = Pv'[y_c]$, as well as FOC_{x_c} and $s[x_c, P] = Pv'[x_c]$, which says that, at the initiation of trade, $\dot{t}c_1 = (1-r)Pv'[0]$. The inequality follows from $\frac{dx_c}{dt}|_{t=\dot{t}} < 0$ (see (112)).

Utility is

$$U = n\left(\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{y_c}{L}\right]g_c + \sum_{c=c_1}^{\bar{c}_x} v\left[\frac{x_c}{L}\right]g_c\right) \ .$$

Totally differentiating with respect to t yields

$$\frac{dU}{dt} = \frac{dn}{dt}\left(\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{y_c}{L}\right]g_c + \sum_{c=c_1}^{\bar{c}_x} v\left[\frac{x_c}{L}\right]g_c\right) + n\left(\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{y_c}{L}\right]\frac{d\left(\frac{y_c}{L}\right)}{dt}g_c + \sum_{c=c_1}^{\bar{c}_x} v'\left[\frac{x_c}{L}\right]\frac{d\left(\frac{x_c}{L}\right)}{dt}g_c\right) \ .$$

At the initiation of trade, $x_c = v[0] = 0 = dy_c/dt|_{t=\dot{t}}$ and $\bar{c}_x = c_1$. Hence,

$$\left.\frac{dU}{dt}\right|_{t=\dot{t}} = \frac{dn}{dt}\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{y_c}{L}\right]g_c + nv'[0]g_{c_1}\frac{1}{L}\left.\frac{dx_{c_1}}{dt}\right|_{t=\dot{t}} \ .$$

Solving (113) for $\frac{dx_{c_1}}{dt}$, substituting and simplifying, we obtain

$$\left.\frac{dU}{dt}\right|_{t=\dot{t}} = \left(\sum_{c=c_1}^{\bar{c}_y}\left(v\left[\frac{y_c}{L}\right] - \frac{1}{1-r}v'\left[\frac{y_c}{L}\right]\frac{y_c}{L}\right)g_c\right)\left.\frac{dn}{dt}\right|_{t=\dot{t}} .$$

Using $dn/dt|_{t=\dot{t}} > 0$, strict concavity of v, and $v\,[0] = 0$,

$$\left.\frac{dU}{dt}\right|_{t=\dot{t}} \overset{(>)}{\underset{(<)}{=}} 0 \iff 1-r \overset{(>)}{\underset{(<)}{=}} \left.\frac{\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{y_c}{L}\right]\frac{y_c}{L}g_c}{\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{y_c}{L}\right]g_c}\right|_{t=\dot{t}} \iff r \overset{(<)}{\underset{(>)}{=}} 1 - \frac{\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{\dot{y}_c}{L}\right]\frac{\dot{y}_c}{L}g_c}{\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{\dot{y}_c}{L}\right]g_c} ,$$

Hence,

$$\left.\frac{dU}{dt}\right|_{t=\dot{t}} \overset{(>)}{\underset{(<)}{=}} 0 \iff r \overset{(<)}{\underset{(>)}{=}} 1 - \tilde{\varepsilon}_v\left[\frac{\dot{y}_c}{L}\right] ,$$

where

$$\tilde{\varepsilon}_v\left[\frac{\dot{y}_c}{L}\right] \equiv \frac{\sum_{c=c_1}^{\bar{c}_y} v'\left[\frac{\dot{y}_c}{L}\right]\frac{\dot{y}_c}{L}g_c}{\sum_{c=c_1}^{\bar{c}_y} v\left[\frac{\dot{y}_c}{L}\right]g_c} .$$

■