Tail risk in production networks

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Abstract

This paper studies the risk of large deviations in GDP in the context of a general nonlinear production model. It derives the probability of extreme events conditional on the structure of the economy and the distribution of the shocks. Tail risk is driven by complementarities in production. Increases in interconnectedness in the presence of complementarity can simultaneously reduce the sensitivity of the economy to small shocks while increasing the sensitivity to large shocks. Tail risk is strongest in economies that display *conditional granularity*, where some sectors become highly influential following negative shocks. For a wide class of shock distributions, all crashes are identical, in the sense that they come with probability one from a particular combination of shocks, which also yields a sufficient statistic for crash risk. The analysis also characterizes what sectors are systemically risky (or conditionally granular): those that produce inputs for a large fraction of final production and have no close substitutes.

1 Introduction

Background

If utility is concave, then the events that have the largest individual impact on welfare are large declines in consumption. A large literature studies large movements in GDP, trying to understand their likelihood, their sources, and their effects on welfare and other features of the economy. Acemoglu et al. (2017), for example, show that large movements in GDP are more likely than predicted by the normal distribution, they and Gabaix (2011) suggest that such events could be caused by shocks to influential sectors or firms, and Barro (2006) studies how such movements in GDP and consumption might affect asset prices.

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One way to motivate this paper's analysis is a quantile-quantile (Q-Q) plot for GDP, which Acemoglu et al. (2017) also study. See the top panel of Figure 1. The Q-Q plot shows that while GDP is well described by a normal distribution for middle quantiles, its left tail is heavier than that of the normal distribution, while the right tail is lighter. At low quantiles, where the normal distribution would be, say, 2 standard deviations below its mean, log GDP was historically about 2.5 standard deviations lower. Conversely, where the normal distribution would be 2 standard deviations above its mean, log GDP was only 1.8 standard deviations higher. This paper studies how and when nonlinear models can generate such behavior – a heavy tail to the left and a normal or lighter tail to the right.

Contribution

The probability distribution of GDP is a function of the distributions of the fundamental shocks hitting the economy and how the economy's structure maps them into final output. What makes understanding that mapping particularly difficult is that, except in highly restricted cases, it is nonlinear. Past work has focused almost exclusively on linear models or linear approximations (or at best, in Baqaee and Farhi (2019), a quadratic approximation).[1] The goal of this paper is to understand the tails of the distribution of GDP (especially the left tail) in general nonlinear models. In particular, how does the structure of the economy determine crash risk, what types of shocks are most likely to cause crashes, and what should we expect crashes to look like?

The paper's core contribution is to answer those questions in the context of a general network production model. It shows, in closed form, how large microeconomic shocks translate into large movements in GDP. The probability distribution of the shocks and the structure of the economy then combine to determine the probability of large deviations in GDP, and the results show formally when the empirical behavior observed in Figure 1 arises. The insights gained from the analysis are significantly different from those from linear and quadratic approximations, which are ill-suited to thinking about extreme events.

As byproducts, the analysis delivers both a novel measure of *tail centrality*, measuring the ability of a shock to a single economic unit, e.g. a sector or firm, to produce a large (as opposed to small) movement in GDP, and also shows precisely what combinations of micro shocks cause crashes. These results help clarify what factors make a firm or sector systemically risky.

Methods

In production networks, economic units produce outputs using as inputs both labor

[1] Jones (2011) and Dew-Becker, Tahbaz-Salehi, and Vedolin (2021) study closed form solutions to nonlinear models, both emphasizing the importance of complementarity and how it causes GDP to be a concave function of primitive shocks.

and the products of other units. The various units interact, propagating and potentially amplifying or attenuating shocks. Importantly, the model allows for arbitrary elasticities of substitution across inputs, which determine how shocks propagate.

Theoretical analyses of structural models typically rely either closed-form solutions in highly restricted models, or on Taylor approximations that are only valid locally. That is, past work has primarily relied on Hulten's (1978) theorem, which says that sensitivity of GDP to a sector's shock depends only on the relative sales of the sector. For large shocks, though, this paper shows that, except for the case of a purely log-linear model, a Taylor series yields errors in this setting that eventually diverge to $\pm\infty$ – sales shares are no longer sufficient statistics.

This paper also studies a series expansion, but one taken at infinity. Formally, for any combination of shocks, the core tool in the analysis is a lemma showing that log GDP has a linear asymptote. The asymptotic slope describes how GDP responds to *large* rather than small shocks (see Figure 2). When combined with an assumption about the distribution of the shocks, those slopes determine the probability of large movements in GDP.

Results

The paper's key result is to show how the probability of large movements in GDP is determined by the structure of the economy. That result has a number of immediate implications. First, left tail risk is driven by complementarity in production. Crashes are more likely when inputs are complements than when they are substitutes, all else equal.

Second, it allows one to find the effect of changes in interconnectedness on tail risk. Interconnectedness increases tail risk in the presence of complementarity, while it decreases it in the presence of substitutability. It is then simple to construct examples in which increased interconnectedness increases the economy's resilience to small shocks while at the same time increasing sensitivity to large shocks.[2]

Third, the paper shows formally that if inputs to production are complements (consistent with other empirical evidence), then we observe exactly the features of the Q-Q plot in Figure 1, a high slope on the left and a low slope on the right. A quantitative version of the model, also displayed in Figure 1 (discussed in section 4.5), is able to match the data well.

For a wide range of shock distributions, all crashes are caused by a single worst-case combination of shocks. The next core question is what determines that worst-case scenario. Which sectors have the most damaging shocks? In past work, the answer has simply been whichever sectors are biggest (due to Hulten's (1978) theorem). But that answer is for linear models (or local approximations) where the relative size of sectors is constant. The

[2]See Acemoglu and Azar (2020) for related work on changes in interconnectedness in production networks.

force underlying all of this paper's results is that what really matters is the relative size of the sectors *in extreme scenarios*. A systemically important sector need not be large on average. Rather, the model shows that what matters is whether the sector would account for a significant fraction of output if it received an extreme shock. That is, whether the economy is *conditionally granular*. As an example, I show that in a fully connected and symmetrical network with N sectors, the average transmission of each sector's shocks to GDP is of order N^{-1} for small shocks, but of order 1 for large shocks. There is no granularity near steady-state, but severe granularity in tail events.

For a more realistic example, take electricity and restaurants. In normal times, those sectors are of similar size – both with Domar weights of about 0.025 – which in a linear approximation would imply that they have similar (and small) effects on GDP. But one lesson of Covid was that shutting down restaurants is not catastrophic for GDP,[3] whereas one might expect that a significant reduction in available electricity would have strongly negative effects – and that those effects would be convex in the size of the decline in available power. Electricity is systemically important not because it is important in good times, but because it *would be* important in bad times.

The fact that it is universal inputs to production that have the largest systemic risk has an important corollary, which is that tail risk in the economy can increase when the production network becomes more interconnected. In the case where production is complementary, every additional link in the network raises tail risk. As a recent practical example, consider the case of semiconductors. Obviously the advent of computer technology has been massively beneficial to the economy, but at the same time it has made essentially every sector sensitive to the supply of semiconductors, making that sector surprisingly influential following a recent negative shock.

The last section of the paper gives a first-pass estimate of tail centrality – the effect on GDP of a large shock to each sector – in the data. The basic finding is that tail centrality and sales shares – which measure local centrality – are only about 60 percent correlated, with numerous sectors with small sales shares having large tail centralities, while many sectors with large sales shares have small tail centralities.

Related literature

As discussed above, this paper contributes to the literature on large movements in aggregate output, including Rietz (1988), Barro (2006), and Acemoglu et al. (2017). Barrot and Sauvagnat (2016) and Carvalho et al. (2020) study the effects of large shocks to individual firms due to natural disasters on production. See also related work by Fujiy,

[3] Consumer spending on food services and accommodations fell by 40 percent, or \$403 billion between 2019Q4 and 2020Q2. Spending at movie theaters fell by 99 percent.

Ghose, and Khanna (2021). Acemoglu et al. (2017) study how heavy tails in GDP can be generated by the production structure of the economy in a linear setting, while Liu and Tsyvinski (2021) consider the effects of shocks in a dynamic but still linear model.

The paper's framework builds on the literature on production networks, going back to Long and Plosser (1983).[4] The closest link is to Baqaee and Farhi (2019), who study higher moments of output in the same nonlinear framework. The key distinction is that that paper analyzes output based on a second-order Taylor series approximation. Its analysis is therefore useful for shedding light on how the structure of the economy generates asymmetry in the *local* response of GDP to small shocks, but it does not speak to large deviations as its approximation has infinitely large errors in the tails. There are also a number of recent papers on the propagation of shocks and distortions in production networks, both empirical and theoretical.[5] A contribution of this paper is to potentially giving a way for work in those areas to get analytic approximations where they were previously unavailable.

Taschereau-Dumouchel (2021) studies an endogenous production network and its effects on the distribution of GDP. There is also a related literature in international trade on endogenous value chains (e.g. Alfaro et al. (2019)).

This paper is also related to recent work on asymmetry in GDP, including Dew-Becker, Tahbaz-Salehi, and Vedolin (2021), Dupraz, Nakamura, and Steinsson (2020), Ilut, Kehrig, and Schneider (2018), and others. While that work primarily focuses on asymmetry at the business cycle frequency, and hence for fairly "typical" sorts of shocks, this paper analyzes the determinants of asymmetry in the tails, giving, for example, a potential explanation why there are occasionally extremely large declines in output but never equally large increases. Asymmetry appears when production features complementarity, and there is significant empirical evidence that production is complementary.[6]

Outline

The remainder of the paper is organized as follows. Section 2 describes the basic structure

[4]That literature is large and work has studied features of networks, e.g. what makes a particular sector or firm central and what determines the behavior of GDP. For recent representative work, in addition to other work discussed, see Liu and Tsyvinski (2021), vom Lehn and Winberry (2021), La'O and Tahbaz-Salehi (2021), and Bigio and La'O (2020).

[5]Liu (2019), Bigio and La'O (2020), and Boehm and Oberfield (2020) study the propagation of distortions in production networks. Costello (2020) and Alfaro, Garcia-Santana, and Moral-Benito (2021) study the propagation of credit supply shocks. Gofman, Segal, and Wu (2020) study the propagation of technology shocks and their effects on firm risk.

[6]Atalay (2017) and Atalay et al. (2018) find evidence for complementarity in sectoral data, while Barrot and Sauvagnat (2016), Boehm, Flaaen, and Pandalai-Nayar (2019), Carvalo et al. (2021), and Peter and Ruane (2020) find complementarity in firm-level data (though in the latter pair of papers the complementarity is only between factors (e.g. labor) and intermediate inputs, not among intermediate. The estimates appear to depend on the time horizon considered for responses (see Ruhl (2008)).

of the economy. Section 3 presents the result on approximating output in terms of the exogenous shocks. That result is then used for deriving the main results on tail risk in section 4. Section 5 uses those results to characterize the risk associated with shocks to different sectors – developing the notion of tail centrality – and section 6 presents empirical estimates of that risk based on the US input-output matrix. Section 7 concludes.

2 Structure of the economy

The model is static and frictionless and takes the form of a standard nested CES production network as studied in Baqaee and Farhi (2019). There are N production units each producing a distinct good. A unit might represent a sector, or a firm, or even just part of a sector or firm, though I will sometimes refer to them as "sectors" as a standard shorthand. Each unit has a CES production function of the form

$$Y_i = Z_i L_i^{1-\alpha} \left(\sum_j A_{i,j}^{1/\sigma_i} X_{i,j}^{(\sigma_i - 1)/\sigma_i} \right)^{\alpha\sigma_i/(\sigma_i - 1)} \tag{1}$$

where Y_i is unit i's output, Z_i its productivity, L_i its use of labor, and $X_{i,j}$ its use of good j as an input (throughout the paper, summations without ranges are taken over $1, ..., N$).[7] The parameters $A_{i,j}$, normalized such that $\sum_j A_{i,j} = 1$, determine the relative importance of different inputs. If $A_{i,j} = 0$, unit i does not use good j.

σ_i is the elasticity of substitution across material inputs for unit i. When $\sigma_i \to 1$, the production function becomes Cobb–Douglas (with the $A_{i,j}$ becoming the exponents). Though I assume a CES specification for simplicity, appendix B.3 shows that the results also hold under much more general conditions. As discussed in Baqaee and Farhi (2019), this structure captures arbitrary substitution patterns through nesting of the production functions.

Last, there is representative consumer whose utility over consumption of the different goods is

$$U(C_1, ..., C_N) = \prod_i C_i^{\beta_i} \tag{2}$$

where $\sum_j \beta_j = 1$ and we define a vector $\beta = [\beta_1, ..., \beta_N]'$. I assume that final consumption has a unit elasticity of substitution because it simplifies the results and because when the

[7]The fact that labor in (1) has a unit elasticity of substitution with material inputs is without loss of generality – one can always specify an additional unit that converts labor into labor services, which are then combined with other inputs with a non-unitary elasticity

elasticity is different from 1, for large shocks the model would imply that a single production unit becomes dominant, accounting for 100 percent of nominal output. The analysis is thus about nonlinearity in production, rather than final demand.[8]

The representative agent purchases C_i units of good i with wages and inelastically supplies a single unit of labor so that $\sum_i L_i = 1$.

Throughout the paper, lower-case letters denote logs, e.g. $z_i = \log Z_i$. I also normalize productivity such that $z_i = 0$ represents, informally, the steady-state or average value.

For the main results I assume labor can be frictionlessly reallocated across sectors. The limits go through identically with fixed labor (appendix B.4), and allowing for an upward sloping aggregate labor supply curve is also straightforward.

Since the economy is frictionless, it can be solved either competitively or from the perspective of a social planner.

Definition. *A competitive equilibrium is a set of prices* $\{P_i\} \cup W$ *and quantities* $\{Y_i\}$, $\{X_{i,j}\}$, $\{C_{i,j}\}$, *and* $\{L_i\}$ *such that each unit* i *maximizes its profits,* $P_i Y_i - W L_i - \sum_j P_j X_{i,j}$, *the representative consumer maximizes utility, producers and the consumer take prices as given, and markets clear:* $Y_i = C_i + \sum_j X_{j,i}$.

Since there is no government spending or investment, GDP is equal to aggregate consumption expenditures. I denote $\log GDP$ by gdp.

The model does not in general have a closed form solution.

3 Large shock behavior

Evaluating the probability of large deviations in GDP requires knowing its response to large shocks. This section describes that response. While the limiting behavior derived here may be interesting on its own, for the purpose of this paper it is a lemma supporting the core probability statements.

Throughout the analysis we will use a polar representation for the vector $\log Z = z = [z_1, z_2, ...]'$, such that

$$z = \theta t \tag{3}$$

where $\theta \in \mathbb{R}^N$, such that $\|\theta\| = 1$ ($\|\cdot\|$ denoting the Euclidean norm), is a unit vector representing a direction in productivity space and t is a scalar determining magnitude. As

[8] That said, this assumption is without loss of generality since one can always add a sector with a non-unitary elasticity of substitution that produces a single final good, with $\beta = 1$ to that sector and equal to zero for all other sectors.

examples, $\theta = [..., 0, 1, 0, ...]$ represents a shock to a single sector, while $\theta = [1, 1, ...] / \sqrt{N}$ represents a common shock to all sectors.

3.1 The large shock limit

Lemma 1. ***Part 1:*** *There exist unique scalar-valued functions $\lambda(\theta)$ and $\mu(\theta)$ independent of t such that*

$$\lim_{t \to \infty} |gdp(\theta t) - (\mu(\theta) + \lambda(\theta) t)| = 0 \tag{4}$$

See appendix A for all proofs.

Since the paper's goal is to characterize the tails of the probability distribution of GDP, it is natural that what will matter is how GDP responds to the very largest shocks. That is what Lemma 1 describes.[9]

To provide intuition, the panels of Figure 2 plot various approximations for log GDP for some arbitrary value of θ, with t varying along the x-axis. The negative side of the axis, for $t < 0$, formally corresponds to reversing the sign of θ – i.e. t runs from 0 to ∞ on each side and θ is replaced with $-\theta$ on the left.

When $\sigma_i = 1$ for all i, the model is fully linear, and there is a vector D (equal to the sales of each sector scaled by GDP) such that $\lambda(\theta) = D'\theta$. When the elasticities differ from 1, the model is nonlinear. That can be locally captured by a Taylor series, as is shown in the left-hand panel. The right-hand panel plots the approximation implied by Lemma 1. As t grows both to the left and right, log GDP approaches the two straight lines, with $\lambda(\theta) \neq -\lambda(-\theta)$. That difference is how the tail approximation captures nonlinearity.

3.2 Characterizing the slope function

Lemma 1. ***Part 2:*** *The slope in equation (4) is $\lambda(\theta) = \beta'\phi$, where $\phi \in \mathbb{R}^N$ is the solution to the system*

$$\phi_i = \theta_i + \alpha f_i(\phi) \text{ for } i \in \{1, ..., N\} \tag{5}$$

[9]Lemma 1 holds under weaker conditions than CES production functions, primarily just requiring constant returns to scale (appendix B.3). An example of a model in which it is violated is one where labor cannot be reallocated across sectors and it has an elasticity of substitution with material inputs smaller than 1 (such a model does not have a solution for all levels of productivity).

and $f_i : \mathbb{R}^N \to \mathbb{R}$ *is defined as*

$$f_i(\phi) = \begin{cases} \max_{j \in S_i} \phi_j \text{ if } \sigma_i > 1 \\ \sum_j A_{i,j} \phi_j \text{ if } \sigma_i = 1 \\ \min \phi_j \text{ if } \sigma_i < 1 \end{cases} \tag{6}$$

for all i*, where* $S_i \equiv \{j : A_{i,j} > 0\}$ *is the set of inputs used by sector* i*.* ϕ *and* λ *are unique and continuous in* θ*.*

λ is determined by a recursion. ϕ_i will turn out to determine the output of each unit (see section 4.6). Each unit's output depends on its own productivity and that of its inputs. When inputs are substitutes ($\sigma_i < 1$), all that matters is the weakest input. When inputs are complements ($\sigma_i > 1$), all that matters is the strongest input. And when the elasticity of substitution is 1, sector i is in the log-linear Cobb–Douglas case.

It is most notable what does *not* affect ϕ and $\lambda(\theta)$: the exact values of the production weights, $A_{i,j}$, and elasticities of substitution, σ_j. The production weights only matter to the extent that they are zero versus positive, determining the inputs used by each sector. Similarly, the elasticities only matter for being greater than, less than, or equal to 1. The form of f is due to the fact that the CES aggregator behaves like a maximum or minimum as the scale of the inputs diverges. Complementarities, with $\sigma_i < 1$, amplify negative and attenuate positive shocks, with substitutability doing the opposite.

Even though the slope $\lambda(\theta)$ is nonlinear in θ, it is continuous, so that two θ's that are close together will have similar impacts on the economy.

3.2.1 Comparative statics

Proposition. ϕ_i *and* $\lambda(\theta)$ *weakly increase when any* σ_i *transitions across the possibilities in (6) (i.e. from* <1 *to* $=1$ *to* >1*)*

When there is more substitutability in the economy, negative shocks become less damaging and positive shocks create more value. Intuitively, substitutability gives greater opportunity to use the output of relatively productive sectors.

Proposition. *When the set of inputs used by sector* i *grows, in the sense that* $S_i \to S_i \cup j$ *for some* $j \not\subset S_i$*,* ϕ_i *and* $\lambda(\theta)$ *weakly increase if* $\sigma_i > 1$ *and decrease if* $\sigma_i < 1$*.*

This result shows how the degree of interconnectedness affects the response of the economy to shocks. Interconnectedness for a given unit is harmful if its inputs are complements, while it is beneficial when they are substitutes. One way to state that result makes it seem obvious:

if the number of inputs needed to produce output grows, then obviously production is more delicate.

There is a less obvious way to put it, though: if a sector discovers an input that strongly increases the marginal product of all of its other inputs, then production is more delicate. Obviously such a discovery will increase output, but it also will make output in the future sensitive to more shocks, since now shocks to the new input will matter, where they did not previously. We will see below that while expanding the number of inputs increases diversification locally, in the sense of making the economy respond less to small shocks, but the result here shows that increasing the number of inputs can simultaneously increase tail risk.

An important feature of both of the comparative statics to note is that ϕ and λ are not continuous in the parameters σ_i and $A_{i,j}$. In particular, when any $A_{i,j}$ moves from zero to some number greater than zero, or when any σ_i moves from < 1 to 1 to > 1, that change either has no effect or a discontinuous effect on ϕ and λ, depending on θ. So even though log GDP is a continuous function of the parameters for any finite value of z, in the large-shock limit, the behavior is qualitatively different, with parameter changes causing discrete changes at the break points of 0 for the production weights and 1 for the elasticities.

3.3 Special cases

3.3.1 A solution for uniform elasticities

The case where the σ_i are all either above or below 1 is an important benchmark that appears in a number of calibrations in the literature. In that special case it is possible to further characterize λ. I take $\sigma_i \leq 1$, with the results going through analogously for $\sigma_i \geq 1$.[10]

Proposition. *If $\sigma_i \leq 1$ for all i, there is a finite set of $N \times 1$ vectors D_k such that*

$$\lambda(\theta) = \min_k D_k' \theta \tag{7}$$

It immediately follows that GDP is concave in that $\lambda(\theta) > 0 \Longleftrightarrow \lambda(\theta) \leq -\lambda(-\theta)$

The vectors D_k are known as Domar weights and give the marginal response of log GDP to a change in productivity (in this case in the direction θ). In a linear model, where the production network is fixed, there is a single slope determining the response to θ, so that $\lambda(\theta) = D'\theta$, where D are the economy's unique set of Domar weights. In a nonlinear model, the Domar weights vary depending on productivity, but the proposition says that in the limit

[10] A slight extension of the argument shows that the same results in fact hold for $\sigma_i \leq 1$.

they only take on a finite set of values. Intuitively, that follows from the recursion defining ϕ and λ in part 2 of Lemma 1. When $\sigma_i < 1$, sector i's output depends only on a single upstream sector. And there are only a finite set of possible upstream sectors.

The proposition again emphasizes how complementarity propagates negative shocks and attenuates positive shocks, giving each the worst possible effect on output, and it will be useful in analyzing specific models later on.[11] The second part captures the concavity of GDP and follows from the concavity of the minimum operator. It simply says that for a given shock that increases GDP, its mirror image reduces GDP by at least as much (and usually more). The fact that GDP is concave in a complementary economy will be an important ingredient in generating the observed asymmetry in GDP growth in Figure 1.

3.3.2 Fully connected economy

Example. *Suppose $\sigma_i < 1$ for all i and every sector uses inputs from every other sector (as in, e.g., the economy in Jones (2011) and Dew-Becker, Tahbaz-Salehi, and Vedolin (2021), among others). Then*

$$\phi_i = \theta_i + \frac{\alpha}{1-\alpha}\theta_{\min} \tag{8}$$

$$\lambda(\theta) = \beta'\theta + \frac{\alpha}{1-\alpha}\theta_{\min} \tag{9}$$

where $\theta_{\min} = \min_i \theta_i$.

In the case of a fully connected production network, each sector's ϕ_i is a linear combination of its own productivity and that of the weakest sector, and GDP then depends on both a linear combination of the θ's and also the minimum. So even if, for example, the economy is fully symmetric, with each good used in equal amounts so that all sectors have identical Domar weights in steady-state, the effect of a shock on GDP in the tail depends additionally on the productivity of the weakest sector. Consistent with the result in Lemma 1, the results in this example do not depend on the exact value of any of the production parameters.

That result is the paper's first view of the importance of conditional granularity. Even though no sector is granular (for large N) when shocks are small, as the shocks become large, the sector with the most negative shock becomes granular in the sense that it becomes a uniquely important determinant of GDP.

[11] The minimization here is reminiscent of the worst-case network analysis in Jiang, Rigobon, and Rigobon (2021).

3.4 Approximation errors

This section compares Lemma 1 to local approximations.

3.4.1 Small shock approximations

A first observation is that if one's goal is to understand asymmetry in the economy, a first-order approximation will never be appropriate. What about a higher-order Taylor series?

Proposition. *As $t \to \infty$, the error from any Taylor series for gdp around any value of z will diverge to $\pm\infty$ for some θ unless $\sigma_i = 1 \ \forall \ i$.*

If the economy has any nonlinearity, the tail approximation is always preferable when the magnitude of shocks is sufficiently large. As $\|z\|$ grows, the error in the tail approximation converges to zero, while it diverges to $\pm\infty$ (in at least some directions) for any Taylor series. That behavior is visualized in Figure 2.

Surprisingly, adding more terms to the Taylor series does not necessarily increase the range of z's for which a local approximation is superior. In general the Taylor series for a CES aggregator has a finite domain of convergence.[12] Outside some finite range for z, as terms are added to the Taylor series the approximation diverges to $\pm\infty$ at any fixed value of z.

3.4.2 The large-shock approximation

While Lemma 1 only guarantees accuracy as $t \to \infty$, its errors also never diverge:

Corollary. *There exists a δ such that, for all θ and t*

$$|gdp\,(\theta t) - (\mu\,(\theta) + \lambda\,(\theta)\,t)| < \delta \tag{10}$$

In addition, there is a stronger form of the main limit for convergence:

Lemma 1. ***Part 3:*** *Equation (4) in part 1 can be replaced by, for any j*

$$\lim_{t\to\infty} |gdp\,(\theta t) - (\mu\,(\theta) + \lambda\,(\theta)\,t)|\, t^j = 0 \tag{11}$$

It is not only the case that a linear approximation for *gdp* exists as $t \to \infty$, but in fact no other powers of t appear in the series expansion. The approximation errors converge to

[12] Specifically, in logs, consider $\log \sum_j w_j \exp(\gamma\theta_j t)$ for some exponent γ and a unit-norm vector θ. The sum inside the log in general has zeros for complex t, meaning that the function has a pole and hence a finite range of convergence for a given θ.

zero exponentially fast as $t \to \infty$, and further one can show that the rate increases with $|\sigma_i - 1|$.[13]

Appendix B.1 describes some results on values of z for which the tail approximation is more accurate than a Taylor series. Overall, when elasticities of substitution are closer to 1, or when the units whose shocks are relevant in the tail (in the sense of being the argument of (6) for some sector) have smaller production weights, the tail approximation will tend to be less accurate for small t.

4 The risk of large deviations in GDP and their source

This section combines the approximation result with assumptions about the probability distribution for shocks to get a probability distribution for GDP.

4.1 Shock distributions

I assume that there is a function $0 \leq s(\theta) < \infty$ that determines the scale of the shocks in direction θ. Specifically, for t greater than some $\bar{t}$, $t/s(\theta)$ has a cumulative distribution function F, with complementary CDF $\bar{F} \equiv 1 - F$ (note $\bar{F}$ is positive and decreasing).

Where $s(\theta)$ is larger, $\|z\|$ tends to be larger. For the purposes of this paper it is only necessary to choose the distribution of z for large t (i.e. when $\|z\|$ is large), with its behavior for $t \leq \bar{t}$ left unrestricted.

I assume θ has a probability measure m_θ. For the main results, the only characteristic of m_θ that matters is its support, and in typical cases it will have full support over θ.

Since $z = \theta t$ is a unique decomposition, we can write the probability distribution equivalently over z or θ and t (with $t = \|z\|$ and $\theta = z/\|z\|$). To formalize the above assumptions, we set, for $t > \bar{t}$

$$\Pr[\theta \in \Theta, t/s(\theta) > x] = m(\Theta)\bar{F}(x) \tag{12}$$

In order to ensure that asymmetry in the distribution of GDP comes from the structure of the economy, in what follows I assume that $m(\theta)$ and $s(\theta)$ are symmetrical ($s(\theta) = s(-\theta)$ and $m(\theta) = m(-\theta)$).

[13] If one wanted to add further terms to the approximation, it would be necessary to approximate $\log(gdp(z) - (\mu(\theta) + \lambda(\theta)t))$.

4.1.1 Examples

The representation in (12) accommodates standard distributions studied in the literature such as multivariate normality, elliptical distributions more generally, transformations of Laplace distributed vectors, and Pareto-tailed distributions (Resnick (2007)).

In the case where $z \sim N(0, \Sigma)$, $s(\theta) = (\theta' \Sigma^{-1} \theta)^{-1/2}$ and $\bar{F}(x) = \exp(-x^2/2)$, while $m(\theta)$ has full support.[14]

If the elements of z are i.i.d. exponential random variables with mean η, then $s(\theta) = 1/\|\theta\|_1$ and $\bar{F}(x) = \exp(-x/\eta)$.

A simple example of a distribution that does not have a representation (12) is the case with $N = 1$ so that z is a scalar and z is distributed normally conditional on being positive but exponentially conditional on being negative. Intuitively, the restriction, which can easily be relaxed, is that the tail shape (as distinct from the scale) is the same for all θ.[15]

See appendix A.7 for derivations.

4.2 General result

Theorem 1. *Given the distribution for z in (12), there exists a function $\varepsilon(x) \geq 0$ with $\lim_{x \to \infty} \varepsilon(x) = 0$ and an $\bar{x}$ such that for $x > \bar{x}$*

$$\int_{\Theta_-} \bar{F}\left(\frac{x - \mu(\theta) + \varepsilon(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \leq \Pr[gdp < -x] \leq \int_{\Theta_-} \bar{F}\left(\frac{x - \mu(\theta) - \varepsilon(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \tag{13}$$

where $\Theta_- = \{\theta : s(\theta)\lambda(\theta) < 0\}$

Theorem 1 says that, as a general matter, the CDF of $\log GDP$ is well approximated by

$$\int_{\Theta_-} \bar{F}\left(\frac{x - \mu(\theta)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \tag{14}$$

and in fact the $\mu(\theta)$ term is for typical cases also irrelevant (since x eventually dominates). Intuitively, this says that the CDF of GDP, in the tail, depends on the average across all shocks ($\int dm(\theta)$), of the probability that each shock (θ) creates a large decline in GDP. This is a general characterization that can be specialized further by specific choices for $\bar{F}$, $m(\theta)$, and $s(\theta)$.

[14] Note that this is an asymptotic representation for $\bar{F}$. The error function has a representation with polynominal terms multiplying the exponential, but they are asymptotically negligible.

[15] For practical purposes, if the tail decays significantly faster in some direction ($z > 0$ in this example), then that can be analyzed by just setting the measure m to zero in that direction.

4.2.1 General properties of the tail of GDP

Even without further specialization, there are general results that follow from Theorem 1.

Determinants of tail risk. First, the probability of large deviations in GDP depends on the probability of large deviations in productivity, scaled by the limiting slope, $\lambda(\theta)$, showing that the tail approximation is the correct way to analyze the economy in this setting. Other aspects of the economy – such as the steady-state Domar weights, the precise values of the elasticities of substitution, or terms in a Taylor expansion – are irrelevant.

A second observation is that the volatility of the shocks in different directions, captured by $s(\theta)$, interacts with $\lambda(\theta)$ to determine tail risk. When the shocks are more volatile – s is larger – tail risk is greater.

Comparative statics. The comparative statics in section 3.2.1 are useful here for showing what makes the economy riskier.

Corollary. *Any factor that weakly reduces $\lambda(\theta)$ for all θ also weakly reduces tail risk in the limiting sense of Theorem 1. In particular, given the propositions from section 3.2.1, tail risk weakly declines:*

1. when any σ_i transitions from < 1 to 1 to > 1

2. when the set of inputs used by any sector i grows if $\sigma_i > 1$ or shrinks if $\sigma_i < 1$

The second part of the corollary shows how changes in interconnectedness affect tail risk – interconnectedness reduces tail risk when it increases the number of substitutes and increases tail risk when it increases the number of complements.

Skewness. We also obtain a general result on skewness.

Corollary. *If the distribution of z is symmetrical,*[16] *then when GDP is concave in the sense that $\lambda(\theta) > 0 \iff -\lambda(-\theta) \geq \lambda(\theta)$, $\Pr[gdp < -x] \geq \Pr[gdp > x]$ for sufficiently large x. In particular, that holds when $\sigma_i < 1$ for all i.*

So under very general (but still only sufficient) conditions, as long as the elasticities are all below 1, the left tail of GDP is heavier than the right. Concavity in production thus robustly generates left skewness in GDP, in the limiting sense of the corollary.

Bayes' Theorem. Fourth, Bayes' theorem can be used to invert the probability distribution to find out what combinations of shocks cause large movements in GDP. Again ignoring the error term, for any subset Θ^* of the unit sphere,

$$\Pr[\theta \in \Theta^* \mid gdp < -x] \to \frac{\int_{\Theta^* \cap \Theta_-} \bar{F}\left(\frac{x-\mu(\theta)}{-s(\theta)\lambda(\theta)}\right) dm(\theta)}{\int_{\Theta_-} \bar{F}\left(\frac{x-\mu(\theta)}{-s(\theta)\lambda(\theta)}\right) dm(\theta)} \tag{15}$$

[16] I.e. $s(\theta) = s(-\theta)$ and $m(\theta) = m(-\theta)$.

as $x \to \infty$. The values of θ most likely to appear when GDP takes an extreme value are, naturally, those for which $s(\theta)\lambda(\theta)$ is large – the shocks are large or GDP is sensitive to them. That is on some level not surprising, but it formalizes the idea that one must be careful to use the correct measure of sensitivity. What matters is not the effect of a shock in regular times – i.e. $D_{ss}'\theta$, where D_{ss} is the vector of steady-state Domar weights – but its effect when things are extreme. The steady-state Domar weights in fact have no bearing at all on the tail probabilities.

Finally, Theorem 1 shows how nonlinearity in the economy generates increases in tail risk. If the economy were linear, the argument of $\bar{F}$ in (13) would be $\frac{x}{-s(\theta)D_{ss}'\theta}$. When $\lambda(\theta)$ is larger in magnitude than $D_{ss}'\theta$, there is a larger chance of a large movement in GDP. Equation (13) shows how that would increase left tail risk in the economy relative to what one would expect based on the steady-state Domar weights.

4.3 Interconnectedness and risk in the economy

As discussed above and in section 3.2.1, when a sector's production is characterized by complementarity, with $\sigma_i < 1$, an increase in its upstream interconnectedness, using more inputs, weakly decreases $\lambda(\theta)$ for all θ. An equivalent version of that statement is that when a sector *sells* to a new downstream sector, left tail risk weakly increases if the new downstream sector has an elasticity of substitution less than 1. In other words, complementarity and interconnectedness combine to increase left tail risk (and at the same time reduce the probability of large booms in GDP).

But obviously the tail probabilities in Theorem 1 are not the only way to evaluate the risk of the economy. Another interesting question is how the economy responds to small shocks, or equivalently, what the variance of $\log GDP$ is in a first-order Taylor approximation.

Hulten's (1978) theorem tells us

$$\frac{d \log GDP}{dz_i} = \frac{P_i Y_i}{\sum_j P_j C_j} \tag{16}$$

When evaluated at $z = 0$, I refer to the vector over i of those derivatives as the steady-state Domar weights, D_{ss}. We then have the well known result that the Domar weights are related to the production network through

$$D_{ss}' = \beta' (I - \alpha A)^{-1} \tag{17}$$

where the matrix A has representative element $A_{i,j}$. The model is set up so that the A and

β parameters determine the Domar weights at steady-state.

When the covariance matrix of the productivity shocks is $\text{var}(z) = \Sigma$, we have, from a first-order approximation,

$$\text{var}(\log GDP) \approx D_{ss}'\Sigma D_{ss} \tag{18}$$

How does a change in interconnectedness affect this local concept of risk? First, we have the simple fact that $D_{ss}'\Sigma D_{ss}$ is continuous in A. So any small change in A – i.e. a change in some $A_{i,j}$ from zero to a small positive number – will cause only a small change in $D_{ss}'\Sigma D_{ss}$. But as discussed above, such a change can cause a discrete shift in the values of the function λ, and hence in tail risk. In other words, local risk is always affected smoothly by A, but tail risk is affected discretely by it.

Second, though, note that an increase in interconnectedness, even though it cannot reduce tail risk, can certainly reduce the sensitivity of GDP to small shocks. Since the sum of the Domar weights, $D_{ss,i}$, is always equal to $(1-\alpha)^{-1}$, we have the following simple example:

Example. *Suppose the shocks are uncorrelated (Σ is diagonal). A marginal increase in the sales share of any sector starting from zero, if it (weakly) reduces the sales shares of all other sectors, will reduce $D_{ss}'\Sigma D_{ss}$.*

The example gives simple sufficient – and far from necessary – conditions for when adding a new sector diversifies the economy. At the same time, though, the results above show that adding new sector will weakly increase tail risk when the elasticity of substitution in production is less than 1.

This section thus shows that in the model increases in interconnectedness – measured here by the number of links in the production network ((i,j) pairs such that $A_{i,j} > 0$) – can diversify the economy, making it less sensitive to small shocks, while at the same time increasing tail risk.

4.4 Weibull tails

This section specializes the result in Theorem 1 to a broad class of distributional forms for the shocks, allowing for more specific results.

A wide range of distributions, including the normal, gamma, exponential, Gumbel, and Fréchet families all can be said to have Weibull tails, up to asymptotically negligible terms, in the following sense:[17]

[17] For the normal distribution in particular, a better approximation for $\bar{F}(t)$ is $c(t-\bar{t})^{-1}\exp\left(-\eta(t-\bar{t})^2\right)$. The t^{-1} term is asymptotically dominated by the exponential, and it is straightforward to show that the proposition in this section also holds when any powers of t multiply the exponential in $\bar{F}$.

Definition. *The shocks have a Weibull-type tail if, for* $t > \bar{t}$*,*

$$\bar{F}(t) = c \exp\left(-\eta\left(t-\bar{t}\right)^{\kappa}\right) \quad (19)$$

$$\text{where } c = \Pr\left(t \leq \bar{t}\right) \quad (20)$$

for parameters $\kappa > 0$ *and* $\eta > 0$*.*

Smaller κ represents heavier tails, with $\kappa = 1$ corresponding to the exponential distribution and $\kappa = 2$ to the normal. In addition, all three types of extreme value distributions (Weibull, Gumbel, and Fréchet) have Weibull-type tails. The Weibull family thus covers a broad range of behaviors, including all but the very lightest (e.g. bounded) and very heaviest (Pareto or Cauchy) tails.

Across that entire family, we have a surprisingly simple result. Denote the essential supremum with respect to the measure m over θ of any function $f(\theta)$ by $\|f(\theta)\|_{\infty}$.[18] In the benchmark case where m has full support, $\|f(\theta)\|_{\infty} = \max_{\theta} f(\theta)$ (note that it is *not* the maximum of $|f(\theta)|$). $\|f(\theta)\|_{\infty;\Theta^*}$ denotes the essential supremum on some subset of the sphere Θ^*.

Proposition. *If the shocks have Weibull tails,*

$$\lim_{x\to\infty} \Pr\left[gdp < -x\right]^{1/(x^{\kappa})} = \exp\left(-\eta\left(\frac{1}{\|-s(\theta)\lambda(\theta)\|_{\infty}}\right)^{\kappa}\right) \quad (21)$$

Furthermore, for any set Θ^* *such that* $\|-s(\theta)\lambda(\theta)\|_{\infty;\Theta^*} < \|-s(\theta)\lambda(\theta)\|_{\infty}$*,*

$$\lim_{x\to\infty} \Pr\left[\theta \in \Theta^* \mid gdp < -x\right] = 0 \quad (22)$$

Analogous results hold for $\Pr\left[gdp > x\right]$*.*

Conditional on the distribution of the shocks, the probability that GDP has an extreme decline is determined by a sufficient statistic: the most negative value of $s(\theta)\lambda(\theta)$. That is, what determines tail risk is the shock with the largest negative impact on GDP – combining both the scale of the shock, $s(\theta)$, and its effects, $\lambda(\theta)$. The exponential form of the distribution for t is what causes only the single most extreme shock to end up mattering, because $\bar{F}$ is effectively infinitely convex as $x \to \infty$.

The second part of the result says that extreme realizations are driven, in probability, by *only* that single most extreme shock – the θ that achieves the maximum for $-s(\theta)\lambda(\theta)$

[18] Formally, $\|f(\theta)\|_{\infty} = \inf\{a \in \mathbb{R} : m(\{\theta : f(\theta) > a\}) = 0\}$.

(under knife-edge conditions, multiple θ's might achieve that maximum). Any θ with $-s(\theta)\lambda(\theta) < \|-s(\theta)\lambda(\theta)\|_\infty$ causes a crash with probability zero asymptotically.

So when shocks have Weibull tails, the most extreme events are caused by, up to knife-edge conditions, a single shock: all crashes (and all booms) are the same. If one wants to evaluate the risk of the economy, it is not necessary to actually know the full network structure and how the economy responds to every possible shock. The impact of the most influential shock, $\|-s(\theta)\lambda(\theta)\|_\infty$, is a sufficient statistic. That result is reminiscent of the finding in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017) that tail risk is determined by the largest Domar weight. In the more general case here, it depends on the product of the Domar weight *in the tail* with the scale of the shock.

The same comparative statics results continue to hold as above, though now really they only matter if they affect $\|-s(\theta)\lambda(\theta)\|_\infty$. That is, any change in the model (the elasticities or the inputs used by sectors) that does not affect that supremum does not affect tail risk.

For the right tail, the probabilities depend on the largest positive shock, $\|s(\theta)\lambda(\theta)\|_\infty$. Asymmetry in λ or s – if it affects their joint maximum – will induce asymmetry in the tail of GDP.

Appendix B.2 describes results for the case where the shocks have Pareto tails, which are heavier than anything in the Weibull family, which still yields a closed form solution, but with differences from the Weibull result.

4.4.1 Tail skewness

In the Weibull case we can significantly sharpen the conditions for tail asymmetry relative to the general case:

Corollary. *When the shocks have Weibull tails, the following necessary and sufficient conditions for tail asymmetry hold:*

$$\|-s(\theta)\lambda(\theta)\|_\infty > \|s(\theta)\lambda(\theta)\|_\infty \Leftrightarrow \lim_{x\to\infty}\frac{\Pr[gdp < -x]}{\Pr[gdp > x]} = \infty \tag{23}$$

$$\|-s(\theta)\lambda(\theta)\|_\infty < \|s(\theta)\lambda(\theta)\|_\infty \Leftrightarrow \lim_{x\to\infty}\frac{\Pr[gdp < -x]}{\Pr[gdp > x]} = 0 \tag{24}$$

When s is symmetrical, a sufficient condition for the first case is $\sigma_i < 1$ for all i, while a sufficient condition for the second case is $\sigma_i > 1$ for all i.

The condition $\|-s(\theta)\lambda(\theta)\|_\infty > \|s(\theta)\lambda(\theta)\|_\infty$ says that the largest negative effects of shocks are larger in magnitude than the largest positive effects. If that is true log GDP is skewed left. When the opposite condition holds, it is skewed right. As in the corollary in section 4.2, a sufficient condition for left tail skewness (in the limiting sense) is that production is complementary, while if production displays substitutability – $\sigma_i > 1$ – then there is right tail skewness. In other words, for a complementary economy with Weibull-type shocks, large booms in GDP are infinitely rare compared to large declines.

In the case of the next two specific examples, it is possible to further characterize the tail distribution of GDP.

4.4.2 Gaussian shocks, conditional granularity, and the Q-Q plot

This example shows how complementarity can generate excess tail risk and the behavior observed in Figure 1. First, note that for small shocks, we have the usual first-order approximation that $gdp \approx D_{ss}'\theta$, implying

$$\operatorname{var}(gdp) \approx D_{ss}'\Sigma D_{ss} \text{ for } z \approx 0 \tag{25}$$

where Σ is the covariance matrix of the shocks.

Example. *Suppose $z \sim N(0, \Sigma)$. If $\sigma_i \leq 1$ for all i, then the left tail of GDP is determined by*

$$\|-s(\theta)\lambda(\theta)\|_\infty = \max_n \sqrt{D_n'\Sigma D_n} \tag{26}$$

$$\Pr[gdp < -x] \to c \exp\left(-\frac{1}{2}\frac{x^2}{D_n'\Sigma D_n}\right) \tag{27}$$

as $x \to \infty$. Denoting the argument for the maximum in (26) by n^*, the shock causing left tail events is $\theta \propto -\Sigma D_{n^*}$.

As noted above, the tail distribution of GDP remains Gaussian. Instead of the local variance, $D_{ss}'\Sigma D_{ss}$, though, the left tail probabilities are as though the variance is $\max_n D_n'\Sigma D_n$. That is, instead of the steady-state Domar weights, the left tail probabilities depend on the worst possible Domar weights, in the sense that they generate the largest variance of any feasible network.[19] Whereas a linear approximation implies that GDP is globally Gaussian, the tail approximation shows that such an approximation misses important deviations from Gaussianity in the tails.

[19] Note also that in the special case of a linear model ($\sigma_i = 0$ for all i), the tail approximation yields the correct result that the effective variance in both the left and right tails is $D_{ss}'\Sigma D_{ss}$.

As an example, suppose $gdp = \sum_j z_j$ and the z_j are i.i.d. normal. Then the results here say that extreme realizations of GDP are due to $\theta = [1, 1, ...] N^{-1/2}$. That result in fact holds for Weibull tails more generally as long as $\kappa > 1$. See Nair, Wierman, and Zwart (2020), Proposition 3.1.

The example also shows that it is possible to calculate the vector of shocks, θ, that causes the largest declines in GDP. Tail events are caused by simultaneous shocks to all sectors, with the magnitude of the shocks lining up with their variances interacted with the worst-case Domar weights.

To see how conditional granularity drives tail risk, note that it is always the case that the sum of the Domar weights in this model is equal to $1/(1-\alpha)$. But tail risk is determined by a quadratic function of the Domar weights. So economies with the largest tail risk maximize a quadratic function of D_n subject to a linear adding up constraint, which will intuitively be maximized when a few of the Domar weights become relatively large.

While it is possible to obtain closed form results for the left tail, the associated optimization for the right tail does not have a closed form solution. However, we can say that

$$\|s(\theta)\lambda(\theta)\|_\infty^2 \leq D_{ss}'\Sigma D_{ss} \leq \|-s(\theta)\lambda(\theta)\|_\infty^2 \tag{28}$$

That is, the mass in the left tail is always at least as large as that implied by the steady-state Domar weights (with the inequality strict when the elasticities are strictly less than 1), while the mass in the right tail is weakly smaller than that implied by D_{ss}, due to the concavity of GDP. That result has important implications for the Q-Q plot in Figure 1, which we return to in a moment. First, though, there is an interesting special case that is solvable by hand.

Example. *Suppose the shocks are i.i.d. normal with variance σ^2, so that $s(\theta)=\sigma$. In addition, assume the network is fully connected and symmetrical, so that $A_{i,j}=\beta_i=1/N$ for all i,j. Then*

$$\lambda(\theta)=\sum_i N^{-1}\theta_i+\frac{\alpha}{1-\alpha}\min_j \theta_j \tag{29}$$

and

1. (Local to steady-state): $D_{ss,i}=N^{-1}/(1-\alpha)$ and $\sqrt{D_{ss}'\Sigma D_{ss}}=N^{-1/2}\sigma/(1-\alpha)$

2. (Right tail): $\|s(\theta)\lambda(\theta)\|_\infty=N^{-1/2}\sigma/(1-\alpha)$, which is attained at $\theta_i=N^{-1/2}$ for all i

3. (Left tail): $\|-s(\theta)\lambda(\theta)\|_\infty=\sigma\alpha/(1-\alpha)+O\left(N^{-1/2}\right)$, which is attained when $\theta_i\propto N^{-1}$ for all i except a single value, where it is proportional to $N^{-1}+\alpha/(1-\alpha)$

Sector i's Domar weight is the coefficient on θ_i in (29).

The assumption of a completely connected network appears elsewhere, and the perfect symmetry, though not necessary, is convenient and has appeared in, for example, Jones (2011).

The example shows both how diversification works in the model, and also how it differs in the tails. Local to steady-state, i.e. where the model is well described by a first order approximation, we end up with the expected result that output has a standard deviation of order $N^{-1/2}$.

For the right tail we obtain exactly the same result – the right tail probabilities match those for the normal distribution with standard deviation $N^{-1/2}\sigma/(1-\alpha)$, the same as near the steady-state. The shock that causes right tail events is one where all sectors have an equal increase in productivity, which is because of the dependence of output on $\theta_{\min}$. Booms occur when all sectors simultaneously receive positive shocks. The core intuition behind diversification is that the odds that all sectors get a big shock simultaneously is small, which applies both near steady state and also for positive shocks, so that large booms are very rare.

On the other hand, for output to fall significantly productivity only needs to fall in a single sector. So in this case as N grows there is actually *no* diversification in the tail at all. The left-tail probabilities are as though the volatility of output is simply $\sigma\alpha/(1-\alpha)$. Booms are rare because productivity rising in all sectors simultaneously is relatively unlikely. But productivity falling in one sector is not surprising at all, making crashes much more likely.

Q-Q plots. Going back to the Q-Q plot again, then, in this model we would expect to see that the plot would have a similar slope at middle and high quantiles, while the slope would be much higher at low quantiles. Both this example and the previous one show how the nonlinearity in the model is able to generate excess tail risk, even with Gaussian shocks, while this section gives an example where the excess risk is entirely at low quantiles. That asymmetry is consistent with Figure 1, and it does not appear in the results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), where symmetrically distributed shocks generate symmetrical tails for GDP (and also where heavy tails in shocks are required for heavy tails in GDP).

Conditional granularity. That said, the mechanism here actually retains an interesting connection to Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), as well as Gabaix (2011) and Acemoglu et al. (2012). Local to the steady-state and in the right tail, the Domar weights are proportional to N^{-1} – each sector is equally important. But in the left tail, the sector with the most negative shock has a weight of $N^{-1}+\alpha/(1-\alpha)$. That is, there is *conditional granularity*. While the model is structurally symmetrical, so that all sectors on average carry the same weight and have the same size, following an extreme shock any given sector can become large and, by itself, have a major impact on the economy.

Overall, this section's examples contain a number of the paper's most important practical results. They show how complementarity generates asymmetry in output, how it can generate excess tail risk and match the Q-Q plot in Figure 1, and why what matters is not just steady-state or average granularity that has been studied in the past, but also conditional granularity.

4.4.3 Exponential tailed shocks and the maximum Domar weight

Next, consider the case of i.i.d. Laplace distributed shocks (i.e. exponentially distributed with a 1/2 probability of a positive or negative sign), again with complementary production.

Example. *Suppose the shocks are i.i.d. Laplace distributed with scale parameter η, so that $s(\theta) = 1/\|\theta\|_1$ (where $\|\cdot\|_1$ here is the vector L_1-norm) and $m(\theta)$ again has full support. If $\sigma_i \leq 1$ for all i, then*

$$\|-s(\theta)\lambda(\theta)\|_\infty = \max_n \max_j D_{n,j} \tag{30}$$

$$\Pr[gdp < -x] \rightarrow \exp\left(-\eta \frac{x}{\max_n \max_j D_{n,j}}\right) \tag{31}$$

where $D_{n,j}$ is the jth element of the vector D_n

The i.i.d. Laplace case is also studied by Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017). In the context of a linear model ($\sigma_i = 1$), they find that the maximum Domar weight, $\max_j D_{ss,j}$, is critical for determining how heavy the tail of the distribution of GDP is. I find that remains true here, but what matters is the maximum value that *any* Domar weight can take *under any circumstances.*[20]

So it need not be the case that $\max_j D_{ss,j}$ is large – the condition in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017). Rather, there just needs to be *some* Domar weight that can be large in some situation. That maximum – the maximum of the maximums – is what determines tail risk in general nonlinear models. A fully linear model is the special case where the set $\{D_n\}$ is just the singleton D_{ss}.

For example, take electricity. Its steady-state Domar weight is not particularly large empirically (see section 6) – it is certainly not the largest sector in the economy. But one can imagine a scenario in which electricity – or some other energy sector – receives a large negative shock, becomes a limiting input in production, and then becomes much more expensive. That is the type of scenario that these limits show is important for driving the largest declines in GDP in this model.

[20] In the case where $\sigma_i = 1$ for all $i \geq 0$, $\{D_n\}$ is just the singleton D_{ss} and we recover their original result.

4.5 Simulated Q-Q plots and conditional granularity

The results so far show that network models can qualitatively match the features of Figure 1. This section asks whether they can plausibly do so quantitatively.

I begin from the quantitative network model simulated by Baqaee and Farhi (2019), which uses a 76-sector input-output matrix to calibrate the production and consumption weights. As in their calibration, I set the elasticity of substitution across inputs to 0.001 (see also Rubbo (2021)). I also assume that labor is combined relatively inelastically with material inputs with an elasticity of 0.3.[21] In addition, following Baqaee and Farhi (2018), I assume that labor cannot be reallocated across sectors. See appendix B.4 describing how the analytic results continue to hold in that case. In order to generate realistically large deviations, it helps to have tails for the shocks that are slightly longer than normal, so I assume that they are Laplace distributed (consistent with empirical evidence from Atalay, Drautzburg, and Wang (2018)). The correlations are set based on the historical covariances of sector level TFP growth, while the scale is set to match the empirical standard deviation for detrended real GDP. Finally, in order to keep the network from being too dense with connections, I set the production weights, $a_{i,j}$, to zero when they are less than 0.05.

The red line in the top panel of Figure 1 shows the Q-Q plot from simulated data, while the blue crosses are the sample Q-Q plot for HP-detrended log GDP for the period 1947–2019. The gray line plots the Q-Q plot for a simulation of a fully linear version of the same model (i.e. with all elasticities set to 1). The curvature of the gray line comes from the fact that the Laplace tails are longer than Gaussian tails.

The model fits the shape of the empirical data well, and the deviation from the gray line shows how complementarity induces concavity in production (that concavity reduces the mean squared fitting error to the data for the nonlinear model by 55 percent compared to the linear model). In both the model and the data, the left tail is heavier than implied by the normal distribution while the right tail is closer to Gaussian or slightly lighter, though the model does not fully capture the lightness of the right tail. The asymmetry in the model and data is inconsistent with the results of Acemoglu, Ozdaglar, and Tahbaz-Salehi (2017), but, as discussed above, arises naturally here in the presence of complementarities in production. Consistent with the predictions from the previous sections, in the middle part of the plot, the model and data quantiles are well described by the normal distribution.

A second notable prediction of the model is that the left tail is associated with increased

[21] As noted above, in the context of the theoretical analysis this corresponds to assuming that there is a sector that produces labor services, and it is labor services that have an elasticity with other inputs of 0.3. As a computational matter, I follow Baqaee anf Farhi's (2019) methods, which allow for general CES patterns.

granularity To see that, the bottom panel of Figure 1 plots the expected value of the maximum Domar weight conditional on output being at a given level. Under the linear model, the Domar weights are constant, so the gray line in the bottom panel is perfectly flat. In the nonlinear model, though, consistent with the theoretical analysis, the maximum Domar weight grows significantly as output falls. Quantitatively, the largest Domar weight is on average equal to 0.21 when output is at low quantiles compared to 0.135 at middle and high quantiles. That rise is concentrated at very low quantiles – starting at 2-3 standard deviations below the mean.

4.6 What do crashes look like?

The analysis so far has discussed only the behavior of aggregate output.

Proposition. *Under the conditions of Lemma 1, unit output and prices follow*

$$\lim_{t\to\infty} t^{-1}\log Y_i = \phi_i(\theta) \tag{32}$$

$$\lim_{t\to\infty} t^{-1}\log P_i = -\phi_i(\theta) \tag{33}$$

where ϕ_i is the same function as in Lemma 1 part 2.

In extreme events, unit output and prices are determined by the same recursion that determines aggregate output. That result is reminiscent of what one obtains in the usual Cobb–Douglas case (and they must match exactly when $\sigma_i = 1\ \forall\ i$). For large shocks, unit output and prices move inversely with each other. As usual, they depend on productivity shocks upstream.

Example. *In the symmetric example from section 4.4.2,*

$$\phi_i = \theta_i + \frac{\alpha}{1-\alpha}\theta_{\min} \tag{34}$$

Whenever $\sigma_i < 1$, sector i's output depends, in the limit, on its own productivity and that of its weakest input. That is the way that productivity shocks propagate. When the network is fully connected, then every sector necessarily has the same weakest input, so their outputs all depend on $\theta_{\min}$. So in addition to $\theta_{\min}$ driving aggregate output, it also acts like a common shock to the output and prices of all other sectors. Even though a large decline in output has as its source a shock to a single sector, one would still observe output falling and prices rising (relative to wages) in all sectors, with the changes somewhat larger in the shocked sector, but not qualitatively different.

5 Which sectors are influential?

The analysis so far has allowed for generic mixtures of shocks through the vector θ. But a natural question is how large shocks to individual sectors affect GDP. Where are large shocks most damaging? In other words, what makes a sector systemically risky, in the sense that a shock to that unit of the production network may be prone to propagate to the rest of the economy? What sectors have the potential to become granular?

As discussed above, Hulten's theorem says that local centrality – the impact of a small shock to a unit on GDP – is exactly the sales share of the unit. This section measures the importance of units based on how *large* shocks to their productivity affect GDP, which is typically closer to the spirit of the question being asked when trying to evaluate systemic risk. Those two measures will often be very different.

Definition. *The left and right tail centralities of unit i are, respectively,*

$$\gamma_i^L \equiv \lim_{\Delta z_i \to -\infty} \frac{\Delta gdp}{\Delta z_i} \tag{35}$$

$$\gamma_i^R \equiv \lim_{\Delta z_i \to \infty} \frac{\Delta gdp}{\Delta z_i} \tag{36}$$

where Δ denotes a deviation from steady-state ($z_i = 0 \ \forall \ i$)

The tail centralities measure how systemic a large shock to a given unit is in the sense of passing through to GDP. A unit with a large left tail centrality is one where a large negative shock has a large effect on GDP. While the tail centralities are defined without any reference to the results developed so far, those results still apply directly to it.

Corollary. *Let e_i denote a vector equal to 1 in element i and zero otherwise. Then in the notation of Lemma 1,*

$$\gamma_i^L = -\lambda(-e_i) \tag{37}$$

$$\gamma_i^R = \lambda(e_i) \tag{38}$$

As in Lemma 1, an immediate consequence of this result is that while tail centralities depend on whether each industry's intermediate inputs are substitutes or complements, they do not depend on the exact values of the corresponding elasticities of substitution. Similarly, except for the knife-edge case of $\sigma_i = 1$, input-output linkages only matter via the identities of each industry's suppliers, and the intensities of such relationships are immaterial for tail centralities.

Comparative statics

More importantly, the recursive characterizations in Lemma 1 yield comparative static results on how various structural features of the economy shape tail centralities. In particular, the comparative statics results from section 3.2.1 for $\lambda(\theta)$ apply here, with results for γ_i^L and γ_i^R just being special cases.

First, a sector's left tail centrality weakly decreases and its right tail centrality weakly increases when any elasticity of substitution – not just its own – transitions from either $\sigma_j < 1$ to $\sigma_j = 1$ or from $\sigma_j = 1$ to $\sigma_j > 1$. That result can be further sharpened: the only elasticities of substitution that affect γ_i^L and γ_i^R are those of units downstream of i (with a sector j being downstream if $i \in S(j)$, or if there is a sector $k \in S(j)$ such that k is downstream of i). A sector that is strictly upstream of i is unaffected by the shock e_i since the recursion defining ϕ looks only upstream – this is the usual result that productivity shocks propagate downstream.

The second result is on interconnectedness. When the number of inputs used by a sector j with $\sigma_j < 1$ increases, or the number used by a sector with $\sigma_j > 1$ decreases, γ_i^L weakly increases and γ_i^R weakly decreases. The next section gives a stronger characterization of how connections matter in the case where $\sigma_i < 1$ for all i.

5.1 Results under complementary production

Consider an economy in which all elasticities of substitution in production are less than 1: $\sigma_i < 1$ for all i. Assume also that $A_{i,i} \in (0,1)$, which guarantees that every unit uses at least two inputs, one of which is its own output (which is true for 88 percent of sectors according to the BEA). The $A_{i,j}$ are otherwise unconstrained.

Using the result from Lemma 1 part 2, it is then immediate that $\gamma_i^R = \beta_i \ \forall\ i$. When a unit gets a sufficiently positive shock, it eventually has no downstream impacts, affecting GDP only through its direct effect on consumption.

While positive unit shocks eventually die out, negative unit shocks propagate, since production in all units is complementary. That implies that

$$\gamma_i^L = \frac{1}{1-\alpha} \sum_{j=1}^{n} \beta_j \alpha^{d_{\min}(j,i)} \tag{39}$$

where $d_{\min}(j,i)$ is the length of the shortest upstream path from i to j.[22] In the complementary economy, a unit's left tail centrality is measured by its average downstream closeness to final

[22] I.e. if $i \neq j$ and $A_{i,j} > 0$, $d_{\min}(i,j) = 1$. If $A_{i,j} = 0$, but there exists a k such that $A_{i,k} > 0$ and $A_{k,j} > 0$, then $d_{\min}(i,j) = 2$. Etc.

consumption. In equation (39), γ_i^L involves the sum across units of each unit's consumption weight times a term, $\alpha^{d_{\min}(j,i)}$, that decreases in the number of upstream steps from that unit back to i.

Equation (39) answers the question of what types of units have high tail centrality under complementarity: those that are direct suppliers to producers of a large fraction of GDP (and that do not have substitutes). Interestingly, these results also imply that tail centralities (and hence fragility) increase when the economy is more connected, as in the fully symmetrical example from above.

More generally, all of the following will increase γ_i^L:

1. An increase in the number of units downstream of i or an increase in their share of GDP
2. A decrease in the number of steps between unit i and the units downstream of it
3. An increase in the share of expenditures on material inputs, α.

Intuitively, the results in this section suggest that the out-degree of a unit – the number of units directly downstream of it – would be closely linked to tail centrality. Define the weighted out-degree of a unit to be

$$\deg_i \equiv \sum_{j:i\in S(j)} \beta_j \tag{40}$$

Proposition. *Left tail centrality satisfies*

$$\frac{1}{1-\alpha}\left(\beta_i + \alpha \deg_i\right) \leq \gamma_i^L \leq \frac{1}{1-\alpha}\left(\beta_i + \alpha \deg_i + \alpha^2\left(1-\deg_i\right)\right) \tag{41}$$

So tail centrality is related to out-degree, with out-degree giving upper and lower bounds for tail centrality.

Finally, we can continue the example of the symmetric economy from above.

Example. *In the symmetric example from section 4.4.2,* $D_{ss,i} = N^{-1}/\left(1-\alpha\right)$*,* $\gamma_i^L = N^{-1} + \alpha/\left(1-\alpha\right)$*, and* $\gamma_i^R = N^{-1}$*.*

In the symmetric complementary economy, every sector, as discussed above, has a small steady-state Domar weight. The right tail centralities are even smaller, though still proportional to N^{-1}. The left tail centrality of every sector is the same, though, and stays large even as N grows. this again emphasizes how a highly interconnected network can be well diversified near steady state but at the same time display effective underdiversification in the tail, with every sector being systemically risky, with the ability to drive down all of GDP.

6 Measuring tail centrality

This section measures left tail centrality empirically and compares it to the Domar weights or sales shares that have been studied in past work. I study the sector detail input-output tables from 2012 reported by the BEA. The tables have 379 private sectors.[23] For this paper's purposes, it is important to use a detailed version of the input-output tables because at higher levels of aggregation, the sectors become very strongly connected. The disaggregated table has much more sparse links.

Define an $A_{i,j}$ coefficient to be positive, so that there is an upstream link, if sector i spends at least 0.5 percent of its expenditures on materials for the output of sector j. We set $\alpha = 1/2$ in calculating γ_i^L. Finally, the β_i parameters are calculated from the fraction of nominal final expenditure going to each sector.

Figure 3 plots Domar weights (nominal output divided by nominal GDP) against left tail centralities. There is a weak positive correlation of 0.23, but the figure makes apparent that the distributions are very different. There are a few sectors, such as Petroleum Refineries, that have sales shares noticeably higher than most others. But there are numerous sectors with tail centralities close to 0.5. 13 sectors have $\gamma_i^L > 0.8 \max\left(\gamma_i^L\right)$, while only two have $D_{ss,i} > 0.8 \max\left(D_{ss,i}\right)$.

One can also see that the top sectors by Domar weight have very different tail centralities – Petroleum Refineries at 0.41, Oil and Gas Extraction at 0.25, and Hospitals at 0.06. Oil and Gas Extraction is lower because it is one more step up the supply chain from refineries. Hospitals are low because they produce almost entirely final output – they are not an important input for any sector.

Table 1 further examines the top sectors sorted by sales and tail centrality. The top panel lists the top by tail centrality. Their most common feature is that they are universal inputs. The first is electricity, which is why it has appeared frequently as an example. The second highest tail centrality is for legal services – again, simply because every sector purchases legal services. Does it make sense to claim that a large negative shock to the legal services sector could cause a crash in GDP? That result is surprising but, on reflection, reasonable. There is ample evidence that legal institutions are necessary for the growth of the economy. All aspects of business rely on property rights and contract enforcement. If, for some reason, the legal system literally shut down and legal services were actually no longer available to firms, it is entirely plausible that there would be massive declines in output.

[23]We exclude customs duties, funds and trusts, real estate sectors, management services, and employment services. Management services are almost entirely offices of holding companies, while employment services represent staffing agencies, so we take both as representing more appropriately relatively generic labor input.

One potential concern with that argument is that the input-output tables do not actually measure things like enforcement of property rights or the use of courts; they just measure expenditures on lawyers by firms. That actually illustrates a key advantage of γ_i^L: measuring it does not require measuring *all* of each sector's expenditures on each input. All that we need to know is that a sector uses some input. And the input-output tables are certainly correct that all sectors directly use legal services.

In addition to utilities (electricity, communications) and professional services like lawyers and accountants, the last major category of sectors that appears repeatedly among the top sources of tail risk is financial institutions. Just as with legal services, all firms use financial services in one way or another (as do essentially all households). The analysis here thus helps explain why the financial sector would be a relevant source of crashes throughout history – when financial services are disrupted, every firm in the economy faces more difficulty in production.

There is past work examining, both in models and in the data, the effects of shocks to the energy sector, financial services, and legal and accounting institutions. The analysis here shows how those shocks are linked: they all represent shocks to inputs that are used nearly universally, which is why they might have effects larger than would be implied just from looking at their average sales shares. Table 1 shows that the tail centralities are at least an order of magnitude larger than the Domar weights, if not more.

The bottom section of table 1 reports the top sectors sorted by sales share. As discussed above, not all have particularly high tail centralities for two reasons: some are too far upstream, like Oil and Gas Extraction, while others produce just final outputs, like Hospitals, Offices of Physicians, Pharmaceuticals, and Scientific R&D.

The empirical analysis overall shows that tail centralities are very different from Domar weights. They are much larger, and among the top sectors closely related to out-degree. The sectors with the highest tail centrality are not necessarily those with the highest sales, but those that sell to the most sectors downstream. They represent utilities, professional and financial services, and petroleum products.

7 Conclusion

This paper studies large deviations in GDP in the context of a general nonlinear network production model. Its core result is to characterize the tail of the distribution of GDP based on the structure of the economy and the distribution of the shocks that affect it. For shocks with Weibull tails, a category that encompasses almost all distributions used in both theoretical and applied work, what determines tail risk in the economy is a single sufficient statistic that measures the most damaging possible combination of shocks.

The simple statement of the core idea is that what determines tail risk is the structure of the economy in the tail. For example, while granularity near steady-state affects the dynamics of the economy, what determines tail risk is whether the economy displays granularity in the tail. The paper shows how that can easily happen even in a perfectly symmetrical economy where all sectors are of equal size at steady-state.

A closely related point is that to understand the systemic risk of a sector – whether a large shock to it will spill over into the rest of the economy – one needs to understand the importance of the sector not on average but rather conditional on the occurrence of a large shock. The analysis shows that it is upstream sectors that produce inputs for a large fraction of GDP that are most systemically risky, while sectors that produce final outputs do not produce systemic risk.

More generally, the paper provides a theoretical foundation for analyzing tail risk in other settings. It shows how to construct an approximation for the dynamics of the economy that, rather than being valid only for small shocks, is valid explicitly for large shocks. That approximation can then be combined with assumptions about the shape of the tail of the shock distribution to yield a description of the tail behavior of the full economy.

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A Proofs

The proofs are organized by section, so that the unnumbered results from the main text are referred to by the section they appear in.

A.1 Lemma 1

The assumption that aggregate labor supply is inelastic and normalized to one implies that real GDP is

$$GDP = W/P_0 \tag{42}$$

where W is the wage and P_0 is the price of the consumption bundle. the index 0 indicates consumption (P_0 might be called a pseudo-price, since it is the cost of the consumption bundle, but not of an actual individual good). I use lower-case letters to denote logs, so $p_0 = \log P_0$, etc. Setting labor to be the numeraire, so that W is normalized to 1, the CES preferences for the consumer immediately imply

$$p_0 = \sum_{j=1}^{N} \beta_i p_i \tag{43}$$

$$gdp = -p_0 \tag{44}$$

Similarly, marginal cost pricing by the producers implies that the log price of good i is

$$p_i = -z_i + \frac{\alpha}{1-\sigma_i} \log \left(\sum_{j=1}^{N} A_{ij} \exp\left((1-\sigma_i)\, p_j \right) \right) \tag{45}$$

Now define $\phi_i = -\lim_{t\to\infty}(1/t)\log p_i$ and set the vector $\phi \equiv [\phi_1, ..., \phi_N]$. If that limit exists and is finite (a claim established below), then diving by t and taking limits of both sides of equations (43) and (45) gives

$$\lim_{t\to\infty} t^{-1} gdp = \beta' \phi \tag{46}$$

$$\phi_i = \theta_i + \alpha f_i(\phi) \tag{47}$$

where the mapping $f_i : \mathbb{R}^N \to \mathbb{R}$ is defined in equation (6). So then as long as the system for ϕ has a unique and finite solution, there is a unique and finite λ in Lemma 1.

To show that the system has a unique solution (guaranteeing that ϕ is also finite), define a mapping $g : \mathbb{R}^N \to \mathbb{R}^N$ such that the ith element of the vector $g(\phi)$ is

$$g_i(\phi) = \theta_i + \alpha f_i(\phi) \tag{48}$$

The set of solutions for ϕ is the set of fixed points for g, so we must just show that g has a unique fixed point. That follows from the Banach fixed point theorem if g_i is a contraction. It is straightforward to confirm the Blackwell's sufficient conditions hold here, giving the result. The continuity of the solution follows from the continuity of g in θ. This completes the proof of **part 2** of the lemma.

To get the constant $\mu(\theta)$, consider a series expansion, $p_i = -\mu_i - \phi_i t + o(1)$ (as $t \to \infty$). Inserting that into (45) taking limits, and using (47) yields

$$-\mu_i - \phi_i t = -z_i + \frac{\alpha}{1-\sigma_i} \log \left(\sum_{j=1}^{N} A_{ij} \exp\left((1-\sigma_i)(-\mu_j - \phi_j t)\right) \right) \tag{49}$$

$$\mu_i = \frac{\alpha}{\sigma_i - 1} \log \left(\sum_{j \in j^*(i)} A_{i,j} \exp\left((\sigma_i - 1)\mu_j\right) \right) \tag{50}$$

where

$$j^*(i) \equiv \begin{cases} \{j : \phi_j = \min_{k \in S_i} \phi_k\} & \text{if } \sigma_i < 1 \\ \{j : \phi_j = \max_{k \in S_i} \phi_k\} & \text{if } \sigma_i > 1 \end{cases} \tag{51}$$

and S_i is the set of inputs of sector i (i.e. $S_i \equiv \{j : A_{i,j} > 0\}$). The results for the Cobb–Douglas case following using similar analysis. It is again straightforward to confirm that μ is the fixed point of a contraction mapping. For GDP, the same analysis gives

$$\mu_0 = \sum_{j=1}^{N} \beta_i \mu_i \tag{52}$$

To show **parts 1 and 3**, consider the next term in a series expansion,[24] $p_i = b_i t^{-1} + \mu_i +$

[24]Formally this is a Laurent series since it has both positive and negative powers of t. The terms in the expansion for t^j with $j > 1$ must be zero because otherwise the first limit in this section would not converge.

One way to think about this is that it is an expansion in $\tau = t^{-1}$ aroud $\tau = 0$. There is a pole at $\tau = 0$ since log GDP and all log prices go to $\pm\infty$. The τ^{-1} terms remove the pole, at which point we just have a standard Taylor series in τ. Part 4 of Lemma 1 says that all terms in that Taylor series of order higher than 0 (i.e. everything but the constant) is equal to zero.

$\phi_i t + o\left(t^{-1}\right)$, and take limits as $t \to \infty$,

$$p_i t - (\mu_i + \phi_i t)\, t \;=\; \left(-\theta_i t + \frac{\alpha}{1-\sigma_i} \log \left(\sum_{j=1}^{n} A_{i,j} \exp\left((1-\sigma_i)\, p_j \right) \right) - \mu_i - \phi_i t \right) t \tag{53}$$

$$b_i \;=\; \lim_{t\to\infty} \left(\begin{array}{c} -\theta_i t + \frac{\alpha}{1-\sigma_i} \log \left(\sum_{j=1}^{n} A_{i,j} \exp\left((1-\sigma_i)\left(b_j t^{-1} + \mu_j + \phi_j t + o\left(t^{-1}\right)\right)\right)\right) \\ -\mu_i - \phi_i t \end{array} \right) t \tag{54}$$

$$b_i = \lim_{t\to\infty} \left[\left\{ \begin{array}{c} \frac{\alpha}{1-\sigma_i} \log \left(\sum_{j=1}^{n} A_{i,j} \exp\left((1-\sigma_i)\left(b_j t^{-1} + \mu_j + \left(\phi_j - f_i(\phi)\right) t + o\left(t^{-1}\right)\right)\right)\right) \\ -\mu_i \end{array} \right\} t \right] \tag{55}$$

The recursion from above for μ_i immediately implies that the limit of the term in braces is zero. Applying L'Hopital's rule then yields the result that the whole limit is equal to zero. The same analysis goes through for terms of any order, so that we have the statement from **part 3** of theorem 1. **Part 1** is a special case. ■

A.2 Theorem 1

We have

$$gdp(z) = \mu(\theta) + \lambda(\theta)\, t + \varepsilon(t,\theta) \tag{56}$$

where $\varepsilon(t,\theta)$ is an error that converges to 0 as $t \to \infty$ (from Lemma 1).

Now define

$$\bar{\varepsilon}(x) = \max_{\theta} \max_{t > \frac{x+\mu(\theta)}{-\lambda(\theta)}} \left|\varepsilon(t,\theta)\right| \tag{57}$$

Consider its limit as $t \to \infty$. Since the right-hand side is continuous in t, the limit can be passed through the maximum and we have

$$\lim_{x\to\infty} \bar{\varepsilon}(x) = 0 \tag{58}$$

Now note that

$$\Pr\left[gdp < -x \mid \theta\right] = \Pr\left[t + \frac{\varepsilon(t,\theta)}{\lambda(\theta)} > \frac{x+\mu(\theta)}{-\lambda(\theta)} \mid \theta \right] \tag{59}$$

where $\lambda(\theta) < 0$. In addition,

$$\begin{aligned}
\Pr\left[t + \frac{\bar{\varepsilon}(x)}{\lambda(\theta)} > \frac{x + \mu(\theta)}{-\lambda(\theta)} \mid \theta\right] &\leq \Pr\left[t + \frac{\varepsilon(t,\theta)}{\lambda(\theta)} > \frac{x + \mu(\theta)}{-\lambda(\theta)} \mid \theta\right] \leq \Pr\left[t - \frac{\bar{\varepsilon}(x)}{\lambda(\theta)} > \frac{x + \mu(\theta)}{-\lambda(\theta)} \mid \theta\right] \\
\Pr\left[t > \frac{x + \mu(\theta) + \bar{\varepsilon}(x)}{-\lambda(\theta)} \mid \theta\right] &\leq \Pr\left[gdp < -x \mid \theta\right] \leq \Pr\left[t > \frac{x + \mu(\theta) - \bar{\varepsilon}(x)}{-\lambda(\theta)} \mid \theta\right] \quad (60)
\end{aligned}$$

By integrating over the measure for θ (i.e. applying Fubini's theorem),

$$\Pr\left[gdp < -x\right] = \int_{\Theta} \Pr\left[gdp < -x \mid \theta\right] dm(\theta) \quad (61)$$

from which the result follows directly. ■

A.3 Propositions in section 3.2.1

Define $f^0 : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ to be the vectorized version of the function in (6). Define a transformation $T^0\phi = \theta + \alpha f^0(\phi)$, with $\phi^0 = T^0\phi^0$ the fixed point of that transformation.

After changing some σ_i, we have a new transformation f^1. First, take the case with σ_i transitioning from below 1 to being equal to 1 or more Then, necessarily,

$$T^1\phi \geq T^0\phi \quad (62)$$

for any ϕ, element-by-element, from which the first proposition follows.

The second proposition holds by the same argument. For example, suppose $\sigma_i < 1$ and the set S_i grows. Again, define a T^2 for the model with the larger S_i. We have

$$T^2\phi \leq T^0\phi \quad (63)$$

for any ϕ, element-by-element, which establishes the second proposition.

A.4 Proposition in section 3.3.1

Define a set of $N \times N$ matrices $\mathbf{A}_n$ representing restricted versions of the production network. For each $\mathbf{A}_n$, each sector is restricted to using just one of its inputs, so that every $\mathbf{A}_n$ has a single value of 1 in each row and is otherwise equal to zero, with links (1's) only appearing where $A_{i,j} > 0$. The set over all n of $\{\mathbf{A}_n\}$ represents every possible restricted network.[25] If

[25] The index n runs from 1 to the product of the number of inputs used by each each sector.

$\sigma_i = 1$, then sector i always uses the same mix of inputs, and the ith row of $\mathbf{A}_n$ is equal to $A_{i,\cdot}$ for every n.

Now define ϕ^* and n^*

$$\begin{aligned} n^* &= \arg\min_n \beta' \left(\mathbf{I} - \alpha \mathbf{A}_n\right)^{-1} \theta & (64) \\ \phi^* &= \left(\mathbf{I} - \alpha \mathbf{A}_{n^*}\right)^{-1} \theta & (65) \end{aligned}$$

where $1_{N\times 1}$ is a vector of 1's. That implies

$$\phi^* = \theta + \alpha \mathbf{A}_{n^*} \phi^* \tag{66}$$

Suppose $\mathbf{A}_{n^*}$ is not the solution to the recursion from Lemma 1 for ϕ^*. Then, clearly, element-by-element $T\phi^* \leq \phi^*$ (where T is the operator $T\phi \equiv \theta + \alpha f(\phi)$), and whatever the solution is for ϕ in Lemma 1, it will be, element-by-element, weakly smaller than ϕ^*. But that solution is always of the form $\left(\mathbf{I} - \alpha \mathbf{A}_n\right)^{-1} \theta$, leading to a contradiction with the original construction of ϕ^*. So ϕ^* must be the solution to the recursion with $T\phi^* = \phi^*$. The result for GDP then follows immediately.

A.5 Proposition in section 3.4.1

The proposition follows from the fact that GDP has a linear asymptote with a constant that is different from zero. Any finite-order Taylor series necessarily diverges infinitely far from the asymptote unless *gdp* is actually linear. This is obvious when the order of the approximation is greater than 1, since eventually the higher order terms dominate. A linear approximation also eventually diverges in the nonlinear case since the slope of *gdp* at $t = 0$ is not the same as at $\pm\infty$.

A.6 Corollary in section 3.4.2

Based on the results above, given δ there exists a t^* such that $|gdp(\theta t) - (\mu(\theta) + \lambda(\theta) t)| < \delta$ for $t > t^*$. For $t < t^*$,

$$\begin{aligned} |gdp(\theta t) - (\mu(\theta) + \lambda(\theta) t)| &\leq |gdp(\theta t)| + |\mu(\theta) + \lambda(\theta) t| & (67) \\ &\leq \max_{t \leq t^*} |gdp(\theta t)| + |\mu(\theta)| + |\lambda(\theta)| t^* & (68) \end{aligned}$$

So the fact that *gdp* is finite for any finite t gives the result. ■

A.7 Distribution examples in section 4.1.1

A.7.1 Multivariate normal

Suppose $z \sim N(0, \Sigma)$. The probability density of z is then proportional to $\exp(-z'\Sigma^{-1}z/2)$. Now note that when $z = \theta t$, we have

$$
\begin{aligned}
t &= \|z\| && (69)\\
\theta &= z/\|z\| && (70)
\end{aligned}
$$

Now consider $\exp\left(-(t/s(\theta))^2/2\right)$. The main text claims that in this case $s(\theta) = (\theta'\Sigma^{-1}\theta)^{-1/2}$. So then

$$
\begin{aligned}
\exp\left(-(t/s(\theta))^2/2\right) &= \exp\left(-\left(\|z\|\left(z'\Sigma^{-1}z'\|z\|^{-2}\right)^{1/2}\right)^2/2\right) && (71)\\
&= \exp\left(-z'\Sigma^{-1}z'/2\right) && (72)
\end{aligned}
$$

as desired.

The last point is that the text claims that the complementary CDF of t is $\exp\left(-(t/s(\theta))^2/2\right)$, even though the results above are for the PDF. It is straightforward to show through integration by parts that the complementary CDF has the form

$$
\int_x^\infty \exp\left(-b^2/2\right) db = \exp\left(-x^2\right)\left(\frac{1}{x} + o\left(x^{-2}\right)\right) \tag{73}
$$

The x^{-1} term can be added to the analysis for the tail probabilities and it has no effect since it is dominated by the exponential.

A.7.2 I.i.d. Exponential distribution

In this case, the probability density is $\exp(-\|z\|_1/\eta)$, where $\|\cdot\|_1$ is the L_1-norm. So in this case,

$$
\begin{aligned}
\exp\left(-(t/s(\theta))/\eta\right) &= \exp\left(-\left(\|z\|\left\|\frac{z}{\|z\|}\right\|_1\right)/\eta\right) && (74)\\
&= \exp\left(-\|z\|_1/\eta\right) && (75)
\end{aligned}
$$

again as desired.

A similar result holds for i.i.d. Weibull variables more generally, with the L_1-norm replaced by the L_κ norm where κ is the Weibull shape parameter.

A.8 Second corollary in section 4.2

Recall the notation from the proof of Theorem 1 that

$$gdp(\theta t) = \mu(\theta) + \lambda(\theta)t + \varepsilon(\theta, t) \tag{76}$$

and that $|\varepsilon(\theta, t)| < \bar{\varepsilon}(x)$ for $t > \frac{x+\mu(\theta)}{-\lambda(\theta)}$. We want to compare $\Pr[gdp < -x]$ with $\Pr[gdp > x]$. Define $\varepsilon'(x) = \max(\bar{\varepsilon}(x), \bar{\varepsilon}(-x))$. We have the bounds

$$\Pr[gdp < -x] \geq \int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x - \mu(\theta) + \varepsilon'(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \tag{77}$$

$$\Pr[gdp > x] \leq \int_{\eta:\lambda(\eta)>0} \bar{F}\left(\frac{x - \mu(\eta) - \varepsilon'(x)}{s(\eta)\lambda(\eta)}\right) dm(\theta) \tag{78}$$

Now first note that, for θ such that $\lambda(\theta) < 0$,

$$\frac{x - \mu(-\theta) - \varepsilon'(x)}{s(-\theta)\lambda(-\theta)} - \frac{x - \mu(\theta) + \varepsilon'(x)}{-s(\theta)\lambda(\theta)} = \left(\frac{1}{s(-\theta)\lambda(-\theta)} - \frac{1}{-s(\theta)\lambda(\theta)}\right)x + \frac{-\mu(-\theta) - \varepsilon'(x)}{s(-\theta)\lambda(-\theta)} - \frac{-\mu(\theta) + \varepsilon'}{-s(\theta)\lambda(} \tag{79}$$

So there exists an $\bar{x}$ such that for $x > \bar{x}$, the argument of $\bar{F}$ in the integral for (77) is smaller than that in (78) for any given θ. In addition,

$$m(\{\eta : \lambda(\eta) > 0\}) \leq m(\{\theta : \lambda(\theta) < 0\}) \tag{80}$$

which yields the result.

A.9 Proposition in section 4.4

The statement of Theorem 1 is

$$\int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x - \mu(\theta) + \varepsilon(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \leq \Pr[gdp < -x] \leq \int_{\theta:\lambda(\theta)<0} \bar{F}\left(\frac{x - \mu(\theta) - \varepsilon(x)}{-s(\theta)\lambda(\theta)}\right) dm(\theta) \tag{81}$$

In this case we have

$$\bar{F}(s) = c\exp\left(-\beta(t - \bar{t})^{\kappa}\right) \tag{82}$$

$$\text{where } c = \Pr(t \leq \bar{t}) \tag{83}$$

If the limits of the two integrals in (81) are the same, then that limit is also the limit for $\Pr[gdp < -x]$. This section gives the derivation for the right-hand side limit, with the arguments holding equivalently on the left with the sign of $\varepsilon(x)$ reversed.

We have

$$\left(\int_{\theta:\lambda(\theta)<0}\bar{F}\left(\frac{x-\mu(\theta)-\varepsilon(x)}{-s(\theta)\lambda(\theta)}\right)dm(\theta)\right)^{1/x^{\kappa}} \tag{84}$$

$$= \left[\int_{\theta\in\Theta}\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}-\frac{\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)}-\frac{\bar{t}}{x}\right)^{\kappa}\right)^{x^{\kappa}}dm(\theta)\right]^{1/x^{\kappa}} \tag{85}$$

Now consider the limit as $x\to\infty$. I show that the limit of the right-hand side is the essential supremum of $\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^{\kappa}\right)$ with respect to the measure $m(\theta)$ (i.e. the measure of the set of θ such that $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right)$ is above the essential supremum is zero). Denote that by $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right)\right\|_{\infty}$.

The structure of this proof is from Ash and Doleans-Dade (2000), page 470, with the addition of the convergence of the argument of the integral with respect to x.

Define, for notational convenience,

$$f(\theta) = \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^{\kappa}\right) \tag{86}$$

$$f(\theta;x) = \exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}-\frac{\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)}-\frac{\bar{t}}{x}\right)^{\kappa}\right) \tag{87}$$

Lemma A1. $\lim_{x\to\infty}\|f(\theta;x)\|_{\infty}=\|f(\theta)\|_{\infty}$.

Proof. $f(\theta;x)\to f(\theta)$ pointwise trivially. The difference $|f(\theta;x)-f(\theta)|$ is bounded due to the facts that $\varepsilon(x)$ and $\mu(\theta)$ are bounded and that $f(\theta;x)$ is decreasing in $s(\theta)\lambda(\theta)$ (for sufficiently large x), which is bounded from above (and below, by zero). $f(\theta;x)$ then converges uniformly to $f(\theta)$, from which $\|f(\theta;x)\|_{\infty}\to\|f(\theta)\|_{\infty}$ follows, since, using the reverse triangle inequality,

$$|\|f(\theta;x)\|_{\infty}-\|f(\theta)\|_{\infty}|\leq\|f(\theta)-f(\theta;x)\|_{\infty} \tag{88}$$

∎

Lemma A2. $\limsup_{x\to\infty}\left[\int_{\theta\in\Theta}f(\theta;x)^{x^{\kappa}}dm(\theta)\right]^{1/x^{\kappa}}\leq\|f(\theta)\|_{\infty}$

Proof. We have (except possibly on a set of measure zero)

$$\|f(\theta;x)\|_{x^{\kappa}}\leq\|\|f(\theta;x)\|_{\infty}\|_{x^{\kappa}}$$

Taking limits of both sides

$$\begin{aligned}\lim_{x\to\infty}\|f(\theta;x)\|_{x^\kappa} &\leq \lim_{x\to\infty}\|\|f(\theta;x)\|_\infty\|_{x^\kappa} &(89)\\ &= \lim_{x\to\infty}\|f(\theta;x)\|_\infty &(90)\\ &= \|f(\theta)\|_\infty &(91)\end{aligned}$$

where the second line follows from the fact that $\|f(\theta;x)\|_\infty$ is constant and the third line uses lemma A1. ■

Lemma A3. $\liminf_{x\to\infty}\left[\int_{\theta\in\Theta} f(\theta;x)^{x^\kappa}\,dm(\theta)\right]^{1/x^\kappa} \geq \|f(\theta)\|_\infty$

Proof. Consider some $\eta>0$, and set $A=\left\{\theta\in\Theta:\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^\kappa\right)\geq\left\|\exp\left(-\left(\frac{1}{-s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty-\eta\right\}$. Consider also the set $A'=\left\{\theta\in\Theta:\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)}-\frac{\bar t}{x}\right)^\kappa\right)\geq\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty-\eta\right\}$. For any η such that A has positive measure, there exists an $\bar x(\eta)$ sufficiently large that A' has positive measure for all $x>\bar x(\eta)$ due to the continuity of $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)}-\frac{\bar t}{x}\right)^\kappa\right)$ and the fact that $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)}\right)^\kappa\right)\to\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)$ as $x\to\infty$.

It is then the case that for $x>\bar x(\eta)$

$$\begin{aligned}&\int_{\theta\in\Theta}\exp\left(-\left(\frac{1}{\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)}-\frac{\bar t}{x}\right)^\kappa\right)^{x^\kappa}dm(\theta) &(92)\\ \geq&\int_{A'}\exp\left(-\left(\frac{1}{\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{s(\theta)\lambda(\theta)}-\frac{\bar t}{x}\right)^\kappa\right)^{x^\kappa}dm(\theta) &(93)\\ \geq&\left(\left\|\exp\left(-\left(\frac{1}{\lambda(\theta)}\right)^\kappa\right)\right\|_\infty-\eta\right)^{x^\kappa}\mu(A') &(94)\end{aligned}$$

Since $\mu(A')>0$ from the definition of $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty$ (ignoring the trivial case of a constant value for $\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)$), and since the above holds for any $\eta>0$,

$$\liminf_{x\to\infty}\left[\int_{\theta\in\Theta}\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}-\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\frac{1}{\lambda(\theta)}-\frac{\bar t}{x}\right)^\kappa\right)^{x^\kappa}\right]^{1/x^\kappa}dm(\theta)\geq\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right.\right. \tag{95}$$

■

Proof of the proposition: Since both the lim inf and lim sup are equal to $\left\|\exp\left(-\left(\frac{1}{s(\theta)\lambda(\theta)}\right)^\kappa\right)\right\|_\infty$, the limit is also.

For the second part, in the set Θ^*, there exists an η such that $|-s(\theta)\lambda(\theta)|<\|-s(\theta)\lambda(\theta)\|_\infty-$

η. Therefore

$$\frac{\int_{\theta\in\Theta^*}\exp\left(-\left(\frac{x+\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)}{\int_{\theta}\exp\left(-\left(\frac{x-\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)} \leq \Pr\left[\theta\in\Theta^* \mid gdp<-x\right] \leq \frac{\int_{\theta\in\Theta^*}\exp\left(-\left(\frac{x-\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)}{\int_{\theta}\exp\left(-\left(\frac{x+\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)} \tag{96}$$

Again, we show that both sides of the inequality have the same limit. For a sufficiently large x,

$$\frac{\int_{\theta\in\Theta^*}\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)}{\int_{\theta\in\Theta}\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)} \leq \frac{\int_{\theta\in\Theta^*}\exp\left(-\left(\frac{x\pm\varepsilon(x)-\mu(\theta)}{\left(\left\|-s(\theta)\lambda(\theta)\right\|_{\infty}-\eta\right)}-\bar{t}\right)^{\kappa}\right)dm(\theta)}{\int_{\theta\in\Theta:|\lambda(\theta)|>|\lambda(\theta)|-\eta/2}\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-s(\theta)\lambda(\theta)}-\bar{t}\right)^{\kappa}\right)dm(\theta)}$$

$$\begin{aligned}
&\leq \frac{\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-\left(\left\|s(\theta)\lambda(\theta)\right\|_{\infty}-\eta\right)}-\bar{t}\right)^{\kappa}\right)}{\exp\left(-\left(\frac{x-\pm\varepsilon(x)-\mu(\theta)}{-\left(\left\|s(\theta)\lambda(\theta)\right\|_{\infty}-\eta/2\right)}-\bar{t}\right)^{\kappa}\right)}\frac{1}{m\left(\left\{\theta\in\Theta:|\lambda(\theta)|>\left\|\lambda(\theta)\right\|_{\infty}-\eta/2\right\}\right)} &(97)\\
&\to 0 &(98)
\end{aligned}$$

■

A.10 First example in section 4.4.2: Gaussian tails

The aim is to find $\max_{\tilde{\theta}:\left\|\tilde{\theta}\right\|_2=1}\left\|-s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\|$. Now note that $b\lambda\left(\tilde{\theta}\right)=\lambda\left(b\tilde{\theta}\right)$, and hence $s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)=\lambda\left(\tilde{\theta}s\left(\tilde{\theta}\right)\right)$. We can then apply a change of variables, with $\theta=\tilde{\theta}s\left(\tilde{\theta}\right)$. Note that $\tilde{\theta}=\theta/\left\|\theta\right\|$, so we have

$$\max_{\tilde{\theta}:\left\|\tilde{\theta}\right\|=1}\left\|-s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\| = \max_{\theta:\left\|\theta/s(\theta/\|\theta\|)\right\|=1}\left\|-\lambda(\theta)\right\| \tag{99}$$

Now in this particular case,

$$\begin{aligned}
\left\|\theta/s\left(\theta/\left\|\theta\right\|\right)\right\| &= \left\|\theta\left(\theta'\Sigma^{-1}\theta\right)^{1/2}\left\|\theta\right\|^{-1}\right\| &(100)\\
&= \left(\theta'\Sigma^{-1}\theta\right)^{1/2} &(101)
\end{aligned}$$

The Lagrangian is then

$$\max_{\theta}\max_{n}\left(-D_n'\theta\right)-\frac{\gamma}{2}\left(\theta'\Sigma^{-1}\theta-1\right) \tag{102}$$

where γ is the multiplier on the constraint on θ. Reversing the order of the optimization gives

$$\min_n \min_\theta D_n'\theta + \frac{\gamma}{2}\left(\theta'\Sigma^{-1}\theta - 1\right) \tag{103}$$

$$\theta = -\gamma^{-1}\Sigma D_n \tag{104}$$

where the second line uses the first-order condition. Solving for the constraint for a given θ yields

$$\theta = -\frac{1}{\sqrt{D_n'\Sigma D_n}}\Sigma D_n \tag{105}$$

(recall that θ here does not have norm 1 but instead satisfies $\theta'\Sigma^{-1}\theta = 1$).

Finally, the value of the objective (which is equal to $\left\|-s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\|$) is

$$\left\|-s\left(\tilde{\theta}\right)\lambda\left(\tilde{\theta}\right)\right\| = -D_n'\theta \tag{106}$$

$$= \sqrt{D_n'\Sigma D_n} \tag{107}$$

A.11 Second example in section 4.4.2: symmetric economy

The complete symmetry of the economy, along with the fact that output is homogeneous of degree $1/(1-\alpha)$ in the vector of productivities immediately implies that $D_{ss,i} = N^{-1}/(1-\alpha)$ for all i.

It is straightforward to confirm that $\phi_i = \theta_i + \frac{\alpha}{1-\alpha}\theta_{\min}$, where $\theta_{\min} = \min_i \theta_i$. Combining that with the final utility function yields

$$\lambda(\theta) = \sum_i N^{-1}\theta_i + \frac{\alpha}{1-\alpha}\theta_{\min} \tag{108}$$

For the right tail, the Lagrangian is

$$\max_\theta \lambda(\theta) - \frac{\gamma}{2}\left(\sum_i \theta_i^2 - 1\right) = \max_\theta \sum_i N^{-1}\theta_i + \frac{\alpha}{1-\alpha}\theta_{\min} - \frac{\gamma}{2}\sum_i \theta_i^2 \tag{109}$$

That problem is nonconvex and is solved at the point $\theta_i = N^{-1/2}$ for all i. That yields

$$\|\sigma\lambda(\theta)\|_\infty = \frac{1}{1-\alpha}\sigma N^{-1/2} \tag{110}$$

For the left tail, the optimization is

$$\max_{\theta} - \sum_{i} N^{-1}\theta_i - \frac{\alpha}{1-\alpha}\theta_{\min} - \frac{\gamma}{2}\left(\sum_{i}\theta_i^2 - 1\right) \tag{111}$$

The first-order condition gives

$$\theta_i = \begin{cases} -\frac{N^{-1}}{\sqrt{(N-1)N^{-2}+\left(N^{-1}+\frac{\alpha}{1-\alpha}\right)^2}} & \text{for } i \neq i\min \\ -\frac{N^{-1}+\frac{\alpha}{1-\alpha}}{\sqrt{(N-1)N^{-2}+\left(N^{-1}+\frac{\alpha}{1-\alpha}\right)^2}} & \text{for } i = i\min \end{cases} \tag{112}$$

where $i\min = \arg\min_i \theta_i$. Note that this solution for θ_i is obviously not unique – $i\min$ can be equal to any integer between 1 and N. To find $\sigma\lambda(\theta)$, we have

$$\begin{aligned} \left\|-\sigma\lambda(\theta)\right\|_{\infty} &= \sigma\frac{N^{-1}+2N^{-1}\frac{\alpha}{1-\alpha}+\left(\frac{\alpha}{1-\alpha}\right)^2}{\sqrt{(N-1)\,N^{-2}+\left(N^{-1}+\frac{\alpha}{1-\alpha}\right)^2}} && (113) \\ &= \sigma\frac{\alpha}{1-\alpha}+O\left(N^{-1/2}\right) && (114) \end{aligned}$$

where $x = O\left(N^{-1/2}\right) \Leftrightarrow |x| \leq MN^{-1/2}$ for all x greater than some x_0 and for some constant M.

A.12 Proposition in section 4.6

To prove this, we will show that the claimed set of limits (along with a third additional result) is consistent with the model's equilibrium conditions. The limits are

$$\lim_{t\to\infty}\frac{y_j}{t} = \lim_{t\to\infty}\frac{c_j}{t} = \lim_{t\to\infty}\frac{-p_j}{t} = \phi_j \tag{115}$$

and the equilibrium conditions are

$$\begin{aligned} Y_i &= \exp(z_i)\, L_i^{1-\alpha}\left(\sum_j A_{i,j}^{1/\sigma_i} X_{i,j}^{(\sigma_i-1)/\sigma_i}\right)^{\alpha\sigma_i/(\sigma_i-1)} && (116) \\ Y_j &= C_j + \sum_i X_{i,j} && (117) \\ P_j &= P_0 C^{1/\sigma_0}\beta_j^{1/\sigma_0} C_j^{-1/\sigma_0} && (118) \\ P_j &= \alpha P_i \exp(z_i)\left(Y_i/\exp(z_i)\right)^{(\alpha-(\sigma_i-1)/\sigma_i)/\alpha} A_{i,j} X_{i,j}^{-1/\sigma_i} && (119) \\ 1 &= (1-\alpha)\, P_i Y_i / L_i && (120) \end{aligned}$$

We first prove some small lemmas. Define

$$j^*(i) \equiv \begin{cases} \arg\min_{j\in S(i)} \phi_j \text{ if } \sigma_i < 1 \\ \arg\max_{j\in S(i)} \phi_j \text{ if } \sigma_i > 1 \end{cases} \tag{121}$$

For $\sigma_i = 1$, set $\phi_{j^*(i)} = f_i(\phi)$.

Lemma A4. *If $\phi_{j^*(i)} + \sigma_i\left(\phi_j - \phi_{j^*(i)}\right) \leq \phi_j$ for all $j \in S(i)$*

Proof. First suppose $\sigma_i < 1$. Then $\phi_j - \phi_{j^*(i)} \geq 0$ and $\sigma_i < 1$, from which the result immediately follows. To see the result for $\sigma_i > 1$, note that

$$\phi_{j^*(i)} + \sigma_i\left(\phi_j - \phi_{j^*(i)}\right) = \phi_j + (1-\sigma_i)\left(\phi_j - \phi_{j^*(i)}\right) \tag{122}$$

Since $1 - \sigma_i > 0$ and $\phi_j - \phi_{j^*(i)} \leq 0$ in this case the result again follows. It holds trivially for $\sigma_i = 1$. ∎

Lemma A5. $f_i\left(\left[\phi_{j^*(i)} + \sigma_i\left(\phi_j - \phi_{j^*(i)}\right)\right]\right) = \phi_{j^*(i)}$

Proof. If $\sigma_i > 1$, then $f_i = \max_{j\in S(i)}$. Note that $\phi_{j^*(i)} + \sigma_i\left(\phi_j - \phi_{j^*(i)}\right) \leq \phi_j$, and $\phi_{j^*(i)} = \max_{j\in S(i)} \phi_j$. Then the result immediately follows. Suppose $\sigma_i < 1$. Then $f_i = \min_{j\in S(i)}$ and $\phi_{j^*(i)} = \min_{j\in S(i)} \phi_j$. In this case, $\phi_{j^*(i)} + \sigma_i\left(\phi_j - \phi_{j^*(i)}\right) \leq \phi_{j^*(i)}$, with equality if $j = j^*(i)$, again giving the result. ∎

This result follows since $f_i(a\phi + c) = af_i(\phi) + c$ for any constants a and c.

To prove the result, we also need the use of inputs. We guess that

$$\lim_{t\to\infty} \frac{x_{i,j}}{t} = \phi_{j^*(i)} + \sigma_i\left[\phi_j - \phi_{j^*(i)}\right] \tag{123}$$

We need to verify that the above, along with the solution in the proposition, satisfies, in the limit, the equilibrium conditions (116)-(120).

We first take limits of the equilibrium conditions. For any variable g_j, define

$$\phi_{g,j} \equiv \lim_{t\to\infty} \frac{g_j}{t} \tag{124}$$

Inspection of equation (120) shows that, given the guesses for $\phi_{p,i}$ and $\phi_{y,i}$, we must have $\phi_{l,i} = 0$.

Dividing the equilibrium conditions (equations (116)-(120), respectively) by t and taking limits as $t \to \infty$ yields

$$
\begin{aligned}
\phi_{y,i} &= \theta_i + (1-\alpha)\,\phi_{l,i} + \alpha f_i\left([\phi_{x,i,j}]\right) && (125)\\
\phi_{y,j} &= \max\left\{\phi_{c,j}, \max_i \phi_{x,i,j}\right\} && (126)\\
\phi_j &= \phi_0 + \sigma_0^{-1}\phi_c - \sigma_0^{-1}\left(\phi_{j^*(0)} + \sigma_0\left[\phi_0 - \phi_{j^*(0)}\right]\right) && (127)\\
\phi_{p,j} &= \phi_{p,i} + \theta_i + \frac{\alpha - (\sigma_i - 1)/\sigma_i}{\alpha}\left(\phi_{y,i} - \theta_i\right) - \sigma_i^{-1}\phi_{x,i,j} && (128)\\
0 &= \phi_{p,i} + \phi_{y,i} - \phi_{l,i} && (129)
\end{aligned}
$$

Equation (125) holds using Lemma A5 and the recursion for ϕ_i. Equation (126) holds trivially using the guesses and Lemma A4. Equations (127)-(129) hold trivially after inserting the various guesses.

A.13 Proposition in section 5.1

The left-hand inequality follows from assuming that the sectors immediately downstream of i have no other downstream users (except final output). The right-hand inequality follows from assuming that the remainder of GDP that is not immediately downstream of sector i's users is a single step further downstream. ■

B Extensions and additional results

B.1 Which is the right approximation to use?

The usual Taylor approximation is around $z = 0$, while this paper focuses on $z \to \infty$. As z grows, the tail approximation is eventually superior, so for any statements about limiting probabilities as $\log gdp \to \pm\infty$, it is the correct representation. But at what point does that transition happen? To shed light on that question, first note that $gdp\,(0) = 0$. So to know the size of the error from using the tail approximation when $z = 0$, we need to know the constants $\mu\,(\theta)$.

Lemma B6. ***Part 4:*** *The constant in equation (4) is $\mu\,(\theta) = \mu_0$, where the vector μ solves the recursion*

$$
\mu_i = -\frac{\alpha}{\sigma_i - 1}\log\left(\sum_{j \in j^*(i)} A_{i,j}\exp\left((\sigma_i - 1)\,\mu_j\right)\right) \tag{130}
$$

with

$$j^*(i) \equiv \begin{cases} \{j : \phi_j = \min_{k \in S_i} \phi_k\} & \textit{if } \sigma_i < 1 \\ \{j : \phi_j = \max \phi_k\} & \textit{if } \sigma_i > 1 \\ \{j\} & \textit{if } \sigma_i = 1 \end{cases} \tag{131}$$

The constant, $\mu(\theta)$, thus increases when the elasticity of substitution is closer to 1 and when the upstream source of shocks is units that are relatively small (have small $A_{i,j}$). Those factors cause the tail approximation to have a relatively larger error as $t \to 0$.

The concave case

In the case where *gdp* is globally concave in the shocks – $\sigma_i \leq 1 \; \forall \; i$ – a stronger result is available. The error for the tail approximation then is smaller than for the first-order Taylor series when

$$t > \frac{\mu(\theta)}{D'_{ss}\theta - \lambda(\theta)} \tag{132}$$

The tail approximation is superior if t is sufficiently large – larger when the constant $\mu(\theta)$ is larger or the gap between the local and tail approximations, $D'_{ss}\theta - \lambda(\theta)$, is smaller. Furthermore, in that case, appendix **??** shows that $\mu(\theta) > 0$ and that it is increasing as any of the σ_i moves closer to 1.

That immediately implies that when any elasticity gets closer to 1, the cutoff point gets larger, since σ_i has no impact on λ and D_{ss}. The closer are the various elasticities to 1, the larger the shocks have to be in order for the tail approximation to be superior to a local approximation.

It is less clear what the effects of the $A_{i,j}$ parameters on the cutoff is because they affect both μ and D_{ss}. Note, though, that (in the concave case), when $\lambda(\theta) < 0$ – i.e. when thinking about shocks that reduce GDP – the tail approximation cannot possibly be the better of the two until $\mu(\theta) + \lambda(\theta)t < 0$, and the point where that happens necessarily increases as the A parameters for the minimizing units (i.e. the units $j \in j^*(i)$ for some i) decline.

B.2 Power law tails

In the case of power law tails, the measure $m(\theta)$ and scale $s(\theta)$ have the same effect on the distribution, so we can normalize $s(\theta) = 1$.

Proposition. *Suppose s is distributed according to a power law for $t > \bar{t}$:*

$$\bar{F}(t) = c(t/\bar{t})^{-\kappa} \tag{133}$$

$$\textit{where } c = \Pr(y \geq \bar{y}) \tag{134}$$

with $s(\theta)=1$. *Define* $\Theta_+ \equiv \{\theta : \lambda(\theta) > 0\}$ *and* Θ_- *analogously. Then*

1.

$$\lim_{x\to\infty} \Pr[gdp < -x] / \left(c\bar{t}^{\kappa} x^{-\kappa}\right) = \int_{\Theta_-} (-s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta) \tag{135}$$

$$\lim_{x\to\infty} \Pr[gdp > x] / \left(c\bar{t}^{\kappa} x^{-\kappa}\right) = \int_{\Theta+} (s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta) \tag{136}$$

2.

$$\lim_{x\to\infty} \Pr[\theta \in \Theta^* \mid gdp < -x] = \frac{\int_{\Theta^*_-} (-s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta)}{\int_{\Theta_-} (-s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta)} \tag{137}$$

$$\lim_{x\to\infty} \Pr[\theta \in \Theta^* \mid gdp > x] = \frac{\int_{\Theta^*_+} (s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta)}{\int_{\Theta_+} (s(\theta)\lambda(\theta))^{\kappa}\, dm(\theta)} \tag{138}$$

So when the shocks have power law tails, equation (135) gives two results for the tail of GDP. First, GDP has, in the limit, a power law tail with the same decay rate as the shocks, κ. Second, the probability of a large deviation in gdp depends on an average (with respect to the measure m) across all possible shocks of the tail slope, $\lambda(\theta)$. When the tail slopes tend to be larger, the probability of a large deviation in GDP is larger.

The integral on the right-hand side of (135) thus gives a formal measure of the fragility of the economy. Recall that $\lambda(\theta)$ comes from theorem 1 and depends only on the structure of the economy, not the shock distribution. So the integral shows how the structure of the economy determines the probability of a large decline in GDP. Economies in which that integral are larger have greater risk of large declines – in a sense are more fragile. The parameter κ affects the relative weighting of the integral. When κ is small, it is essentially an average of $\lambda(\theta)$ (with respect to the measure $m(\theta)$). When κ is larger, on the other hand, the integral is determined more and more by the largest values of $\lambda(\theta)$.

Equation (136) gives the probability of a large increase in GDP. The tail shape is again κ. Differences in the probability of large increases in large declines are determined by the average value of $\lambda(\theta)$ when $\lambda(\theta)$ is positive versus negative. Asymmetry in the distribution of GDP comes from asymmetry in the tail slopes.

Applying Bayes' rule as above yields the second part of the result. The probability of any a large deviation in GDP being caused by any particular combination of shocks, θ, depends on the value of $\lambda(\theta)$ relative to the average value, again scaling by κ and with respect to the measure m. When κ is relatively small, shocks are very heavy tailed, so that in some sense any combination becomes close to equally likely (that is the $\kappa \to 0$ limit). On the other hand, as $\kappa \to \infty$, only the θ that has the largest $\lambda(\theta)$ has any likelihood of causing a large deviation in GDP.

A special case within the general Pareto tail is if the unit shocks are i.i.d.. Then the measure $m(\theta)$ puts mass only on the axes – $\Pr[\theta = e_i] = 1/N$, and $\Pr[\theta \in \Theta^*] = 0$ for any set Θ^* that does not contain one of the e_i (Resnick (2007), section 6.5.1). In that case, the above formulas specialize to

$$\lim_{x\to\infty} \Pr[gdp < -x] / \left(c\bar{s}^{\kappa} x^{-\kappa}\right) = N^{-1} \sum_i \left(-\gamma_i^L\right)^{\kappa} \tag{139}$$

$$\lim_{x\to\infty} \Pr[\theta = -e_i \mid gdp < -x] = \frac{\left(-\gamma_i^L\right)^{\kappa}}{N^{-1}\sum_i \left(-\gamma_i^L\right)^{\kappa}} \tag{140}$$

With independent Pareto tails, then, the left tail distribution for GDP depends on the left tail centralities, and the right tail of GDP on the right tail centralities. That immediately yields a prediction for asymmetry in GDP: any time the left tail centralities are uniformly larger than the right tail centralities – e.g. if all elasticities of substitution are less than 1 – the distribution of gdp will be skewed left in the sense that the $\Pr[gdp < -x] / \Pr[gdp < x] > 1$.

B.2.1 Proof of Proposition B.2

We have

$$\bar{F}(s) = c\,(t/\bar{t})^{-\kappa} \tag{141}$$

$$\text{where } c = \Pr(t \geq \bar{t}) \tag{142}$$

Inserting those into the formula from theorem 1, we again show that the integrals have the same bound. The bound is now

$$\int_{\theta\in\Theta:\lambda(\theta)<0} \left(\frac{x + \varepsilon(x) + \mu(\theta)}{-xs(\theta)\lambda(\theta)}\right)^{-\kappa} dm(\theta) \leq \Pr[gdp < -x] / \left(c\bar{t}^{\kappa} x^{-\kappa}\right) \leq \int_{\theta\in\Theta:\lambda(\theta)<0} \left(\frac{x - \varepsilon(x) + \mu(\theta)}{-xs(\theta)\lambda(\theta)}\right)^{-\kappa} dm(\theta) \tag{143}$$

with limit

$$\lim_{x\to\infty} \int_{\theta\in\Theta:s(\theta)\lambda(\theta)<0} \left(-\left(s(\theta)\lambda(\theta)\right)^{-1} + x^{-1}\frac{\pm\varepsilon(x) + \mu(\theta)}{-s(\theta)\lambda(\theta)}\right)^{-\kappa} dm(\theta) \tag{144}$$

Again, recall that the $\pm\varepsilon(x)$ term is bounded, as are $\lambda(\theta)$ and $\mu(\theta)$ (since Θ is compact). The argument of the integral therefore converges uniformly, since

$$\left\|\left(-\left(s(\theta)\lambda(\theta)\right)^{-1}+x^{-1}\frac{\pm\varepsilon(x)+\mu(\theta)}{-s(\theta)\lambda(\theta)}\right)^{-\kappa}-\left(-\left(s(\theta)\lambda(\theta)\right)^{\kappa}\right)\right\|_{\infty}\leq\left\|\left(-\lambda(\theta)^{-1}+x^{-1}\frac{\pm\varepsilon(x)+\mu(\theta)}{-\left(s(\theta)\lambda(\theta)\right)}\right)^{-1}\right.$$

$$\leq \left\|-s(\theta)\lambda(\theta)\left(1+\frac{\pm\varepsilon(x)+\mu(\theta)}{x}\right)^{-1}\right\|_{\infty}^{\kappa}+\left\|-s(\theta)\lambda(\theta)\right\|_{\infty}^{\kappa} \tag{145}$$

$$\leq \left\|-s(\theta)\lambda(\theta)\right\|_{\infty}^{\kappa}\left\|\frac{x}{x+\inf_{\theta\in\Theta}\{\mu(\theta)\}}\right\|_{\infty}^{\kappa}+\left\|-s(\theta)\lambda(\theta)\right\|_{\infty}^{\kappa} \tag{146}$$

with the last line being bounded. Passing the limit through the integral yields the result from the text. The second claim is again an application of Bayes' rule. ■

B.3 Relaxing the CES assumption

This section extends the baseline result to a broader class of production functions and shows that theorem 1 holds with no modification.

Consider the same competitive economy as in the main analysis, with the only difference that each sector's production need not be CES. Rather, just assume that it each sector has constant returns to scale. Again, without loss of generality, assume that labor and materials are combined with a unit elasticity of substitution. Those assumptions imply that, in competitive equilibrium, the price of good i is given by

$$P_i=\frac{1}{Z_i}W^{1-\alpha}(C_i(P_1,\ldots,P_n))^{\alpha} \tag{147}$$

where Z_i is the productivity shock to industry i, C_i is a homogenous function of degree one, and $\alpha<1$. In addition to the intermediate-input-producing industries, there is also an industry with cost function C_0 that produces a final good, which is then sold to the representative consumer. Therefore, the final good price, P_0, also satisfies equation (147), with the convention that $\alpha_0=1$ and $Z_0=1$.

The only additional assumption imposed on C_i is that

$$\lim_{t\to\infty}\frac{1}{t}\log C_i\left(\exp\left(\phi_l t\right),\exp\left(\phi_1 t\right),...,\exp\left(\phi_n t\right)\right)=\tilde{f}_i\left(\phi_l,\phi_1,...,\phi_n\right) \tag{148}$$

for some function $\tilde{f}_i$. A sufficient condition for that limit to exist is that

$$\lim_{t\to\infty} \frac{d}{dt} C_i \left(\exp\left(\phi_l t\right), \exp\left(\phi_1 t\right), ..., \exp\left(\phi_n t\right)\right) \tag{149}$$

exists. That is, it is sufficient that the gradients of the cost functions have limits, but even that is not strictly necessary. The restriction of C_i to the CES family leads to the set of functions f_i that appear in theorem 1.

Theorem 2. *Under the assumptions of this section, and with $z = \theta t$,*

$$\lim_{t\to\infty} \left(gdp\left(z\right) - \lambda\left(\theta\right) t\right) t^{-1} = 0 \tag{150}$$

where $\lambda(\theta) = \phi_0$ and $\phi \in \mathbb{R}^{N+1}$ is a function of θ that is implicitly defined by the system of equations

$$\phi_i = \theta_i + \alpha \tilde{f}_i\left(\phi\right) \text{ for } i \in \{0, 1, ..., N\} \tag{151}$$

This result shows that what ultimately determines the behavior of GDP for extreme shocks is the limiting slope of the sector-level cost functions.

Proof. The price of good i is

$$p_i = -\log z_i + \alpha_i \log C_i\left(\exp\left(p\right)\right) \tag{152}$$

Let

$$\phi_i = -\lim_{t\to\infty} t^{-1} p_i \tag{153}$$

we maintain for the moment that this limit exists and is finite and verify that later. Then

$$\begin{aligned} t^{-1} p_i &= -\theta + \alpha_i t^{-1} \log C_i\left(\exp\left(p\right)\right) && (154)\\ \phi_i &= -\theta + \alpha_i \lim_{t\to\infty} t^{-1} \log C_i\left(\exp\left(p\right)\right) && (155)\\ &= -\theta + \alpha_i f_i\left(\phi\right) && (156) \end{aligned}$$

where the second line takes the limit as $t \to \infty$ and the third line uses the definition of f_i along with the continuity of C_i and the price function.

Note also that the price of the final good is

$$\log GDP = -\log P = f_0\left(\phi\right) t + o\left(t\right) \tag{157}$$

Finally, to show that a solution to the system exists, define

$$\hat{g}_i(\phi) = \theta_i + \alpha_i f_i(\phi) \tag{158}$$

This has a unique solution if $\hat{g}$ is a contraction. To see why that is true, we just check Blackwell's sufficient conditions of monotonicity and discounting. Monotonicity holds simply because the cost function itself is assumed to be monotone. Constant returns in the function C_i also imply that $f(\phi + a) = f(\phi) + a$. Since $\alpha_i < 1$, $\hat{g}_i$ has the discounting property, making it a contraction, so we can then apply the Banach fixed point theorem. ■

B.4 Fixed labor

Assume labor is normalized to 1 and the elasticity of substitution at the household level is 1. Then the production function, resource constraint, and FOCs are

$$Y_i = \exp(z_i)\left(\sum_j a_{i,j} x_{i,j}^{\gamma_i}\right)^{\alpha/\gamma_i} \tag{159}$$

$$\begin{aligned} Y_j &= c_j + \sum_i x_{i,j} && (160)\\ p_j &= b_j c_j^{-1} && (161)\\ p_j &= \alpha p_i \exp(z_i)\left(y_i/\exp(z_i)\right)^{(\alpha-\gamma_i)/\alpha} a_{i,j} x_{i,j}^{\gamma_i - 1} && (162) \end{aligned}$$

We assume productivity in each sector is

$$\log z_i = \theta_i t \tag{163}$$

with $t \to \infty$. Define ϕ to be the solution to the recursion

$$\phi_i = \theta_i + \alpha f_i(\phi) \tag{164}$$

Proposition. *We have the following limits,*

$$\lim_{t\to\infty} \frac{\log y_j}{t} = \lim_{t\to\infty} \frac{\log c_j}{t} = \lim_{t\to\infty} \frac{-\log p_j}{t} = \phi_j \tag{165}$$

Proof. The result can be proven by simply verifying that it satisfies the equilibrium conditions.

An additional result we need is the use of inputs, $x_{i,j}$. The limit is

$$\lim_{t\to\infty}\frac{\log x_{i,j}}{t}=\phi_{j^*(i)}+\frac{1}{(1-\gamma_i)}\left[\phi_j-\phi_{j^*(i)}\right] \tag{166}$$

where

$$j^*(i)=\begin{cases}\arg\min_{j\in S(i)}\phi_j \text{ if } \gamma_i<0\\ \arg\max_{j\in S(i)}\phi_j \text{ if } \gamma_i>0\end{cases} \tag{167}$$

We need to verify that the above, along with the solution in the proposition, satisfies the limits of the two FOCs, the resource constraint, and the production function.

We first take limits of the equilibrium conditions. For any variable g_j, define

$$\phi_{g,j}\equiv\lim_{t\to\infty}\frac{\log g_j}{t} \tag{168}$$

First, a small lemma:

$$\phi_{x,i,j}\le\phi_j \tag{169}$$

To see why, first suppose $\gamma_i<0$. Then $\phi_j-\phi_{j^*(i)}\ge 0$ and $\frac{1}{(1-\gamma_i)}<1$, from which the result immediately follows. To see the result for $\gamma_i>1$, note that

$$\begin{aligned}\phi_{x,i,j} &= \phi_{j^*(i)}+\frac{1}{(1-\gamma_i)}\left(\phi_j-\phi_{j^*(i)}\right) && (170)\\ &= \phi_j+\frac{\gamma_i}{(1-\gamma_i)}\left(\phi_j-\phi_{j^*(i)}\right) && (171)\end{aligned}$$

Since $\frac{\gamma_i}{1-\gamma_i}>0$ and $\phi_j-\phi_{j^*(i)}\le 0$ in this case, the result again follows. It holds trivially for $\gamma_i=0$. Furthermore, note that

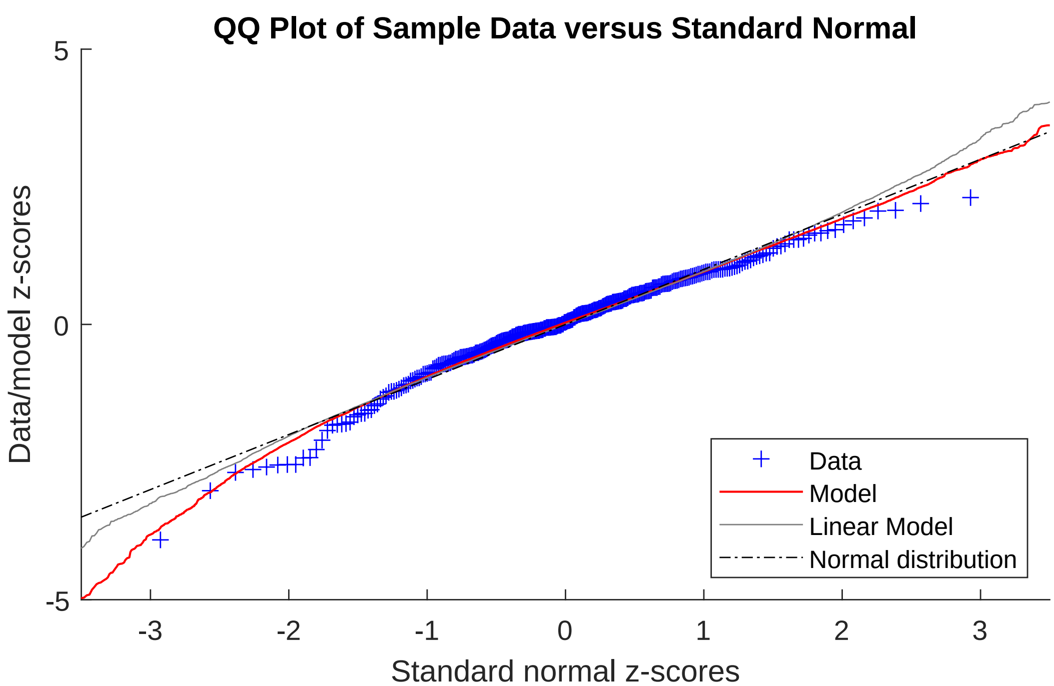
$$f_i\left(\phi_{x,i,j}\right)=\phi_{j^*(i)} \tag{172}$$

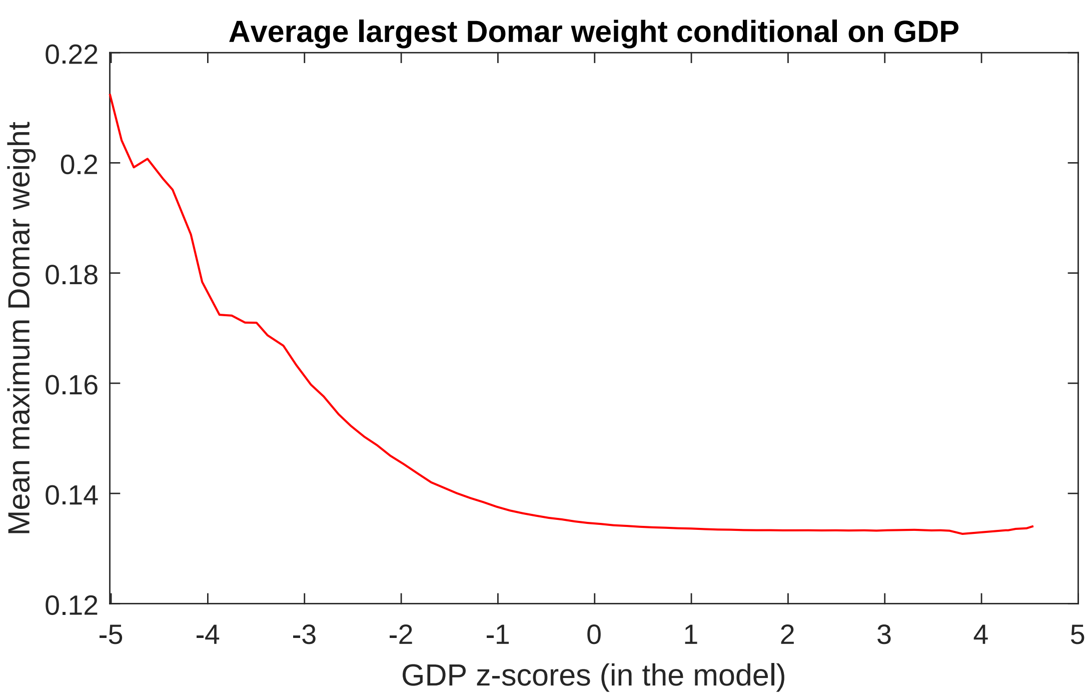
Then the limits of the three equilibrium conditions and the production function are (equations (159) to (162), respectively)

$$\begin{aligned}\phi_{y,i} &= \zeta_i+\alpha_i f_i\left(\phi_{x,i,j}\right) && (173)\\ \phi_{y,j} &= \max\left\{\phi_{c,j},\max_i\phi_{x,i,j}\right\} && (174)\\ \phi_{p,j} &= -\phi_{c,j} && (175)\\ \phi_{p,j} &= \phi_{p,i}+\zeta_i+\frac{\alpha_i-\gamma_i}{\alpha_i}\left(\phi_{y,i}-\zeta_i\right)+\left(\gamma_i-1\right)\phi_{x,i,j} && (176)\end{aligned}$$

where the first equation uses equation (169). The first and second equations hold because of (172). The third is trivial. The fourth holds by substituting in the various ϕ terms and using the recursion defining ϕ. ■

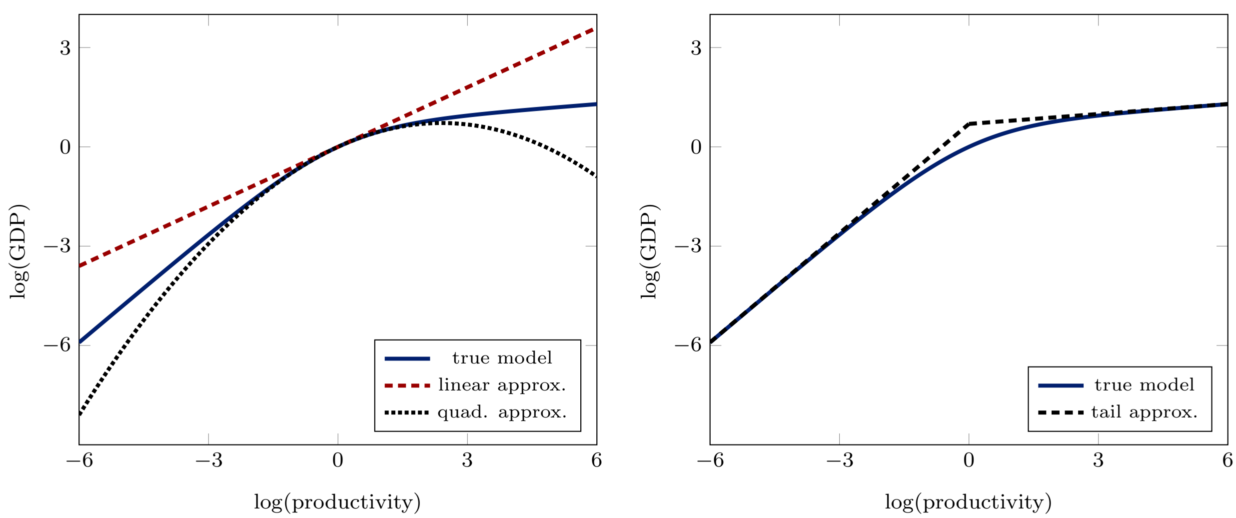
Figure 1: Normal quantile-quantile plot





Notes: The empirical series is for HP-detrended log GDP, 1947–2019.

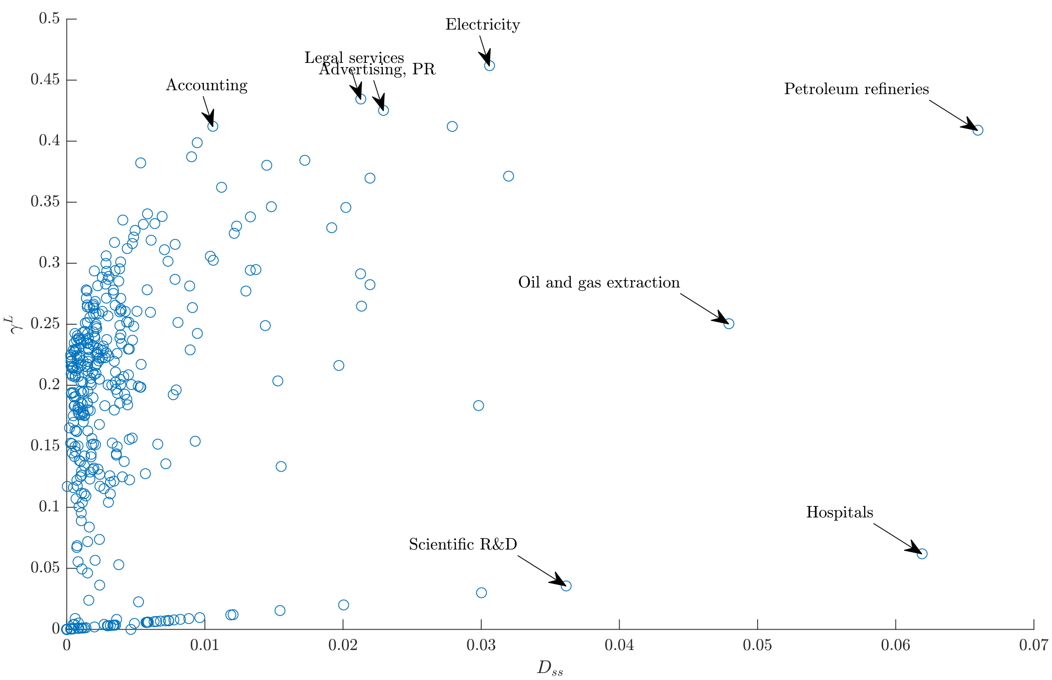
Figure 2: Linear, quadratic, and tail approximations



(a) Small-shock approximations (b) Large-shock approximation

Notes: The x-axis is log productivity and the y-axis log aggregate output. The x-axis may represent productivity in a single sector, or it could be the scale of a shock that affects productivity in multiple sectors. The concavity in GDP in this example is consistent with an economy featuring complementarities.

Figure 3: Domar weights and tail centralities



Notes: The x-axis is the Domar weight of each sector. The y-axis is the left tail centrality. The data is the 2012 BEA input-output table. The top four sectors according to both centrality measures are labeled.

Table 1: **Top sectors by left tail centrality and sales share**

Largest by left tail centrality		
Sector	γ_i^L	*Sales share*
Electric power generation, transmission, and distr.	0.462	0.031
Legal services	0.435	0.021
Advertising, public relations, and related services	0.425	0.023
Accounting, tax prep., bookkeeping, and payroll serv.	0.412	0.011
Monetary authorities and depository credit intermed.	0.412	0.028
Petroleum refineries	0.409	0.066
Other plastics product manufacturing	0.399	0.009
Services to buildings and dwellings	0.387	0.009
Architectural, engineering, and related services	0.384	0.017
All other misc. prof., scientific, and tech. serv.	0.382	0.005
Management consulting services	0.380	0.014
Largest by sales share		
Sector	γ_i^L	*Sales share*
Petroleum refineries	0.409	0.066
Hospitals	0.062	0.062
Oil and gas extraction	0.250	0.048
Scientific research and development services	0.036	0.036
Insurance carriers, except direct life	0.371	0.032
Electric power generation, transmission, and distr.	0.462	0.031
Offices of physicians	0.030	0.030
Pharmaceutical preparation manufacturing	0.183	0.030
Monetary authorities and depository credit intermed.	0.412	0.028
Advertising, public relations, and related services	0.425	0.023
Other financial investment activities	0.282	0.022

Notes: Sales shares and tail centralities calculated from the 2012 BEA input-output tables.