Information Acquisition and the Pre-Announcement Drift

Hengjie Ai, Ravi Bansal, and Leyla Jianyu Han*

July 10, 2022

Abstract: We present a dynamic Grossman-Stiglitz model with endogenous information acquisition to explain the pre-FOMC announcement drift. Because FOMC announcements reveal substantial information about the economy, investors' incentives to acquire information are particularly strong days ahead of the announcements. Information acquisition partially resolves the uncertainty for uninformed traders. Under generalized risk sensitive preferences (Ai and Bansal, 2018), resolution of uncertainty is associated with realizations of risk premium, generating a pre-FOMC announcement drift. Because our theory does not rely on leakage of information about the contents of the announcement, it can simultaneously explain the high average return and the low realized volatility during the pre-FOMC announcement period.

Keywords: Macroeconomic Announcement Premium, Pre-FOMC Announcement Drift, Asymmetric Information, Ambiguity.

JEL Code: D83, D84, G11, G12, G14

*Hengjie Ai (hengjie.ai@wisc.edu) is from Wisconsin School of Business, University of Wisconsin-Madison; Ravi Bansal (ravi.bansal@duke.edu) is affiliated with Fuqua School of Business, Duke University and NBER, and Leyla Jianyu Han (leylahan@bu.edu) is associated with Questrom School of Business, Boston University. We thank our conference discussants Daniel Andrei, Charles Martineau, Haoxiang Zhu, Liyan Yang, Kerry Back, Efstathios Avdis, Mariana Khapko, Savitar Sundaresan, Andrea Buffa, Jun Li, as well as participants at Boston University, Adam Smith Workshop 2022, 6th Annual Young Scholars Finance Consortium, 7th Annual University of Connecticut Finance Conference, Western Finance Association 2021, European Finance Association 2021, Northern Finance Association 2021, Society for Economic Dynamics 2021, China International Conference in Finance 2021, China International Conference in Macroeconomics 2021, Midwest Finance Association 2021. We also thank Snehal Banerjee, Anna Cieslak, Dobrislav Dobrev, and Yang Liu for helpful comments on the paper.

1 Introduction

This paper presents a noisy rational expectations equilibrium (REE) model with endogenous information acquisition to explain the pre-FOMC announcement drift documented by Lucca and Moench (2015). Information is publicly available but costly to acquire. Because FOMC announcements resolve substantial uncertainty of the aggregate economy and have a significant impact on the stock market, informed traders have particularly large information advantages in trading over uninformed traders before announcements are made. As a result, it is optimal for uninformed traders to start to acquire information days ahead of the announcements. Due to generalized risk sensitivity (GRS) (Ai and Bansal, 2018) in preferences, as uncertainty resolves, equity market risk premium realizes shortly before announcements. More importantly, because the newly acquired information is from publicly available sources rather than leakage of the content of the upcoming announcement, our theory can simultaneously explain the high average return and the low realized volatility during the pre-FOMC announcement period.

Stock market returns earned on FOMC announcement days account for almost 100% of the overall equity market risk premium since the mid-1990s. Ai and Bansal (2018) demonstrate that this phenomenon can be consistent with general equilibrium asset pricing models if investors have generalized risk sensitive preferences. The puzzling aspect of the FOMC announcement premium is that it is mostly realized during the trading day before the actual announcements. If one is willing to assume that most of the time, the contents of FOMC announcements are leaked to the market days before the announcements, the example in Ai and Bansal (2018), illustrated in Figure 4 of their paper, provides a direct explanation for the pre-FOMC announcement drift. However, information leakage-based models are inconsistent with the low realized volatility during the pre-announcement period.[1]

We define information leakage as the arrival of new information that is correlated with the upcoming announcement but has not been incorporated in market prices. Because arrivals of new information trigger immediate stock market responses, information leakage-based models typically imply a counter-factually high level of realized volatility during the drift period. Empirically, however, the realized volatility of market returns during the pre-announcement period is slightly lower than their counterparts on non-FOMC announcement days. It is the coexistence of low volatility and high average return during the pre-announcement period that makes this phenomenon particularly puzzling.

We develop a noisy REE model to explain the above puzzling pattern. In our model, the long-run growth rate of the economy is governed by a latent state variable that is unobservable to all investors but periodically announced by the central bank. Information, modeled as noisy signals about the latent variable, is available but costly to acquire. There are two groups of investors, informed and uninformed. Informed traders have zero cost of information acquisition and always observe noisy

[1]From an institutional point of view, evidence for information leakage is mostly anecdotal. In addition, the pre-FOMC announcement drift accounts for almost 100% of the FOMC announcement premium. Attributing the pre-FOMC announcement drift to leakage of information requires most of the information to be leaked before announcements. As argued by Lucca and Moench (2015), this extreme form of information leakage is implausible.

signals about the latent growth rate. Uninformed investors do not observe the signals until they pay a cost to acquire them. In our model, uninformed investors normally pay less attention to stock market dynamics than informed traders but may choose to increase their attention when the benefit exceeds the cost of information acquisition.

Our model has three key ingredients. The first is endogenous information acquisition. In our model, uninformed investors endogenously choose to acquire information a few days ahead of FOMC announcements. The endogenous information acquisition in our model is consistent with the evidence documented by Fisher, Martineau, and Sheng (2020) that investors' attention peaks roughly three days before pre-scheduled FOMC announcements.

Second, uniformed investors' preference satisfies generalized risk sensitivity. To maintain tractability of the noisy REE setup, we develop a risk sensitive operator that extends the T^2 operator of Hansen and Sargent (2007) and Hansen and Sargent (2011). In particular, this formulation allows us to model investors' ambiguity aversion about the hidden state variable, and at the same time, to keep the closed-form solutions for the standard CARA-normal setup. Due to the generalized risk sensitivity of preferences, resolution of uncertainty in our model is associated with the realization of risk premium during the pre-announcement period, producing a pre-FOMC announcement drift.

The third key ingredient of our is asymmetric information. The information acquired by uninformed investors is not leakage about the upcoming announcement; rather, it is the information that is already known to informed investors and has been incorporated into equilibrium prices. As a result, the degree of asymmetric information drops and noise gets eliminated from prices. More importantly, information acquisition lowers, rather than increases, the realized volatility during the pre-announcement period. This feature of our model generates the low realized volatility during the pre-announcement period consistent with the empirical evidence.

Related Literature Our paper builds on the literature of macroeconomic announcement premium. Savor and Wilson (2013, 2014) are among the first to document the macroeconomic announcement premium. Ai and Bansal (2018) provide a revealed preference theory for the macroeconomic announcement premium. Wachter and Zhu (2020) develop a quantitative model of the macroeconomic announcement premium based on rare disasters. Ai, Bansal, Im, and Ying (2020) provide evidence for the impact of announcements on macroeconomic quantities as well as asset markets and develop a production-based asset pricing model to explain these facts. Ernst, Gilbert, and Hrdlicka (2019) present additional evidence for the macroeconomic announcement premium.

Within the above broader literature, our paper is more closely related to the FOMC announcement premium. Lucca and Moench (2015) document the pre-FOMC announcement drift and Cieslak, Morse, and Vissing-Jorgensen (2019) provide evidence for stock returns over the FOMC announcement cycles. Morse and Vissing-Jorgensen (2020) provide a study for the information transmission mechanism for Fed policies. Both Laarits (2020) and Ying (2020) provide models of pre-announcement drifts. Both papers rely on the arrival of new information during the pre-announcement period as in the example of Ai and Bansal (2018). Cocoma (2020) develops a general equilibrium with disagreement to explain the pre-FOMC announcement drift.

Several recent empirical work document important facts related to investor attention and trading activities around FOMC announcement which provide a basis for the development of the theoretical model in this paper. Fisher, Martineau, and Sheng (2020) develop a macroeconomic attention index and provide a systematic study of the pattern of investor attention around macroeconomic announcements. Boguth, Grégoire, and Martineau (2018) emphasize the importance of press conferences in shaping market expectations. Hu, Pan, Wang, and Zhu (2020) document the dynamics of implied volatility around FOMC announcements. Ai, Bansal, Guo, and Yaron (2020) link the dynamics of implied volatility around announcements to investors' preference for early resolution of uncertainty. Bollerslev, Li, and Xue (2018) study the relationship between realized volatility and trading volume around FOMC announcements.

From the theoretical point of view, this paper builds on the noisy rational expectations literature pioneered by Grossman and Stiglitz (1980), Grossman (1981), and Hellwig (1980). The continuous-time and dynamic setup are directly related to Wang (1993, 1994), and the setup of the macroeconomic announcement is related to Han (2020). An incomplete list of recent applications of the dynamic Grossman-Stiglitz models include Breon-Drish (2015), Bond and Goldstein (2015), Banerjee and Green (2015), Goldstein and Yang (2017), Albuquerque and Miao (2014), Avdis (2016), Andrei and Cujean (2017), Andrei, Cujean, and Wilson (2018), Sockin (2019), Buffa, Vayanos, and Woolley (2019).

This paper is also related to the literature on endogenous information acquisition and information choice. Veldkamp and Van Nieuwerburgh (2010) study a joint decision problem of portfolio choice and information acquisition. Banerjee and Breon-Drish (2020) analyze endogenous information acquisition problems in an environment with strategic trading. Veldkamp (2011) provides an excellent review of the literature of information choice and attention allocation.

From the perspective of general equilibrium asset pricing, this paper belongs to the large literature that studies various aspects of equity market risk and risk compensation based on preferences with generalized risk sensitivity. To incorporate generalized risk sensitivity in a tractable way in the Grossman-Stiglitz setup, we use the recursive multiple prior setup of Chen and Epstein (2002). See also, Epstein and Schneider (2007). This preference is also related to the robust control preference of Hansen and Sargent (2007, 2008, 2011). We do not attempt to survey this large literature but refer the readers to Ai and Bansal (2018) for the references of preferences that satisfy generalized risk sensitivity and their applications in asset pricing.

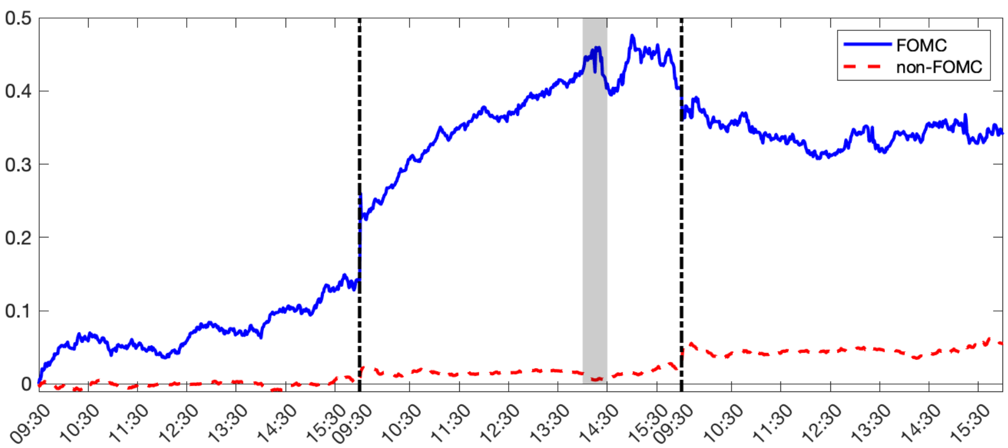
The rest of the paper is organized as follows. In Section 2, we summarize stylized facts related to the FOMC announcement premium and the pre-FOMC announcement drift. We present our model in Section 4 and study its implications in Section 5. Section 6 concludes.

2 Stylized Facts

We begin by summarizing the four stylized facts about stock market dynamics around pre-scheduled FOMC announcements. All of the facts we list here are well established in the literature, and we simply use them as a guidance for the development of the model. See Appendix 6.1 for a detailed data description.

1. The aggregate stock market exhibits high average returns starting from the previous trading day until the release of the FOMC announcement. In Figure 1, we plot the cumulative return around FOMC announcement starting from one trading day before the announcement until one trading day afterwards. The solid line stands for announcement days and the dashed line is the non-FOMC announcement day cumulative returns. The shaded area, 14:00-14:30 p.m., depicts the timing of most prescheduled FOMC meetings. Consistent with Lucca and Moench (2015), we find that the 24-hour return before the pre-scheduled FOMC announcement during the period of January 1994 to September 2020 is about 32 basis points on average.

Figure 1: The Pre-FOMC Announcement Drift

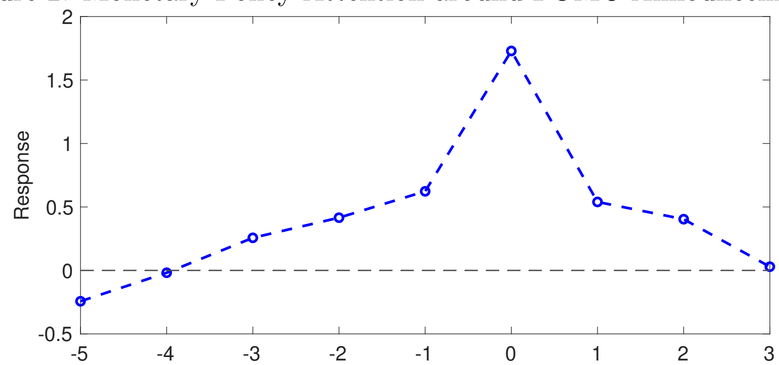


This figure plots the average three-day cumulative return (in percentage) around FOMC and non-FOMC announcement days. The solid line displays the average cumulative return during regular trading hours from 9:30 a.m. on one trading days before the FOMC announcements to 16:00 p.m. on days afterward. The dashed line is the average cumulative return on all three trading consecutive days that do not include any FOMC announcement (Note that there exits overlapping among these three-day windows). The shaded area, 14:00-14:30 is the half an hour window containing most of the FOMC releases. The sample period is from January 1994 to September 2020.

2. Investors' attention rises three days before FOMC announcements and peaks right after FOMC announcements. Fisher, Martineau, and Sheng (2020) develop a macroeconomic attention index based on news article counts. In Figure 2, we plot their constructed macroeconomic attention index around FOMC announcements. Investor's attention about monetary policy starts to increase three days ahead of announcements. It spikes at one because there is one day delay for the news article to be printed. This is the motivating evidence for our endogenous information acquisition-based theory.

3. The realized volatility during the pre-FOMC announcement period is slightly lower than the realized volatility during the same hours on non-announcement days. Realized volatility peaks right after FOMC announcements. In Figure 3, we plot the 30-minute realized volatility over the three days around FOMC announcements. The dotted line stands for announcement days

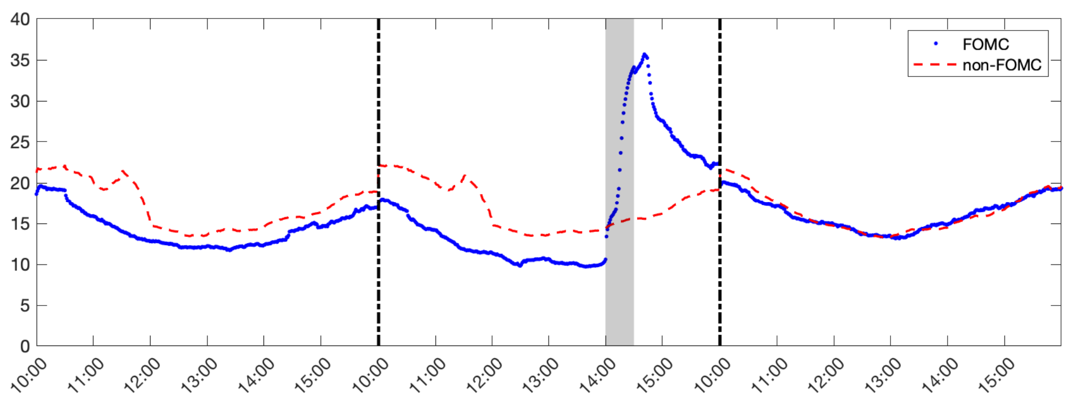
Figure 2: Monetary Policy Attention around FOMC Announcements



This figure replicates the bottom right panel of Figure 4 in Fisher, Martineau, and Sheng (2020). The horizontal axis is the number of days since the announcement. The vertical axis shows the lag and forward coefficients by regressing the daily demeaned macroeconomic attention of composite monetary index on dummy variables of days since the announcements, controlling for day-of-the-week fixed effects. The sample period is from January 1994 to December 2020.

and the dashed line depicts non-announcement days. Compared to non-announcement days, realized volatility is lower before FOMC announcements, and peaks right after announcements.

Figure 3: Realized Volatility



This figure plots the intraday average market realized volatility during the three days around FOMC and non-FOMC announcement days. The dotted line is the realized volatility for FOMC announcement days, and the dashed line is that for non-FOMC announcement days. Realized volatility (annualized in percentage) is the average rolling sum of squared log returns on the S&P 500 E-mini futures over the past 30 minutes. The dashed line is the same calculation on all three consecutive trading days that do not include any FOMC announcement (Note that there exits overlapping among these three-day windows). We calculate the realized volatility for each minute from 10:00 to 16:00. The sample period is from September 1997 to September 2020. The shaded area, 14:00-14:30 is the half an hour window containing most of the FOMC releases.

This evidence is inconsistent with information leakage-based story, which will trigger stock market reactions and result in a high realized volatility during the pre-FOMC announcement drift period.

In the following section, we show that a dynamic noisy rational expectations (REE) model with endogenous information acquisition, after incorporating generalized risk sensitive preferences, provides a unified explanation for the above facts.

3 An Example of the Pre-FOMC Announcement Drift

In this section, we reproduce the simple example in Figure 4 of Ai and Bansal (2018) to illustrate how combining generalized risk sensitivity and information leakage can generate a pre-FOMC announcement drift. More importantly, we use this example to illustrate the difficulty for a representative agent model to simultaneously explain the low volatility and high return during the pre-FOMC announcement period.

The Ai and Bansal (2018) model assumes a continuous-time setup where the aggregate consumption follows $\frac{dC_t}{C_t} = [x_t dt + \sigma dB_{C,t}]$, where σ is the volatility of consumption growth. The expected consumption growth is driven by a hidden state variable x_t , which follows

$$dx_t = a_x \left(\bar{x} - x_t\right) dt + \sigma_x dB_{x,t}, \tag{1}$$

where $\bar{x}$ is the long-run mean of x_t, b is the rate of mean reversion, σ_x is the volatility of the hidden state x_t, and $B_{x,t}$ is a standard Brownian motion independent of $B_{C,t}$. At time $t = T, 2T, 3T, \cdots$, pre-scheduled FOMC announcements reveal the true values of x_t. To model information leakage, we assume that starting from time $\tau < T$, the representative investor observes an additional signal l_t, which carries information about the content of upcoming announcement x_t:

$$l_t = x_t dt + \sigma_l\left(t\right) dB_{l,t}. \tag{2}$$

where $\sigma_l\left(t\right)$ is the inverse of signal precision and $B_{l,t}$ is a mutually independent Brownian motion noise. Ai and Bansal (2018) show that the posterior mean of x_t, denoted $\hat{x}_t$ can be written as:

$$d\hat{x}_t = a_x \left(\bar{x} - \hat{x}_t\right) dt + \frac{q_t}{\sigma} d\hat{B}_{C,t} + \frac{q_t}{\sigma_l\left(t\right)} d\hat{B}_{l,t}, \tag{3}$$

where q_t is the posterior variance of x_t, defined as $q_t = \mathbb{E}_t\left[\left(\hat{x}_t - x_t\right)^2\right]$. $d\hat{B}_{C,t} = \frac{1}{\sigma}\left(\frac{dC_t}{C_t} - \mathbb{E}_t\left[\frac{dC_t}{C_t}\right]\right)$ and $d\hat{B}_{l,t} = \frac{1}{\sigma_l(t)}\left(dl_t - \mathbb{E}_t\left[dl_t\right]\right)$ are innovations in the observation processes relative to the investor's belief.

Assume that the representative investor has a recursive preference with a subjective discount rate of ρ, a unit IES, and a risk aversion of γ, the pricing kernel can be written as:

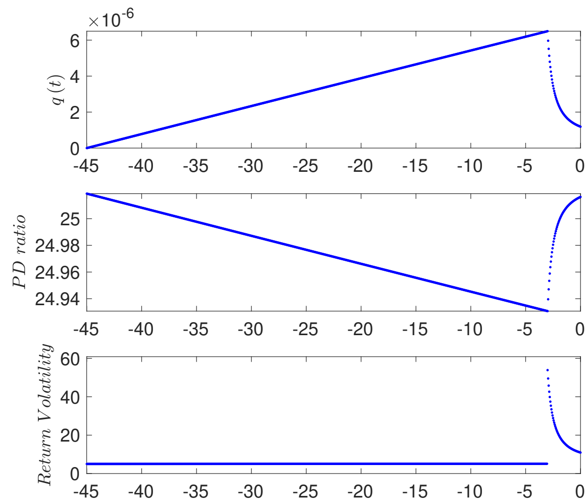
$$d\pi_t = -r_t dt - \sigma d\hat{B}_{C,t} - (\gamma - 1)\left[\left(1 + \frac{q_t}{\left(a_x + \rho\right)\sigma^2}\right)\sigma d\hat{B}_{C,t} + \frac{q_t}{\left(a_x + \rho\right)\sigma_l\left(t\right)} d\hat{B}_{l,t}\right], \tag{4}$$

where the first term is the risk free rate, and the second term $\sigma dB_{C,t}$ comes from the standard expected utility with log preference, and the term in the square bracket arises due to generalized risk sensitivity: $\gamma > 1$.

The case of information leakage can be modeled by assuming $\sigma_l\left(t\right) = \infty$ for $t < \tau$ and $\sigma_l\left(t\right) =$

0.01% for $t \in [\tau, T]$. That is, during the pre-announcement period $[\tau, T]$, the investor suddenly starts to observe a very precise signal about the true contents of the upcoming announcement, x_t. In Figure 4 below, we plot the posterior variance (top panel), the average price-to-dividend ratio (middle panel), and the volatility of the market return (bottom panel) implied by the above model. Because the information is very precise, the posterior variance q_t drops sharply at $t = \tau$. At the same time, the average price-to-dividend ratio rises sharply. This is because leakage of information is associated with a high volatility of the stochastic discount factor: the term $\frac{q_t}{(a_x+\rho)\sigma_l(t)}$ in equation (4) rises sharply after τ because $\sigma_l(t)$ is close to zero. This mechanism generates a large risk premium in the short period ahead of announcements.

Figure 4: Equilibrium without and with Information Acquisition



This figure plots $\hat{q}_t$, the posterior variance for $\hat{x}_t$ (top panel), the average price-to-dividend ratio (middle panel), and the return volatility (annualized in percentage) (bottom panel) over one announcement cycle at the steady state value $\hat{x}_t = \bar{x}$. The horizontal axis is the number of days before the upcoming announcement, which is normalized to 0. The information starts to leak at time $\tau < T$. Here, $\tau = 42$, three days before the announcement and $T = 45$. We refer to Table S.I in Ai and Bansal (2018) for the rest of the parameter values.

The high volatility of the stochastic discount factor, however, is associated with the high volatility of the posterior belief $\frac{q_t}{\sigma_l(t)}$ in equation (3). In fact, the high volatility of $\hat{x}_t$ is the reason for the high volatility of the stochastic discount factor. As shown in Figure 4, the realized volatility rises sharply and simultaneously as the price-to-dividend ratio increases with leakage of information.

The above example illustrates a key difficulty for models that generate a pre-FOMC announcement drift based on the arrival of new information to the market, or leakage of information. In the data, the average excess return during the pre-FOMC announcement period is roughly 30 bps per trading day, and that on non-announcement days is less than 2 bps. Holding the Sharpe ratio constant, to account for a 30 bps premium, the information leakage-based story requires a realized market volatility of twenty times higher during the pre-announcement period, whereas in the data, the realized market volatility in this period is in fact lower than that on non-announcement days. In the rest of the paper, we develop a noisy rational expectations model with information acquisition to resolve the above puzzle.

4 Dynamic Model

This section develops a continuous-time noisy REE model with endogenous information acquisition and generalized risk-sensitive preferences.

4.1 Model Setup

The asset market Time is continuous and infinite. There is a unit measure of investors. An ω fraction of them are uninformed investors and $1-\omega$ fraction are informed investors. There are two assets available for trading, a stock and a risk-free bond. We assume that the risk-free return r is constant. The stock is the claim to the following dividend process:

$$dD_t = (x_t - D_t)\, dt + \sigma_D dB_{D,t}, \tag{5}$$

where D_t is the dividend flow, x_t is the long-run trend for the dividend flow, σ_D is the volatility of the dividend flow, and $B_{D,t}$ is an i.i.d. shock to the dividend payment modeled as a standard Brownian motion. We model the expected dividend flow as $x_t - D_t$, so that the dividend process is stationary. The assumption that the mean reversion rate equals to 1 is not important and can be relaxed without affecting most parts of the model. The long-run trend of the dividend flow, x_t, is itself mean reverting, modeled as an Ornstein-Uhlenbeck (OU) process as in equation (1). In addition, as is standard in the noisy REE literature, we assume that the total equity supply is a stochastic process and denote it as θ_t, where

$$d\theta_t = a\left(\bar{\theta} - \theta_t\right) dt + \sigma_\theta dB_{\theta,t}. \tag{6}$$

In the above equation, a is the rate of mean reversion, $\bar{\theta}$ is the long-run mean for θ_t, and σ_θ is the noisy supply volatility. We assume that Brownian motions $B_{D,t}$, $B_{x,t}$, and $B_{\theta,t}$ are mutually independent.

Information and preference of informed investors We assume that the dividend process, D_t, is observable to all investors, but its long-run trend x_t and the total risky asset supply θ_t are not. At pre-scheduled times, $t = nT$, for $n = 1, 2, \cdots$, the monetary authority (central bank) makes periodic announcements that reveal the true value of x_t. Both informed and uninformed investors can observe D_t and the pre-scheduled FOMC announcements and use them to update their beliefs about the latent variable that drives the economic growth, x_t.

We assume that market research can produce a signal that is informative about x_t, denoted as s_t:

$$ds_t = x_t dt + \sigma_s dB_{s,t}, \tag{7}$$

where σ_s is the signal volatility and $B_{s,t}$ is a Brownian motion independent of $B_{D,t}$, $B_{x,t}$, and $B_{\theta,t}$. We think of s_t as information available in the public domain but costly to acquire. Informed investors have a comparative advantage relative to uninformed investors in terms of information acquisition. For simplicity, we assume that they have zero information acquisition cost and observe s_t at all times.

Both types of investors maximize CARA utilities represented by $\left[\mathbb{E}\int_0^{\infty} -e^{-\rho t-\gamma C_t}dt\right]$, where C_t is the consumption at time t, ρ is the subjective time discount rate and γ is the absolute risk aversion.

Information and beliefs Informed investors observe three sources of information about the latent variable x_t that drives the economic growth: the dividend process D_t, pre-scheduled FOMC announcements at $t = nT, n = 1, 2, \cdots$, and the signal process s_t obtained from market research. Denote $\hat{x}_t \equiv \hat{\mathbb{E}}_t[x_t]$ and $\hat{q}(t) \equiv \hat{\mathbb{E}}_t\left[(\hat{x}_t - x_t)^2\right]$ as the posterior mean and variance of the informed investors about x_t, where $\hat{\mathbb{E}}$ indicates the belief of the informed. If the informed investors' prior for x_t is a Gaussian distribution, then their posterior distribution for x_t is also Gaussian and can be characterized by the standard Kalman filter. We assume FOMC announcements convey information about the economic growth and fully reveal the true value of x_t, we then have $\hat{x}_t = x_t$ and $\hat{q}_t = 0$ at prescheduled announcements $t = nT$. After announcements, because $\hat{x}_t$ process evolves according to equation (8), $\hat{x}_t$ drifts away from the true value of x_t and $\hat{q}_t$ increases above zero, up until the next announcement. Standard Kalman filter implies that the dynamics of $\hat{x}_t$ can be computed by:

$$d\hat{x}_t = b(\bar{x} - \hat{x}_t)\,dt + \frac{\hat{q}(t)}{\sigma_D}d\hat{B}_{D,t} + \frac{\hat{q}(t)}{\sigma_s}d\hat{B}_{s,t}, \tag{8}$$

where $d\hat{B}_{D,t} = dD_t - \hat{\mathbb{E}}_t[dD_t]$ and $d\hat{B}_{s,t} = ds_t - \hat{\mathbb{E}}_t[ds_t]$ are innovations in the observation processes relative to informed investors' expectations.

In contrast, uninformed investors do not observe s_t, unless they pay a cost to acquire such information. To keep the structure simple, we assume that they have an option to acquire information at time τ by paying a one-time fixed cost K. Exercising the option allows all uninformed investors to observe a common noisy signal at time $t \geq \tau$ about the publicly available information $\{s_v\}_{v=-\infty}^{t}$ by paying a constant flow cost k. Note that $\{s_v\}_{v=-\infty}^{t}$ summarizes the history of information that informed investors have already observed up to time t. Since informed investors' posterior belief $\hat{x}_t$ contains all the information in $\{s_v\}_{v=-\infty}^{t}$ that is relevant for forecasting x_t, learning from $\{s_v\}_{v=-\infty}^{t}$ is equivalent to learning about $\hat{x}_t$. Because the stock price (see equation (10) below) is a function of $\hat{x}_t$, it is more convenient to model the newly acquired information as a signal for $\hat{x}_t$:

$$ds_{u,t} = \hat{x}_t dt + \sigma_u(t)\,dB_{u,t}, \tag{9}$$

where $B_{u,t}$ is independent of $B_{s,t}$, $B_{D,t}$, $B_{x,t}$, and $B_{\theta,t}$. We focus on the symmetric equilibrium where all uninformed investors start to acquire information at time τ. In Section 4.2, we show that uninformed investors solve an optimal information acquisition problem by choosing the optimal stopping time τ.

We interpret s_t as the frontier market research, that is, the best information any agent in the economy has about the latent state variable x_t. Following Sims (2003), we can interpret information acquisition as a form of rational inattention. Before τ, uninformed investors are rationally inattentive to the frontier market research as acquiring such information is costly. We can conveniently denote this case as $\sigma_u(t) = \infty$ for $t < \tau$. After the option of information acquisition is exercised at time τ, uninformed investors start to allocate attention towards the information in s_t until the next announcement. We write $\sigma_u(t) = \sigma_u$ for $t \in [\tau, T]$. Different from the example in Section 3 where the signal is about the true contents of the announcement x_t, the newly acquired information here is about the publicly available information s_t that anyone could obtain by paying more attention to the frontier market research.

It is convenient to denote the posterior mean of an uninformed investor as $\tilde{x}_t = \tilde{\mathbb{E}}_t[\hat{x}_t]$ and the posterior variance as $\tilde{q}(t) \equiv \tilde{\mathbb{E}}_t\left[(\tilde{x}_t - \hat{x}_t)^2\right]$, where $\tilde{\mathbb{E}}$ captures the belief of the uninformed investor. We conjecture and later verify that the equilibrium price takes the following form

$$P_t = \phi(t) + \phi_D D_t - \phi_\theta(t)\,\theta_t + \phi_x(t)\,\hat{x}_t + \phi_\Delta(t)\,\tilde{x}_t, \tag{10}$$

where $\phi_\theta(t)$, $\phi_x(t)$, and $\phi_\Delta(t)$ are time-varying sensitivities of price to θ_t, $\hat{x}_t$ and $\tilde{x}_t$, respectively, and the sum of the two coefficients, $\bar{\phi}_x \equiv \phi_x(t) + \phi_\Delta(t)$ is a constant.[2] Clearly, if we define $\Delta_t \equiv \hat{x}_t - \tilde{x}_t$ to be the difference between the beliefs of the informed and uninformed investors, price can therefore be written as:

$$P_t = \phi(t) + \phi_D D_t - \phi_\theta(t)\,\theta_t + \bar{\phi}_x \hat{x}_t - \phi_\Delta(t)\,\Delta_t. \tag{11}$$

Learning from prices Here we describe the beliefs of uninformed investors in our model, which is the key to understanding the model's implications for the pre-FOMC announcement drift. Note that due to the presence of the noisy supply, the equilibrium price is only partially revealed to the uninformed but still contains information about the best predictions for x_t. Uninformed investors would benefit from learning from the equilibrium price. It is convenient to define $\xi_t = \phi_x(t)\,\hat{x}_t - \phi_\theta(t)\,\theta_t - \frac{\hat{q}_t}{\sigma_D^2}\phi_x(t)\,D_t$ as the information content of prices, as observing ξ_t is the same as observing the equilibrium price. The uninformed investors observe three sources of information about the informed investors' belief $\hat{x}_t$: the dividend process, the equilibrium price (or ξ_t), and the signal $s_{u,t}$ after paying the information acquisition cost. Standard Kalman filter implies that the dynamics of $\tilde{x}_t$ can be written as:[3]

$$d\tilde{x}_t \quad = \quad b(\bar{x} - \tilde{x}_t)\,dt + \frac{\hat{q}(t) + \tilde{q}(t)}{\sigma_D} d\tilde{B}_{D,t} + \nu(t)\,\sigma_\xi(t)\,d\tilde{B}_{\xi,t} + \frac{\tilde{q}(t)}{\sigma_u(t)} d\tilde{B}_{u,t}, \tag{12}$$

[2] As is standard in this literature, we use a guess-and-verify approach to prove the functional form of P_t and the property that $\phi_x(t) + \phi_\Delta(t) = \bar{\phi}_x$ is a constant.

[3] See Appendix 6.3 for the proof.

where $d\tilde{B}_{D,t} = dD_t - \tilde{\mathbb{E}}_t[dD_t]$, $d\tilde{B}_{\xi,t} = d\xi_t - \tilde{\mathbb{E}}_t[d\xi_t]$ and $d\tilde{B}_{u,t} = ds_{u,t} - \tilde{\mathbb{E}}_t[ds_{u,t}]$ are innovations in the observation processes relative to expectations. In the above expression, $\nu(t)$ is defined in equation (51) and the volatility of $d\xi_t$, $\sigma_\xi(t)$ is defined in equation (46) in Appendix 6.3.

Before τ, uninformed traders can only learn about $\hat{x}_t$ from the dividend process and the equilibrium price. After the information acquisition, they can also learn from the newly acquired information, $s_{u,t}$. It is important to note that in our setup, the endogenously acquired signal, $s_{u,t}$ is about the information about $\hat{x}_t$, which has already been incorporated into the market price through informed investors' trading activities. The newly acquired information is not informative about the difference between the true value of x_t and $\hat{x}_t$, which is only revealed through announcements. In other words, the content of the upcoming announcement, x_t, is not revealed until right after the announcement. This feature of our model is essential in accounting for the low volatility during the pre-announcement period.

GRS through recursive CARA preferences In order to account for the equilibrium announcement premium, we assume that the uninformed investors' preference satisfies the property of GRS in Ai and Bansal (2018). To maintain tractability and at the same time to allow for GRS, we extend the T^1 and T^2 operators in Hansen and Sargent (2007, 2011) and define the preference as a stochastic differential utility. Denote the continuation utility of the uninformed investor at time t as $\tilde{V}_t$. Given a consumption process $\{C_t\}_{t=0}^{\infty}$, the associated continuation utility $\tilde{V}_t$ is a stochastic differential utility of the form

$$d\tilde{V}_t = \mathcal{L}\tilde{V}_t dt + \sigma_V(t)\, d\tilde{B}_t, \tag{13}$$

where $\tilde{B}_t = \left[\tilde{B}_{D,t}, \tilde{B}_{\xi,t}, \tilde{B}_{u,t}\right]$ is a vector of standard Brownian motions relative to uninformed investors' information, and $\sigma_V(t)$ is vector of diffusions defined in equation (103) in Appendix 6.4. The Dynkin operator $\mathcal{L}[\centerdot]$ is defined as

$$\mathcal{L}\tilde{V}_t = \lim_{\Delta\to 0}\frac{1}{\Delta}\mathbb{E}_t\left[\tilde{V}_{t+\Delta} - \tilde{V}_t\right]. \tag{14}$$

We show that $\tilde{V}_t$ must satisfy an HJB equation of the form

$$\rho\tilde{V}_t = -e^{-\gamma C_t} + \mathcal{L}\tilde{V}_t + \mathcal{T}\tilde{V}_t, \tag{15}$$

where the operator $\mathcal{T}[\centerdot]$ is defined as:

$$\mathcal{T}\tilde{V}_t = -\frac{1}{2}\frac{\kappa}{\left|\tilde{V}_t\right|}\lim_{\Delta\to 0}\frac{1}{\Delta^2}Var\left[\mathbb{E}\left[\tilde{V}_{t+\Delta}\middle|\hat{x}_t\right]\middle|\tilde{x}_t\right]. \tag{16}$$

In the above formulation, κ is a parameter that describes the investors' ambiguity aversion. The case $\kappa = 0$ corresponds to the expected utility without GRS, and a positive κ implies that investors are ambiguity averse with respect to the unknown state variable, $\hat{x}_t$. In settings under robust control with hidden Markov state variables, Hansen and Sargent (2007, 2011) use the T^1 operator to model robustness concerns about the conditional distribution of signals given hidden state variables (model uncertainty) and the T^2 operator to model robustness concerns about misspecification of the distribution of the hidden state variable (state uncertainty). The $\mathcal{T}$ operator defined above has the same interpretation as the T^2 operator in Hansen and Sargent (2011), although we use a different functional form for tractability. We provide the details of the development of these operators for CARA utility in the appendix. For simplicity, we assume that the ambiguity aversion parameter for the T^1 operator is zero and use the $\mathcal{T}$ operator defined above to focus on ambiguity aversion about the hidden state. κ therefore summarizes uninformed investors' degree of state uncertainty.

4.2 Equilibrium and Equilibrium Conditions

For simplicity, we will focus on stationary equilibria in which equilibrium prices satisfy $P_t = P_{t \ mod \ T}$ where *mod* denotes the modulo operator, and so do equilibrium quantities. That is, all equilibria are identical across different announcement cycles. Without loss of generality, we can therefore focus on prices and quantities over the closed time interval $[0, T]$, because they repeat themselves within each announcement cycle. We use T^+ and T^- to denote the moment right after announcements and right before announcements, respectively. Whenever there is any confusion, time 0 should be understood as T^+ and T should be understood as T^-.

Below we construct an equilibrium in which there exists a $\tau \in (0, T)$ such that all uninformed investors find it suboptimal to acquire any information before τ, and after $t > \tau$, they optimally choose to acquire the signal $s_{u,t}$ until the next announcement.

Definition of the equilibrium A stationary equilibrium consists of a collection of pricing functions $\{\phi(t), \phi_D, \phi_\theta(t), \phi_\Delta(t)\}$, demand functions of the informed investors, $\alpha(t, \theta_t, \Delta_t) = \alpha_0(t) + \alpha_\theta(t)\theta_t + \alpha_\Delta(t)\Delta_t$, and demand functions for uninformed investors, $\beta\left(t, \tilde{\theta}_t\right) = \beta_0(t) + \beta_\theta(t)\tilde{\theta}_t$ such that:

1. Given the pricing functions $\{\phi(t), \phi_D, \phi_\theta(t), \phi_x(t), \phi_\Delta(t)\}$, $\{\alpha_0(t), \alpha_\theta(t), \alpha_\Delta(t)\}$ represents the optimal portfolio demand for the informed investors.

2. Uniformed investors strictly prefer not to acquire information for all $t < \tau$. After time τ, uninformed investors prefer to acquire information.

3. Given their information set, $\{\beta_0(t), \beta_\theta(t)\}$ represents the optimal portfolio demand of uninformed investors.

4. Markets clear, that is,

$$(1-\omega)\alpha(t, \theta_t, \Delta_t) + \omega\beta\left(t, \tilde{\theta}_t\right) = \theta_t \tag{17}$$

for all $t \in [0, T]$.

Equilibrium beliefs Because the information set of uninformed investors is a subset of the information set of the informed investors, informed investors can infer the belief of uninformed investors, $\tilde{x}_t$ and compute the difference, $\Delta_t \equiv \hat{x}_t - \tilde{x}_t$. We combine equations (8) and (12) to derive the difference in belief as a diffusion process:

$$d\Delta_t = -a_\Delta(t)\,\Delta_t dt - \frac{\tilde{q}(t)}{\sigma_D} d\hat{B}_{D,t} + \frac{\hat{q}(t)}{\sigma_s}\left[1 - \phi_x(t)\,\nu(t)\right] d\hat{B}_{s,t} + \phi_\theta(t)\,\nu(t)\,\sigma_\theta dB_{\theta,t} - \frac{\tilde{q}(t)}{\sigma_u(t)} dB_{u,t}, \quad (18)$$

where $a_\Delta(t)$ is defined in equation (57) in Appendix 6.3. Given the pricing equation (11), we define the excess return process as $dQ_t = (D_t - rP_t)\,dt + dP_t$. Using the law of motion of the state variables, we can write the excess return as a diffusion process from the perspective of the informed investors:

$$dQ_t = \left[e_0(t) + e_\theta(t)\,\theta_t + e_\Delta(t)\,\Delta_t\right] dt + \varrho_D(t)\,d\hat{B}_{D,t} + \varrho_\xi(t)\,d\hat{B}_{\xi,t} + \varrho_u(t)\,dB_{u,t}, \quad (19)$$

where the coefficients $e_0(t)$, $e_\theta(t)$, $e_\Delta(t)$, $\varrho_D(t)$, $\varrho_\xi(t)$, $\varrho_u(t)$ are given in equation (60) in Appendix 6.3, and $\sigma_\xi(t)\,d\hat{B}_{\xi,t} = d\xi_t - \hat{\mathbb{E}}_t[d\xi_t]$ is the innovation in ξ_t relative to the informed investors' information. This implies the local variance of excess return is of the following form

$$\sigma_P(t) = \varrho_D^2(t) + \varrho_\xi^2(t) + \varrho_u^2(t)\,. \quad (20)$$

Uninformed investors, however, cannot distinguish Δ_t from θ_t. Because they observe the prices, rational expectations imply $P_t = \tilde{\mathbb{E}}_t[P_t]$. This allows us to rewrite the equilibrium price (11) as:

$$P_t = \phi(t) + \phi_D D_t - \phi_\theta(t)\,\tilde{\theta}_t + \bar{\phi}_x \tilde{x}_t. \quad (21)$$

The law of motion of $\tilde{x}_t$ is given in equation (12). To derive the law of motion for $\tilde{\theta}_t$, recall that observing prices is equivalent to observing $\xi_t = \phi_x(t)\,\hat{x}_t - \phi_\theta(t)\,\theta_t - \frac{\hat{q}(t)}{\sigma_D^2}\phi_x(t)\,D_t$. Taking conditional expectation $\tilde{\mathbb{E}}_t$ on both sides, we have $\xi_t = \tilde{\mathbb{E}}_t[\xi_t]$. Therefore,

$$\xi_t = \phi_x(t)\,\tilde{x}_t - \phi_\theta(t)\,\tilde{\theta}_t - \frac{\hat{q}(t)}{\sigma_D^2}\phi_x(t)\,D_t. \quad (22)$$

We have: $\tilde{\theta}_t = \frac{\phi_x(t)}{\phi_\theta(t)}\tilde{x}_t - \frac{1}{\phi_\theta(t)}\xi_t - \frac{\hat{q}(t)}{\sigma_D^2}\frac{\phi_x(t)}{\phi_\theta(t)}D_t$. The law of motion of $\tilde{\theta}_t$ can therefore be written as:

$$d\tilde{\theta}_t = a\left(\bar{\theta} - \tilde{\theta}_t\right) dt + \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D} d\tilde{B}_{D,t} + \left[\phi_x(t)\,\nu(t) - 1\right]\frac{\sigma_\xi(t)}{\phi_\theta(t)} d\tilde{B}_{\xi,t} + \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)} d\tilde{B}_{u,t}. \quad (23)$$

This allows us to write the excess return process dQ_t in terms of a diffusion process adapted to the information set of the uninformed investors:

$$dQ_t = \left[e_0(t) + e_\theta(t)\,\tilde{\theta}_t\right] dt + \varrho_D(t)\,d\tilde{B}_{D,t} + \varrho_\xi(t)\,d\tilde{B}_{\xi,t} + \varrho_u(t)\,d\tilde{B}_{u,t}. \quad (24)$$

Portfolio selection and information acquisition Informed investors in our model solve an optimal portfolio selection problem. At time t, they maximize life-time utility, $\hat{\mathbb{E}}_t\left[\int_0^\infty -e^{-\rho s-\gamma\hat{C}_{t+s}}ds\right]$ by choosing consumption and portfolio holdings, $\left\{\hat{C}_{t+s},\alpha_{t+s}\right\}_{s=0}^\infty$, subject to the following law of motion of wealth:

$$d\hat{W}_t=\left(\hat{W}_t r-\hat{C}_t\right)dt+\alpha_t dQ_t, \tag{25}$$

where the excess return process dQ_t is given in equation (19). As a result, the value function for informed investors, denoted as $\hat{V}(t,W,\theta,\Delta)$ satisfies the following HJB equation:

$$\hat{V}\left(t,\hat{W},\theta,\Delta\right)=\max_{\hat{C},\alpha}\left\{u\left(\hat{C}\right)+\mathcal{L}^{\hat{C},\alpha}\hat{V}\left(t,\hat{W},\theta,\Delta\right)\right\}, \tag{26}$$

where the Dynkin operator $\mathcal{L}^{\hat{C},\alpha}$ is defined in equation (14), and the superscripts indicate that the expectation is taken under the probability law associated with the policy functions $\left\{\hat{C},\alpha\right\}$.

Uninformed investors solve both an optimal consumption-investment problem and an optimal information acquisition problem. They maximize the life-time utility by choosing the optimal stopping time τ, optimal consumption and portfolio holding $\left\{\tilde{C}_{t+s},\beta_{t+s}\right\}_{s=0}^\infty$, subject to the law of motion of wealth: $d\tilde{W}_t=\left(\tilde{W}_t r-\tilde{C}_t\right)dt+\beta_t dQ_t$. Consider an announcement cycle, $[0,T]$, we focus on symmetric equilibria where all uninformed investors exercise the option of information acquisition at time $\tau\in[0,T]$. For tractability, we allow the information acquisition cost, $K\left(\tilde{\theta}\right)$ to depend on the posterior belief $\tilde{\theta}_t$. This allows us to choose the functional form of $K\left(\tilde{\theta}\right)$ so that the equilibrium optimal stopping time τ is deterministic.

For $t<\tau$, we denote the value function as $\tilde{V}\left(t,W,\tilde{\theta}|\infty\right)$, where the notation $(\cdot|\infty)$ indicates $\sigma_u(t)=\infty$ of $t<\tau$. The value function must satisfy the following HJB equation:

$$\rho\tilde{V}\left(t,W,\tilde{\theta}|\infty\right)=\max_{C,\beta}\left[u(C)+\mathcal{L}_\infty^{C,\beta}\tilde{V}\left(t,W,\tilde{\theta}|\infty\right)+\mathcal{T}_\infty^{C,\beta}\tilde{V}\left(t,W,\tilde{\theta}|\infty\right)\right], \tag{27}$$

where the $\mathcal{L}_\infty^{C,\beta}$ and $\mathcal{T}_\infty^{C,\beta}$ operators are defined in equations (14) and (16), respectively. The superscripts indicate that the expectation is taken under the probability law associated with the policy functions $\left(\tilde{C},\beta\right)$ and the belief associated with $\sigma_u(t)=\infty$.

After the exercising the option for information acquisition, for $t>\tau$, we denote the value function as $\tilde{V}\left(t,W,\tilde{\theta}|\sigma_u\right)$. Optimality of consumption-portfolio choice requires that the following HJB equation must be satisfied:

$$\rho\tilde{V}\left(t,W,\tilde{\theta}|\sigma_u\right)=\max_{C,\beta}\left[u(C-k)+\mathcal{L}_{\sigma_u}^{C,\beta}\tilde{V}\left(t,W,\tilde{\theta}|\sigma_u\right)+\mathcal{T}_{\sigma_u}^{C,\beta}\tilde{V}\left(t,W,\tilde{\theta}|\sigma_u\right)\right], \tag{28}$$

where the rate of consumption is $C-k$ because uninformed investors need to pay a flow cost of k to keep observing the signals $s_u(t)$. Here, the subscripts of the $\mathcal{L}_{\sigma_u}^{C,\beta}$ and $\mathcal{T}_{\sigma_u}^{C,\beta}$ operators indicate the expectations are taken with respect to an information set that includes the newly acquired signal, $s_u(t)$.

Because uninformed investors optimally exercise the option of information acquisition at time $t = \tau$ by paying the cost $K\left(\tilde{\theta}\right)$, value functions must satisfy the following value matching condition. For all W and $\tilde{\theta}$,

$$\tilde{V}\left(\tau, W, \tilde{\theta}|\sigma_u\right) = \tilde{V}\left(\tau, W - K\left(\tilde{\theta}\right), \tilde{\theta}|\infty\right). \tag{29}$$

In addition, as shown in the appendix, the following conditions are sufficient conditions for $t = \tau$ being an optimal stopping time.

Lemma 1. *Suppose $\tilde{V}\left(t, W, \tilde{\theta}|\sigma_u\right)$ satisfies the HJB equations, (27) and (28), and the value matching condition (29). Suppose also, the following two conditions hold:*

1. *For all $t < \tau$, the following inequality holds:*

$$\rho\tilde{V}\left(t, W + K\left(\tilde{\theta}\right), \tilde{\theta}|\infty\right) \geq \max_{C,\beta}\left\{u\left(C - k\right) + \mathcal{L}_{\sigma_u}^{C,\beta}\tilde{V}\left(t, W + K\left(\tilde{\theta}\right), \tilde{\theta}|\infty\right)\right\}. \tag{30}$$

2. *For all $t > \tau$,*

$$\rho\tilde{V}\left(t, W - K\left(\tilde{\theta}\right), \tilde{\theta}|\sigma_u\right) \geq \max_{C,\beta}\left\{u\left(C\right) + \mathcal{L}_{\infty}^{C,\beta}\tilde{V}\left(t, W - K\left(\tilde{\theta}\right), \tilde{\theta}|\sigma_u\right)\right\} \tag{31}$$

Then τ is the optimal time for information acquisition, and $\tilde{V}\left(t, W, \tilde{\theta}|\sigma_u\right)$ is the associated value function.

Proof. See Appendix 6.6. □

Market clearing In our model, the equilibrium price is pinned down by the market clearing condition in equation (17). Using equation (22), $\phi_x(t)\hat{x}_t - \phi_\theta(t)\theta_t = \phi_x(t)\tilde{x}_t - \phi_\theta(t)\tilde{\theta}_t$, we could obtain the following identity,

$$\tilde{\theta}_t = \theta_t - \frac{\phi_x(t)}{\phi_\theta(t)}\Delta_t. \tag{32}$$

Intuitively, because uninformed investors observe prices, they can make mistakes about $\hat{x}_t$ and θ_t separately, but will not make a mistake about $\phi_x(t)\hat{x}_t - \phi_\theta(t)\theta_t$. This restriction implies that the only reason for the uninformed to be relatively more pessimistic about $\hat{x}_t$ is that they believe that the higher level of price is not justified by higher fundamentals, $\hat{x}_t$, but by a lower noisy supply θ_t. That is, $\hat{x}_t - \tilde{x}_t$ and $\theta_t - \tilde{\theta}_t$ must have the same sign to capture the relative pessimism between informed and uninformed investors.

Using the above to replace $\tilde{\theta}_t$ in the market clearing condition (17), we obtain the following restrictions on the portfolio decisions:

$$(1-\omega)\alpha_0(t) + \omega\beta_0(t) = 0, \tag{33}$$

$$(1-\omega)\alpha_\theta(t) + \omega\beta_\theta(t) = 1, \tag{34}$$

$$(1-\omega)\alpha_\Delta(t) - \omega\frac{\phi_x(t)}{\phi_\theta(t)}\beta_\theta(t) = 0. \tag{35}$$

In Appendix 6.3, we show that investors' optimality problems and the above market clearing conditions jointly pin down the pricing functions $\{\phi(t), \phi_\theta(t), \phi_\Delta(t)\}$.

5 Model Implications

Thanks to the CARA-normal setup, our model allows for closed-form solutions. We provide details of our model solutions and derivations in Appendix 6.5. In this section, we calibrate our model and demonstrate the main implications of our model under our chosen calibration. Here we summarize the main implications of the model.

1. Uninformed investors' incentive to acquire information increases monotonically over time and peaks before the announcements. Because information acquisition is costly, it is optimal to acquire information shortly before announcements.

2. As uninformed investors start to acquire information, stock returns and their stochastic discount factor become more correlated. Under generalized risk sensitivity, this higher correlation translates into a higher risk premium and leads to an increase in expected returns, or the pre-FOMC announcement drift.

3. Because newly acquired information (about $\hat{x}_t$) has already been incorporated into the market price through informed investors' trading activities, information acquisition by uninformed investors does not trigger a high realized volatility. Instead, it eliminates noise in the equilibrium price and leads to a lower realized volatility during the pre-announcement period.

4. Upon the announcement, the true value of x_t is revealed, and as a result, realized volatility spikes.

We begin by analyzing the incentives for the endogenous information acquisition.

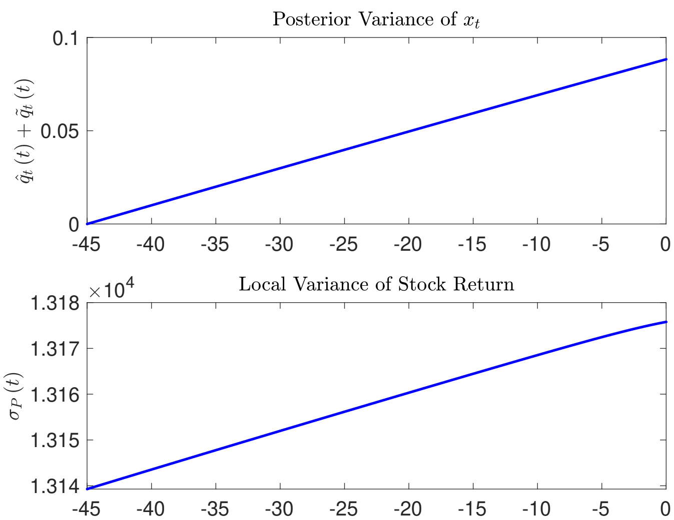
Timing of the information acquisition In our model, as in the data, periodical announcements are pre-scheduled. Uninformed investors do not find it optimal to acquire information until close to the upcoming announcements for two reasons. First, because announcements fully reveal the true value of x_t, initially after the previous announcement, both the informed and the uninformed investors have little uncertainty about x_t, so there is no need to acquire additional information. As t increases from 0, x_t drifts away from its previous value due to lack of information. From the perspective of uninformed traders, uncertainty slowly builds up and the benefit of information acquisition rises over time.

In the model, we measure uncertainty in two ways, the posterior variance for x_t from the perspective of uninformed investors, $\tilde{\mathbb{E}}\left[(\tilde{x}_t - x_t)^2\right] = \tilde{q}(t) + \hat{q}(t)$, and the stock return volatility.[4] To illustrate the buildup of uncertainty over time, in Figure 5, we plot the posterior variance of uninformed investors, $\tilde{q}(t) + \hat{q}(t)$ in the top panel and the variance of stock returns, $Var[dQ_t]$ in

[4] See Appendix 6.3 for the derivations of the joint distributions in beliefs.

the bottom panel, in an economy without information acquisition. As in the data, our example features eight FOMC announcements per year and therefore each announcement cycle is 45 days. Clearly, without information acquisition, both measures of uncertainty increase over time until the announcement. As a result, the benefit for the uninformed traders to acquire information also increases over time.

Figure 5: Uncertainty Dynamics in an Economy without Information Acquisition



This figure plots two measures of uncertainty in an economy without information acquisition within one announcement cycle. The top panel is the uninformed investors' posterior variance about x_t, $\hat{q}_t + \tilde{q}_t$ and the bottom panel is the local variance of stock market returns, $\sigma_P(t)$ defined in equation (20). The horizontal axis is the number of days before the upcoming announcement, which is normalized to 0. A -5, for example, stands for five days before the announcement.

The second reason for the increasing pattern of the incentive for information acquisition is that due to the asymmetric information, the information disadvantage of the uninformed investors rises over time, and so does their trading losses. To see this, note that the expected excess return of the uninformed investors' portfolio is $\beta\left(\tilde{\theta}_t\right) dQ_t$, where we denote $\beta\left(\tilde{\theta}_t\right) = \beta_0(t) + \beta_\theta(t)\,\tilde{\theta}_t$ is the equilibrium portfolio policy for the uninformed investors. We can intuitively define $\mathbb{E}\left[\beta(\theta_t)\,dQ_t - \beta\left(\tilde{\theta}_t\right) dQ_t\right]$ as the expected trading loss due to asymmetric information. That is, the expected gain uninformed investors could obtain if he knows the true state variable θ_t. Note that

$$\left[\beta(\theta_t) - \beta\left(\tilde{\theta}_t\right)\right] dQ_t = \beta_\theta(t)\left(\theta_t - \tilde{\theta}_t\right) dQ_t. \tag{36}$$

Using the identities $P_t = \tilde{\mathbb{E}}_t[P_t]$ and equation (32), $\theta_t - \tilde{\theta}_t = -\frac{\phi_x(t)}{\phi_\theta(t)}\Delta_t$. We first take conditional expectation of (36) given the informed investors' information set:

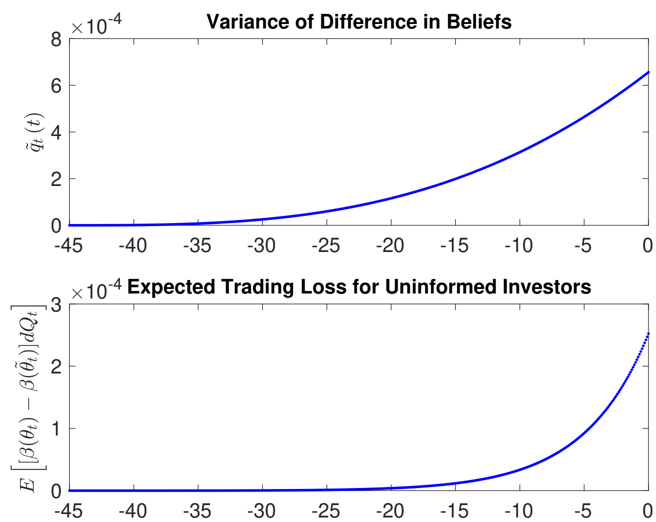
$$\begin{aligned}
\hat{\mathbb{E}}_t\left[\beta_\theta(t)\left(\theta_t-\tilde{\theta}_t\right)dQ_t\right] &= \beta_\theta(t)\left(\theta_t-\tilde{\theta}_t\right)\left[e_0(t)+e_\theta(t)\theta_t+e_\Delta(t)\Delta_t\right]dt \\
&= \beta_\theta(t)\left(\theta_t-\tilde{\theta}_t\right)\left[e_0(t)+e_\theta(t)\theta_t\right]dt+\beta_\theta(t)\left(\theta_t-\tilde{\theta}_t\right)e_\Delta(t)\Delta_t dt \\
&= \beta_\theta(t)\left(\theta_t-\tilde{\theta}_t\right)\left[e_0(t)+e_\theta(t)\theta_t\right]dt-\beta_\theta(t)\frac{\phi_x(t)}{\phi_\theta(t)}e_\Delta(t)\Delta_t^2 dt,
\end{aligned}$$

where the last line uses the identity $\theta_t-\tilde{\theta}_t=-\frac{\phi_x(t)}{\phi_\theta(t)}\Delta_t$. Now, taking the unconditional expectation of the above equations, we have

$$\mathbb{E}\left[\left[\beta(\theta_t)-\beta\left(\tilde{\theta}_t\right)\right]dQ_t\right]=-\beta_\theta(t)\frac{\phi_x(t)}{\phi_\theta(t)}e_\Delta(t)\tilde{q}_t dt. \tag{37}$$

The term $\beta_\theta(t)\frac{\phi_x(t)}{\phi_\theta(t)}e_\Delta(t)Var[\Delta_t]$ can therefore be interpreted as the unconditional expectation of trading losses per unit of investment in the market portfolio for uninformed investors due to information disadvantage.

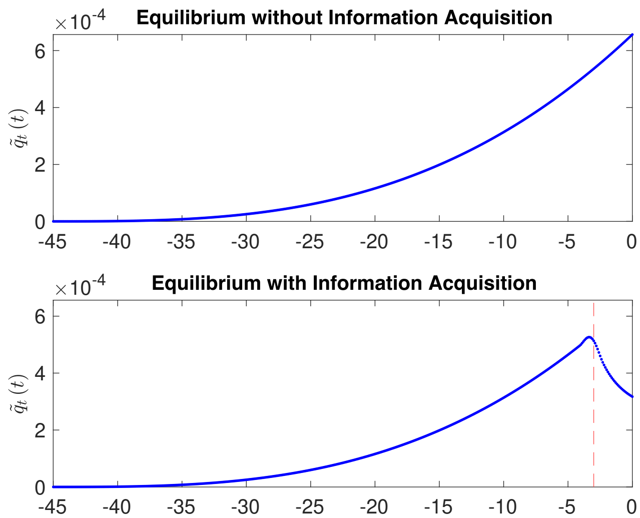
Figure 6: Information Disadvantage of Uninformed Investors



This figure plots two measures of the information disadvantage for uninformed investors over one announcement cycle without information acquisition. The top panel plots the unconditional variance of difference in beliefs, $\tilde{q}_t$. The bottom panel is the expected trading loss due to the information gap, $\beta_\theta(t)\frac{\phi_x(t)}{\phi_\theta(t)}e_\Delta(t)Var[\Delta_t]$, defined in equation (37). The horizontal axis is the number of days before the upcoming announcement, which is normalized to 0. A -5, for example, stands for five days before the announcement.

In Figure 6, we plot two measures of the information disadvantage for uninformed investors. The top panel plots the unconditional variance of difference in beliefs: $\tilde{q}_t=Var[\Delta_t]$. The bottom panel is the expected trading loss due to information gap, $\frac{\phi_x(t)}{\phi_\theta(t)}e_\Delta(t)Var[\Delta_t]$, as defined in equation (37). As shown in the figure, the information disadvantage for uninformed investors, as measured by $\tilde{q}_t$ is relatively small on non-announcement days and increases over time until the announcement. The expected trading loss exhibits the same pattern. In an economy without information acquisition, the information advantage of informed investors increases over time and peaks right before the announcement. At the announcement, the true value of x_t is revealed and a large amount of new information arrives at the market in a short period. Information suddenly becomes homogeneous and the posterior variance for both informed and uninformed investors jumps to zero. Therefore, information acquisition prior to announcements is particularly valuable for uninformed investors because the information disadvantage is particularly costly right before the announcements.

Figure 7: Uninformed Investors' Posterior Variance, $\tilde{q}_t$

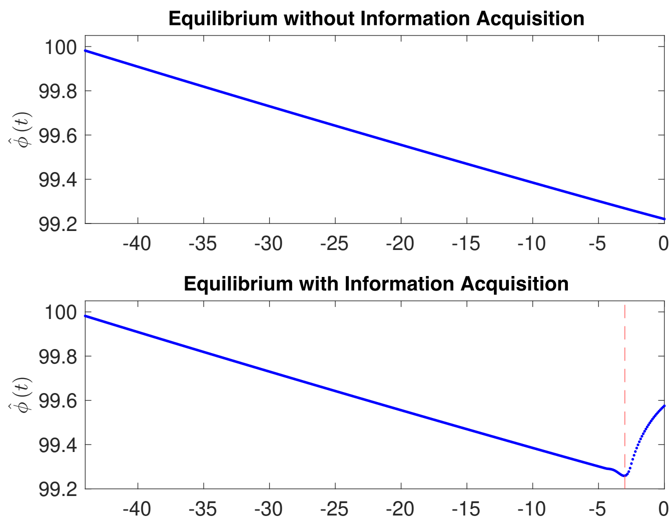


This figure plots $\tilde{q}_t$, the posterior variance of the uninformed investor's belief of $\hat{x}_t$ over one announcement cycle. The top panel is a model without information acquisition and the bottom penal is our benchmark economy with endogenous information acquisition. The vertical line indicates the timing when uninformed investors start to acquire information. The horizontal axis is the number of days before the upcoming announcement, which is normalized to 0. A -5, for example, stands for five days before the announcement.

In Figure 7, we plot $\tilde{q}_t$, which is the uninformed investors' posterior variance of $\hat{x}_t$. The top panel is the path of $\tilde{q}_t$ in equilibrium without information acquisition, where $\tilde{q}_t$ increases monotonically from day -45 to day 0, the announcement day. The bottom panel of Figure 7 is $\tilde{q}_t$ in our benchmark model with endogenous information acquisition, in which the uninformed decide to acquire information starting from 3 days before the announcement. As uninformed investors start to acquire information, the price becomes more informative, and $\tilde{q}_t$ drops sharply from day -3 to day 0. The fact that investors start to acquire information endogenously in our model days ahead of the FOMC announcement provides a rational explanation for the increasing patterns of investors' attentions around macroeconomic announcements documented by Fisher, Martineau, and Sheng (2020).

Pre-FOMC announcement drift To understand the model's implications on pre-FOMC announcement drift, in Figure 8, we plot the unconditional expectation of equilibrium price: $\hat{\phi}(t) = \mathbb{E}[P_t] = \phi(t) + \left[\bar{\phi}_x + \phi_D\right]\bar{x} - \phi_\theta(t)\bar{\theta}$ as a function of time for a model without information acquisition (top panel) and that for a model with information acquisition (bottom panel). To illustrate the quantitative implication on the magnitude of the pre-announcement drift, we normalize the level of $\hat{\phi}(t)$ at time -45 to 100. Therefore, an increase of $\hat{\phi}(t)$ from 99 to 100, for instance, corresponds to 100 basis points of return. Our model generates a pre-announcement drift of 31.6 bps.

Figure 8: Expected level of price, $\hat{\phi}(t)$



This figure plots the expected level of price, $\hat{\phi}(t)$ of a model without information acquisition (top panel) and that for our benchmark economy with endogenous information acquisition (bottom panel). The vertical line indicates the timing when uninformed investors start to acquire information. The horizontal axis is the number of days before the upcoming announcement, which is normalized as 0. A -5 for example, stands for five days before the announcement.

In the model without information acquisition, the expected level of price monotonically decreases until T and jumps upwards upon the announcement. The fact that generalized risk sensitivity produces an announcement premium is the same as in Ai and Bansal (2018). In the model with information acquisition, the function $\hat{\phi}(t)$ reaches its minimum at time τ, as the uninformed investors start to acquire information. From its minimum at time $\tau = -3$ to the announcement time, the drift is about 20 basis points, which is similar in magnitude to the pre-FOMC announcement drift we computed in Section 2. At the announcement, the expected return increases again as the new information arrives, giving an announcement premium about 22 basis points.

The above pattern of drift in price is also reflected in the pattern of expected returns. From equation (19), the unconditional expected excess return of the stock is

$$\mathbb{E}[dQ_t] / \mathbb{E}[P_t] = \left[e_0(t) + e_\theta(t)\bar{\theta}\right] / \mathbb{E}[P_t]. \tag{38}$$

In Figure 9, we plot the expected return of the stock as a function of time. Consistent with the pattern of prices, expected return increases sharply starting from $\tau = -3$, as uninformed investors start to acquire information.

Equation (150) in Appendix 6.5 shows explicitly that the unconditional expected return depends on two terms, one comes from the standard expected utility and the other term is proportional to the ambiguity aversion κ and the continuation utility reduction $\mathcal{T}\tilde{V}_t$ defined in (16). Intuitively, after information acquisition, the variance of conditional expectation increases sharply, as newly arrived information has a large impact on the belief of uninformed investors. When uninformed investors start to acquire information, they become more pessimistic hence require higher compensation in returns because the newly acquired information creates significant variations in their continuation utilities due to ambiguity. This feature of our model provides a rational explanation for the pre-FOMC announcement drift.

As in Ai and Bansal (2018), due to GRS in preferences, resolution of uncertainty is associated with realizations of risk premium. In the context of our model, GRS is captured by the $\mathcal{T}$ operator defined in equation (16). Note that the magnitude of this term depends on the variance of conditional expectation. Intuitively, uninformed investors are ambiguity averse about the hidden state $\hat{x}_t$, or equivalently, θ_t.[5] Intuitively, higher variations in the conditional expectation about the hidden state trigger more pessimism from the uninformed investors due to ambiguity aversion and lower their level of continuation utility. In equilibrium, this ambiguity requires risk compensation and demands a higher return from the stock market.

From equation (12), the local variance of the conditional expectation is given by

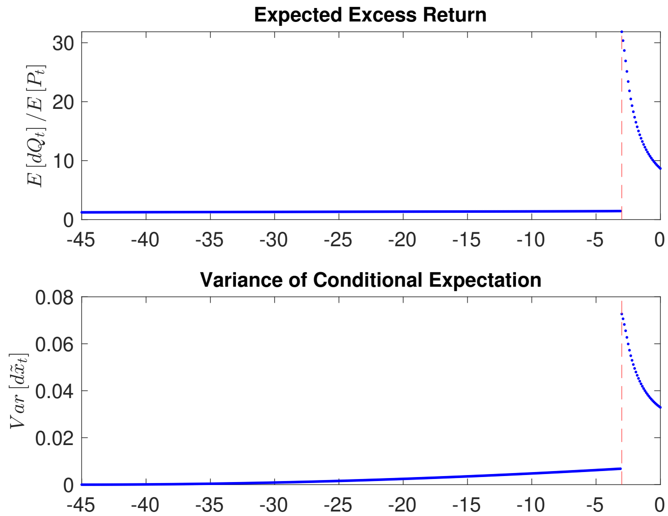
$$Var\left[d\tilde{x}_t\right]=\left[\frac{\hat{q}\left(t\right)+\tilde{q}\left(t\right)}{\sigma_D}\right]^2+\left[\nu\left(t\right)\sigma_\xi\left(t\right)\right]^2+\left[\frac{\tilde{q}\left(t\right)}{\sigma_u\left(t\right)}\right]^2. \tag{39}$$

As we show in the appendix, the equilibrium volatility of the stochastic discount factor is proportional to $Var\left[d\tilde{x}_t\right]$. Before information acquisition, for $t<\tau$, $\sigma_u\left(t\right)=\infty$. As a result, $Var\left[d\tilde{x}_t\right]=\left[\frac{\hat{q}(t)+\tilde{q}(t)}{\sigma_D}\right]^2+\left[\nu\left(t\right)\sigma_\xi\left(t\right)\right]^2$ is determined by the first two terms. After information acquisition, because uninformed investors pay more attention to learn about the public available information, they start to obtain a very precise signal, i.e., a small σ_u. Most of the variance of $d\tilde{x}_t$ starts to be driven by the last term $\left[\frac{\tilde{q}(t)}{\sigma_u(t)}\right]^2$. As a result, the variance of the stochastic discount increases sharply during the pre-announcement period, generating a significant pre-announcement drift.

In Figure 9, we plot the expected excess return of the stock (top panel) and the variance of conditional expectation, $Var\left[d\tilde{x}_t\right]$ (bottom panel) in the economy with information acquisition. After information acquisition, the variance of conditional expectation increases sharply, as newly arrived information has a large impact on the belief of uninformed investors. Because the variance of the stochastic discount factor is proportional to $Var\left[d\tilde{x}_t\right]$, as the variance of conditional expectation increases, the expected return of the stock also rises. Intuitively, when uninformed investors start to acquire information, they become more pessimistic hence require higher compensation in returns because the newly acquired information creates significant variations in their continuation utilities due to ambiguity. This feature of our model provides a rational explanation for the pre-FOMC announcement drift.

[5] Recall that the identity $P_t=\tilde{\mathbb{E}}_t\left[P_t\right]$ implies that $\phi_x\left(t\right)\tilde{x}_t-\phi_\theta\left(t\right)\tilde{\theta}_t=\phi_x\left(t\right)\hat{x}_t-\phi_\theta\left(t\right)\theta_t$, where the sum $\phi_x\left(t\right)\hat{x}_t-\phi_\theta\left(t\right)\theta_t$ can be inferred from prices. As a result, concerns about misspecification of the distribution of $\hat{x}_t$ is equivalent to concerns about robustness of θ_t.

Figure 9: Expected Excess Return of the Stock



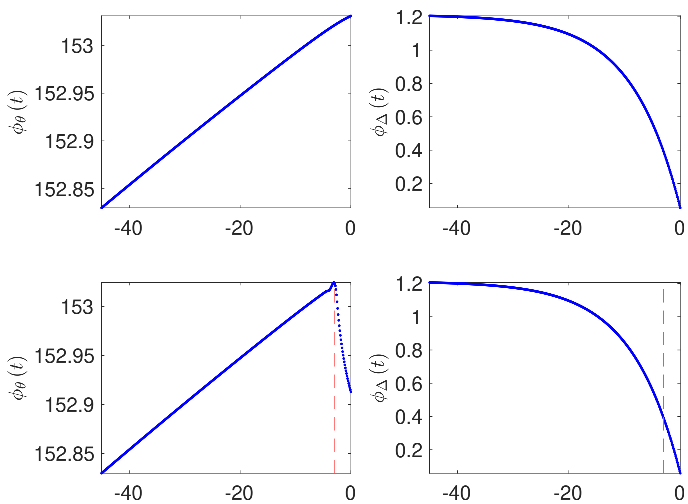
This figure plots the expected excess return of the stock $\mathbb{E}[dQ_t]/\mathbb{E}[P_t]$ (top panel) and the variance of conditional expectation $Var[d\tilde{x}_t]$ (bottom panel), as a function of time in our benchmark model with information acquisition. The horizontal axis is the number of days before the upcoming announcement, which is normalized to 0. A -5, for example, stands for five days before the announcement.

Equilibrium asset prices To better understand the asset pricing implications of ambiguity and endogenous information acquisition, in this section, we discuss the equilibrium pricing functions $\phi_\theta(t)$ and $\phi_\Delta(t)$. We plot pricing functions $\phi_\theta(t)$ and $\phi_\Delta(t)$ in Figure 10 for an economy without information acquisition in top panels and those for our benchmark model with endogenous information acquisition in bottom panels. As shown in Figure 10, the function $\phi_\theta(t)$ monotonically increases in the model without information acquisition. $\phi_\theta(t)$ is the impact of noisy supply on the stock price from equation (10). In our model, stock price decreases in θ_t for two reasons. First, increases in supply lower the equilibrium price due to a downward sloping demand curve as in standard equilibrium models. This effect does not depend on the uncertainty or the asymmetric information. Second, the information asymmetry and learning amplify the responses of prices to supply shocks, therefore an increase in θ_t further lowers the price. Because the uninformed investors cannot infer the true value of θ_t and x_t from prices, they attribute part of the price drop as deteriorations in fundamentals and downwardly revise their beliefs about x_t. The uninformed investors reduce their holdings of the stock because of their distorted pessimistic beliefs. This lowers the demand of the asset and the price has to drop further to clear the market.

Clearly, the second effect is stronger when uninformed investors are more uncertain about x_t. At time $t = -45$, right after an announcement, uninformed investors know the true value of x_0 and the information asymmetry is temporarily eliminated. As t increases, the uncertainty about x_t builds up, and changes in prices have stronger impacts on uninformed investors' beliefs because they have to rely more and more on learning from prices. Therefore, prices become more sensitive to supply shocks, θ_t.

In the economy with information acquisition, after time $\tau = -3$, as the uninformed investors

Figure 10: Pricing functions



This figure plots the pricing functions $\phi_\theta(t)$ and $\phi_\Delta(t)$ of a model without information acquisition (the top panels) and those for our benchmark economy with endogenous information acquisition (the bottom panels). The vertical line indicates the timing when uninformed investors start to acquire information. The horizontal axis is the number of days before the upcoming announcement, which is normalized as 0. A -5 for example, stands for five days before announcements.

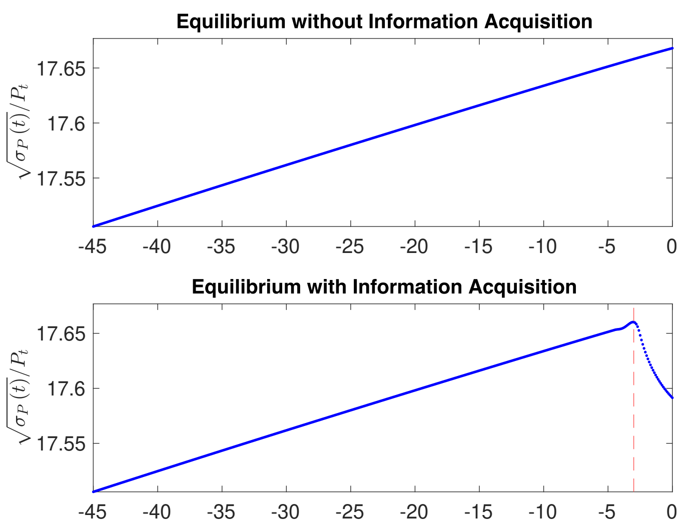
acquire more information, their uncertainty drops and the amplification effect from information asymmetry reduces. As a result, the function $\phi_\theta(t)$ starts to drop until time 0. The drop of $\phi_\theta(t)$ function after τ is important for our model to account for the lower realized volatility during the pre-announcement drift period. After time τ, the impact of noise traders reduces, and so does the realized volatility of stock returns.

In both models, with and without information acquisition, $\phi_\Delta(t)$ monotonically decreases over time. The function $\phi_\Delta(t)$ has a clear interpretation in equation (10): it is the price impact of uninformed investors. In the economy without information acquisition, because the posterior variance, $\tilde{q}(t)$, monotonically increases over time, uninformed investors become more and more uncertain about $\hat{x}_t$ and, as a result, they trade less and less aggressively, and exert a lower and lower price impact over time.

In the model with information acquisition, $\phi_\Delta(t)$ decreases monotonically over time until time -3, when information acquisition starts. The information acquisition at time $\tau = -3$ has two effects on uninformed investors' price impact, $\phi_\Delta(t)$. On one hand, information acquisition lowers uncertainty and uninformed traders have an incentive to trade more aggressively. On the other hand, information acquisition creates an additional correlation between stock returns and the wealth of the uninformed investors due to ambiguity aversion. As a result, they trade less aggressively due to the hedging demand channel. In Figure (10), the hedging demand channel dominates, and the price impact $\phi_\Delta(t)$ keeps falling after τ.

Return volatility As we emphasize in Section 2, a particularly puzzling aspect of the pre-FOMC announcement drift is the coexistence of high average returns and low realized volatility during the pre-announcement period. While the low realized volatility during the pre-announcement drift period is difficult to reconcile with an information leakage-based explanation, it is a robust feature of our model with asymmetric information and endogenous information acquisition. Because the newly acquired information is about the private information of informed investors, which has already been incorporated into prices through their trading activities, information acquisition is not associated with a higher realized volatility. On the contrary, information acquisition reduces uncertainty, information asymmetry, and in particular, the price impact of noise traders. As we show in Figure 10, the price impact of noise traders drops after τ. As a result, information acquisition by uninformed investors is a process of eliminating noise in stock prices and therefore associated with a lower realized volatility of stock returns.

Figure 11: Return Volatility



The top and bottom panel plot the return volatility (annualized in percentage) in our benchmark economy without and with endogenous information acquisition, respectively. The vertical line indicates the timing when uninformed investors start to acquire information. The horizontal axis is the number of days before the upcoming announcement, which is normalized as 0. A -5 for example, stands for five days before announcements.

In Figure 11, we plot the realized volatility of stock returns in the economy without information acquisition (top panel) and that in an economy with information acquisition (bottom panel) implied by our model. Consistent with the pattern of pricing functions in Figure 10, realized volatility is low in our model during the pre-announcement period. This feature of our model provides a coherent explanation for the coexistence of high average return and low realized volatility during the pre-announcement period.

In addition, because there is no information leakage during the pre-announcement period, the actual announcements are associated with arrival of substantial new information to the market. As a result, in our model, realized volatility spikes upon announcements. This pattern is also consistent with the empirical evidence that we document in Section 2.

6 Conclusion

In this paper, we develop a noisy rational expectations model with endogenous information acquisition and periodic announcements to account for the pre-FOMC announcement puzzle. We show that the endogenous information acquisition together with the generalized risk sensitive preference allow us to provide an equilibrium interpretation of the coexistence of the puzzling pattern of a high average return and low realized market volatility during the pre-FOMC announcement period. The endogenous information acquisition in our model is consistent with the pattern of investor attention documented by Fisher, Martineau, and Sheng (2020). Our model does not assume information leakage and matches the empirical patterns of the FOMC announcement returns and volatility dynamics in the data quite well.

References

Ai, Hengjie, and Ravi Bansal, 2018, Risk Preferences and the Macroeconomic Announcement Premium, *Econometrica* 86, 1383–1430.

Ai, Hengjie, Ravi Bansal, Hongye Guo, and Amir Yaron, 2020, Identifying Preference for Early Resolution from Asset Prices, *Unpublished Working Paper*.

Ai, Hengjie, Ravi Bansal, Jay Im, and Chao Ying, 2020, A Model of the Macroeconomic Announcement Premium with Production, *Unpublished Working Papers*.

Albuquerque, Rui, and Jianjun Miao, 2014, Advance Information and Asset Prices, *Journal of Economic Theory* 149, 236–275.

Andrei, Daniel, and Julien Cujean, 2017, Information Percolation, Momentum and Reversal, *Journal of Financial Economics* 123, 617–645.

Andrei, Daniel, Julien Cujean, and Mungo Ivor Wilson, 2018, The Lost Capital Asset Pricing Model, *Unpublished Working Paper*.

Avdis, Efstathios, 2016, Information Tradeoffs in Dynamic Financial Markets, *Journal of Financial Economics* 122, 568–584.

Banerjee, Snehal, and Bradyn Breon-Drish, 2020, Strategic trading and unobservable information acquisition, *Journal of Financial Economics* 138, 458–482.

Banerjee, Snehal, and Brett Green, 2015, Signal or Noise? Uncertainty and Learning about Whether Other Traders are Informed, *Journal of Financial Economics* 117, 398–423.

Barndorff-Nielsen, O. E., P. Reinhard Hansen, A. Lunde, and N. Shephard, 2009, Realized Kernels in Practice: Trades and Quotes, *The Econometrics Journal* 12, C1–C32.

Boguth, Oliver, Vincent Grégoire, and Charles Martineau, 2018, Shaping Expectations and Coordinating Attention: The Unintended Consequences of FOMC Press Conferences, *Journal of financial and quantitative analysis* 54, 2327–2353.

Bollerslev, Tim, Jia Li, and Yuan Xue, 2018, Volume, Volatility, and Public News Announcements, *The Review of Economic Studies* 85, 2005–2041.

Bond, Philip, and Itay Goldstein, 2015, Government Intervention and Information Aggregation by Prices, *Journal of Finance* 70, 2777–2812.

Breon-Drish, Bradyn, 2015, On Existence and Uniqueness of Equilibrium in a Class of Noisy Rational Expectations Models, *The Review of Economic Studies* 82, 868–921.

Brownlees, C.T, and G.M Gallo, 2006, Financial Econometric Analysis at Ultra-high Frequency: Data Handling Concerns, *Computational statistics & Data Analysis* 51, 2232–2245.

Buffa, Andrea, Dimitri Vayanos, and Paul Woolley, 2019, Asset Management Contracts and Equilibrium Prices, *Unpublished Working Paper*.

Chen, Zengjing, and Larry Epstein, 2002, Ambiguity, Risk, and Asset Returns in Continuous Time, *Econometrica* 70, 1403–1443.

Cieslak, Anna, Adair Morse, and Annette Vissing-Jorgensen, 2019, Stock Returns over the FOMC Cycle, *The Journal of Finance* 74, 2201–2248.

Cocoma, Paula, 2020, Explaining the Realized Pre-Announcement Drift, *Unpublished Working Paper*.

Epstein, Larry G, and Martin Schneider, 2007, Learning under Ambiguity, *The Review of Economic Studies* 74, 1275–1303.

Ernst, Rory, Thomas Gilbert, and Christopher M Hrdlicka, 2019, More than 100% of the Equity Premium: How Much is Really Earned on Macroeconomic Announcement Days?, *Unpublished Working Paper*.

Fisher, Adlai J, Charles Martineau, and Jinfei Sheng, 2020, Macroeconomic Attention and the Stock Market, *Unpublished Working Paper*.

Goldstein, Itay, and Liyan Yang, 2017, Information Disclosure in Financial Markets, *Annual Review of Financial Economics* 9, 101–125.

Grossman, Sanford J., 1981, An Introduction to the Theory of Rational Expectations Under Asymmetric Information, *The Review of Economic Studies* 48, 541–559.

Grossman, Sanford J., and Joseph E. Stiglitz, 1980, On the Impossibility of Informationally Efficient Markets, *American Economic Review* 70, 393–408.

Han, Leyla Jianyu, 2020, Announcements, Expectations, and Stock Returns with Asymmetric Information, *Unpublished Working Paper*.

Hansen, Lars Peter, and Thomas J. Sargent, 2007, Recursive Robust Estimation and Control without Commitment, *Journal of Economic Theory* 136, 1–27.

Hansen, Lars Peter, and Thomas J Sargent, 2008, *Robustness*. (Princeton University Press).

Hansen, Lars Peter, and Thomas J Sargent, 2011, Robustness and Ambiguity in Continuous Time, *Journal of Economic Theory* 146, 1195–1223.

Hellwig, Martin F, 1980, On the Aggregation of Information in Competitive Markets, *Journal of Economic Theory* 22, 477–498.

Hu, Grace Xing, Jun Pan, Jiang Wang, and Haoxiang Zhu, 2020, Premium for Heightened Uncertainty: Solving the FOMC Puzzle, *Unpublished Working Paper*.

Laarits, Toomas, 2020, Pre-Announcement Risk, *Unpublished Working Paper*.

Liptser, Robert S, and Albert N Shiryaev, 2001, *Statistics of Random Processes II: Applications* vol. 6. (Springer Berlin) 2nd edn.

Lucca, David O, and Emanuel Moench, 2015, The pre-FOMC announcement drift, *Journal of Finance* 70, 329–371.

Luo, Yulei, Jun Nie, and Haijun Wang, 2022, Ignorance, Pervasive Uncertainty, and Household Finance, *Journal of Economic Theory* 199, 105204.

Maenhout, Pascal J, 2004, Robust Portfolio Rules and Asset Pricing, *Review of financial studies* 17, 951–983.

Morse, Adair, and Annette Vissing-Jorgensen, 2020, Information Transmission from the Federal Reserve to the Stock Market: Evidence from Governors Calendars, Working paper, University of California at Berkeley.

Savor, Pavel, and Mungo Wilson, 2013, How Much Do Investors Care About Macroeconomic Risk? Evidence from Scheduled Economic Announcement, *Journal of Financial and Quantitative Analysis* 48, 343–375.

Savor, Pavel, and Mungo Wilson, 2014, Asset Pricing: A Tale of Two Days, *Journal of Financial Economics* 113, 171–201.

Sims, Christopher, 2003, Implications of Rational Inattention, *Journal of Monetary Economics* 50, 665–690.

Sockin, Michael, 2019, Not So Great Expectations: A Model of Growth and Informational Frictions, *Unpublished Working Paper.*

Veldkamp, Laura, 2011, *Information Choice in Macroeconomics and Finance.* (Princeton University Press Princeton, N.J.).

Veldkamp, Laura, and Stijn Van Nieuwerburgh, 2010, Information Acquisition and Under-Diversification, *Review of Economic Studies* 77, 779–805.

Wachter, Jessica A., and Yicheng Zhu, 2020, A Model of Two Days: Discrete News and Asset Prices, *Forthcoming, The Review of Financial Studies.*

Wang, Jiang, 1993, A Model of Intertemporal Asset Prices under Asymmetric Information, *Review of Economic Studies* 60, 249–282.

Wang, Jiang, 1994, A Model of Competitive Stock Trading Volume, *Journal of Political Economy* 102, 127–168.

Ying, Chao, 2020, The Pre-FOMC Announcement Drift and Private Information: Kyle Meets Macro-Finance, *Unpublished Working Paper.*

Appendix

6.1 Data

We obtain the pre-scheduled FOMC announcement days from Bloomberg. It includes both the dates and the exact release time. Following Lucca and Moench (2015), we focus on pre-scheduled FOMC meetings, and extend the sample period to September 2020. There are in total 213 scheduled FOMC meetings between January 1994 and September 2020. Before 2011, most FOMC announcements were scheduled around 14:15 p.m. Between 2011 to 2012, there were eight FOMC meetings arranged around 12:30 p.m. After March 2013, all the FOMC announcements were scheduled around 14:00.

We use high frequency data on E-mini S&P 500 index futures from the Chicago Mercantile Exchange (CME) which start from 11:30 a.m. EST, September 9, 1997.[6,7] We focus on the Emini data because it reports the trading volume and it is tradable over 24 hours. Before that, we use S&P 500 index futures instead from CME, available from April 21, 1982. On each day of the E-mini futures, there may be multiple contract delivery dates. We choose the delivery date with the highest volume within each calendar day as the most active futures contract, which is usually the nearest-term contract and occasionally the next contract during rolling forward weeks. We then convert the time zone to EST as the original time stamp is in CST. The raw data are cleaned following the standard procedures described in Barndorff-Nielsen, Hansen, Lunde, and Shephard (2009). First, we delete those entries outside regular trading hours (9:30 to 16:00). Second, we delete the invalid prices with missing values or equal to zero (or 9999.75). Third, we delete entries with canceled or corrected prices (exclude "CAN" = "C" or "X"). Last, within each time stamp (in seconds), we use the median price. If there are two median prices, we use the mean of these two medians. In this way, we obtain the time series of prices in one second. Fourth, in order to mitigate the microstructure noise, we sample the price into one minute frequency. The sampling method follows the "Last" scheme of Brownlees and Gallo (2006), where we pick the last entry of the period ending immediately prior to the timestamp. For example, 10:30 represents the last data from 10:29:00 to 10:29:59. After September 1997 when the Emini S&P 500 futures are available, we obtain the trading volume as the total contracts traded within the 1-min sampling interval. We delete the all the entries with 0 or missing trading volume.

We use log return on the futures from 24 hours before to five minutes before FOMC announcements as the pre-announcement drift. To measure the post-announcement return, we use log return from five minutes before FOMC announcements to one hour afterwards. For instance, the pre-announcement drift for the meeting at 14:00 on 2019 Dec.11 is defined as the log return from 13:55 on 2019 Dec.10 to 13:55 on 2019 Dec.11, whereas the post-announcement return is calculated as the log return from 13:55 to 14:55 on 2019 Dec.11. We report the summary statistics for pre- and post- announcement return in Table 1.

[6] https://www.cmegroup.com/confluence/display/EPICSANDBOX/Time+and+Sales. We use the calendar date (Entry Date) instead of the adjusted trading date (Trade Date).

[7] There are three missing dates from E-mini future data: October 29,1997, January 28 and 29, 2014. The last one is a pre-scheduled FOMC release day. We exclude these days in our analysis.

The average pre-FOMC announcement drift is 32.28 basis points with a Newey-West t-stat of 4.88. The coefficient of regressing pre-announcement drift on post-announcement return is -0.106, with a robust standard error of 0.048 (t-stat of -2.21). The ex-ante return is significantly negative correlated with ex-post announcement return.

We calculate realized variance as the sum of 30-minutes log squared return. The realized volatility at the minute t can be estimated by $\sigma_t = \sqrt{\frac{1}{N}\sum_{j=0}^{N-1} r_{t-j}^2}$, where N equals 30 if there is no missing values and r_t is the log return at time t (defined as the log price difference). For the trading volume M, we simply average over past 30 minutes total number of contracts traded: $M_t = \frac{1}{N}\sum_{j=0}^{N-1} volm_{t-j}$. We calculate the rolling realized volatility and trading volume for each minute.

Table 1: Summary Statistics

	Mean	St.Dev.	Min	Max	Obs.	Time
Pre-Ann Drift (%)	0.323	0.965	-3.098	8.639	212	1994Jan-2020Sep
Post-Ann Return (%)	0.056	0.056	-2.131	2.901	212	1994Jan-2020Sep
Realized Volatility (annualized in %)	13.360	9.634	1.865	144.628	184	1997Sep-2020Sep
Trading Volume (1000 shares)	2.473	2.397	0.004	26.011	184	1997Sep-2020Sep

This table reports summary statistics of pre-announcement drift, post-announcement return, realized volatility and trading volume on FOMC announcement days. We obtain log returns on S&P 500 futures during regular trading hours (9:30-16:00) from January 1994 to September 2020. Pre-Ann Drift stands for the log return in 24-hour windows from one day before the FOMC announcement to five minutes before the meeting. Post-Ann Return is the log return from 5 minutes before the FOMC announcement to one hour afterward. Realized volatility (annualized in percentage) is the average sum of squared returns over the past 30 minutes ($t = [-29, 0]$) and the trading volume is the average contracts traded during the past 30 minutes on FOMC days. We calculate the rolling realized volatility and trading volume for each minute from 10:00 to 16:00. The sample period is from September 1997 to September 2020.

6.2 Calibration

Table 2: Parameters

Para.	Value	Description	Para.	Value	Description
r	0.012	risk-free rate	σ_θ	0.75	volatility of total equity supply
ρ	0.03	time discount factor	σ_u	0.002	inverse of acquired information precision
$\bar{x}$	80	mean level of dividend flow	κ	300	ambiguity aversion
b	0.15	persistence of hidden state	γ	2	risk aversion
a	0.1	persistence of total equity supply	$\bar{\theta}$	35	unconditional mean of aggregate supply
σ_d	1	dividend flow volatility	ω	0.95	fraction of uninformed investor
σ_s	0.7	inverse of signal precision	k	10	flow cost of information acquisition
σ_x	0.85	volatility of hidden state			

6.3 Equilibrium Beliefs

The Filtering Problem of Informed The optimal learning for the informed investor is a standard Kalman filter problem with the unobserved state variable given in equation (1) and the observed processes in equations (5), (6), and (7). Applying Theorem 10.3 from Liptser and Shiryaev (2001), we can show that the law of motion of the posterior mean satisfies equation (8) where the innovation processes for (5) and (7) are given by

$$d\hat{B}_{D,t} = \frac{1}{\sigma_D}\left[dD_t - \left(\hat{x}_t - D_t\right)dt\right], \text{and } d\hat{B}_{s,t} = \frac{1}{\sigma_s}\left(ds_t - \hat{x}_t dt\right). \tag{40}$$

The law of motion of the conditional variance $\hat{q}_t$ must satisfy the Riccati equation

$$d\hat{q}\left(t\right) = \left[\sigma_x^2 - 2b\hat{q}\left(t\right) - \left(\frac{1}{\sigma_D^2} + \frac{1}{\sigma_s^2}\right)\hat{q}^2\left(t\right)\right]dt. \tag{41}$$

We can solve $\hat{q}\left(t\right) = \frac{\sigma_x^2\left(1-e^{-2\hat{b}(t+t^*)}\right)}{\left(\hat{b}-b\right)e^{-2\hat{b}(t+t^*)}+b+\hat{b}}$, where $\hat{b} = \sqrt{b^2 + \sigma_x^2\left(\frac{1}{\sigma_D^2} + \frac{1}{\sigma_s^2}\right)}$ and $t^* = \frac{1}{2\hat{b}}\ln\frac{\sigma_x^2+\left(\hat{b}-b\right)\hat{q}(0)}{\sigma_x^2-\left(\hat{b}+b\right)\hat{q}(0)}$. We assume announcements fully resolve the uncertainty, so that $\hat{q}\left(0\right) = 0$.

Information Content of Prices In addition to observing the dividend, the uninformed trader also observes the equilibrium price. We have assumed that the price process takes the form of equation (10). Because the uninformed know D_t and $\tilde{x}_t$, observing the price is equivalent to observing $\zeta_t \equiv \phi_x\left(t\right)\hat{x}_t - \phi_\theta\left(t\right)\theta_t$. Here, ζ_t can be represented as a Markov process given the state variable $\hat{x}_t, \zeta_t$:

$$\begin{aligned} d\zeta_t \;=\; & \left[b\bar{x}\phi_x\left(t\right) - a\bar{\theta}\phi_\theta\left(t\right) + \left(\left(a - b - \frac{\phi_\theta'\left(t\right)}{\phi_\theta\left(t\right)}\right)\phi_x\left(t\right) + \phi_x'\left(t\right)\right)\hat{x}_t + \left(\frac{\phi_\theta'\left(t\right)}{\phi_\theta\left(t\right)} - a\right)\zeta_t\right]dt \\ & + \frac{\hat{q}\left(t\right)}{\sigma_D}\phi_x\left(t\right)d\hat{B}_{D,t} + \frac{\hat{q}\left(t\right)}{\sigma_s}\phi_x\left(t\right)d\hat{B}_{s,t} - \sigma_\theta\phi_\theta\left(t\right)dB_{\theta,t}, \end{aligned} \tag{42}$$

It is convenient to define $\xi_t = \zeta_t - \frac{\hat{q}(t)}{\sigma_D^2}\phi_x\left(t\right)D_t$ so that $\left(\hat{x}_t, D_t, \xi_t\right)$ has a state space representation. The dynamics of ξ_t is

$$d\xi_t = \left[b\bar{x}\phi_x\left(t\right) - a\bar{\theta}\phi_\theta\left(t\right) + m_x\left(t\right)\hat{x}_t + \left(\frac{\phi_\theta'\left(t\right)}{\phi_\theta\left(t\right)} - a\right)\xi_t + m_D\left(t\right)D_t\right]dt + \sigma_\xi\left(t\right)d\hat{B}_{\xi,t}, \tag{43}$$

where the coefficients,

$$m_x\left(t\right) = \left(a - b - \frac{\phi_\theta'\left(t\right)}{\phi_\theta\left(t\right)} - \frac{\hat{q}\left(t\right)}{\sigma_D^2}\right)\phi_x\left(t\right) + \phi_x'\left(t\right), \tag{44}$$

$$m_D\left(t\right) = \frac{1}{\sigma_D^2}\left[\hat{q}\left(t\right)\phi_x\left(t\right)\left(1 - a + \frac{\phi_\theta'\left(t\right)}{\phi_\theta\left(t\right)}\right) - \hat{q}'\left(t\right)\phi_x\left(t\right) - \hat{q}\left(t\right)\phi_x'\left(t\right)\right], \tag{45}$$

and the volatility of ξ_t is

$$\sigma_\xi\left(t\right) = \sqrt{\frac{\hat{q}^2\left(t\right)}{\sigma_s^2}\phi_x^2\left(t\right) + \sigma_\theta^2\phi_\theta^2\left(t\right)}, \tag{46}$$

and $\hat{B}_{\xi,t}$ is a standard Brownian motion that is independent of $\hat{B}_{D,t}$:

$$d\hat{B}_{\xi,t} = \frac{1}{\sigma_\xi(t)}\left[\frac{\hat{q}(t)}{\sigma_s}\phi_x(t)\,d\hat{B}_{s,t} - \sigma_\theta\phi_\theta(t)\,dB_{\theta,t}\right]. \tag{47}$$

We will call ξ_t the information content of price, as observing price is equivalent to observing ξ_t. From the informed investor's perspective, dividend flow follows

$$dD_t = (\hat{x}_t - D_t)\,dt + \sigma_D d\hat{B}_{D,t}. \tag{48}$$

To apply the Kalman-Bucy filter from Liptser and Shiryaev (2001), we treat (8) as the unobserved state variable and (48), (43) and (9) as the observations. The uninformed investors' posterior beliefs can be characterized as follows

$$\begin{aligned} d\tilde{x}_t &= b(\bar{x} - \tilde{x}_t)\,dt + \frac{\hat{q}(t) + \tilde{q}(t)}{\sigma_D}d\tilde{B}_{D,t} + \nu(t)\,\sigma_\xi(t)\,d\tilde{B}_{\xi,t} + \frac{\tilde{q}(t)}{\sigma_u(t)}d\tilde{B}_{u,t}, && (49)\\ d\tilde{q}(t) &= \left[\left(\frac{1}{\sigma_D^2} + \frac{1}{\sigma_s^2}\right)\hat{q}_t^2 - 2b\tilde{q}_t - \frac{1}{\sigma_u^2(t)}\tilde{q}_t^2 - \frac{(\hat{q}_t + \tilde{q}_t)^2}{\sigma_D^2} - \left(\frac{m_x(t)\,\tilde{q}_t + \frac{\phi_x(t)\hat{q}_t^2}{\sigma_s^2}}{\sigma_\xi}\right)^2\right]dt && (50) \end{aligned}$$

where

$$\nu(t) = \frac{1}{\sigma_\xi^2(t)}\left[\frac{\phi_x(t)}{\sigma_s^2}\hat{q}^2(t) + m_x(t)\,\tilde{q}(t)\right]. \tag{51}$$

and $d\tilde{B}_{D,t} = \frac{1}{\sigma_D}\left(dD_t - \tilde{\mathbb{E}}_t[dD_t]\right)$, $d\tilde{B}_{\xi,t} = \frac{1}{\sigma_\xi(t)}\left(d\xi_t - \tilde{\mathbb{E}}_t[d\xi_t]\right)$ and $d\tilde{B}_{u,t} = \frac{1}{\sigma_u(t)}\left(ds_{u,t} - \tilde{\mathbb{E}}_t[ds_{u,t}]\right)$ are innovations in the observation processes relative to expectations. More specifically,

$$\begin{aligned} d\tilde{B}_{D,t} &= \frac{1}{\sigma_D}[dD_t - (\tilde{x}_t - D_t)\,dt] && (52)\\ d\tilde{B}_{\xi,t} &= \frac{1}{\sigma_\xi(t)}[d\xi_t - \tilde{\mu}_\xi(t)\,dt] && (53)\\ d\tilde{B}_{u,t} &= \frac{1}{\sigma_u(t)}[ds_{u,t} - \tilde{x}_t dt]. && (54) \end{aligned}$$

where $\tilde{\mu}_\xi(t) = b\bar{x}\phi_x(t) - a\bar{\theta}\phi_\theta(t) + m_x(t)\,\tilde{x}_t + \left(\frac{\phi_\theta'(t)}{\phi_\theta(t)} - a\right)\xi_t + m_D(t)\,D_t$.

Joint Distributions From the perspective of the informed, $x_t|s^t \sim N(\hat{x}_t, \hat{q}_t)$, and both $\hat{x}_t$ and θ_t are observable. Below, we derive the joint distribution of $[x_t, \hat{x}_t, \theta_t]$ from the perspective of the uninformed investors.

We deal with the interior and the boundary separately. In the interior, beliefs are continuous and there is no probability distortion over an infinitesimal interval. Obviously, under the belief of the uninformed, $\hat{x}_t|s^t \sim \mathcal{N}(\tilde{x}_t, \tilde{q}_t)$. By law of iterated expectation, $\tilde{\mathbb{E}}(x_t) = \tilde{x}_t$. In addition, $\tilde{V}ar[x_t] = \tilde{q}_t + \hat{q}_t$. That is, from the perspective of the uninformed, $x_t|s^t \sim \mathcal{N}(\tilde{x}_t, \tilde{q}_t + \hat{q}_t)$.

We are now ready to derive the conditional distribution for θ_t. Note that the learning identity implies $\zeta_t \equiv \phi_x(t)\hat{x}_t - \phi_\theta(t)\theta_t$, that is, $\theta_t = \frac{1}{\phi_{\theta,t}}(\phi_{x,t}\hat{x}_t - \zeta_t)$. Hence, $\tilde{\mathbb{E}}(\theta_t) = \frac{1}{\phi_{\theta,t}}(\phi_{x,t}\tilde{x}_t - \zeta_t)$. In addition, $\tilde{V}ar[\theta_t] = \left(\frac{\phi_{x,t}}{\phi_{\theta,t}}\right)^2 \tilde{q}_t$.

We can also compute covariance.

$$\tilde{C}ov(x_t, \hat{x}_t) = \tilde{C}ov(x_t, \hat{x}_t) = \tilde{q}_t.$$

In addition,

$$\tilde{Cov}(x_t, \theta_t) = \tilde{Cov}\left[x_t, \frac{1}{\phi_{\theta,t}}(\phi_{x,t}\hat{x}_t - \zeta_t)\right] = \frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_t, \tag{55}$$

and $\tilde{Cov}(\hat{x}_t, \theta_t) = \frac{\phi_{x,t}}{\phi_{\theta,t}}\tilde{q}_t$.

On the boundary, from the perspective of the uninformed,

$$\hat{x}_T | s^T \sim \mathcal{N}(\tilde{x}_T, \tilde{q}_T)\,;\; x_T | s^T \sim \mathcal{N}(\tilde{x}_T, \tilde{q}_T + \hat{q}_T)\,.$$

To compute the distribution of θ_T, we have $\tilde{\mathbb{E}}\left(\theta_T | \hat{x}_T^-\right) = \theta_T$ because once the uninformed know $\hat{x}_t$, they will know θ_t from the learning identity, and

$$\tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left(\theta_T | \hat{x}_T^-\right)\right] = \frac{1}{\phi_{\theta,T}}\tilde{\mathbb{E}}\left(\phi_{x,T}\hat{x}_T^- - \zeta_T\right).$$

This is to say, there is probability distortion in the first step, but no probability distortion after we conditioning on $\hat{x}$. In the first step, we have:

$$\tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left(\theta_T | \hat{x}_T^-\right)\right] = \frac{1}{\phi_{\theta,T}}\tilde{\mathbb{E}}\left(\phi_{x,T}\hat{x}_T^- - \zeta_T\right) = \frac{1}{\phi_{\theta,T}}(\phi_{x,T}\tilde{x}_T - \zeta_T) = \tilde{\theta}_T^-$$

The last equality is true, because in the interior, the equality $\zeta_T^- \equiv \phi_x(T^-)\,\tilde{x}_T^- - \phi_\theta(T^-)\,\tilde{\theta}_T^-$.

Difference in Beliefs Define the difference in beliefs $\Delta_t \equiv \hat{x}_t - \tilde{x}_t$. Because the informed do not have ambiguity, the law of motion of Δ_t under the informed investors information set is

$$d\Delta_t = -a_\Delta(t)\,\Delta_t dt - \sigma_{\Delta D}(t)\,d\hat{B}_{D,t} + \sigma_{\Delta s}(t)\,d\hat{B}_{s,t} + \sigma_{\Delta\theta}(t)\,dB_{\theta,t} - \sigma_{\Delta u}(t)\,dB_{u,t}, \tag{56}$$

where the coefficients are:

$$a_\Delta(t) = b + \frac{\hat{q}(t) + \tilde{q}(t)}{\sigma_D^2} + \nu(t)\,m_x(t) + \frac{\tilde{q}_t}{\sigma_u(t)}, \tag{57}$$

$$\sigma_{\Delta D}(t) = \frac{\tilde{q}(t)}{\sigma_D},$$

$$
\begin{aligned}
\sigma_{\Delta s}(t) &= \frac{\hat{q}(t)}{\sigma_s}\left[1-\phi_x(t)\,\nu(t)\right], \\
\sigma_{\Delta\theta}(t) &= \phi_\theta(t)\,\nu(t)\,\sigma_\theta. \\
\sigma_{\Delta u}(t) &= \frac{\tilde{q}(t)}{\sigma_u(t)}. \qquad (58)
\end{aligned}
$$

Note that compared to Han (2020), $d\Delta_t$ has a downward trend. The uninformed think that the informed are over optimistic, and therefore, $\tilde{x}_t$ typically increases faster than what $\hat{x}_t$ actually is.

Excess Returns In this subsection, we use the results from the filtering problem derived above to derive the excess return of the stock as diffusion processes under two types of investors' beliefs. We have conjectured that the equilibrium price is of the form (10). In order to solve for the optimal portfolio choice, we need to compute investors' beliefs about the return process. In the interior, this means we need to represent instantaneous excess return $dQ_t = dP_t + D_t dt - rP_t dt$ as functions of investors' own Brownian motions. On the boundary, we need to compute the conditional distribution of $P_T^+ - P_T^-$ from investors' own beliefs. Consider first the informed investors. Equations (48), (6), (8), and (18) represent the variables D_t, θ_t, $\hat{x}_t$, and Δ_t in terms of Brownian motions with respect to their information set. These give

$$
\begin{aligned}
dQ_t &= \left\{e_0(t) + \left[1-(1+r)\,\phi_D(t)\right]D_t + e_\theta(t)\,\theta_t + \left[\phi_D - (b+r)\,\phi_x\right]\hat{x}_t + e_\Delta(t)\,\Delta_t\right\}dt \\
&\quad + \varrho_D(t)\,d\hat{B}_{D,t} + \varrho_s(t)\,d\hat{B}_{s,t} + \varrho_\theta(t)\,dB_{\theta,t} + \varrho_u(t)\,dB_{u,t}, \qquad (59)
\end{aligned}
$$

where

$$
\begin{aligned}
e_0(t) &= \phi'(t) - r\phi(t) + b\bar{x}\bar{\phi}_x - a\bar{\theta}\phi_\theta(t) \\
e_\theta(t) &= (a+r)\,\phi_\theta(t) - \phi_\theta'(t), \\
e_\Delta(t) &= (a_\Delta(t)+r)\,\phi_\Delta(t) - \phi_\Delta'(t), \\
\varrho_D(t) &= \phi_D\sigma_D + \bar{\phi}_x\frac{\hat{q}(t)}{\sigma_D} + \phi_\Delta(t)\,\sigma_{\Delta D}(t), \qquad (60) \\
\varrho_s(t) &= \left[1+\phi_\Delta(t)\,\nu(t)\right]\phi_x(t)\,\frac{\hat{q}_t}{\sigma_s}, \\
\varrho_\theta(t) &= -\left[1+\phi_\Delta(t)\,\nu(t)\right]\phi_\theta(t)\,\sigma_\theta, \\
\varrho_u(t) &= \phi_\Delta(t)\,\frac{\tilde{q}_t}{\sigma_u(t)}.
\end{aligned}
$$

Note that before the information acquisition, all uninformed have homogeneous information, and price is of the form (21). Further define the variance of excess return as

$$
\sigma_P(t) = \varrho_D^2(t) + \varrho_s^2(t) + \varrho_\theta^2(t) + \varrho_u^2(t). \qquad (61)
$$

The market clearing condition implies that the expected return of the stock cannot depend on D_t, $\hat{x}_t$ and the constant. As a result, the coefficients them must be 0, implying

$$\phi_D = \frac{1}{1+r}, \text{ and } \bar{\phi}_x = \frac{\phi_D}{b+r}. \tag{62}$$

The excess return could be simplified as

$$\begin{aligned} dQ_t &= [e_0(t) + e_\theta(t)\theta_t + e_\Delta(t)\Delta_t]\,dt + \varrho_D(t)\,d\hat{B}_{D,t} + \varrho_s(t)\,d\hat{B}_{s,t} + \varrho_\theta(t)\,dB_{\theta,t} + \varrho_u(t)\,dB_{u,t}, \\ &= [e_0(t) + e_\theta(t)\theta_t + e_\Delta(t)\Delta_t]\,dt + \varrho_D(t)\,d\hat{B}_{D,t} + \varrho_\xi(t)\,d\hat{B}_{\xi,t} + \varrho_u(t)\,dB_{u,t}, \end{aligned}$$

where

$$\varrho_\xi(t) = -\frac{\sigma_\xi(t)}{\sigma_\theta\phi_\theta(t)}\varrho_\theta(t). \tag{63}$$

Similarly, we can use equations (23), and (12) to write the excess return in terms of Brownian motions with respect to the uninformed investor's information set. This gives

$$dQ_t = \left[e_0(t) + e_\theta(t)\tilde{\theta}_t\right]dt + \varrho_D(t)\,d\tilde{B}_{D,t} + \varrho_\xi(t)\,d\tilde{B}_{\xi,t} + \varrho_u(t)\,d\tilde{B}_{u,t}. \tag{64}$$

6.4 Generalized Risk Sensitivity Preference with Ambiguity

Recursive preference with GRS In this section, we develop the ambiguity preference that satisfies the generalized risk sensitivity (GRS). We first start from a general recursive preference. Then we take a continuous-time limit to form the HJB equation. Finally, we develop two-stage operators that introduce the robustness concerns about model uncertainty and hidden state uncertainty, respectively.

In general, a recursive preference can be specified in the following way for a small time interval Δ:

$$V_t = u(C_t)\Delta + e^{-\rho\Delta}h^{-1}\left(\mathbb{E}\left[h\left(V_{t+\Delta}\right)\right]\right). \tag{65}$$

Ai and Bansal (2018) shows that GRS requires h to be a non-decreasing concave function. In the continuous-time limit, we could write the the value function as the form of the stochastic differential utility:

$$dV_t = \mathcal{L}V_t dt + \sigma_V(t)\,dB_t, \tag{66}$$

where $\mathcal{L}V_t$ summarizes the drift and $\sigma_V(t)$ contains the vector of diffusions, and dB_t is a standard vector of Brownian motions. Note that in continuous-time, we can write $V_{t+dt} = V_t + dV_t$. Using Taylor's expansion around V_t, we have:

$$h(V_{t+dt}) = h(V_t) + h'(V_t)\left[\mathcal{L}V_t dt + \sigma_V(t)\,dB_t\right] + \frac{1}{2}h''(V_t)\left[\mathcal{L}V_t dt + \sigma_V(t)\,dB_t\right]^2 + o\,\|dt\|, \tag{67}$$

where $o\left\|dt\right\|$ indicates the higher order terms than dt. Taking expectations, we have

$$\mathbb{E}\left[h\left(V_{t+dt}\right)\right] = h\left(V_t\right) + h'\left(V_t\right)\mathcal{L}V_t dt + \frac{1}{2}h''\left(V_t\right)\left\|\sigma_V\left(t\right)\right\|^2 dt, \tag{68}$$

and now, we apply the inverse function h^{-1} on both sides and use Taylor's expansion around $h\left(V_t\right)$:

$$\begin{aligned} h^{-1}\left\{\mathbb{E}\left[h\left(V_{t+dt}\right)\right]\right\} &= h^{-1}\left(h\left(V_t\right)\right) + \left[h^{-1}\left(h\left(V_t\right)\right)\right]'\left[h'\left(V_t\right)\mathcal{L}V_t + \frac{1}{2}h''\left(V_t\right)\left\|\sigma_V\left(t\right)\right\|^2\right]dt + o\left\|dt\right\| \\ &= V_t + \left[\mathcal{L}V_t + \frac{1}{2}\frac{h''\left(V_t\right)}{h'\left(V_t\right)}\left\|\sigma_V\left(t\right)\right\|^2\right]dt + o\left\|dt\right\|. \end{aligned} \tag{69}$$

Then the recursion (65) could be approximated as:

$$e^{\rho dt}V_t = e^{\rho dt}u\left(C_t\right)dt + V_t + \left[\mathcal{L}V_t + \frac{1}{2}\frac{h''\left(V_t\right)}{h'\left(V_t\right)}\left\|\sigma_V\left(t\right)\right\|^2\right]dt. \tag{70}$$

Subtracting V_t from both sides, dividing by dt, and taking the limit as $dt \to 0$, we obtain the following lemma.

Lemma 2. *The continuous-time limit of the HJB (65) can be written as:*

$$\rho V_t = u\left(C_t\right) + \mathcal{L}V_t + \frac{1}{2}\frac{h''\left(V_t\right)}{h'\left(V_t\right)}\left\|\sigma_V\left(t\right)\right\|^2. \tag{71}$$

Now we specify the functional form of h. We consider the following concave function:

$$h\left(V_t\right) = -\frac{1}{1+\kappa}\left(-V_t\right)^{1+\kappa}. \tag{72}$$

where κ measures the degree of ambiguity aversion or preference for robustness. There are three key features about this specification.

First, $h'\left(V\right) = \left(-V\right)^{\kappa} > 0$, and $h''\left(V\right) = -\kappa\left(-V\right)^{\kappa-1} < 0$ indicate that h is a strictly increasing and strictly concave function. This concavity ensures that our recursive preference in (65) satisfies GRS by Ai and Bansal (2018), which is a sufficient condition to generate the announcement premium.

Second, the functional form of h preserves the homogeneity property of preferences. In particular, it maintains the close-form solution for the value function as an exponential function. Our derivation is equivalent to Luo, Nie, and Wang (2022) who use recursive exponential utility to keep the tractability for CARA utility and adopt the two-stage procedures of optimal robust control and robust filtering problems as in our paper.

Third, our specification coincides with the normalized version of HJB advocated by Maenhout (2004). In order to keep the homogeneity in HJB equation with robustness, he scales the ambiguity κ by the value function: $\hat{\kappa} = \kappa/V_t$. Here, we are essentially giving a recursive utility interpretation of the normalization without the scaling.

If we plug (72) in (71), we will get

$$\rho V_t = u(C_t) + \mathcal{L}V_t + \frac{\kappa}{V_t}\|\sigma_V(t)\|^2. \tag{73}$$

Two-stage robustness Now we provide a generalization of the T^1 and T^2 operator in Hansen and Sargent (2007, 2011) to our recursive setup. Assume the law of motion of x follows a Markov diffusion process of the form

$$dx_t = \mu_x(x_t)\,dt + \sigma_x(t)\,dB_{x,t}.$$

We assume that a subset of x, which we denote as z are unobservable. We assume that z_t is normally distributed and we observe a vector of signals for z_t:

$$ds_t = \mu_s(z_t)\,dt + \sigma_s(t)\,dB_{s,t}. \tag{74}$$

Here, we assume that s is L dimensional signal process, and $B_{s,t}$ are L dimensional Brownian Motions. That is, for simplicity, we are assuming that the noise in different signal processes are independent. Also, we assume $\mu_s(z_t) = \mu_{s,0} + \mu_s z_t$ is linear, where $\mu_{s,0}$ and μ_s are both $L\times 1$ dimensional. For simplicity, we assume that $\sigma_s(t) = diag(\sigma_1(t), \sigma_2(t), \cdots, \sigma_L(t))$ is diagonal. We assume that Kalman filter can be applied. Because the hidden state z_t is contained in x_t, the new state variable needs to replace z_t with its estimate (assuming normality, so that $\hat{z}_t$ is the sufficient statistic for the posterior distribution of z_t). We denote the new state variable (after replacing the hidden state with its point estimate) as $\hat{x}_t$. After this replacement, $\hat{z}_t$ is contained in $\hat{x}_t$. We assume that the law of motion of the state variables $\hat{x}_t$ can be written as:

$$d\hat{x}_t = \mu_x(\hat{x}_t)\,dt + \sigma_x(t)\,d\hat{B}_{x,t}, \tag{75}$$

where $\hat{B}_{x,t}$ is a $L\times 1$ dimensional vector of innovation process of the form:

$$d\hat{B}_{x,t} = \sigma_s^{-1}(t)\left[ds_t - \mu_s(z_t)\,dt\right] = \sigma_s^{-1}(t)\left[\mu_s(t)(z_t - \hat{z}_t)\,dt + \sigma_s(t)\,dB_s(t)\right]. \tag{76}$$

In our formulation, $\mu_x(\hat{x}_t, \hat{z}_t)$ is $J\times 1$, $\sigma_x(t)$ is $J\times L$, $\mu_s(t)$ is $L\times 1$, and $\sigma_s(t)$ is $L\times L$.

We will work with the simple case in which the value function does not depend on the hidden state, in the Language of Hansen and Sargent (2011). In this formulation, the value function is a function of $\hat{x}_t$ and the law of motion of utility (in the interior) is:

$$\begin{aligned} dV(\hat{x}_t) &= \mathcal{L}V(\hat{x}_t)\,dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,d\hat{B}_{x,t} \\ &= \mathcal{L}V(\hat{x}_t)\,dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\left[\underbrace{\mu_s(t)(z_t-\hat{z}_t)\,dt}_{\text{misspecified state}} + \underbrace{\sigma_s(t)\,dB_s(t)}_{\text{misspecified dynamics}}\right], \end{aligned} \tag{77}$$

The standard robust control problem concerns the misspecification of the future shocks $\hat{B}_{x,t}$. The two-stage robust control problem concern about misspecification of $\mu_s(t)(z_t - \hat{z}_t)\,dt$ and $dB_s(t)$

separately. The distortion with respect to $dB_s(t)$ represents ambiguity about the law of motion of signals given the true state variable (T^1 operator), and distortion with respect to $\mu_s(t)(z_t - \hat{z}_t)\,dt$ represents ambiguity about the true distribution of the hidden state z_t arising from filtering (T^2 operator).

Consider the recursive preference with two-stage robustness:

$$V(\hat{x}_t) = u(C_t)\,dt + e^{-\rho dt} T^2 \left\{ T^1 \left[V(\hat{x}_{t+dt}) \middle| x_t \right] \middle| \hat{x}_t \right\} \tag{78}$$

where $T^1(V) = h_1^{-1}\mathbb{E}[h_1(V)]$, and $T^2(V) = h_2^{-1}\mathbb{E}[h_2(V)]$, where $h_1 = -\frac{1}{1+\kappa_1}(-V)^{1+\kappa_1}$ and $h_2 = -\frac{1}{1+\kappa_2/dt}(-V)^{1+\kappa_2/dt}$. T^1 and T^2 operators reflect investors' concerns about misspecified dynamics given the true state and misspecified state estimation, respectively. Note that we follow Hansen and Sargent (2011) and scale the ambiguity aversion by $1/dt$ to prevent the ambiguity from vanishing in the continuous time limit. Using (77) we can write

$$\begin{aligned}
V_{t+dt} &= V_t + \mathcal{L}V(\hat{x}_t)\,dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,d\hat{B}_{x,t} && (79)\\
&= V_t + \mathcal{L}V(\hat{x}_t)\,dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\left[\mu_s(t)(z_t - \hat{z}_t)\,dt + \sigma_s(t)\,dB_s(t)\right] \\
&= V_t + \left[\mathcal{L}V(\hat{x}_t) + \partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)(z_t - \hat{z}_t)\right]dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,dB_s(t). && (80)
\end{aligned}$$

Using Taylor expansion similar to (69), we obtain

$$\begin{aligned}
T^1\left[V_{t+dt}\middle| z_t\right] &= h_1^{-1}\mathbb{E}\left[h_1(V_{t+dt})\middle| z_t\right] \\
&= V_t + \left[\mathcal{L}V(\hat{x}_t) + \partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)(z_t - \hat{z}_t)\right]dt + \frac{1}{2}\frac{h_1''(V_t)}{h_1'(V_t)}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\right\|^2 dt.
\end{aligned}$$

To save notation, we denote $\eta_t = \mathcal{L}V(\hat{x}_t) + \frac{1}{2}\frac{h_1''(V_t)}{h_1'(V_t)}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\right\|^2$. Given our assumption on the functional form of h_1, we have $\eta_t = \mathcal{L}V(\hat{x}_t) - \frac{1}{2}\frac{\kappa_1}{|V_t|}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\right\|^2$.

Note that η_t is known given $\hat{x}_t$. We linearize around $V_t + \eta_t dt$ and using Taylor expansion to write

$$\begin{aligned}
T^2\left\{T^1\left[V_{t+dt}\middle| z_t\right]\right\} &= h_2^{-1}\mathbb{E}\left\{h_2\left(V_t + \eta_t dt + \partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)(z_t - \hat{z}_t)\,dt\right)\right\} \\
&= V_t + \eta_t dt + \frac{1}{2}\frac{h_2''(V_t)}{h_2'(V_t)}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)\right\|^2 q_t\,(dt)^2. && (81)
\end{aligned}$$

where q_t denotes the posterior variance of z_t. Note the the ambiguity term is of order $(dt)^2$. Therefore, as remarked by Hansen and Sargent (2011), in order to sustain an ambiguity adjustment in continuous time, we increase the curvature of h_2 as we have diminished the sampling interval by $1/dt$. Using the assumed functional form $h_2 = -\frac{1}{1+\kappa_2/dt}(-V)^{1+\kappa_2/dt}$, we have $\frac{1}{2}\frac{h_2''(V_t)}{h_2'(V_t)}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)\right\|^2 q_t\,(dt)^2 = -\frac{1}{2}\frac{\kappa_2}{|V_t|}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)\right\|^2 q_t dt$. We denote $T^1V_t = -\frac{1}{2}\frac{\kappa_1}{|V_t|}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\right\|^2$, and $T^2V_t = -\frac{1}{2}\frac{\kappa_2}{|V_t|}\left\|\partial V(\hat{x}_t)\,\sigma_x(t)\,\sigma_s^{-1}(t)\,\mu_s(t)\right\|^2 q_t$. Multiply equation (78) by $e^{\rho dt}$ and substituting the above equation and η_t back, we have

$$\left(e^{\rho dt}-1\right)V_t=\left[e^{\rho dt}u\left(C_t\right)+\mathcal{L}V\left(\hat{x}_t\right)+T^1V_t+T^2V_t\right]dt.$$

Dividing both sides by dt and take the limit as $dt\to 0$, we obtain the following HJB equation:

$$\rho V_t=u\left(C_t\right)+\mathcal{L}V_t+T^1V_t+T^2V_t, \tag{82}$$

which is the generalization of Equations (8) and (9) in Hansen and Sargent (2011). It is clear that T^1V_t characterizes the continuation reduction due to model uncertainty and T^2V_t shows the continuation reduction due to state uncertainty, and κ_1 and κ_2 penalizes each term.

Note that we can write

$$T^1V_t=-\frac{1}{2}\frac{\kappa_1}{|V_t|}\frac{1}{dt}Var_t\left[\partial V\left(\hat{x}_t\right)\sigma_x\left(t\right)d\hat{B}_s\left(t\right)\Big|\,z_t\right]=-\frac{1}{2}\frac{\kappa_1}{|V_t|\,dt}Var_t\left[V_{t+dt}|\,z_t\right] \tag{83}$$

and

$$\begin{aligned} T^2V_t &= -\frac{1}{2}\frac{\kappa_2/dt}{|V_t|}\frac{1}{dt}Var_t\left\{\partial V\left(\hat{x}_t\right)\sigma_x\left(t\right)\sigma_s^{-1}\left(t\right)\mu_s\left(t\right)\left(z_t-\hat{z}_t\right)dt\right\} \\ &= -\frac{1}{2}\frac{\kappa_2/dt}{|V_t|}\frac{1}{dt}Var_t\left\{\mathbb{E}\left[\partial V\left(\hat{x}_t\right)\sigma_x\left(t\right)d\hat{B}_s\left(t\right)\Big|\,z_t\right]\right\}=-\frac{1}{2}\frac{\kappa_2}{|V_t|\left(dt\right)^2}Var_t\left\{\mathbb{E}\left[V_{t+dt}|\,z_t\right]\right\} \end{aligned} \tag{84}$$

In our model, we assume the uninformed investors do not have robust concerns about model uncertainty. They care about the misspecification in the distributions of the hidden state. Therefore, from now on, we assume $\kappa_1=0$ and $\kappa_2=\kappa$.

6.5 Optimal Portfolio Choice Decisions

Portfolio Demand for the Informed: Interior The optimization problem for the informed investor in the interior is written as

$$\begin{aligned} \hat{V}\left(t,\hat{W},\theta,\Delta\right) &= \max_{\alpha_t,\hat{C}_t}\hat{\mathbb{E}}\left[\int_0^{T-t}-e^{-\rho s-\gamma\hat{C}_{t+s}}ds+e^{-\rho(T-t)}V^-\left(T,\hat{W}_T,\theta_T,\Delta_T\right)\right] \\ s.t.\ d\hat{W}_t &= \left(\hat{W}_tr-\hat{C}_t\right)dt+\alpha_tdQ_t \\ dQ_t &= \left[e_0\left(t\right)+e_\theta\left(t\right)\theta_t+e_\Delta\left(t\right)\Delta_t\right]dt+\varrho_D\left(t\right)d\hat{B}_{D,t}+\varrho_s\left(t\right)d\hat{B}_{s,t}+\varrho_\theta\left(t\right)dB_{\theta,t}+\varrho_u\left(t\right)dB_{u,t}, \\ d\theta_t &= a\left(\bar{\theta}-\theta_t\right)dt+\sigma_\theta dB_{\theta,t}, \\ d\Delta_t &= -a_\Delta\left(t\right)\Delta_tdt-\sigma_{\Delta D}\left(t\right)d\hat{B}_{D,t}+\sigma_{\Delta s}\left(t\right)d\hat{B}_{s,t}+\sigma_{\Delta\theta}\left(t\right)dB_{\theta,t}-\sigma_{\Delta u}\left(t\right)dB_{u,t}. \end{aligned}$$

Conjecture the informed investor's value function takes the form of $\hat{V}\left(t,\hat{W},\theta,\Delta\right)=-e^{-r\gamma\hat{W}-g(t,\theta,\Delta)}$, where

$$g\left(t,\theta,\Delta\right)=g\left(t\right)+g_\theta\left(t\right)\theta_t+\frac{1}{2}g_{\theta\theta}\left(t\right)\theta_t^2+g_\Delta\left(t\right)\Delta_t+\frac{1}{2}g_{\Delta\Delta}\left(t\right)\Delta_t^2+g_{\theta\Delta}\left(t\right)\theta_t\Delta_t. \tag{85}$$

Using Ito's Lemma, the HJB equation is:

$$\begin{aligned}\rho\hat{V} &= -e^{-\gamma\hat{C}}+\hat{V}_t+\hat{V}_W\left[r\hat{W}-\hat{C}+\alpha\left(e_0(t)+e_\theta(t)\theta+e_\Delta(t)\Delta\right)\right]+\frac{1}{2}\hat{V}_{WW}\alpha^2\sigma_P(t)+\alpha\hat{V}_{W\theta}\sigma_\theta\varrho_\theta(t)\\ &\quad+\alpha\hat{V}_{W\Delta}\sigma_{Q\Delta}(t)+\hat{V}_\theta a\left(\bar{\theta}-\theta\right)-\hat{V}_\Delta\left(a_\Delta(t)\Delta+b_\Delta(t)\right)+\frac{1}{2}\hat{V}_{\theta\theta}\sigma_\theta^2+\frac{1}{2}\hat{V}_{\Delta\Delta}\sigma_\Delta(t)+\hat{V}_{\Delta\theta}\sigma_\theta\sigma_{\Delta\theta}(t),\end{aligned}$$

where

$$\begin{aligned}\sigma_\Delta(t) &= \sigma_{\Delta D}^2(t)+\sigma_{\Delta s}^2(t)+\sigma_{\Delta\theta}^2(t)+\sigma_{\Delta u}^2(t)\\ \sigma_{Q\Delta}(t) &= -\varrho_D(t)\sigma_{\Delta D}(t)+\varrho_s(t)\sigma_{\Delta s}(t)+\varrho_\theta(t)\sigma_{\Delta\theta}(t)-\varrho_u(t)\sigma_{\Delta u}(t),\end{aligned} \tag{86}$$

Under the guessed value function form, the first order conditions (FOCs) with respect to $\hat{C}$ and α are

$$\hat{C}_t = r\hat{W}+\frac{1}{\gamma}\left[g(t,\theta,\Delta)-\ln r\right], \tag{87}$$

$$\alpha_t = \frac{\left[\begin{array}{c} e_0(t)+e_\theta(t)\theta+e_\Delta(t)\Delta-\left(g_\theta(t)+g_{\theta\theta}(t)\theta_t+g_{\theta\Delta}(t)\Delta_t\right)\sigma_\theta\varrho_\theta(t)\\ -\left(g_\Delta(t)+g_{\Delta\Delta}(t)\Delta_t+g_{\theta\Delta}(t)\theta_t\right)\sigma_{Q\Delta}(t)\end{array}\right]}{r\gamma\sigma_P(t)} \tag{88}$$

substituting expressions in (86) yields the demand function of the form:

$$\alpha_t = \alpha_0(t)+\alpha_\theta(t)\theta_t+\alpha_\Delta(t)\Delta_t, \tag{89}$$

where

$$\alpha_0(t) = \frac{e_0(t)-g_\theta(t)\sigma_\theta\varrho_\theta(t)-\sigma_{Q\Delta}(t)g_\Delta(t)}{r\gamma\sigma_P(t)} \tag{90}$$

$$\alpha_\theta(t) = \frac{e_\theta(t)-\varrho_\theta(t)\sigma_\theta g_{\theta\theta}(t)-\sigma_{Q\Delta}(t)g_{\theta\Delta}(t)}{r\gamma\sigma_P(t)} \tag{91}$$

$$\alpha_\Delta(t) = \frac{e_\Delta(t)-\varrho_\theta(t)\sigma_\theta g_{\theta\Delta}(t)-\sigma_{Q\Delta}(t)g_{\Delta\Delta}(t)}{r\gamma\sigma_P(t)}. \tag{92}$$

Matching coefficients of the value function, and use $\alpha_0(t)$, $\alpha_\theta(t)$ and $\alpha_\Delta(t)$ to simplify, we have the following odes system,

$$\begin{aligned}
g'(t) &= r-\rho-r\ln r+rg(t)-\frac{1}{2}r^2\gamma^2\sigma_P(t)\alpha_0^2(t)+\frac{1}{2}\sigma_\theta^2\left[g_\theta^2(t)-g_{\theta\theta}(t)\right] \\
&\quad +\frac{1}{2}\sigma_\Delta(t)\left[g_\Delta^2(t)-g_{\Delta\Delta}(t)\right]+\sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_\Delta(t)-g_{\theta\Delta}(t)\right]-a\bar{\theta}g_\theta(t), && (93)\\
g'_{\theta\theta}(t) &= rg_{\theta\theta}(t)-r^2\gamma^2\sigma_P(t)\alpha_\theta^2(t)+2ag_{\theta\theta}(t)+\sigma_\theta^2g_{\theta\theta}^2(t)+\sigma_\Delta(t)g_{\theta\Delta}^2(t)+2\sigma_\theta\sigma_{\Delta\theta}(t)g_{\theta\theta}(t)g_{\theta\Delta}(t), && (94)\\
g'_{\Delta\Delta}(t) &= rg_{\Delta\Delta}(t)-r^2\gamma^2\sigma_P(t)\alpha_\Delta^2(t)+2a_\Delta(t)g_{\Delta\Delta}(t)+\sigma_\theta^2g_{\theta\Delta}^2(t)+\sigma_\Delta(t)g_{\Delta\Delta}^2(t) \\
&\quad +2\sigma_\theta\sigma_{\Delta\theta}(t)g_{\theta\Delta}(t)g_{\Delta\Delta}(t), && (95)\\
g'_{\theta\Delta}(t) &= rg_{\theta\Delta}(t)-r^2\gamma^2\sigma_P(t)\alpha_\theta(t)\alpha_\Delta(t)+ag_{\theta\Delta}(t)+a_\Delta(t)g_{\theta\Delta}(t)+\sigma_\theta^2g_{\theta\theta}(t)g_{\theta\Delta}(t) \\
&\quad +\sigma_\Delta(t)g_{\Delta\Delta}(t)g_{\theta\Delta}(t)+\sigma_\theta\sigma_{\Delta\theta}(t)\left[g_{\theta\theta}(t)g_{\Delta\Delta}(t)+g_{\theta\Delta}^2(t)\right]; && (96)\\
g'_\theta(t) &= rg_\theta(t)-r^2\gamma^2\sigma_P(t)\alpha_0(t)\alpha_\theta(t)+ag_\theta(t)+\sigma_\theta^2g_\theta(t)g_{\theta\theta}(t) \\
&\quad +\sigma_\Delta(t)g_\Delta(t)g_{\theta\Delta}(t)+\sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_{\theta\Delta}(t)+g_{\theta\theta}(t)g_\Delta(t)\right]-a\bar{\theta}g_{\theta\theta}, && (97)\\
g'_\Delta(t) &= rg_\Delta(t)-r^2\gamma^2\sigma_P(t)\alpha_0(t)\alpha_\Delta(t)+a_\Delta(t)g_\Delta(t)+\sigma_\theta^2g_\theta(t)g_{\theta\Delta}(t) \\
&\quad +\sigma_\Delta(t)g_\Delta(t)g_{\Delta\Delta}(t)+\sigma_\theta\sigma_{\Delta\theta}(t)\left[g_\theta(t)g_{\Delta\Delta}(t)+g_{\theta\Delta}(t)g_\Delta(t)\right]-a\bar{\theta}g_{\theta\Delta}, && (98)
\end{aligned}$$

Portfolio Demand for the Uninformed: Interior The optimization problem of the uninformed investors is characterized as:

$$\begin{aligned}
\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right) &= \max_{\beta_t,\tilde{C}_t}\tilde{\mathbb{E}}\left[\int_0^{T-t}-e^{-\rho s-\gamma\tilde{C}_{t+s}}ds+e^{-\rho(T-t)}\tilde{V}^-\left(T,\tilde{W}_T,\tilde{\theta}_T\right)\right] \\
s.t.\ d\tilde{W}_t &= \left(\tilde{W}_tr-\tilde{C}_t\right)dt+\beta_tdQ_t \\
dQ_t &= \left[e_0(t)+e_\theta(t)\tilde{\theta}_t\right]dt+\varrho_D(t)d\tilde{B}_{D,t}+\varrho_\xi(t)d\tilde{B}_{\xi,t}+\varrho_u(t)d\tilde{B}_{u,t}, \\
d\tilde{\theta}_t &= a\left(\bar{\theta}-\tilde{\theta}_t\right)dt+\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}d\tilde{B}_{D,t}+\left[\phi_x(t)\nu(t)-1\right]\frac{\sigma_\xi(t)}{\phi_\theta(t)}d\tilde{B}_{\xi,t}+\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}d\tilde{B}_{u,t}.
\end{aligned}$$

Conjecture the uninformed investor's value function would be of the form: $\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)=-e^{-r\gamma\tilde{W}-f\left(t,\tilde{\theta}\right)}$, where

$$f\left(t,\tilde{\theta}\right)=f(t)+f_\theta(t)\tilde{\theta}_t+\frac{1}{2}f_{\theta\theta}(t)\tilde{\theta}_t^2. \tag{99}$$

The stochastic differential equation of the value function in equation (13) takes the following form:

$$d\tilde{V}=\mathcal{L}\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)+\mathcal{D}\left[d\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)\right] \tag{100}$$

where

$$\begin{aligned}
\mathcal{L}\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right) &= \tilde{V}_t+\tilde{V}_W\left[r\tilde{W}-\tilde{C}+\beta\left(e_0(t)+e_\theta(t)\tilde{\theta}\right)\right] \\
&\quad +\frac{1}{2}\tilde{V}_{WW}\beta^2\sigma_P(t)+\beta\tilde{V}_{W\theta}\sigma_{Q\theta}(t)+\tilde{V}_\theta a\left(\bar{\theta}-\tilde{\theta}\right)+\frac{1}{2}\tilde{V}_{\theta\theta}\sigma_{\theta\theta}(t)
\end{aligned}$$

where

$$\sigma_{Q\theta}(t) = \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}\varrho_D(t) + (\phi_x(t)\nu(t)-1)\frac{\sigma_\xi(t)}{\phi_\theta(t)}\varrho_\xi(t) + \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}\varrho_u(t) \tag{101}$$

$$\sigma_{\theta\theta}(t) = \left[\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}\right]^2 + \left[(\phi_x(t)\nu(t)-1)\frac{\sigma_\xi(t)}{\phi_\theta(t)}\right]^2 + \left[\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}\right]^2. \tag{102}$$

$\mathcal{D}[\cdot]$ is the diffusion operator and

$$\begin{aligned}
\mathcal{D}\left[d\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)\right] &= \tilde{V}_W\beta\left[\varrho_D(t)\,d\tilde{B}_{D,t} + \varrho_\xi(t)\,d\tilde{B}_{\xi,t} + \varrho_u(t)\,d\tilde{B}_{u,t}\right] \\
&\quad + \tilde{V}_\theta\left[\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}d\tilde{B}_{D,t} + [\phi_x(t)\nu(t)-1]\frac{\sigma_\xi(t)}{\phi_\theta(t)}d\tilde{B}_{\xi,t} + \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}d\tilde{B}_{u,t}\right] \\
&= -\tilde{V}\left\{\left[r\gamma\beta\varrho_D(t) + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}\right]d\tilde{B}_{D,t} + \left[r\gamma\beta\varrho_\xi(t) + \frac{\partial f}{\partial\tilde{\theta}}[\phi_x(t)\nu(t)-1]\frac{\sigma_\xi(t)}{\phi_\theta(t)}\right]d\tilde{B}_{\xi,t}\right. \\
&\quad \left. + \left[r\gamma\beta\varrho_u(t) + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}\right]d\tilde{B}_{u,t}\right\}
\end{aligned}$$

Therefore, the vector $\sigma_V(t)$ is defined as

$$\sigma_V(t) = \left[-\tilde{V}\left(r\gamma\beta\varrho_{D,t} + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_{x,t}}{\phi_{\theta,t}}\frac{\tilde{q}_t}{\sigma_D}\right), -\tilde{V}\left(r\gamma\beta\varrho_{\xi,t} + \frac{\partial f}{\partial\tilde{\theta}}(\phi_{x,t}\nu_t-1)\frac{\sigma_{\xi,t}}{\phi_{\theta,t}}\right), -\tilde{V}\left(r\gamma\beta\varrho_{u,t} + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_{x,t}}{\phi_{\theta,t}}\frac{\tilde{q}_t}{\sigma_{u,t}}\right)\right]^\top. \tag{103}$$

Rewrite the above in terms of the informed investors' information set,

$$\begin{aligned}
\mathcal{D}\left[d\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)\right] &= -\tilde{V}\left\{\left[r\gamma\beta\varrho_D(t) + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}\right]\left(\frac{1}{\sigma_D}\Delta_t dt + d\hat{B}_{D,t}\right)\right. \\
&\quad + \left[r\gamma\beta\varrho_\xi(t) + \frac{\partial f}{\partial\tilde{\theta}}[\phi_x(t)\nu(t)-1]\frac{\sigma_\xi(t)}{\phi_\theta(t)}\right]\left(\frac{m_x(t)}{\sigma_\xi(t)}\Delta_t dt + d\hat{B}_{\xi,t}\right) \\
&\quad \left. + \left[r\gamma\beta\varrho_u(t) + \frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}\right]\left(\frac{1}{\sigma_u(t)}\Delta_t dt + dB_{u,t}\right)\right\}
\end{aligned}$$

Therefore, we apply the $\mathcal{T}$ operator in equation (16) on the value function to obtain

$$\begin{aligned}
\mathcal{T}\left[\tilde{V}\right] &= -\frac{1}{2}\kappa\frac{1}{\left|\tilde{V}\right|}Var\left[\mathbb{E}\left[d\tilde{V}\mid\hat{x}\right]\mid\tilde{x}\right]\\
&= \frac{1}{2}\kappa\tilde{V}\tilde{q}(t)\left\{\left[r\gamma\beta\varrho_D(t)+\frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D}\right]\frac{1}{\sigma_D}\right.\\
&\quad\left.+\left[r\gamma\beta\varrho_\xi(t)+\frac{\partial f}{\partial\tilde{\theta}}\left[\phi_x(t)\nu(t)-1\right]\frac{\sigma_\xi(t)}{\phi_\theta(t)}\right]\frac{m_x(t)}{\sigma_\xi(t)}+\left[r\gamma\beta\varrho_u(t)+\frac{\partial f}{\partial\tilde{\theta}}\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u(t)}\right]\frac{1}{\sigma_u(t)}\right\}^2\\
&= \frac{1}{2}\kappa\tilde{V}\tilde{q}(t)\left\{r\gamma\beta\left[\frac{\varrho_D(t)}{\sigma_D}+\varrho_\xi(t)\frac{m_x(t)}{\sigma_\xi(t)}+\frac{\varrho_u(t)}{\sigma_u(t)}\right]\right.\\
&\quad\left.+\frac{\partial f}{\partial\tilde{\theta}}\left[\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D^2}+\left[\phi_x(t)\nu(t)-1\right]\frac{m_x(t)}{\phi_\theta(t)}+\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u^2(t)}\right]\right\}^2\\
&= \frac{1}{2}\kappa\tilde{V}\tilde{q}(t)\left[(r\gamma\chi_D)^2\beta^2+\left(\frac{\partial f}{\partial\tilde{\theta}}\right)^2\chi_\theta^2+2r\gamma\beta\chi_D\frac{\partial f}{\partial\tilde{\theta}}\chi_\theta\right]
\end{aligned}$$

where

$$\chi_D = \frac{\varrho_D(t)}{\sigma_D}+\varrho_\xi(t)\frac{m_x(t)}{\sigma_\xi(t)}+\frac{\varrho_u(t)}{\sigma_u(t)} \tag{104}$$

$$\chi_\theta = \frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_D^2}+\left[\phi_x(t)\nu(t)-1\right]\frac{m_x(t)}{\phi_\theta(t)}+\frac{\phi_x(t)}{\phi_\theta(t)}\frac{\tilde{q}(t)}{\sigma_u^2(t)}. \tag{105}$$

The HJB of the above problem is written as:

$$\rho\tilde{V} = -e^{-\gamma\tilde{C}}+\mathcal{L}\tilde{V}\left(t,\tilde{W},\tilde{\theta}\right)+\mathcal{T}\left[\tilde{V}\right] \tag{106}$$

The FOCs wrt $\tilde{C}_t$ and β_t are

$$\begin{aligned}
0 &= \gamma e^{-\gamma\tilde{C}_t}-\tilde{V}_W\\
0 &= \tilde{V}_W\left(e_0(t)+e_\theta(t)\tilde{\theta}\right)+\tilde{V}_{WW}\beta\sigma_P(t)+\tilde{V}_{W\theta}\sigma_{Q\theta}(t)+\kappa\tilde{V}\tilde{q}(t)\left[(r\gamma\chi_D)^2\beta+r\gamma\frac{\partial f}{\partial\tilde{\theta}}\chi_D\chi_\theta\right]
\end{aligned}$$

which gives

$$\beta = -\frac{\tilde{V}_W\left(e_0(t)+e_\theta(t)\tilde{\theta}\right)+\tilde{V}_{W\theta}\sigma_{Q\theta}(t)+\kappa\tilde{V}\tilde{q}(t)r\gamma\frac{\partial f}{\partial\tilde{\theta}}\chi_D\chi_\theta}{\tilde{V}_{WW}\sigma_P(t)+\kappa\tilde{V}\tilde{q}(t)(r\gamma\chi_D)^2}.$$

Under the guessed value function form, $\tilde{V}_t = -\frac{\partial f}{\partial t}\tilde{V}$, $\tilde{V}_W = -r\gamma\tilde{V}$, $\tilde{V}_{WW} = (r\gamma)^2\tilde{V}$, $\tilde{V}_\theta = -\frac{\partial f}{\partial\theta}\tilde{V}$,

$\tilde{V}_{\theta\theta} = \left[\left(\frac{\partial f}{\partial \theta}\right)^2 - \frac{\partial^2 f}{\partial \theta^2}\right]\tilde{V}$, $\tilde{V}_{W\theta} = r\gamma\frac{\partial f}{\partial \theta}\tilde{V}$. The FOCs are therefore

$$\tilde{C} = r\tilde{W} + \frac{1}{\gamma}\left[f\left(t, \tilde{\theta}_t\right) - \ln r\right], \tag{107}$$

$$\beta = \frac{e_0(t) + e_\theta(t)\tilde{\theta} - \left(f_\theta(t) + f_{\theta\theta}(t)\tilde{\theta}\right)\left[\sigma_{Q\theta}(t) + \kappa\chi_D\chi_\theta\tilde{q}(t)\right]}{r\gamma\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]}. \tag{108}$$

We then obtain the demand function of the form

$$\beta_t = \beta_0(t) + \beta_\theta(t)\tilde{\theta}_t. \tag{109}$$

where

$$\beta_0(t) = \frac{e_0(t) - f_\theta(t)\left[\sigma_{Q\theta}(t) + \kappa\chi_D\chi_\theta\tilde{q}(t)\right]}{r\gamma\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]} \tag{110}$$

$$\beta_\theta(t) = \frac{e_\theta(t) - f_{\theta\theta}(t)\left[\sigma_{Q\theta}(t) + \kappa\chi_D\chi_\theta\tilde{q}(t)\right]}{r\gamma\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]}. \tag{111}$$

Substituting this into HJB gives

$$\begin{aligned} 0 &= r - \rho - \frac{\partial f}{\partial t} + rf - r\ln r - r\gamma\beta\left(e_0(t) + e_\theta(t)\tilde{\theta}\right) + \frac{1}{2}r^2\gamma^2\beta^2\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right] \\ &\quad + \beta r\gamma\frac{\partial f}{\partial \tilde{\theta}}\left[\sigma_{Q\theta}(t) + \kappa\chi_D\chi_\theta\tilde{q}(t)\right] - \frac{\partial f}{\partial \tilde{\theta}}a\left(\bar{\theta} - \tilde{\theta}\right) + \frac{1}{2}\left[\left(\frac{\partial f}{\partial \tilde{\theta}}\right)^2 - \frac{\partial^2 f}{\partial \tilde{\theta}^2}\right]\sigma_{\theta\theta}(t) + \frac{1}{2}\kappa\tilde{q}(t)\left(\frac{\partial f}{\partial \tilde{\theta}}\right)^2\chi_\theta^2. \end{aligned}$$

Matching coefficients of the value function, and use $\beta_0(t)$ and $\beta_\theta(t)$ to simplify, we have

$$\begin{aligned} 0 &= r - \rho - \frac{\partial f}{\partial t} + rf - r\ln r - \frac{1}{2}r^2\gamma^2\beta^2\left[\sigma_P(t) + \phi_2\chi_D^2\tilde{q}(t)\right] \\ &\quad - \frac{\partial f}{\partial \tilde{\theta}}a\left(\bar{\theta} - \tilde{\theta}\right) + \frac{1}{2}\left[\left(\frac{\partial f}{\partial \tilde{\theta}}\right)^2 - \frac{\partial^2 f}{\partial \tilde{\theta}^2}\right]\sigma_{\theta\theta}(t) + \frac{1}{2}\kappa\tilde{q}(t)\left(\frac{\partial f}{\partial \tilde{\theta}}\right)^2\chi_\theta^2. \end{aligned}$$

Finally we obtain the ODEs for the uninformed investors' value function coefficients as follows:

$$\begin{aligned} f'(t) &= r - \rho - r\ln r + rf(t) - \frac{1}{2}(r\gamma)^2\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]\beta_0^2(t) + \frac{1}{2}\sigma_{\theta\theta}\left[f_\theta^2(t) - f_{\theta\theta}(t)\right] \\ &\quad - a\bar{\theta}f_\theta(t) + \frac{1}{2}\kappa\tilde{q}(t)\chi_\theta^2 f_\theta^2(t) \end{aligned} \tag{112}$$

$$f'_{\theta\theta}(t) = (2a + r)f_{\theta\theta}(t) - (r\gamma)^2\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]\beta_\theta^2(t) + \left(\sigma_{\theta\theta} + \kappa\tilde{q}(t)\chi_\theta^2\right)f_{\theta\theta}^2(t) \tag{113}$$

$$f'_\theta(t) = (a + r)f_\theta(t) - (r\gamma)^2\left[\sigma_P(t) + \kappa\chi_D^2\tilde{q}(t)\right]\beta_0(t)\beta_\theta(t) + \left(\sigma_{\theta\theta} + \kappa\tilde{q}(t)\chi_\theta^2\right)f_\theta(t)f_{\theta\theta}(t) - a\bar{\theta}f_{\theta\theta}(t) \tag{114}$$

Market Clearing Conditions Market clearing condition is

$$(1 - \omega)\alpha_t + \omega\beta_t = \theta_t, \tag{115}$$

Using the relationship $\tilde{\theta}_t = \theta_t - \frac{\phi_x(t)}{\phi_\theta(t)}\Delta_t$ gives

$$0 = (1-\omega)\,\alpha_0(t) + \omega\beta_0(t) \tag{116}$$

$$1 = (1-\omega)\,\alpha_\theta(t) + \omega\beta_\theta(t), \tag{117}$$

$$0 = (1-\omega)\,\alpha_\Delta(t) - \omega\beta_\theta(t)\,\frac{\phi_x(t)}{\phi_\theta(t)}. \tag{118}$$

The ODEs for the pricing functions $\phi(t)$, $\phi_\theta(t)$ and $\phi_\Delta(t)$ can be characterized as follows:

$$\phi'(t) = r\phi + a\bar{\theta}\phi_\theta - b\bar{x}\bar{\phi}_x + \frac{(1-\omega)\left(\kappa\tilde{q}\chi_D^2 + \sigma_P\right)\left(g_\theta\sigma_\theta\varrho_\theta + g_\Delta\sigma_{Q\Delta}\right) + \sigma_P\omega f_\theta\left(\kappa\tilde{q}\chi_D\chi_\theta + \sigma_{Q\theta}\right)}{(1-\omega)\,\kappa\tilde{q}\chi_D^2 + \sigma_P} \tag{119}$$

$$\phi_\theta'(t) = (a+r)\phi_\theta - \frac{(1-\omega)\left(\kappa\tilde{q}\chi_D^2 + \sigma_P\right)\left(g_{\theta\theta}\sigma_\theta\varrho_\theta + g_{\theta\Delta}\sigma_{Q\Delta}\right) + \sigma_P\omega f_{\theta\theta}\left(\kappa\tilde{q}\chi_D\chi_\theta + \sigma_{Q\theta}\right) + r\gamma\sigma_P\left(\kappa\tilde{q}\chi_D^2 + \sigma_P\right)}{(1-\omega)\,\kappa\tilde{q}\chi_D^2 + \sigma_P} \tag{120}$$

or simplify the term $e_\theta(t) = (a+r)\,\phi_\theta(t) - \phi_\theta'(t)$ in $\beta_\theta(t)$ in (118) gives

$$\begin{aligned}\phi_\Delta'(t) &= (a_\Delta + r)\,\phi_\Delta(t) - \left(g_{\theta\Delta}\sigma_\theta\varrho_\theta + g_{\Delta\Delta}\sigma_{Q\Delta}\right)\\ &\quad - \frac{\bar{\phi}_x - \phi_\Delta(t)}{\phi_\theta(t)}\left[\frac{\omega\sigma_P\left(\frac{1}{1-\omega}r\gamma\sigma_P + g_{\theta\Delta}\sigma_{Q\Delta} + g_{\theta\theta}\sigma_\theta\varrho_\theta - f_{\theta\theta}\sigma_{Q\theta} - \kappa f_{\theta\theta}\tilde{q}\chi_D\chi_\theta\right)}{(1-\omega)\,\kappa\tilde{q}\chi_D^2 + \sigma_P}\right]\end{aligned} \tag{121}$$

Portfolio Demand for the Informed: Boundary First, we derive boundary conditions for the informed investor's value function coefficients. The informed investor's optimization problem at the boundary can be written as

$$\begin{aligned}-e^{-r\gamma\hat{W}^- - g(T,\theta_T,\Delta_T)} &= \max_{\alpha_T}\left\{-\hat{\mathbb{E}}_T\left[e^{-r\gamma\hat{W}^+ - g(0,\theta_T,0)}\right]\right\}\\ &= e^{-r\gamma\hat{W}^-}\max_{\alpha_T}\left\{-\hat{\mathbb{E}}_T\left[e^{-r\gamma\alpha_T\left(P_T^+ - P_T^-\right) - g(0,\theta_T,0)}\right]\right\},\end{aligned} \tag{122}$$

where $x_T \sim \mathcal{N}(\hat{x}_T, \hat{q}_T)$. Solving the exponent part within the expectation operator yields:

$$-r\gamma\alpha_T\left(P_T^+ - P_T^-\right) - g(0,\theta_T,0) = -\Phi_0 - \Phi_1 x_T,$$

where $\Phi_0 = r\gamma\alpha_T\left\{[\phi(0) - \phi(T)] - [\phi_\theta(0) - \phi_\theta(T)]\,\theta_T - \bar{\phi}_x\hat{x}_T + \phi_\Delta(t)\,\Delta_T\right\} + g(0) + g_\theta(0)\,\theta_T + \frac{1}{2}g_{\theta\theta}(0)\,\theta_T^2$ and $\Phi_1 = r\gamma\alpha_T\bar{\phi}_x$. Then

$$\hat{\mathbb{E}}_T\left[e^{-r\gamma\alpha_T\left(P_T^+ - P_T^-\right) - g(0,\theta_T,0)}\right] = e^{-\Phi_0 - \left(\Phi_1\hat{x}_T - \frac{1}{2}\Phi_1^2\hat{q}_T\right)} = e^{Term^i}, \tag{123}$$

where

$$\begin{aligned}Term^i &= -r\gamma\alpha_T\left\{[\phi(0) - \phi(T)] - [\phi_\theta(0) - \phi_\theta(T)]\,\theta_T + \phi_\Delta(t)\,\Delta_T\right\}\\ &\quad - g(0) - g_\theta(0)\,\theta_T - \frac{1}{2}g_{\theta\theta}(0)\,\theta_T^2 + \frac{1}{2}r^2\gamma^2\alpha_T^2\bar{\phi}_x^2\hat{q}_T.\end{aligned} \tag{124}$$

Optimization implies

$$\alpha_T = \alpha_0(T) + \alpha_\theta(T)\,\theta_T + \alpha_\Delta(T)\,\Delta_T, \tag{125}$$

where

$$\alpha_0(T) = \frac{\phi(0) - \phi(T)}{r\gamma\bar{\phi}_x^2\hat{q}_T},\ \alpha_\theta(T) \quad = \quad \frac{\phi_\theta(T) - \phi_\theta(0)}{r\gamma\bar{\phi}_x^2\hat{q}_T},\ \text{and}\ \alpha_\Delta(T) = \frac{\phi_\Delta(T)}{r\gamma\bar{\phi}_x^2\hat{q}_T}. \tag{126}$$

Therefore, $g(T, \theta_T, \Delta_T) = -Term^i$ gives

$$\begin{aligned} & g(T) + g_\theta(T)\,\theta_T + \frac{1}{2}g_{\theta\theta}(T)\,\theta_T^2 + g_\Delta(T)\,\Delta_T + \frac{1}{2}g_{\Delta\Delta}(T)\,\Delta_T^2 + g_{\theta\Delta}(T)\,\theta_T\Delta_T \\ = \quad & \frac{\left[\phi(0) - \phi(T) + \phi_\Delta(T)\,\Delta_T + (\phi_\theta(T) - \phi_\theta(0))\,\theta_T\right]^2}{\hat{q}_T\bar{\phi}_x^2} + \frac{1}{2}g_{\theta\theta}(0)\,\theta_T^2 + g_\theta(0)\,\theta_T + g(0). \end{aligned} \tag{127}$$

Matching the coefficients yields the boundary conditions summarized as follows

$$\begin{aligned} g(T) - g(0) &= \frac{[\phi(T) - \phi(0)]^2}{2\hat{q}_T\bar{\phi}_x^2},\ g_{\theta\theta}(T) - g_{\theta\theta}(0) = \frac{[\phi_\theta(T) - \phi_\theta(0)]^2}{\hat{q}_T\bar{\phi}_x^2}, \\ g_\theta(T) - g_\theta(0) &= \frac{-[\phi(T) - \phi(0)]\,[\phi_\theta(T) - \phi_\theta(0)]}{\hat{q}_T\bar{\phi}_x^2},\ g_{\Delta\Delta}(T) = \frac{\phi_\Delta^2(T)}{\hat{q}_T\bar{\phi}_x^2}, \\ g_\Delta(T) &= -\frac{[\phi(T) - \phi(0)]\,\phi_\Delta(T)}{\hat{q}_T\bar{\phi}_x^2},\ g_{\theta\Delta}(T) = \frac{\phi_\Delta(T)\,[\phi_\theta(T) - \phi_\theta(0)]}{\hat{q}_T\bar{\phi}_x^2}. \end{aligned} \tag{128}$$

Portfolio Demand for the Uninformed: Boundary Second, we derive boundary conditions for the uninformed investor's value function coefficients. The uninformed investor's optimization problem at the boundary is

$$V\left(T, \tilde{W}_T^-, \tilde{\theta}_T\right) \quad = \quad h^{-1}\left[\max_{\beta_T}\left\{\tilde{\mathbb{E}}_T\left[h\left(V\left(0, \tilde{W}_T^+, \theta_T\right)\right)\right]\right\}\right]$$

where $h[V] = -\frac{1}{1+\kappa}(-V)^{1+\kappa}$. This gives

$$\begin{aligned} e^{\left[-r\gamma\tilde{W}_T^- - f\left(T,\tilde{\theta}_T\right)\right](1+\kappa)} &= e^{-r\gamma\tilde{W}^-(1+\kappa)}\max_{\beta_T}\tilde{\mathbb{E}}_T\left[e^{\left[-r\gamma\beta_T\left(P_T^+ - P_T^-\right) - f(0,\theta_T)\right](1+\kappa)}\right], \\ e^{-f\left(T,\tilde{\theta}_T\right)(1+\kappa)} &= \max_{\beta_T}\tilde{\mathbb{E}}_T\left[e^{\left[-r\gamma\beta_T\left(P_T^+ - P_T^-\right) - f(0,\theta_T)\right](1+\kappa)}\right], \end{aligned} \tag{129}$$

where $\begin{pmatrix} x_T \\ \theta_T \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \tilde{x}_T \\ \tilde{\theta}_T \end{pmatrix}, \begin{pmatrix} \hat{q}_T + \tilde{q}_T & \frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_T \\ \frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_T & \frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_T \end{pmatrix}\right)$, in which we use the variance-covariance relationship derived before. Note, $x_T - \tilde{x}_T = x_T - \hat{x}_T + \hat{x}_T - \tilde{x}_T$. Recall the learning identity: $\phi_x(t)\,\hat{x}_t - \phi_\theta(t)\,\theta_t = \phi_x(t)\,\tilde{x}_t - \phi_\theta(t)\,\tilde{\theta}_t$. Use this to write $\hat{x}_T - \tilde{x}_T$ as a function of $\theta_T - \tilde{\theta}_T$, we obtain: $\hat{x}_T - \tilde{x}_T = \frac{\phi_{\theta,T}}{\phi_{x,T}}\left(\theta_T - \tilde{\theta}_T\right)$. Therefore,

$$\begin{aligned}
P_T^+ - P_T^- &= [\phi(0) - \phi(T)] - \left[\phi_\theta(0)\,\theta_T - \phi_\theta(T)\,\tilde{\theta}_T\right] + \bar{\phi}_x(x_T - \tilde{x}_T) \\
&= [\phi(0) - \phi(T)] - \left(\phi_{\theta,0} - \frac{\phi_{\theta,T}}{\phi_{x,T}}\bar{\phi}_x\right)\left(\theta_T - \tilde{\theta}_T\right) + (\phi_{\theta,T} - \phi_{\theta,0})\,\tilde{\theta}_T + \bar{\phi}_x(x_T - \hat{x}_T).
\end{aligned}$$

Note that $x_T \sim \mathcal{N}(\hat{x}_T, \hat{q}_T)$, or $x_T - \hat{x}_T \sim \mathcal{N}(0, \hat{q}_T)$. We also know that $\theta_T - \tilde{\theta}_T \sim \mathcal{N}(0, \Omega)$, where $\Omega \equiv \frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_T$. The joint distribution $\begin{pmatrix} x_T - \hat{x}_T \\ \theta_T - \tilde{\theta}_T \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{q}_T & 0 \\ 0 & \Omega \end{pmatrix}\right)$. Hence, we would like to rewrite the above equation in terms of $x_T - \hat{x}_T$ and $\theta_T - \tilde{\theta}_T$:

$$P_T^+ - P_T^- = n_0(T) + \mu_1(T)\left(\theta_T - \tilde{\theta}_T\right) + n_\theta(T)\,\tilde{\theta}_T + \bar{\phi}_x(x_T - \hat{x}_T)$$

where

$$n_0(T) \equiv \phi(0) - \phi(T) \tag{130}$$

$$n_\theta(T) \equiv \phi_\theta(T) - \phi_\theta(0) \tag{131}$$

$$\mu_1(T) \equiv \bar{\phi}_x\frac{\phi_\theta(T)}{\phi_x(T)} - \phi_\theta(0). \tag{132}$$

Also,

$$\begin{aligned}
f(0, \theta_T) &= f(0) + f_\theta(0)\,\theta_T + \frac{1}{2}f_{\theta\theta}(0)\,\theta_T^2 \\
&= \nu_0(T) + \nu_1(T)\left(\theta_T - \tilde{\theta}_T\right) + \frac{1}{2}f_{\theta\theta,0}\left(\theta_T - \tilde{\theta}_T\right)^2
\end{aligned}$$

where

$$\nu_0(T) \equiv f(0) + f_\theta(0)\,\tilde{\theta}_T + \frac{1}{2}f_{\theta\theta}(0)\,\tilde{\theta}_T^2 \tag{133}$$

$$\nu_1(T) \equiv f_\theta(0) + f_{\theta\theta}(0)\,\tilde{\theta}_T. \tag{134}$$

Solving the exponent part within the expectation operator in the optimization problem gives:

$$(1+\kappa)\left\{-r\gamma\beta_T\left[n_0(T) + \mu_1(T)\left(\theta_T - \tilde{\theta}_T\right) + n_\theta(T)\,\tilde{\theta}_T + \bar{\phi}_x(x_T - \hat{x}_T)\right] - \left[\nu_0 + \nu_1\left(\theta_T - \tilde{\theta}_T\right) + \frac{1}{2}f_{\theta\theta,0}\left(\theta_T - \tilde{\theta}_T\right)\right.\right.$$

Then we need to compute the expectation. We first integrate out

$$\tilde{\mathbb{E}}_T\left[e^{-r\gamma(1+\kappa)\beta_T\bar{\phi}_x(x_T - \hat{x}_T)}\right] = e^{\frac{1}{2}(r\gamma)^2(1+\kappa)^2\bar{\phi}_x^2\hat{q}_T\beta_T^2}.$$

Hence,

$$
\begin{aligned}
& \tilde{\mathbb{E}}_T\left[e^{\left[-r\gamma\beta_T\left(P_T^{+}-P_T^{-}\right)-f(0,\theta_T)\right](1+\kappa)}\right] \\
= \;& e^{(-r\gamma\beta_T\mu_0-\nu_0)(1+\kappa)+\frac{1}{2}(r\gamma)^2(1+\kappa)^2\bar{\phi}_x^2\hat{q}_T\beta_T^2}\tilde{\mathbb{E}}_T\left[e^{\left[-(r\gamma\beta_T\mu_1+\nu_1)\left(\theta_T-\tilde{\theta}_T\right)-\frac{1}{2}f_{\theta\theta,0}\left(\theta_T-\tilde{\theta}_T\right)^2\right](1+\kappa)}\right],
\end{aligned}
$$

where

$$
\mu_0(T) \equiv n_0(T) + n_\theta(T)\,\tilde{\theta}_T. \tag{135}
$$

Lemma 3. *Let $X \sim \mathcal{N}(0,\Omega)$, then*

$$
\mathbb{E}\left[e^{-\frac{1}{2}aX^2+bX}\right] = \frac{1}{\sqrt{1+a\Omega}}e^{\frac{1}{2}\frac{b^2\Omega}{1+a\Omega}} = e^{\frac{1}{2}\left[\frac{b^2\Omega}{1+a\Omega}-\ln(1+a\Omega)\right]}. \tag{136}
$$

Applying the above lemma, we have

$$
\tilde{\mathbb{E}}_T\left[e^{\left[-(r\gamma\beta_T\mu_1+\nu_1)\left(\theta_T-\tilde{\theta}_T\right)-\frac{1}{2}f_{\theta\theta,0}\left(\theta_T-\tilde{\theta}_T\right)^2\right](1+\kappa)}\right] = e^{\frac{\Omega(r\gamma\beta_T\mu_1+\nu_1)^2(1+\kappa)^2}{2\left(1+f_{\theta\theta,0}(1+\kappa)\Omega\right)}-\frac{1}{2}\ln\left(1+f_{\theta\theta,0}(1+\kappa)\Omega\right)}.
$$

Therefore,

$$
\tilde{\mathbb{E}}_T\left[e^{\left[-r\beta_T\left(P_T^{+}-P_T^{-}\right)-f(0,\theta_T)\right](1+\kappa)}\right] \;=\; e^{Term^u},
$$

where

$$
\begin{aligned}
Term^u \;=\;& \frac{1}{2}(r\gamma)^2(1+\kappa)\Lambda\beta_T^2 - r\gamma(1+\kappa)\left[\mu_0-\Gamma(1+\kappa)\mu_1\nu_1\right]\beta_T \\
&+\left[-\nu_0(1+\kappa)+\frac{\Gamma}{2}(1+\kappa)^2\nu_1^2-\frac{1}{2}\ln\left(1+f_{\theta\theta,0}(1+\kappa)\Omega\right)\right]
\end{aligned}
$$

where

$$
\Gamma \equiv \frac{\Omega}{1+f_{\theta\theta}(0)\,\Omega} \tag{137}
$$

$$
\Lambda \equiv \bar{\phi}_x^2\hat{q}_T + \Gamma\mu_1^2(T). \tag{138}
$$

The FOC with respect to β_T gives

$$
\begin{aligned}
\beta_T &= \frac{1}{r\gamma\Lambda}\left[\mu_0(T)-\Gamma(1+\kappa)\mu_1(T)\nu_1(T)\right] \\
&= \frac{1}{r\gamma\Lambda}\left[m_0(T)+m_\theta(T)\,\tilde{\theta}_T\right]
\end{aligned}
$$

where

$$
m_\theta(T) \equiv n_\theta(T) - \Gamma(1+\kappa)\mu_1(T)\,f_{\theta\theta}(0) \tag{139}
$$

$$
m_0(T) \equiv n_0(T) - \Gamma(1+\kappa)\mu_1(T)\,f_\theta(0). \tag{140}
$$

Denote

$$\beta_T = \beta_0(T) + \beta_\theta(T)\tilde{\theta}_T, \tag{141}$$

where

$$\beta_0(T) = \frac{1}{r\gamma\Lambda} m_0(T) \tag{142}$$

$$\beta_\theta(T) = \frac{1}{r\gamma\Lambda} m_\theta(T). \tag{143}$$

Plugging back,

$$Term^u = \left[-\nu_0(1+\kappa) + \frac{\Gamma}{2}\nu_1^2(1+\kappa)^2 - \frac{1}{2}\ln\left(1 + f_{\theta\theta,0}(1+\kappa)\Omega\right) \right] - \frac{1+\kappa}{2\Lambda}\left(m_0 + m_\theta\tilde{\theta}_T\right)^2.$$

Substituting $\nu_0(T)$ and $\nu_1(T)$ into $f\left(T, \tilde{\theta}_T\right) = -\frac{Term^u}{1+\kappa}$ gives

$$\begin{aligned} f(T) + f_\theta(T)\tilde{\theta}_T + \frac{1}{2}f_{\theta\theta}(T)\tilde{\theta}_T^2 &= f(0) + f_\theta(0)\tilde{\theta}_T + \frac{1}{2}f_{\theta\theta}(0)\tilde{\theta}_T^2 - \frac{\Gamma}{2}\left[f_\theta(0) + f_{\theta\theta}(0)\tilde{\theta}_T\right]^2(1+\kappa) \\ &+ \frac{1}{2(1+\kappa)}\ln\left(1 + f_{\theta\theta,0}(1+\kappa)\Omega\right) + \frac{1}{2\Lambda}\left(m_0 + m_\theta\tilde{\theta}_T\right)^2 \end{aligned}$$

and matching the coefficients yields the boundary conditions summarized as follows

$$f(T) - f(0) = -\frac{\Gamma}{2}f_\theta^2(0)(1+\kappa) + \frac{1}{2\Lambda}m_0^2(T) + \frac{1}{2(1+\kappa)}\ln\left(1 + f_{\theta\theta}(0)(1+\kappa)\Omega\right) \tag{144}$$

$$f_\theta(T) - f_\theta(0) = -\Gamma f_{\theta\theta}(0) f_\theta(0)(1+\kappa) + \frac{1}{\Lambda}m_0(T)m_\theta(T) \tag{145}$$

$$f_{\theta\theta}(T) - f_{\theta\theta}(0) = -\Gamma f_{\theta\theta}^2(0)(1+\kappa) + \frac{1}{\Lambda}m_\theta^2(T). \tag{146}$$

Market Clearing Note that market clearing requires: $(1-\omega)\alpha_T + \omega\beta_T = \theta_T$ at the announcement. This implies

$$\begin{aligned} (1-\omega)\alpha_0(T) + \omega\beta_0(T) &= 0, \\ (1-\omega)\alpha_\theta(T) + \omega\beta_\theta(T) &= 1, \\ (1-\omega)\alpha_\Delta(T) - \omega\beta_\theta(T)\frac{\phi_{x,T}}{\phi_{\theta,T}} &= 0. \end{aligned}$$

Substituting expressions in equations (126) and (141) gives

$$\begin{aligned}
(1-\omega)\frac{\phi_0-\phi_T}{\bar{\phi}_x^2\hat{q}_T}+\omega\frac{m_0}{\Lambda} &= 0,\\
(1-\omega)\frac{\phi_{\theta,T}-\phi_{\theta,0}}{\bar{\phi}_x^2\hat{q}_T}+\omega\frac{m_\theta}{\Lambda} &= r\gamma,\\
(1-\omega)\frac{\phi_{\Delta,T}}{\bar{\phi}_x^2\hat{q}_T}-\omega\frac{m_\theta}{\Lambda}\frac{\phi_{x,T}}{\phi_{\theta,T}} &= 0.
\end{aligned}$$

Eventually we can pin down the boundary conditions for the pricing function coefficients.

$$\phi(T)-\phi(0) = -\frac{\omega(1+\kappa)\bar{\phi}_x^2\hat{q}_T}{(1-\omega)\Lambda+\omega\bar{\phi}_x^2\hat{q}_T}\Gamma\mu_1(T)f_\theta(0) \tag{147}$$

$$\phi_\theta(T)-\phi_\theta(0) = \frac{\bar{\phi}_x^2\hat{q}_T}{(1-\omega)\Lambda+\omega\bar{\phi}_x^2\hat{q}_T}\left[r\gamma\Lambda+\omega(1+\kappa)\Gamma\mu_1(T)f_{\theta\theta}(0)\right] \tag{148}$$

$$\phi_\Delta(T) = \frac{\omega\bar{\phi}_x^2\hat{q}_T}{(1-\omega)\Lambda}\frac{\phi_x(T)}{\phi_\theta(T)}m_\theta(T). \tag{149}$$

Unconditional Expected Return Now we derive the unconditional expected return from our model. Using equations (110) and (111), we can write

$$\begin{aligned}
\mathbb{E}(dQ_t)/\mathbb{E}(P_t) &= \frac{\mathbb{E}\left(e_0(t)+e_\theta(t)\tilde{\theta}_t\right)}{\phi(t)+\left(\phi_D+\bar{\phi}_x\right)\bar{x}-\phi_\theta(t)\bar{\theta}}\\
&= \underbrace{\frac{r\gamma\left(\beta_0(t)+\beta_\theta(t)\bar{\theta}\right)\sigma_P(t)+\left(f_\theta(t)+f_{\theta\theta}(t)\bar{\theta}\right)\sigma_{Q\theta}(t)}{\phi(t)+\left(\phi_D+\bar{\phi}_x\right)\bar{x}-\phi_\theta(t)\bar{\theta}}}_{\text{expected return due to expected utility}}\\
&\quad \underbrace{+\kappa\tilde{q}(t)\chi_D\frac{r\gamma\left(\beta_0(t)+\beta_\theta(t)\bar{\theta}\right)\chi_D\tilde{q}(t)+\left(f_\theta(t)+f_{\theta\theta}(t)\bar{\theta}\right)\chi_\theta}{\phi(t)+\left(\phi_D+\bar{\phi}_x\right)\bar{x}-\phi_\theta(t)\bar{\theta}}}_{\text{expected return due to ambiguity}}.
\end{aligned} \tag{150}$$

The first term in the expected return comes from the standard expected utility, whereas the second term appears only because of ambiguity aversion. Note that χ_D defined in equation (104) arises from the variance of conditional expectation in the $\mathcal{T}$ operator and κ represents the degree of ambiguity aversion.

6.6 Optimal Information Acquisition

In this section, we provide a proof for the sufficient conditions for optimal information acquisition time in Lemma 1. For simplicity, we focus on the case of expected utility. Using the notation in the lemma, we let $\tilde{V}\left(t,W,\tilde{\theta}|\infty\right)$ be the value function before information acquisition and $\tilde{V}\left(t,W,\tilde{\theta}|\sigma_u\right)$ be the value function after information acquisition. To prove the optimality of τ, it is enough to show that for all $t < \tau$,[8]

$$\tilde{V}\left(t, W, \tilde{\theta}|\infty\right) \geq \tilde{V}\left(t, W - K\left(\tilde{\theta}\right), \tilde{\theta}|\sigma_u\right), \tag{151}$$

and for all $t > \tau$, and any consumption plan $\{C_s\}_{s=\tau}^{t}$,

$$\tilde{V}\left(\tau, W - K\left(\tilde{\theta}\right), \tilde{\theta}|\sigma_u\right) \geq \mathbb{E}_{\tau}^{\infty}\left[\int_{\tau}^{t} e^{-\rho(s-\tau)} u\left(C_s\right) ds + e^{-\rho(t-\tau)} \tilde{V}\left(t, W_t - K\left(\tilde{\theta}_t\right), \tilde{\theta}_t|\sigma_u\right)\right]. \tag{152}$$

Inequality (151) guarantees that exercising the option at any time $t < \tau$ is dominated by exercising at time τ. And inequality (152) implies that exercising the option at time τ is better than postponing option exercise to t for any $t > \tau$. The notation $\mathbb{E}_{\tau}^{\infty}$ indicates that the expectation is taken with respect to the belief such that $\sigma_u(s) = \infty$ for all $s \in (\tau, t)$.

We first establish (151). Given the negative exponential form of the value function, (151) is equivalent to

$$\tilde{V}\left(t, W + K\left(\tilde{\theta}\right), \tilde{\theta}|\infty\right) \geq \tilde{V}\left(t, W, \tilde{\theta}|\sigma_u\right). \tag{153}$$

Let $\{C_s\}_{s=t}^{\tau}$ be the consumption policy associated with the value function $\tilde{V}\left(t, W_t, \tilde{\theta}_t|\sigma_u\right)$,

$$\tilde{V}\left(t, W_t, \tilde{\theta}_t|\sigma_u\right) = \mathbb{E}_{t}^{\sigma_u}\left[\int_{t}^{\tau} e^{-\rho(s-t)} u\left(C_s - k\right) ds + e^{-\rho(\tau-t)} \tilde{V}\left(\tau, W_\tau, \tilde{\theta}_\tau|\sigma_u\right)\right]. \tag{154}$$

Replacing the term $\tilde{V}\left(\tau, W_\tau, \tilde{\theta}_\tau|\sigma_u\right)$ using the value matching condition at τ, $\tilde{V}\left(\tau, W_\tau, \tilde{\theta}_\tau|\sigma_u\right) = \tilde{V}\left(\tau, W_\tau + K\left(\theta_\tau\right), \tilde{\theta}_\tau|\infty\right)$, we can write the inequality (153) as

$$\tilde{V}\left(t, W_t + K\left(\tilde{\theta}_t\right), \tilde{\theta}_t|\infty\right) \geq \mathbb{E}_{t}^{\sigma_u}\left[\int_{t}^{\tau} e^{-\rho(s-t)} u\left(C_s - k\right) ds + e^{-\rho(\tau-t)} \tilde{V}\left(\tau, W_\tau + K\left(\theta_\tau\right), \tilde{\theta}_\tau|\infty\right)\right], \tag{155}$$

or

$$e^{-\rho t} \tilde{V}\left(t, W + K\left(\tilde{\theta}_t\right), \tilde{\theta}_t|\infty\right) \geq \mathbb{E}_{t}^{\sigma_u}\left[\int_{t}^{\tau} e^{-\rho s} u\left(C_s - k\right) ds + e^{-\rho \tau} \tilde{V}\left(\tau, W_\tau + K\left(\theta_\tau\right), \tilde{\theta}_\tau|\infty\right)\right]. \tag{156}$$

Using Dynkin's formula,

$$\begin{aligned}\mathbb{E}_{t}^{\sigma_u}\left[e^{-\rho\tau} \tilde{V}\left(\tau, W_\tau + K\left(\theta_\tau\right), \tilde{\theta}_\tau|\infty\right)\right] &= e^{-\rho t} \tilde{V}\left(t, W_t + K\left(\tilde{\theta}_t\right), \tilde{\theta}_t\middle|\infty\right) \\ &+ \mathbb{E}_{t}^{\sigma_u}\left[\int_{t}^{\tau} \mathcal{L}^{\sigma_u}\left\{e^{-\rho s} \tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s\middle|\infty\right)\right\} ds\right].\end{aligned} \tag{157}$$

[8]Here we allow the value function $\tilde{V}\left(t, W, \tilde{\theta}|\sigma_u\right)$ to be defined for all $t \in [0, T]$ through equation (28). For $t < \tau$, $\tilde{V}\left(t, W, \tilde{\theta}|\sigma_u\right)$ is interpreted as investor's utility if he were to decide to acquire information at t.

The term $\mathcal{L}^{\sigma_u}\left\{e^{-\rho s}\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right)\right\} = e^{-\rho s}\left[-\rho\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right) + \mathcal{L}^{\sigma_u}\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right)\right.\right.$
Note that

$$\begin{aligned}
&-\rho\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right) + \mathcal{L}^{\sigma_u}\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right) \\
=&-\rho\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right) + \mathcal{L}^{\sigma_u}\tilde{V}\left(s, W_s + K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \infty\right) + u\left(C_s - k\right) - u\left(C_s - k\right) \\
\leq&-u\left(C_s - k\right),
\end{aligned}$$

where the last inequality uses the condition (30). As a result, Equation (157) implies

$$\mathbb{E}_t^{\sigma_u}\left[e^{-\rho\tau}\tilde{V}\left(\tau, W_\tau + K\left(\theta_\tau\right), \tilde{\theta}_\tau|\infty\right)\right] \leq e^{-\rho t}\tilde{V}\left(t, W_t + K\left(\tilde{\theta}_t\right), \tilde{\theta}_t \middle| \infty\right) - \mathbb{E}_t^{\sigma_u}\left[\int_t^\tau e^{-\rho s}u\left(C_s - k\right) ds\right], \tag{158}$$

which is (156).

Similarly, inequality (152) can be written as:

$$e^{-\rho\tau}\tilde{V}\left(\tau, W_\tau - K\left(\tilde{\theta}_\tau\right), \tilde{\theta}_\tau \middle| \sigma_u\right) \geq \mathbb{E}_\tau^\infty\left[\int_\tau^t e^{-\rho s}u\left(C_s\right) ds + e^{-\rho t}\tilde{V}\left(t, W_t - K\left(\tilde{\theta}_t\right), \tilde{\theta}_t \middle| \sigma_u\right)\right]. \tag{159}$$

Using Dynkin's formula,

$$\begin{aligned}
\mathbb{E}_\tau^\infty\left[e^{-\rho t}\tilde{V}\left(t, W_t - K\left(\tilde{\theta}_t\right), \tilde{\theta}_t \middle| \sigma_u\right)\right] =& e^{-\rho\tau}\tilde{V}\left(\tau, W_\tau - K\left(\tilde{\theta}_\tau\right), \tilde{\theta}_\tau \middle| \sigma_u\right) \\
&+ \mathbb{E}_\tau^\infty\left[\int_\tau^t \mathcal{L}^\infty\left\{e^{-\rho s}\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right)\right\} ds\right].
\end{aligned} \tag{160}$$

We can write the term $\mathcal{L}^\infty\left\{e^{-\rho s}\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right)\right\}$ as $e^{-\rho s}\left\{-\rho\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right) + \mathcal{L}^\infty\tilde{V}\left(s, W\right.\right.$
Note that

$$\begin{aligned}
&-\rho\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right) + \mathcal{L}^\infty\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right) \\
=&-\rho\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right) + \mathcal{L}^\infty\tilde{V}\left(s, W_s - K\left(\tilde{\theta}_s\right), \tilde{\theta}_s \middle| \sigma_u\right) + u\left(C_s\right) - u\left(C_s\right) \\
\leq&-u\left(C_s\right),
\end{aligned}$$

where the last inequality uses the condition (152). As a result, (160) implies

$$\mathbb{E}_\tau^\infty\left[e^{-\rho t}\tilde{V}\left(t, W_t - K\left(\tilde{\theta}_t\right), \tilde{\theta}_t \middle| \sigma_u\right)\right] \leq e^{-\rho\tau}\tilde{V}\left(\tau, W_\tau - K\left(\tilde{\theta}_\tau\right), \tilde{\theta}_\tau \middle| \sigma_u\right) - \mathbb{E}_\tau^\infty\left[\int_\tau^t e^{-\rho s}u\left(C_s\right) ds\right], \tag{161}$$

which is (159), as needed.

6.7 Implied Volatility

Implied Variance We would like to compute $Var_0\left[P_t - P_0\right] = Var_0\left[P_t\right]$. First consider the case in which $t < T$. We solve the three components separately. First, we solve for $\tilde{x}_t$. Using the law of motion (12), we have:

$$\tilde{x}_t = e^{-bt}\int_0^t e^{bs} b\bar{x} ds + e^{-bt}\int_0^t e^{bs}\frac{\hat{q}_s + \tilde{q}_s}{\sigma_D} d\tilde{B}_{D,s} + e^{-bt}\int_0^t e^{bs}\nu\left(s\right)\sigma_\xi\left(s\right) d\tilde{B}_{\xi,s} + e^{-bt}\int_0^t e^{bs}\frac{\tilde{q}_s}{\sigma_{u,s}} d\tilde{B}_{u,s}.$$

Therefore, compute the diffusion

$$\mathcal{D}\left[\tilde{x}_t\right] = \int_0^t e^{b(s-t)}\frac{\hat{q}_s + \tilde{q}_s}{\sigma_D} d\tilde{B}_{D,s} + \int_0^t e^{b(s-t)}\nu\left(s\right)\sigma_\xi\left(s\right) d\tilde{B}_{\xi,s} + \int_0^t e^{b(s-t)}\frac{\tilde{q}_s}{\sigma_{u,s}} d\tilde{B}_{u,s}. \tag{162}$$

Next, we compute D_t. Using law of motion: $dD_t = \left(\tilde{x}_t - D_t\right) dt + \sigma_D d\tilde{B}_{D,t}$, we obtain

$$D_t = e^{-t}\left[\int_0^t e^s \tilde{x}_s ds + \int_0^t e^s \sigma_D d\tilde{B}_{D,s}\right].$$

The term

$$\int_0^t e^u \tilde{x}_u du = \int_0^t e^{(1-b)u}\int_0^u \left\{e^{bs} b\bar{x} ds + \int_0^u e^{bs}\frac{\hat{q}_s + \tilde{q}_s}{\sigma_D} d\tilde{B}_{D,s} + \int_0^u e^{bs}\nu\left(s\right)\sigma_\xi\left(s\right) d\tilde{B}_{\xi,s} + \int_0^u e^{bs}\frac{\tilde{q}_s}{\sigma_{u,s}} d\tilde{B}_{u,s}\right\} du.$$

We focus on the diffusion part:

$$\begin{aligned}
\int_0^t \int_0^u e^{bs+(1-b)u}\frac{\hat{q}_s + \tilde{q}_s}{\sigma_D} d\tilde{B}_{D,s} du &= \int_0^t \int_s^t e^{bs+(1-b)u}\frac{\hat{q}_s + \tilde{q}_s}{\sigma_D} du d\tilde{B}_{D,s} \\
&= \frac{1}{(1-b)\,\sigma_D}\int_0^t \left[e^{(1-b)t+bs} - e^s\right]\left(\hat{q}_s + \tilde{q}_s\right) d\tilde{B}_{D,s}.
\end{aligned}$$

Similarly, we have:

$$\int_0^t \int_0^u e^{bs+(1-b)u}\nu\left(s\right)\sigma_\xi\left(s\right) d\tilde{B}_{\xi,s} du = \frac{1}{(1-b)}\int_0^t \left[e^{(1-b)t+bs} - e^s\right]\nu\left(s\right)\sigma_\xi\left(s\right) d\tilde{B}_{\xi,s}.$$

$$\int_0^t \int_0^u e^{bs+(1-b)u}\frac{\tilde{q}_s}{\sigma_{u,s}} d\tilde{B}_{u,s} du = \frac{1}{(1-b)}\int_0^t \left[e^{(1-b)t+bs} - e^s\right]\frac{\tilde{q}_s}{\sigma_{u,s}} d\tilde{B}_{u,s}.$$

The diffusion part of D_t is

$$\begin{aligned}
\mathcal{D}\left[D_t\right] &= \int_0^t \left[\left(e^{b(s-t)} - e^{s-t}\right)\frac{\hat{q}_s + \tilde{q}_s}{(1-b)\,\sigma_D} + e^{s-t}\sigma_D\right] d\tilde{B}_{D,s} \\
&\quad + \int_0^t \left(e^{b(s-t)} - e^{s-t}\right)\frac{\nu\left(s\right)\sigma_\xi\left(s\right)}{1-b} d\tilde{B}_{\xi,s} + \int_0^t \left(e^{b(s-t)} - e^{s-t}\right)\frac{\tilde{q}_s}{(1-b)\,\sigma_{u,s}} d\tilde{B}_{u,s}
\end{aligned} \tag{163}$$

Finally, we deal with $\tilde{\theta}_t$. Using law of motion of $\tilde{\theta}_t$ in (23),

$$\begin{aligned}\mathcal{D}\left[\tilde{\theta}_t\right] &= \int_0^t e^{a(s-t)}\frac{\phi_x(s)}{\phi_\theta(s)}\frac{\tilde{q}_s}{\sigma_D}d\tilde{B}_{D,s}+\int_0^t e^{a(s-t)}\left[\phi_x(s)\nu(s)-1\right]\frac{\sigma_\xi(s)}{\phi_\theta(s)}d\tilde{B}_{\xi,s}\\ &\quad+\int_0^t e^{a(s-t)}\frac{\phi_x(s)}{\phi_\theta(s)}\frac{\tilde{q}_s}{\sigma_{u,s}}d\tilde{B}_{u,s}. \end{aligned} \tag{164}$$

Summing up (162), (163), and (164), we can represent the price in the form of

$$\mathcal{D}[P_t]=\int_0^t Term_D(s)\,d\tilde{B}_{D,s}+\int_0^t Term_\xi(s)\,d\tilde{B}_{\xi,s}+\int_0^t Term_u(s)\,d\tilde{B}_{u,s}, \tag{165}$$

where

$$Term_D(s) = \phi_D\left[\left(e^{b(s-t)}-e^{s-t}\right)\frac{\hat{q}_s+\tilde{q}_s}{(1-b)\sigma_D}+e^{s-t}\sigma_D\right]-\phi_\theta(t)e^{a(s-t)}\frac{\phi_x(s)}{\phi_\theta(s)}\frac{\tilde{q}_s}{\sigma_D}+\bar{\phi}_x e^{b(s-t)}\frac{\hat{q}_s+\tilde{q}_s}{\sigma_D} \tag{166}$$

$$Term_\xi(s) = \phi_D\left[e^{b(s-t)}-e^{s-t}\right]\frac{\nu(s)\sigma_\xi(s)}{1-b}-\phi_\theta(t)e^{a(s-t)}\left[\phi_x(s)\nu(s)-1\right]\frac{\sigma_\xi(s)}{\phi_\theta(s)}+\bar{\phi}_x e^{b(s-t)}\nu(s)\sigma_\xi(s) \tag{167}$$

$$Term_u(s) = \phi_D\left[e^{b(s-t)}-e^{s-t}\right]\frac{\tilde{q}_s}{(1-b)\sigma_{u,s}}-\phi_\theta(t)e^{a(s-t)}\frac{\phi_x(s)}{\phi_\theta(s)}\frac{\tilde{q}_s}{\sigma_{u,s}}+\bar{\phi}_x e^{b(s-t)}\frac{\tilde{q}_s}{\sigma_{u,s}}. \tag{168}$$

and compute the variance as:

$$Var_0[P_t]=\int_0^t Term_D^2(s)\,ds+\int_0^t Term_\xi^2(s)\,ds+\int_0^t Term_u^2(s)\,ds. \tag{169}$$

Next, consider the general case where we need to compute $Var_t[P_{t+\tau}]$. If $t+\tau\le T$, that is, if we compute implied variance within an announcement cycle, we use the above formula. If $t+\tau>T$. We first compute $Var_t[P_{T-}]$ using the above formula. It is also easy to compute the variance at the announcement $Var_{T^-}[P_{T^+}-P_{T^-}]$. We then compute $Var_{T^+}[P_{t+\tau}]$. The reason we can add up variance is because these different components are independent.

$$Var_t[P_{t+\tau}]=\int_t^{t+\tau}\left[Term_D^2(s)\,ds+Term_\xi^2(s)+Term_u^2(s)\right]ds. \tag{170}$$

$$P_T^+-P_T^- = \bar{\phi}_x(x_T-\tilde{x}_T)-\phi_\theta(0)\left[\theta_T-\tilde{\theta}_T\right]-\left[\phi_\theta(0)-\phi_\theta(T)\right]\tilde{\theta}_T,$$

where $\begin{pmatrix} x_T\\ \theta_T\end{pmatrix}\sim\mathcal{N}\left(\begin{pmatrix}\tilde{x}_T\\ \tilde{\theta}_T\end{pmatrix},\begin{pmatrix}\hat{q}_T+\tilde{q}_T & \frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_T\\ \frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_T & \frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_T\end{pmatrix}\right)$, gives

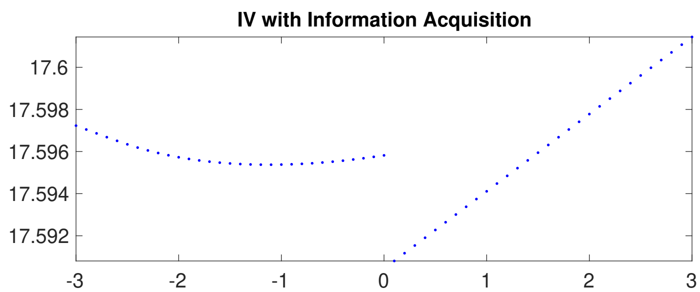
$$Var_{T^-}[P_{T^+}-P_{T^-}] = \bar{\phi}_x^2(\hat{q}_T+\tilde{q}_T)+\phi_\theta^2(0)\frac{\phi_x^2(T)}{\phi_\theta^2(T)}\tilde{q}_T-2\bar{\phi}_x\phi_\theta(0)\frac{\phi_x(T)}{\phi_\theta(T)}\tilde{q}_T. \tag{171}$$

Therefore, the total variance is obtained by

$$\begin{aligned}
&\underbrace{Var_t\left[P_{T^-}\right]}_{\text{IV before announcement}} + \underbrace{Var_{T^-}\left[P_{T^+} - P_{T^-}\right]}_{\text{IV at announcement}} + \underbrace{Var_{T^+}\left[P_{t+\tau}\right]}_{\text{IV after announcement}} \\
= &\int_t^{T^-} \left[Term_D^2(s) + Term_\xi^2(s) + Term_u^2(s)\right] ds + \int_{T^+}^{t+\tau} \left[Term_D^2(s) + Term_\xi^2(s) + Term_u^2(s)\right] ds \\
&+ \bar{\phi}_x^2 \left(\hat{q}_T + \tilde{q}_T\right) + \phi_\theta^2(0) \frac{\phi_x^2(T)}{\phi_\theta^2(T)} \tilde{q}_T - 2\bar{\phi}_x \phi_\theta(0) \frac{\phi_x(T)}{\phi_\theta(T)} \tilde{q}_T \qquad (172)
\end{aligned}$$

In Figure 12, we plot the implied variance around the announcement. It is clear that the implied variance reduces during the drift period and drops significantly after the announcements. This is consistent with the empirical evidence documented by Hu, Pan, Wang, and Zhu (2020).

Figure 12: Implied Volatility



The figure plots the implied volatility (annualized in percentage) in our benchmark economy with endogenous information acquisition. The horizontal axis is the number of days around the announcement, which is normalized as 0. A -1 for example, stands for five days before announcements.