# Pollution Permits: Efficiency by Design

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**Abstract**

The annual adverse effects of pollution are on the order of 10% of world GDP. Many approaches are used or have been proposed to control the growing pollution problem, but none of them allows for efficient pollution control in settings in which the marginal cost of pollution is increasing and polluters are better informed than regulators about the costs of abatement. We propose a simple primary market mechanism, *True-Cost Pay as Bid*, that implements efficient pollution control and does not depend on how much information the regulators have about the abatement costs.

## 1 Introduction

Multi-unit auctions are frequently used to allocate emissions permits and other goods. In many such markets, the auctioneer is interested in the social efficiency of the outcome. For example, the auctioneer may want to induce market participants to limit their emissions to the point at which the cost of further emissions reductions (abatement) crosses the social marginal cost of emissions (the marginal damage curve). Less abatement is environmentally costly because pollution is socially undesirable, and more abatement is economically costly because it reduces production.

We propose a simple mechanism that we call True-Cost Pay as Bid (TCPAB) because it is a pay-as-bid auction with elastic supply set equal to the true expected marginal social cost curve and we show that this mechanism guarantees efficiency in this problem, while

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uniform-price auctions and other standard permit market designs do not despite all of them being frequently employed in environmental contexts. In particular, in environments where bidders have common information an optimally-designed pay-as-bid auction will generically outperform any uniform-price auction. Furthermore, pay-as-bid auctions approximately outperform uniform-price auctions even when bidders are asymmetric. The good properties of our proposed mechanism obtain whether there is an active resale market (cap and trade) or not.

We allow polluters to be asymmetric in their cost of abatement; our substantive assumption is that the polluters are close to symmetrically informed. We focus on a seller who cares only about the quantities allocated to various bidders, conditional on the information revealed by bids. Thus, in the benchmark model, the auctioneer does not care about revenue, but can care about allocative efficiency. We relax this assumption in an extension, and show that sellers who are partially interested in revenue still prefer the pay-as-bid auction.

The case for price discovery through auctions, instead of relying on regulators' expectations of what the right price should be, was aptly summarized by *The Economist* (February 27, 2021):

> "Working out costs is tricky. The International Energy Agency (IEA), for instance, has routinely underestimated the pace of deployment of renewables. And because economies of scale drive down prices, that means that it has overestimated the costs of switching, too. In 2010 the lowest the IEA expected solar prices to drop to over the next decade was about \$195 per megawatt hour. Today the price in America and Europe is \$30-60."

The problem with changing costs while the allocation mechanism remains in place is commonly recognized, cf. Schmalensee et al. (1998) (history of sulphur dioxide in the US) and Blanchard et al. (2021) (central bank for permits).

The mechanism we propose is sufficiently flexible to accommodate innovation policy. Gillingham and Stock (2018) and Gates 2021 observe that a long-term view of abatement calls for discrimination among abatement approaches and de facto subsidizing technologies that will reduce emissions more effectively in the long-term. Such discrimination can be incorporated into the auctions we propose through preferential treatment of policymaker-favored technologies, e.g. by subsidizing them. Such discrimination, however, would need to be accompanied by limitations on secondary market permit trades.

We propose the TCPAB mechanism focusing on pollution types that can be controlled by a single social planner, such as sulphur dioxide or particulate matter. An important property of our mechanism is that it is well-suited to multilateral negotiations and hence to controlling the global carbon pollution. Cramton, MacKay, et al. (2017) show experimentally that multilateral negotiations over a tax are easier than negotiations over other control mechanisms such as quantity caps because negotiating over taxes allows the negotiating parties to agree on the minimal of all proposed taxes. One could run multilateral negotiations in the context of TCPAB: negotiators would propose elastic supply curves and the outcome of the negotiation would be the pointwise minimum of the elastic supply curves proposed by individual negotiators. We think that this is an advantage of our mechanism over the discretionary central bank type solutions (of course, pollution banks could sell pollution permits using our mechanism, thus inheriting its good multilateral negotiation properties).

The advantages of elastic supply have been recognized in many markets (Cramton and Stoft (2005) electricity; P. Klemperer (2010), toxic assets).[1] In the context of pollution, the potential advantages of elastic supply were recognized before us by Karp and Traeger (2021), who recognized its beneficial properties in the context of perfect Walrasian competition. We go beyond their work by proposing a well-behaved auction mechanism and studying its properties.

We contribute as well to the large literature on multi-unit auctions. Equilibrium existence for our setting was studied in Woodward (2016) and Pycia and Woodward (2022a). The latter paper also studies revenue maximization in multi-unit auctions; while they focus on auctions with inelastic supply, they also establish the essential uniqueness of equilibria of pay as bid with deterministic elastic supply. For earlier work on multi-unit auctions, see e.g. Ausubel et al. (2014).

We also contribute to the literature on trade-offs between price and quantity initiated by Weitzman (1974): we show that by using TCPAB one can achieve the unconstrained optimum, thus eliminating the need for the trade off.

Finally, note that by achieving efficiency in equilibrium, TCPAB provides an immediate tatonnement to Walrasian prices.

## 2 Canonical Pollution Control Mechanisms

In this paper we show that the True-Cost Pay as Bid auction is efficient and approximately robust to asymmetric information. The True-Cost Pay as Bid auction is a specific implementation of a pay as bid auction, with elastic supply set equal to the true marginal damage curve associated with emissions. We argue in favor of True-Cost Pay as Bid auction in part

[1] LiCalzi and Pavan (2005) discuss the role elastic supply might play in eliminating tacit collusion in uniform price; cf. also McAdams (2007), who proposed ex post adjustable supply as a mean to fight collusion. Our pay as bid proposal does not run into tacit collusion problems that might occur in uniform-price auctions.

by showing that other real-world permit mechanisms do not share its desirable qualities. To illustrate these points, we consider standard pollution mechanisms in a simple example of an emissions permit market with two polluters. Our polluters procure emissions permits to offset their emissions from each year's production. When the bidders' common signal is $s$, a polluter's value for emissions quantity $q$ is

$$v\left(q; s\right) = \left(s - q\right)_+$$

where $\left(x\right)_+$ denotes $\max\{x, 0\}$. The social cost of emissions is $C(Q; \omega) = Q$, and does not depend on the state of the world. Conditional on polluters' private information, the efficient level of emissions is $Q = 2s/3$. This introductory section is mostly self-contained; for formal definitions of e.g. uniform-price and pay-as-bid auctions, see Section 3 that follows.

## 2.1 Command and Control

Command and control is used to control pollution in many markets, including car emissions in the U.S., Europe, Mexico, and many other countries. In a command and control framework, the regulator sets explicit conditions on emissions: maximum levels, reduction technologies, etc. We consider a regulator who sets an annual emissions quantity $Q$. Because the marginal value of emissions is weakly positive, we may assume that polluters emit the entire allocated quantity, and therefore this is a straightforward decision problem for the regulator,

$$\max_Q \mathbb{E}\left[2\int_0^{\frac{1}{2}Q} v\left(q; s\right) dq - \int_0^Q C\left(Q'; \omega\right) dQ'\right].$$

This leaves

$$\max_Q \left(\mathbb{E}\left[s\right] + 1\right) Q - \frac{3}{4}Q^2.$$

Thus optimal supply is

$$Q^\star = \frac{2}{3}\left(1 + \mathbb{E}\left[s\right]\right). \tag{1}$$

In implementation, command and control has an observed weakness of inflexibility, cf. Weitzman (1974). This can be readily seen in 1. The mechanism's optimal level of emissions depends on bidders' expected information, and there will be an efficiency gap associated with the regulator's incomplete information.

## 2.2 Taxes

Taxes are used—or scheduled to be used—to control carbon dioxide pollution in many markets including the U.K., France, Canada, Argentina, and many other countries (cf., e.g., Cramton, MacKay, et al. 2017, Blanchard et al. 2021, and Santikarn et al. 2021). Under taxation, the regulator sets a fixed price at which polluters may purchase an arbitrary quantity of emissions permits. Profit maximization implies that if the tax on pollution is $\tau^\star$, each polluter will purchase $q = (s - \tau^\star)_+$ emissions permits. As under command and control, this is a straightforward decision problem for the regulator,

$$\max_{\tau} \mathbb{E}\left[2\int_0^{(s-\tau)_+} v\left(q;s\right)dq - \int_0^{2(s-\tau)_+} C\left(Q';\omega\right)dQ'\right].$$

From this, it is straightforward to see that

$$\mathbb{E}\left[(s-\tau)_+ - \frac{1}{2}s\right] = 0. \tag{2}$$

Implementation of optimal taxation faces the difficulty of determining optimal taxes. In environments which are dynamically variant or statically unstable, a single optimal tax will not be able to enforce efficient market outcomes, cf. Weitzman (1974). This can be readily seen in 2, where the single-dimensional tax must aggregate all expected information about polluters' signals.

## 2.3 Pollution Permits

Pollution permits are allocated via auctions, the two main design being uniform price and pay as bid.

### 2.3.1 Uniform-Price

Uniform-price auctions are used—or scheduled to be used—to allocate pollution permits (e.g., to control carbon dioxide pollution) in many markets including the U.S., China, Germany, Canada, and many others (Santikarn et al. 2021). While the implemented auctions are mostly inelastic let us consider the more general class of uniform-price auctions allowing elastic supply.[2] Unlike command and control or taxation, elastic supply permits the regulator to implement a mechanism with more than a single parameter: the elastic supply curve

[2] In practice, supply is typically either fixed or subject to a *price collar*, a fixed supply with the addition of a price floor and a price ceiling.

corresponds to a continuum of parameters chosen against individual prices. While this can allow the permit efficient allocations to arise, it also introduces ambiguity into the allocation procedure because of the multiplicity of equilibria (c.f., Wilson (1979), P. D. Klemperer and Meyer (1989), and Wang and Zender (2002)). We say that an auction *implements allocation* $\tilde{q}$ if, for all bidding equilibria $b^{\star}$, $q(b^{\star}(s)) = \tilde{q}(s)$; we then also write $q \circ b^{\star} = q^{\star}$.

**Lemma 1.** **[Uniform-Price Equilibrium Selection]** *Fix an optimal allocation correspondence $q^{\star}$. If aggregate optimal supply $Q^{\star}$ is nonconstant, then for any supply curve $S$ the uniform-price auction admits bidding equilibria in which $q \circ b^{\star} \neq q^{\star}$.*

A few notes are in order. First, we assume that the seller's marginal cost curve is nonconstant. If it were, a fixed reserve price (letting $S(Q) = c$ for all $Q$) would yield efficient implementation in the uniform-price auction; this would be equivalent to a tax. Thus in settings where the seller's marginal cost curve is roughly constant, optimal taxation will achieve near efficiency. Second, we assume that the support of optimal aggregate supply is convex and nondegenerate. If it were degenerate, fixed supply would yield efficient outcomes in the uniform-price auction. To see the role of convexity, consider an optimal aggregate supply which may take only two values. In this case, provided the two quantities are sufficiently far apart relative to the seller's change in marginal cost (and also taking into consideration bidders' values), the seller can implement efficient outcomes by setting supply equal to a step function with jumps at each of the two relevant quantities. Third, the optimal uniform-price auction depends on the selected bidding equilibrium; cf. the discussion of equilibrium multiplicity above.

### 2.3.2 Inelastic Pay as Bid

Pay-as-bid auctions are used to sell e.g. sulphur dioxide permits. They are implemented with fixed and inelastic supply. The resulting pricing problems of this approach can be seen in the prices of U.S. SO2 permits sold by U.S. EPA. The inelastic supply is unable to generate sensible prices when demand for permits substantially changes (for an analysis of the demand change, see, e.g., Schmalensee et al. (1998)).

The nice properties we derive for the True-Cost Pay-as-Bid auction depend both on the incentives induced by the pay-as-bid mechanism, and on the flexibility afforded by elastic supply. Removing one or the other feature of the mechanism allows suboptimal outcomes to arise again.

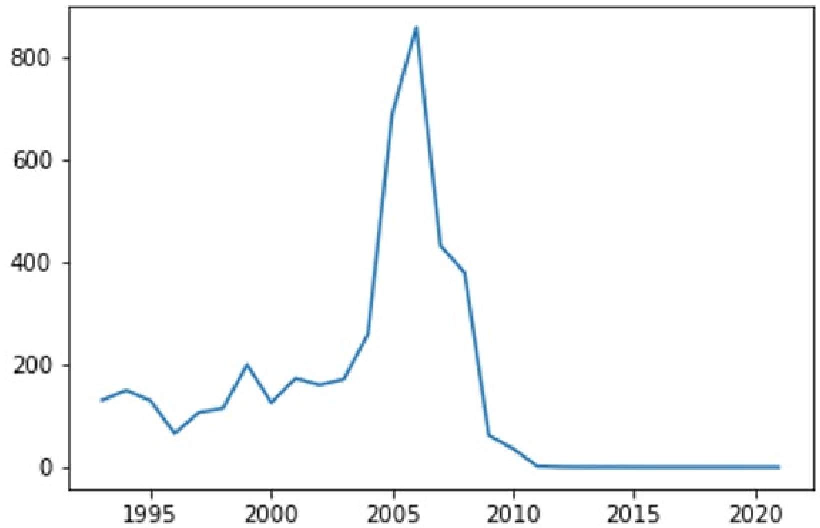


Figure 1: US SO2 spot permit prices in EPA annual auctions (vertical axis: market claring prices in USD; horizontal axis: auction years). Data: US Environmental Protection Agency.

# 3 Model

There are $n \geq 2$ polluters, $i \in I = \{1, ..., n\}$; we allow the polluters to be ex ante asymmetric. Polluter $i$'s marginal value for $q$-th permit is denoted $v^i(q; s_i)$, where $s_i$ is a signal privately known to polluter $i$; in particular, we treat polluters' marginal values for receiving $q$ permits as the primitive parameter. For a given polluter $i$ with status quo emissions $E_i$, the marginal value for receiving $q$ permits is the marginal cost of abating $E_i - q$ units. We decompose polluter $i$'s signal as $s_i = (s, \theta_i)$, where $s$ is common to all polluters and $\theta_i$ is private to polluter $i$. We assume that each $v^i(\cdot; s_i)$ is strictly decreasing where it is strictly positive, Lipschitz continuous, and almost-everywhere differentiable in $q$. We allow arbitrary dimensionality of $s_i$, and an arbitrary integrable $v^i(q; \cdot)$. We focus on environments in which $s_i$ are highly correlated across bidders but the regulator does not know the polluters' information. Our results take the simplest form when the correlation across polluters is perfect, $s_1 = ... = s_n = (s, 0)$, without imposing any further assumptions on the distribution of $s$. Under perfect correlation, signal $s$ has no strategic importance for bidders participating in an auction, and thus when studying the equilibrium among such bidders we fix $s$ and denote the bidders' marginal valuation by $v^i(q; s_i) = v^i(q)$.[3]

[3]The seller may not know the bidders' information if, for example, the seller needs to commit to the auction mechanism before this information is revealed. Alternatively, the seller may want to fix a single design for multiple auctions.

To avoid market corners, we assume that with probability $\pi > 0$ the signal profile is such that there exists $\underline{q} > 0$ such that in the efficient allocation of permits at least two bidders receive quantity $\underline{q}$ or higher. We call such bidders *active* and we call $\pi$ the probability of no market corners.

The regulator's value for a social allocation $(q_i)_{i=1}^n$ is $\pi(q; s, \omega)$, and depends on the level of emissions, the bidders' information $s$, and an unknown state of the world $\omega \sim F_\omega$. Note that the regulator's value can depend on polluters' common information. This will be the case when, for example, the seller is interested in allocative efficiency. Given polluters' common signal $s$, we denote the seller's optimal allocation by $q^\star(s) = \arg\max_q \mathbb{E}_\omega[\pi(q; s, \omega)]$, and we assume that each $q^\star(s)$ is single-valued. We allow the seller's value to depend on endogenous equilibrium outcomes, such as the revenue obtained from permit allocation, but for the bulk of our analysis we assume that the seller is interested in efficiency, so that

$$\pi\left(q; s, \omega\right) = \sum_{i=1}^{n} \int_0^{q_i} v^i\left(x; s\right) dx - C\left(\sum_{i=1}^{n} q_i; \omega\right),$$

where $C(Q; \omega)$ is the economic damage associated with emissions $Q$ when the state of the world is $\omega$.[4] We say that a mechanism is efficient if it all its equilibria are efficient. Setting $Q^\star\left(s\right) = \sum_{i=1}^{n} q_i^\star\left(s\right)$, we assume throughout that the set of welfare-maximizing aggregate quantities $\{Q^\star\left(s\right) : s \in \mathcal{S}\}$ is a non-trivial interval. In particular, this assumption rules out discrete bidder signal spaces. We also assume that the seller's marginal cost curve $c = dC/dQ$ is weakly increasing on $\operatorname{Supp} Q^\star$.

## 3.1 Elastic Pay as Bid

In an elastic pay-as-bid auction the regulator first announces a supply curve $S$, then solicits decreasing bid curves $b^i : \mathbb{R}_+ \to \mathbb{R}_+$ from each polluter. After receiving bid curves, the regulator computes the market-clearing price given reported aggregate demand and announced supply,

$$p^\star = \inf\left\{p \colon \exists q \geq 0 \text{ s.t. } b^i\left(q_i\right) \leq p \; \forall i, \text{ and } S\left(\sum_{i \in I} q_i\right) \leq p\right\}.$$

[4]Following the prior literature we understand efficiency as equalizing the abatement cost and the social cost of pollution. This standard approach ignores two effects: (i) there might be benefits to production (e.g. consumer surplus) that are not captured by the polluters, and (ii) polluters might also care about the social cost of pollution. Our framework can be adapted to handle both. For instance, if polluters capture half of the surplus associated with each emission, we can adjust TCPAB and use PAB with elastic supply equal to half of the social cost. Similarly, we can adjust the supply curve to account for polluters caring about the social cost of pollution.

The regulator allocates to each polluter the number of permits they demand at the market-clearing price, $q_i = \varphi^i(p^\star)$, where $\varphi^i$ is the demand function of polluter $i$ (the inverse of the submitted bid function $b^i$.[5] In exchange, the polluter transfers to the regulator the entire area under their submitted bid curve, $t_i = \int_0^{q_i} b^i(x)dx$.

# 4 True-Cost Pay as Bid

*True-Cost Pay as Bid* is pay as bid with supply equal to the expected marginal social cost curve, $S = \mathbb{E}_\omega [C(\cdot;\omega)]$. We say that an allocation is efficient if it maximizes expected welfare over all allocations that are independent of the social cost parameter $\omega$, which is unknown to all parties.

**Theorem 1. [Efficiency of True-Cost Pay as Bid]** *Suppose that polluters are symmetrically informed. True-Cost Pay as Bid has an essentially unique bidding equilibrium and this equilibrium implements the efficient allocation of pollution permits.*

For a heuristic argument, suppose that the seller knows $\omega$. In any deterministic pay-as-bid auction, equilibrium bids are flat, and the resulting allocations equate marginal value with the market price. By setting the supply to the realized true cost $C(\cdot;\omega)$ ,the seller then equates marginal value with the true cost, thus achieving the efficient allocation in equilibrium.[6]

Because True-Cost Pay as Bid does not depend on the distribution of bidders' values, Theorem 1 implies

**Corollary 1. [True-Cost Pay as Bid Is Detail-Free]** *If polluters are symmetrically informed, then the efficient allocation can be implemented by a seller who does not know the distribution of bidders' values.*

In contrast with the efficiency of True-Cost Pay as Bid, many common pollution-control mechanisms fail to yield efficient outcomes. To formalize this gap, we define polluters' signals to be *rich* if there is no market allocation $q^\star$ that is efficient regardless of polluter signal $s$. The analysis in Section 2 leads us to the following

[5]Tie-breaking is handled pro-rata on the margin. We assume here that the bid function $b^i$ is locally invertible, but in analysis we handle technical details relating to locally non-invertible (flat) bids.

[6]If the polluters are symmetrically informed, then for any allocation correspondence $q^\star$ that equalizes polluters' marginal costs and is monotonic in the common marginal cost, there is a supply curve such that the resulting Elastic Pay as Bid has an essentially unique bidding equilibrium and this equilibrium implements $q^\star$. (Note that if the seller knew bidders' marginal values, then by using heterogenous subsidies the seller could implement any desired allocation.)

**Theorem 2. [Inefficiency under non-TCPAB]** *Suppose the signal space is rich and the marginal social cost of pollution strictly monotonic. Then, quantity regulation, taxes, uniform price auctions, as well as inelastic pay as bid (that is, the mechanisms discussed in Section 2) are not efficient.*

This theorem follows from Lemma 1 because quantity regulation and taxes can be interpreted as special cases of Uniform Price. Under quantity regulation and taxes, there are unique equilibria and they are inefficient. In Uniform Price more generally there are many equilibria and generic equilibrium is inefficient; furthermore, the unique equilibrium selected in line with the canonical proposal of P. D. Klemperer and Meyer 1989 is always inefficient.[7]

# 5 Asymmetric Information

When bidders are asymmetrically informed our approach gives us bounds on efficiency.[8] The bounds' tightness relies on the asymmetry of information being small. We are assuming in this section that $\gamma > 0$ is a lower bound on the slope of the true cost of pollution.[9]

**Theorem 3. [Efficiency guarantee with small variance]** *For every $\varepsilon > 0$ there is $\delta > 0$ such that if, for each relevant $q$, the variance of $v(q, s_i)$ is bounded by $\delta$, then the per-permit efficiency loss in TCPAB is bounded by $\varepsilon$ with probability $1 - \varepsilon$ and with probability $1 - \varepsilon$ the total welfare loss is bounded by $4\varepsilon^2/\gamma$.*

This theorem is implied by Lemma 2 below because Chebyshev's inequality tells us that small variance leads to concentration. In this lemma, for each $q \geq 0$ we fix an interval $I(q) \ni q$ such that if $q > 0$ then $q$ is in the interior of $I(q)$.

**Lemma 2. [Efficiency guarantee for concentrated values]** *Let $(s_i^*)_{i \in I}$ be a reference signal profile and let $(q_i^\star(s^*))_{i \in I}$ be the vector of efficient assignments for this reference signal profile. Suppose that, for at least probability $1 - \varepsilon$ of signal profiles $(s_i)_{i \in I}$, for all relevant $q \in I(q_i^\star(s^*))$ we have $|v^i(q; s_i) - v^i(q; s_i^*)| < \varepsilon$. The efficiency loss is then small provided $\varepsilon$ is small: for each $\delta > 0$ we can find $\varepsilon_\delta > 0$ such that if $\varepsilon \in [0, \varepsilon_\delta]$ then with probability $1 - \delta$ the per-permit and total efficiency loss are lower than $\delta$.*

[7]For more inefficiency of Uniform Price see e.g. Marszalec, Teytelboym, and Laksá 2020. For more on P. D. Klemperer and Meyer 1989 selection, see e.g. Pycia and Woodward 2022a.

[8]Pycia and Woodward (2022a) established related bounds on revenue in terms of the size of the support of uncertainty; we provide the bounds for efficiency in terms of variance. In particular, we do not require the support to be bounded.

[9]We could alternatively assume that the slope of bidders valuations for the permits is bounded away from 0.

The bounded support argument works conditional on the event in which all polluters know that the support is small; the proof provided in the appendix shows that common knowledge is not necessary. The analogues of Theorem 3 and Lemma 2 also hold true conditional on the common signal $s$.

If, in the theorem, the slope of the true cost of pollution curve is bounded below by some $\gamma > 0$, then the total efficiency loss is bounded from above by $\varepsilon$. A key role in the proof of Lemma 2 is played by our Theorem 5, stated in Appendix A.1. This last result also allows us to prove the following.

**Theorem 4.** **[Efficiency guarantee with locally small support]** *Let $s = (s_i)_{i \in I}$ be a reference signal profile and let $(q_i^\star(s))_{i=\in I}$ be the vector of efficient assignments for this reference signal profile. Suppose that there is $\delta \geq 0$ such that for all signal profiles $s'$ and all bidders $i$, the permit value $v^i(q_i^\star(s); s_i')$ is within $\delta$ of $v^i(q_i^\star(s); s_i)$. Then the ex-post per-permit inefficiency of TCPAB is bounded above by $2\delta$ and the total welfare loss is bounded by $4\delta^2/\gamma$.*

**Corollary 2.** **[Efficiency guarantee with small support]** *If the range of marginal values for each relevant quantity $q$ is bounded by $\delta$, then the ex-post inefficiency of TCPAB per permit purchased is bounded by $2\delta$ and the total welfare loss is bounded by $4\delta^2/\gamma$.*

# 6 Other policy dimensions

## 6.1 Heterogenous Technology and Innovation

The True-Cost Pay as Bid can be adapted to promote innovation by selling different permits for use with different technologies and making permits not tradeable across technologies. The price paid by polluters using a promising technology can then be subsidized.

## 6.2 Multilateral Bargaining

For local pollutants such as particulate matter or SO2, one regulator is in charge of the design. For global pollutants, such as CO2, a multilateral agreement needs to be reached. Our auction can be negotiated over in a similar way to multilateral carbon tax negotiations (cf. Cramton, MacKay, et al. (2017)): we can have each party propose the elastic supply curve of permits and we can take the minimum of this curves and use it in a global TCPAB auction. The advantage of the tax approach is that it can be more easily decentralized.

# 7 Conclusion

We have proposed a simple auction mechanism to sell pollution permits. The auction—True-Cost Pay as Bid—achieves efficiency without the need for the designer to know the abatement costs. Thanks to this property the mechanism self-adjusts to changing abatement costs. Furthermore the auction works well whether the polluters are symmetric or not.

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# A A Bound on Market Price in Pay as Bid

Our analysis of optimal bidding relies on the following key theorem; in this theorem we impose no restrictions on bidders' information and we allow mixed-strategy equilibria.

**Theorem 5. [A Bound on Market Price]** *In any mixed-strategy equilibrium of the pay-as-bid auction, for any signal profile $(s, (\theta_1, \ldots, \theta_n))$ all realizations of the market clearing price for the realized market quantity $Q^\star(s, \cdot)$ are bounded between the smallest and largest marginal value at the largest quantity feasible in equilibrium,*

$$\min_i \operatorname*{ess\,inf}_{\tilde{\theta}_i} v^i\left(\overline{q}^i\left(s, \tilde{\theta}_i\right); s, \tilde{\theta}_i\right) \leq p\left(Q^\star\left(s, \theta\right); s, \theta\right) \leq \max_i \operatorname*{ess\,sup}_{\tilde{\theta}_i} v^i\left(\overline{q}^i\left(s, \tilde{\theta}_i\right); s, \tilde{\theta}_i\right),$$

*where $\overline{q}^i\left(s, \tilde{\theta}_i\right)$ is the maximum quantity purchased by bidder $i$ with type $\left(s, \tilde{\theta}_i\right)$.*

The proof of Theorem 5 shows a slightly stronger claim: for any realization of $(s, \theta)$, the equilibrium bid for the maximum quantity bidder $i$, with type $s_i = (s, \theta_i)$, can obtain equals the bidder's marginal value for this quantity; or, bids equal values at the maximum feasible quantity. The intuition for this claim is that if a bidder has strictly positive margin at the maximum feasible quantity, they can slightly increase their bid and obtain a non-negligible additional quantity at minimally higher price, which is a profitable deviation; the proof of Theorem 5 formalizes this intuition and takes care of technical complications related to tie-breaking, flat bids, and binding monotonicity constraints. Note that this intuition applies only to the maximum quantity at which the increased bid is paid only when it is marginal; at any lower quantity the increased bid would need to be paid also when inframarginal.

Because bids are decreasing in quantity, the equilibrium market-clearing price paid by bidder $i$ is minimized when their realized quantity is maximized, $q^i(s,\theta) = \overline{q}^i(s,\theta)$. Thus, the theorem provides bounds on the minimum market price. In the special case where bidders are symmetric and have only common and no idiosyncratic information, then, for any $i$,

$$\operatorname*{ess\,inf}_{\tilde{\theta}_i} v^i\left(\frac{1}{n}Q^\star(s); s, \tilde{\theta}_i\right) = v\left(\frac{1}{n}Q^\star(s); s\right) = \operatorname*{ess\,sup}_{\tilde{\theta}_i} v^i\left(\frac{1}{n}Q^\star(s); s, \tilde{\theta}_i\right)$$

That is, the market price at the maximum quantity is exactly equal to each bidder's marginal value at the last unit they receive.

Because bids equal values at bidders' maximum feasible quantities, and these quantities are sold when a bidder's opponents submit their lowest bids, the equality of market price and a bidder's marginal value obtains only when other bidders submit their lowest bids. For such a bid profile the market-clearing price will be minimized, but the lower bound of Theorem 5 remains valid irrespective of the realization of supply. The market-clearing price at other bid profiles can (and frequently does) rise above $\max_i \operatorname{ess\,sup}_{\tilde{\theta}_i} v^i(\overline{q}^i(s,\tilde{\theta}_i); s, \tilde{\theta}_i)$.

## A.1 Proof of Theorem 5 (Bound on Market Price)

Our equilibrium analysis relies on the identification of the minimum equilibrium market clearing price. In this appendix we prove Theorem 5, which bounds this price. The arguments do not depend on the presence (or absence) of idiosyncratic private information or mixed strategies. We consolidate all bidder-known uncertainty into $\zeta_i = (s, \theta_i, \xi_i)$, where $s$ is the signal observed by all bidders, $\theta_i$ is bidder $i$'s idiosyncratic private information, and $\xi_i$ is a term parameterizing bidder $i$'s potentially-mixed strategy; thus bidder $i$'s bid $b^i : [0,\overline{Q}] \times \operatorname{Supp}\zeta_i \to \mathbb{R}_+$.[10] Where useful, we consider $\zeta_i|s$ to hold fixed the common signal $s$ while letting $\theta_i$ and $\xi_i$ vary.

We also introduce notation for the (essential) minimum market clearing price $\underline{p}$ and (essential) maximum receivable quantity $\overline{q}^i$, conditional on strategy profile $(b^j)_{j=1}^n$,

$$\underline{p}(s) = \operatorname*{ess\,inf}_{\zeta|s} p\left(Q; \left(b^j(\cdot;\zeta_j)\right)_{j=1}^n\right);$$
$$\overline{q}^i(\zeta_i) = \operatorname*{ess\,sup}_{\zeta_{-i}|s} q^i\left(Q; b^i(\cdot;\zeta_i), b^{-i}(\cdot,\zeta_{-i})\right).$$

Thus, when the bidding strategy profile is $(b^j)_{j=1}^n$, the market clearing price is almost never below $\underline{p}(s)$ when the common signal is $s$, and bidder $i$'s allocation is almost never above

[10] For compactness we also write $v(\cdot;\zeta_i) = v(\cdot; s, \theta_i)$, but we do not imply that a bidder's marginal value may vary with her action selection from a mixed strategy.

$\overline{q}^i(\zeta_i)$ when her type is $\zeta_i$.

**Lemma 3.** *In any equilibrium, conditional on common signal $s$, at least $n-1$ bidders, with probability 1, bid their true value for their maximum receivable quantity. That is,*

$$\#\left\{i\colon \Pr\left(b^i\left(\overline{q}^i\left(\zeta\right);\zeta\right) = v\left(\overline{q}^i\left(\zeta\right);\zeta\right)\middle|s\right) = 1\middle|s\right\} \geq n-1.$$

*Proof.* For a given agent $i$, common signal $s$, and $\lambda > 0$, consider an alternative bidding strategy $b^\lambda$ defined by

$$b^\lambda\left(q;\zeta_i\right) = \begin{cases} b^i\left(q;\zeta_i\right) & \text{if } b^i\left(q;\zeta_i\right) \geq b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) + \lambda, \\ \min\left\{b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) + \lambda, v\left(q;\zeta_i\right)\right\} & \text{otherwise.} \end{cases}$$

Since $b^i(\cdot;\zeta_i)$ is left-continuous, for small $\lambda$ this deviation will award the agent all excess quantity above $\sum_{j\neq i}\varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i) + \lambda;\zeta_j)$. Let $q^\star(\lambda;\zeta)$ be the quantity obtained under this deviation when, under the original strategy, $q^i(\zeta)$ units would be obtained. Explicitly,

$$q^\star\left(\lambda;\zeta\right) = Q - \sum_{j\neq i}\varphi^i\left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) + \lambda;\zeta_j\right) = Q - \sum_{j\neq i}\underline{q}^{ji}\left(\lambda;\zeta\right),$$

where $\underline{q}^{ji}(\lambda;\zeta) = \varphi^j(b^i(\overline{q}^i(\zeta_i);\zeta_i) + \lambda;\zeta_j)$ is the quantity bidder $j$ receives when the aggregate signal profile is $\zeta$ and bidder $i$ implements bid $b^\lambda$; note that $\underline{q}^{ii}(\lambda;\zeta)$ is the maximum quantity for which bidder $i$ bids above $b^i(\overline{q}(\zeta_i);\zeta_i) + \lambda$, which does not depend on $\zeta_{-i}$, and denote this quantity by $\underline{q}^i_\lambda(\zeta_i)$. We will use the quantity $q^\star(\lambda;\zeta)$ to analyze the additional quantity the deviation yields above baseline,

$$\Delta^i_L\left(\lambda;\zeta\right) = q^i\left(\zeta\right) - \underline{q}^{ii}\left(\lambda;\zeta\right), \quad \Delta^i_R\left(\lambda;\zeta\right) = q^\star\left(\lambda;\zeta\right) - q^i\left(\zeta\right),$$
$$\Delta^i\left(\lambda;\zeta\right) = \Delta^i_L\left(\lambda;\zeta\right) + \Delta^i_R\left(\lambda;\zeta\right).$$

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

$$\mathbb{E}_{\zeta|s}\left[\int_{\underline{q}^i_\lambda(\zeta_i)}^{q^i(\zeta)} b^\lambda\left(x;\zeta_i\right) - b^i\left(x;\zeta_i\right)dx\middle|q_i \geq \underline{q}^i_\lambda\left(\zeta_i\right)\right]$$
$$\geq \mathbb{E}_{\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v\left(x;\zeta_i\right) - b^\lambda\left(x;\zeta_i\right)dx\middle|q_i \geq \underline{q}^i_\lambda\left(\zeta_i\right)\right].$$

Importantly, this inequality must hold both ex ante and interim, unconditional on $\theta_i$. Because bids are weakly decreasing, the left-hand expectation is bounded above by

$$\begin{aligned}
&\mathbb{E}_{\zeta|s}\left[\int_{\underline{q}_\lambda^i(\zeta_i)}^{q^i(\zeta)} b^\lambda(x;\zeta_i) - b^i(x;\zeta_i)\,dx \middle| q_i \geq \underline{q}_\lambda^i(\zeta)\right] \\
&\leq \mathbb{E}_{\zeta|s}\left[\int_{\underline{q}_\lambda^i(\zeta_i)}^{q^i(\zeta)} b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) + \lambda - b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&= \lambda \mathbb{E}_{\zeta_{-i}|s}\left[\Delta_L^i(\lambda;\zeta) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].
\end{aligned}$$

As marginal values are Lipschitz in quantity and $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v^i(\overline{q}^i(\zeta_i);\zeta_i)$ by assumption, the right-hand expectation is bounded above by ($M$ is the Lipschitz modulus of $v$)

$$\begin{aligned}
&\mathbb{E}_{\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda;\zeta)} v(x;\zeta_i) - b^\lambda(x;\zeta_i)\,dx \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&\geq \mathbb{E}_{\zeta|s}\left[\int_{q^i(\zeta)}^{q^\star(\lambda,\zeta)} \left(v^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) - \left(x - q^i(\zeta)\right)M - \left(b^i\left(\overline{q}^i(\zeta_i);\zeta_i\right) + \lambda\right)\right)_+ dx \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \\
&\geq \mathbb{E}_{\zeta|s}\left[\frac{1}{2}\left(\mu(\zeta_i) - \lambda\right)\min\left\{\Delta_R^i(\lambda;\zeta), \frac{\mu(\zeta_i) - \lambda}{M}\right\} \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right],
\end{aligned}$$

where $\mu(\zeta_i) = v^i(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$. If it is the case that $(\mu(\zeta_i) - \lambda)/M \leq \Delta_R^i(\lambda;\zeta)$ for all $\lambda$, then it is impossible that the overall inequality is satisfied for all $\lambda$ (its left-hand side converges to zero in $\lambda$, while the right-hand side converges to a strictly positive value) and incentive compatibility is violated. Therefore we assume that the $\min\{\cdot,\cdot\}$ resolves to $\Delta_R^i(\lambda;\zeta)$. Then the overall inequality implies

$$\lambda \mathbb{E}_{\zeta|s}\left[\Delta_L^i(\lambda;\zeta) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\frac{1}{2}\left(\mu(\zeta_i) - \lambda\right)\Delta_R^i(\lambda;\zeta) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].$$

Since $\Delta_R^i(\lambda;\zeta)$ is bounded, there is $m^i(\lambda)$ such that

$$\lambda \mathbb{E}_{\zeta|s}\left[\Delta_L^i(\lambda;\zeta) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right] \geq \frac{1}{2}\left(m^i(\lambda) - \lambda\right)\mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i(\lambda;\zeta) \middle| q_i \geq \underline{q}_\lambda^i(\zeta_i)\right].$$

For any $i$, any $\lambda$, and any $\kappa > 0$, there is $\Lambda^i(\lambda,\kappa) > 0$ such that

$$\Lambda^i(\lambda,\kappa) < \frac{1}{2}\left(m^i(\lambda) - \lambda\right)\kappa.$$

The term $m^i(\lambda)$ can be specified so that $m^i(\lambda) - \lambda$ is decreasing in $\lambda$, so if $\Lambda^i(\lambda;\kappa) < (m^i(\lambda) - \lambda)\kappa/2$, then $\Lambda^i(\lambda,\kappa) < (m^i(\lambda') - \lambda')\kappa/2$ for all $\lambda' > \lambda$. Then let $\overline{\Lambda} = \min\{\Lambda^i(\lambda,\kappa)\colon\ \Pr_{\zeta_i}(b^i(\overline{q}^i(\zeta_i);\zeta_i) >$

$v^i(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0\}$. For any such $\kappa$, $\overline{\Lambda}$, it must be that

$$\kappa \mathbb{E}_{\zeta|s}\left[\Delta_L^i\left(\overline{\Lambda};\zeta\right)\middle|q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right] \geq \mathbb{E}_{Q,\zeta|s}\left[\Delta_R^i\left(\overline{\Lambda};\zeta\right)\middle|q_i \geq \underline{q}_\lambda^i\left(\zeta_i\right)\right].$$

Define bidder $j$ with type $\zeta_j$ to be *relevant* given price $p$ (and common signal $s$) if $b^j(\overline{q}^j(\zeta_j);\zeta_j) \leq p < v^j(\overline{q}^j(\zeta_j);\zeta_j)$. Fixing price $p$ and summing the above incentive inequality over all relevant agents gives

$$\begin{aligned}
&\kappa \sum_{j \text{ relevant}} \mathbb{E}_{\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&\geq \sum_{j \text{ relevant}} \mathbb{E}_{\zeta|s}\left[\Delta_R^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \\
&= \sum_{j \text{ relevant}} \mathbb{E}_{\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] - \mathbb{E}_{Q,\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].
\end{aligned}$$

Thus,

$$(\kappa+1) \sum_{j \text{ relevant}} \mathbb{E}_{\zeta|s}\left[\Delta_L^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right] \geq \sum_{j \text{ relevant}} \mathbb{E}_{Q,\zeta|s}\left[\Delta^j\left(\overline{\Lambda};\zeta\right)\middle|q_j \geq \underline{q}_\lambda^j\left(\zeta_j\right)\right].$$

By definition, $\Delta^j(\overline{\Lambda};\zeta) = Q - \underline{q}_\lambda^j(\zeta_j) - \sum_{k\neq j} \underline{q}^{kj}(\overline{\Lambda};\zeta) \equiv Q - \underline{Q}^j(\overline{\Lambda};\zeta)$ and $\Delta_L^j(\overline{\Lambda};\zeta) = q^j(\zeta) - \underline{q}_\lambda^j(\overline{\Lambda};\zeta)$. Furthermore,

$$\sum_{j \text{ relevant}} q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) \leq \sum_j q^j\left(\zeta\right) - \underline{q}_\lambda^j\left(\zeta_j\right) = Q - \underline{Q}\left(\underline{p}+\delta\right).$$

Then it follows that

$$\kappa + 1 \geq \#\left\{j \text{ relevant}\right\}.$$

Since $\kappa > 0$ may be arbitrarily small, it follows that there is at most one relevant bidder; i.e., there is at most a single bidder $i$ such that $\Pr(b^i(\overline{q}^i(\zeta);\zeta) < v(\overline{q}^i(\zeta);\zeta)) < 1$. □

**Lemma 4.** *For all bidders $i$ and all bidder-common signals $s$,*

$$\Pr\left(b^i\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right) = v\left(\overline{q}^i\left(\zeta_i\right);\zeta_i\right)\middle|s\right) = 1.$$

*Proof.* Fix a common signal $s$. Lemma 3 shows that at least $n-1$ bidders $j$ are such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ with probability 1. If all $n$ bidders' bids satisfy this condition, the desired result follows immediately from market clearing. Otherwise, there is some bidder $i$ such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ with $\zeta_i|s$-strictly positive probability. We show that (i) this bidder's bid must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, (ii) with $\zeta_{-i}|s$-positive probability, opposing bidders' bids are asymptotically flat near $\overline{q}^j(\zeta_j)$, and (iii) this implies that bidder $i$ has a strict incentive to increase her (flat) bid near $\overline{q}^i(\zeta_i)$.

Let bidder $i$ and parameter $\zeta_i$ be such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) = \underline{p} < v(\overline{q}^i(\zeta_i);\zeta_i)$, and assume that $b^i$ is strictly decreasing in a neighborhood to the left of $\overline{q}^i(\zeta_i)$. For $\lambda > 0$, define an alternate bid $b^\lambda$,

$$b^\lambda(q) = \begin{cases} b^i(q;\zeta_i) & \text{if } b^i(q;\zeta_i) \geq \underline{p} + \lambda, \\ \underline{p} + \lambda & \text{otherwise.} \end{cases}$$

Since $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and we analyze small $\lambda > 0$, we may assume that $\lambda$ is small enough that for any feasible quantity $q$, $b^\lambda(q) \leq v(q;\zeta_i)$. Then whenever the market clearing price would be $p < \underline{p} + \lambda$ if bidder $i$ submitted bid $b^i$, the market clearing price will be $\underline{p} + \lambda$ if she submits bid $b^\lambda$ instead. Further, bidder $i$ receives the full residual supply,

$$q_i^\lambda = Q - \sum_{j \neq i} \overline{\varphi}^j\left(\underline{p} + \lambda; \zeta_j\right).$$

The utility gain associated with bid $b^\lambda$ versus bid $b^i$ is bounded below by

$$\mathbb{E}_{\zeta_{-i}}\left[\begin{array}{l} \displaystyle\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v(x;\zeta_i) - \left(\underline{p}+\lambda\right) dx \\ \displaystyle - \int_{\overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)}^q \left(\underline{p}+\lambda\right) - b^i(x;\zeta_i)\, dx \end{array}\middle| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right]. \tag{3}$$

Because bidder $i$'s opponents all have $\Pr(b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)) = 1$, and bids are below values and values are Lipschitz continuous, there is $M > 0$ such that $\overline{q}^j(\zeta_j) - \overline{\varphi}^j(\underline{p}+\lambda;\zeta_j) > M\lambda$ with probability 1 for all $j \neq i$. Then, letting $\lambda < v(\overline{q}^i(\zeta_i);\zeta_i) - b^i(\overline{q}^i(\zeta_i);\zeta_i)$, the bound in 3 is in turn bounded below by

$$\begin{aligned}
&\mathbb{E}_{\zeta_{-i}|s}\left[\int_q^{\left(Q-\sum_{j\neq i}\overline{q}^j(\zeta_j)\right)+(n-1)M\lambda} v(x) - \left(\underline{p}+\lambda\right) dx - \left(q - \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right)\lambda \middle| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&\geq \mathbb{E}_{\zeta_{-i}|s}\left[\left(\left[\left(Q - \sum_{j\neq i}\overline{q}^j(\zeta_j)\right) + (n-1)M\lambda\right] - q\right)\lambda - \left(\left[Q - \overline{Q}\right] + (n-1)M\lambda\right)\lambda \middle| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right] \\
&= \mathbb{E}_{\zeta_{-i}|s}\left[\overline{Q} - \sum_{j\neq i}\overline{q}^j(\zeta_j) - q \middle| q \geq \overline{\varphi}^i\left(\underline{p}+\lambda;\zeta_i\right)\right]\lambda > 0.
\end{aligned}$$

In the above we rely on the fact that the minimum market clearing price is obtained when aggregate supply is maximized. Since $b^\lambda$ yields higher expected utility than $b^i$ when $\lambda > 0$

is small, $b^i$ is not a best response, and therefore any best response $b^i$ must be constant in a neighborhood of $\overline{q}^i(\zeta_i)$, if $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$.

Define $\check{q}^i(\zeta_i) = \underline{\varphi}^i(\underline{p};\zeta_i)$ to be the left endpoint of the flat interval of bidder $i$'s bid, containing $\overline{q}^i(\zeta_i)$. Without loss of generality, we may assume that $b^i(q;\zeta_i) = \underline{p}$ for all $q > \check{q}^i(\zeta_i)$ whenever $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$: extending the flat portion of the bid function either does not affect allocation, or (by market clearing) increases allocation to some $q$ such that $v(q;\zeta_i) > \underline{p}$. Since $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$ for all $\zeta_i$ with $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$, it follows that $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$. Consider a bidder $j \neq i$ and type $\zeta_j$ such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p}$; since $\Pr(p(Q,\zeta) = \underline{p}|s) > 0$, it must be that $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$. If the bid $b^j(\cdot;\zeta_j)$ is optimal, it must not be utility-improving to decrease the bid to $b^{\lambda\mu}$, where[11]

$$b^{\lambda\mu}(q) = \begin{cases} b^j(q;\zeta_j) & \text{if } q < \overline{q}^j(\zeta_j) - \lambda, \\ \underline{p} + \mu & \text{otherwise.} \end{cases}$$

The bid $b^{\lambda\mu}$ saves payment $\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - (\underline{p}+\mu)dq$ whenever $q_j = \overline{q}^j(\zeta_j)$, but potentially reduces quantity when $q_j \in (\overline{q}^j(\zeta_j) - \lambda, \overline{q}^j(\zeta_j))$. The change in utility from implementing bid $b^{\lambda\mu}$ instead of bid $b^j(\cdot;\zeta_j)$ is bounded below by

$$\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} b^j(q;\zeta_j) - \left(\underline{p}+\mu\right) dq \Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right)$$
$$-\int_{\overline{q}^j(\zeta_j)-\lambda}^{\overline{q}^j(\zeta_j)} \int_{\overline{q}^j(\zeta_j)-\lambda}^{q} v(x;\zeta_j) - b^j(x;\zeta_j)\, dx dG^i\left(q;b^j\right).$$

The derivative of this expression with respect to $\lambda$ must be weakly negative,

$$\left(b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - \left(\underline{p}+\mu\right)\right)\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right)$$
$$-\left(v\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)\right)\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \leq 0.$$

This inequality holds for all $\mu > 0$. Letting $M$ be the Lipschitz modulus of $v$, substituting in for $b^j(\overline{q}^j(\zeta_j);\zeta_j) = \underline{p}$ means that the previous inequality implies

$$\left(b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right) - \underline{p}\right)\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right) - M\lambda \Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right) \leq 0$$
$$\iff \quad -\frac{b^j\left(\overline{q}^j(\zeta_j);\zeta_j\right) - b^j\left(\overline{q}^j(\zeta_j)-\lambda;\zeta_j\right)}{\lambda} \leq \frac{M\Pr\left(q_j \in \left(\overline{q}^j(\zeta_j)-\lambda, \overline{q}^j(\zeta_j)\right)\middle|s\right)}{\Pr\left(q_j = \overline{q}^j(\zeta_j)\middle|s\right)}.$$

[11]The $\mu$ term ensures that bidder $j$ wins ties against the flat portion of bidder $i$'s bid; this term will be taken to zero and thus will have no marginal effect on utility.

Taking the limit as $\lambda \searrow 0$, we obtain that $b_q^j(\overline{q}^j(\zeta_j);\zeta_j) = 0$. Thus any bidder $j \neq i$ with type $\zeta_j$ such that $b^j(\overline{q}^j(\zeta_j);\zeta_j) = v(\overline{q}^j(\zeta_j);\zeta_j)$ and $\Pr(q_j = \overline{q}^j(\zeta_j)|s) > 0$ is such that $b_q^j(\overline{q}^j(\zeta_j);\zeta_j) = 0$.

Now return to bidder $i$ with type $\zeta_i$ such that $b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)$ and $\check{q}^i(\zeta_i) < \overline{q}^i(\zeta_i)$, and consider the alternate bid function $b^\lambda$ defined in the first portion of this proof. We now place a slightly different bound on the utility gained by implementing bid $b^\lambda$ versus bid $b^i(\cdot;\zeta_i)$. Payments increase by at most $\overline{Q}\lambda$, with at most probability 1; and, whenever $q_i > \check{q}^i(\zeta_i)$ under $b^i(\cdot;\zeta_i)$, bidder $i$ receives the full residual quantity $Q - \sum_{j\neq i}\overline{\varphi}^j(\underline{p}+\lambda;\zeta_j)$. Then a lower bound on the utility improvement generated by the alternate bid $b^\lambda$ (versus $b^i(\cdot;\zeta_i)$) is

$$\mathbb{E}_{\zeta_{-i}}\left[\int_q^{Q-\sum_{j\neq i}\overline{\varphi}^j\left(\underline{p}+\lambda;\zeta_j\right)} v\left(x;\zeta_i\right) - \underline{p}dx - \overline{Q}\lambda \middle| q \geq \check{q}^i\left(\zeta_i\right)\right].$$

For $b^\lambda$ to not be utility-improving, this expectation must be weakly negative. Dividing through by $\lambda$ and taking the limit at $\lambda \searrow 0$ gives

$$\mathbb{E}_{\zeta_{-i}}\left[-\left(v\left(Q-\sum_{j\neq i}\overline{q}^j\left(\zeta_j\right);\zeta_i\right)-\underline{p}\right)\sum_{j\neq i}\overline{\varphi}_p^j\left(\underline{p};\zeta_j\right)-\overline{Q}\middle| q\geq \check{q}^i\left(\zeta_i\right)\right]\leq 0.$$

By assumption, $v(Q - \sum_{j\neq i}\overline{q}^j(\zeta_j);\zeta_j) > \underline{p}$, and from the previous paragraph we have that $\overline{\varphi}_p^j(\underline{p};\zeta_j) = -\infty$ with strictly positive probability. Then the above inequality cannot be satisfied. It follows that there is no bidder $i$ such that $\Pr(b^i(\overline{q}^i(\zeta_i);\zeta_i) < v(\overline{q}^i(\zeta_i);\zeta_i)|s) > 0$. □

**Corollary 3. [Bid pasting under symmetry]** *When bidders are symmetric and have symmetric information, $(s,\theta_i) = (s,0)$ for all bidders $i$, the equilibrium market clearing price equals the marginal value for efficient quantity,*

$$p^\star(s) = \hat{v}\left(nq^\star(s);s\right).$$

**Corollary 4. [Efficient bid pasting at minimum price]** *Let $q_i^{\mathit{eff}}(s)$ be the efficient allocation to bidder $i$ when the signal is $s$. The minimum market-clearing price $\underline{p}$ is*

$$\underline{p} = \operatorname*{ess\,inf}_{i,s'} v^i\left(q_i^{\mathit{eff}}(s');s_i'\right).$$

*Proof.* First we show that $\underline{p} \leq \operatorname{ess\,inf}_{i,s'} v^i(q_i^{\text{eff}}(s');s_i')$. Otherwise, Theorem 5 implies that there is a bidder $i$ and signal $s$ such that $\overline{q}^i(s) < q_i^{\text{eff}}(s)$. Let $Q^{\text{eff}}(s) = \sum_{i\in I} q_i^{\text{eff}}(s)$. By assumption,

$$p^{\star}(s) > v^{i}\left(q_i^{\text{eff}}(s); s_i\right) = C\left(Q^{\text{eff}}(s)\right).$$

Then $Q(s) = \sum_{i \in I} q_i(s) \geq Q^{\text{eff}}(s)$, and there is a bidder $j$ for whom $q_j(s) \geq q_j^{\text{eff}}(s)$. For this bidder,

$$v^{j}(q_j(s); s_j) \leq v^{j}\left(q_j^{\text{eff}}(s); s_j\right) = v^{i}\left(q_i^{\text{eff}}(s); s_i\right) < p^{\star}(s).$$

Thus bidder $j$ is bidding above their marginal value, which cannot occur in equilibrium.

Second we show that $\underline{p} \geq \text{ess inf}_{i,s'} v^i(q_i^{\text{eff}}(s'); s_i')$. Otherwise, there is a signal $s$ such that $p^{\star}(s) < \text{ess inf}_{i,s'} v^i(q_i^{\text{eff}}(s'); s_i')$; Theorem 5 further implies that there is a signal $\tilde{s}$ with $v^i(q_i^{\star}(\tilde{s}); \tilde{s}_i) \leq v^i(q_i^{\text{eff}}(\tilde{s}); \tilde{s}_i)$. Since marginal values are decreasing, it must be that $Q^{\star}(\tilde{s}) = \sum_{i \in I} q_i^{\star}(\tilde{s}) > \sum_{i \in I} q_i^{\text{eff}}(s') = Q^{\text{eff}}(s')$. Market clearing then implies that $p^{\star}(Q^{\star}(\tilde{s})) \geq p^{\star}(Q^{\text{eff}}(\tilde{s}))$, since the supply curve $S$ is increasing, a contradiction. Thus $\underline{p} \geq \text{ess inf}_{i,s'} v^i(q_i^{\text{eff}}(s'); s_i')$. $\square$

# B Proofs for Section 2

*Proof of Lemma 1.* Consider a supply curve $S$ such that the uniform-price auction admits a bidding equilibrium $b^{\star}$ such that $q \circ b^{\star} = q^{\star}$ (if no such $S$ exists, the claim is trivially true). By assumption, there is a level of aggregate supply $\hat{Q}$ such that $S$ is locally nonconstant at $\hat{Q}$; without loss of generality, assume that $S(Q') < S(\hat{Q})$ for all $Q' < \hat{Q}$ (otherwise, the result goes through assuming that $S(\hat{Q}) < S(Q')$ for all $Q' > \hat{Q}$). Fix $\delta > 0$, and let $\varepsilon = S(\hat{Q}) - S(\hat{Q} - n\delta) > 0$. Without loss of generality we may assume that there is $\tilde{Q} \in [\hat{Q} - n\delta, \hat{Q})$ such that for all $Q' \in (\tilde{Q}, \hat{Q}]$, $S(Q') - S(\tilde{Q}) \geq (Q' - \tilde{Q})\varepsilon/\delta$.[12]

Let the bidder signal $\hat{s}$ be such that optimal aggregate supply is equal to $\hat{Q}$, that is $Q^{\star}(\hat{s}) = \hat{Q}$. Define $\tilde{\delta} = (\hat{Q} - \tilde{Q})/n$, and consider a bid function $\hat{b}$ where

$$\hat{b}(q) = \begin{cases} v(0; \hat{s}) & \text{if } q < \frac{1}{n}\hat{Q} - \tilde{\delta} \\ 0 & \text{otherwise.} \end{cases}$$

If all bidders submit the bid function $\hat{b}$, aggregate supply is $\hat{Q} - n\tilde{\delta}$, the market-clearing price is $S(\hat{Q} - n\tilde{\delta})$, and each bidder receives quantity $(\hat{Q}/n) - \tilde{\delta}$. Because supply is deterministic, it is sufficient to show that there is no constant bid $\tilde{b}$ that yields any deviating bidder greater utility. By submitting the alternate bid $\tilde{b} \in (S(\hat{Q} - n\tilde{\delta}), S(\hat{Q})]$, the deviating bidder guarantees herself quantity $q = S^{-1}(\tilde{b}) - [(n-1)\hat{Q}/n] + (n-1)\tilde{\delta}$, and her utility increases

[12] An exception to this condition arises when $S$ is discontinuous at $\hat{Q}$. In this case we may analyze $\hat{Q}'$ just above $\hat{Q}$, taking limits where appropriate.

by

$$\int_{\frac{1}{n}\hat{Q}-\tilde{\delta}}^{S^{-1}\left(\tilde{b}\right)-(n-1)\left[\frac{1}{n}\hat{Q}-\tilde{\delta}\right]} v\left(x;s\right)-S\left(\hat{Q}-n\tilde{\delta}\right)dx-\left[\tilde{b}-S\left(\hat{Q}-n\tilde{\delta}\right)\right]\left[S^{-1}\left(\tilde{b}\right)-(n-1)\left[\frac{1}{n}\hat{Q}-\tilde{\delta}\right]\right].$$

The left-hand term is the increase in gross utility, and the right-hand term is the loss from increased payments. By construction, $S^{-1}(\tilde{b})$ is bounded by

$$\hat{Q}-n\delta \leq S^{-1}\left(\tilde{b}\right) \leq \hat{Q}-n\tilde{\delta}+\left(\tilde{b}-S\left(\hat{Q}-n\tilde{\delta}\right)\right)\frac{\tilde{\delta}}{\varepsilon}.$$

It follows that the effect of the deviation $\hat{b}$ on the bidder's utility is bounded above by

$$\int_{\frac{1}{n}\hat{Q}-\tilde{\delta}}^{\frac{1}{n}\hat{Q}-\tilde{\delta}+\left(\tilde{b}-S\left(\hat{Q}-n\tilde{\delta}\right)\right)\frac{\delta}{\varepsilon}} v\left(x;s\right)-S\left(\hat{Q}-n\tilde{\delta}\right)dx-\left[\tilde{b}-S\left(\hat{Q}-n\tilde{\delta}\right)\right]\left[\frac{1}{n}\hat{Q}-\tilde{\delta}\right].$$

The derivative of this bound with respect to $\tilde{b}$ is

$$\left[v\left(\frac{1}{n}\hat{Q}-\tilde{\delta}+\left(\tilde{b}-S\left(\hat{Q}-n\tilde{\delta}\right)\right)\frac{\delta}{\varepsilon}\right)-S\left(\hat{Q}-n\tilde{\delta}\right)\right]-\left[\frac{1}{n}\hat{Q}-\tilde{\delta}\right].$$

Since $\tilde{\delta}\leq\delta$, when $\delta$ is small this derivative is strictly negative (recall that $v(\hat{Q}/n;s)=S(\hat{Q})$). Then there is no profitable deviation when the submitted bid profile is $(\hat{b})_{i=1}^{n}$, and $q(\hat{b})\neq q^{\star}(s)$ by construction. □

# C Proof of Theorem 1

*Proof.* Let $S$ be a strictly increasing supply curve.[13] If $S$ is optimal, there cannot be a quantity $\hat{q}$ and a perturbation $\varepsilon>0$ such that $S^{\varepsilon}$ yields greater efficiency, where

$$S^{\varepsilon}\left(q\right)=\begin{cases}S\left(q\right) & \text{if } q\notin\left[\hat{q},\hat{q}+\varepsilon\right],\\ S\left(\hat{q}+\varepsilon\right) & \text{if } q\in\left[\hat{q},\hat{q}+\varepsilon\right].\end{cases}$$

The expected inefficiency induced by $S$ is

$$\mathbb{E}_{s,\omega}\left[\int_{nq^{\star}(s)}^{nq^{\text{eff}}(s,\omega)}\hat{v}\left(x;s\right)-c\left(x;\omega\right)dx\right].$$

[13] The case of locally-constant supply curves may be handled by examining the endpoints of the locally-constant region. If the supply curve is globally constant, the analysis must be adjusted but the result does not change.

Here, $nq^{\star}(s)$ is the equilibrium (aggregate) quantity allocated when the polluters' signal is $s$, and $nq^{\text{eff}}(s,\omega)$ is the ex post efficient aggregate quantity given polluter signal $s$ and realized marginal damage curve $c(\cdot;\omega)$. Letting $nq^{\varepsilon}(s)$ be the equilibrium aggregate quantity under supply curve $S^{\varepsilon}$, the expected inefficiency induced by $S^{\varepsilon}$ is

$$\mathbb{E}_{s,\omega}\left[\int_{nq^{\varepsilon}(s)}^{nq^{\text{eff}}(s,\omega)} \hat{v}\left(x;s\right)-c\left(x;\omega\right)dx\right].$$

The difference between the two is

$$\mathbb{E}_{s,\omega}\left[\int_{nq^{\varepsilon}(s)}^{nq^{\star}(s)} \hat{v}\left(x;s\right)-c\left(x;\omega\right)dx\right].$$

If $S$ is optimal, it must be that the derivative of the above with respect to $\varepsilon$ is zero:

$$\mathbb{E}_{s,\omega}\left[\left(\hat{v}\left(q^{\varepsilon}\left(s\right);s\right)-c\left(q^{\varepsilon}\left(s\right);\omega\right)\right)n\frac{dq^{\varepsilon}\left(s\right)}{d\varepsilon}\right]\Bigg|_{\varepsilon=0}=0.$$

At $\varepsilon=0$, we have $q^{\varepsilon}(s)=\hat{q}$; and since we are examining pay as bid, we have $\hat{v}(\hat{q};s)=S(\hat{q})$. Then the above is

$$\mathbb{E}_{s,\omega}\left[\left(S\left(\hat{q}\right)-c\left(\hat{q};\omega\right)\right)n\frac{dq^{\varepsilon}\left(s\right)}{d\varepsilon}\middle|\, q^{\star}\left(s\right)=\hat{q}\right]=0.$$

Note that the left-hand multiplicand is independent of $\theta$, and the right-hand multiplicand is independent of $\omega$. Thus we have

$$n\mathbb{E}_{s}\left[\frac{dq^{\varepsilon}\left(s\right)}{d\varepsilon}\middle|\, q^{\star}\left(s\right)=\hat{q}\right]\mathbb{E}_{\omega}\left[S\left(\hat{q}\right)-c\left(\hat{q};\omega\right)\right]=0.$$

It follows that $S$ is optimal only if $S(\hat{q})=\mathbb{E}_{\omega}[c(\hat{q};\omega)]$, for all $\hat{q}$. $\square$

# D Proofs for Section 5

**Lemma 5. [Range of market price given small support]** *Let $s=((s,\theta_i))_{i\in I}$ be a signal profile, and let $(q_i^{\star}(s))_{i\in I}$ be its associated efficient quantity allocation. If there is $\varepsilon$ such that, for all signals $s'$ and all bidders $i$, $|v^i(q_i^{\star}(s);s')-v^i(q_i^{\star}(s);s)|<\varepsilon$, then for all signals $s'$ in TCPAB, $|p^{\star}(s')-p^{\star}(s)|<\varepsilon$.*

*Proof of Theorem 4.* First we show that $p^{\star}(s')<p^{\star}(s)+\varepsilon$ for all $s'$. To establish a contradiction, suppose to the contrary that there exists a signal $s'$ such that $p^{\star}(s')\geq p^{\star}(s)+\varepsilon$.

Because aggregate supply is weakly increasing in market price, it must be that $\sum_{i\in I} q_i^\star(s) \leq \sum_{i\in I} q_i^\star(s')$, hence there is some bidder $i$ such that $q_i^\star(s) < q_i^\star(s')$. By assumption,

$$v^i\left(q_i^\star\left(s'\right); s'\right) \leq v^i\left(q_i^\star\left(s\right); s'\right) < v^i\left(q_i^\star\left(s\right); s\right) + \varepsilon < p^\star\left(s'\right).$$

Thus this bidder is bidding above their marginal value, which is not feasible in equilibrium.

Second we show that $p^\star(s') > p^\star(s) - \varepsilon$ for all $s'$. To establish a contradiction, suppose to the contrary that there exists a signal $s'$ such that $p^\star(s') \leq p^\star(s) - \varepsilon$. Since the minimum market-clearing price $\underline{p}$ is weakly below $p^\star(s')$, Corollary 4 implies that there is a signal $s''$ with $v^i(q_i^\star(s''); s'') < p^\star(s) - \varepsilon$ for all bidders $i$. By the assumption of the theorem, it must be that $\sum_{i\in I} q_i^\star(s'') > \sum_{i\in I} q^\star(s)$; since aggregate supply decreases when the market-clearing price falls, this contradicts the assumption that $q^\star(s'')$ is efficient.

Theorem 5 shows that the equilibrium market-clearing price is bounded below by

$$\underline{p} = \operatorname*{ess\,inf}_{i,s'} v^i\left(q_i^\star\left(s'\right); s_i'\right) \geq \operatorname*{ess\,inf}_{i,s'} v^i\left(q_i^\star\left(s\right); s_i'\right) - \delta.$$

Since marginal damages are increasing in quantity, Theorem 5 further implies that the equilibrium market-clearing price is bounded above by $\underline{p} + 2\delta$. By Lemma 5, the efficient market-clearing price will be $p^\star(s') \in [\underline{p}, \underline{p} + 2\delta]$; this establishes the first claim of the theorem. The second claim, bounding efficiency loss, follows by analyzing the area under curve with height at most $2\delta$ and width at most $\delta/\gamma$. □

*Proof of Lemma 2.* An analogous argument to that for small bounded support works provided we show that the small probability that others have low values cannot suppress a bidder's bid to the extent that it induces even bidders near the concentration value $v_i^*\left(s_i^*\right)$ to bid low on the last unit won $q_i^*\left(s^*\right)$ when all bidders have common knowledge that the profile of types is $(s_1^*, ...s_n^*)$, which consists of their concentration types.[14] Note that $v_i^*\left(q_i^*\left(s^*\right), s_i^*\right) = v^*$ is the same for all active bidders, where active bidder is a bidder who wins strictly positive amount in the above common knowledge situation. Because $\gamma > 0$ is the lower bound on the slope of supply and the bidders' values are concentrated, in order to show that bids are close to values with probability close to 1, it is enough to show that bids are close to values with probability close to $\pi$. Recall that we assumed that with probability at least $\pi$ there are two or more active bidders. We focus on this event and show that each active bidder $i$'s bids on $q_i^*\left(s^*\right)$ are concentrated around $v^*$.

For simplicity suppose that the equilibrium is in pure strategies.[15] Let $p_j\left(x\right)$ be the

[14]Note that there is no concern that the bidders bid too high as on relevant units (those the bidder has positive probability of buying) it is a dominant strategy to bid below one's value.

[15]To extend the argument to mixed strategies we would focus in the extension on high value types who

probability that bidder $j$ bids below $x \in (0, v^*)$ on the quantity $q_j^*(s^*)$. Let $x < y < z < v^*$ are such that $|v^* - x|$ is small. We want to show that bidders with values above $z$ bid above $x$ for sufficiently small $\varepsilon$. Suppose thus that in the equilibrium some bidder $i$ bids below $x$ on the quantity $q_i^*(s^*)$ with positive probability bounded away from 0. By the concentration of values, we can assume that at least $1 - \varepsilon \in (0, 1)$ mass of types of other bidders $j$ have values on $q_j^*(s^*)$ above $z$. We may further assume that there is a strictly positive probability $p > 0$ such that $\Pr(p^\star \in [x, y) \mid s_i^\star) > p$.[16] Consider two cases.

Case 1. As $\varepsilon \searrow 0$, the expected quantity gained by polluter $i$ with type $s_i^\star$ from increasing bids between $x$ and $y$ to $y$ is bounded below by $r > 0$. The gain from such a deviation is bounded below by $pr \times |v^\star - z|$, and the cost is bounded above by $(pr + \varepsilon\overline{Q}) \times |y - x|$. Since $y - x < v^\star - z$ by assumption, as $\varepsilon$ becomes small this deviation becomes profitable.

Case 2. As $\varepsilon \searrow 0$, the expected quantity gained by polluter $i$ with type $s_i^\star$ from increasing bids between $x$ and $y$ to $y$ goes to zero. This is only possible if opponents' bids are asymptotically inelastic at prices in this interval. But if this is the case, then polluter $i$'s bid must also be inelastic between $x$ and $y$; it follows that all bidders face inelastic residual supply between prices $x$ and $y$, and any bidder can reduce their bid near $q_j^\star(s^\star)$, saving payment without sacrificing quantity.

□

# E Beyond Efficiency

Finally, it is known from (Pycia and Woodward 2022b) that the revenue-maximizing pay as bid auction has deterministic supply and yields more revenue than all uniform-price auctions. If the regulator cares about revenue as well as efficiency—as when they reinvest auction proceeds into social projects—this dominance of pay as bid carries over into the current context provided bidders' values are regular in the sense of Pycia and Woodward (2022a).

**Definition 1.** **[Regularity]** Let $M$ be the set of monopoly-optimal quantities and prices,

$$M = \left\{ (p^\star, q^\star) : \exists s,\ p^\star \in \arg\max_p p v^{-1}(p; s) \text{ and } q^\star = v^{-1}(p; s) \right\}.$$

A value function $v^i$ is *regular* if $M$ is monotone: if $(p, q) \in M$ and $(p', q') \in M$, then $p > p'$ implies $q > q'$.

bid with positive probability in the interval $(x - \varepsilon, x)$.

[16] Otherwise, the market-clearing price is never between $x$ and $y$; since, by assumption, $\Pr(p^\star < v^\star - \delta) > \delta$, there are some $x'$ and $y'$ and a polluter-$i$ type $\tilde{s}_i^\star$ such that this inequality holds.

**Theorem 6. [Optimality of Elastic Pay as Bid]** *Suppose that bidders are symmetric, their values are regular, and that the auctioneer's objective is a monotone combination of auction revenue and welfare:*

$$\pi\left(q; s, \omega\right) = f\left(R\left(q; s\right), C\left(\sum_{i=1}^{n} q_i; \omega\right)\right),$$

*where $f$ is increasing in its first argument and decreasing in its second argument. If values $v$ are concave in quantity and monotone in signal, and costs $C$ are convex, then there exists elastic supply such that pay as bid with this supply weakly outperforms all command-and-control schemes, all tax schemes, and all uniform-price auctions.*

*Proof.* It is sufficient to show that properly designed Pay as Bid dominates uniform-price auctions.

We first show that the claim holds in the full-information context, where $s$ is known. Consider an equilibrium $(b^i)_{i=1}^n$ of the full-information uniform-price auction with supply curve $S$. If the market-clearing price associated with this equilibrium is $p^\star$, at the equilibrium allocation $(q_i)_{i=1}^n$ it must be that $v^i(q_i; s) \leq p^\star$ for all $i$. Now consider the allocation $(q_i')_{i=1}^n$ where $\sum_{i=1}^n q_i = \sum_{i=1}^n q_i'$ and $v^i(q_i'; s) = v^j(q_j'; s)$ for all bidders $i$ and $j$; by market clearing, it must be that $v^i(q_i'; s) \equiv p' \geq p^\star$. Consider a pay-as-bid auction with supply curve $\hat{S}(Q') = p'$. By Theorem 5, the equilibrium in this allocation will have market-clearing price $p' \geq p^\star$ and allocation $(q_i')_{i=1}^n$, where $\sum_{i=1}^n q_i' = \sum_{i=1}^n q_i$. Then revenue in the pay-as-bid auction is weakly higher than in the uniform-price auction, and welfare is unchanged. It follows that the regulator's objective is weakly improved by implementing a pay-as-bid auction with supply curve $S'$.

As in Pycia and Woodward (2022a), if the efficient quantity and price are in monotone correspondence with one another—that is, if for all $(Q, p), (Q', p') \in \{(Q^\star(s), \mathbb{E}C(Q^\star(s); \omega) : s \in \operatorname{Supp} s\}$ it is the case that $(Q - Q')(p - p') \geq 0$—we can trace out the optimal supply curve pointwise. Thus an adapted form of regularity is sufficient. Note that, fixing $s$, the regulator's objective can be written as

$$\max_{Q} \mathbb{E}\left[\pi\left(Q; s, \omega\right)\right] = \lambda Q \hat{v}\left(Q; s\right) + (1 - \lambda)\left(\int_0^Q \hat{v}\left(x; s\right) ds - \mathbb{E}\left[C\left(Q; \omega\right)\right]\right).$$

The first-order condition is

$$\hat{v}\left(Q; s\right) + \lambda Q \hat{v}_Q\left(Q; s\right) = (1 - \lambda)\, \mathbb{E}\left[C_Q\left(Q; \omega\right)\right]. \tag{4}$$

When $v$ is monotone in signal and concave in quantity, and $C$ is convex in signal, the implied correspondence between $p^\star(s) = \hat{v}(Q^\star(s); s)$ and $Q^\star(s)$ will be monotone.[17] Following Pycia and Woodward (2022a), this establishes the desired result. $\square$

[17] Suppose otherwise. Then without loss of generality there are signals $s$ and $s'$ with $s' > s$ and such that $p^\star(s') > p^\star(s)$ and $Q^\star(s') < Q^\star(s)$. Then $\mathbb{E}[C_Q(Q^\star(s'); \omega] < \mathbb{E}[C_Q(Q^\star(s); \omega]$, $\hat{v}(Q^\star(s'); s') > \hat{v}(Q^\star(s); s)$, and $Q^\star(s')\hat{v}_Q(Q^\star(s'); s')) \geq Q^\star(s)\hat{v}_Q(Q^\star(s); s)$ (recall that $\hat{v}_Q \leq 0$), establishing a contradiction.