Endogenous Market Structure: Over-the-Counter versus Exchange Trading *

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Abstract

For many assets, traders favor either over-the-counter (OTC) or centralized markets. This paper examines how traders' choice between these trading venues depends on asset and trader characteristics. Market structures are endogenously determined by traders' individual choices of market and counterparty. Traders choose OTC markets over centralized exchanges when the idiosyncratic component dominates in their individual asset valuations and their private information is sufficiently inaccurate. It is due to the benefit of learning the asset value from the price and lowering the price impact. Endogenizing traders choices on the trading venues derives rich market structures: either only OTC or centralized market, or both. The OTC and centralized markets coexist only when traders' asset values are heterogeneously correlated and when the centralized market is imperfectly competitive.

KEYWORDS: Noncompetitive trading, Over-the-counter markets, Exchanges, Price impact, Liquidity, Efficiency

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1 Introduction

Over-the-counter markets have been an essential alternative trading venue in financial markets. In over-the-counter markets, buyers and sellers are paired and privately choose their own trading terms, while public exchanges use a centralized trading mechanism such as uniform-price auctions. Certain types of assets appear to be traded mostly in over-the-counter markets, whereas others have been traded in centralized exchanges. Corporate bonds, interest rate swaps, index derivatives, and many liquid financial products are traded in over-the-counter markets despite their high volumes of trade. Furthermore, the over-the-counter and centralized markets coexist for some assets. For instance, foreign exchange is traded in both over-the-counter spot markets and centralized futures markets.[1] The goal of this paper is to examine traders' incentives to enter different trading venues and explains why some assets are traded in the over-the-counter markets, centralized markets, or both.

The main finding of this paper is that heterogeneity in trader and asset characteristics matter for understanding traders' market choice and endogenous market structure. Heterogeneity in characteristics affects *liquidity* and *learning* for traders. Liquidity is measured by the price impact, which is endogenously defined in equilibrium as the change in price with respect to a one-unit increase in trader's demand.[2] Larger price impacts, i.e., lower liquidity, reduce traders' demands and lower their utilities. Also, traders choose trading strategies based on their private information on assets and market prices, so the demand schedule incorporates inference about the asset values. Heterogeneity in traders' asset valuation and the precision of private information incorporated in this paper jointly affect the liquidity and learning, and thus, determine the incentives for traders to choose different trading venues.

This paper endogenizes the market structure in a two-period model. In the first period $t = 0$, each trader chooses to enter either a centralized market or an over-the-counter market that opens at $t = 1$. In the *centralized market*, all traders' demand and supply schedules determine the single market price. In the *over-the-counter market*, traders trade bilaterally at a pair-specific price after forming a pairwise stable matching. Traders are uncertain about the value of a risky asset but receive a private signal about the value. Signal precision can be heterogeneous among traders. The individual asset valuations are interdependent and consist of two components: a common component, which is the same for all traders, and an idiosyncratic component, which can be heterogeneously correlated across traders. In both the centralized market and over-the-counter bilateral trades, traders play a Bayesian Nash equilibrium in the

[1] A good portion of over-the-counter FOREX trading is in fast venues, such as Currenex, EBS, and Reuters, in which the foreign exchange spot price is generally the same as the price in centralized futures markets.

[2] The model can incorporate the frictions considered in literature (e.g., search costs, a chance that bilateral trades fail, bid-ask spreads by dealers, etc.). Such frictions would increase traders' incentives to choose centralized markets over over-the-counter markets, but the effects studied in this paper would still be present.

uniform-price double auction: all traders simultaneously submit their (net) demand schedules $q_i(\cdot) : p \mapsto q_i(p)$. The trades clear at price p^* such that $\sum_i q_i(p^*) = 0$. The endogenous market structure is immune to an individual or pairwise deviation from equilibrium market choices.

The uniform-price double auction is the canonical model for non-competitive markets for divisible goods (e.g., Kyle (1989), Vives (2011), and Rostek and Weretka (2012)). This trading mechanism allows an explicit treatment of price impacts, which are the equilibrium objects that determine traders' trading behaviors as well as the market choices. The literature based on the uniform-price mechanism so far has maintained a joint symmetry assumption on traders risk aversions, the correlation in traders' asset values, and variance of the values and uncertainty. Rostek and Yoon (2018) dispense with any symmetry restrictions to allow heterogeneity in all characteristics and primitives. This paper follows the model of asymmetric markets in Rostek and Yoon (2018).

The results in this paper help explain which traders choose the over-the-counter or centralized markets, which assets are traded in either type of trading venues, and when centralized and over-the-counter markets can coexist.

First, over-the-counter markets are more attractive to traders in both learning and liquidity when the idiosyncratic component dominates the common value, in the sense that the dispersion of correlations between traders' asset valuation is larger than the average level of correlations with the large idiosyncratic component.[3] In centralized markets, equilibrium price is a weighted average of traders' signals and thus aggregates *out* the idiosyncratic components. An over-the-counter market can improve his learning about this idiosyncratic component of the asset value. In addition, the over-the-counter market allows the trader to choose a counterparty who would more likely have the opposite trading needs (i.e., more negatively correlated asset values). Such counterparty provides the trader a lower price impact and a larger trading amounts in the over-the-counter market. On the other hand, the centralized price is an average of asset values of both buyers and sellers and thus mitigates trading surplus. These effects on learning and price impact do not necessarily create a trade-off and encourage traders to trade in the over-the-counter market when a trader's value relies more on the idiosyncratic value compared to the common component.

The incentives to choose over-the-counter versus centralized markets differ between informed and uninformed traders. Traders with low information precision (i.e., *uninformed traders*) benefit from an over-the-counter market because it helps them learn counterparties' information. On the other hand, the over-the-counter market discourages those whose asset values are less idiosyncratic or those with high information precision (i.e., *informed traders*) to participate

[3] Suppose that $\rho_{i,j}$ denotes the correlation between individual asset values of two traders i, j. A trader i's value is correlated with all other traders. The heterogeneity in $\{\rho_{i,j}\}_{j \neq i}$ represents the idiosyncratic component in trader i's asset value, while the average of correlations represents the common component. See Section 2.

since it decreases the likelihood of meeting a counterparty to trade with and also may increase price impact. The trade-off between information and liquidity incentives in over-the-counter markets creates a cutoff level of information precision. If a trader's information precision is higher than this cutoff level, the liquidity incentive dominates and he chooses to trade in the centralized market. Likewise, with a precision lower than the cutoff, the learning incentive dominates and traders enter the over-the-counter market. Hence, the over-the-counter trading occurs between traders who have relatively larger idiosyncratic value components or lower information precisions.

Second, endogenizing market structure is determined by a fixed point problem between traders' incentives in market choices and the distribution of traders in two trading venues. This paper shows that heterogeneity is crucial for understanding the endogenous market structure. When traders' asset values are interdependent with the same correlations[4] and their information precisions are the same, the trading strategies and incentives in the market choices are symmetric for all traders. With these symmetric incentives, the endogenous distribution of traders in two trading venues has a corner solution in the sense that all traders choose the centralized market or the over-the-counter market. Equivalently, the centralized and over-the-counter market can coexit only when traders are asymmetric in how their asset values are correlated or in information precision.

The over-the-counter market can exist through traders' trading venue choices, even when there is no difference in prices between an over-the-counter trade and centralized market. This is because despite the lack of a difference in prices, for certain trader and asset characteristics, trading over-the-counter offers the benefit of improving learning and lowering price impacts. Moreover, traders can choose the over-the-counter market when the centralized exchange is competitive and traders' trading needs can be fully exhausted. These results, based on endogenous price impact and learning in trades of divisible assets, complement the literature on traders' incentives to trade a single unit of assets in the over-the-counter market.[5]

Lastly, this paper identifies the types of over-the-counter market matching in equilibrium, taking into account traders' choice of market. In the over-the-counter market, there are two types of matching structure depending on a dominant incentive: When the information precision is sufficiently heterogenous for all traders, the dominant incentive, learning or liquidity, differs between informed and uninformed. It creates a *cross-type* (i.e., negative assortative) matching occurs. With symmetric information precision, a *same-type* (i.e., positive assortative) match-

[4] Rostek and Weretka (2012) define a symmetric interdependence in traders' asset valuation by an *equicommonal model*: $\frac{1}{I-1}\sum_{j\neq i}\rho_{ij} = \bar{\rho}$ for all i. The symmetry condition in this paper is stronger than the equicommonal model. The correlations ρ_{ij} are the same for all pairs (i, j).

[5] Babus and Palatore (2018) and Babus and Hachem (2022) study over-the-counter trades for divisible assets among symmetric traders and analyze the role of price impact in the efficiency of market and security design. The heterogeneity of traders in this paper creates asymmetric learning incentives in over-the-counter markets.

ing occurs. With an available centralized market, however, the cross-type matching does not occur in equilibrium. When an informed trader values low price impact more than improving learning, he is better off trading in the centralized market than in the over-the-counter trading with uninformed counterparties. Hence, taking into account the endogenized market choices, the over-the-counter market will attract uninformed traders or both informed and uninformed traders.

The predictions in this paper are consistent with empirical results: Biais and Green (2019) show that transaction costs and liquidity are vital determinants on why most trades for bonds are held in over-the-counter markets, while Attanasi, Centorrino, and Moscati (2016) explore the effects of lack of information in the over-the-counter market on efficiency. Forward contracts, interest rate swaps, or credit linked securities are traded in over-the-counter markets, even though their trading volumes (liquidity) are substantial. The purpose of trading these financial products is often to hedge traders' outside portfolios so that they are idiosyncratically valued. On the other hand, centralized markets attract assets traded by many arbitrageurs or short-term investors, such as stocks or bonds with short maturity. These assets are valued at the future prices that are common to all traders. High-yield bonds that have low credit ranking can be traded in the over-the-counter markets (e.g., Hendershott and Madhavan (2015)) because low past trading volume and volatile return prevent the traders' access to quality information (i.e., low information precision) and increase the heterogeneity in information precision between insiders and other traders.

Related Literature: A growing theoretical literature has studied over-the-counter trading in several aspects. Assuming a fixed market structure, a strand of literature on over-the-counter markets look at how liquidity affects traders' behavior and efficiency (Duffie, Gârleanu, and Pedersen (2005), Vayanos and Weill (2008), Weill (2008), Atkeson, Eisfeldt, and Weill (2015)). Other studies show how over-the-counter markets aggregate private information (Duffie, Malamud, and Manso (2014), Back, Liu, and Teguia (2017), Babus and Kondor (2018), Maurin (2022)). In addition, observing that over-the-counter markets can dominate centralized markets in welfare terms, several authors have explored possible determinants: such as default, search friction, price impacts, and information asymmetry between sellers and buyers (Acharya and Bisin (2010), Malamud and Rostek (2014), Glode and Opp (2016), Duffie and Wang (2017)). Praz (2015) and Zhu (2014) studies how the presence of an alternating trading venue affect equilibrium in centralized markets.

The objective of this paper is to understand endogenous market structures when centralized and over-the-counter markets are both available. Kirilenko (2000), Viswanathan and Wang (2002), Bolton, Santos, and Scheinkman (2016), Lee and Wang (2019) have explored the choice of some traders between centralized and over-the-counter markets. In this paper, all buyers and sellers, informed and uninformed, strategically choose a trading venue. Endogenizing the market choices of all traders lets the incentives of trading in over-the-counter markets to be functions of endogenized participation rather than functions of fixed market structures. Dugast, Üslü, and Weill (2019) studies endogenous market structures for traders who are heterogeneous in their trading needs and capacity to take large positions, while this paper focus on the heterogeneity is in traders' asset valuations and information precision.

Another strand of literature endogenizes the over-the-counter structure itself by studying incentives to choose a counterparty in over-the-counter markets (Duffie, Gârleanu, and Pedersen (2005), Zhu (2012), Golosov, Lorenzoni, and Tsyvinski (2014), Farboodi (2021)). This paper shows that an available centralized market can change predictions in the over-the-counter markets studied in the literature. In addition to the literature for the counterparty choices, Babus and Parlatore (2021) endogenize over-the-counter networks when trading is based on a uniform-price double auction as in this paper. While they study traders' choice of a dealer, this paper explores financial markets where there is no explicit notion of dealers.

2 Model

This paper considers a static economy where two trading venues open simultaneously for the same assets: a centralized market where all traders' bids are cleared at a single market price and an over-the-counter market where a pair of traders are matched and they trade bilaterally at a pair-specific price. Figure 1 summarizes the economy. Before the markets are open ($t = 0$), traders choose which market they would trade in. If a trader chooses the over-the-counter market, then he also chooses a counterparty with whom he would like to trade. The market choice and bilateral over-the-counter matching occur once and for all in that traders can trade only once in one market and with one counterparty if they are in the over-the-counter market. At trading period $t = 1$, two assets – a risky asset (asset) and a riskfree asset (numeraire) – are traded in both markets. The assets are perfectly divisible. Traders submit their demands to the market they chose at the entering period, and each market clears independently. The details are described below including (1) traders and payoffs, (2) information, (3) markets, (4) strategies, and (5) equilibrium.

Strategic Traders: There are $I < \infty$ strategic traders. Each trader has a quadratic utility on quantity traded q_i of the risky asset net of payment $-pq_i$ where p is the price in the market he participates in.[6]

$$u_i(q_i, p) = \tilde{\theta}_i q_i - \frac{\mu}{2} q_i^2 - pq_i. \tag{1}$$

[6] The quadratic utility (mean-variance utility) is the standard of the non-competitive market literature (see Kyle (1989), Vives (2011), and Rostek and Weretka (2012)). The quadratic utility is equivalent to the constant absolute risk-aversion (CARA) utility $v_i(q_i, p) = -\exp(-\mu(\tilde{\theta}_i q_i - pq_i))$: The results on trading behavior and market choices carry over in the model with CARA utilities. See Section 6 for more discussion.

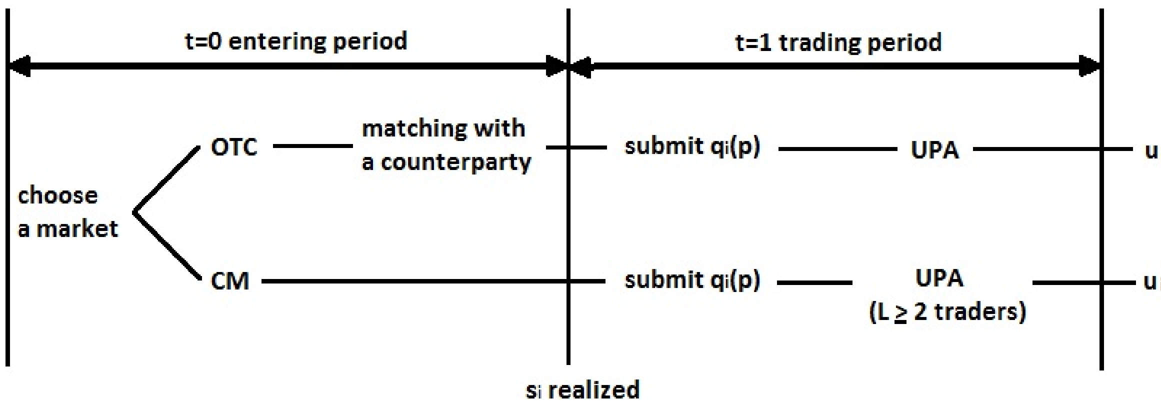

Figure 1: Timing of economy. $q_i(p)$ is his quantity demand at a market price p. Trader i has a private information s_i before trading. Their individual asset value $\tilde{\theta}_i$ is realized at the end of economy and so does the utility u_i evaluated at the realization of $\tilde{\theta}_i$ and equilibrium outcomes $q_i^* = q_i(p^*)$ and p^*.

Here, $\mu > 0$ is the risk-aversion that is common for all traders, and $\tilde{\theta}_i$ is the individual valuation of the risky asset for trader i. The value of numeraire is normalized by one. The individual asset value $\tilde{\theta}_i$ is uncertain and is randomly drawn from $\tilde{\theta}_i \sim \mathcal{N}(E[\tilde{\theta}], \sigma_\theta^2)$. The prior distribution of asset values $\{\tilde{\theta}_i\}_i$ is the same for all i. The homogeneous mean and variance of asset value across traders captures that the ex-ante trading needs for all traders are the same.[7]

Traders' individual asset values $(\tilde{\theta}_i)_{i\in I}$ are *interdependent*. The model allows an arbitrary Gaussian structure for traders' asset values. The correlation matrix for $(\tilde{\theta}_i)_i$ is denoted by $\Sigma_\theta = (\tilde{\rho}_{ij})_{i,j\in I}$ with $\tilde{\rho}_{ij} := Corr(\tilde{\theta}_i, \tilde{\theta}_j)$. The interdependence of $(\tilde{\theta}_i)_i$ is decomposed into a *common value* component, which is attributed to the future asset return in the market, and an *idiosyncratic value* component, which comes from individual portfolio return consisting of other assets that are correlated to the trading asset in the market. The idiosyncratic value component is independent to the common value component, but it is *correlated* to other traders' idiosyncratic value components.

The decomposition of asset value $\tilde{\theta}_i$ is formalized as follows:

$$\tilde{\theta}_i = \omega_{i,cv}\theta + \omega_{i,iv}\delta_i, \quad \forall\, i \in I, \tag{2}$$

where θ is the common value component, and δ_i is the idiosyncratic value component. The common and idiosyncratic value components are independent and drawn from normal distribution, $\theta \sim \mathcal{N}(E[\theta], \sigma_\theta^2)$ and $(\delta_i)_i \sim \mathcal{N}(0, \sigma_\theta^2\Sigma_\delta)$. The interdependence of idiosyncratic values

[7] The flexibility of this model can allow traders to have ex-ante different trading needs by introducing the heterogeneous mean and variance. The heterogeneity in ex-ante trading needs strengthens the benefits of trading in over-the-counter markets for some traders.

$(\delta_i)_i$ is captured by the correlation matrix,

$$\Sigma_\delta = \big(Corr(\delta_i, \delta_j)\big)_{i,j} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1I} \\ \rho_{12} & 1 & \cdots & \rho_{2I} \\ \vdots & & \ddots & \vdots \\ \rho_{1I} & \rho_{2I} & \cdots & 1 \end{bmatrix}.$$

The correlations ρ_{ij} are *heterogeneous across pairs of traders* (i, j), which is the key heterogeneity in this paper. The model imposes an assumption that the sum of correlations in a row $\sum_{j \neq i} \rho_{ij}$ is normalized to zero for all i, without loss of generality. This is in order to prevent the heterogeneity from being additional source of common value and thus allows a clear separation of information aggregation. Lastly, $(\omega_{i,cv}, \omega_{i,iv}) \in [0, 1]^2$ are weights of his valuation on the common and idiosyncratic value components, respectively, such that $\omega_{i,cv}^2 + \omega_{i,iv}^2 = 1$.

The model incorporates a wide range of the heterogeneous interdependence in asset values: the common value $\tilde{\theta}_i = \theta$ for all traders i (Kyle (1989)), fundamental value in which the correlations are homogeneous $\tilde{\rho}_{ij} = \tilde{\rho}$ for all (i, j) (Vives (2011)), independent private value with $\tilde{\rho}_{ij} = 0$ for all (i, j), and equicommonal models in which $(\omega_{i,cv}, \omega_{i,iv})$ are homogeneous for all i (Rostek and Weretka (2012)). This paper and Rostek and Yoon (2018) relax the symmetry assumptions and incorporate arbitrary interdependence of traders' asset valuations.

Example 1 shows how the common and idiosyncratic value components determine Σ in a simplest and intuitive setting.

Example 1 (Symmetric Interdependence in Asset Values) There are two groups of traders – *buyers and sellers* – with equal group sizes.[8] Each trader has individual asset value that is decomposed into two independent random variables:

$$\tilde{\theta}_i = \omega_{cv}\theta + \omega_{iv}\delta_i = \begin{cases} \omega_{cv}\theta + \omega_{iv}\delta & \text{if } i \text{ is a buyer,} \\ \omega_{cv}\theta - \omega_{iv}\delta & \text{if } i \text{ is a seller.} \end{cases}$$

Here, $\omega_{cv}^2 + \omega_{iv}^2 = 1$. Then, the correlation matrix of idiosyncratic values $(\delta_i)_i$ is

$$\Sigma_\delta = \left[\begin{array}{c|c} \mathbf{1} & \mathbf{-1} \\ \hline \mathbf{-1} & \mathbf{1} \end{array}\right],$$

where each block represents $(\frac{I}{2} \times \frac{I}{2})$ matrix. From the distributions of common and idiosyncratic

[8] Buyers and sellers are not explicitly determined in this model. Because two groups in this example have negatively correlated idiosyncratic asset values, it is without loss of generality to call two groups buyers and sellers.

components, the correlation matrix of *total asset values* $(\tilde{\theta}_i)_i$ is

$$\Sigma_\theta = \frac{1}{\omega_{cv}^2 + \omega_{iv}^2} \left[\begin{array}{c|c} \omega_{cv}^2 + \omega_{iv}^2 & \omega_{cv}^2 - \omega_{iv}^2 \\ \hline \omega_{cv}^2 - \omega_{iv}^2 & \omega_{cv}^2 + \omega_{iv}^2 \end{array} \right] = \left[\begin{array}{c|c} \mathbf{1} & \omega_{cv}^2 - \omega_{iv}^2 \\ \hline \omega_{cv}^2 - \omega_{iv}^2 & \mathbf{1} \end{array} \right]. \quad (3)$$

The interdependence of traders asset values is *symmetric* in this example, in the sense that the set of correlations in each row is the same, $\{\rho_{ik}\}_{k\neq i} = \{\rho_{jk}\}_{k\neq j}$ for any $i \neq j$. However, the correlations $(\rho_{ij})_{i,j}$ are still heterogeneous across pairs of traders (i, j). Section 4 will consider this example for further analysis and show the effects of the heterogeneous correlations across pairs and asymmetric interdependence across traders on endogenous market structures. □

INFORMATION: Each strategic trader receives a private signal on his own asset value: $s_i = \tilde{\theta}_i + \varepsilon_i$ with an independent noise $\varepsilon_i \sim \mathcal{N}(0, \sigma_{i,\varepsilon}^2)$. The information precision $\phi_i := \frac{1}{\sigma_{i,\varepsilon}^2/\sigma_\theta^2}$ can differ across traders. The private signal s_i is realized at the beginning of trading period $t = 1$ and privately observed by trader i. The realizations of other traders' signals $(s_j)_{j\neq i}$ and prices in all markets, including the market where trader i participates in, cannot be observed by trader i until all trades are done.

Heterogeneity across traders, which is the critical component in the model, is in both the correlation structure Σ_θ and information precision $(\phi_i)_i$. Throughout this paper, this pair of asset and trader characteristics is called a *type*. Each trader's type represents his identity, such as buyers or sellers in Example 1.[9] It is worth remarking that the type defined in this paper does not represent the realization of private signal s_i, and thus, this definition of type is different from the conventional definition in incomplete information games. Traders' types and prior distribution of asset values and signals are common knowledge.[10]

2.1 Centralized Market (CM) Mechanism

The centralized market is a large market where many buyers and sellers trade at a single price. This paper designs the centralized market as a canonical uniform-price double auction with divisible goods. A strategic trader i who enters the centralized exchange submits his net demand schedule $q_i(p) : \mathbb{R} \to \mathbb{R}$ (i.e., a combination of limit orders) as a continuous function of

[9] The types of traders are more general than their identity. Traders are indifferent between the identity of other traders $j \neq i$ who are with the same type. For instance, in Example 1, each trader's counterparty choice in the over-the-counter market depends on other traders' types, i.e., buyers or sellers, rather than their identity, i.e., a particular trader $j \neq i$ in one group.

[10] In non-competitive markets, trades are dominated by large institutions, who have a better knowledge on the market and other agents; often the number of relevant agent is indeed small, as Oehmke and Zawadowski (2017) and Avellaneda and Cont (2010) report for CDS dealer markets. Perfect information on the other traders' types is useful for tractability but not critical to the results. With some uncertainty, the results carry over.

price.

$$\max_{q_i(\cdot):\mathbb{R}\to\mathbb{R}} E[u_i(q_i,p)|s_i] = \max_{\{q_i(p);\forall p\in\mathbb{R}\}} \Big\{E[\tilde{\theta}_i|s_i,p]q_i - pq_i - \frac{\mu}{2}q_i^2\Big\}. \tag{4}$$

There are $L \geq 2$ traders, called *liquidity traders* in the centralized market who are not given a choice to enter the over-the-counter market. The presence of two or more liquidity traders in the centralized market ensures that a trader $i \in I$ who is currently in the over-the-counter market can consider a non-cooperative deviation to the centralized market. Each liquidity trader $l \in L$ is strategic and submits a demand schedule $q_l(p)$ that maximizes the utility (4). The precision of private information for $l \in L$ is equal to the least informed traders: $\sigma^2_{lq,\varepsilon} = \max_{i\in I}\sigma^2_{i,\varepsilon}$, and he values the risky asset by $\tilde{\theta}_l = \omega_{lq,cv}\theta + \omega_{lq,iv}\delta_l$ with an independent idiosyncratic value component: $Corr(\delta_l,\delta_j) = 0$ for any $j \neq l \in I \cup L$.

After demands of all traders $i \in I$ and $l \in L$ are submitted, the centralized market is cleared at a price of which the total demand of traders is equal to zero; p^* such that $\sum_{i\in I} q_i(p^*) + \sum_{l\in L} q_l(p^*) = 0$. The equilibrium allocation is determined by the demand schedule traders submitted, $q_i^* = q_i(p^*)$ for any $i \in I$ and $q_l^* = q_l(p^*)$ for any $l \in L$.

2.2 Over-the-Counter Market (OTC) Mechanism

An over-the-counter market is an off-exchange trading venue in which bilateral trades occur between large institutions. The over-the-counter matching is *pairwise stable*, in the sense that no pair of traders have a positive incentive to leave their current matches and create a new match. Each trader has an individual ranking on other traders based on their types: information precision and correlations. Traders' ranking determines a pairwise stable matching by the algorithm of Irving (1985). If two traders are matched, they trade and leave the market. The over-the-counter market ends when all traders participate in exactly one bilateral trade or when only a single trader is left.

Once the matching occurs, each bilateral trade is operated by the uniform price double auction. Two traders simultaneously submit their demand schedules $q_i(p)$ as functions of price p by solving the optimization problem (4). The equilibrium price p^*_{ij} is determined by the market clearing condition: $q_i(p^*_{ij}) + q_j(p^*_{ij}) = 0$. The price p^*_{ij} is pair-specific in the over-the-counter market. If an equilibrium price does not exist, then there is no trade and the over-the-counter market ends without any further trade. The utility of traders in such a case are set to be the autarky utility $u_i(q_i = 0) = 0$.

2.3 Market Choice and Trading

Based on the trading mechanisms in two markets described in Section 2.1 and 2.2, the strategies of traders and equilibrium are defined in this section.

Strategies: At $t = 0$, each trader $i \in I$ chooses a market where he enters, $m_i \in \{OTC, CM\}$ and a type of counterparty τ_i upon his entering to the over-the-counter exchange $m_i = OTC$. When the market choice of a trader is $m_i = CM$, his counterparty choice is notated by $\tau_i = \emptyset$ for the convenience. At $t = 1$, the trader chooses his demand function $q_i(\cdot : m_i, \tau_i)$ in market (m_i, τ_i). Therefore, the strategy profile of trader i is $\{(m_i, \tau_i), q_i(\cdot : m_i, \tau_i)\}$. A liquidity trader $l \in L$ in the centralized market has a strategy $\{(CM, \emptyset), q_l(\cdot; CM, \emptyset)\}$ since he cannot enter the over-the-counter market.

Equilibrium: Definition 1 provides three conditions for equilibrium: (i) Bayesian Nash equilibrium in the double auction in each market, (ii) no incentive to deviate from the over-the-counter market to the centralized market, and (iii) pairwise stable market structure including a pairwise deviation from the centralized to over-the-counter market and from one over-the-counter matching to another. For each trader $i \in I$, $E[u_i(m_i, \tau_i)]$ denotes the expected utility with (m_i, τ_i) for given equilibrium distribution of traders in both markets.

Definition 1 (Equilibrium) *An equilibrium is defined by the profile of schedules of all traders* $\{(m_i, \tau_i), q_i(\cdot : m_i, \tau_i)\}_{i \in I}$ *such that*

(i) traders' optimal bid schedules $\{q_i(\cdot : m_i, \tau_i)\}_i$ *solving the optimization problem (4) characterize a Bayesian Nash equilibrium in each market;*

(ii) no trader in the over-the-counter market has a strictly positive incentive to deviate to the centralized market: i.e., if $(m_i^*, \tau_i^*) = (OTC, \tau_i^*)$ *for trader* i*, then*

$$E[u_i(m_i^*, \tau_i^*)] \geq E[u_i(CM, \emptyset)], \quad \forall i \in I; \quad \text{and}$$

(iii) the market structure is pairwise stable: i.e., there exists no pair of traders (i, j) *such that both traders* i *and* j *strictly benefit from deviating from their current market or match and creating a new over-the-counter match between them: i.e.,*

$$E[u_i(m_i^*, \tau_i^*)] \geq E[u_i(OTC, j)] \ \text{ or } \ E[u_j(m_j^*, \tau_j^*)] \geq E[u_j(OTC, i)], \quad \forall i \neq j \in I.$$

The inequalities in Definition 1 (iii) include the case where either trader i or j (or both) choose the centralized market, $m_i^* = CM$ or $m_j^* = CM$, in equilibrium. This condition ensures that the over-the-counter matching is immune to an entry of a trader from the centralized market as well as counterparty choices within the over-the-counter market. Furthermore, with this pairwise deviation from the centralized market to the over-the-counter market, a market structure where all traders choose the centralized market, $m_i^* = CM$ for all i, is not a trivial equilibrium.

The following sections characterize equilibrium defined in Definition 1: Equilibrium bid strategies and outcomes in Bayesian Nash equilibrium for a given market - part (i) - is characterized in Section 3. The characterization allows us to develop comparative statics on traders' expected utilities over the market, asset, or traders characteristics. Section 4 and 5 show endogenous market structures that are formed by traders' market and counterparty choice - part (ii) and (iii) - and analyze influences of the characteristics in traders' market choices and thus in endogenous market structures.

3 Equilibrium in Double Auctions

This section shows traders' bidding strategies in a given market. Suppose that there are N traders in a market. This includes $N = 2$ in an over-the-counter bilateral trade. In a centralized market, $N \geq 3$, including $N = I + L$ when all traders participate in the centralized market. The number of traders N^* in equilibrium will be endogenously determined by traders' market choices in Section 4, while this section takes N as exogenously given. A correlation structure of asset values is $\Sigma_\theta \in \mathbb{R}^{N\times N}$ and information precision is $\{\phi_i\}_{i\in N}$.

Each trader i chooses his demand schedule $q_i(p) : \mathbb{R} \to \mathbb{R}$ to maximize his expected utility $E[u_i|s_i]$ as in equation (1) for given private information s_i. The point-wise optimization for each realization of price $p \in \mathbb{R}$ characterizes the first order condition as follows:

$$E[\tilde{\theta}_i|s_i, p] - \mu q_i = p + \lambda_i q_i, \qquad \forall\ p \in \mathbb{R}, \tag{5}$$

where $\lambda_i := \frac{dp}{dq^i}$ is the *price impact* that represents the change in price when trader i increases his demand by one unit. A larger price impact implies that each unit of a trader's demand leads to a further increase in price so that the trader's demand is reduced more by higher price impact. This demand reduction due to the price impact represents *market illiquidity* that is endogenously determined by the traders' strategies. A competitive market with infinitely many traders is perfectly liquid, and the price impact is zero.

In the first-order condition (5), trader i takes an expectation of his asset value $\tilde{\theta}_i$ conditioning on price to-be-realized p as well as on his private information s_i. The trader chooses a bid at each potential realization of price, and thus, his behavior incorporates the information revealed by the price as if he observed the price. Therefore, even in the static model, traders make inference about their asset values by the schedule bidding. The benefit or cost of learning information from the market price is valued in terms of equilibrium utility change.

Endogenizing price impact (illiquidity) and inference (learning) in this model allows us to consider a market choice without exogenous frictions.[11] Proposition 1 shows the equilibrium

[11] The model does not contain a search friction or transaction cost, and thus, the price impact is the only

conditions, including the price impact and inference, for a given market with N traders whose asset values are correlated by Σ_θ and whose information precisions are $\{\phi_i\}_i$. The characterization consists of three conditions: (i) a trader' strategy optimizes his utility for a given price impact and inference on asset values; (ii) the price impact is correct; and (iii) the inference is correct.[12] The model imposes the positive price impacts for all traders, equivalently, the submitted bidding function should be strictly decreasing in price.

Proposition 1 (Equilibrium Representation in a Market) *In a market, a profile of demand schedules $\{q_i(\cdot)\}_i$ is a linear Bayesian Nash equilibrium (hereafter, equilibrium) if*

(i) a demand schedule $q_i(\cdot : \lambda_i)$ maximizing trader i's utility is

$$q_i = \frac{E[\tilde{\theta}_i|s_i, p] - p}{\mu + \lambda_i} = \frac{c_{\theta,i}E[\tilde{\theta}] + c_{s,i}s_i - (1 - c_{p,i})p}{\mu + \lambda_i}, \tag{6}$$

where $E[\tilde{\theta}_i|s_i, p] = c_{\theta,i}E[\theta_i] + c_{s,i}s_i + c_{p,i}p$,

(ii) price impacts satisfy the consistency condition

$$\lambda_i = -\Big(\sum_{j\neq i} \frac{\partial q_j(\cdot)}{\partial p}\Big)^{-1} = \Big(\sum_{j\neq i} \frac{1 - c_{p,j}}{\mu + \lambda_j}\Big)^{-1} \geq 0, \quad \forall i, \tag{7}$$

(iii) inference coefficients $\{c_{\theta,i}, c_{s,i}, c_{p,i}\}$ in $E[\tilde{\theta}_i|s_i, p]$ are determined by the Projection Theorem, with equilibrium price distribution following

$$p = (\sum_i \frac{1 - c_{p,i}}{\mu + \lambda_i})^{-1} \sum_i \frac{c_{\theta,i}E[\tilde{\theta}] + c_{s,i}s_i}{\mu + \lambda_i}. \tag{8}$$

In equilibrium, a trader' indirect utility is written as a function of his price impact λ_i and expected asset value $E[\tilde{\theta}_i|s_i, p]$ conditioning on (s_i, p). The ex-ante expected utility of trader i is

$$E[u_i] = \frac{1}{2\mu} \underbrace{\frac{\mu(\mu + 2\lambda_i)}{(\mu + \lambda_i)^2}}_{\text{liquidity effect}} \underbrace{Var\big(E[\tilde{\theta}_i|s_i, p] - p\big)}_{\text{learning effect}}, \quad \forall i. \tag{9}$$

Equation (9) decomposes the trader i's utility $E[u_i]$ into two parts: the liquidity effect and the learning effect. The *liquidity* effect $\frac{\mu(\mu+2\lambda_i)}{(\mu+\lambda_i)^2} = 1 - (\frac{\lambda_i}{\mu+\lambda_i})^2$ is defined by the benefit

source of illiquidity in both centralized and over-the-counter markets. Incorporating a friction would not change the qualitative results.

[12] A linear Bayesian Nash equilibrium uniquely exists under conditions. In general, there is no closed-form characterization for equilibrium, outside of models with symmetric correlation and symmetric precision.

of improving liquidity in the utility term. Recall that trader i's demand is reduced from the demand in the perfectly liquid (competitive) market by the fraction $\frac{\lambda_i}{\mu+\lambda_i}$:

$$q_i = \Big(1 - \frac{\lambda_i}{\mu + \lambda_i}\Big)\frac{1}{\mu}\big(E[\tilde{\theta}_i|s_i, p] - p\big) = \Big(1 - \frac{\lambda_i}{\mu + \lambda_i}\Big)q_i^{**}(p),$$

where $q_i^{**}(p)$ is the demand of trader i in a competitive market for a given price p. The demand reduction lowers the equilibrium utility following this fraction, and thus, the liquidity effect increases as the price impact λ_i decreases. The *learning* effect is captured by $Var(E[\tilde{\theta}_i|s_i, p]-p)$ in the expected utility (9), i.e., the variance of the marginal revenue of buying a unit of the risky asset, $E[\tilde{\theta}_i|s_i, p] - p$. Equilibrium price aggregates all market participants' private information on asset values. Traders learn the aggregated information from conditioning on price. Such information on the marginal revenue $E[\tilde{\theta}_i|s_i, p] - p$ influences the trader's expected utility.

3.1 Learning and Price Impact

This section develops a comparative statics for the liquidity effect and learning effect in equilibrium utilities with respect to three characteristics: market size (market characteristic), interdependence of traders' asset values (asset), and precision of private information (traders). For illustrative purposes, this section focuses on symmetric traders. Asymmetric traders with asymmetric correlations and information precision will be considered in Section 5.[13]

Definition 2 (Symmetric Traders) *Traders are symmetric, if*

(i) the weights $(\omega_{i,cv}, \omega_{i,iv})$ on two components in asset values are the same for all i;

(ii) the profile of correlations $\{\rho_{ij}\}_{j\neq i}$ is the same for all i; and

(iii) the information precision ϕ_i is the same for all i.

With the symmetric traders defined in Definition 2, traders submit symmetric strategies in each market, but $Corr(\delta_i, \delta_j) = \rho_{ij}$ differ for each pair (i, j). Example 2 and Proposition 2 show how the three key characteristics affect the ex-ante utility of each trader in symmetric markets.

Example 2 (Symmetric Interdependence and Precision) Consider a market with N traders. All traders have a symmetric information precision $\phi_i = \sigma_\theta^2/\sigma_\varepsilon^2 = \phi$ and a symmetric average

[13] In asymmetric markets, in the sense that the profiles of correlations $\{\rho_{ij}\}_{j\neq i}$ are heterogeneous across traders and/or that information precision ϕ_i are heterogeneous, traders' optimal trading strategies are asymmetric. The effects of characteristics on traders' behavior in this section and the comparative statics on endogenous market structures in Section 4 are also applied to asymmetric markets.

correlation to the residual market $\bar{\rho}_i := \frac{1}{N-1}\sum_{j\neq i}\tilde{\rho}_{ij} = \bar{\rho}$ for all i. Each trader's optimal schedule and equilibrium price are

$$q_i \;=\; \frac{c_\theta E[\tilde{\theta}] + c_s s_i - (1-c_p)p}{\mu+\lambda}, \quad \forall\ i; \qquad p \;=\; \frac{1}{1-c_p}(c_\theta E[\tilde{\theta}] + c_s \frac{1}{N}\sum_i s_i).$$

Here, trader i gets the ex-ante utility $E[u_i]$ as in equation (9). With the closed form of inference parameters and price impact in the proof of Proposition 2 in Appendix A, the *liquidity effect* on utility is

$$\frac{\mu(\mu+2\lambda_i)}{(\mu+\lambda_i)^2} = 1 - \Big(\frac{1}{(N-1)(1-c_{p,i})}\Big)^2, \quad \text{where} \quad c_{p,i} = \frac{N\bar{\rho}\phi^{-1}}{(1+(N-1)\bar{\rho})(1-\bar{\rho}+\phi^{-1})}.$$

The liquidity term increases as N increases or $\bar{\rho}$ decreases. When the information precision ϕ increases, the liquidity increases if $\bar{\rho} > 0$, and decreases if $\bar{\rho} < 0$. The *effect of learning* from the price on utility is measured by

$$Var(E[\tilde{\theta}_i|s_i,p]-p) = c_{s,i}^2 Var(s_i - \bar{s}) = \frac{N-1}{N}\frac{(1-\bar{\rho})^2}{1+\phi^{-1}-\bar{\rho}}\sigma_\theta^2.$$

which is increasing in information precision ϕ and in the number of traders N; and decreasing with respect to $\bar{\rho}$. □

Proposition 2 characterizes the effects of each characteristic - market size, correlations, and information precision - on traders' expected utilities through learning and liquidity, when the other characteristics are fixed.

Proposition 2 (Liquidity and Learning Effects) *In a sufficiently symmetric market, the equilibrium utility $E[u_i]$ of a trader i increases as*

(i) the number of traders in market N increases; or

(ii) asset value is more negatively correlated to price, i.e., $Corr(\tilde{\theta}_i, p)$ is more negative.

(iii) information precision $\phi_{-i}^* \in [0,\infty)$ of other traders maximizes trader i's utility.

In a sufficiently symmetric market, with more traders participating in the market (part (i)), price reveals more accurate information and price impact is small in large markets. When the market size and information precision are fixed, Proposition 2 (ii) suggests that if the equilibrium price exhibits greater negative correlations with trader i's asset valuation, then it provides new information that is not captured in trader i's private information. The correlation structure also affects the liquidity through the endogenous price impacts λ_i. With more negative correlations, other traders $j \neq i$ would rely more on the price for his inference (i.e., $c_{p,j}$ is more negative). These more elastic demands reduce trader i's price impact. Hence, his equilibrium utility increases due to *both* learning and liquidity as the correlation between his asset values and price is more negative.[14]

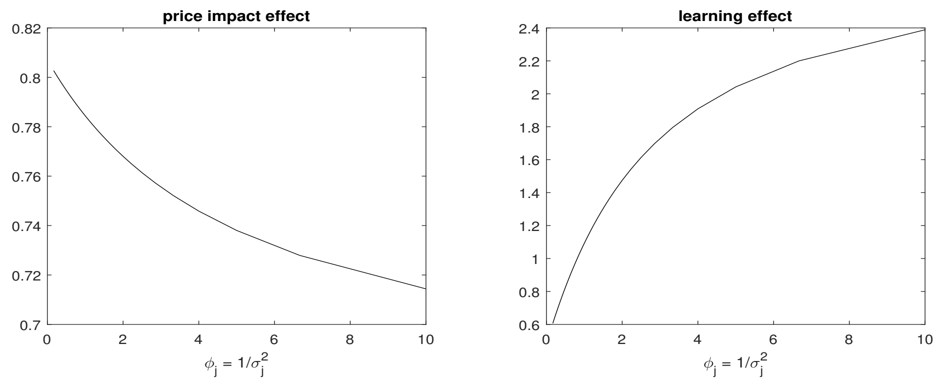

Figure 2: Liquidity and learning effects with respect to other traders' information precision ϕ_{-i}. A trader i has a information precision $\phi_i = /\sigma_i^2 = 1$. Two effects are monotone over the other traders' information precision $\phi_{-i} = 1/\sigma_{-i}^2 := avg(1/\sigma_j^2)_{j \neq i}$.

Information precision has an ambiguous effect on traders' expected utilities (part (iii)). The learning effect increases when the trader's own information precision is lower or when the (weighted) average of other traders' information precision is higher. At the same time, the price impact increases, and thus, the liquidity decreases. It creates a *trade-off* between learning and liquidity when the precision of information from the price changes. This trade-off is shown in Figure 2. If his precision is sufficiently low, the learning effect dominates the liquidity effect, so that his utility is monotonically increasing in others' precision.

4 Endogenous Market Structure

This section characterizes endogenous market structures by using the comparative statics developed in the previous section. First, Section 4.1 considers individual market choices for markets and counterparties for a given residual market. In Section 4.2, traders' individual choice forms a *market structure*, which is defined by a distribution of traders' types in centralized and over-the-counter markets and also an over-the-counter matching. The number of traders who choose the centralized market will be denoted by N_{cm}. For convenience, a market structure where traders I who have a choice of markets and counterparties choose the over-the-counter is called

[14] The joint effects of the number of traders and aggregate correlation on learning and price impact have been studied in the literature of centralized market trading. Rostek and Weretka (2012) show that the benefits of large markets in learning and liquidity are not necessarily true, when the correlation $corr(\tilde{\theta}_i, p)$ between price and asset values depends on the market size N.

a structure with only the over-the-counter market, even though liquidity traders L still trade in the centralized market.

4.1 Individual Choice of Market

Expected equilibrium utility of a trader is larger when more traders participate in the market or as the correlation between his asset value and the market price is more negative (Proposition 2). With the presence of $L \geq 2$ liquidity traders in the centralized market, the market size is larger in the centralized market, which encourages traders to enter into the centralized market. The effect of correlation $Corr(\tilde{\theta}_i, p)$ between asset value and price supports traders choosing the over-the-counter market by improving both learning and liquidity. In the over-the-counter market, a trader can *target* a counterparty who has the most negative correlation, while the correlation with the centralized market price is determined by the average correlation over all participants.

	OTC		CM
number of traders	2	$<$	$N_{cm} + L$
$Corr(\tilde{\theta}_i, p)$	$\min_{j \neq i} Corr(\tilde{\theta}_i, \tilde{\theta}_j)$	$<$	$\frac{1}{I-1} \sum_{j \neq i} Corr(\tilde{\theta}_i, \tilde{\theta}_j)$

Table 1: Traders' market choices and trade-off between market sizes and correlations

The effects of market size and correlation create a trade-off in the traders' choice of market. Theorem 1 examines when the correlation effect would dominate and provides a sufficient and necessary condition such that the trader prefers to trade in the over-the-counter market.

Theorem 1 (When A Trader Chooses OTC Markets) *Suppose that the correlations between traders' asset valuations are heterogeneous across pairs of traders, i.e., there exists $j \neq k$ who are not i such that $Corr(\tilde{\theta}_i, \tilde{\theta}_j) \neq Corr(\tilde{\theta}_i, \tilde{\theta}_k)$. The over-the-counter market opens in equilibrium by some traders entering into the market, if and only if*

$$\omega_{iv}^2 > \hat{\kappa}(\{\phi_i\}_i, \Sigma, N_{cm} + L) \quad \text{and} \quad \phi_i = \sigma_\theta^2 / \sigma_{i,\varepsilon}^2 < \hat{\phi}_i(\omega_{cv}, \omega_{iv}, \Sigma, N_{cm} + L) \tag{10}$$

for some bounds $\hat{\kappa} < \infty$ and $\hat{\phi}_i > 0$.

The first inequality in Theorem 1 implies that traders choose the over-the-counter market when their asset values are sufficiently heterogeneous by depending more on idiosyncratic value component, i.e., ω_{iv}^2 is sufficiently large. Under this inequality condition on ω_{iv}^2, the targeted bilateral trades in the over-the-counter market improve both learning and liquidity. The dominance of correlation effect is strengthened by low information precision, as the second inequality condition in Theorem 1 shows. This is because the trader's inference depends more on the price, so the difference between correlations is emphasized more in the trade-off between the effects of market size and correlations. The low information precision, *a characteristic of traders*, creates a joint condition with the sufficiently heterogeneous asset valuation, *a characteristic of assets*, for the over-the-counter market to be chosen by traders in endogenous market structure.

The predictions in Theorem 1 are consistent with financial markets. Many over-the-counter products, such as forward contracts or corporate bonds, are traded for hedging. The portfolio they need to diversify is idiosyncratic so that asset values are valued by idiosyncratic value components. On the other hand, stocks or options are often traded by speculators. The speculators value these assets by the common values, which are the future prices in the market. It should be noted that the condition for over-the-counter markets to exist is a joint condition on asset valuation and information precision. Bonds with low credit rating, which traders would not have precise information due to its volatile value, are often traded in over-the-counter markets. Alternatively, treasuries and high ranked bonds are also traded in centralized futures markets, even though they are valued idiosyncratically.

Literature has shown that the over-the-counter markets can be beneficial for (i) providing an additional trading opportunity to traders who could not entirely clear their trading needs in centralized markets (e.g., Bessembinder and Venkataraman (2014), Ready (2014), Degryse, Jong, and Kervel (2015)); and for (ii) allowing traders to search for better prices than the centralized market price (e.g., Zhu (2013), Vayanos and Wang (2007), Vayanos and Weill (2008), Lee and Wang (2019)). Despite these benefits of trading in over-the-counter markets being present in this model, Corollary 1 and 2 show that improved learning and lower price impacts also motivate traders to choose the over-the-counter market.

Corollary 1 (OTC with a Competitive CM) *Suppose that the centralized market is competitive, i.e., the number of liquidity traders is large, $L \to \infty$. There exists a set of correlation Σ and precision ϕ, satisfying the inequalities (10), such that some traders choose the over-the-counter market over the competitive centralized market.*

Traders may prefer to trade in the over-the-counter market even when the centralized market is perfectly liquid (Corollary 1). This is because, traders' improved learning about heterogeneous asset values can favor the choice of an over-the-counter market, even when entering to the centralized market is a dominant strategy in the aspect of liquidity. Under a condition on primitives, in equilibrium all I traders choose to trade over-the-counter and only liquidity traders participate in the competitive centralized market. See Example 3 in Section 4.2.

Second, Corollary 2 shows that traders may choose the over-the-counter market even when the prices in two markets are the same.

Corollary 2 (No Price Difference in OTC and CM) *There exists a set of primitives such that the expected prices are the same in two markets and traders choose the over-the-counter market.*

Even when traders cannot find any better price in the over-the-counter markets, by which the literature motivates the over-the-counter markets so far, the over-the-counter market can open due to the endogenized learning and liquidity within trades. In particular, for certain traders and asset characteristics, trading in the over-the-counter market offers a lower price impact and thus a larger trading amount at the same price relative to trading in the centralized market. The results of a larger trading volume in the over-the-counter market in Corollary 2 is complementary to the literature on the over-the-counter trading mechanism for a unit demand for the risky asset.[15]

4.2 Endogenous Market Structure in Equilibrium

For some intuition on endogenous market structures, Example 3 examines considers a perfectly competitive centralized market with $L \to \infty$. When the centralized exchange is competitive, the liquidity incentive strongly derives traders to avoid higher price impacts in over-the-counter bilateral trades. Hence, participation to the over-the-counter market occurs only when the benefit of learning from the market is high enough to dominate the loss from illiquidity. Example 3 examines when the benefit of learning is sufficiently stronger in the over-the-counter market.

Example 3 (Competitive Symmetric Centralized Market) Example 2 characterizes the expected utility in a closed-form when there are N symmetric traders. The utility in the competitive centralized trading is derived by taking $N = N_{cm} + L \to \infty$, while the one from a bilateral trades in the over-the-counter market is by setting $N = 2$. From the expected utilities in two markets, trader i enters the over-the-counter exchange if and only if

$$E[u_i^{cm}] < E[u_i^{otc}] \quad \Leftrightarrow \quad \frac{(1-\bar{\rho}_{cm})^2}{(1+\phi^{-1}-\bar{\rho}_{cm})} < \frac{-2\phi^{-1}\rho_{otc}(1+\phi^{-1}-\rho_{otc}^2)}{(1+\phi^{-1}+\rho_{otc})^2(1+\phi^{-1}-\rho_{otc})}\mathbf{1}_{\{\rho_{otc}<0\}}.$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Here, $\bar{\rho}_{cm} = \omega_{cv}^2$ is the average correlation in the centralized market and $\rho_{otc} = \omega_{cv}^2 + \omega_{iv}^2\rho_{ij}$ is the correlation between two traders (i,j) who are matched in the over-the-counter market.

Let us consider the following canonical assumptions on traders' asset valuation:

- Fundamental Value (Vives (2011)): $\omega_{cv} = \sqrt{1-\omega_{iv}^2}, \omega_{iv} \in (0,1)$, and $\rho_{ij} = 0$ for all $i \neq j$. The fundamental value model includes *common values* by taking $\omega_{iv} = 0$, i.e., $\tilde{\theta}_i = \theta$ for all i, and *independent private values* by taking $\omega_{iv} = 1$, i.e., $\tilde{\theta}_i = \delta_i$ for all i.

[15] Babus and Palatore (2018) and Babus and Hachem (2022) consider the over-the-counter trades for divisible assets when the correlations of traders' asset values are symmetric, i.e., $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \rho$ for all (i,j).

The over-the-counter trade provides no gain-to-trade since $\rho_{otc} = \omega_{cv}^2 > 0$, while traders get a strictly positive expected utility $E[u_i^{cm}] = \frac{\omega_{iv}^4}{\omega_{iv}^2+\phi^{-1}} > 0$. Only the centralized market opens in equilibrium. Fundamental value model does not incorporate heterogeneous correlations across pairs of traders, in the sense that $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \omega_{cv}^2$ for any pair (i, j). This concludes that the heterogeneous correlation is necessary for a trader to choose the over-the-counter market.

- Two-Sided Market with Buyers and Sellers (Example 1): $\omega_{cv} = \sqrt{1 - \omega_{iv}^2}, \omega_{iv} \in (0, 1)$, and $\rho_{ij} = \pm 1$. Traders choose the over-the-counter market if and only if

$$\frac{\omega_{iv}^4}{\omega_{iv}^2 + \phi^{-1}} < \frac{-2\phi^{-1}(1 - 2\omega_{iv}^2)(\phi^{-1} + 4\omega_{iv}^2 - 4\omega_{iv}^4)}{(2 + \phi^{-1} - 2\omega_{iv}^2)^2(\phi^{-1} + 2\omega_{iv}^2)} \mathbf{1}_{\{\omega_{cv}^2 - \omega_{iv}^2 < 0\}}. \tag{11}$$

The inequality holds if and only if

$$\underbrace{\omega_{iv}^2 \ > \ 2 - \sqrt{2}}_{\text{iv is sufficiently large}} \quad and \quad \underbrace{\phi \ < \ \hat{\phi}(\omega_{iv})}_{\text{precision is sufficiently low}} \quad , \tag{12}$$

where $\hat{\phi}(\omega_{iv})$ is the unique positive solution that equates (11):

$$(\omega_{iv}^4 - 4\omega_{iv}^2 + 2) + 2\omega_{iv}^2(1 - \omega_{iv}^2)(5 - 7\omega_{iv}^2)\phi + 12\omega_{iv}^4(1 - \omega_{iv}^2)^2\phi^2 + 8\omega_{iv}^6(1 - \omega_{iv}^2)^2\phi^3 = 0.$$

The over-the-counter market opens if the idiosyncratic value component in asset values is sufficiently large and the information precision is sufficiently low.

The inequality (11) is independent of other traders' market choices. The market size of the centralized market is independent of traders' market choices and fixed by $N + L \to \infty$. Furthermore, the correlation between asset values and centralized market price is symmetric for all i, i.e., $Corr(\tilde{\theta}_i, p) = Corr(\tilde{\theta}_i, \bar{s}_L)$ when $\bar{s}_L = \lim_{L\to\infty} \frac{1}{L} \sum_{l \in L} s_l$. Thus, the utility benefit of trading in the centralized or over-the-counter market is symmetric to all traders and independent of other traders' market choices. In equilibrium, all traders $i \in I$ choose the centralized market ($N_{cm}^* = I$) or the over-the-counter market ($N_{cm}^* = 0$), and the equilibrium market structure consists of only one market. □

When the centralized market is imperfectly competitive ($L < \infty$), a trader's choice of market and counterparty depend on other traders' choices which affects the market size and correlation between asset value and price in the centralized market. The equilibrium market structure is determined as a fixed point. There are three types of endogenous market structure: (a) only the over-the-counter market opens (i.e., only liquidity traders trade in the centralized market), (b) only the centralized market opens, and (c) the over-the-counter and centralized market coexist.

The heterogeneity among traders is crucial for understanding the endogenous market structure. Theorem 2 show that two dimensions of trader heterogeneity are important. First, their trading needs differ. This is captured by traders' individual asset values are heterogeneously correlated. Second, not all traders are equally informed about the asset value, which is represented as heterogeneous precision in signals about values. Without such heterogeneity, the centralized and over-the-counter markets cannot coexist.

Theorem 2 (Coexistence of OTC and CM) *Equilibrium market structure consists of only one market, either the over-the-counter market ($N^*_{cm} = 0$) or the competitive centralized market ($N^*_{cm} = I$), if one of the following conditions holds:*

(i) the centralized market is competitive, i.e., $L \to \infty$; or

(ii) traders' asset values are homogeneously correlated, i.e., $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \tilde{\rho}$ for all (i, j), and their information precision is the same, i.e., $\phi_i = \phi$ for all i.

When the correlations of asset values and the information precision are symmetric across traders (Theorem 2 (ii)), the price informativeness is the same in the centralized and over-the-counter market: $\bar{\rho}_{cm} = \rho_{otc} = \tilde{\rho}$, independently of traders' choices of market and counterparty. Suppose that a trader gains from deviating from the over-the-counter to centralized market. His deviation increases the market size in the centralized market. With the equivalent learning effect, the incentive to deviate from the over-the-counter market to the centralized market for the next trader is even stronger. Recursively, all traders end up choosing the centralized market. The deviation from the centralized market to the over-the-counter market follows the similar argument, by recursively increasing incentives to deviate due to decreasing market size in the centralized market. It concludes that the endogenous distribution of traders in two trading venues has a *corner* solution when traders' asset values are homogeneously correlated. Equivalently, the over-the-counter and centralized markets can coexist in equilibrium only if traders are asymmetric: in the sense that either (a) the correlations $\{\rho_{ij}\}_{j \neq i}$ is heterogeneous among pairs (i, j) or (b) the information precision ϕ_i is asymmetric across traders.

The predictions in Example 3 can be generally extended to the model outside of Theorem 2, i.e., the model with the heterogeneity on correlations or information precision. Traders choose the over-the-counter market over the centralized market if the idiosyncratic value component in asset values is sufficiently large and the information precision is sufficiently small. Unlike in Theorem 2, however, traders' market choice and endogenous market structure create a fixed point problem in equilibrium. The incentives of traders in the choice of the market are functions of distribution of traders in two markets as well as of endogenized market structure. In particular, both the liquidity and learning effects depend on who chooses the centralized market, which is captured by $\bar{\rho}_{cm} = \frac{1}{N_{cm}+L-1}\sum_{j\neq i\in N_{cm}\cup L}\rho_{ij}$, and how many traders are in the centralized market, captured by $N_{cm}+L$.

Proposition 3 (Endogenous Market Structure) *Suppose that all players are symmetric and there is a finite number of liquidity providers $L<\infty$. When the idiosyncratic value component is sufficiently large, i.e., $\omega_{iv}^2>\kappa$, there exists ϕ_{max} and ϕ_{min} such that only over-the-counter market opens if $\phi<\phi^{min}$ and only the centralized market opens if $\phi>\phi^{max}$.*

When the information precision ϕ of traders are between the two bounds, i.e., $\phi_{min}<\phi<\phi_{max}$, the equilibrium market structure is determined by the interdependence of $\bar{\rho}_{cm}$ and $I_{cm}=N_{cm}+L$. Recall the sufficient and necessary condition (10) for a trader to choose the over-the-counter market. Suppose that the idiosyncratic component is sufficiently large: $\omega_{iv}^2>-\frac{N_{cm}+L-2}{2(N_{cm}+L-1)\min_{j\neq i}\rho_{ij}}$. The information precision should be small enough, i.e., $\sigma^2>\hat{\sigma}^2$ for *a marginal trader* to choose the over-the-counter market. The cutoff point $\hat{\sigma}^2$ depends on the number of traders in the centralized market and the aggregate correlation $\bar{\rho}_{cm}$, which are functions of other traders' market choices. In particular, $\frac{\partial\hat{\sigma}^2}{\partial N_{cm}}>0$ when $\bar{\rho}_{cm}$ is fixed, and $\frac{\partial\hat{\sigma}^2}{\partial\bar{\rho}_{cm}}>0$ when N_{cm} is fixed.

Example 1 - cont'd (Joint Effect of N_{cm} and $\bar{\rho}_{cm}$ in Choice of Markets) Suppose that two groups of symmetric traders - buyers and sellers - have the interdependent asset values with correlation matrix (3). When $\frac{I-N_{cm}}{2}$ pairs of a buyer and a seller enter the over-the-counter market, and N_{cm} traders enter the centralized market, the average correlation in the centralized market is characterized as a function of N_{cm}:

$$\bar{\rho}_{cm}(N_{cm})=\frac{1}{N_{cm}+L-1}\sum_{j\neq i\in N_{cm}\cup L}Corr(\tilde{\theta}_i,\tilde{\theta}_j)=\omega_{cv}^2-\omega_{iv}^2\frac{1}{N_{cm}+L-1}. \tag{13}$$

Hence, the average correlation in the centralized market decreases as the centralized market size increases. The equation (13) shows that the interdependence of average correlation $\bar{\rho}_{cm}$ and the market size N_{cm} based on other traders' market choices can create a trade-off in a trader i's individual decision of which market he participates in. When more traders enter the over-the-counter market, the precision cutoff $\hat{\phi}$ increases if the effect of the market size dominates, while it decreases if the effect of average correlation dominates. □

Figure 3 shows the trade-off between $\frac{\partial\hat{\sigma}^2}{\partial N_{cm}}$ and $\frac{\partial\hat{\sigma}^2}{\partial\bar{\rho}_{cm}}$ when $\hat{\sigma}^2=\hat{\phi}^{-1}$. In the left panel, the effect of market size $\frac{\partial\hat{\sigma}^2}{\partial N_{cm}}$ dominates the effect of change in average correlation $\frac{\partial\hat{\sigma}^2}{\partial\bar{\rho}_{cm}}$, and thus, the cutoff $\hat{\sigma}^2=\phi^{-1}$ of the marginal trader's market choice increases when more traders choose the centralized market. The marginal trader i choose the over-the-counter market if his noise variance σ_i^2 is higher than the cutoff $\hat{\sigma}^2(N_{cm})$, and he choose the centralize market if σ_i^2 is lower

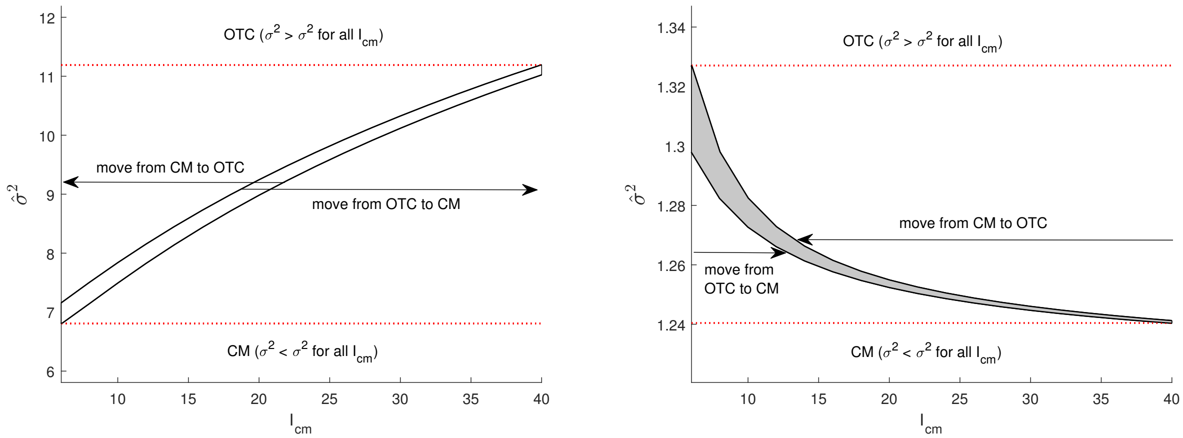

Figure 3: Endogenous Market Structure. $\omega_{iv}^2 = 0.7$ (left) $\omega_{iv}^2 = 0.9$ (right), $\rho_{ij} = \pm 0.8, L = 4, I = 36$. For the left panel, only corner solution exists, i.e., OTC only or CM only is stable. In the right panel, for the middle $\hat{\sigma}^2(N = I) < \sigma^2 < \hat{\sigma}^2(N = L)$, OTC and CM coexists: In particular, the endogenous market structure is in the shaded region.

than the cutoff $\hat{\sigma}^2(N_{cm} + 1)$. Due to $\hat{\sigma}^2(N_{cm})$ increasing in N_{cm}, if the marginal trader i chose the centralized market, the next trader who has the symmetric noise variance has even stronger incentive to enter the centralized market. Similarly, when i chose the over-the-counter market, the next trader also chooses the over-the-counter market. Hence, the market structure with the over-the-counter market only and the one with the centralized market only are both stable equilibrium structure.

The right panel of Figure 3 shows that the over-the-counter market and centralized market can coexist when $\hat{\sigma}^2(N_{cm})$ is decreasing in the number of traders N_{cm} in the centralized market. When a marginal trader chooses the centralized market, the next trader's incentive to enter the centralized market is weaker than his. This implies that there exists $N_{cm}^* \in (0, I)$ such that $\hat{\sigma}^2(N_{cm} + 1) < \sigma^2 < \hat{\sigma}^2(N_{cm})$ and thus the marginal trader does not have an incentive to deviate in his market choice.

In an endogenous market structure in which both trading venues coexist, some strategic traders choose the over-the-counter market while other traders choose the centralized market. The next sections examine which types of traders are more attracted to the over-the-counter market versus which are more attracted to the centralized market. Section 5.1 explores the effect of heterogeneous correlation profiles and Section 5.2 examines the heterogeneous information precision in both endogenous over-the-counter matching and market structure.[16]

[16] Since the incentives to enter each market are functions of endogenized market structure, traders' market choice and endogenous market structure create a fixed point problem in equilibrium. Due to complexity in the fixed point problem, the two-dimensional asymmetry - heterogeneous correlation profiles and heterogeneous information precision - will be considered separately. The results from each heterogeneity can be jointly studied in a model with an heterogeneous correlation structure and information precisions.

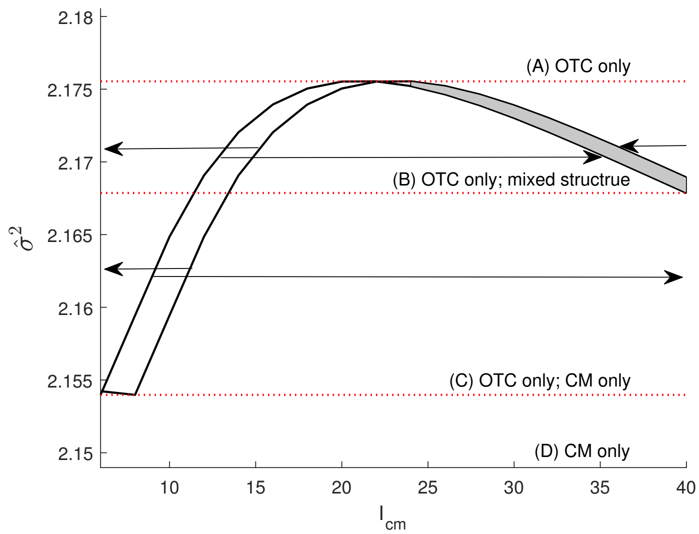

Figure 4: Endogenous Market Structure. $\omega_{iv}^2 = 0.732, \rho_{ij} = \pm 0.9, L = 4, I = 36$. There can be multiple equilibria, depending on how I and $\bar{\rho}$ move together. In particular, region (C) shows two corner solutions, i.e., OTC only and CM only are both stable market structures, and region (D) shows that OTC only and a market structure with coexisting OTC and CM are stable.

4.3 Welfare Implications of Introducing OTC Markets

An important implication related to the over-the-counter market is whether introducing an over-the-counter market is beneficial to total welfare. The model in this paper provides a flexible framework within which one could determine an appropriate policy intervention, such as the effects of closing or opening over-the-counter markets and transparency regulation. In particular, the policy perspective often promotes efficiency. One would want to know whether the market structure we observe is indeed efficient or whether some and if so which type of regulation would be welcome.

The welfare is defined as the sum of expected utility of all traders, including the liquidity traders in the endogenous market structure:

$$W \equiv \sum_{i \in I \setminus N_{cm}^*} E[u_i^{otc}] + \sum_{i \in N_{cm}^*} E[u_i^{cm}] + \sum_{l \in L} E[u_l].$$

Allowing traders to choose with whom they want to trade and in which market, the resulting market structure is not necessarily efficient. The inefficiency arises due to the trade-off between traders who choose the over-the-counter market and those who stay in the centralized market: introducing an over-the-counter market reduces the trading volume of those who choose to trade in an exchange, although the traders who choose the over-the-counter market can trade more. This can lead to allocative inefficiency.

The trade-off between traders in different markets implies that introducing an over-the-counter market weakly improves the total welfare when the endogenous market structure is unique and consists of only only market, either centralized or over-the-counter markets (see Proposition 3). As an example, Figure 4 examines the trade-off in welfare depending on the information precision of traders' signals, $\sigma_{i,\varepsilon}^2 = 1/\phi_i$. There are four regions:

- The over-the-counter market only (A) or the centralized market only (D): allowing traders to enter an over-the-counter market weakly improves the welfare.[17]
- Two corner solution in equilibrium (C): The market structure with only the over-the-counter market is a stable equilibrium but Pareto dominated by the market structure with only the centralized market. Allowing an over-the-counter trading (weakly) decreases the welfare.
- Coexisting over-the-counter and centralized markets (B): welfare implication of introducing an over-the-counter market is ambiguous: the welfare improves when the utility gains of traders who enter the over-the-counter market dominate the utility loss of traders who stay in the centralized market.

Section 5.2 will discuss another source of inefficiency: an informational inefficiency can arise when an over-the-counter market is introduced, because private information is shared only between informed traders in OTC markets.

5 Market Structure with Asymmetric Traders

Theorem 1 examined a trader's choice between the over-the-counter and centralized markets and the resulting endogenous market structures. The key characteristics are the heterogeneous correlations in asset valuation $Corr(\tilde{\theta}_i, \tilde{\theta}_j)$ across pairs of traders (i, j). This section considers the effects of other types of heterogeneities: namely, heterogeneous correlations across traders, not only across pairs, and heterogeneous information precision. A model with such heterogeneities across traders is called an *asymmetric* model, and a model without the heterogeneities is called a *symmetric* model (See Definition 2).

In a symmetric market, all strategic traders are symmetric in the market choice. On the other hand, the incentives to choose either market can differ by traders in asymmetric markets,

[17] Lee and Wang (2017) study welfare implications of closing an over-the-counter market in a model with endogenized bid-ask spreads. They show that closing the over-the-counter market increases welfare if uninformed hedgers dominate informed speculator in the population, due to dealers' screening against speculators. On the other hand, in this paper, closing the over-the-counter market can decrease the welfare with more uninformed traders, because they learn less in the exchange and may face higher price impact. The joint effects of endogenized bid-ask spreads and learning and price impacts will determine the benefits of introducing or closing over-the-counter markets.

and thus, who enters to the over-the-counter market matters in addition to how many traders are in the market.

5.1 Asymmetric Interdependence of Asset Valuations

Suppose that traders are asymmetric in the sense that their profiles of correlation $\{\rho_{ij}\}_{j\neq i}$ differ. In the over-the-counter market, each trader can choose his counterparty based on the type, i.e., correlation structure. The matching fails if the choice of the counterparty is not mutual. In addition, even when two traders are matched, the trade does not occur if the traders have positively correlated asset values. In that, if $Corr(\theta_i, \theta_j) = \rho_0 > 0$, traders optimal bid function becomes inelastic so that there is no trade. No trader chooses a counterparty whose values are positively correlated with his. Consequently, if a trader i's asset value satisfies $Corr(\tilde{\theta}_i, \tilde{\theta}_j) > 0$ for any $j \neq i$, i.e., his asset valuations are positively correlated with all other traders, then he gets zero gains-to-trade in the over-the-counter market which makes him enter the centralized market.

Recall that the over-the-counter matching is determined by traders' ranking on counterparties and that a trader prefers to trade with the one whose value is more negatively correlated with his. Traders who have relatively less negative correlations may not be matched with whom he wants, or he may not be matched with anyone. These traders will not be chosen by the ones that they want to be matched with. This lowers the benefit of the over-the-counter market and makes them choose the centralized market.

Proposition 4 (OTC Matching with Heterogeneous Correlations) *Suppose that traders are asymmetric in correlation structures but symmetric in information precision. There exists a pairwise stable over-the-counter matching determined by the ranking in the negative correlations* $(-Corr(\tilde{\theta}_i, \tilde{\theta}_j))$ *of pairs of traders.*

The pair-wise stable over-the-counter matching is determined by the following algorithm:

STEP-1. Two traders, who have the most negative correlation among all pairs, are matched and the matching is denoted by (i_1, j_1). If there are multiple pairs that have the most negative correlation, select one pair randomly.

STEP-2. Eliminating the selected traders, select the most negative correlation among the remaining paper and create another pair: called (i_2, j_2).

STEP-3. Repeating this procedure until there is at most one remaining trader in the over-the-counter market, and then the over-the-counter matching is determined $\{(i_t, j_t)\}_{t=1,2,\ldots}$.

In general, a stable equilibrium may fail to exist in one-sided matching problems. However, if traders are asymmetric only in correlations, a pairwise stable over-the-counter matching always exists in endogenous market structure with the endogenized choice of market and counterparty. First, the ranking based on negative correlations in Proposition 4 guarantees the transitive property on ranking. It is shown by the fact that the rankings of traders do not create any circular preferences. Suppose that there exists traders i, j, and k whose rankings are circular: i prefers j to k, j prefers k to i, and k prefers i to j. By Proposition 4, it implies that $Corr(\tilde{\theta}_i, \tilde{\theta}_j) \leq Corr(\tilde{\theta}_i, \tilde{\theta}_k) < 0$, $Corr(\tilde{\theta}_j, \tilde{\theta}_k) \leq Corr(\tilde{\theta}_j, \tilde{\theta}_i) < 0$, and $Corr(\tilde{\theta}_k, \tilde{\theta}_i) \leq Corr(\tilde{\theta}_k, \tilde{\theta}_j) < 0$. Hence, the correlations between the three traders satisfy $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = Corr(\tilde{\theta}_j, \tilde{\theta}_k) = Corr(\tilde{\theta}_k, \tilde{\theta}_i)$, and thus, each trader is indifferent between the other traders in his counterparty choice. In addition to the stability, the individual market choice keeps a trader from not being matched in the over-the-counter market. A trader who fails to find a counterparty in the over-the-counter market deviates to the centralized market. Therefore, a stable over-the-counter matching exists in equilibrium due to the ranking mechanism in Proposition 4 and the endogenized market choice between over-the-counter and centralized markets.

5.2 Asymmetric Information Precision

Suppose that traders are asymmetric only in information precisions but symmetric in the correlation structure, i.e., ϕ_i is different across i but the profile $\{\rho_{ij}\}_{j \neq i}$ is the same for all i. Proposition 2 (iii) shows that a trader's equilibrium utility is non-monotone over other traders' information precision. A trader prefers an informed counterparty to improve learning, but he would prefer an uninformed counterparty to have a lower price impact. In the trade-off between learning and price impact, which effect dominates depends on the trader's precision. If trader i's information precision is already high, lower price impact is more valuable than better learning so that he chooses a relatively less informed counterparty. On the other hand, if his precision is low he prefers a more informed counterparty for better learning. This asymmetric ranking of traders on counterparty's information precision leads to a potential over-the-counter trading between traders who informed unequally. With a sufficient heterogeneity of information precision, an informed trader prefers an uninformed counterparty for lower price impact while an uninformed trader prefers an informed counterparty for higher learning benefit.

It is useful to consider a model with two types of information precisions, informed and uninformed types. The information precisions of two types are assumed by $\phi_U = 1/\sigma_U^2 < \phi_I = 1/\sigma_I^2$. With two informational types, the over-the-counter matching is either positive assortative matching (i.e., *same-type* matching) or negative assortative matching (i.e., *cross-*

type matching).[18] The same-type matching is when traders are matched with other traders with the same type, and cross-type matching is when informed and uninformed traders are matched to each other. Proposition 5 and Figure 5 show endogenous choice of counterparty when all traders are in the over-the-counter market.

Proposition 5 (OTC Matching with Heterogeneous Information Precision) *Suppose that all traders $i \in I$ enters the over-the-counter markets. The pair-wise stable matching in the over-the-counter market is negative assortative (i.e., the cross-type matching), if and only if ϕ_I is sufficiently high and sufficiently larger than ϕ_U.*

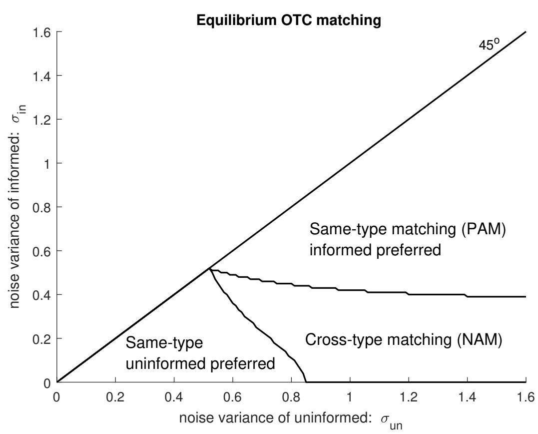

Figure 5: Equilibrium matching in OTC subgame. Each region shows an endogenous over-the-counter matching, either same-type matching or cross-type matching between informed and uninformed traders. All traders are assumed to be in the over-the-counter market. The x-axis is the noise variance of uninformed traders while the y-axis is of informed traders.

With the presence of the centralized market, however, an over-the-counter market in which two informational types match does not exist. When the informed type follows lower price impact rather than better learning, he can be better off trading in the centralized market than in the over-the-counter trading with uninformed counterparties.

Theorem 3 (Non-Existence of Cross-Type OTC Matching) *Suppose that there are two precision types $\{\phi_I, \phi_U\}$. With a sufficiently large common value component ω_{cv} and a sufficiently many liquidity players L, there is no over-the-counter trade between informed and uninformed traders (i.e., cross-type matching).*

Figure 6 presents an example of endogenous market structures through traders' market and counterparty choice. The left panel shows threpes of equilibrium: all traders choose the centralized market (when both σ_I and σ_U are small, and learning is not sufficiently valuable to

[18] The non-monotone ranking on counterparty's precision may prevent an assortative matching in over-the-counter markets with more than two types of heterogeneous information precisions. However, I am expecting that there exists an endogenous statistics in which the matching is assortative.

either of them); only uninformed traders choose to trade in the over-the-counter market (large σ_U but small σ_I); and all traders enter the over the counter market (both types of traders have inaccurate information). Since learning incentive is a dominant incentive for uninformed traders, there is no equilibrium where only informed traders enter the over-the-counter market.

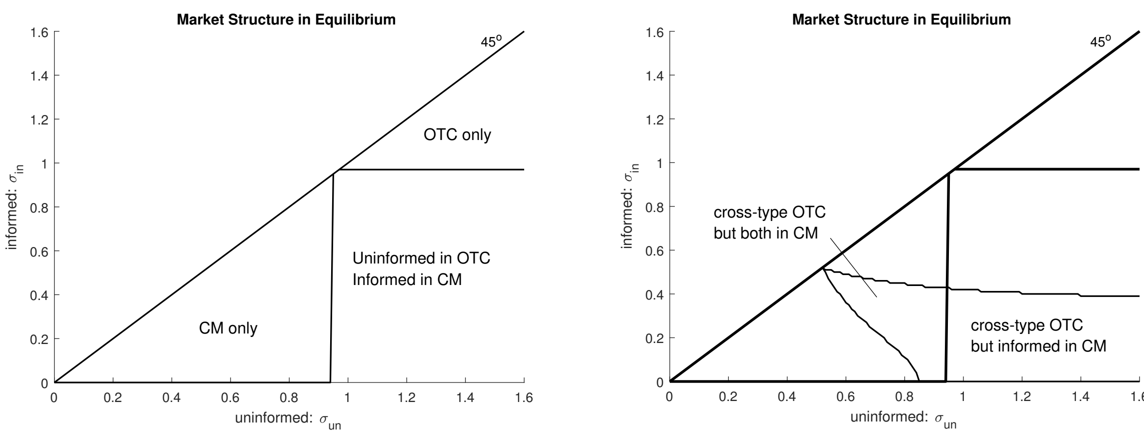

Figure 6: Endogenous Market Structure. The x-axis is the noise variance of uninformed traders while the y-axis is of informed traders.

Endogenizing the market choice, the over-the-counter market will attract either the uninformed or both.[19] Furthermore, traders in the over-the-counter market always trade with the same-type counterparty. The non-existence of matching between informed and uninformed attributes to aggravating information asymmetry in the over-the-counter markets. With a random match mechanism, information can be transmitted from the informed to the uninformed trader when they met, and thus information asymmetry disappears or diminishes over time. However, when traders choose their own counterparty based on information precision, the informed traders do not want to be matched with uninformed traders. Information is shared only within each type, and the asymmetry between types increases after trades take place. Consequently, the informational inefficiency by allowing an over-the-counter market into the economy.

6 Discussion

Connection to Markets. I show that an over-the-counter market opens when the size of the centralized market is small, the asset values are closer to idiosyncratic than common, and private information of traders is less precise. Many financial derivatives such as forward

[19] A conventional wisdom that informed traders are more likely to trade in the over-the-counter market for keeping their private information from the public. Since this model considers only a static trading, such privacy incentive does not exist. The incentives to trade in the over-the-counter market due to the heterogeneity across traders is a separate effect. In dynamic models, the privacy incentives would interact with the effects of heterogeneities.

contracts, interest rate swaps, or equity or credit linked securities are traded in over-the-counter markets, even though their trading volumes (liquidity) are substantial. When these products are required to be held by traders until the maturity, it suggests that the purpose of trading can be hedging of traders' outside portfolios. This paper suggests that idiosyncratically valued assets tend to be traded in the over-the-counter markets. On the other hand, centralized markets attract assets traded mostly by speculators, such as stocks or bonds with short maturity, which are valued by future prices that are common to all traders. High-yield bonds that have low credit ranking are often traded in the over-the-counter markets (e.g., Hendershott and Madhavan (2015)). The volatile return prevents the traders' access to quality information, and hence the information precision is low. This is consistent with this paper's prediction that low information precision encourages traders to choose over-the-counter markets.

CARA Utility. The utility with constant absolute risk aversion (CARA utility) is often assumed as traders' preferences in non-competitive trading literature:

$$E[u_i(q_i,p)|\cdot] = -\exp\Big(-\mu\big(E[\tilde{\theta}_i|\cdot]q_i - pq_i - \frac{\mu}{2}Var(\tilde{\theta}_i|\cdot)q_i^2\big)\Big). \tag{14}$$

Under the CARA-Gaussian setting, the conditional variance $V(\tilde{\theta}_i|\cdot)$ is a non-random constant independent of q_i or any realizations in the market. Therefore, the expected utility (14) is equivalent to a *quadratic* utility with the endogenously determined quadratic coefficient:

$$v_i(q_i,p) := -\frac{1}{\mu}\log(-E[u_i(q_i)|\cdot]) = E[\tilde{\theta}_i|\cdot]q_i - pq_i - \frac{\mu}{2}Var(\tilde{\theta}_i|\cdot)q_i^2.$$

As same as with the mean-variance utility (1), the CARA utility provides linear equilibrium. However, the endogenous quadratic coefficient $V(\tilde{\theta}_i|\cdot)$ for CARA utility, instead of the constant coefficient Σ in (1), provides a different prediction on traders' incentives in the choice of market and counterparty. In particular, a trader i's expected CARA utility is decomposed into *three* effects:

$$E[u_i] = -\Big(1 + \frac{1}{2\mu}\underbrace{\frac{1}{Var(\tilde{\theta}_i|s_i,p)}}_{\text{risk-reduction}}\underbrace{\frac{(1+2\widehat{\lambda}_i)}{(1+\widehat{\lambda}_i)^2}}_{\text{liquidity effect}}\underbrace{Var\big(E[\tilde{\theta}_i|s_i,p]-p\big)}_{\text{learning effect}}\Big)^{-1/2}, \quad \forall i. \tag{15}$$

where $\hat{\lambda}_i = \lambda_i/Var(\tilde{\theta}_i|s_i,p)$ is the normalized price impact. The additional term $Var(\tilde{\theta}_i|s_i,p)^{-1}$, which is called a *risk-reduction effect*, captures a utility benefit of reducing the risk in uncertain asset value $\tilde{\theta}_i$ by conditioning on the price to-be-realized.

Proposition 6 (Dominance of Learning over Price Impact with CARA Utilities) *In a bilateral matching between traders i and j, the equilibrium utility (9) of trader i increases in*

ϕ_j, i.e., the learning incentive dominates the liquidity effect for all players i.

For traders who has the constant-absolute-risk-aversion utility (14), the risk-reduction and learning incentive unambiguously dominate the liquidity effect in traders' expected utilities. When the counterparty j's information precision $\phi_j = 1/\sigma_j^2$ decreases, the risk $Var(\tilde{\theta}_i|s_i, p)$ of trader i in the bilateral trade or in a market becomes higher. This inherently leads the trader to have a higher effective risk aversion $\mu Var(\tilde{\theta}_i|s_i, p)$, and thus, his demand $q_{i,t}(p)$ is reduced.

The dominance of learning effect provides that traders have the same preference over counterparty's information precision: a trader prefers a counterparty who has the highest information precision $\phi^*_{-i} = \max_{j\neq i} \phi_j$, independently of his own precision ϕ_i. As a result, only the same-type matching occurs in equilibrium, and, with the presence of the centralized market, less informed traders may move to the centralized market. This concludes that the results in Theorem 1-3 still hold in the CARA setting.

Alternative Over-the-Counter Designs. I have endogenized centralized and over-the-counter markets assuming their prices and allocations based on an uniform-price double auction. This allows us to understand the implications of price impact without having to incorporate search frictions and to analyze traders' incentives in static markets. This allows me to explicitly analyze equilibrium price impact in the over-the-counter market as well as centralized markets, and consider traders' learning about asset values even with static trading. In particular, I allow but do not assume competitive markets or efficient surplus division. That is another innovation relative to the literature.

One might consider an alternative mechanism for the determination of over-the-counter prices and trades. However, what matters for the results is that the allocation is not fully efficient due to either market noncompetitiveness or two-sided private information. In particular, with random matching, searching with frictions, or other mechanisms in literature introduced in over-the-counter markets, the effects in this paper continue to be present. Introducing random matching mechanism in the over-the-counter market does not affect the endogenous market structure qualitatively, but it can increase traders' incentive to enter the over-the-counter market when the heterogeneity across traders is present. Exogenous frictions in the over-the-counter markets - a probability that a trader does not trade, cost of waiting, etc. - can decrease traders' incentive to trade in over-the-counter markets.

Extension to Dynamic Markets. Traders' individual asset values θ_i can be interpreted as future returns of the asset in the markets and/or future returns of traders' individual portfolios. This provides a natural extension of the results in the static model in this paper to dynamic trading and market choice problem. Dynamics itself creates new effects in both the analysis of the choice of the trading venue and design. First, traders who have better information have an incentive to maintain their information advantage for future trades. This incentive favors the over-the-counter market. On the other hand, the idiosyncratic value component would be less significant in value functions (compared to its significance in the primitive asset valuations), so traders' asset valuation and dynamic inference might create a trade-off that is not present in static trading.

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A Proofs

Proof of Proposition 1 (Equilibrium Representation in a Market). See Appendix B.1 and Rostek and Yoon (2018). ■

Proof of Proposition 2 (Benefits of Learning and Liquidity). In equilibrium for a given market, subject to existence, the ex-ante indirect utility of a trader i is

$$E[u_i] = \frac{1}{2\mu}\frac{1+2\hat{\lambda}_i}{(1+\hat{\lambda}_i)^2}Var(E[\theta_i|s_i,p]-p).$$

Here, $\hat{\lambda}_i = \lambda_i/\mu$ is the normalized price impact. Suppose that the correlation of traders' asset values and information precision is symmetric across traders. We show that more negative asset correlation increases the variance of difference between individual asset value and price, $Var(E[\theta_i|s_i,p]-p)$, the larger market size decreases price impact λ_i.

Let us define a weighted average of correlation: $\bar{\rho}_i := \frac{\mu+\lambda_i}{c_{s,i}}\frac{1}{I-1}\sum_{j\neq i}\frac{c_{s,j}}{\mu+\lambda_j}\rho_{ij}$ for each i. In symmetric market, $\bar{\rho}_i = \frac{1}{I-1}\sum_{j\neq i}\rho_{ij}$ for each i. For each i,

$$c_{s,i} = \frac{1-\bar{\rho}}{1-\bar{\rho}+\sigma^2}, \qquad c_{p,i} = \frac{I\bar{\rho}\sigma^2}{(1+(I-1)\bar{\rho})(1-\bar{\rho}+\sigma^2)},$$
$$\lambda_i = \frac{\mu}{(I-1)(1-c_p)-1} = \frac{\mu(1+(I-1)\bar{\rho})(1-\bar{\rho}+\sigma^2)}{(I-1)(1-\bar{\rho})(1+\sigma^2+(I-1)\bar{\rho})-(1+(I-1)\bar{\rho})(1-\bar{\rho}+\sigma^2)}.$$

Furthermore, the inference coefficients characterize the learning effect in expected utility.

$$Var(E[\theta_i|s_i,p]-p) = \sigma_\theta^2 c_s^2\frac{I-1}{I}(1+\sigma^2-\bar{\rho}) = \frac{I-1}{I}\frac{(1-\bar{\rho})^2}{1+\sigma^2-\bar{\rho}}\sigma_\theta^2.$$

The above characterization for the symmetric markets provides the following comparative statics of the three characteristics (i) the market size, (ii) the average correlation $\bar{\rho}$, and (iii) information precision $\phi_i = 1/\sigma^2$: The liquidity effect on utility is captured by the term $\frac{1+2\widehat{\lambda}_i}{(1+\widehat{\lambda}_i)^2}$. With this closed-form solution of inference parameters and price impact,

$$\frac{1+2\widehat{\lambda}_i}{(1+\widehat{\lambda}_i)^2} = 1-(\frac{\widehat{\lambda}_i}{1+\widehat{\lambda}_i})^2 = 1-\Big(\frac{(1+\sigma^2-\bar{\rho})(1+(I-1)\bar{\rho})}{(I-1)(1-\bar{\rho})(1+\sigma^2+(I-1)\bar{\rho})}\Big)^2.$$

By taking a derivative with respect to $\sigma^2, \bar{\rho}$, and I,

$$w.r.t.\ \sigma^2: \quad -2\Big(\frac{(1+\sigma^2-\bar{\rho})(1+(I-1)\bar{\rho})^2}{(I-1)^2(1-\bar{\rho})^2(1+\sigma^2+(I-1)\bar{\rho})}\Big)\frac{I\bar{\rho}}{(1+\sigma^2+(I-1)\bar{\rho})^2} > 0, \quad \text{iff } \bar{\rho} < 0,$$

$$w.r.t.\ \bar{\rho}: \quad -2\Big(\frac{(1+\sigma^2-\bar{\rho})(1+(I-1)\bar{\rho})}{(I-1)(1-\bar{\rho})(1+\sigma^2+(I-1)\bar{\rho})}\Big)\frac{I\sigma^2(1+\sigma^2+(I-1)\bar{\rho}^2)}{(I-1)(1-\bar{\rho})^2(1+\sigma^2+(I-1)\bar{\rho})^2}<0,$$

$$w.r.t.\ I: \quad 2\Big(\frac{(1+\sigma^2-\bar{\rho})(1+(I-1)\bar{\rho})}{(I-1)(1-\bar{\rho})(1+\sigma^2+(I-1)\bar{\rho})}\Big)\frac{\sigma^2+(1+(I-1)\bar{\rho})^2}{(I-1)^2(1+\sigma^2+(I-1)\bar{\rho})^2}>0.$$

Larger market size and/or more negative correlation with others on average results in more liquidity, and thus higher utility for traders. When the information precision $\phi = 1/\sigma^2$ increases, the endogenous liquidity of the market increases if $\bar{\rho} > 0$, and decreases if $\bar{\rho} < 0$.

The effect of learning from the price on utility is measured by

$$\frac{1}{\sigma_\theta^2}Var(E[\tilde{\theta}_i|s_i,p]-p)=\frac{I-1}{I}\frac{(1-\bar{\rho})^2}{1+\sigma^2-\bar{\rho}},$$

which is increasing in information precision $\phi = 1/\sigma^2$ and the number of traders I, and decreasing with respect to $\bar{\rho}$.

$$w.r.t.\ \sigma^2: \quad -\frac{I-1}{I}\frac{(1-\bar{\rho})^2}{(1+\sigma^2-\bar{\rho})^2}<0,$$

$$w.r.t.\ \bar{\rho}: \quad -\frac{I-1}{I}\frac{(1-\bar{\rho})(1+2\sigma^2-\bar{\rho})}{(1+\sigma^2-\bar{\rho})^2}<0,$$

$$w.r.t.\ I: \quad \frac{1}{I^2}\frac{(1-\bar{\rho})^2}{1+\sigma^2-\bar{\rho}}>0.$$

Expected utility $E[u_i]$ increases as the number of traders in market increases (I increases) and the correlation between asset values decreases ($\bar{\rho}$ decreases). When $\bar{\rho} > 0$, the expected utility increases as traders' information precision increases ($\phi := 1/\sigma^2$ increases). When $\bar{\rho} < 0$, as the precision ϕ increases, two components in the expected utility change differently: $\frac{1+2\hat{\lambda}_i}{(1+\hat{\lambda}_i)^2}$ decreases but $\frac{Var(E[\tilde{\theta}_i|s_i,p]-p)}{Var(\tilde{\theta}_i|s_i,p)}$ increases. ■

Proof of Theorem 1 (When A Trader Chooses OTC Markets). This proof is under the condition that equilibrium exists. Suppose that only a centralized market opens in equilibrium but no over-the-counter market does. Since no pair of traders have an incentive to switch their market choices to the over-the-counter market, his expected utilities $E[u_i^{CM};\text{all in CM}]$, when all traders are in the centralized market, is higher than a potential utility in the over-the-counter market. The potential utility in the over-the-counter market $E[u_i;OTC(i,j)]$ is pair-specific, i.e., it depends on the pair (i,j). A sufficient and necessary condition on the endogenous market structure consists of only the centralized market is that there is no trader who has a positive incentive to deviate to the over-the-counter market with his best counterparty. This condition is equivalent to that the utility in centralized market is higher than the maximum utility trader i would get in over-the-counter market:

$$E[u_i^{CM};\text{all in CM}] > \max_{j\neq i} E[u_i;OTC(i,j)], \quad \forall i.$$

Under the symmetry assumption, $\frac{1}{I-1}\sum_{j\neq i}\rho_{ij} = \bar{\rho}$ and $\phi_i = \phi = 1/\sigma_\varepsilon^2$ for all i, the equilibrium utility in each market is characterized as follows. Suppose that $\sigma = \sigma_\varepsilon/\sigma_\theta$. In the centralized market, the equilibrium utility is characterized by

$$
\begin{aligned}
E[u_i^{cm}] &= \frac{\sigma_\theta^2}{\mu}\frac{(I-1)^2(1-\bar\rho)^2(1+\sigma^2+(I-1)\bar\rho)^2-(1+\sigma^2-\bar\rho)^2(1+(I-1)\bar\rho)^2}{I(I-1)(1+\sigma^2+(I-1)\bar\rho)^2(1+\sigma^2-\bar\rho)} \\
&= \frac{\sigma_\theta^2}{\mu}\frac{((I-2)(1+\sigma^2+(I-2)\bar\rho-(I-1)\bar\rho^2)-2(I-1)\sigma^2\bar\rho)(1+\sigma^2+(I-2)\bar\rho-(I-1)\bar\rho^2)}{(I-1)(1+\sigma^2+(I-1)\bar\rho)^2(1+\sigma^2-\bar\rho)}.
\end{aligned}
$$

On the other hand, the over-the-counter market provides the equilibrium utility:

$$
E[u_i^{otc}] = \frac{\sigma_\theta^2}{\mu}\frac{-2\sigma^2\rho_{ij}(1+\sigma^2-\rho_{ij}^2)}{(1+\sigma^2+\rho_{ij})^2(1+\sigma^2-\rho_{ij})}.
$$

Comparing the equilibrium utility in two markets,

$$
E[u_i^{cm}] > E[u_i^{otc}] \quad\Leftrightarrow\quad \frac{1+2\widehat{\lambda}_i^{cm}}{(1+\widehat{\lambda}_i^{cm})^2}Var(E[\theta_i|s_i,p]-p) > \frac{1+2\widehat{\lambda}_i^{otc}}{(1+\widehat{\lambda}_i^{otc})^2}Var(E[\theta_i|s_i,s_j]-p), \quad (16)
$$

which is equivalent to the following inequality:

$$
\frac{(I-1)^2(1-\bar\rho)^2(1+\sigma^2+(I-1)\bar\rho)^2-(1+\sigma^2-\bar\rho)^2(1+(I-1)\bar\rho)^2}{I(I-1)(1+\sigma^2+(I-1)\bar\rho)^2(1+\sigma^2-\bar\rho)} > \frac{-2\sigma^2\rho_{ij}(1+\sigma^2-\rho_{ij}^2)}{(1+\sigma^2+\rho_{ij})^2(1+\sigma^2-\rho_{ij})}.
$$

Under the positive definiteness of correlation matrices, the above inequality is simplified into an inequality:

$$
f(\sigma^2) > 0, \quad (17)
$$

when $f(\sigma^2)$ is a fifth-order polynomial,

$$
\begin{aligned}
f(\sigma^2) \;:=\; & ((I-2)(1+\sigma^2+(I-2)\bar\rho-(I-1)\bar\rho^2)-2(I-1)\sigma^2\bar\rho) \quad (18)\\
& \times(1+\sigma^2+(I-2)\bar\rho-(I-1)\bar\rho^2)(1+\sigma^2+\rho_{ij})^2(1+\sigma^2-\rho_{ij}) \\
& +(I-1)(1+\sigma^2+(I-1)\bar\rho)^2(1+\sigma^2-\bar\rho)2\sigma^2\rho_{ij}(1+\sigma^2-\rho_{ij}^2).
\end{aligned}
$$

In order to solve the inequality (17), let us first denote the coefficients of function $f(\sigma^2)$ by $f(\sigma^2) := a_5\sigma^{10}+a_4\sigma^8+a_3\sigma^6+a_2\sigma^4+a_1\sigma^2+a_0$. Lemma 1 shows a property of coefficients $(a_0,a_1,a_2,a_3,a_4,a_5)$ of function $f(\sigma^2)$, that is useful to decide the number of positive solution for $f(\sigma^2)=0$:

Lemma 1 *For any $I \geq 3$ and $\bar\rho \geq 0 > \rho_{ij}$, if $a_n > 0$ then $a_{n-1} > 0$ for any $n = 1, \cdots, 5$. In addition, $a_0 > 0$ always holds.*

Proof. See Appendix B.2 ■

From Lemma 1, if the highest-power coefficient $a_5 = (I-2-2(I-1)(\bar\rho-\rho_{ij}))$ is positive, the coefficients of all powers $(a_0,a_1,a_2,a_3,a_4,a_5)$ in $f(\sigma^2)$ are positive. In this case, Descartes' Sign Rule implies that there is no positive solution $\sigma^2 > 0$ for $f(\sigma^2)=0$, i.e., the inequality $f(\sigma^2) > 0$ holds for all $\sigma^2 > 0$. Since this inequality (17) is equivalent to $\xi_i^{CM} > \xi_i^{OTC}$, a trader prefers to trade in the centralized market than in the over-the-counter market, independent of his information precision $\phi = 1/\sigma^2 > 0$. If $a_5 = (I - 2 - 2(I-1)(\bar{\rho} - \rho_{ij})) < 0$, Lemma 1 shows that there exists only one sign change in the sequence of coefficients $(a_0, a_1, a_2, a_3, a_4, a_5)$. From Descartes' Sign Rule, there exists a unique positive solution $\hat{\sigma}^2 > 0$ such that $f(\hat{\sigma}^2) = 0$. Hence, when $(I - 2 - 2(I-1)(\bar{\rho} - \rho_{ij})) < 0$, $\xi_i^{CM} > \xi_i^{OTC}$ if and only if $\sigma^2 < \hat{\sigma}^2$.

This shows that the sufficient and necessary condition for $\xi_i^{CM} < \xi_i^{OTC}$ is

$$\frac{I-2}{2(I-1)} + \rho_{ij} < \bar{\rho}, \quad \text{and} \quad \sigma^2 > \hat{\sigma}^2.$$

Without loss of generality, we denote the correlations between asset values $(\tilde{\theta}_i)_i$ by $\bar{\rho} = \frac{\sigma_{cv}^2}{\sigma_{cv}^2 + \sigma_{iv}^2}$ and $\rho_{ij} = \min_{j \neq i} Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \frac{\sigma_{cv}^2 - \sigma_{iv}^2}{\sigma_{cv}^2 + \sigma_{iv}^2}$.

$$\frac{\sigma_{iv}^2}{\sigma_{cv}^2 + \sigma_{iv}^2} > \frac{I-2}{2(I-1)} := \hat{\kappa} \;\; \text{and} \;\; \phi_i = \frac{1}{\sigma_\varepsilon^2} = \frac{\sigma_{cv}^2 + \sigma_{iv}^2}{\sigma^2} < \frac{\sigma_{cv}^2 + \sigma_{iv}^2}{\widehat{\sigma}^2} := \hat{\phi}(\sigma_{cv}^2, \sigma_{iv}^2, \Sigma);$$

when $\hat{\sigma}^2$ is the unique positive solution of $f(\sigma^2) = 0$ in (18). ■

Proof of Corollary 1 (OTC Existence with a Competitive CM). From the proof of Theorem 1, the inequality (16) is a sufficient and necessary condition for a trader to prefer the centralized market to the over-the-counter market. Taking the number of traders in the centralized market I to infinity, the inequality is written as

$$\frac{(1-\bar{\rho})^2}{(1+\sigma^2-\bar{\rho})} > \frac{-2\sigma^2\rho_{ij}(1+\sigma^2-\rho_{ij}^2)}{(1+\sigma^2+\rho_{ij})^2(1+\sigma^2-\rho_{ij})}.$$

and it is simplified into

$$\begin{aligned} 0 \;\; < \;\; & ((1-\bar{\rho})^2 + 2\rho_{ij})\sigma^6 + ((1-\bar{\rho})^2(3+\rho_{ij}) + (2-\bar{\rho}-\rho_{ij}^2)2\rho_{ij})\sigma^4 \\ & + ((1-\bar{\rho})^2(3+2\rho_{ij}-\rho_{ij}^2) + (1-\bar{\rho})(1-\rho_{ij}^2)2\rho_{ij})\sigma^2 + (1-\bar{\rho})^2(1-\rho_{ij})(1+\rho_{ij})^2. \end{aligned} \tag{19}$$

Because Lemma 1 is applied to any number of traders I, if $((1-\bar{\rho})^2 + 2\rho_{ij}) > 0$ then all coefficients are positive. In that case, the inequality (19) holds for any $\sigma^2 >$, and thus, all traders choose to enter the centralized market over the over-the-counter market. If $((1-\bar{\rho})^2 + 2\rho_{ij}) < 0$, Lemma 1 shows that there is one sign change in the r.h.s of inequality (19). Hence, there exists a unique positive solution $\hat{\sigma}^2 > 0$ such that the equality of (19) holds. The trader choose to enter to the over-the-counter market rather than to the centralized market, if and only if

$$(1-\bar{\rho})^2 + 2\rho_{ij} < 0 \quad \text{and} \quad \sigma^2 > \hat{\sigma}^2.$$

Without loss of generality, the above joint condition can be rewritten by

$$1-\bar{\rho}=\frac{\sigma_{iv}^2}{\sigma_{cv}^2+\sigma_{iv}^2}>2-\sqrt{2}\quad\text{and}\quad \phi=\frac{1}{\sigma_\varepsilon^2}<\frac{\sigma_{cv}^2+\sigma_{iv}^2}{\hat{\sigma}^2}.$$

by using $\rho_{ij}=\min_{j\neq i}Corr(\tilde{\theta}_i,\tilde{\theta}_j)=\frac{\sigma_{cv}^2-\sigma_{iv}^2}{\sigma_{cv}^2+\sigma_{iv}^2}=2\bar{\rho}-1$. ■

Proof of Corollary 2 (No Price Difference in OTC and CM). Suppose that traders' asset values ($\tilde{\theta}_i=\theta+\delta_i$) follow the distributions $\theta\sim\mathcal{N}(E[\theta],\sigma_{cv}^2)$ and $\delta_i\sim\mathcal{N}(0,\sigma_{iv}^2\Sigma)$ with $\Sigma=\left[\begin{array}{c|c}\mathbf{1} & -\mathbf{1}\\ \hline -\mathbf{1} & \mathbf{1}\end{array}\right]$ where $\mathbf{1}=(1)_{I/2\times I/2}$ is a $(\frac{I}{2}\times\frac{I}{2})$-matrix with all elements being one, as in Example 1. In this model, the equilibrium price in the centralized market with a sufficiently large number of traders is $p_{cm}=\frac{1}{I}\sum_{i\in I}E[\theta+\delta_i|s_i,p]\approx\frac{c_\theta E[\theta]+c_s\theta}{c_\theta+c_s}$, since the average correlation of the idiosyncratic component (δ_i) is zero. On the other hand, the equilibrium price is determined in each over-the-counter matching between two traders whose correlation is $Corr(\delta_i,\delta_j)=-1$.

$$p_{otc}=\frac{c_\theta E[\theta_i]+c_s\frac{1}{2}(\theta+\delta_i+\varepsilon_i+\theta+\delta_j+\varepsilon_j)}{1-c_p}=\frac{c_\theta E[\theta]+c_s\theta}{c_\theta+c_s}+\frac{c_s}{c_\theta+c_s}\big((\delta_i+\delta_j)+(\varepsilon_i+\varepsilon_j)\big)=p_{cm}+(\text{noise}).$$

Hence, the price p_{otc} in each over-the-counter market follows a normal distribution with the mean equal to the centralized market price p_{cm}. It shows that there exists a model where the over-the-counter and centralized market prices are same in expectation. Even in this case, a trader choose to trade in the over-the-counter market if the relative variance of idiosyncratic value component, $\frac{\sigma_{iv}^2}{\sigma_{cv}^2+\sigma_{iv}^2}$, satisfies the inequality (19). ■

Proof of Theorem 2 (Endogenous Market Structure with Competitive Exchange). Suppose that traders are symmetric in the sense that the profile of correlation in each row is same and that the information precision is same across traders (See Definition 2). The centralized market is competitive with infinitely many liquidity providers $L\to\infty$.

(i) First, we show that with the symmetric correlation structure, a trader is matched with the counterparty whose asset valuations has the minimum correlation: i.e., the over-the-counter matching (i,j) occurs such that $Corr(\tilde{\theta}_i,\tilde{\theta}_j)=\min_{k\neq i}Corr(\tilde{\theta}_i,\tilde{\theta}_k)=\rho_{min}$, upon traders i and j's participation in the over-the-counter market.

It can be shown as a contradiction. Suppose that there is an over-the-counter matching (i,j) such that $Corr(\tilde{\theta}_i,\tilde{\theta}_j)\gneq\rho_{min}$. Since the profile of correlations are same across traders, there exists another trader $k\neq i,j$ who has the minimum correlation with trader i, $Corr(\tilde{\theta}_i,\tilde{\theta}_k)=\rho_{min}$. If trader k's current matching in the over-the-counter market is not with the minimum correlation, traders i and k have a positive incentive to deviate from their current matchings, and thus the current matchings are not pair-wise stable.Therefore, trader k's current matching (k,l) has to be such that $Corr(\tilde{\theta}_k,\tilde{\theta}_l)=\rho_{min}$. It implies that trader k has two other traders i and l that provides the minimum correlation ρ_{min}. By the symmetry, the profile of correlations of trader i, $\{Corr(\tilde{\theta}_i,\tilde{\theta}_m)\}_{i,m}$, contains two or more ρ_{min}, in that there exists another trader $m\neq i,j,k$ such that $Corr(\tilde{\theta}_i,\tilde{\theta}_m)=\rho_{min}$. With the same argument, the current matching for trader m has the minimum correlation, while implies that there are three or more ρ_{min} in the correlation profile. Recursively, the symmetricity of traders and pair-wise stable matching concludes that all pairs of traders have $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \rho_{min}$, which is a contradiction to the assumption that there exists (i,j)-match with $Corr(\tilde{\theta}_i, \tilde{\theta}_j) \gneq \rho_{min}$. Therefore, all over-the-counter matching in equilibrium satisfies $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \min_{k \neq i} Corr(\tilde{\theta}_i, \tilde{\theta}_k) = \rho_{min}$, $\forall (i,j)$.

(ii) By the part (i) and by the competitiveness of the centralized market, the utility comparison between centralized and over-the-counter market is same for all traders. More formally, ρ^{otc}_{ij} and $\bar{\rho}^{cm}$ in two markets, and thus, the difference of these two correlations are fixed and symmetric for all traders. In addition, the individual trader's choice of market does not change the market size in the centralized market $N_{cm} + L \to \infty$ for any $N_{cm} \geq |I \setminus L|$. With the fixed correlation difference and market size, symmetric traders' incentive to enter either market is symmetric independently of other traders' choice of markets. Therefore, the endogenous distribution of traders in the over-the-counter and centralized market has a corner solution. Equivalently, it concludes that two trading venues coexist only if traders are asymmetric or the centralized market is not competitive $L < \infty$. ■

Proof of Proposition 3 (Endogenous Market Structure). There are three types of endogenous market structure: (i) only OTC market opens, (ii) only CM opens, (iii) OTC and CM coexists. The equilibrium market structure depends on $\frac{\partial \hat{\sigma}^2}{\partial N_{cm}}$ and $\frac{\partial \hat{\sigma}^2}{\partial \bar{\rho}_{cm}}$. In particular, how the number of traders in the centralized market N_{cm} and the average correlation in $\bar{\rho}_{cm}$ co-move based on the given traders' types $\Sigma, \{\sigma^2_{i,\varepsilon}\}_i$.

From the proof of Theorem 1, for given $(\bar{\rho}, \rho_{ij}, N_{cm})$, a trader i prefers the over-the-counter market to the centralized market, if and only if,

$$\frac{I-2}{2(N_{cm}-1)} + \rho_{ij} < \bar{\rho}, \quad \text{and} \quad \sigma^2 > \hat{\sigma}^2(\bar{\rho}, \rho_{ij}, N_{cm}),$$

where $\hat{\sigma}^2$ is the unique positive solution of equation (17), or equivalently, a trader who has the noise variance $\hat{\sigma}^2$ is indifferent between two markets:

$$E[u^{cm}_i(\hat{\sigma}^2; \bar{\rho}, N_{cm})] = E[u^{otc}_i(\hat{\sigma}^2; \rho_{ij})] \tag{20}$$

for given $(\bar{\rho}, \rho_{ij}, N_{cm})$ such that $(N_{cm} - 2) - 2(N_{cm} - 1)(\bar{\rho} - \rho_{ij}) < 0$. By implicitly differentiating equation (20) with respect to N_{cm}, while $(\bar{\rho}, \rho_{ij})$ are fixed, we get

$$\frac{\partial E[u^{cm}_i]}{\partial N_{cm}} + \frac{\partial E[u^{cm}_i]}{\partial \sigma^2}\frac{\partial \hat{\sigma}^2}{\partial N_{cm}} = \frac{\partial E[u^{otc}_i]}{\partial \sigma^2}\frac{\partial \hat{\sigma}^2}{\partial N_{cm}}, \quad i.e., \quad \Big(\frac{\partial E[u^{cm}_i]}{\partial \sigma^2} - \frac{\partial E[u^{otc}_i]}{\partial \sigma^2}\Big)\frac{\partial \hat{\sigma}^2}{\partial N_{cm}} = -\frac{\partial E[u^{cm}_i]}{\partial N_{cm}}$$

From Proposition 2, the expected utility $E[u^{cm}_i]$ in the centralized market is increasing in N_{cm}, and thus, $\frac{\partial E[u^{cm}_i]}{\partial N_{cm}} > 0$. From the uniqueness of $\hat{\sigma}^2$ and the the proof of Theorem 1, the difference in expected utilities in the two markets is decreasing in σ^2, i.e., $\frac{\partial E[u^{cm}_i]}{\partial \sigma^2} - \frac{\partial E[u^{otc}_i]}{\partial \sigma^2} < 0$. This proves that $\frac{\partial \hat{\sigma}^2}{\partial N_{cm}} > 0$. Similarly, by implicitly differentiating equation (20) with respect to $\bar{\rho}$, while (N_{cm}, ρ_{ij}) are fixed,

$$\Big(\frac{\partial E[u^{cm}_i]}{\partial \sigma^2} - \frac{\partial E[u^{otc}_i]}{\partial \sigma^2}\Big)\frac{\partial \hat{\sigma}^2}{\partial \bar{\rho}} = -\frac{\partial E[u^{cm}_i]}{\partial \bar{\rho}}.$$

From Proposition 2, $\frac{\partial E[u_i^{cm}]}{\partial \bar{\rho}} < 0$, and thus, $\frac{\partial \hat{\sigma}^2}{\partial \bar{\rho}} < 0$. The over-the-counter market and centralized market can coexist when $\frac{\partial \hat{\sigma}^2}{\partial N_{cm}}$ and $\frac{\partial \hat{\sigma}^2}{\partial \bar{\rho}_{cm}}$ makes a trade-off so that $\hat{\sigma}^2$ is decreasing in the number of traders in the centralized market. ■

Proof of Proposition 4 (OTC Matching with Heterogeneous Correlations). The pair-wise stable over-the-counter matching is determined by the following algorithm: (i) Upon traders' entry to the over-the-counter market, two traders, who have the most negative correlation among all pairs, are matched: we call this pair (i_1, j_1). (ii) If there are multiple pairs that have the most negative correlation, select one pair randomly. (iii) Eliminating the selected traders, select the most negative correlation among the remaining paper and create another pair: called (i_2, j_2). (iv) Repeating this procedure until there is at most one remaining trader in the over-the-counter market, and then the over-the-counter matching is determined $\{(i_t, j_t)\}_{t=1,2,\cdots}$.

It suffices to show that the matching $\{(i_t, j_t)\}_{t=1,2,\cdots}$ from this algorithm is pairwise stable. Suppose that there exists two traders who have a strictly positive incentive to deviate from their current matching and create their own matching. Formally, there exists i_t and j_s, for some $t \lneq s$, such that $E[u_{i_t}; OTC, (i_t, j_s)] \gneq E[u_{i_t}; OTC, (i_t, j_t)]$ and $E[u_{j_s}; OTC, (i_t, j_s)] \gneq E[u_{j_s}; OTC, (i_s, j_s)]$. From Proposition 2, it implies that $Corr(i_t, j_s) \lneq Corr(i_t, j_t)$ and $Corr(i_t, j_s) \lneq Corr(i_s, j_s)$, since traders are symmetric in information precision and any bilateral trade has the equal market size $N = 2$. It is a contradiction to the algorithm: at the step (i) for t, the matching at t is created between (i_t, j_t), and hence, $Corr(i_t, j_t) \leq Corr(i_t, j_s)$ for any $s > t$. It is contradicted to the assumption that trader i_t has a profitable deviation by having another counterparty j_s, $Corr(i_t, j_s) \lneq Corr(i_t, j_t)$. The proof is complete. ■

Proof of Proposition 5 (OTC Matching with Heterogeneous Information Precision). First, we characterize the equilibrium in a given over-the-counter bilateral matching between two traders i and j who have heterogeneous information precision. The correlation between two traders' asset values are $\rho := \rho_{ij} < 0$ and information precision is $\phi_i = 1/\sigma_i^2$ and $\phi_j = 1/\sigma_j^2$.

In the bilateral trades, the equilibrium inference coefficients $\{c_{s,i}, c_{p,i}\}$, price impacts $\{\lambda_i\}$, and residual uncertainty in asset values $\{Var(\theta_i|s_i, p)\}$ are characterized as follows: when $\Gamma_i := \frac{c_{s,j}}{\mu+\lambda_j} / \frac{c_{s,i}}{\mu+\lambda_i}$ and $\Gamma_j := 1/\Gamma_i$, and thus $\bar{\rho}_i := \Gamma_i \rho$,

$$c_{s,i} = \frac{(1+\sigma_j^2 - \rho^2) - \Gamma_i^{-1}\sigma_i^2\rho}{(1+\sigma_i^2)(1+\sigma_j^2) - \rho^2}; \quad \lambda_i = \frac{\mu + \lambda_j}{1 - c_{p,j}} = \frac{\mu(2 - c_{p,i})}{(1-c_{p,i})(1-c_{p,j}) - 1};$$

$$c_{p,i} = \frac{(\frac{1-c_{p,i}}{c_{s,i}}\Gamma_i^{-1} + \frac{1-c_{p,j}}{c_{s,j}})\rho\sigma_i^2}{(1+\sigma_i^2)(1+\sigma_j^2) - \rho^2} = \frac{\rho\sigma_i^2((1+\sigma_j^2+\rho) + \Gamma_i^{-1}(1+\sigma_i^2+\rho))}{(1+\rho)((1+\sigma_i^2)(1+\sigma_j^2) - \rho^2)}.$$

We remark that

$$1 - c_{p,i} = \frac{(1+\sigma_i^2+\rho)(1+\sigma_j^2 - \rho^2 - \Gamma_i^{-1}\sigma_i^2\rho)}{(1+\rho)((1+\sigma_i^2)(1+\sigma_j^2) - \rho^2)} = \frac{(1+\sigma_i^2+\rho)c_{s,i}}{(1+\rho)}.$$

By plugging $\{c_{s,i}, c_{p,i}, \lambda_i\}_i$ into the definition of Γ_i, we get a closed-form solution of the unique solution

Γ_i:

$$\Gamma_i = \frac{(1+\sigma_i^2+\rho)}{(1+\sigma_j^2+\rho)}\frac{2-c_{p,j}}{2-c_{p,i}} = \frac{(1+\sigma_i^2+\rho)}{(1+\sigma_j^2+\rho)}\frac{2((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+\rho(\sigma_i^2-\sigma_j^2)}{2((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+\rho(\sigma_j^2-\sigma_i^2)}.$$

For given Γ_i, the price impact is

$$\lambda_i = \frac{(1+\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)\mu}{\Gamma_i(1+\sigma_j^2+\rho)(1+\sigma_j^2-\rho^2)-(1+\sigma_j^2+\rho)\sigma_i^2\rho-(1+\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)}.$$

Furthermore, for future analysis, we have

$$\frac{\lambda_i}{\mu+\lambda_i} = \frac{2-c_{p,i}}{(1-c_{p,i})(2-c_{p,j})} = \frac{1+\sigma_i^2+\rho}{\Gamma_i(1+\sigma_j^2+\rho)(1-c_{p,i})} = \frac{1+\rho}{(1+\sigma_j^2+\rho)\Gamma_i c_{s,i}}.$$

$$Var(E[\theta_i|s_i,p]-p) = \frac{\Gamma_i^2 c_{s,i}^2 Var((1+\sigma_j^2+\rho)s_i-(1+\sigma_i^2+\rho)s_j)}{((1+\sigma_i^2+\rho)+(1+\sigma_j^2+\rho)\Gamma_i)^2}.$$

The equilibrium utility is derived as follows:

$$E[u_i;(i,j)] = \frac{1}{2\mu}\frac{\mu(\mu+2\lambda_i)}{(\mu+\hat{\lambda}_i)^2}Var(E[\theta_i|s_i,p]-p),$$

where

$$\begin{aligned}\frac{\mu(\mu+2\lambda_i)}{(\mu+\lambda_i)^2} &= 1-\Big(\frac{(1+\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)}{(\Gamma_i(1+\sigma_j^2-\rho^2)-\sigma_i^2\rho)(1+\sigma_j^2+\rho)}\Big)^2,\\ Var(E[\theta_i|s_i,p]-p) &= \sigma_\theta^2\frac{(\Gamma_i(1+\sigma_j^2-\rho^2)-\sigma_i^2\rho)^2((1+\sigma_i^2+\rho)+(1+\sigma_j^2+\rho))}{((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)((1+\sigma_i^2+\rho)+\Gamma_i(1+\sigma_j^2+\rho))^2}.\end{aligned}$$

By plugging the closed form solution of Γ_i into the two components in the utility, we get

$$\begin{aligned}\frac{\mu(\mu+2\lambda_i)}{(\mu+\lambda_i)^2} &= 1-\Big(\frac{(1+\rho)(2((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+\rho(\sigma_j^2-\sigma_i^2))}{2(1-\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+4\rho(1-\rho)(1+\sigma_i^2+\rho)+\rho(3-\rho)(\sigma_j^2-\sigma_i^2)}\Big)^2,\\ Var(E[\theta_i|s_i,p]-p) &= \sigma_\theta^2\frac{((1+\sigma_i^2+\rho)+(1+\sigma_j^2+\rho))}{16((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)(1+\sigma_i^2+\rho)^2(1+\sigma_j^2+\rho)^2}\\ &\quad\times(2(1-\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+4\rho(1-\rho)(1+\sigma_i^2+\rho)+\rho(3-\rho)(\sigma_j^2-\sigma_i^2))^2.\end{aligned}$$

Let us take derivatives of liquidity and learning effect with respect to the counterparty j's noise variance σ_j^2. The liquidity component in player i's utility increases as σ_j^2 increases:

$$\frac{\partial}{\partial\sigma_j^2}\frac{\lambda_i}{\mu+\lambda_i} = \frac{4\rho(1+\rho)(1+\sigma_i^2+\rho)((1-\rho)(1+\sigma_i^2+\rho)-\rho\sigma_i^2)}{(2(1-\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+4\rho(1-\rho)(1+\sigma_i^2+\rho)+\rho(3-\rho)(\sigma_j^2-\sigma_i^2))^2} < 0,$$

and thus, $\frac{\partial}{\partial\sigma_j^2}\frac{\mu(\mu+2\lambda_i)}{(\mu+\lambda_i)^2} = -\frac{2\lambda_i}{\mu+\lambda_i}\frac{\partial}{\partial\sigma_j^2}\big(\frac{\lambda_i}{\mu+\lambda_i}\big) > 0$. The learning component in player i's utility decreases as σ_j^2 increases:

$$\frac{\partial Var(E[\theta_i|s_i,p]-p)}{\partial\sigma_j^2} = \frac{-(1+\sigma_i^2+\rho)^2 Var(E[\theta_i|s_i,p]-p)}{((1+\sigma_i^2+\rho)+(1+\sigma_j^2+\rho))((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)}$$
$$+\frac{2\rho(1+\rho)(1+\sigma_i^2+\rho)}{(1+\sigma_j^2+\rho)(2(1-\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)+4\rho(1-\rho)(1+\sigma_i^2+\rho)+\rho(3-\rho)(\sigma_j^2-\sigma_i^2))} < 0.$$

This proves the trade-off between liquidity and learning effects with respect to the counterparty j's information precision $\phi_j = 1/\sigma_j^2$.

Lastly, by multiplying the two components, the equilibrium utility $E[u_i]$ is

$$E[u_i] = \frac{\sigma_\theta^2}{2\mu}\frac{-\rho((1+\sigma_i^2+\rho)+(1+\sigma_j^2+\rho))(1+\sigma_j^2-\rho^2)((1+\sigma_j^2+\rho)\sigma_i^2+(1+\sigma_i^2+\rho)\sigma_j^2)}{2(1+\sigma_i^2+\rho)((1+\sigma_i^2)(1+\sigma_j^2)-\rho^2)(1+\sigma_j^2+\rho)^2}. \quad (21)$$

By taking a derivative of $E[u_i]$ with respect to σ_j^2, we get

$$\frac{\partial E[u_i]}{\partial\sigma_j^2}\frac{1}{E[u_i]} = \frac{-2(\sigma_i^2\sigma_j^2-(1+\rho)^2)(\sigma_i^2+1+\rho)}{((1+2\sigma_i^2+\rho)\sigma_j^2+(1+\rho)\sigma_i^2)(\sigma_j^2+(\sigma_i^2+2+2\rho))(\sigma_j^2+1+\rho)}$$
$$+\frac{\sigma_i^2\rho^2}{((1+\sigma_i^2)\sigma_j^2+(1+\sigma_i^2-\rho^2))(\sigma_j^2+(1-\rho^2))}.$$

When the noise variance of the counterparty's signal σ_j^2 is small such that $\sigma_i^2\sigma_j^2-(1+\rho)^2<0$, the expected utility $E[u_i]$ increases as σ_j^2 increases, i.e., $\frac{\partial E[u_i]}{\partial\sigma_j^2}>0$. When σ_j^2 is sufficiently large, $E[u_i]$ decreases as σ_j^2 increases, i.e., $\frac{\partial E[u_i]}{\partial\sigma_j^2}<0$. More precisely, $\frac{\partial E[u_i]}{\partial\sigma_j^2}<0$ if and only if $g(\sigma_j^2)<0$, when $g(\sigma_j^2)$ is defined as follows for given σ_i^2 and ρ:

$$g(\sigma_j^2) := -2(\sigma_i^2\sigma_j^2-(1+\rho)^2)(\sigma_i^2+1+\rho)((1+\sigma_i^2)\sigma_j^2+(1+\sigma_i^2-\rho^2))(\sigma_j^2+(1-\rho^2))$$
$$+\sigma_i^2\rho^2((1+2\sigma_i^2+\rho)\sigma_j^2+(1+\rho)\sigma_i^2)(\sigma_j^2+(\sigma_i^2+2+2\rho))(\sigma_j^2+1+\rho). \quad (22)$$

From the definition of $g(\sigma_j^2)$,

$$g(0) = 2(1+\rho)^3(1-\rho)(\sigma_i^2+1+\rho)(1+\sigma_i^2-\rho^2)+(1+\rho)^2\rho^2\sigma_i^4(\sigma_i^2+2+2\rho)>0;$$
$$\lim_{\sigma_j^2\to\infty}\frac{g(\sigma_j^2)}{\sigma_j^6} = -2\sigma_i^2(1+\sigma_i^2+\rho)(1+\sigma_i^2-\rho^2)-\sigma_i^2\rho^2(1+\rho)<0.$$

In addition, let us denote the coefficients for the cubic function $g(\sigma_j^2)$ by $\{b_3, b_2, b_1, b_0\}$:

$$\begin{aligned} g(\sigma_j^2) &:= b_3\sigma_j^6 + b_2\sigma_j^4 + b_1\sigma_j^2 + b_0 \\ &:= -[2(1+\sigma_i^2+\rho)(1+\sigma_i^2-\rho^2)+\rho^2(1+\rho)]\sigma_i^2\sigma_j^6 \\ &\quad -(1+\rho)[4(1-\rho)\sigma_i^6 + 2(3-3\rho-5\rho^2)\sigma_i^4 - 3\rho(1+\rho)(2+\rho)\sigma_i^2 - 2(1+\rho)^2]\sigma_j^4 \\ &\quad +(1+\rho)[(-2+2\rho+3\rho^2)\sigma_i^6 + 2(1+\rho)\rho(3+2\rho)\sigma_i^4 + 2(1+\rho)^2(3-\rho^2)\sigma_i^2 + 4(1+\rho)^3(1-\rho)]\sigma_j^2 \\ &\quad +2(1+\rho)^3(1-\rho)(1+\sigma_i^2+\rho)(1+\sigma_i^2-\rho^2) + \rho^2(1+\rho)^2(\sigma_i^2+2+2\rho)\sigma_i^4. \end{aligned}$$

Then, $b_2 > 0$ implies $b_1 > 0$, because

$$\begin{aligned} \frac{-b_2}{(1+\rho)\sigma_j^4} &= 4(1-\rho)\sigma_i^6 + 2(3-3\rho-5\rho^2)\sigma_i^4 - 3\rho(1+\rho)(2+\rho)\sigma_i^2 - 2(1+\rho)^2 < 0 \\ \frac{b_1}{(1+\rho)\sigma_j^2} &= (-2+2\rho+3\rho^2)\sigma_i^6 + 2(1+\rho)\rho(3+2\rho)\sigma_i^4 + 2(1+\rho)^2(3-\rho^2)\sigma_i^2 + 4(1+\rho)^3(1-\rho). \end{aligned}$$

This shows that there exists a unique positive solution $\tilde{\sigma}_j^2(\sigma_i^2, \rho) > 0$ such that $g(\sigma_j^2) = 0$ holds for given $\sigma_i^2 > 0$ and $\rho < 0$. Hence, $\frac{\partial E[u_i]}{\partial \sigma_j^2} > 0$ if and only if

$$\sigma_j^2 < \widetilde{\sigma}_j^2(\sigma_i^2, \rho).$$

When the information precision of the counterparty is large, the expected utility $E[u^i]$ increases as $\phi_j = 1/\sigma_j^2$ increases (i.e., σ_j^2 decreases). If $\phi_j = 1/\sigma_j^2$ is small, the expected utility $E[u_i]$ decreases as ϕ_j increases. ■

Proof of Theorem 3 (Market Structure with Heterogeneous Precision). Suppose that there are two groups of traders in the market: buyers and sellers. The correlation between two traders is $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \rho_0 > 0$ if traders i and j are both buyers or both sellers. The correlation $Corr(\tilde{\theta}_i, \tilde{\theta}_j) = \rho < 0$ if one trader is a buyer and the other trader is a seller. I parameterize these correlations $\bar{\rho}_0 = \omega_{cv}^2$ and $\rho = \omega_{cv}^2 - \omega_{iv}^2 = 2\omega_{cv}^2 - 1$, as described in Example 1.

The proof consists of two steps: (i) there exists a unique $\check{\sigma}^2(\rho_{ij})$ such that $\check{\sigma}^2 = \widetilde{\sigma}_j^2(\check{\sigma}^2, \rho_{ij})$, (ii) there exists $\overline{I}$ and $\overline{\omega}_{cv}(I)$ such that $\check{\sigma}^2(\rho_{ij}) < \hat{\sigma}^2(I, \bar{\rho}, \rho_{ij})$ for any $\omega_{cv} > \overline{\omega}_{cv}(I)$ and $I > \overline{I}$. By (i) and (ii), an informed trader who would prefer an less informed traders in the over-the-counter market, i.e., $\phi_I > 1/\check{\sigma}^2$, chooses to enter the centralized market rather than the over-the-counter market, i.e., $\phi_I > 1/\check{\sigma}^2 \geq 1/\hat{\sigma}^2$. (iii) Lastly, $\overline{\omega}_{cv}(I) \in [0, \frac{1}{2})$ is decreasing in I, which implies that the condition on $(I, \bar{\rho}, \rho_{ij})$ to satisfy $\check{\sigma}^2(\rho_{ij}) < \hat{\sigma}^2(I, \bar{\rho}, \rho_{ij})$ is general.

(i) $\widetilde{\sigma}_j^2(\sigma_i^2, \rho)$ is the unique solution of $g(\sigma_j^2) = 0$, where $g(\cdot)$ is defined by (22). Plugging $\sigma_i^2 = \sigma_j^2$ into $g(\sigma_j^2) = 0$ shows that the solution $\check{\sigma}^2$ of equation $\check{\sigma}^2 = \widetilde{\sigma}_j^2(\sigma_i^2, \rho)$ must satisfy

$$0 = \sigma^6 + (1-3\rho)(1+\rho)\sigma^4 - (1-\rho^2)(1+2\rho)\sigma^2 - (1-\rho^2)^2. \tag{23}$$

Since $(1-3\rho)(1+\rho) > 0$, by the Descartes' Sign Rule, the above equation has a unique positive solution $\check{\sigma}^2 > 0$.

(ii) Now, let us a find a sufficient and necessary condition for $\check{\sigma}^2(\rho) < \hat{\sigma}^2(I, \bar{\rho}, \rho_{ij})$. By the definition of $\hat{\sigma}^2(I, \bar{\rho}, \rho_{ij})$ in equation (17) and the monotonicity of $f(\cdot)$, it is sufficient to find a sufficient and necessary condition for $f(\check{\sigma}^2) > 0$. The implicit equation (23) of $\check{\sigma}^2$ simplifies $f(\check{\sigma}^2)$ into a quadratic formula: $f(\check{\sigma}^2) = c_2\check{\sigma}^4 + c_1\check{\sigma}^2 + c_0$. If there is no solution of $c_2x^2 + c_1x + c_0 = 0$ and $c_2 > 0$, then $f(\check{\sigma}^2) > 0$ is satisfied. When there exist two solutions of the quadratic equation $c_2x^2 + c_1x + c_0 = 0$ and they are denoted by $x_1 \leq x_2$, $f(\check{\sigma}^2) > 0$ is satisfied if and only if

$$c_2 < 0 \text{ and } x_1 < \check{\sigma}^2 < x_2; \text{ or } c_2 > 0 \text{ and } \check{\sigma}^2 < x_1; \text{ or } c_2 > 0 \text{ and } \check{\sigma}^2 > x_2.$$

By plugging two solution x_1 and x_2 into equation (23), the sufficient and necessary condition on $f(\check{\sigma}^2) > 0$ is characterized.

With the market choice between the centralized and over-the-counter markets is endogenized, any trader who cannot be matched in the over-the-counter market chooses the centralized markets. Hence, the number of buyers and sellers in the over-the-counter market is the same in equilibrium. From Theorem 1, a sufficiently less informed traders choose the over-the-counter market while a sufficiently more informed traders choose the centralized market. ∎

B Additional Materials

B.1 Equilibrium Representation in a Market

For a given price impact $\lambda_i > 0$ and inference $E[\tilde{\theta}_i|s_i, p] = c_{\theta,i}E[\tilde{\theta}_i] + c_{s,i}s_i + c_{p,i}p$, trader i's first order condition gives his best response, i.e. demand schedule.

$$q_i = \frac{E[\tilde{\theta}_i|s_i, p] - p}{\mu + \lambda_i} = \frac{c_{\theta,i}E[\tilde{\theta}_i] + c_{s,i}s_i - (1 - c_{p,i})p}{\mu + \lambda_i}.$$

With positive price impacts, the second order condition holds for all i:

$$-\mu - 2\lambda_i < 0.$$

The market clearing condition $\sum_i q_i(\cdot) = 0$ determines equilibrium price from the demand function.

$$p = (\sum_i \frac{1 - c_{p,i}}{\mu + \lambda_i})^{-1} \sum_i \frac{c_{\theta,i}E[\tilde{\theta}_i] + c_{s,i}s_i}{\mu + \lambda_i}.$$

Since the price is a linear function of traders' private information $\{s_i\}_i$, it follows a normal distribution as well as the signals. This Gaussian-linear structure allows us to use the Projection Theorem in order to derive traders' conditional expectation on asset value. First, the unconditional expectation of price is equal to $E[\tilde{\theta}_i] = E[\tilde{\theta}]$ which is same across traders. It results in $c_{\theta,i} + c_{s,i} + c_{p,i} = 1$ for any i. The inference coefficient $\{c_{s,i}, c_{p,i}\}$ is

$$\begin{bmatrix} c_{s,i} \\ c_{p,i} \end{bmatrix} = \begin{bmatrix} Var(s_i) & Cov(s_i, p) \\ Cov(s_i, p) & Var(p) \end{bmatrix}^{-1} \begin{bmatrix} Cov(\tilde{\theta}_i, s_i) \\ Cov(\tilde{\theta}_i, p) \end{bmatrix}. \tag{24}$$

We denote $\sigma_i^2 = \sigma_{i,\varepsilon}^2/\sigma_\theta^2$, the relative variance of noise in private information compared to variance of asset values. By plugging the following variance and covariance of (s_i, p) into equation (24),

$$\begin{aligned} Var(p) &= (\sum_i \frac{1 - c_{p,i}}{\mu + \lambda_i})^{-2} \Big(\frac{c_{s,i}}{\mu + \lambda_i}\Big)_i \cdot (\sigma_\theta^2 \Sigma + diag(\sigma_{i,\varepsilon}^2)_i) \Big(\frac{c_{s,i}}{\mu + \lambda_i}\Big)_i, \\ Cov(s_i, p) &= (\sum_j \frac{1 - c_{p,j}}{\mu + \lambda_j})^{-1} \Big(\frac{c_{s,j}}{\mu + \lambda_j}\Big)_j \cdot \Big(\sigma_\theta^2 \rho_{ij} + \sigma_{i,\varepsilon}^2 \mathbf{1}_{j=i}\Big)_j, \end{aligned}$$

we get a fixed point problem for the inference coefficients $\{c_{s,i}, c_{p,i}\}_i$,

$$c_{s,i} = \frac{\sum_{j,k} \frac{c_{s,j}c_{s,k}(\rho_{jk} + \sigma_j^2 \mathbf{1}_{j=k}) - c_{s,j}c_{s,k}\rho_{ij}(\rho_{ik} + \sigma_i^2 \mathbf{1}_{i=k})}{(\mu + \lambda_j)(\mu + \lambda_k)}}{\sum_{j,k} \frac{(1+\sigma_i^2)c_{s,j}c_{s,k}(\rho_{jk} + \sigma_j^2 \mathbf{1}_{j=k}) - c_{s,j}c_{s,k}(\rho_{ij} + \sigma_i^2 \mathbf{1}_{i=j})(\rho_{ik} + \sigma_i^2 \mathbf{1}_{i=k})}{(\mu + \lambda_j)(\mu + \lambda_k)}}, \quad \forall i, \tag{25}$$

$$c_{p,i} = \frac{\sum_j \frac{1 - c_{p,j}}{\mu + \lambda_j} \sum_j \frac{(1+\sigma_i^2)c_{s,j}\rho_{ij} - c_{s,j}(\rho_{ij} + \sigma_i^2 \mathbf{1}_{i=j})}{\mu + \lambda_j}}{\sum_{j,k} \frac{(1+\sigma_i^2)c_{s,j}c_{s,k}(\rho_{jk} + \sigma_j^2 \mathbf{1}_{j=k}) - c_{s,j}c_{s,k}(\rho_{ij} + \sigma_i^2 \mathbf{1}_{i=j})(\rho_{ik} + \sigma_i^2 \mathbf{1}_{i=k})}{(\mu + \lambda_j)(\mu + \lambda_k)}} \quad \forall i. \tag{26}$$

In addition, the price impacts are characterized by

$$\lambda_i = \Big(\sum_{j \neq i} \frac{1 - c_{p,j}}{\mu + \lambda_j}\Big)^{-1}, \quad \forall i, \tag{27}$$

for a given inference coefficients $\{c_{p,j}\}_j$. Equations (25) - (27) solves $\{c_{s,i}, c_{p,i}, \lambda_i\}_i$, and thus, characterizes equilibrium.

With the equilibrium characterization, a trader's indirect interim utility is

$$E[u_i|s_i, p] = -pq_i + E[\tilde{\theta}_i|s_i, p]q_i - \frac{\mu}{2}q_i^2 = \frac{\mu + 2\lambda_i}{2(\mu + \lambda_i)^2}(E[\tilde{\theta}_i|s_i, p] - p)^2,$$

and his ex-ante utility is

$$E[u_i] = \frac{\mu + 2\lambda_i}{2(\mu + \lambda_i)^2} E\big[(E[\tilde{\theta}_i|s_i, p] - p)^2\big] = \frac{\mu + 2\lambda_i}{2(\mu + \lambda_i)^2} Var(E[\tilde{\theta}_i|s_i, p] - p).$$

The last inequality holds because $E[E[\tilde{\theta}_i|s_i, p] - p] = 0$ when $E[\tilde{\theta}_i] = E[\tilde{\theta}]$ for all i.

B.2 Proofs of Lemmas

Lemma 1 *For any $I \geq 3$ and $\bar{\rho} \geq 0 > \rho_{ij}$, if $a_n > 0$ then $a_{n-1} > 0$ for any $n = 1, \cdots, 5$. In addition, $a_0 > 0$ always holds.*

Proof of Lemma 1. From the definition of function $f(\sigma^2) := a_5\sigma^{10} + a_4\sigma^8 + a_3\sigma^6 + a_2\sigma^4 + a_1\sigma^2 + a_0$ in equation (18), each coefficient is characterized by

$$\begin{aligned}
a_5 &:= X + Y; \\
a_4 &:= X((1 + (I-1)\bar{\rho})(1 - \bar{\rho}) + 2(1 + \rho_{ij}) + (1 - \rho_{ij})) \\
&\quad +Y(1 - \rho_{ij}^2 + 2(1 + (I-1)\bar{\rho}) + (1 - \bar{\rho})) \\
&\quad +(I-2)(1 + (I-1)\bar{\rho})(1 - \bar{\rho}); \\
a_3 &:= X((1 + (I-1)\bar{\rho})(1 - \bar{\rho})(2(1 + \rho_{ij}) + (1 - \rho_{ij})) + (1 + \rho_{ij})^2 + 2(1 - \rho_{ij}^2)) \\
&\quad +Y((1 - \rho_{ij}^2)(2(1 + (I-1)\bar{\rho}) + (1 - \bar{\rho})) + (1 + (I-1)\bar{\rho})^2 + 2(1 + (I-1)\bar{\rho})(1 - \bar{\rho})) \\
&\quad +(I-2)(1 + (I-1)\bar{\rho})(1 - \bar{\rho})((1 + (I-1)\bar{\rho})(1 - \bar{\rho}) + 2(1 + \rho_{ij}) + (1 - \rho_{ij})); \\
a_2 &:= X((1 + (I-1)\bar{\rho})(1 - \bar{\rho})((1 + \rho_{ij})^2 + 2(1 - \rho_{ij}^2)) + (1 + \rho_{ij})(1 - \rho_{ij}^2)) \\
&\quad +Y((1 - \rho_{ij}^2)(2(1 + (I-1)\bar{\rho})(1 - \bar{\rho}) + (1 + (I-1)\bar{\rho})^2) + (1 + (I-1)\bar{\rho})^2(1 - \bar{\rho})) \\
&\quad +(I-2)(1 + (I-1)\bar{\rho})(1 - \bar{\rho})((1 + (I-1)\bar{\rho})(1 - \bar{\rho})(2(1 + \rho_{ij}) + (1 - \rho_{ij})) + (1 + \rho_{ij})^2 + 2(1 - \rho_{ij}^2)); \\
a_1 &:= X(1 + (I-1)\bar{\rho})(1 - \bar{\rho})(1 + \rho_{ij})(1 - \rho_{ij}^2) \\
&\quad +Y(1 - \rho_{ij}^2)(1 + (I-1)\bar{\rho})^2(1 - \bar{\rho}) \\
&\quad +(I-2)(1 + (I-1)\bar{\rho})(1 - \bar{\rho})((1 + (I-1)\bar{\rho})(1 - \bar{\rho})((1 + \rho_{ij})^2 + 2(1 - \rho_{ij}^2)) + (1 + \rho_{ij})(1 - \rho_{ij}^2)); \\
a_0 &:= (I-2)(1 - \bar{\rho})^2(1 + (I-1)\bar{\rho})^2(1 - \rho_{ij})(1 + \rho_{ij})^2 > 0.
\end{aligned}$$

where $X := ((I-1)(1-\bar{\rho}) - (1+(I-1)\bar{\rho}))$ and $Y := (I-1)((1+\rho_{ij}) - (1-\rho_{ij}))$. Each coefficient a_n is a linear function of (X, Y). It is worth to remark that X and Y are the determinants of the sign of a_n for each n, because X and Y can be positive or negative while all other terms (weights on X and Y and the constant term) are always positive. In particular, $-I < X \leq I-2$ and $-2(I-1) < Y < 0$ since $\bar{\rho} \geq 0$ and $\rho_{ij} < 0$. The coefficient of zero power a_0 does not contain X or Y and it is always positive.

We now show that $a_5 = X + Y > 0$ implies $a_4 > 0$. The weights on (X, Y) in a_4 are positive and the weight on X is smaller than the one on Y:

$$\begin{aligned} & ((1+(I-1)\bar{\rho})(1-\bar{\rho}) + 2(1+\rho_{ij}) + (1-\rho_{ij})) - (1-\rho_{ij}^2 + 2(1+(I-1)\bar{\rho}) + (1-\bar{\rho})) \\ = \quad & -(I-1)\bar{\rho}(1+\bar{\rho}) + \rho_{ij}(1+\rho_{ij}) < 0. \end{aligned}$$

This implies that a boundary line (X, Y) such that $a_5 = 0$ has a more negative slope $\frac{dY}{dX}$ than the slope of a boundary line (X, Y) such that $a_4 = 0$[20]:

$$\frac{dY}{dX}\Big|_{a_5=0} = -1 \quad < \quad \frac{dY}{dX}\Big|_{a_4=0} = -\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho}) + 2(1+\rho_{ij}) + (1-\rho_{ij}))}{(1-\rho_{ij}^2 + 2(1+(I-1)\bar{\rho}) + (1-\bar{\rho}))}. \tag{28}$$

Furthermore, the boundary point $(\bar{\rho} = 0, \rho_{ij} = -\frac{I-2}{2(I-1)})$ such that $a_5 = 0$ and $\bar{\rho} \geq 0$ satisfies

$$a_4\big(\bar{\rho} = 0, \rho_{ij} = -\frac{I-2}{2(I-1)}\big) = (I-2)(1 - \frac{I-2}{2(I-1)} + (\frac{I-2}{2(I-1)})^2) \; > \; 0.$$

This implies that the intersection of two lines (X, Y) such that $a_5 = 0$ and $a_4 = 0$ is unique and it is outside of support $\rho_{ij} < 0$. Hence, $a_5 > 0$ implies $a_4 > 0$.

Similarly, we show that the ratio of the weight on X relative to the weight on Y in a_3 is smaller than the ratio in a_4:

$$\begin{aligned} & \frac{dY}{dX}\Big|_{a_4=0} = -\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho}) + 2(1+\rho_{ij}) + (1-\rho_{ij}))}{(1-\rho_{ij}^2 + 2(1+(I-1)\bar{\rho}) + (1-\bar{\rho}))} \\ & < \frac{dY}{dX}\Big|_{a_3=0} = -\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho})(2(1+\rho_{ij}) + (1-\rho_{ij})) + (1+\rho_{ij})^2 + 2(1-\rho_{ij}^2))}{((1-\rho_{ij}^2)(2(1+(I-1)\bar{\rho}) + (1-\bar{\rho})) + (1+(I-1)\bar{\rho})^2 + 2(1+(I-1)\bar{\rho})(1-\bar{\rho}))}, \end{aligned}$$

or equivalently,

$$1 < \frac{((1-\rho_{ij}^2)(3+(2I-3)\bar{\rho}) + (1+(I-1)\bar{\rho})(3+(I-3)\bar{\rho}))((1+(I-1)\bar{\rho})(1-\bar{\rho}) + 3 + \rho_{ij})}{(1-\rho_{ij}^2 + 3 + (2I-3)\bar{\rho})((1+(I-1)\bar{\rho})(1-\bar{\rho})(3+\rho_{ij}) + (1+\rho_{ij})(3-\rho_{ij}))}. \tag{29}$$

[20] The boundary of $a_n > 0$, i.e., $\{(X, Y); a_n = 0\}$ is not a straight line on the two-dimensional plane of (X, Y): (X, Y) are the functions of $(\bar{\rho}, \rho_{ij})$ that also determines the weights on (X, Y) and the constant term in each coefficient a_n. However, the non-linearity of $\{(X, Y); a_n = 0\}$ does not affect the argument in the proof, because the non-constant slope $\frac{dY}{dX}\big|_{a_n=0}$ satisfies the inequality (28) for any (X, Y) in the support, or equivalently, for any $\bar{\rho} \geq 0$ and $\rho_{ij} < 0$.

When $\rho_{ij} = 0$, this inequality (29) holds for any $\bar{\rho} \geq 0$ and for any I.

$$1 < 1 + \frac{(1+(I-1)\bar{\rho})(1-\bar{\rho})(I-1)(I-3)\bar{\rho}^2 + 3((1+(I-1)\bar{\rho})^2 - 1)}{3(4+(2I-3)\bar{\rho})((1+(I-1)\bar{\rho})(1-\bar{\rho})+1)}.$$

Since the right-hand-side of inequality (29) is decreasing in $\rho_{ij} < 0$, for given $\bar{\rho} \geq 0$ and I:

$$\begin{aligned}\frac{\partial rhs}{\partial \rho_{ij}} \quad \propto \quad & \frac{-2\rho_{ij}}{1-\rho_{ij}^2 + (1+(I-1)\bar{\rho})\frac{3+(I-3)\bar{\rho}}{3+(2I-3)\bar{\rho}}} - \frac{-2\rho_{ij}}{1-\rho_{ij}^2 + 3 + (2I-3)\bar{\rho}} \\ & - \frac{(1+(I-1)\bar{\rho})(1-\bar{\rho}) - 1 - 3\rho_{ij}}{(1+(I-1)\bar{\rho})(1-\bar{\rho})(3+\rho_{ij}) + (1+\rho_{ij})(3-\rho_{ij})} \\ & + \frac{1}{((1+(I-1)\bar{\rho})(1-\bar{\rho}) + 3 + \rho_{ij})} - \frac{1}{(1+(I-1)\bar{\rho})(1-\bar{\rho}) + (1+\rho_{ij})\frac{3-\rho_{ij}}{3+\rho_{ij}}} \quad \leq 0.\end{aligned}$$

the inequality (29) holds for any $\rho_{ij} > 0$, $\bar{\rho} \geq 0$, and for any I. Thus, the boundary line (X, Y) such that $a_4 = 0$ has a slope $\frac{dY}{dX}$ more negative than the slope of the boundary line (X, Y) for $a_3 = 0$. Furthermore, at a point $(\bar{\rho}, \rho_{ij})$ on the boundary of $a_4 > 0$ and $\bar{\rho} \geq 0$, the value of a_3 evaluated at such $(\bar{\rho}, \rho_{ij})$ is positive: with ρ_{ij} such that $a_4(\bar{\rho} = 0, \rho_{ij}) = (I-2)(5+\rho_{ij}) + 2(I-1)\rho_{ij}(4-\rho_{ij}^2) = 0$,

$$a_3(\bar{\rho} = 0, \rho_{ij}) \quad = \quad -\rho_{ij}\{2(I-1)(\rho_{ij} + \frac{I-2}{4(I-1)})^2 + \frac{15I^2 - 12I - 4}{8(I-1)}\} \;>\; 0.$$

Hence, $a_4 > 0$ implies $a_3 > 0$.

The similar argument shows that $a_3 > 0$ implies $a_2 > 0$. The boundary for $a_3 > 0$, i.e., the line (X, Y) such that $a_3 = 0$, has a more negative slope $\frac{dY}{dX}$ than the slope of the boundary for $a_2 > 0$, if and only if $(\rho_{ij}, \bar{\rho})$ satisfies the following inequality for given I:

$$\begin{aligned} & -\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho})(2(1+\rho_{ij}) + (1-\rho_{ij})) + (1+\rho_{ij})^2 + 2(1-\rho_{ij}^2))}{((1-\rho_{ij}^2)(2(1+(I-1)\bar{\rho}) + (1-\bar{\rho})) + (1+(I-1)\bar{\rho})^2 + 2(1+(I-1)\bar{\rho})(1-\bar{\rho}))} \\ < \quad & -\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho})((1+\rho_{ij})^2 + 2(1-\rho_{ij}^2)) + (1+\rho_{ij})(1-\rho_{ij}^2))}{((1-\rho_{ij}^2)(2(1+(I-1)\bar{\rho})(1-\bar{\rho}) + (1+(I-1)\bar{\rho})^2) + (1+(I-1)\bar{\rho})^2(1-\bar{\rho}))},\end{aligned}$$

or equivalently,

$$\begin{aligned}1 \quad < \quad & \frac{((1+(I-1)\bar{\rho})(1-\bar{\rho})(3+\rho_{ij}) + (1+\rho_{ij})(3-\rho_{ij}))}{((1-\rho_{ij}^2)(3+(2I-3)\bar{\rho}) + (1+(I-1)\bar{\rho})(3+(I-3)\bar{\rho}))} \\ & \times \frac{((1-\rho_{ij}^2)(3+(I-3)\bar{\rho}) + (1+(I-1)\bar{\rho})(1-\bar{\rho}))}{((1+(I-1)\bar{\rho})(1-\bar{\rho})(3-\rho_{ij}) + (1-\rho_{ij}^2))}\frac{1+(I-1)\bar{\rho}}{1+\rho_{ij}}. \end{aligned} \tag{30}$$

The right-hand-side of inequality (30) is decreasing in $\rho_{ij} < 0$, for given $\bar{\rho} \geq 0$ and I,

$$\frac{\partial rhs}{\partial \rho_{ij}} \propto -\frac{-2\rho_{ij}}{(1-\rho_{ij}^2)+(1+(I-1)\bar{\rho})\frac{3+(I-3)\bar{\rho}}{3+(2I-3)\bar{\rho}}} + \frac{-2\rho_{ij}}{(1-\rho_{ij}^2)+(1+(I-1)\bar{\rho})\frac{1-\bar{\rho}}{3+(I-3)\bar{\rho}}} - \frac{1}{1+\rho_{ij}}$$
$$+\frac{(1+(I-1)\bar{\rho})(1-\bar{\rho})+2-2\rho_{ij}}{(1+(I-1)\bar{\rho})(1-\bar{\rho})(3+\rho_{ij})+(1+\rho_{ij})(3-\rho_{ij})} + \frac{(1+(I-1)\bar{\rho})(1-\bar{\rho})+2\rho_{ij}}{(1+(I-1)\bar{\rho})(1-\bar{\rho})(3-\rho_{ij})+(1-\rho_{ij}^2)}.$$

When $\bar{\rho}=0$, the inequality (30) holds for any $\rho_{ij} < 0$,

$$1 < 1+\frac{-\rho_{ij}(6-14\rho_{ij}-15\rho_{ij}^2+3\rho_{ij}^2(1+\rho_{ij})^2)}{3((1-\rho_{ij}^2)+1)((3-\rho_{ij})+(1-\rho_{ij}^2))(1+\rho_{ij})}.$$

Suppose that ρ_{ij} is the solution that satisfies $a_3 = 0$ and $\bar{\rho} = 0$. With ρ_{ij} such that $a_3(\bar{\rho}=0, \rho_{ij}) = (I-2)(10+4\rho_{ij}-\rho_{ij}^2)+6(I-1)\rho_{ij}(2-\rho_{ij}^2) = 0$, the coefficient a_2 is positive.

$$a_2 = -\rho_{ij}((I-2)(\rho_{ij}+1)^2+I+2) > 0.$$

In addition, when $\bar{\rho}$ is the solution that satisfies $a_3 = 0$ and $\rho_{ij} = 0$, i.e.,

$$a_3 = (I-2)(1+(I-1)\bar{\rho})^2(1-\bar{\rho})^2+6((I-2)-(I-1)\bar{\rho})(1+(I-1)\bar{\rho})(1-\bar{\rho})+3((I-2)-2(I-1)\bar{\rho}) = 0,$$

then, with such $\bar{\rho}$, we get

$$a_2 = 2(I-2)((1+(I-1)\bar{\rho})^2(1-\bar{\rho})^2-1)+4(I-1)\bar{\rho} > 0.$$

This concludes that if $a_3 > 0$ then $a_2 > 0$.

Lastly, suppose that $a_2 > 0$. The slope of boundary for $a_2 > 0$ is more negative than the slope of boundary for $a_1 > 0$, if and only if

$$-\frac{((1+(I-1)\bar{\rho})(1-\bar{\rho})((1+\rho_{ij})+2(1-\rho_{ij}))+(1-\rho_{ij}^2))}{((1-\rho_{ij}^2)(2(1-\bar{\rho})+(1+(I-1)\bar{\rho}))+(1+(I-1)\bar{\rho})(1-\bar{\rho}))}\frac{(1+\rho_{ij})}{(1+(I-1)\bar{\rho})}$$
$$< -\frac{(1+(I-1)\bar{\rho})(1-\bar{\rho})(1+\rho_{ij})(1-\rho_{ij}^2)}{(1-\rho_{ij}^2)(1+(I-1)\bar{\rho})^2(1-\bar{\rho})} = -\frac{(1+\rho_{ij})}{(1+(I-1)\bar{\rho})}.$$

The above inequality is equivalent to

$$\bar{\rho} < \frac{((I-1)-\rho_{ij})+\sqrt{((I-1)-\rho_{ij})^2-4(I-1)(2-\rho_{ij})\rho_{ij}(1-2\rho_{ij})}}{2(I-1)(2-\rho_{ij})},$$

or equivalently,

$$\rho_{ij} < \frac{(1+(I-1)\bar{\rho})(1-\bar{\rho})-\sqrt{(1+(I-1)\bar{\rho})^2(1-\bar{\rho})^2-4(2+(I-3)\bar{\rho})(I-1)\bar{\rho}(1-2\bar{\rho})}}{2(2+(I-3)\bar{\rho})}.$$

The slope $\frac{dY}{dX}|_{a_2=0}$ is more negative than $\frac{dY}{dX}|_{a_1=0}$ if and only if X is larger and Y is smaller than the cutoffs, and thus, the boundary $\{(X,Y) : a_1 = 0\}$ for $a_1 > 0$ is more convex than the boundary $\{(X,Y) : a_2 = 0\}$ for $a_2 > 0$. Furthermore, at another boundary condition $\bar{\rho} = 0$, if ρ_{ij} satisfies $a_2 = (I-2)(10 + 6\rho_{ij} - 3\rho_{ij}^2 - \rho_{ij}^3) + 2(I-1)\rho_{ij}(4 - 3\rho_{ij}^2) = 0$, then

$$2a_1 \quad = \quad -\rho_{ij}(2I + 3(I-2)\rho_{ij} + (I-4)\rho_{ij}^2) \quad > \quad 0.$$

Also, at the boundary condition $\rho_{ij} = 0$, if $\bar{\rho}$ satisfies $a_1 \propto 2((I-2) - (I-1)\bar{\rho}) + 3(I-2)(1 + (I-1)\bar{\rho})(1-\bar{\rho}) = 0$, then

$$3(I-2)a_2 \quad = \quad -8((I-1)\bar{\rho} - \frac{5}{8}(I-2))^2 - \frac{15}{8}(I-2)^2 < 0.$$

This concludes that $\{(X,Y) : a_2 \geq 0, \bar{\rho} \geq 0, \rho_{ij} < 0\}$ is a subset of $\{(X,Y) : a_1 \geq 0, \bar{\rho} \geq 0, \rho_{ij} < 0\}$, i.e., $a_2 > 0$ implies $a_1 > 0$.

Taking $I \to \infty$: Similarly to Lemma 1, we can be proved that if the coefficient of σ^{2n} is positive, the coefficient of $\sigma^{2(n-1)}$ is positive in the r.h.s. of inequality (19) for any $n = 1, 2, 3$.
When $(1-\bar{\rho})^2 + 2\rho_{ij} > 0$,

$$(1-\bar{\rho})^2(3+\rho_{ij}) + (2 - \bar{\rho} - \rho_{ij}^2)2\rho_{ij} > -(\frac{3}{4} + \bar{\rho} + (\rho_{ij} + \frac{1}{2})^2)2\rho_{ij} > 0.$$

When $(1-\bar{\rho})^2(3+\rho_{ij}) + (1-\bar{\rho})2\rho_{ij} + (1-\rho_{ij}^2)2\rho_{ij} > 0$,

$$(1-\bar{\rho})(3 + 2\rho_{ij} - \rho_{ij}^2) + (1-\rho_{ij}^2)2\rho_{ij} > (1-\bar{\rho})(\bar{\rho}(3+\rho_{ij}) - \rho_{ij}(1+\rho_{ij})) > 0.$$

The last coefficient $(1-\bar{\rho})^2(1-\rho_{ij})(1+\rho_{ij})^2$ is always positive. ■