Measuring the Welfare Gains from Cardinal-Preference Mechanisms in School Choice

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Welfare Measurements

- Measure **welfare improvements** of **cardinal mechanisms** over **ordinal mechanisms**
- We show theory alone does **not** predict whether welfare gains **always positive**
- **Empirical issue** how big gains from cardinal mechanisms are
- Public school choice setting, Seattle

Overview

1. **Cardinal-preference** mechanisms
2. Hylland & Zeckhauser (1979) **pseudomarket cardinal-preference mechanism** (no real money)
   - **Almost never been implemented**, even in artificial setting on a computer to our knowledge (except recent CS working paper)
   - **Operationalize**: modify mechanism for practical reasons & provide computer algorithm
3. **Envy-free cardinal-preference mechanism** related to Nguyen, Peivandi & Vohra (2015)
   - **Linear program**, less challenging computationally
4. Seattle school choice: **estimate cardinal preferences** of students
5. Seattle: **measure welfare gains** from **cardinal-preference mechanisms** over ordinally efficient ordinal mechanism

Many Ordinal Mechanisms Coincide In Large Markets

- **Ordinal mechanisms**: each student submits **list of ranked schools**
- **Algorithm** on school district computer then **assigns students to schools**
- Liu and Pycia (2016) prove

All asymptotically ordinally efficient, symmetric, and asymptotically strategy-proof **ordinal mechanisms lead to the same allocation in large markets.**

- **Limited benefits** from **new ordinal mechanisms** in large markets
- Ordinally efficient: no stochastic dominance (not important for today)
- (earlier work by Che and Kojima 2010)

Room for Welfare Improvements

- Table 8 in Abdulkadiroglu, Agarwal and Pathak (2017)
- Demand estimation of preferences for **NYC students**
- Assignment that maximizes **utilitarian welfare** is baseline
  - Not mechanism with truthtelling incentives
- Compare both utilitarian, **deferred acceptance** to "go to closest school"
  - No school choice program
- **Deferred acceptance** achieves **80%** of (utilitarian minus "go to closest school")
- **20% of welfare** for **better ordinal, cardinal mechanisms**
- **We** compare cardinal mechanisms to **ordinally efficient ordinal mechanism**

Cardinal Mechanisms

- School choice: each student goes to one school

1. Describe model notation, **cardinal preferences, cardinal mechanisms**
2. Outline *our* **pseudomarket mechanism**
   - Later compare to Hylland & Zeckhauser (1979)
3. **Envy-free mechanism**
   - Special case of Nguyen, Peivandi & Vohra (2015)

Model Basics

- $i \in I$ **students** (I also # of students)
- $j \in J$ **schools** (J also # of schools)
- q_j **capacity** of school j
- $\pi_i = (\pi_{i1}, \ldots, \pi_{iJ})$ **lottery** facing student i
- Each $\pi_{ij} \geq 0$, $\sum_{j \in J} \pi_{ij} = 1$
- $\pi = (\pi_{ij})_{i \in I, j \in J}$ **assignment matrix** from mechanism
- $\sum_i \pi_{ij} \leq q_j \, \forall j$ **feasible** assignment
- Decompose feasible assignment π into convex combination of deterministic assignments
- Decomposition code freely available, not discuss further
- Can allow undercapacity & unacceptable schools, not in this paper except estimation

Cardinal Preferences

- Vector of **school utilities** for student i

$$v_i = (v_{i1}, \ldots, v_{iJ})$$

- v_i encodes **preferences** over **lotteries**

$$\pi_i = (\pi_{i1}, \ldots, \pi_{iJ})$$

- **Expected utility** of lottery

$$\pi_i \cdot v_i = \sum_{j=1}^{J} \pi_{ij} v_{ij}$$

- **Positive affine transformation** of v_i encodes same preference
- Our mechanism can be made to give **same assignment** from **positive affine transformations**
- Can compute **ordinal preference** (rank order) from v_i

Cardinal Mechanisms

- Website solicits **numeric scores for schools** for each student i

$$v_i = (v_{i1}, \ldots, v_{iJ})$$

- Numeric scores used in **algorithm run on school district computer**
- **Algorithm** uses numeric scores to compute **expected utility preferences over lotteries**
- Could use more sophisticated website to solicit potentially more accurate preferences over lotteries
- Would need to experimentally evaluate website designs in lab, field
- Clarification: no pre-Budish business school course-bidding mechanisms (Sönmez & Unver 2010), or Budish (2011) itself

Pseudomarket Mechanism as a Mathematical Program

- Matrix of **student utilities** $v = (v_{ij})_{i \in I, j \in J}$ is input
- Vector of **school capacities** $q = (q_j)_{j \in J}$ is input
- Output is stochastic **assignment matrix** $\pi = (\pi_{ij})_{i \in I, j \in J}$
- **Pseudomarket mechanism** as mathematical program

$$\max_{\pi, p} W(\pi, v) \text{ subject to}$$

$$(\pi, p) \in \text{Equilibria}(v, q)$$

- Maximize **social welfare** $W(\pi, v)$ subject to **school prices** $p = (p)_{j \in J}$ and π being a **pseudomarket equilibrium**

Example Social Welfare Functions

- Choice of school district to **select equilibria** with desired properties
- **Nash product** in expected utilities

$$W(\pi, v) = \prod_{i=1}^{I} (\pi_i \cdot v_i - \tilde{q} \cdot v_i)$$

- $\tilde{q} = \left(q_j / \sum_{k \in J} q_k\right)_{j \in J}$ vector of percentage capacities
- **Sum of expected utilities**

$$W(\pi, v) = \sum_{i=1}^{I} \frac{\pi_i \cdot v_i - \tilde{q} \cdot v_i}{\max_j v_{ij} - \min_j v_{ij}}$$

- Same assignment π for **positive affine transformations** of school utilities
- (Or use common location, scale normalizations for all students after submission)

Pseudomarket Equilibria

- Equilibria (v, q) set of **competitive equilibria** (π, p) to pseudomarket
- **Competitive equilibrium** is pair (π, p) where

1. Assignment matrix π is **feasible**: $\sum_i \pi_{ij} \leq q_j \, \forall j \in J$
2. For price vector p, assignment vector π_i for each student i **maximizes expected utility** subject to **unit budget**

$$\pi_i \in \arg\max_{\tilde{\pi}_i} (\tilde{\pi}_i \cdot v_i)$$
$$\text{subject to } \tilde{\pi}_i \cdot p \leq 1$$

- Equilibria (v, q) set same for **positive affine transformations** of school utilities

Pseudomarket Incentive Properties

- Students' **incentives to misreport** vanish in series of replica economies
  - He, Miralles, Pycia & Yan (2018)
- **Strategy proof in large**
  - Azevedo & Budish (2019)

Pseudomarket Pareto Efficiency

- Computationally find that our mechanism produces **Pareto efficient** assignments π
  - If social welfare strictly increasing in all expected utilities
- Hylland & Zeckhauser (1979) impose additional restrictions to **prove** that equilibria are Pareto efficient
- **Exact capacity**
  - HZ1: At least one school has **zero price**
  - HZ2: Student picks a **least cost** lottery π_i when multiple lotteries maximize expected utility
- **Excess capacity**, HZ2 and
  - HZ3: All **underdemanded** schools have prices of **zero**

Equilibrium Selection Example in Paper

- Paper has a **non-computational example** of four equilibria to same pseudomarket

1. Social welfare is Nash product
2. Social welfare is sum of expected utilities
3. Sum of expected utilities plus least cost property and underdemanded schools have zero price
4. Minimize sum of expected utilities

- 1,2,3 are **Pareto efficient**
- 1,2 have **higher social welfare** than 3
- Only 3 satisfies HZ2, HZ3, fits into **Hylland & Zeckhauser** (1979)
- We consider more equilibria than **Hylland & Zeckhauser**

Pseudomarket Numerical Implementation

- **Mechanism** as mathematical program

$$\max_{\pi,p} W(\pi, v) \text{ subject to}$$
$$(\pi, p) \in \text{Equilibria}(v, q)$$

- **Bilevel optimization** (like subgame perfection)
- **Single level reduction**: maximize welfare subject to **KKT conditions** for all student problems
- Student linear programs: KKT conditions have **complementarity constraints**
- **MPCC**: mathematical program with complementarity constraints
- $I \cdot (J + 1)$ complementarity constraints (not toy examples)
- After much investigation & online benchmarks, KNITRO's approach to MPCCs fastest for our mechanism
- Benchmarks in paper

Envy-Free Mechanism

- **Envy-free mechanism**

$$\max_{\pi} W(\pi, v) \text{ subject to}$$
$$\pi \in \text{Envy Freeness}(v)$$

- i **not envying** j means

$$\pi_i \cdot v_i \geq \pi_j \cdot v_i$$

- Special case of bundle-choice mechanism in Nguyen, Peivandi & Vohra (2015)
- **Linear program** if $W(\pi, v)$ sum of expected utilities
- Easy to compute
- Except number of envy-free constraints is $I \cdot (I - 1)$
- **All pseudomarket equilibria (with equal budgets) are envy free**
- So social welfare weakly higher under envy-free mechanisms

Seattle Public Schools

- Study **ninth grade**
- School choice optional, all students have **default school**
- 11 schools, 964 students
- **Non-strategy proof**, ordinal mechanism
- Students face non-trivial **lotteries** from submitting **ranked lists of schools**
- Not interested in Seattle mechanism
- Lack IT department's code for mechanism
- Use to **identify cardinal preferences**
- Agarwal and Somaini (2018), Kapor, Neilson and Zimmerman (2020)

Estimating Assignment Probabilities

- Data
  - m_i **ranked list of schools** submitted by student i
  - x_i **priorities** (sibling, residence) of student i
  - $y_{ij} = 1$ if school j **enrolled in fall**
- Estimate **statistical model** with random forest, some cross-validated tuning parameters

$$\Pr\left(y_{ij} = 1 \mid x_i, m_i\right) \approx h_j\left(x_i, m_i\right)$$

- **Lottery** facing student i for list m_i

$$h\left(x_i, m_i\right) = \left(h_j\left(x_i, m_i\right)\right)_{j \in J}$$

Partial Identification of School Utilities

- Assume student submits **list** m_i that maximizes **expected utility**

$$v_i \cdot h(x_i, m) = \sum_{j \in J} v_{ij} h_j(x_i, m)$$

- $\tilde{M}_i$ set of **counterfactual lists** m chosen to balance run time and identifying power
- Non-sharp **identified set** of **school utilities** for student i

$$\tilde{V}_i = \left\{ v_i \mid v_i \cdot h(x_i, m_i) \geq v_i \cdot h(x_i, m) \; \forall \, m \in \tilde{M}_i \right\}$$

- Following Bajari & Benkard (2005), impose **uniform distribution** on $\tilde{V}_i$ and sample from it
- Uniform distribution is **assumption**
- Robustness: alter uniform distribution in mechanism comparison
- Rejection sampler
- Each **replication of mechanisms** has one draw from $\tilde{V}_i$ for each student i

Advantages of Partial Identification

- Conditions for nonparametric **point identification** of distribution of school utilities likely not satisfied
- Lotteries may not vary across students **continuously**
  - Priorities discrete in Seattle data
- Lotteries may **not** be **independently distributed** from school utilities
  - **Sibling** enrollment related to parent preferences
  - **Parents** may choose **where to live** based on school preferences
  - Students growing up in **neighborhood** may hear about **nearby** schools
  - Just **spatial patterns** of preferences
- Impose normalizations that $\sum_{j \in J} v_{ij} = 1$, & $\min_{j \in J} v_{ij} = 0 \, \forall j \in J$ for student i

Another Behavioral Assumption

- Partial identification can be **expanded** to **other behavioral assumptions**
- Use following as **robustness check**

All listed schools preferable to default school, **all schools not listed** unacceptable compared to default school

- Uniform draws over cardinal preferences consistent with above
- Requires full list length (five) to not be used
- No cardinal information as no lotteries

Seattle Schools & Estimates

High School	# Submitted First Choice	# Estimated Top Utility	Capacity (Under Exact)
Ballard	51	66	79
Cleveland	243	194	233
Franklin	55	82	72
Garfield	85	73	58
Roosevelt	120	161	84
Chief Sealth	17	24	31
West Seattle	50	49	44
Ingraham	198	173	165
Ranier Beach	8	10	47
Nathan Hale	54	58	84
Center	83	74	66

- **One replication** of **one school utility draw** for each student
- **Capacities**: proportional to fall enrollment of choice students

Cardinal vs Ordinal Mechanisms

- Compare **two cardinal mechanisms** to **probabilistic serial**
  - Bogomolnaia & Moulin (2001)
- All three mechanisms
  - **Lotteries** π_i for student i
  - **Strategy proof in the large**
- Our pseudomarket **Pareto efficient** in simulations
- Envy free **not always** Pareto efficient
- Probabilistic serial **ordinally efficient**

Gain Measure

- Explain **gain measure** for pseudomarket, also applies to envy-free
- π_i^{PM} pseudomarket assignment for student i
- π_i^{PS} probabilistic serial assignment for student i
- **Gain** for student i

$$\frac{v_i \cdot (\pi_i^{\mathrm{PM}} - \pi_i^{\mathrm{PS}})}{\max_j v_{ij} - \min_j v_{ij}}$$

- Invariant to positive affine transformations of v_i
- Papers with **different normalizations** can report **same gain** measure
- Can **drop denominator** if happy with normalizations, worried about denominator

Gain Measure in Top School Probabilities

- For student i, find **lottery** over **top school**, **worst school** giving **same utility** as **pseudomarket** assignment
- $\bar{\pi}_i^{(1,\mathrm{PM})}$ probability i assigned to **top school** in **equivalent lottery**
- $1 - \bar{\pi}_i^{(1,\mathrm{PM})}$ probability i assigned to **worst school**
- Algebra shows $\bar{\pi}_i^{(1,\mathrm{PM})}$ satisfies

$$\frac{v_i \cdot \pi_i^{\mathrm{PM}}}{\max_j v_{ij} - \min_j v_{ij}} = \bar{\pi}_i^{(1,\mathrm{PM})}$$

- Same for **probabilistic serial** $\bar{\pi}_i^{(1,\mathrm{PS})}$
- To relate equivalent lotteries to **gain measure**

$$\frac{v_i \cdot \pi_i^{\mathrm{PM}}}{\max_j v_{ij} - \min_j v_{ij}} - \frac{v_i \cdot \pi_i^{\mathrm{PS}}}{\max_j v_{ij} - \min_j v_{ij}} = \frac{v_i \cdot \left(\pi_i^{\mathrm{PM}} - \pi_i^{\mathrm{PS}}\right)}{\max_j v_{ij} - \min_j v_{ij}} = \bar{\pi}_i^{(1,\mathrm{PM})} - \bar{\pi}_i^{(1,\mathrm{PS})}$$

Gain Measure in Top School Probabilities

- **Gain measure** satisfies

$$\frac{v_i \cdot (\pi_i^{\mathrm{PM}} - \pi_i^{\mathrm{PS}})}{\max_j v_{ij} - \min_j v_{ij}} = \bar{\pi}_i^{(1,\mathrm{PM})} - \bar{\pi}_i^{(1,\mathrm{PS})}$$

- Say gain measure is 0.05 for student i
- Use gain in a sentence:
- "A **gain** or normalized differences in expected utilities between the mechanisms for student i of 0.05 is equal to an **increase in top school probabilities** of 0.05, when the needed 0.05 reduction occurs for the worst choice school."

Computational Bounds on Gain Measure

- How big, small can **gains** of **pseudomarket over probabilistic serial** be?
- Consider market with four students, four schools
- Each school has capacity of one
- Search over four **school utility vectors** v_1, v_2, v_3, v_4
  - $16 = 4 \cdot 4$ scalar school utilities
- **Upper bound** on **gain** is **0.29**
- **Lower bound** on **gain** is **negative** and likely smaller than 0.29 in absolute value
  - (Found bug in lower bound code recently)
- **Asymmetry** in absolute value of bounds
- **Large difference** between upper, lower bounds

Empirical issue: sign and magnitude of gain

Benchmark Without Truthtelling Incentives

- Compare gains from our **two cardinal mechanisms** to **benchmark**
  - Pseudomarket, envy-free
- Benchmark maximizes welfare **without truthtelling incentives**

$$\max_{\pi} W(\pi, v)$$

- Calculate gain (**versus probabilistic serial**) for benchmark
- **Divide gains** (versus probabilistic serial) of two cardinal mechanisms by **gains from benchmark**
- Get **percentage of possible gain** from **any cardinal mechanism** that each cardinal mechanism with truthtelling incentives achieves

Mechanism Comparisons: Exact Capacity

	Envy Free			Sum Expect Util PM			Nash Product PM		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Indifferent	0.10	0.08	0.13	0.12	0.09	0.14	0.12	0.10	0.15
Prefer Cardinal	0.67	0.63	0.71	0.71	0.67	0.75	0.71	0.66	0.75
Prefer Ordinal	0.22	0.17	0.27	0.17	0.13	0.21	0.18	0.13	0.22
Mean Gain	0.045	0.036	0.052	0.016	0.012	0.020	0.016	0.011	0.021
Gain No Indiff.	0.049	0.041	0.057	0.018	0.014	0.023	0.018	0.013	0.024
No-Incentives Gain	0.077	0.066	0.085		Same			Same	

- **Exact capacity**: proportional to fall enrollment
- 400 replications: new draws of school utilities for each student
- Compare **envy-free & pseudomarket** (PM) to **probabilistic serial**
- Two different pseudomarket **welfare** functions

Mechanism Comparisons: Exact Capacity

	Envy Free			Sum Expect Util PM			Nash Product PM		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Indifferent	0.10	0.08	0.13	0.12	0.09	0.14	0.12	0.10	0.15
Prefer Cardinal	0.67	0.63	0.71	0.71	0.67	0.75	0.71	0.66	0.75
Prefer Ordinal	0.22	0.17	0.27	0.17	0.13	0.21	0.18	0.13	0.22
Mean Gain	0.045	0.036	0.052	0.016	0.012	0.020	0.016	0.011	0.021
Gain No Indiff.	0.049	0.041	0.057	0.018	0.014	0.023	0.018	0.013	0.024
No-Incentives Gain	0.077	0.066	0.085		Same			Same	

- **Envy-free mechanism** captures **58% of the gain** that **any cardinal mechanism** could hope to have over an ordinally efficient ordinal mechanism, at mean gain
  - 58% is 0.045 divided by 0.077
- **Pseudomarket mechanism** captures **21% of the possible gain**
  - 21% is 0.016 divided by 0.077

Mechanism Comparisons: Excess Capacity

	Envy Free			Sum Expect Util PM			Nash Product PM		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Indifferent	0.12	0.09	0.27	0.14	0.10	0.31	0.13	0.10	0.31
Prefer Cardinal	0.72	0.56	0.77	0.61	0.33	0.76	0.62	0.18	0.75
Prefer Ordinal	0.16	0.12	0.21	0.25	0.12	0.54	0.25	0.13	0.68
Mean Gain	0.032	0.023	0.038	0.006	-0.012	0.018	0.006	-0.032	0.017
Gain No Indiff.	0.036	0.029	0.042	0.007	-0.013	0.020	0.007	-0.037	0.017
No-Incentives Gain	0.047	0.037	0.056		Same			Same	

- **Excess capacity**: proportional to fall enrollment plus 10%
- **56–77% of students** prefer pseudomarket assignment
- **Gain of 0.023–0.038** for envy-free mechanism
  - At mean, $0.032/0.047 =$ **68% of possible gain from cardinal mechanisms**
- Pseudomarket captures 13% of possible gain, at mean

Weaker Preference Restrictions

	Envy Free			Sum Expect Util PM			Nash Product PM		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Indifferent	0.05	0.02	0.10	0.06	0.03	0.11	0.06	0.04	0.11
Prefer Cardinal	0.79	0.72	0.84	0.78	0.67	0.84	0.78	0.67	0.83
Prefer Ordinal	0.17	0.11	0.23	0.16	0.11	0.22	0.16	0.12	0.23
Mean Gain	0.022	0.019	0.027	0.009	0.006	0.012	0.009	0.006	0.012
Gain No Indiff.	0.024	0.020	0.029	0.010	0.006	0.012	0.034	0.006	0.013
No-Incentives Gain	0.033	0.028	0.038		Same			Same	

- "**All listed schools** preferable to default school, all schools not listed unacceptable compared to default school"
- 78-79% prefer cardinal assignment
- Envy-free achieves **67%** of **possible gain from cardinal mechanisms**
- Pseudomarket achieves **27%** of benchmark

Non-Uniform Distributions

	Envy Free		Sum Expect Util PM		Nash Product PM	
	Max Gap	Min Gap	Max Gap	Min Gap	Max Gap	Min Gap
Indifferent	0.14	0.10	0.14	0.10	0.14	0.10
Prefer Cardinal	0.72	0.72	0.50	0.50	0.72	0.50
Prefer Ordinal	0.13	0.17	0.14	0.39	0.14	0.39
Mean Gain	0.044	0.060	0.044	0.004	0.044	0.004
No-Incentives Gain	0.063	0.069	Same		Same	

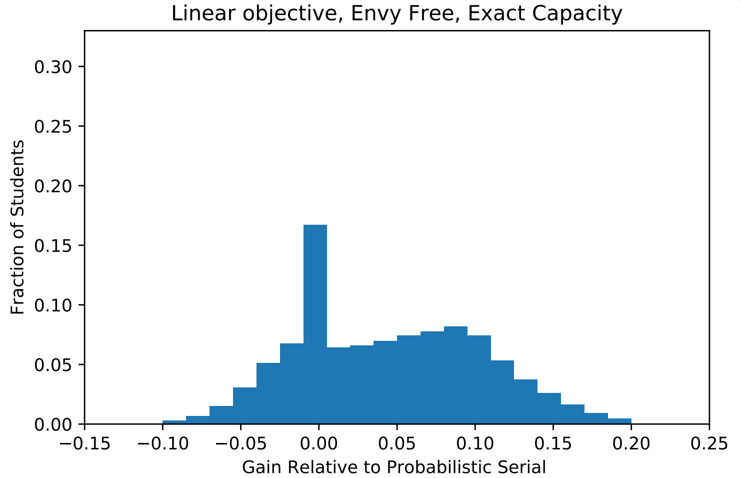
- Consider extremely **non-uniform distributions** over identified set
- **Utility gap** between **first, second choice schools for** i is $v_{i(1)} - v_{i(2)}$
- **Min gap** is for each student, take **minimum** of $v_{i(1)} - v_{i(2)}$ across previous 400 replications
- **Max gap** takes **maximum** of $v_{i(1)} - v_{i(2)}$ across 400 replications

Serial Dictatorship

	Envy Free			Sum Expect Util PM			Nash Product PM		
	Mean	Min	Max	Mean	Min	Max	Mean	Min	Max
Indifferent	0.10	0.08	0.14	0.12	0.10	0.15	0.12	0.10	0.16
Prefer Cardinal	0.67	0.63	0.71	0.68	0.63	0.72	0.67	0.61	0.73
Prefer Ordinal	0.23	0.18	0.28	0.20	0.16	0.24	0.20	0.16	0.24
Mean Gain	0.042	0.033	0.049	0.013	0.009	0.017	0.013	0.008	0.019
No-Incentives Gain	0.074	0.063	0.082		Same			Same	

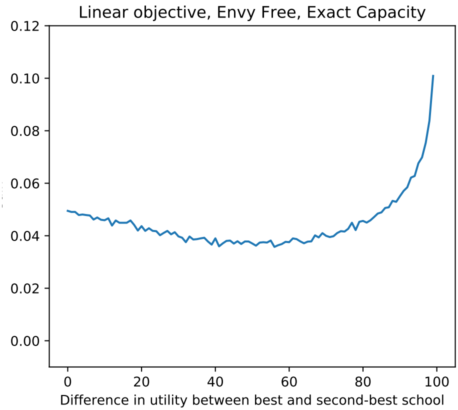
- Similar table as baseline except **random serial dictatorship** as **ordinal mechanism**
- Same 400 replications
- RSD used in practice
- Equivalent to **student-proposing deferred acceptance** with random tie breakers
- RSD outperforms probabilistic serial, slightly

Student Heterogeneity in Mechanism Gains



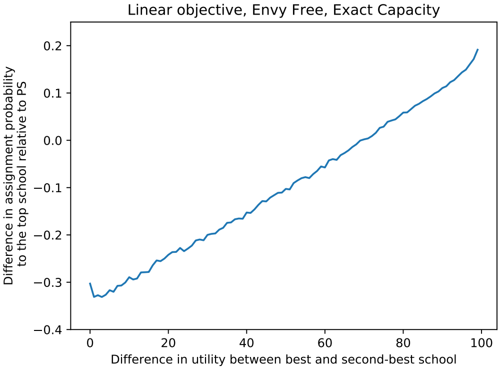
- Some big winners
- Mass near zero of indifferent students
- Fewer losers than winners

Gains by Utility Gap Between Top Two Schools



- Percentiles of **utility gaps** between top two schools, $v_{i(1)} - v_{i(2)}$
- Those with **high utility gaps gain more** from cardinal mechanisms
  - Versus probabilistic serial

Increases in Top School Probabilities by Utility Gap Between Top Two Schools



- Percentiles of **utility gaps** $v_{i(1)} - v_{i(2)}$ between top two schools
- Vertical axis is increase in **top school probability** $\pi_{i(1)}$ from **cardinal minus ordinal** mechanism
- Indifferent students excluded from figure

Conclusions

- Goal: measure **welfare gains** from **cardinal mechanisms**
- **Envy-free** mechanism
- **Pseudomarket** mechanism
  - Modified **pseudomarket** with **equilibrium selection** through **social welfare**
  - Consider **more equilibria** than Hylland & Zeckhauser (1979)
  - **Numerical method** to compute pseudomarket mechanism

Estimate school utilities using Seattle data

- **63-75% of students** prefer **cardinal mechanisms** to probabilistic serial
- **Gain** of **envy-free** over probabilistic serial: **0.036–0.052**
- Envy-free mean achieves **58% of possible gain** of **any** cardinal mechanism

- Cardinal mechanisms give **higher probabilities** to **first choice schools**, particularly for students with **higher gaps** between **utilities for first, second choice schools**