Capital Commitment *

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January 2022

Abstract

Over ten trillion dollars are allocated to private market funds that require outside investors to commit to transferring capital on demand; most of these funds are Private Equity (PE). We show within a novel dynamic portfolio allocation model that ex-ante commitment has large effects on investors' portfolios and welfare, and we quantify those effects. Investors are under-allocated to PE and are willing to pay a larger premium to adjust the quantity committed than to eliminate other frictions, like timing uncertainty and limited tradability. Perhaps counter-intuitively, commitment risk premiums increase with secondary market liquidity and they do not disappear even if investments are spread over many funds.

JEL Classification: G11, G12

Keywords: Capital commitment, private equity, commitment risk, liquidity premium

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1 Introduction

Institutional investors' exposure to private market funds amounts to over \$10 trillion.[1] These funds span a wide range of investments from real estate to leveraged buyouts and venture capital. A defining feature of private market funds, irrespective of their focus, is that they require investors to commit capital to fund managers before it is used ("called"), and thus to relinquish control over their portfolio allocation. Capital commitments have doubled since 2008 to a total of \$3 trillion, and the delay between commitments and calls is about three years. This paper studies the effect of capital commitments on portfolio allocation decisions and investors' welfare.

We solve the dynamic portfolio optimization problem of a risk-averse investor with an infinite horizon and access to stocks, bonds, and private equity funds. We model investments in Private Equity (PE) as follows. At time 0, the investor commits a positive amount to a PE fund; they do not know when the capital will be called or when investment proceeds will be distributed. We use Poisson processes to model the stochastic timing of capital calls and distributions. The first jump triggers the capital call, which is when the investor transfers the committed amount to the fund manager. If the investor is unable or unwilling to make this transfer, they default on their commitment. If the investor makes the transfer, the capital is invested by the fund manager. The second jump of the Poisson process marks the time at which the fund manager distributes the proceeds from the fund back to the investor. Then, a new capital commitment can be made to a new PE fund, and the process is repeated.

Our model is flexible enough to incorporate both strategic default – the investor can skip a capital call at the cost of lost future opportunities – and access to a secondary market – the investor can sell the claim on their invested capital at a discount. We also add liquidity cycles: two persistent states in which the bad state has lower returns, higher volatility, higher correlation between public and private equity, longer average commitment and holding periods, and a larger haircut on the secondary market. By peeling back different parts of the model, we can not only analyze and compare the different frictions induced by capital commitment, but also study the interaction between these frictions.

We provide an additional contribution by analyzing *stochastic liquidity management* within a portfolio of funds. Our model compares investor behavior and welfare across economies with one, two, and an infinity of PE funds. These comparisons allow us to quantify the benefits of diversification across both cash flows and liquidity shocks. Our results

[1] As is customary, the \$10 trillion figure represents the sum of the Net Asset Value of all existing funds and the sum of all committed but uncalled capital ("dry powder"). This information is from the Preqin Pro website, as of November 2021.

are particularly relevant for practice because there is a wide dispersion in the number of contemporary fund commitments held by institutional investors, from one to fifty or more.[2]

We conduct a unique and thorough calibration of our model. We jointly estimate the set of parameters that best capture the empirically observed speed of capital calls and the cross-sectional distribution of fund performance as measured by the Kaplan-Schoar Public Market Equivalent (PME). We believe this is the first time a comprehensive structural model of private equity investments is brought to the data. We show that our relatively stylized model can closely and simultaneously match the whole empirical distribution of both capital calls and fund performance.

Our calibration leads to an optimal commitment to one PE fund of 5.2% of wealth. With access to two PE funds, the optimal commitment increases to 8.8% of wealth. With an infinity of PE funds, at the steady state, 21.9% of wealth is allocated to PE.[3]

Our model design allows us to define and decompose the liquidity frictions related to commitment. In addition to the existence of a delay between capital commitment and call, there are two additional stochastic frictions: i) the commitment delay is of uncertain length; and ii) because of public market movements during the commitment delay, the amount of capital that will be called *as a fraction of wealth* is unknown.[4] We refer to the first stochastic friction as *commitment-timing* risk, and the second as *commitment-quantity* risk. For both frictions, we define the welfare cost as the one-off amount of wealth the investor would give up to remove the friction from the economy. Similarly, the return premium is the permanent PE return loss the investor would accept to remove the friction.

Our first result is that the cost of *commitment-quantity* risk is large. With our calibrated parameters, investors are willing to pay 1.21% of their initial wealth (i.e., 23% of their initial PE allocation), to switch to an economy without this liquidity friction. Alternatively, they will accept a permanent loss (i.e., return premium) of 1.05% to private equity returns. This large cost is explained by two effects. First, liquid wealth is volatile, and investors have an optimal PE portfolio that they do not want to deviate from. Second, to avoid default on

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[2] In the sample of Cavagnaro et al. (2019), the average number of funds held by investors is 23 over a 30-year period. At the high end, CalPERS, which is one of the most active PE investors, reports a total of 311 commitments to PE over the last 23 years, i.e., about 65 simultaneously active commitments.

[3] Aggregate asset allocation across endowments and foundations as of March 2020 according to the Bank of New York Mellon Corporation was 31.1% in listed equity, 16% in fixed income, 18.8% in hedge funds, and 16.9% in private equity. From https://www.pionline.com/interactive/larger-endowments-foundations-lean-private-equity-allocations

[4] During the time period between commitment and capital call, investors face stock volatility, causing the fraction of wealth committed to be sub-optimal at the time of the capital call. For example, following a decline in stock prices during the commitment period, the amount called is larger than the optimal amount.

commitment and the loss of future opportunities, investors commit less to PE than they otherwise would. Thus, commitment-quantity risk is asymmetric: investors almost always want to increase their investment when their capital is called. This result offers a novel rationalization for the increased offering of co-investment opportunities to investors by fund managers at the time of capital call. Co-investment opportunities are valuable options to increase PE exposures post commitment, whereas the literature mostly presents them as tools to reduce fees.

Second, perhaps surprisingly, *commitment-timing* risk carries a cost close to zero. In fact, this cost can even be negative for large values of the subjective discount factor. Intuitively, the investor receives a discounted utility gain from the capital call, and exponential discounting is a convex function – certainty is not valuable. This result holds irrespective of whether we switch commitment-quantity risk on or off, highlighting a lack of interaction between commitment-quantity and commitment-timing risk.

Third, the secondary market, as a tool to liquidate PE positions, is relatively unimportant to investors.[5] Since investors already under-commit, they do not find themselves very often with excess holdings that they would like to sell. Yet, we find that secondary market and commitment liquidity are complements. The direct effect of making the secondary market liquid is a small welfare gain and an increase in PE allocation. The indirect effect is an increase of the welfare cost and return premiums associated with commitment-quantity risk. This liquidity complementarity is surprising because one might think that making an investment more liquid in one way would satiate investor needs. Instead, the opposite result holds: secondary market liquidity allows the investor to take a larger position in PE and therefore increases their willingness to pay to remove other liquidity frictions.

Increasing the number of funds to two and infinity allows the investor to diversify their holdings. Since part of PE cash flow risk and investment timing risk are idiosyncratic, the investor will be able to smooth both cash flow shocks and investment timing. However, there are two reasons to think that diversification will not be as useful in the PE context as it is in the context of public equities. First, commitment-quantity risk is driven by the denominator of the commitment-to-wealth ratio – commitments are constant, but the investor's liquid wealth is volatile – and the denominator is the same for all funds. Second, investing in multiple funds creates the potential for a *funding mismatch*, or funding externality. Increasing investment across multiple funds means using distributions from earlier funds to meet later

[5]Our secondary market allows investors to sell their holdings in PE funds that have called their capital. We do not allow for pure unfunded commitments to be traded, nor for investors to buy funds that have called their capital. In practice, secondary markets are most active for funds that have passed their capital commitment period and thus are fully invested (Nadauld et al. (2019)).

capital calls. However, if one fund distributes late and another calls early, the investor may be short of liquid assets.

We find that the benefits of diversification are moderate, and their impact on commitment risk is small, due to both the correlated denominator risk and the funding mismatch problem. Thus, capital commitment remains relevant, even when the investor has access to multiple funds. First, investors would accept a permanent return reduction of 0.91% to go from one fund to two funds. This is *less* than the 1.05% premium associated with commitment-quantity risk, meaning that investors value the ability to change their commitment more than the ability to diversify their investment to two funds. Second, going from one fund to two funds decreases the commitment-quantity premium to 0.89%, and access to an infinity of funds brings it down to 0.74%, which are both small changes.

Finally, we find that liquidity cycles amplify the cost of commitment-quantity risk when the investor has access to a small number of funds. In fact, the commitment-quantity risk premium is higher in *both* high and low liquidity states than in an economy with no cycles. The intuition is that the investor anticipates the possibility of a state change, and would be willing to pay to change their commitments in that event. With an infinity of funds, this logic only holds in the bad state. In the good state, the investor can easily increase their allocation by committing a small amount to many different funds.

To sum up, our key finding is that ex-ante capital commitment is an important inefficiency for investors. Understanding the ultimate source of this inefficiency is a worthwhile research agenda, but beyond the objective of this paper. Capital commitment is a contractual relationship between funds and outside investors. However, the very structure of private equity deals would make it difficult, if not impossible, to have markets where investors could deploy any amount they wish into private equity at any point in time. First, transactions take several months to be executed.[6] Second, capital calls depend on the needs of the target companies. For example, in financing an early stage company, capital is called if and when the company expands, not when PE investors decide to increase their exposure to venture capital. Nonetheless, we observe some partial solutions in practice. Two important phenomena are consistent with the predictions of our model: as investors increase their allocation to PE, as seen over the last two decades, we do witness i) an increase in co-investment invitations and ii) an increase in the use of secondary markets as a tool to increase PE allocations more quickly, rather than as a liquidation tool. We discuss these features of private equity, along with other key institutional details in the next section.

[6] For example, the public-to-private transaction on Hilton Hotels in 2007, which is quite typical, took twelve months from initiation to completion. See Phalippou (2021) for more details.

2 Institutional Setup & Literature Review

2.1 Fund Structure

Private Equity (PE) funds are set up as private limited partnerships and pursue a wide range of investment strategies (e.g. leveraged buyout, real estate, venture capital, infrastructure, mezzanine etc.). A PE firm acts as the General Partner (GP) for the fund, and capital is provided by the Limited Partners (LPs). "Investing in a PE fund" means becoming an LP.

During a fund-raising period that spans three to eighteen months, the PE firm (e.g., KKR) seeks capital for its fund (e.g., KKR millennium fund). LPs commit some amount of capital, meaning they agree to provide cash on demand to the fund, up to that committed amount ("capital calls"). When the PE firm ends its fundraising, it has its "final close", and the year this occurs is called the fund vintage year. The vintage year marks the beginning of the "investment period", during which most capital calls should occur.

Capital calls are made by the GP on the LPs and are in connection to either a fee payment or investments. The timing of the capital calls is uncertain; LPs only know an ex-ante specified investment period, during which most, but not all, capital calls should occur. For example, leveraged buyout funds typically have a five year investment period, but some capital may be called afterwards for fee payments or follow-on investments in existing portfolio companies. For venture capital funds, the investment period is typically longer than for buyout funds to allow for large follow-on investments in portfolio companies that are successful. As shown in Axelson et al. (2009), ex-ante capital commitments can be a second best optimal contracting solution in the presence of conflicts of interests for the PE fund managers.[7]

Some capital calls are pooled to reduce the frequency of capital calls to LPs. To do so, a GP uses credit facilities with LP commitments as collateral. In addition, investors may participate in a private equity fund via intermediary vehicles (e.g. fund-of-funds, separately managed accounts), and those intermediaries may also partially pool capital calls.

When an investment is exited, the payout is usually distributed to LPs and cannot be

[7] In their model, fund managers have skills in identifying and managing potentially profitable investments, but have to rely on external capital. Because fund managers have limited liability, they have an incentive to overstate the quality of potential investments when they try to raise financing. Giving fund managers capital ex-ante preserves their incentives to avoid bad deals in good times. That is, the agency problem is minimized when funds are raised to finance a number of future projects rather than when they raise capital on a deal-by-deal basis. This model, therefore, provides a rationale for why investors would accept commitment risk despite the cost associated to it. Naturally, there are exceptions; a small number of PE funds call capital separately for each deal, and some new funds invest so often and so quickly that they allow investors to add capital regularly over time (e.g. Tiger Global).

recycled to make a new investment, but there are some exceptions. The distribution period is flexible, spanning the entire life of the fund, including an overlap with the investment period. A fund's life is often 10 years but there are multiple circumstances under which funds obtain extensions. Many funds are not fully liquidated in their fifteenth year and beyond.

See Korteweg and Westerfield (2021) for a comprehensive survey of the institutional details associated with the LP's problem and the associated academic literature.

2.2 Capital Commitments and Commitment Default

There is a wide dispersion in the number of contemporary fund commitments held by institutional investors. In the sample of Cavagnaro et al. (2019), which only includes investors with more than 4 funds, there are 27,283 fund commitments from 1,209 LPs over 30 years, or about 23 commitments per LP over a 30-year period. At the high end, CalPERS, which is one of the most active PE investors, reports a total of 311 commitments to PE over the last 23 years, i.e., about 65 simultaneously active commitments. Thus, institutions differ greatly in their degree of diversification within PE, with many institutions facing lumpy stochastic capital calls.

There are two facts regarding commitment default that are particularly important. First, default is rare. We do not know of any major PE investor that has defaulted on their PE commitment. Second, the stated penalties for default, specified in PE partnership agreements, are high (Banal-Estañol et al. (2017)). Penalties include forfeiture of some or all existing investments in the fund, and impossibility to invest in PE going forward. As these two salient facts suggest, PE investors are willing to take significant and costly actions to avoid default. These costly actions include, for example, redeeming capital from other investments despite low overall liquidity, selling their fund stakes on the secondary market at large discounts, and issuing high yield bonds.

To illustrate, here is a typical account of how PE investors fared during the 2008 crisis: "A growing set of limited partners find themselves short on cash amid the financial crisis – and thus are scrambling for ways to make good on undrawn obligations to private equity vehicles. Among those in the same boat: Duke University Management, Stanford Management, University of Chicago and University of Virginia... Brown, whose \$2.3 billion endowment has a 15% allocation for private equity products, is apparently thinking about redeeming capital from hedge funds to raise the money it needs to meet upcoming capital calls from private equity firms... Carnegie, a \$3.1 billion charitable foundation, is also in a squeeze. Its managers have been calling on commitments faster than expected, while distributions from older funds have slowed down, creating a cash shortfall. As for Duke, the university's endowment has been named as one of the players most likely to default on private equity fund commitments. That partly explains a massive secondary-market offering that the school floated last month, as it sought to raise much-needed cash and get off the hook for undrawn obligations by unloading most of its $2 billion of holdings in the sector... Some of the bigger investors are considering tapping credit facilities to meet near-term capital calls."[8]

In the end, no well-known PE investor defaulted on their PE commitment. However, the extent and cost of the default avoidance strategies demonstrate the large perceived penalty of default.

2.3 The Secondary Market, Direct and Co- Investments

Over the last fifteen years, private equity investments have changed in ways that give LPs more control over their PE exposures. The most important changes have been the growth in the secondary market and the growth in LP co-investments and direct investments.

The secondary market for PE fund stakes grew significantly from an annual turnover of $10 billion in 2006 to over $100 billion in 2021 (projected).[9] Yet, the $100 billion volume still represents only 1% of the $10 trillion allocated to PE. In addition, if we simply measure illiquidity by the observed discount to the reported Net Asset Value (NAV), PE investing is quasi-liquid because the average discount is 13.8% overall and 9% since 2010. However, and importantly, liquidity varies greatly across fund types and fund age. For example, there are few transactions of funds during their investment period, and most transactions are for relatively large funds (Nadauld et al. (2019)).

The other major transformation of the PE industry is the growth in LP co-investments and direct investments. Co-investment opportunities allow LPs to add additional capital to a deal when the GP makes a capital call. Direct investment refers to a situation in which an LP directly executes a PE transaction: e.g., a pension fund buys an operator of nursing homes. Both co-investments and direct investments offer the opportunity to LPs to adjust upwards, but not downwards, their allocation to PE.

Co-investments remain a somewhat small fraction of the total amount committed to PE

[8]From the magazine 'Private Equity Insider' in its issue of November 5, 2008. See also Barron's, 6/29/2009,"The Big Squeeze"; Forbes, 10/24/2009, "Did Harvard Sell At the Bottom?"; Institutional Investor, 11/4/2009, "Lessons Learned: Colleges Lose Billions in Endowments."

[9]Prior to 2008, the secondary market was much less active due to contractual restrictions on transfers (see Lerner and Schoar (2004)). For turnover, see https://www.jefferies.com/CMSFiles/Jefferies.com/Files/IBBlast/Jefferies-Global-Secondary-Market-Review.pdf

but their prevalence is growing fast. Pre-2008, co-investment invitations were limited to large global LPs and to other PE firms in so-called club deals (Officer et al. (2010)). Post-2008, co-investment invitations are more widespread but mostly restricted to LPs in the PE fund that makes the transaction. More generally, Lerner et al. (2021) show that discretionary vehicles – i.e., vehicles which do not ask LPs for ex-ante capital commitments – grew to about 15% of all PE allocations in 2015. They show that another 25% of PE allocations is made via vehicles other than those described in Section 2.1; however, these vehicles also require ex-ante capital commitments.

2.4 Literature Review

Several papers study the optimal portfolio choice problem in the presence of illiquid assets. Illiquidity is often defined as the inability to trade an asset during a given period of time, e.g., Longstaff (2001); Kahl et al. (2003); Longstaff (2009); Gârleanu (2009); Dai et al. (2015).

More recently, papers have attempted to model the specific illiquidity features of private equity funds. Korteweg (2019) and Korteweg and Westerfield (2021) provide surveys of this literature. In Sorensen et al. (2014), a (single) private equity fund is acquired at time 0, hence the capital is immediately invested, but this investment cannot be traded. The fund is liquidated at maturity T, which is finite and known ex ante. The main objective of the paper is to extend the valuation of the performance-related fee (carry) charged by general partners solved in Metrick and Yasuda (2010) to a setting where funds are not continuously traded.

In Ang et al. (2014), an illiquid asset cannot be traded during stochastic periods of time. They illustrate how trading illiquidity can create funding illiquidity, and the resulting portfolio effects and welfare costs are found to be large. Dimmock et al. (2019) allow the agent to liquidate their positions in the illiquid asset on a secondary market at a cost and evaluate the "endowment model" used by some institutions that invest in alternative assets. Bollen and Sensoy (2021) extend the analysis of Sorensen et al. (2014) by also allowing for a secondary market for partnership interests.

In all of the papers cited above, capital committed to the illiquid asset is immediately invested. Thus, the central feature of private market funds, ex ante capital commitment, is not modelled. These papers focus on the effect of illiquidity by constraining an investor to hold an illiquid asset over a period of time, which is either deterministic or stochastic.

In a contemporaneous paper, Giommetti and Sorensen (2020) model a private equity portfolio in which capital is gradually called and distributed from a composite private equity fund. Capital commitments are therefore implicitly embedded in their model. Their central finding is that the optimal allocation to private equity is not sensitive to risk aversion due to the nature of private equity funds' illiquidity. They also find that this result depends on the liquidity of the secondary market: a liquid secondary market makes risk aversion more relevant.

Both our objective and our model are distinct from those of Giommetti and Sorensen (2020). We focus on a different question: the impact of capital commitments on investor welfare. Furthermore, we solve for optimal policies for one, two and an infinity of funds with stochastic periods of illiquidity. These features allow us to analyze stochastic liquidity management policies and to decompose commitment risk into its component parts.

Finally, there is an empirical literature on private equity illiquidity. Ljungqvist et al. (2019) propose a hazard/duration regression approach to model the speed at which capital is called as a function of market conditions for a sample of US buyout funds. In a similar exercise, Robinson and Sensoy (2016) find that fund net cash flows are pro-cyclical, i.e. distributions tend to decrease and capital calls tend to increase in bad times, suggesting high liquidity risk. However, they argue that this liquidity risk might be diversifiable.[10]

3 Model

We model investment portfolios that combine private equity with liquid risky and riskless assets. Our setup is designed to capture important institutional details from Section 2, and it allows for one, two, or an infinity of private equity funds.

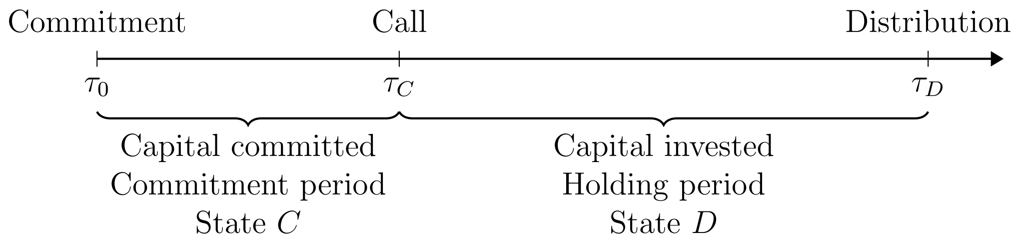
3.1 The Liquid Assets

There are two liquid assets in the economy that can be rebalanced continuously at no cost: a risk-free bond, which captures the fixed-income market, and a risky stock, which captures the public equity market.[11]

[10] Also related is Franzoni et al. (2012) who empirically quantify the liquidity risk premium for private equity in the four factor model of Pastor and Stambaugh (2003). They find that private equity returns are particularly sensitive to changes in aggregate liquidity conditions during the life of the investments.

[11] We consider an information structure that obeys standard assumptions. There exists a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ supporting the vector of four independent Brownian motions $Z_t = (Z_t^L, Z_t^{PE}, Z_t^{1\perp}, Z_t^{2\perp})$, and two independent Poisson processes $M_t = (M_t^1, M_t^2)$. Some stochastic processes may be unused, depending on the number of private equity funds. $\mathcal{P}$ is the corresponding measure and $\mathcal{F}$ is a right-continuous increasing filtration generated by $Z \times M$. Following Dybvig and Huang (1988) and Cox and Huang (1989), we restrict the set of admissible strategies to those that satisfy the standard integrability

Figure 1: Timeline of a fund's life.



The price B_t of the bond appreciates at constant rate r,

$$dB_t = rB_t dt, \tag{1}$$

and the stock price P_t follows a geometric Brownian motion,

$$\frac{dP_t}{P_t} = \mu \, dt + \sigma dZ_t^L, \tag{2}$$

where Z_t^L is a standard Brownian motion associated with liquid public markets, μ is the return drift, and σ is the return volatility.

The investor's liquid wealth W_t is the sum of their holdings in the stock and bond.

3.2 Modelling Private Equity

Investors can allocate capital to private equity funds. We begin by describing the entire investor's problem for one fund and then extend the model to include multiple funds.

As illustrated in Figure 1, the fund manager collects capital commitments from the investor at time τ_0, calls the committed capital and invests it at time $t = \tau_C$, and then distributes the value of the investment at time $t = \tau_D$. We call the *commitment period* the time period $[\tau_0, \tau_C]$ (or the fund is in state C), and the *holding period* the time period $[\tau_C, \tau_D]$ (the fund is in state D). After the fund distribution, the investor makes their next commitment and the process repeats to infinity.

At time τ_0, the investor commits a positive amount $X_{\tau_0} \geq 0$ to the private equity fund. This commitment is a promise to make capital available when the manager calls it at time τ_C. During the commitment period $[\tau_0, \tau_C]$, fees are paid out of the investor's liquid wealth

conditions. All policies are appropriately adapted to $\mathcal{F}_t$.

to the fund manager at rate $fX_{\tau_0}dt$, but no investment is made. Because the commitment cannot be changed, $dX_t = 0$ until the committed capital is called and invested at τ_C.

We use a Poisson process to model the timing of capital transfers between the investor and the fund manager. The process has intensity λ_C during the commitment period. A jump triggers the capital call and the end of the commitment period, at which time the investor transfers X_{τ_0} of liquid wealth to the fund manager.

In a slight abuse of notation, we denote by X_t the amount of capital committed between times τ_0 and τ_C, and we use X_t again to refer to the net-of-fee amount of capital invested in the fund after τ_C.

After capital is transferred and invested, the value of the private equity asset, net of all fees, evolves as a geometric Brownian Motion, with drift ν, and volatility ψ:[12]

$$\frac{dX_t}{X_t} = \nu dt + \psi dZ_t^X, \tag{3}$$

where $dZ_t^X = \rho_L dZ_t^L + \sqrt{1-\rho_L^2}dZ_t^{1\perp}$. This specification implies that the correlation between public and private equity is ρ_L, and the beta of a private equity fund is

$$\beta = \rho_L \frac{\psi}{\sigma}. \tag{4}$$

We again use the Poisson process to model the timing of capital distributions. During the holding period $[\tau_C, \tau_D]$, the intensity of the Poisson process is λ_D, and a jump triggers capital distribution. The private equity investment is fully exited, and the investor receives the value of the fund, $X_{\tau_D -}$. The Poisson process resets, and the investor is immediately able to make a new capital commitment to a new private equity fund.

In our setup, the uncertainty around capital calls and distributions is modelled with two random times for the private equity fund, τ_C and τ_D. These two random times represent two sources of market incompleteness. Even if the liquid asset and the private equity fund had fully correlated returns, the investor would not be able to hedge the risk coming from the random times and the market would still be incomplete.

Our model with one fund relies on two assumptions that ensure analytical tractability. First, there is a single capital call equal to the committed amount, as opposed to having capital calls spread across the investment period. Second, there is a single payout.

[12] To keep the model parsimonious, during the holding period we do not model the management fee and carried interest separately from returns. Instead, we assume that the net-of-fees value X_t follows a geometric Brownian motion.

Below, we will generalize this basic model in two important directions. First, we allow for multiple funds, including an infinite-fund limit. Second, we allow for private equity cycles in which parameters, including returns and waiting times, are allowed to vary over time. In all cases, our representation allows us to use numerical methods based on Markov chain approximations to solve the ODEs and PDEs associated with the portfolio allocation problem.

Our model setup is flexible enough to allow for the existence of a secondary market. We assume that during the holding period, the investor can sell their invested private equity on a secondary market, receiving αX_t. $0 \leq \alpha \leq 1$, and $1-\alpha$ is the haircut. This sale advances τ_D at the cost of a haircut on the investor's asset value. After the sale, the investor can immediately make a new commitment, starting the process over.

If the investor is unable or unwilling to pay for the capital call at time τ_C, then they default on their commitment. We allow for strategic default. The consequence of a default is that the investor does not turn over the capital associated with commitment, but is banned from accessing private equity – which is a realistic feature (see Section 2.2).

3.3 The Investor's Problem

The investor continuously rebalances their liquid wealth between the two liquid assets, and consumes out of liquid wealth at rate $c_t = C_t/W_t$. We denote with θ_t the fraction of liquid wealth allocated to stocks, so the evolution of the investor's liquid wealth is given by:

$$\frac{dW_t}{W_t} = \left(r + (\mu - r)\,\theta_t - c_t\right)dt - \mathbb{1}_{S=C} f \frac{X_{\tau_0}}{W_t} dt + \theta_t \sigma dZ_t^L - \frac{dI_t}{W_t} \tag{5}$$

where dI_t denotes any transfer between liquid wealth and illiquid wealth. Throughout the paper we will use $\mathbb{1}$ as an indicator variable. Thus $\mathbb{1}_{S=C} f \frac{X_{\tau_0}}{W_t}$ denotes fees that are paid during the commitment period.

Given our assumptions, $W > 0$ and $X \geq 0$ a.s. The value function is given by

$$F(W_t, X_t, S_t) = \max_{\{\theta, X, c\}} \mathrm{E}_t \left[\int_t^{\infty} e^{-\delta(u-t)} U(C_u) du \right], \tag{6}$$

subject to (3) and (5), with δ denoting the subjective discount factor, and $S_t = \{C, D\}$ denoting the state. The investor has a standard power utility, i.e., $U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$, with $\gamma > 1$. We extend our results to Epstein-Zin utility in Online Appendix C.

At $t = \tau_C$, and if the investor does not default on their commitment, we have $dI_t =$

$dI_{\tau_C} = X_{\tau_0}$; i.e. committed capital is called, and X_{τ_0} is transferred out of liquid wealth to private equity. The state changes from $S = C$ (commitment period) to $S = D$ (holding period), and the value function jumps discretely from $F(W, X, S = C)$ to $F^{+C}(W, X, S = C) \equiv F(W - X, X, S = D)$. The investor strategically defaults if the welfare value of the standard Merton problem – our model with access to the liquid stock and bond but without the private equity sector – exceeds the continuation value with PE. We denote the value function of a Merton investor $F^{Merton} = W^{1-\gamma} H^{Merton}$. The investor defaults if $F(W, X, S = C) < F^{Merton}$. At time τ_C, default occurs if $F^{+C}(W, X, S = C) < F^{Merton}$.

During the holding period, the investor can sell their stakes on the secondary market and re-commit, and they do so if $F(W, X, S = D) < \max_{X'} F(W + \alpha X, X', S = C)$.

When $t = \tau_D$ is reached, we have $dI_{\tau_D} = -X_{\tau_D -}$; i.e. capital is paid out and X_{τ_D} is transferred from private equity to liquid wealth. The investor chooses their new level of committed capital, the state changes from D to C. The value function jumps discretely from $F(W, X, S = D)$ to $F^{+D}(W, X, S = D) \equiv \max_{X'} F(W + X, X', S = C)$.

At all other times, $dI_t = 0$.

The investor's optimal consumption and allocation solve the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
\delta F = \max_{\{\theta, c\}} \Big\{ & U(cW) + \left(r + (\mu - r)\theta - c - \mathbb{1}_{S=C} f \frac{X}{W} \right) W F_W + \frac{1}{2}\theta^2 \sigma^2 W^2 F_{WW} \\
& + \nu X F_X \mathbb{1}_{S=D} + \frac{1}{2}\psi^2 X^2 F_{XX} \mathbb{1}_{S=D} + \theta\sigma\psi\rho_L W X F_{WX} \mathbb{1}_{S=D} \\
& + \lambda_C \left(F^{+C} - F \right) \mathbb{1}_{S=C} + \lambda_D \left(F^{+D} - F \right) \mathbb{1}_{S=D} \Big\}. \qquad (7)
\end{aligned}$$

We use TW to denote total wealth and ξ to denote the fraction of total wealth either committed or invested, so that

$$TW = W + X \mathbb{1}_{S=D} \qquad (8)$$

$$\xi = \frac{X}{TW} \qquad (9)$$

Since $W > 0$, we require $\xi < 1$. Because the utility function is homothetic and the return processes have constant moments, it must be the case that the value function F is homogenous of degree $1 - \gamma$ in total wealth. Thus, the value function can be written as the product of a power function of total wealth and a function of the wealth composition:

$$F(W, X, S) = TW^{1-\gamma} H(\xi, S). \qquad (10)$$

The optimal commitment is given by the following Proposition:

Proposition 1 *The investor's value function can be written as in (10), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = C)$, which exists.*

The function H is characterized by the set of ODEs shown in Online Appendix A.1, and our method for generating numerical results is detailed in Online Appendix B.

3.4 The Illiquidity Stack

We refer to the model presented in Sections 3.2 and 3.3 as the **baseline economy (Economy 0)**, i.e. an economy where all the sources of private equity liquidity risk are present. We now define five other economies, each of which corresponds to a situation where one or more of the frictions of our baseline model are modified. These modifications allow us to assess theoretical counter-factuals and isolate the impact of the various private equity investment frictions. The solution to the investor's problem in these five economies is given in the Online Appendices A.2 to A.6.

First, we turn off *commitment-timing risk* by making the call time deterministic. The agent commits capital at τ_0, but instead of waiting for a random delay of expected length $\frac{1}{\lambda_C}$, capital is called deterministically at $\tau_C = \tau_0 + \frac{1}{\lambda_C}$. Thus we maintain the average delay but remove the uncertainty about this delay:

Economy 1: Deterministic call time, choose quantity when committing

Next, we turn off *commitment-quantity risk*. In the baseline economy, committed capital is fixed, but liquid wealth evolves randomly before the committed capital is called. Thus, *the relative size of the commitment changes*, and investors do not know the fraction of their wealth that they will be required to turn over to the PE fund. To turn this commitment-quantity risk off, we let the investor choose the quantity invested in private equity when the capital is called, instead of at commitment time. So, the commitment-to-wealth ratio ξ^* is chosen at call time τ_C rather than commitment time τ_0. This change also removes any risk of default, but the timing of the capital call remains stochastic:

Economy 2: Stochastic call time, choose quantity when called

If neither commitment-timing nor commitment-quantity risk is present, there is a commitment delay but *no commitment risk*:

Economy 3: Deterministic call time, choose quantity when called

Absent commitment risk, our model still features a *commitment delay*. The investor needs to wait until call time to access the private equity returns: their active investment time is the holding period of the fund. This restriction can be lifted and the commitment period time brought to zero. Hence, at time τ_D, the investor freely allocates capital between bonds, stocks and private equity and directly enters the fund's holding period:

Economy 4: Immediate private equity access, no commitment risk

Economies 1-4 constitute a peeling back of the institutional details associated with commitment risk. We remove the commitment-timing and quantity risks separately and then together, and then we remove the commitment delay as well. For comparison, we also consider making the stochastic distribution time deterministic. In this economy, the investor's capital is called at random time τ_C, but the holding period duration is deterministic, with $\tau_D = \tau_C + \frac{1}{\lambda_D}$. Thus we maintain the average holding period but remove the risk. This economy still contains commitment risk:

Economy 5: Deterministic payout time

3.5 Measuring the Cost of Illiquidity

We define two measures to quantify the costs of the different liquidity frictions described in Section 3.4.

First, the **welfare cost to the investor** of any economy A with respect to any economy B, denoted $\zeta^{A,B}$, is the fraction of wealth the investor would be willing to pay at the time of commitment to switch from economy A to economy B while simultaneously adjusting their capital commitment. If $\zeta^{A,B} = 0$, then the investor is indifferent between the two economies. $\zeta^{A,B}$ is the solution to the following equation:

$$H^A(\xi^{A*}, S) = (1 - \zeta^{A,B})^{1-\gamma} H^B(\xi^{B*}, S), \tag{11}$$

where ξ^{A*} and ξ^{B*} denote, respectively, the optimal allocations in economies A and B. We evaluate welfare at time the agent chooses the allocation, meaning the commitment period ($S = C$) for Economies 0, 1 and 5 and the holding period ($S = D$) for Economies 2, 3 and 4.

Second, we define the **return premium** of any economy A with respect to economy B as the additional return of the PE funds that would be needed in economy A to make the investor indifferent between the two economies. The return premium applies to all PE funds, both current and future. If the investor is indifferent between economy B with PE expected returns ν and economy A with expected returns $\nu + \epsilon_{AB}$, then the return premium is ϵ_{AB}.

3.6 Liquidity Diversification: Investing in Two Funds

PE investors benefit from having access to multiple funds. First, investors can diversify cash flow risk. Each fund has idiosyncratic risk, and a standard diversification intuition indicates than an investor would prefer to spread their wealth across multiple assets.

Second, by investing in multiple funds, investors reduce the lumpiness of capital calls and distributions. However, as we will see, the investor cannot fully smooth the risk of capital calls despite timing shocks being fully independent. The reason is that any shock to public equity is a shock to all commitment-to-wealth ratios through the denominator. So while diversifying across funds allows for multiple smaller commitments and smoother calls, the outstanding commitment-to-wealth ratio is not less volatile.

Third, the existence of multiple funds enables investors to have commitments to one fund while investing in another. Even if the two funds start out together, they will rapidly de-synchronize because of the stochastic call and distribution timing; one should think of the steady state in this economy as one in which calls and distributions randomly overlap one another. This allows investors to try to smooth out their quantity invested by balancing allocations across multiple funds. It also allows investors to increase the time during which their capital is invested and earning returns in PE – their active investment time – by using distributions from earlier investments to fund later investments. As we will see, taking advantage of this feature creates a potential funding mismatch (or, a funding externality), the risk that earlier distributions will be late or insufficient to fund capital calls.

We extend the model of Sections 3.2 and 3.3 to include two PE funds. The investor chooses an optimal commitment X^i when the previous fund i distributes its capital. During each fund's holding period, returns follow (3), with the same expected return ν and volatility ψ. We assume that the Brownian motions that drive fund returns have correlation ρ_L with public market equities and correlation $\rho_{PE} > \rho_L$ with each other. Thus the returns of each fund i during its holding period are given by

$$\frac{dX_t^i}{X_t^i} = \nu dt + \psi dZ_t^i, \tag{12}$$

where $dZ_t^i = \rho_L dZ_t^L + \sqrt{\rho_{PE}^2 - \rho_L^2} dZ_t^{PE} + \sqrt{1-\rho_{PE}^2} dZ_t^{i\perp}$, and Z^L is the public market shock, Z^{PE} is a common PE shock, and $Z^{i\perp}$ is the idiosyncratic shock associated with fund i. The solution to this problem is given in Online Appendix A.7.

Welfare costs and return premiums are computed from the value function evaluated at $\xi^1 = \xi^{1*}$ and $\xi^2 = \xi^{2*}$, i.e. assuming that commitments to both funds are optimal.

3.7 Limiting Case: Investing in an Infinity of Funds

We extend our model to allow for an infinity of funds. Appealing to the law of large numbers, we assume that the PE funds make calls and distributions continuously. Since individual funds have commitment periods that are exponentially distributed with parameter λ_C, we have that a fraction $\lambda_C dt$ of funds call capital over the interval dt. Similarly, a fraction λ_D of funds make distributions over dt. Because all funds are ex-ante identical, the investor spreads their commitments evenly: a fraction $\lambda_C dt$ of the investor's commitments is called and a fraction $\lambda_D dt$ of the invested capital is distributed over dt.

These assumptions imply that commitment-timing risk disappears in the limit. However, commitment-quantity risk remains: the investor's commitments are called over time, and liquid wealth is fluctuating randomly.

Our two state variables are i) the aggregate capital invested in all PE funds (X_t^∞), and ii) the aggregate committed amount (Y_t^∞). We label the investor's new commitments as $dJ_t \geq 0$. Then, extending (5) and (12), we have the following dynamics:

$$\frac{dY_t^\infty}{Y_t^\infty} = \frac{dJ_t}{Y_t^\infty} - \lambda_C dt \tag{13}$$

$$\frac{dX_t^\infty}{X_t^\infty} = \lambda_C \frac{Y_t^\infty}{X_t^\infty} dt - \lambda_D dt + \nu dt + \psi^\infty dZ_t^\infty \tag{14}$$

$$\frac{dW_t}{W_t} = \left(r + (\mu - r)\,\theta_t - c_t\right) dt + \theta_t \sigma dZ_t^L - f \frac{Y_t^\infty}{W_t} dt - \lambda_C \frac{Y_t^\infty}{W_t} dt + \lambda_D \frac{X_t^\infty}{W_t} dt. \tag{15}$$

where $dZ_t^\infty = \rho_L^\infty dZ_t^L + \sqrt{1 - \rho_L^{\infty 2}} dZ_t^{PE}$.

The parameters driving equation (14) are those of an equally weighted portfolio of PE funds, taking the limit as the number of funds in the portfolio goes to infinity. As with the two-fund case, we assume that the shocks of each PE fund have correlation ρ_L with public markets and ρ_{PE} with each other. Then we can calculate the volatility ψ^∞ and correlation with the stock market ρ_L^∞ of an equally weighted portfolio analytically:

$$\psi^\infty = \psi \sqrt{\rho_{PE}} \quad ; \quad \rho_L^\infty = \frac{\rho_L}{\sqrt{\rho_{PE}}}. \tag{16}$$

The investor maximizes their expected discounted utility as in (6), subject to the budget constraints (13)-(15) and the constraint that J_t is non-decreasing.

Next, we define the ratio of committed wealth to total wealth, and the ratio of invested illiquid wealth to total wealth to be

$$\pi_t \equiv \frac{Y_t^\infty}{X_t^\infty + W_t} \quad , \quad \xi_t \equiv \frac{X_t^\infty}{X_t^\infty + W_t}$$

As in the case of a finite number of funds, the investor's value function can be decomposed into the effect of total wealth and the effect of wealth composition on the continuation utility:

$$F^\infty(W, X^\infty, Y^\infty) = (W + X^\infty)^{1-\gamma} H^\infty(\pi, \xi), \tag{17}$$

where the function $H^\infty(\pi, \xi)$ satisfies ODEs given in Online Appendix A.8. In contrast with the one-fund case, there is an optimal PE commitment for each level of PE investment.

Furthermore, the investor can sell a fraction ω of their invested capital at any time on the secondary market, if $F^\infty(W, X^\infty, Y^\infty) < F^\infty(W + \alpha\omega X^\infty, X^\infty(1-\omega), Y^\infty)$. This corresponds to selling complete positions in some set of the infinite funds.

We conduct our welfare analysis at the steady state of the aggregate investment X_t^∞, i.e., when $E[dX_t^\infty] = 0$.[13] We denote this steady state investment by X^∞. Using the laws of motion, the steady state aggregate commitment Y^∞ is linked to the investment X^∞ by:

$$\frac{Y^\infty}{X^\infty} = \frac{\lambda_D - \nu}{\lambda_C}, \tag{18}$$

Thus, the steady state ratio of committed to allocated capital is a simple function of the rates of calls and distributions and the mean PE return.

Because commitment-timing risk is eliminated with an infinity of funds, our study of the illiquidity stack collapses to the comparison between the baseline economy and Economy 2 in which the investor can choose their commitment at the time of a capital call. With an infinity of funds, Economy 2 allows an investor to immediately add to their PE assets (X_t^∞) to reach the optimal level of invested capital. Equation (14) becomes:

$$\frac{dX_t^\infty}{X_t^\infty} = \frac{dJ_t}{X_t^\infty} - \lambda_D dt + \nu dt + \psi^\infty dZ_t^\infty \tag{19}$$

Because negative calls are not allowed, an investor above their optimal allocation must wait for distributions for their invested capital to decline, or they must sell a fraction of their invested capital on the secondary market at a discount.

[13]One might also be interested in the steady state for the X^∞/W ratio: $E[d\ln(X_t^\infty/W_t)] = 0$. This change implies using a different location in $\{X, Y, W\}$ space to do the welfare analysis. The overall allocation to PE is slightly higher, with more capital committed and less invested, and the welfare and return premiums are almost the same.

4 Model Calibration & Portfolio Allocation

In this section we provide a detailed calibration of private equity return dynamics. We calibrate for economies with one-, two-, and an infinite- number of PE funds and provide the resulting portfolio and consumption policies in the baseline economy.

4.1 Model Calibration

We use the past thirty years of data (1991-2020) to calibrate our model. We use standard parameters for public market moments: $r = 0.03$, $\mu = 0.08$, and $\sigma = 0.15$. These correspond to the average 3-Month Treasury Bill and the mean and volatility of the S&P 500 index log returns at monthly frequency over the past thirty years. In addition, we use standard values for the investor risk aversion and discount factor: $\gamma = 4$ and $\delta = 0.05$. The discount for PE-funds secondary market sales is set to 13.8%, which is the average reported in Nadauld et al. (2019). Management fees during the commitment period are set to $f = 2\%$ of the committed amount (see Metrick and Yasuda (2010)). Our calibration of the private equity return dynamics uses the Preqin dataset with fund cash flows as of the end of year 2020. We select all private equity funds (venture capital, growth equity, leveraged buyout) focused on the US and raised between 1991 and 2015, so that we have at least five years of investment activity for each fund.[14]

We construct two cumulative distribution functions for fund cash inflows: i) the empirical one, and ii) the model-implied one. The former is derived directly from the Preqin dataset. Preqin records all cash inflows into any given private equity fund, without a distinction between fee payments and capital invested. The latter is calculated analytically and verified with simulations.

In our model, PE-fund cash *inflows* consist of the management fees during the commitment period plus capital called for investment at time τ_C. Assume that 1 dollar is committed to each one of the N funds. The wait from τ_0 to τ_C has an exponential distribution. Therefore, at any time t, the proportion of funds across simulations that have not called yet is $e^{-\lambda_C t}$. The total fees paid by these funds is $Nfe^{-\lambda_C t}$. Imposing the law of large numbers, the cumulative cash inflow across is the the sum of the cumulative fees paid by each fund until its capital call $\int_{\tau_0}^{t}(Nf)(e^{-\lambda_C t})dt = \frac{Nf}{\lambda_C}\left(1 - e^{-\lambda_C t}\right)$, plus the amount of capital already called is $(1 - e^{-\lambda_C t})N$. As $t \to \infty$, i.e. after all funds have exited their investments, the total cash inflows approach $N\left(\frac{f}{\lambda_C} + 1\right)$. Thus, the cumulative cash inflow at time t, as a fraction

[14] We select funds with a size of at least \$10 million, at least two capital calls, and at least two capital distributions. The resulting sample contains 1398 funds.

of the total, is

$$\frac{\frac{Nf}{\lambda_C}\left(1-e^{-\lambda_C t}\right)+(1-e^{-\lambda_C t})N}{N\left(\frac{f}{\lambda_C}+1\right)}=1-e^{-\lambda_C t},$$

which implies an exponential distribution with parameter λ_C.

We search for the λ_C that minimizes the least-square distance between the model-implied and empirical cumulative distributions. The best fit is obtained for $\lambda_C = 0.344$, which corresponds to an average commitment period of about three years. Panel A of Figure 1 shows that the two cumulative distribution functions are very close to one another, with an RMSE of 1.9×10^{-2}, which validates our modeling choice.

To calibrate the PE cash-*outflows*, we do not directly use the fund distributions observed in Preqin, as we do for the inflows. The reason is that it takes about fifteen years to observe the complete time-series of fund distributions, and we would then be restricted to using a sample of funds raised before 2005. Instead, we use the same sample as above – funds raised up to 2015 – and match the distribution of their performance as of end of year 2020. To measure fund performance, we adopt the most common measure: the Kaplan-Schoar Public Market Equivalent (PME). That is, for each fund, we compute the present value of cash inflows and cash outflows, each discounted using the realized S&P 500 index returns, and we value unexited investments at their reported Net Asset Value (NAV). As we are interested in the distribution of fund-level PMEs, we assign equal weight to all funds.

The model-implied PMEs are obtained by simulating the cash flows of 100,000 private equity funds using i) the return dynamics of equation (3), and ii) draws from Poisson distributions to trigger capital calls and distributions. There are four free parameters in our model: PE expected return (ν), PE volatility (ψ), the intensity of capital distributions (λ_D), and the correlation between private and public equity (ρ_L).

We choose the parameters $(\nu, \psi, \lambda_D, \rho_L)$ which minimize the least-square distance between the model-implied and the empirical cumulative distributions. The cumulative distribution of PMEs is shown in Panel B of Figure 1. The blue bars represent the empirical distribution and the red line the model-implied distribution.

The two curves are remarkably close, with an RMSE of 3×10^{-2} for PME $\in [0.5, 2.5]$, and the best fit is obtained with a combination of a relatively high ν and ψ:

$$\nu = 14\%; \quad \psi = 33.5\%; \quad \lambda_D = 0.174; \quad \rho_L = 0.66$$

Our calibrated parameters produce an implied private equity β of 1.47:

$$\beta = \rho_L \frac{\psi}{\sigma} = 0.66 \left(\frac{0.335}{0.15} \right) = 1.47.$$

This β is nearly the same as the 1.43 estimate obtained by Ang et al. (2018) using a completely different methodology. Ang et al. (2018) use Bayesian methods to extract the risk exposures that are most consistent with the observed panels of cash flows.

Our calibrated λ_D implies an average holding period of about 5.7 years, which is close to the median holding period of 5.3 years that Brown et al. (2020) find in a large sample of individual transactions.[15]

In the two-fund case, we use the set of parameters described above for the dynamics of each fund, but need to also calibrate the correlation between the two PE funds: ρ_{PE}. We draw 5000 portfolios of two funds each, taken randomly and with replacement from the Preqin dataset. We calculate the PME for each of these portfolios, and match the PME distribution to its model-implied counterpart again minimizing the least-squared distance between distributions. We obtain a correlation between PE funds of $\rho_{PE} = 0.68$.

We calibrate the infinite-fund problem by using analytic extensions of the parameters above. In the infinite-fund problem, the investor is continuously active and earning PE returns. We assume an equally weighted portfolio and take limits as the number of funds in the portfolio goes to infinity. The volatility and the correlation of the PE portfolio with the stock market can be calculated following equation (16):

$$\psi^\infty = \psi \sqrt{\rho_{PE}} = 0.335 \times \sqrt{0.68} \approx 0.276 \quad ; \quad \rho_L^\infty = \frac{\rho_L}{\sqrt{\rho_{PE}}} = \frac{0.66}{\sqrt{0.68}} \approx 0.80. \tag{20}$$

This correlation estimate matches the most commonly used estimate in practice, which is the one given by Blackrock on their capital market assumption webpage.[16]

Table 1 summarizes the parameter values that we use. It is remarkable that although we use a parsimonious model, we not only match the distributions of PMEs to their empirical counterparts, but also generate calibrated parameters that are in line with the literature. These results give us additional confidence in our model and the associated counter-factuals.

[15] They do not report an average, but the skewness of the distribution indicates that it would be higher than the median.

[16] https://www.blackrock.com/institutions/en-us/insights/charts/capital-market-assumptions. The model used by Blackrock to generate this correlation is not publicly available, but they also use PE fund cash flows to infer the correlation.

4.2 Portfolio Allocation in the Baseline Economy with 1 Fund

In this section, we solve the model with one fund using the calibrated parameters we just described. At time $t = \tau_0$ and at the end of each PE fund's life, the investor chooses the optimal commitment to private equity relative to total wealth, ξ^*. We refer to ξ^* as the optimal PE commitment. At each point in time, the investor chooses an allocation to stocks and bonds and a consumption level.

Each panel of Figure 2 displays a function that describes the solution to the model as a function of ξ. Recall that ξ denotes the amount *committed* to PE divided by total wealth during the commitment period, and denotes the amount *invested* in PE divided by total wealth during the holding period. We refer to ξ as the PE allocation in both cases.

The four functions displayed on the figure are: i) the value function solving equation (10) (Panel A), ii) the probability density function of the PE allocation ξ (Panel B), iii) the optimal consumption rate c (Panel C) and iv) the optimal allocation to public equity θ (Panel D). The dashed lines represent the functions during the commitment period and the solid lines the functions during the holding period.

4.2.1 The Private Equity Allocation

Following Proposition 1, the optimal PE commitment ξ^* is chosen so that it maximizes the value function that prevails during the commitment period. We illustrate the results in Figure 2. The investor optimally chooses to commit $\xi^* = 5.2\%$ of wealth.

During the commitment period, the investor's liquid wealth moves stochastically. This movement causes the PE allocation ξ to change as well, even though the amount committed to PE is fixed and PE investments have not been made yet. The investor's welfare moves along the dashed line (Panel A), going to the right if liquid wealth decreases (ξ increases) and to the left if wealth increases (ξ decreases).

The shape of the value function shows that during the commitment period the region around the optimal allocation is fairly flat (Panel A): as long as the PE allocation remains between 0% and 10%, welfare does not vary much. If the PE allocation grows beyond 10% of wealth, welfare decreases more rapidly. The investor strategically defaults on their commitment if the PE allocation reaches 21% of wealth during the commitment period (black circle). The consequence is a permanent loss of access to the private equity market. However, Panel B shows that the likelihood of reaching a 21% PE allocation (dashed line) is nearly zero, so default hardly ever happens. In fact, the PE allocation does not vary much during the commitment period and remains smaller than 7.3% with a 99% probability. Thus, strategic default is both very rare and also important for welfare: welfare decreases rapidly as the investor moves closer to default, but the investor almost never reaches default. This matches the institutional details from Section 2.2.

When capital is called, the value function jumps up to the solid black line (Panel A). At that point in time, the optimal PE allocation is 9.1%, compared to an earlier optimal capital commitment of 5.2% and a 99th percentile realized commitment value of 7.3%. Thus, the investor chooses an optimal initial commitment that results in being significantly under-allocated to PE with high probability. This is despite the fact that liquid wealth drifts up, meaning that the investor's commitment as a fraction of wealth declines from 5.2% on average. Given the desire to avoid default, finding under-allocation is not surprising, but the size of the under-commitment is striking.

A direct implication of under-allocation is that the demand for changing the amount allocated is asymmetric. Investors will not frequently exercise an option to reduce their allocation to PE, and they will not value that option. In contrast, they will highly value the option to top-up their allocation when capital is called. For example, they highly value the possibility to co-invest, which is consistent with the institutional details reviewed in Section 2.3.

The secondary market is hardly ever used to liquidate the PE position, even with a relatively small haircut. With a 13.8% discount, the investor sells their stakes on the secondary market only when the PE allocation ξ reaches 58.4% of wealth (black square on Panel A), which is a rare event. Perhaps unexpectedly, the investor also sells on the secondary market when the PE allocation is too low (below 0.5% of wealth) because they want more PE and to do so need to get out of current fund in order to commit to a new one (not pictured).[17]

Table 3 reports the welfare cost of changing the haircut in each economy. In the baseline economy, the investor is only willing to give up only 0.1% of wealth to decrease the secondary market haircut from 13.8% to 0%. These features are consistent with the institutional details in Section 2.3: secondary market volume is low relative to aggregate allocations.

4.2.2 Liquid Portfolio and Consumption Policies

Panel C of Figure 2 shows how the investor alters their consumption policy if they move too far from their optimal portfolio composition. During the commitment period (dashed line), consumption is relatively insensitive to the PE allocation, varying between 4.5% and 4.6%

[17]In our figures, we focus on upper-bound exits as they are more relevant and interesting, but lower-bound exits also exist in all one- and two-fund sub-models.

of wealth. In contrast, during the holding period (solid line), consumption is higher at the optimal portfolio composition, but more sensitive to the PE allocation, dropping to 4.2% before the investor accesses the secondary market.

Panel D describes the liquid asset allocation. The stock allocation fluctuates around 55.5% during the commitment period, i.e., 44.5% of the portfolio is invested in the bond. This large allocation to the bond would be only slightly smaller (44.2%) without PE. However, there is a near convexity at default, as is usual with default options, so the investor increases their stock allocation, which reaches 58.7% of wealth in a narrow region. In contrast, during the PE holding period, the investor's allocation to public market equity declines strongly and monotonically with their PE exposure.

Both consumption and liquid asset policies are consistent with private equity changing the concavity of the investor's value function. Capital calls are good news, and consumption is higher during the holding period than the commitment period. However, an unbalanced portfolio reduces the investor's welfare, and consumption is more sensitive to market movements during the holding period. The investor holds a portion of their liquid assets in the bond to finance their commitments, but gambles to avoid default. Once the investor's commitment has been called, they modify their stock holdings to control their overall investment risk exposure.

Online Appendix C describes the portfolio allocation of an investor with recursive utility instead of power utility. We find that the rate of consumption increases for lower elasticity of intertemporal substitution, in line with the intuition of a weaker desire to smooth consumption. But most importantly, the changes in the allocations to the risky assets are negligible.

4.3 Sensitivity of Portfolio Allocation to Model Parameters

We evaluate the sensitivity of the optimal PE commitment (ξ^*) to the key parameters of our model: PE expected return, volatility and correlation with the stock portfolio, investors' risk aversion, secondary market liquidity, and the intensity of capital calls and distributions.

We begin by varying parameters associated with asset returns, comparing how outcomes change in our model to how they would change in a Merton two-asset model, i.e. a model in which PE is completely liquid and traded alongside public equity. The results are shown in Figure 3, Panels A to D. The solid line corresponds to our model, while the dashed line corresponds to a Merton two-asset model. In all four cases, PE allocations are less sensitive to parameter changes than they would be in a fully liquid model.

As expected, the optimal PE commitment is sensitive to PE expected returns. When PE expected returns are smaller than 12.5%, the optimal commitment to PE is zero. Above that threshold, the PE commitment increases linearly with expected returns: each additional percentage of expected return increases the allocation by 2.2%. However, if PE were a fully liquid asset, the increase would be nearly twice as fast: 3.8% for each additional 1% return.

This striking result can be explained using the specific institutional features of private equity. The key point is that private equity cannot be easily exited. Hence, to achieve the optimal portfolio composition in the PE holding period, the investor needs to account for the drift over time in the value of invested capital. The higher the expected return on PE, the more the value of invested capital drifts up, the more the investor pulls down their initial commitment. This effect reduces the sensitivity of the optimal allocation to returns. Higher returns make PE funds more desirable and pull up the optimal allocation, but not as much as when the investment is liquid.

A similar result holds for the PE return volatility (Panel B): the optimal PE commitment is much less sensitive to volatility than if PE were fully liquid. In the baseline model, varying the volatility of private equity from 40% to 20% changes the optimal commitment from 0% to 20% of wealth. In the fully liquid model, the corresponding allocation varies from 6% to 50% – the allocation is higher and varies more. Similar results hold for changes to the correlation between private equity and public markets and for changes in the investor's risk aversion, as shown in Panels C and D.[18]

The intuition for these results relies on the stochastic illiquidity of private equity. Stochastic capital call and distribution times generate unspanned risks, making private equity a bad hedge for public market risk. More, the investor cannot easily exit their investment and so is less willing to take larger positions, even when opportunities are better. Thus, the allocation to PE is lower and less sensitive to changes in parameters.

We next examine the sensitivity of the optimal PE allocation to three liquidity parameters; λ_C, λ_D and the secondary market haircut. The optimal commitment is increasing with respect to the intensity of capital calls (λ_C; Panel E). Increasing λ_C amounts to making the commitment period shorter on average and less risky, reducing both commitment risk and commitment delay. In the cases of one-year and a five-year average commitment periods, the optimal PE commitments are respectively 7.6% and 3.1%. In the limit of λ_C going to infinity, the economy reduces to Economy 3, in which there is no commitment period and the capital is invested at commitment time. The optimal PE allocation is then 9.1% of wealth.

[18] The correlation result is consistent with Ang et al. (2014), who examine a model of illiquidity without commitment, and the risk aversion result is similar to the result in Giommetti and Sorensen (2020).

The optimal PE commitment is hump-shaped with respect to the intensity of capital distributions (λ_D; Panel F). It is slightly increasing for low intensities, i.e., holding periods that are longer than 6.5 years on average, and decreasing for higher intensities, i.e., holding periods that are shorter than 6.5 years. High intensities (short holding periods) are associated with short active investment times – for any given level of commitment risk or delay, the investor earns the high PE returns for only a short time. This leads allocations to decrease as λ_D increases (holding periods become shorter), and it is the dominant effect when λ_D is high. However, a different effect dominates when λ_D is low (holding periods are long). In this region, there is a higher realized average allocation to PE for any given level of commitment because the illiquid asset has time to appreciate in value. This means that a small commitment to PE is optimal if the holding period is sufficiently long, and this is the dominant effect when λ_D is low.

Finally, the optimal PE commitment increases slightly when the secondary market becomes more liquid: from 5.2% in our base calibration (haircut of 13.8% of investment value) to 6% with a fully liquid secondary market (zero haircut). Similarly, the allocation decreases slightly when the secondary market becomes more illiquid than the baseline case. The direction of this effect is clear – the investor makes larger allocations when PE is easier to exit – and the magnitude is consistent with our earlier results that the secondary market is relatively unimportant for welfare.

4.4 Portfolio Allocation with Two Funds

We solve the model with two funds described in Section 3.6. This was, we can analyse the diversification benefits from expanding the opportunity set from one fund to two. Intuitively, the investor takes advantage of liquidity diversification by spreading their allocation across the funds. Doing so allows them to smooth their capital inflows and outflows. As with all risk-averse investors, smoothing wealth variation should result in higher total risky asset allocation. Similarly, the investor can diversify asset return risk because the two private equity funds have only partially correlated cash flows.

Despite this standard argument for liquidity diversification, there are two effects that indicate that diversification will be less effective here. First, fluctuations in public equity prices affect *both* fund allocations because liquid wealth is the denominator of both ξ^1 and ξ^2. The investor cares about the fraction of wealth they have committed to both funds, and those fractions are perfectly correlated through the denominator. Second, the investor faces a *funding mismatch* from the random timing of calls and distributions. The investor cannot reliably smooth capital calls and distributions because their order in time is random.

We find that the investor does benefit somewhat from liquidity diversification, but the effect is greatly reduced by the correlated shocks and the potential for funding mismatch. Neither liquidity shocks nor cash flows diversify as easily as the simple intuition would imply.

Figure 4, Panel A shows the optimal commitment to the second fund as a function of the existing commitment to the first fund. The dashed line illustrates the relationship when the first fund is in the commitment period, the solid line when the first fund is in the holding period. If both funds are in the commitment period, the welfare maximizing commitment is 4.4% in each fund, or 8.8% total. This is an increase over the 5.2% commitment when there only exists one fund. After time zero, when a new fund is raised, the larger the allocation to the ongoing fund, the less the investor optimally commits to the new fund. If the ongoing fund has already called the committed capital (solid line), there is a chance that it will distribute capital back to the investor before the new fund's call. Thus, the investor commits more capital when the ongoing fund is in the holding period rather than the commitment periods.

Panels B and C illustrate the effect of correlated denominator risk. Panel B shows two comparative statics in the commitment period. We start by keeping the allocation to fund 2 at its optimal level and varying the allocation to fund 1 (thin line). The penalty for having an allocation to fund 1 that becomes too large (moving right on the thin line) is small. Then, we force ξ^2 to vary together with ξ_1 ($\xi^2 = \xi^1$) to reflect the joint effect of a change in the stock price on fund allocations (thick line). The value function becomes more concave, with a larger penalty when the allocation to fund 1 increases. Default is triggered earlier and welfare is lost from the correlated shocks. These results are even more striking when both funds are in their holding period (Panel C). Accounting for the correlated shocks ($\xi^2 = \xi^1$, thick line versus thin line) triggers earlier access to the secondary market and a much more severe loss of welfare.

Finally, in Panel D, we analyze the funding externality created by the second fund on the value of the first one. We look at the investor's utility change when their commitment to fund 1 is called, depending on whether the second fund is in the commitment period (dashed line) or holding period (solid line). When the second fund is in the commitment period, the capital call is mostly *good news* – the welfare change is positive and the investor has smoothed the calls and now has one fund in the commitment period and one in the distribution period. However, when the second fund is in the holding period, the capital call is *bad news* – the welfare change is negative because both funds have called their capital, putting a double strain on the investor's liquid wealth.

4.5 Portfolio Allocation with an Infinity of Funds

We solve the infinite-fund problem described in Section 3.7. Figure 5 represents the optimal commitment to the new funds that are continuously being raised, $\frac{Y^{\infty *}}{W}$, as a function of the capital that is already invested in PE, $\frac{X^{\infty}}{W}$. Putting numbers into (18), we see that the steady state is reached when committed capital is 9.9% of invested capital:

$$\frac{Y^{\infty}}{X^{\infty}} = \frac{\lambda_D - \nu}{\lambda_C} = \frac{.174 - .14}{.344} = .099, \tag{21}$$

This corresponds to the point at which 19.8% of wealth has been invested and 2% of wealth has been committed (dotted vertical line). The total PE allocation (i.e., the sum of the invested amount and the new commitment) amounts to 21.9% of total wealth.

As in the two-fund problem, the aggregate commitment and the aggregate investment as functions of liquid wealth – $\frac{Y^{\infty}}{W}$ and $\frac{X^{\infty}}{W}$ – are correlated. However, unlike in the two-fund case, in the infinite-fund limit, calls and distributions occur continuously and deterministically, so there is no funding externality.

Most importantly, the investor's ability to manage their investment timing means that they can sustain levels of investment that are much higher than their levels of commitment. With one fund, there was no such control; the optimal commitment was 5.2%, the optimal invested allocation was 9.1%, and the investor was under-allocated. In contrast, with an infinity of funds, commitments can be kept around the steady state value of 9.9% of invested capital (2% committed versus 21.9% invested). The amount of commitment required to sustain a given level of investment is therefore much lower than in the one-fund case.

5 Timing and Quantity Risks

In this section, we quantify the impact of the different liquidity frictions on the investor's welfare, solving the investor's problem in the five economies described in Section 3.4. We show that commitment-timing risk – associated with uncertainty in the timing of the capital call – carries a small cost. On the other hand, commitment-quantity risk – associated with liquid asset movements during the commitment period – carries a large cost. We then investigate how the latter cost changes with the ability to diversify commitments across PE funds.

5.1 Timing Risk in the One-Fund Problem

We start by examining the difference between Economies 0 and 1, i.e., we turn off call timing risk by making the capital call time deterministic instead of stochastic. Results in Table 2 and Figure 6 show that eliminating timing risk has little impact on the optimal commitment, consumption policy, or stock allocation. The welfare cost and return premiums associated with commitment-timing risk are close to zero. In fact, Figure 7 shows that for larger values of the subjective discount factor δ, the welfare cost becomes negative, implying that the agent *prefers* uncertainty about the timing of capital calls.

Two competing forces can explain this change of sign. On the one hand, when capital is called, the value function jumps up, as shown by Panel A of Figure 2. A stochastic capital call time implies that there is uncertainty about the timing of this utility gain. The key fact is that the expected present value of a utility gain is *increasing* in uncertainty over its timing because discounting, $e^{-\delta t}$, is a convex function of time. Jensen's inequality implies $\mathbb{E}e^{-\delta\tau_C}U(1) > e^{-\delta\mathbb{E}[\tau_C]}U(1)$. This feature pulls the cost of commitment risk down, even to being negative, and the effect is increasing in δ.

On the other hand, a stochastic capital call time induces uncertainty in the ability of the investor to fund both the capital call and intermediate consumption in the commitment period. The investor is more likely to default on the call or be forced to reduce consumption as the commitment-to-liquid-wealth ratio accumulates variation. This effect may be small if the commitment delay cannot become too large. It is driven by variations in ξ which are small during the commitment period (Figure 2, Panel B).

Importantly, whether it is positive or negative, the cost of timing risk remains close to zero, of the order of 10^{-4}.

5.2 Quantity Risk in the One-Fund Problem

Next, we compare Economies 0 and 2, meaning we enable the investor to adjust their committed amount at the moment the capital call occurs. This ability includes both upward and downward adjustments. This comparison allows us to evaluate the impact of quantity risk. Results in Table 2 show that switching off quantity risk leads to a significant increase of the PE allocation: from 5.2% to 9.2% of total wealth. The rest of the policies are hardly affected: consumption increases only slightly (Figure 6, Panel B), and the stock-bond split is unaffected (Panel D).

Along with the increase in allocation, there is a corresponding welfare gain of 1.21% of wealth (Table 2, Panel A). This amount is large: it corresponds to 23% of initial commitment to PE. Equivalently, the investor is willing to give up a return premium of 1.05% out of PE's expected return, forever.

5.3 Interaction between Timing and Quantity Risks

We next examine the interaction of timing and quantity risks. A priori, one might think this interaction effect could be large. The reason is that the variation in the PE allocation during the commitment period depends on both the volatility of liquid wealth and the length of the commitment period. The longer the commitment period, the more the PE allocation can fluctuate away from its mean with fluctuations in the stock market, and the larger the quantity risk. On the other side, a random call time also makes it possible to have a short commitment period, during which the PE allocation will not move much away from its optimal initial level, inducing low quantity risk. Is it more valuable to be able to adjust the PE allocation at call time if this time is random or deterministic?

To quantify the interaction of quantity and timing risk, we solve the investor's problem in Economy 3, which has a deterministic commitment period with the ability to adjust commitments at the time of the capital call. Thus we have turned off both quantity and timing risk. Comparing Economies 0 and 1 means turning off timing risk while quantity risk is on. Comparing Economies 2 and 3 means turning off timing risk while quantity risk is off. The welfare differences are similar and almost zero, with 0.01% welfare cost and return premium (Table 2, Panel A). Therefore, interestingly, there is no interaction between timing risk and quantity risk: timing risk carries a negligible premium irrespective of whether quantity risk is present in the economy. This reinforces the conclusion that commitment-timing risk and commitment-quantity risk are two independent features.

5.4 Removing the Commitment Period

Next we remove the commitment delay entirely (Economy 4), so that the investor's capital is invested as soon as it is committed. This allows us to make two useful comparisons. First, by comparing Economy 4 to the baseline (Economy 0), we can assess the overall effect of capital commitment on asset allocations and welfare. Second, by comparing Economies 2 and 4, we can decompose the impact of capital commitment into the impact of the commitment *delay* versus the impact of commitment *risk*.

Interestingly, the optimal portfolio allocation changes between Economy 0 and 2, but it is very similar in Economies 2 and 4. This means that the commitment delay has almost no effect on the allocation once quantity risk has been removed. The investor increases their consumption slightly, in anticipation of the distributions that they will now receive earlier. The stock-bond split does not change. Thus, it is commitment-quantity risk, not the commitment delay, that impacts the PE allocation.

In contrast to the allocation decision, removing the commitment delay does have a large effect on welfare. The investor is willing to pay 2.77% of their wealth to switch from Economy 0 to Economy 4, i.e., to remove the commitment delay. This is more than twice the amount they are willing to pay to switch from Economy 0 to Economy 2, i.e. to adjust their allocation at call time. Equivalently, the return premium associated with moving from Economy 0 to Economy 4 is 2.08%, almost exactly twice that of moving to Economy 2. This premium is an effect of increasing active investment time: more time is spent with assets invested in PE, as opposed to waiting in the commitment period, and this change has a large effect on welfare.

5.5 Distribution Timing Risk

In this section, we draw a comparison between timing risk applied to commitment and applied to distribution. To do so, we solve Economy 5, which has the baseline level of commitment risk, but a deterministic holding period: the time between the capital call and distribution is fixed at $\frac{1}{\lambda_D}$. We find that distribution-timing risk is meaningful, and substantially more important than commitment-timing risk, but less impactful than commitment-quantity risk.

Table 2 shows that the optimal PE commitment increases from Economy 0 to Economy 5, from 5.2% to 6.8%. The investor also increases their consumption during the commitment period (Figure 6, Panel A), and they default with higher PE allocations than in the baseline economy. However, the stock-bond allocation remains similar to the one in the baseline economy.

Distribution timing risk has a positive cost: the investor is willing to pay 0.43% of their initial wealth to remove this risk, or equivalently to accept a permanent decrease in the PE fund return of 0.41%. This result is in line with what is documented in Ang et al. (2014): The investor prefers certainty in the timing of distributions. However, the welfare cost is less than half the cost of commitment-quantity risk, highlighting the importance of accounting for commitment risk.[19]

The magnitude of the cost of distribution timing risk is very different from that of commitment-timing risk. When offered a deterministic call time (Economy 1), the investor

[19]The discrepancies between our values of welfare costs and the ones presented in Ang et al. (2014) have two sources. First, the Ang et al. (2014) model does not have a commitment period. Second, our parameters are chosen from a detailed calibration to private equity data described in Section 4.1.

may prefer receiving the utility gain at a stochastic time because discounting is convex and the magnitude of the utility gift does not vary much with ξ (Figure 2, Panel A). With deterministic holding periods (Economy 5), the distributions that are received are stochastic, and depend on the evolution of the Geometric Brownian motion driving the PE dynamics. Because ξ is very volatile during the holding period, so is the value of these distributions. The investor then prefers certainty on the distribution timing, which allows them to better handle their consumption stream.

5.6 Changing model parameters

Investors require negligible compensation to bear call timing risk, but the compensation for quantity risk is substantial: the investor is willing to pay 1.21% of initial wealth (23% of PE commitment), or, equivalently, to lose 1.05% in PE returns to remove this risk. We now assess the sensitivity of this result to changes in parameters.

Figure 8 illustrates the sensitivity of the welfare costs (solid lines) and return premiums (dashed lines) of quantity risk to different model parameters. The investor is willing to pay more to eliminate quantity risk when PE expected returns increase, PE volatility decreases, when the correlation between the stock and PE decreases or when their risk aversion decreases. However, the increase in the welfare cost closely follows the increase in the optimal PE allocation displayed on Figure 3. In contrast, there is very little compensation required per unit of allocation when any of these four parameters changes – the return premium does not vary much when changing the model parameters. We conclude that the change in welfare cost induced by a change of these parameters operates mostly through the allocation channel, meaning it is driven by a substitution effect.

The welfare premium associated with quantity risk is negatively associated with λ_C. This is intuitive: when the intensity of capital calls is higher, the commitment period is shorter and all forms of commitment risk are lower. Thus, portfolio allocations increase (Figure 3, Panel E) while the premiums associated with alleviating quantity risk decline (Figure 8, Panel E).

In contrast, the welfare premium on quantity risk only weakly increases as distributions become more intense (λ_D increases), but the return premium increases more quickly. To understand the small welfare premium, we observe that a shorter holding period implies that the desirable PE returns and the undesirable risk are both reduced, roughly in proportion to keep the investor's overall optimal allocation flat (Figure 3, Panel F). The larger return premium changes stem from active time: as λ_D increases, the holding period becomes shorter and the economy spends more time in the commitment period. Since private equity returns are only relevant in the holding period, the return premium must be correspondingly higher when the holding period is shorter.

All results still hold with recursive preferences. In particular, the welfare costs of quantity risk are reported for different values of the intertemporal elasticity of substitution in Online Appendix C.

5.7 Secondary Market & Quantity Risk Complementarity

In our model, the secondary market allows investors to sell their PE investment at any time during the holding period, with a haircut of 13.8% of investment value. While the secondary market is valuable, the actual welfare benefits are small: from Table 3, in the baseline economy changing the secondary market haircut from 0% to 40% generates a welfare cost of only 0.16% of wealth.

Surprisingly, the secondary market has a larger *indirect* effect: it raises the premiums associated with capital commitment, particularly the premiums associated with quantity risk and the commitment delay. Table 2, Panels A-C, show how the compensation for the different liquidity risks changes if the secondary market is perfectly liquid (haircut of 0%), or illiquid (haircut of 40%). The investor is willing to pay more to remove quantity risk when the secondary market has a smaller haircut (is more liquid). The welfare cost of quantity risk increases from 1.21% to 1.53% as the haircut declines from 13.6% to 0%. The return premium associated with quantity risk also increases from 1.05% to 1.49% as the secondary market haircut declines. The willingness to pay to eliminate the commitment period increases as well, with a cost going from 2.77% to 3.21%.

Similarly, improving the liquidity of the secondary market increases welfare more when other commitment frictions are lower (Table 3).

Putting the two results together, we conclude that the two types of liquidity – ease of commitment and ease of access in the secondary market – are complements, not substitutes. Increasing liquidity along one dimension increases the willingness to pay to remove frictions (welfare cost) along the second dimension. Moreover, the indirect effect of the secondary market on the welfare cost of capital commitment is larger than the direct welfare effect of the secondary market itself. Thus, our model implies that the development of a PE secondary market would increase investors' desire to alleviate other liquidity frictions, rather than satiate that desire. Intuitively, the investor is more willing to invest in PE if they have an easier out, so this out becomes more valuable when the other PE frictions are absent.

5.8 Is Commitment Risk Diversifiable? Multiple PE Funds

We now assess the investor's ability to diversify away the costs of capital commitment. As discussed in Section 3.6, there are two primary ways in which splitting commitments across funds could benefit the investor. First, diversification of timing shocks and the ability to fund capital commitments from another fund's distributions; second, diversification of cash flow shocks. There were two corresponding reasons to think diversification would not be so useful. First, there is a potential funding mismatch generated by random call and distribution times. Second, public market shocks (denominator risk) impact all portfolio composition ratios at the same time.

To examine the impact of diversification, we solve the baseline economy with two funds, then we remove quantity risk and then the commitment delay (so, we solve Economies 0, 2, and 4 with two funds). Table 4 displays the optimal PE commitment to each PE fund in the two-fund case, in the baseline economy and in Economies 2 and 4.

First, diversification (access to a second fund) is only moderately valuable to the investor. They would be willing to give up 1.04% of their wealth, or accept a permanent reduction in PE returns of 0.91% in order to gain access to a second fund. The corresponding welfare value of going from two funds to an infinity of funds is 5.43% of wealth or a return premium of 2.85%.

Second, the investor with access to two funds is willing to pay more to eliminate quantity-risk than the investor with access to one fund. The welfare cost is 1.64% with two funds and 1.21% with one fund. However, the return premium is lower: 0.89% with two funds versus 1.05% with one fund. This difference is a result of the differing allocations: with two funds, the investor commits more in total to PE, so the investor is willing to pay more to eliminate frictions. However, the welfare premiums associated with quantity risk overall are not too different between the one-fund and two-fund economies, and the return premium is only slightly lower with multiple funds.

Importantly, the welfare gain from going from one fund to two funds (0.91% return premium) is less than the gain from eliminating commitment-quantity risk in the one-fund economy (1.05% return premium). So, the ability to adjust committed capital is worth more than access to a second fund and the corresponding ability to diversify both liquidity shocks and cash flow shocks. However, there is a larger value in going from two funds to an infinity of funds (equal to a 2.85% return premium) as the law of large numbers removes the stochastic feature of capital calls.

Table 5 displays the welfare cost of quantity risk as well as the return premium in the limiting case of an infinity of funds. We see similar results as in the two-fund case. The investor makes more commitments to private equity and obtains a higher allocation. The welfare premium associated with eliminating quantity risk is large (4.32% of wealth). However, because the allocation is so large, the return premium is only slightly smaller with an infinity of funds (0.74%) than one fund (1.05%). This reduction likely reflects a removal of the funding mismatch – it is no longer the case that some funds can call or distribute early relative to the average. However, public market movements still drive volatility in the fraction of wealth that is committed, and that risk is removed when commitments can be adjusted.

Finally, note that our results likely *overstate* the value of diversification because our timing shocks are purely idiosyncratic – they are driven by independent Poisson processes. Empirically, Robinson and Sensoy (2016) show that the timing of capital calls and distributions is only mostly idiosyncratic.[20] We conclude that diversification is only a moderately useful tool when managing PE liquidity shocks.

5.9 Summary and Discussion

We have broken that risk down into three parts: a fixed delay before call, variation around the fixed delay, and public market movements before the call. In principle, all three components might impose welfare losses on investors, which would require compensation. In fact, public market movements – the source of commitment-quantity risk – and the average delay before a capital call are costly for investors. Variation around the average delay – commitment-timing risk – is not.

For both one and two funds, investors under-commit – they respond to commitment risk by reducing their optimal commitment level below their optimal investment level. Once capital has been called and commitment risk has been resolved, the welfare maximizing allocation is higher. This means that investors will value a top-up option. Even without under-commitment on average, investors are subject to public market movements during the commitment delay, and so they will value the ability to change their investment when capital is called.

The desire to change capital allocations on call – most often to increase the allocation

[20]From Robinson and Sensoy (2016): "most variation in fund-level cash flows is purely idiosyncratic across funds of a given age at a given point in time (...) this suggests that liquidity shocks arising from the uncertain timing of calls and distributions can be significantly mitigated by holding a portfolio of investments diversified both across different funds of the same age and across funds of different ages. For example, for buyout funds the standard deviation of quarterly net cash flows averages 11.57% of committed capital, and this standard deviation shrinks to 4.54% in a portfolio of all buyout funds in the sample."

– is consistent with institutional development within the PE sector. As discussed in Section 2.3, Lerner et al. (2021) show that investments made with little or no pre-commitment ("discretionary vehicles") grew to about 15% of all PE commitments by 2015. In addition, Braun et al. (2020) describe their sample as one in which the demand for co-investment opportunities, which is a subset of discretionary opportunities, exceeds availability in many cases. In other words, institutions have developed that allow some investors to alleviate commitment-quantity risk for themselves, separate from diversification techniques.

A natural question that arises is "Why do LPs not simply diversify the problem away?" Making multiple different capital commitments reduces the size of each commitment and call, removing the lumps. Instead, we have shown that this diversification is only weakly helpful in reducing commitment risk. Investors care about the fraction of their wealth that they have committed, and public market movements change all the denominators at once, so the key risk – commitment-quantity risk – cannot be diversified. Worse, without perfect and full diversification, there is always a potential funding mismatch (externality) in which holdings in one fund impact the welfare value – through future capital calls – of commitments in other funds.

6 Liquidity Cycles

Leibowitz and Bova (2009) report that in 2008 "The horrendous declines presented liquidity problems even for many portfolio managers who were long-term oriented, had modest payment schedules, and a seemingly ample percentage of liquid assets. This perfect liquidity storm, layered on top of a perfect asset storm, resulted from a toxic combination of: 1) a need to fulfill prior commitments to private equity, venture capital, real estate, and hedge funds, 2) reduced distributions from these asset classes..." In this section, we examine how time varying liquidity impacts our core results. Specifically, we examine the impact of adding liquidity cycles on commitment risk.

6.1 Model and Calibration

In our baseline model, the Poisson processes triggering capital calls and distributions are independent of each other and independent of PE fund returns. However, our setup enables us to relate the intensity of capital calls and payouts to the (expected) performance of the fund. To create this relationship, we incorporate time-varying liquidity and asset return moments.

At any time t, the economy can be in one of two states $s_t = \{L, H\}$. State L corresponds to periods of low liquidity and state H to high liquidity. The state of liquidity s_t follows a continuous time Markov process with a transition probability matrix between t and $t + dt$ given by

$$M = \begin{pmatrix} 1 - \chi^L \, dt & \chi^L \, dt \\ \chi^H \, dt & 1 - \chi^H \, dt \end{pmatrix}. \tag{22}$$

We assume that the probability of being in the bad state is 30%, which corresponds to three bad years out of ten. We choose extreme parameter values in order to analyze the effects of severe liquidity cycles on our results. We characterize the low liquidity state by lower expected PE returns, higher PE volatility and a higher correlation between PE and listed assets and between PE funds. With an infinity of funds, we adjust the correlation between PE and the stock market as well as the volatility of PE as in equation (16). Calls and distributions occur more rarely: the average commitment period becomes 3 years (instead of 2.9 years in the baseline model) and the average holding period 7 years (instead of 5.7 years).

To obtain economies that are comparable with and without cycles, we set the parameters in the high liquidity state so that their state-weighted average match their value absent cycles. In line with Nadauld et al. (2019), we set the secondary market haircut to 28% in the low liquidity state and 9% in the high liquidity state. Our chosen parameter values are summarized in Table 6.

6.2 Effect of Liquidity Cycles on Quantity Risk Diversification

Results in Table 7 show the impact of liquidity states on private equity allocations and welfare for the one-fund, two-fund and infinity-of-fund models. Panel A displays the allocations, which are given in an economy without cycles (repeating our earlier results) and compared to the allocations in the low and high liquidity states of an economy with cycles. We also compute the time-weighted average allocation across high and low states. As one expects, the PE allocations are higher in the good liquidity state and lower in the bad liquidity state; the weighted average PE allocations obtained with cycles are comparable to those without cycles when the investor has access to one or two funds. It is slightly lower with an infinity of funds.

Panel B displays the welfare cost as well as the return premium of switching from the baseline economy (Economy 0) to an economy without commitment-quantity risk (Economy

2). With one and two funds, both the welfare cost and the return premium are larger in both liquidity states than the corresponding values in an economy without cycles.

Figure 9 illustrates why commitment-quantity risk and the compensation investors require for it are exacerbated by liquidity cycles. We plot the investor's value function, before and after a capital call, in both liquidity states. Points A and B are low- and high-liquidity optimal allocations during the commitment period, while C and D are welfare maximising allocations during the holding period.

With time-varying liquidity, the investor is willing to pay a premium to avoid commitment-quantity risk for two reasons. The first reason is the same as in the baseline model: the investor allocates less during the commitment period than they would like to have during the holding period. Commitment risk leads to under-allocation in the absence of cycles, and under-allocation-on-average is still present with cycles. The second reason is a direct effect of cycles: when liquidity levels change, so does the optimal allocation, and the investor is willing to pay to adjust it. This effect drives up the welfare premium in both the high and low states.

Our result – that the welfare costs and the return premiums are *larger* in *both* liquidity states than in an economy without cycles – is stronger with two funds than with one. Liquidity diversification relies on capital calls and distributions being spread over time, such that calls can partly be funded by distributions. In the low liquidity regime, the intensities of capital call and distribution are low, but the intensity of distributions decreases more than the intensity of calls. This difference means that the investor, on average, faces additional capital outflows in the low-liquidity period. This effect makes it more difficult to use the proceeds of the previous fund when the capital of the current fund is called. Therefore, in the low liquidity regime, liquidity diversification is more hampered by the funding mismatch.

The relative sizes of welfare costs and return premiums for commitment-quantity risk change with an infinity of funds. Capital calls and distributions are deterministic, so the funding externality has vanished. Instead, the investor experiences asymmetrically adjustment speeds. When transitioning from the low to the high state, the investor can increase their allocation immediately by increasing their commitment. When transitioning from the high to the low state, the investor faces slow adjustment: The committed amount slowly declines at rate λ_C, and the investment slowly declines at rate $\lambda_D - \nu - \lambda_C \frac{Y_t^\infty}{X_t^\infty}$. This makes the ability to arbitrarily adjust allocations much more valuable in the low state.

We conclude that liquidity cycles exacerbate commitment-quantity risk. This is true in both states of the economy when the investor has access to a small number of funds, and in the low liquidity regime with an infinity of funds. However, even with an infinity of funds and a high liquidity regime, the welfare cost of commitment-quantity risk is still 1.96% of total wealth.

7 Conclusion

This paper proposes an optimal dynamic portfolio allocation model that includes capital commitments and distributions with uncertain timing. We calibrate this model to cash flow data and show that ex-ante commitment has large effects on investors' portfolios and welfare. Investors are under-allocated to PE and willing to pay a large premium to adjust the quantity committed upon capital call. With one fund, this premium amounts to 1.21% of the investors' initial wealth, or equivalently to a permanent loss of 1.05% of private equity returns. It is larger than the premium investors are willing to pay to eliminate other liquidity frictions: timing uncertainty and the limited tradability of PE investments. Furthermore, commitment risk premiums increase with secondary market liquidity, and increasing the number of funds does not allow the investor to diversify commitment risk, particularly when liquidity is time varying. Our results also provide a new explanation to recent developments of the PE market such as the growth in LP co-investments.

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Table 1: Calibrated Parameters

This table displays the values of parameters obtained from the calibration described in Section 4.

Panel A: Calibrated parameters in the model with a finite number of funds

Parameter	Symbol	Parameter value
Risk-free rate	r	0.03
PE expected returns	ν	0.14
Stocks' expected returns	μ	0.08
PE fund volatility	ψ	0.335
Stocks' volatility	σ	0.150
Correlation stock & PE	ρ_L	0.66
Correlation between PE funds	ρ_{PE}	0.68
Intensity of capital call	λ_C	0.344
Intensity of capital distribution	λ_D	0.174
Secondary market haircut	α	13.8%
Investor's time discounting	δ	0.05
Investor's risk aversion	γ	4
Fee on commitment	f	2%

Panel B: Calibrated parameters in the model with an infinite number of funds

Parameter	Symbol	Parameter value
Correlation between stock & PE portfolios	ρ_L^∞	0.80
PE portfolio volatility	ψ^∞	0.276

Table 2: Optimal PE Allocation & Liquidity Frictions

The baseline economy (E0) is our central model, i.e. an economy in which investors need to commit ex-ante on the amount that will be called, wait for capital to be called and receive capital calls and distributions at random times. These frictions are removed one at a time in Economies 1 through 4, as described in Section 3.4. Economy 5 contains commitment risk, but the distribution time is deterministic. As defined in Section 3.5, the welfare cost is the amount investors are willing to pay to switch from E0 to a given economy, i.e. the willingness to pay to remove a given friction. The return premium is the additional return that PE should deliver (ν) in Economy 0 for the economy under consideration to be equivalent to E0, i.e. the return premium associated with a given friction. Panel A shows the results when the haircut on the secondary market is 13.8%. Panels B and C show the results when the haircut is 0% and 40%, respectively.

Panel A: Default secondary market haircut of 13.8%

		PE allocation	Welfare cost	Return premium
E0	Baseline (All frictions)	5.23%		
E1	Deterministic call time	5.18%	0.01%	0.01%
E2	Choose quantity on call	9.16%	1.21%	1.05%
E3	E1 ∩ E2	8.32%	1.22%	1.06%
E4	No commitment period	9.07%	2.77%	2.08%
E5	Deterministic payout time	6.79%	0.43%	0.41%

Panel B: Secondary market haircut of 0% (liquid market)

		PE allocation	Welfare cost	Return premium
E0	Baseline (All frictions)	5.96%		
E1	Deterministic call time	5.90%	0.01%	0.01%
E2	Choose quantity on call	12.12%	1.53%	1.49%
E3	E1 ∩ E2	10.72%	1.54%	1.50%
E4	No commitment period	12.62%	3.21%	3.06%
E5	Deterministic payout time	6.99%	0.33%	0.33%

Panel C: Secondary market haircut of 40% (illiquid market)

		PE allocation	Welfare cost	Return premium
E0	Baseline (All frictions)	4.98%		
E1	Deterministic call time	4.93%	0.01%	0.01%
E2	Choose quantity on call	8.46%	1.12%	0.95%
E3	E1 ∩ E2	7.80%	1.14%	0.97%
E4	No commitment period	8.37%	2.58%	2.15%
E5	Deterministic payout time	6.79%	0.47%	0.40%

Table 3: Secondary Market

This table reports the marginal impact of changing the secondary market haircut on investor welfare, in a given economy. The welfare cost of the default haircut, i.e., the amount that the investor is willing to pay to change this haircut from 13.8% to 0% (resp., from 13.8% to 40%) is displayed in the second (resp., third) column. Each row references an economy with a different liquidity friction.

		Default (h=13.8%)	Liquid (h=0%)	Illiquid (h=40%)
E0	Baseline economy	0	0.12%	-0.04%
E1	Deterministic call time	0	0.12%	-0.04%
E2	Choose quantity on call	0	0.45%	-0.13%
E3	E1 $\cap$ E2	0	0.44%	-0.13%
E4	No commitment-to-call time	0	0.56%	-0.24%
E5	Deterministic payout time	0	0.02%	0.00%

Table 4: Two-Fund Allocations, Welfare Costs and Return Premiums

Panel A reports in the first column the total optimal PE commitment in Economy 0 (baseline economy; first line), Economy 2 (the investor can update their commitment upon capital call; second line) and Economy 4 (no commitment period; third line), when the investor has access to two private equity funds. Columns 2 and 3 display the welfare costs and return premiums of Economies 2 and 4 compared to the baseline economy. Panel B reports the welfare cost and return premium of having access to one fund instead of two.

Panel A: Two-fund allocations and costs of commitment-quantity risk

	PE allocation	Welfare cost	Return premium
Baseline economy	8.84%		
Choose quantity on call	12.60%	1.64%	0.89%
No commitment period	11.84%	4.08%	1.96%

Panel B: Benefits of going from one fund to two

	Welfare cost	Return premium
Baseline economy	1.04%	0.91%

Table 5: Allocation, Welfare Cost and Return Premiums for an Infinity of Funds

This table reports optimal commitments in the baseline economy (Panel A) and in Economy 2, absent quantity risk (Panel B), when the investor has access to an infinity of private equity funds. The associated welfare cost of Economy 2 compared to the baseline economy is reported in Panel B. Panel C reports the welfare cost and return premium of having access to 2 funds instead of an infinity. All results are in the steady state. The steady state is defined as the state in which the expected change in invested capital is zero: $E[dX_t^{\infty}] = 0$, where X_t^{∞} follows the dynamics given in equation (14).

Panel A: Baseline economy	
Committed amount	2.24%
Invested amount	19.62%
Total PE allocation	21.86%

Panel B: Economy 2 - No commitment-quantity risk	
PE allocation	28.50%
Welfare cost	4.32%
Return premium	0.74%

Panel C: Benefits of going from 2 funds to an infinity in the baseline economy	
Welfare cost	5.42%
Return premium	2.85%

Table 6: Calibrated Parameters with Cycles

This table displays the values of parameters obtained in our extension of the model with liquidity cycles, following the calibration described in Section 6.1. The values are given in the low and high liquidity states.

Panel A: Calibrated parameters in the model with a finite number of funds

Parameter	Symbol	Low liquidity	High liquidity
Probability to switch from state H to L	χ^H	-	0.143
Probability to switch from state L to H	χ^L	0.333	-
PE expected returns	ν	0.120	0.149
PE fund volatility	ψ	0.350	0.329
Correlation between stocks & PE	ρ_L	0.850	0.579
Correlation between PE funds	ρ_{PE}	0.90	0.586
Intensity of capital call	λ_C	0.333	0.349
Intensity of capital distribution	λ_D	0.143	0.187
Secondary market haircut	α	28%	9%

Panel B: Calibrated parameters in the model with an infinite number of funds

Parameter	Symbol	Low liquidity	High liquidity
PE portfolio volatility	ψ^∞	0.332	0.251
Correlation between stock & PE portfolios	ρ_L^∞	0.896	0.756

Table 7: PE Allocation and Cost of Quantity Risk with Liquidity cycles

This table reports the private equity allocations (Panel A) and the welfare costs and return premiums associated with commitment-quantity risk (Panel B). Each of these variables is given in the baseline economy without cycles, and in the low and high liquidity states with cycles. The allocation with one fund and two funds corresponds to the optimal total commitment to PE, whereas the allocation with an infinity of funds is the sum of the invested capital and the optimal pledge at the steady state.

Panel A: Private equity allocation

	No cycles	Low liq.	High liq.	Weighted average
1 fund	5.23%	4.02%	5.76%	5.24%
2 funds	8.84%	6.80%	9.66%	8.80%
Infinity of funds	21.86%	1.39%	26.26%	18.80%

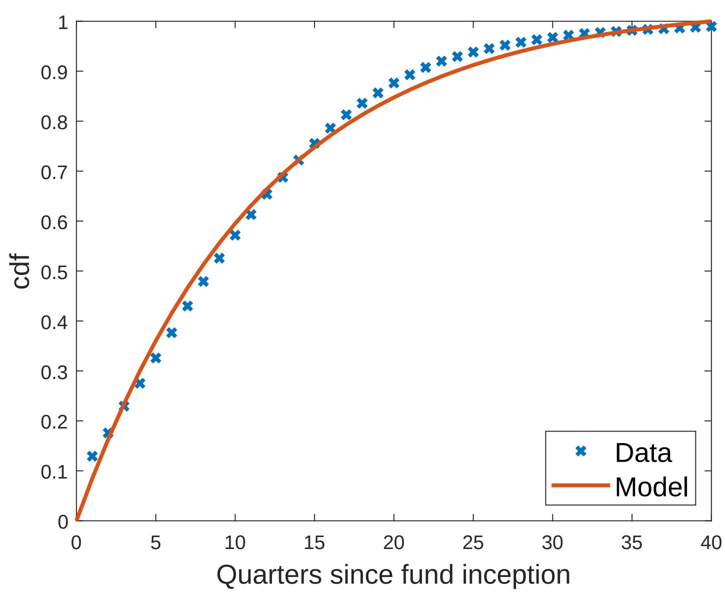
Panel B: Cost of quantity risk

	Welfare cost			Return premium		
	No cycles	Low liq.	High liq.	No cycles	Low liq.	High liq.
1 fund	1.21%	1.23%	1.31%	1.05%	1.17%	1.14%
2 funds	1.64%	1.90%	2.01%	0.89%	1.11%	1.09%
Infinity of funds	4.32%	6.41%	1.96%	0.74%	1.38%	0.34%

Figure 1: Model validation

This figure illustrates the two steps of our calibration procedure for the parameters of the private equity fund dynamics. In Panel A, the blue curve represents the empirical fraction of called capital after n quarters since capital commitment, for n between 1 and 40. As the exact time of capital commitment differs across investors in a fund and is unknown, the first investment is taken as a proxy for the capital commitment time. The red curve represents the model-implied fraction of capital calls for the our calibrated λ_C of 0.344. The calibration of the intensity parameter is described in Section 4. Panel B displays the empirical cumulative distribution function (cdf) of PMEs in our data sample, and the model-implied cdf. The parameters ν, ϕ, λ_D and ρ are calibrated in order for the model-implied cdf of PMEs to match its empirical counterpart.

Panel A: Fraction of called capital



Panel B: Distribution of PME

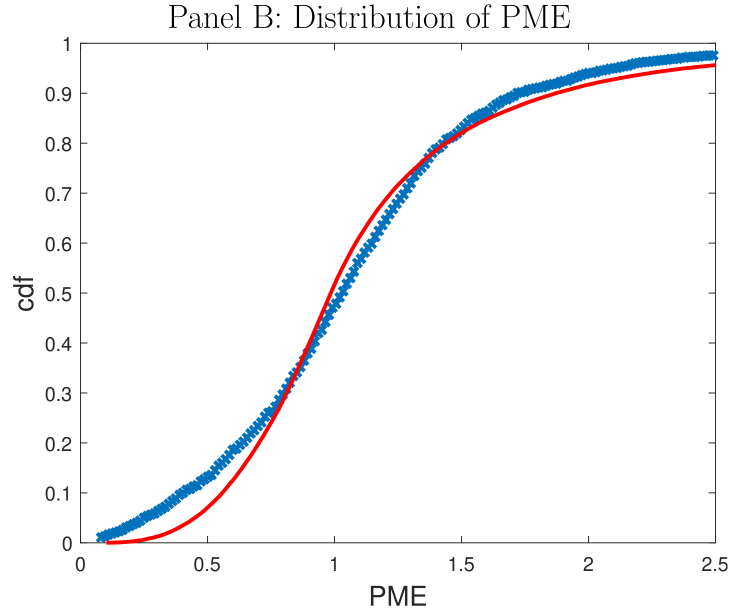


Figure 2: Optimal Allocation and Policies in the Baseline Economy

Panel A represents the value function of the investor during the commitment period (dashed line) and the holding period (plain line). Default is represented as a circle, sale on the secondary market as a square. Panel B displays the distribution of the PE allocation during the commitment period (dashed line) and holding period (plain line). Panel C displays the optimal consumption of the investor given their PE allocation. Panel D displays the optimal stock allocation. The vertical dotted line represents the optimal PE commitment ξ^*.

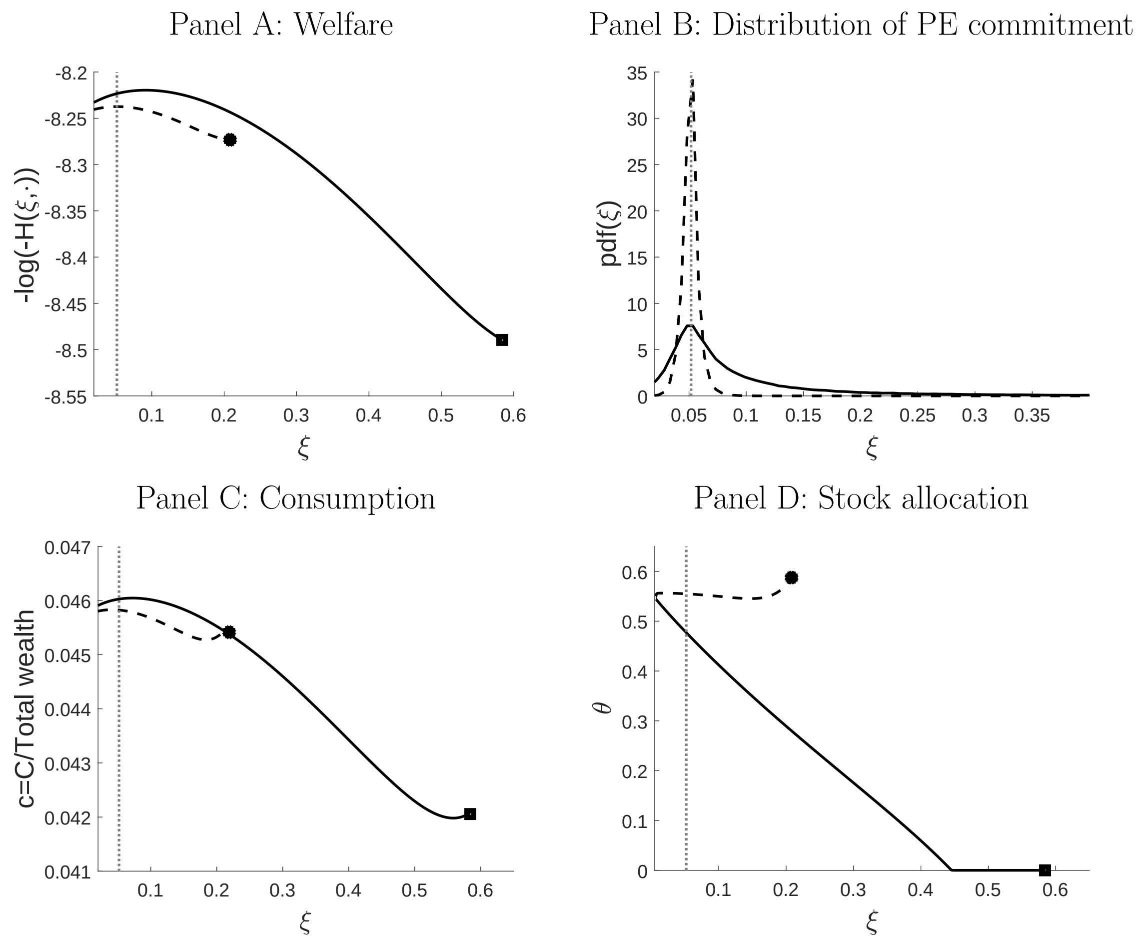
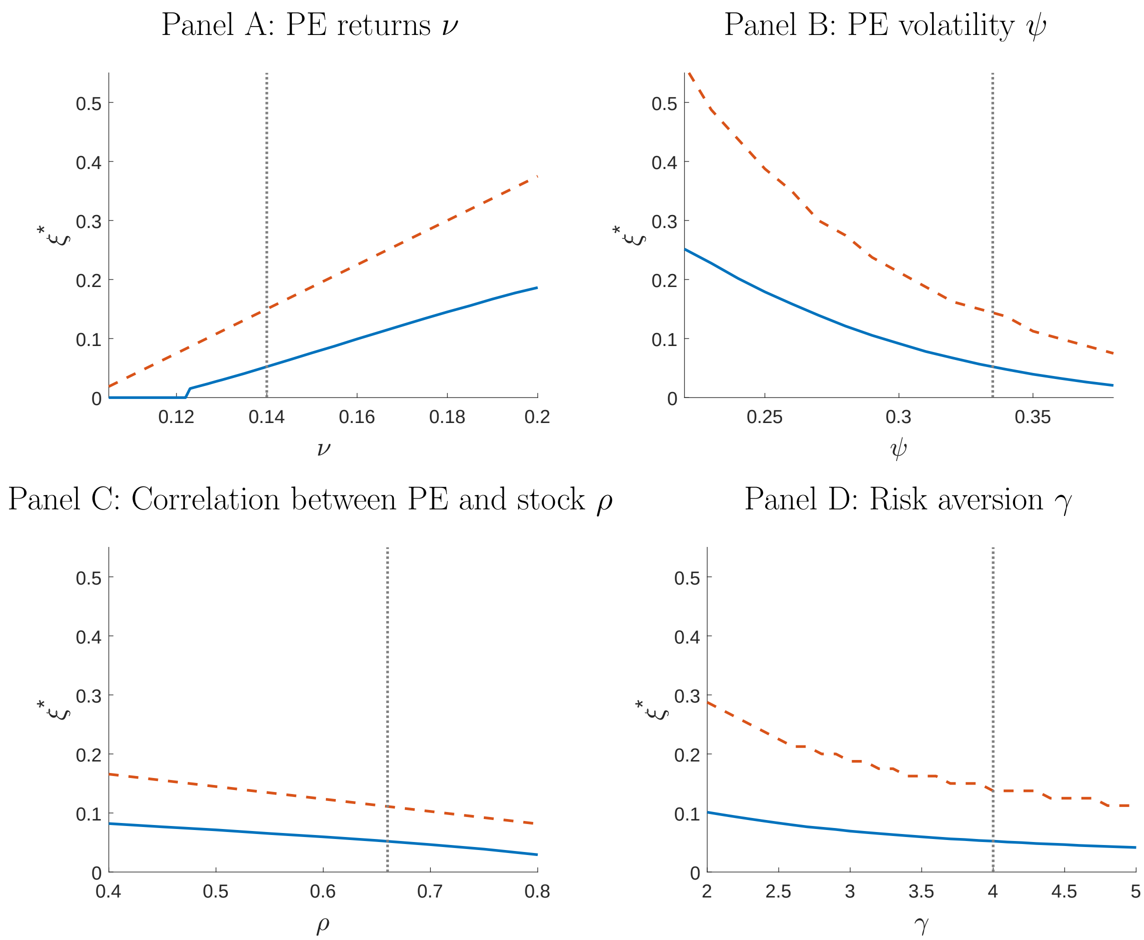


Figure 3: Sensitivity of Optimal PE Allocation to Model Parameters

This figure represents the optimal PE commitment ξ^* as a function of the different parameters of the model (solid lines). The dashed lines in Panels A to D represent the optimal allocation in the Merton two-asset model, i.e., if the PE fund were freely traded. We only vary one parameter while keeping the other parameters constant. The fixed parameters are those of our standard calibration (Table 1). The dotted vertical lines represent the value used in the standard calibration for the parameter that we vary.



Panel E: Intensity of calls λ_C

Panel F: Intensity of distributions λ_D

Panel G: Secondary Market Haircut

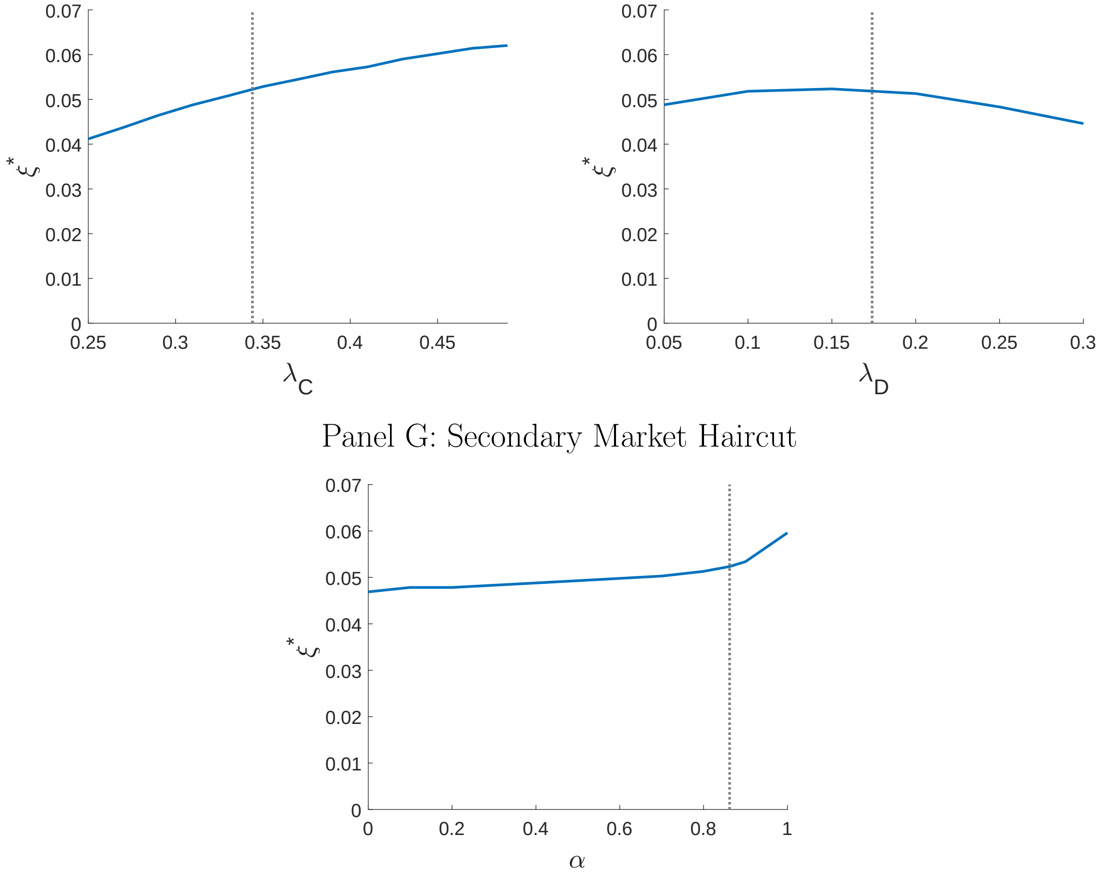


Figure 4: PE Allocation and Welfare in the Two-Fund Model

Panel A represents the optimal commitment of the investor to the second fund, ξ^{2*}, as a function of their ongoing allocation to the first fund ξ^1, if the first fund is (i) in the commitment period (dashed line) and (ii) in the holding period (solid line). The circle indicates strategic default in fund 1. The square indicates fund 1 being sold on the secondary market. The dotted vertical line marks the optimal allocation in the first fund, ξ^{1*}. Panels B and C display the value function of the investor as a function of the allocation in fund 1, (B) when both funds are in the commitment period, (C) when both funds are in the holding period. Thin lines correspond to the case in which the allocation in fund 2 is optimal (B) at inception of the fund and (C) at capital call. Thick lines correspond to the case in which the same fraction of wealth is allocated to both funds. Panel D represents the utility gain when fund 1 calls, if fund 2 is in its commitment period (dashed line) and if it is in its holding period (solid line).

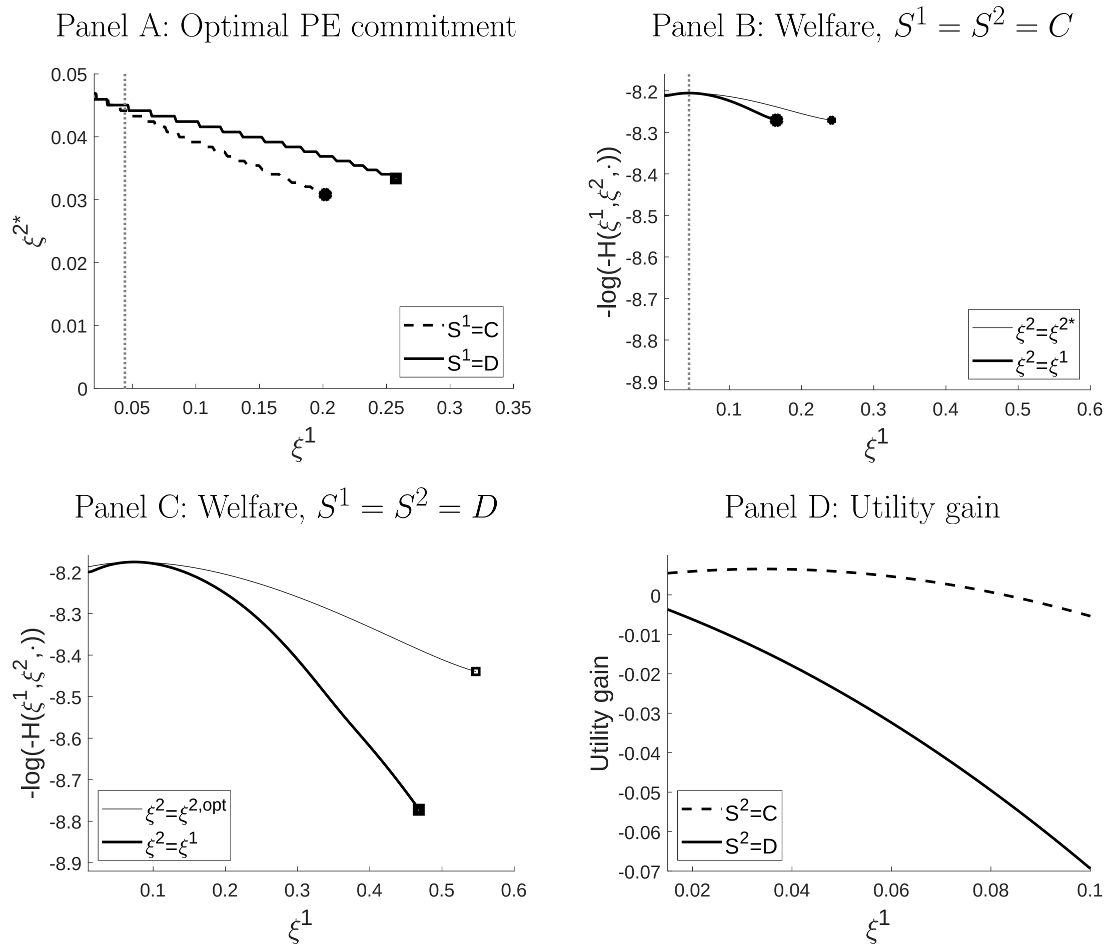


Figure 5: PE Allocation in the Infinite-Fund Case

This figure represents the optimal aggregate commitment of the investor, $Y^{\infty *}/W$, as a function of the capital that is already invested, X^{∞}/W, in the infinite-fund model. The diagonal line represents the committed capital as a function as the invested capital in the steady state, see equation (21). The steady state is defined as the state in which the expected change in invested capital is zero: $E[dX_t^{\infty}] = 0$, where X_t^{∞} follows the dynamics given in equation (14). The dotted vertical line marks the optimal investment in the steady state.

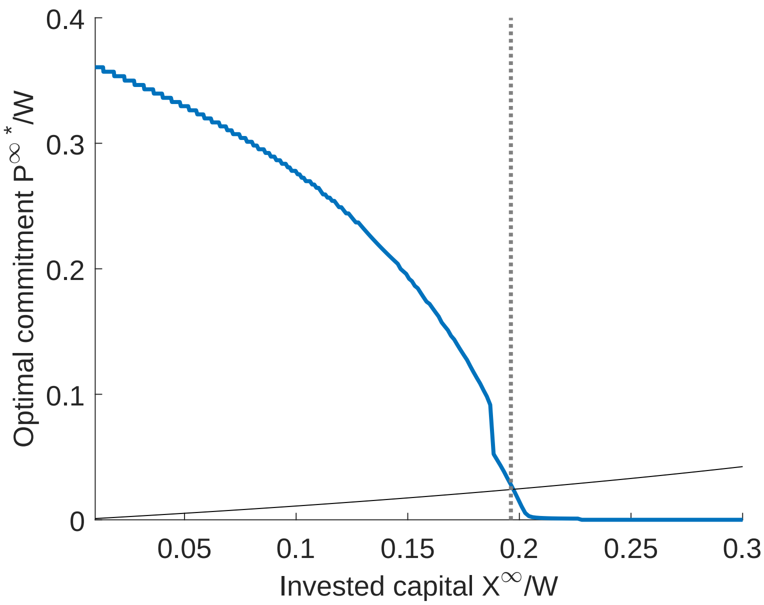


Figure 6: Consumption and Stock Allocation in Economies E0 to E5

This figure represents the optimal consumption rate and stock allocation before capital call (Panels A and C) and after capital call (Panel B), in the different economies. In Economies 2 to 4, the amount of capital invested in private equity is chosen at capital call therefore the relation between consumption (resp. stock allocation) and PE allocation before capital call is not displayed. After capital call, stock allocations overlap in economies E0 to E5.

Economies are summarized below:

Baseline economy E0	Model described in Section 3	All risks on
Economy E1	Deterministic call time	Commitment-timing risk off
Economy E2	Choose quantity when called	Commitment-quantity risk off
Economy E3	Choose quantity when called + deterministic call time	Commitment risk off
Economy E4	No commitment delay	Commitment risk off
Economy E5	Deterministic payout delay	Distribution-timing risk off

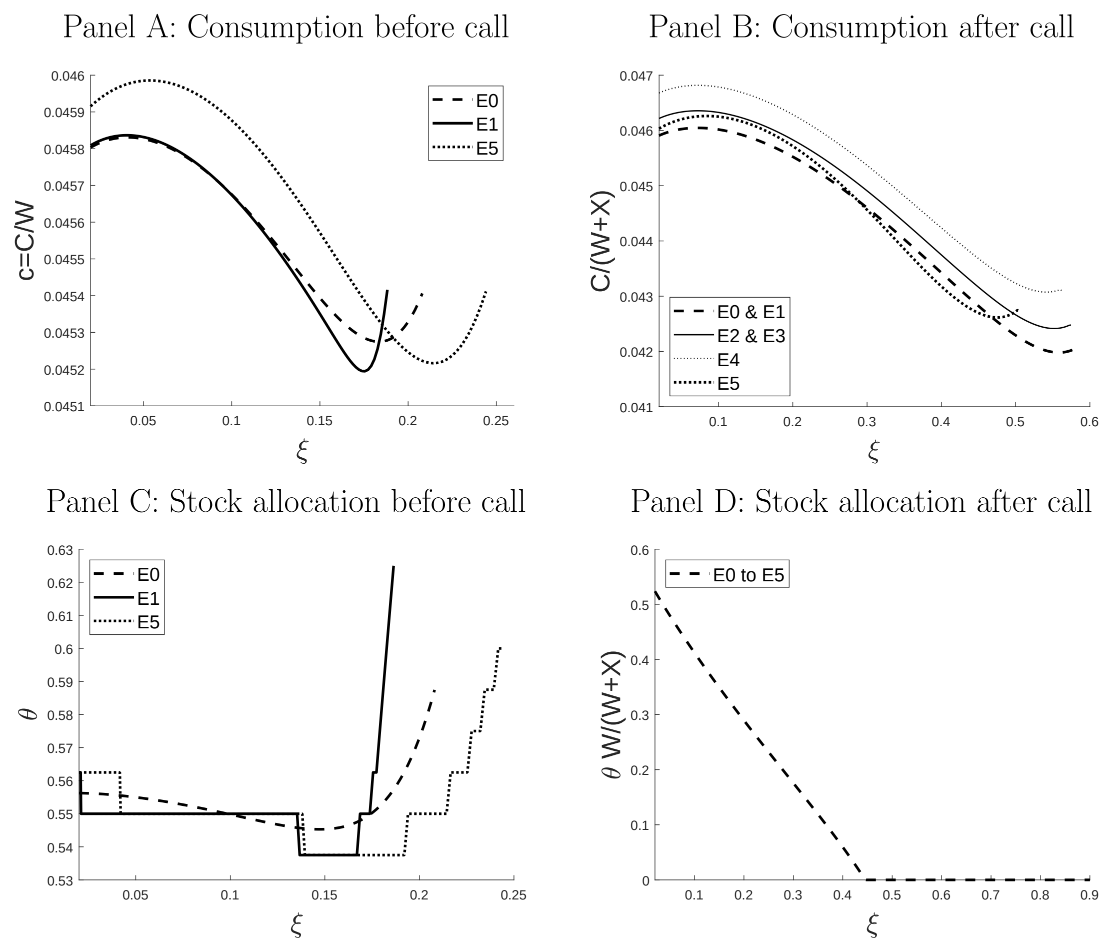


Figure 7: Welfare cost of timing risk

This figure represents the welfare cost of commitment-timing risk as a function of the subjective discount factor δ. We only vary δ while keeping the other parameters constant, as in our standard calibration (Table 1). In the standard calibration we use $\delta = 0.05$ (vertical dotted line). The horizontal line is at zero.

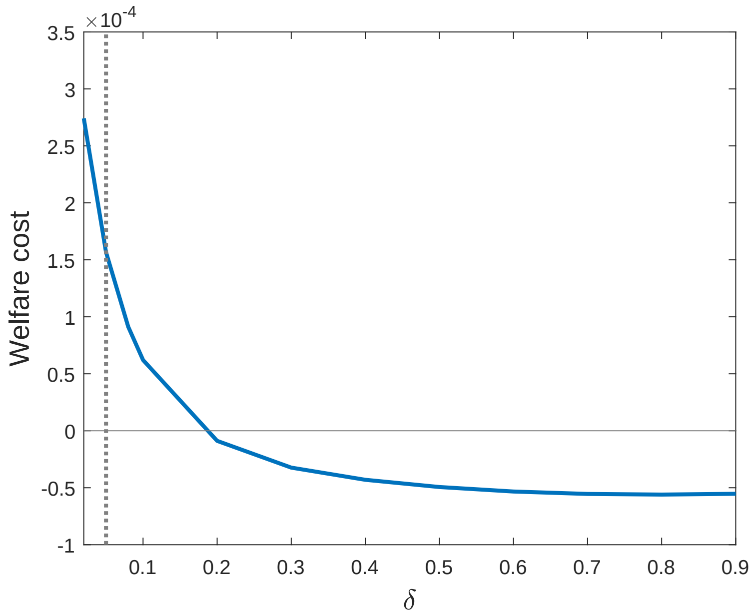


Figure 8: Sensitivity of Return Premium and Welfare Cost of Commitment-Quantity Risk to Model Parameters

This figure represents the return premium and the welfare cost of commitment-quantity risk, as functions of the different parameters of the model. We only vary one parameter while keeping the other parameters constant. The fixed parameters are those of our standard calibration (Table 1). The vertical dotted lines represent the value used in the standard calibration for the parameter that we vary.

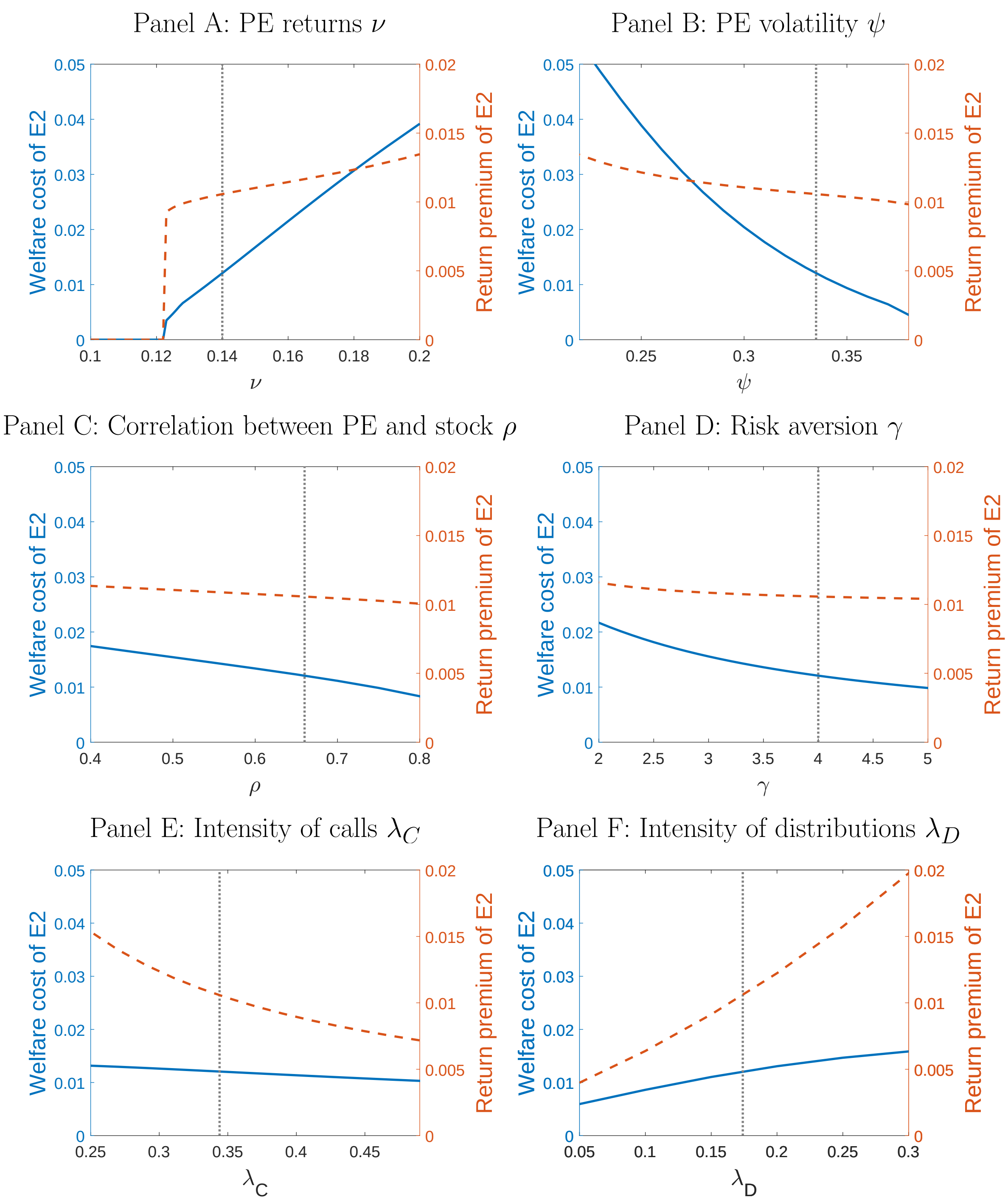
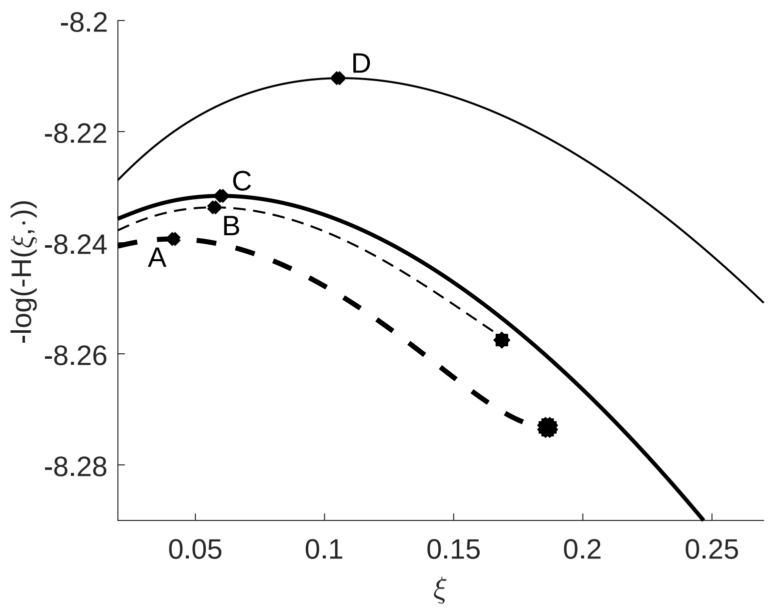


Figure 9: Welfare in the baseline economy with cycles

This figure represents the value function of the investor in the low liquidity state during the commitment period (thick dashed line) and the holding period (thick plain line), and in the high liquidity state (thin dashed line and thin plain line, respectively). Default is represented as a circle. The thick (resp. thin) dotted line marks the optimal PE allocation in the low (resp. high) liquidity state.



ONLINE APPENDIX

A Model solutions

A.1 Solution in Economy 0 - 1 fund

The investor solves problem (6). The function F solves the HJB equation (7) and the value function F can be written as (10). We give below the reduced HJB equation solved by H.

Proposition 2 (Economy 0, 1 fund) *The investor's value function can be written as in (10), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = C)$, which exists. During the commitment period, $H(\xi, S = C)$ is characterized by*

$$0 = \max_{c,\theta}\Big[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = C)H + A_1(\xi, c, \theta, S = C)H_\xi + A_2(\xi, c, \theta, S = C)H_{\xi\xi} + \lambda_C(H^{+C} - H)\Big] \tag{23}$$

where

$$A_0(\xi, c, \theta, S = C) = (1-\gamma)(r + \theta(\mu - r) - c - f\xi) + \frac{\gamma}{2}(\gamma - 1)\sigma^2\theta^2 \tag{24}$$

$$A_1(\xi, c, \theta, S = C) = \left[-(r + \theta(\mu - r) - c - f\xi) + \sigma^2\theta^2\gamma\right]\xi \tag{25}$$

$$A_2(\xi, c, \theta, S = C) = \frac{1}{2}\sigma^2\theta^2\xi^2 \tag{26}$$

$$H^{+C}(\xi) = H(\xi, S = D). \tag{27}$$

The investor strategically defaults at time $t \in (\tau_0, \tau_C]$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by $H(\xi_{def}, S = C) = H^{Merton}$.[21] *The smooth pasting and super contact conditions are given by:*

$$\lim_{\xi \to \xi_{def}} H_\xi(\xi, S = C) = 0 \tag{28}$$

$$\lim_{\xi \to \xi_{def}} H_{\xi\xi}(\xi, S = C) = 0. \tag{29}$$

During the holding period, $H(\xi, S = D)$ is characterized by

$$0 = \max_{c,\theta}\Big[\frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = D)H + A_1(\xi, c, \theta, S = D)H_\xi + A_2(\xi, c, \theta, S = D)H_{\xi\xi} + \lambda_D(H^{+D} - H)\Big] \tag{30}$$

[21] Recall that we define H^{Merton} in Section 3.3 from the standard Merton problem with access to the liquid stock and bond but without the private equity sector. We denote the value function of a Merton investor $F^{Merton} = W^{1-\gamma}H^{Merton}$.

where

$$
\begin{aligned}
&A_0(\xi, c, \theta, S = D) = (1-\xi)(1-\gamma)(r + \theta(\mu - r) - c) + \xi(1-\gamma)\nu \\
&\qquad + \frac{\gamma}{2}(\gamma - 1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) &(31)\\
&A_1(\xi, c, \theta, S = D) = -\xi(1-\xi)(r + \theta(\mu - r) - c) + \xi(1-\xi)\nu \\
&\qquad + \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) &(32)\\
&A_2(\xi, c, \theta, S = D) = \frac{1}{2}\xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right) &(33)\\
&H^{+D}(\xi) = \max_{\xi'} H(\xi', S = C). &(34)
\end{aligned}
$$

The investor strategically sells their PE stakes on the secondary market at $t \in [\tau_C, \tau_D)$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi', S = C)$. The smooth pasting and super contact conditions are given by

$$
\begin{aligned}
\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S = D) = &\quad (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi', S = C) &(35)\\
\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S = D) = &\quad (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}}\left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] \\
&\quad \max_{\xi'} H(\xi', S = C). &(36)
\end{aligned}
$$

Proposition 1 is a corollary to Proposition 2. The propositions in this Online Appendix are all straightforward applications of the HJB equation, combined with the usual verification arguments. As such, the proofs are omitted.

A.2 Solution in Economy 1 - 1 fund

In Economy 1, the time of capital call τ_C is known and equal to $\tau_0 + \frac{1}{\lambda_C}$. The commitment period has duration fixed at $\frac{1}{\lambda_C}$ and we use $t \in [0, 1/\lambda_C]$ during the commitment period. In an abuse of notation, we will write the agent's value function as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the holding period because the duration is controlled by a Poisson process.

The following proposition characterizes the solution to the investor's problem:

Proposition 3 (Economy 1, 1 fund) *The investor's value function can be written as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, where $H(t, \xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(0, \xi, S = C)$, which exists. Between private equity commitments and capital* *calls, $H(t, \xi, S = C)$ is characterized by*

$$0 = \max_{c,\theta} \Big[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(\xi, c, \theta, S = C)H + A_1(\xi, c, \theta, S = C)H_\xi + A_2(\xi, c, \theta, S = C)H_{\xi\xi} \Big] \tag{37}$$

with A_0, A_1 and A_2 given by equations (24)-(26).

The investor strategically defaults at time $t \in (0, \tau_C]$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by $H(t, \xi_{def}, S = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{def}} H_\xi(t, \xi, S = C) = 0 \tag{38}$$

$$\lim_{\xi \to \xi_{def}} H_{\xi\xi}(t, \xi, S = C) = 0. \tag{39}$$

.

At capital call,

$$H(1/\lambda_C, \xi, S = C) = H(\cdot, \xi, S = D). \tag{40}$$

Between capital calls and distributions, H is characterized by

$$0 = \max_{c,\theta} \Big[\frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = D)H + A_1(\xi, c, \theta, S = D)H_\xi + A_2(\xi, c, \theta, S = D)H_{\xi\xi} + \lambda_D(H^{+D} - H) \Big] \tag{41}$$

where A_0, A_1 and A_2 are given by equations (31)-(33). H^{+D} is given by

$$H^{+D}(\cdot, \xi) = \max_{\xi'} H(0, \xi', S = C). \tag{42}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\cdot, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(0, \xi', S = C)$. The smooth pasting and super contact conditions are given for $S = D$ by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\cdot, \xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(0, \xi', S = C) \tag{43}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\cdot, \xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma \right] \max_{\xi'} H(0, \xi', S = C). \tag{44}$$

A.3 Solution in Economy 2 - 1 fund

In Economy 2, the investor can update their commitment to PE at time of capital call τ_C. This implies that the agent commits $\xi^* = 0$ to avoid fees and increases the commitment fully at the time of the capital call. We abuse notation by writing H as a function of ξ in $S = C$ even though ξ does not vary. The following proposition characterizes the solution to the investor's problem:

Proposition 4 (Economy 2, 1 fund) *The investor's value function can be written as in (10), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* = 0$. Upon capital call, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = D)$, which exists. During the commitment period, $H(\xi = 0, S = C)$ is given by*

$$0 = \max_{c,\theta} \left[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi = 0, c, \theta, S = C)H + \lambda_C(H^{+C} - H) \right] \tag{45}$$

where $A_0(\xi = 0, c, \theta, S = C)$ is characterized by equation (24) and

$$H^{+C} = \max_{\xi'} H(\xi', S = D). \tag{46}$$

During the holding period, $H(\xi, S = D)$ is characterized by equations (30)-(33). Equation (34) becomes

$$H^{+D} = H(\xi = 0, S = C). \tag{47}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H(0, S = C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} H(0, S = C) \tag{48}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma \right] H(0, S = C). \tag{49}$$

A.4 Solution in Economy 3 - 1 fund

In Economy 3, the time of capital call τ_C is known and equal to $\tau_0 + \frac{1}{\lambda_C}$. As in the previous two economies, we abuse notation in two ways. The commitment period has duration fixed at $\frac{1}{\lambda_C}$ and we use $t \in [0, 1/\lambda_C]$ during the commitment period. We write the agent's value function as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the holding period because the duration is controlled by a Poisson process. In addition, the investor can update their commitment to PE at time of capital call τ_C. This implies that the agent commits $\xi^* = 0$ to avoid fees and increases the commitment fully at the time of the capital call. We write H as a function of ξ in $S = C$ even though ξ does not vary.

The following proposition characterizes the solution to the investor's problem:

Proposition 5 (Economy 3, 1 fund) *The investor's value function can be written as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, where $H(t, \xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* = 0$. Between private equity commitments and capital calls, $H(t, \xi = 0, S = C)$ is characterized by*

$$0 = \max_{c,\theta} \left[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(c, \theta, S = C)H \right] \tag{50}$$

with

$$A_0(\xi, c, \theta, S = C) = (1-\gamma)(r + \theta(\mu - r) - c) + \frac{\gamma}{2}(\gamma - 1)\sigma^2\theta^2 \tag{51}$$

Upon capital call, the investor selects $\xi^* \equiv \arg\max_\xi H(\cdot, \xi, S = D)$, which exists.

Between capital calls and distributions, H is characterized by equation (41) with A_0, A_1 and A_2 given by equations (31)-(33). Equation (34) becomes

$$H^{+D}(\cdot, \xi) = H(0, \xi = 0, S = C). \tag{52}$$

The investor strategically sells their PE stakes on the secondary market in $S = D$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\cdot, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H(0, \xi = 0, S = C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\cdot, \xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1 + \alpha\xi_{sec\_mkt})^{-\gamma}}{(1 + \xi_{sec\_mkt})^{2-\gamma}} H(0, \xi = 0, S = C) \tag{53}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\cdot, \xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1 + \alpha\xi_{sec\_mkt})^{-\gamma}}{(1 + \xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1 + \xi_{sec\_mkt})}{1 + \alpha\xi_{sec\_mkt}} - 2 + \gamma \right] H(0, \xi = 0, S = C). \tag{54}$$

A.5 Solution in Economy 4 - 1 fund

In Economy 4, the investor immediately invests in PE, there is no commitment period. $S = C$ does not exist, and we have only $S = D$. The following proposition characterizes the solution to the investor's problem:

Proposition 6 (Economy 4, 1 fund) *The investor's value function can be written as in (10), where $H(\xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\xi, S = D)$, which exists. $H(\xi, S =$*

D) is characterized by equations (30)-(33). Equation (34) becomes

$$H^{+D}(\xi) = \max_{\xi'} H(\xi', S = D). \tag{55}$$

The investor strategically sells their PE stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(\xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi', S = D)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi', S = D) \tag{56}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi, S = D) = (1-\gamma)(\alpha - 1)\frac{(1+\alpha\xi_{sec\_mkt})^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha\xi_{sec\_mkt}} - 2 + \gamma\right] \max_{\xi'} H(\xi', S = D). \tag{57}$$

A.6 Solution in Economy 5 - 1 fund

In Economy 5,the payout time τ_D is deterministic and happens at time $\tau_C + \frac{1}{\lambda_D}$. The holding period has duration fixed at $\frac{1}{\lambda_D}$ and we use $t \in [0, 1/\lambda_D]$ during the holding period. We abuse notation by writing the agent's value function as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, with $S \in \{C, D\}$. However, H is not a function of t during the commitment period because the duration is controlled by a Poisson process.

The following proposition characterizes the solution to the investor's problem:

Proposition 7 (Economy 5, 1 fund) *The investor's value function can be written as $F(t, W, X, S) = TW^{1-\gamma}H(t, \xi, S)$, where $H(t, \xi, S)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$. Whenever the investor can commit capital, they select $\xi^* \equiv \arg\max_\xi H(\cdot, \xi, S = C)$, which exists. Between private equity commitments and capital calls, H is characterized by*

$$0 = \max_{c,\theta}\Big[\frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi, c, \theta, S = C)H + A_1(\xi, c, \theta, S = C)H_\xi + A_2(\xi, c, \theta, S = C)H_{\xi\xi} + \lambda_C(H^{+C} - H)\Big] \tag{58}$$

where A_0, A_1 and A_2 are given by equations (24)-(26). H^{+C} is given by

$$H^{+C}(\cdot, \xi) = H(0, \xi, S = D). \tag{59}$$

The investor strategically defaults in $S = C$ if $\xi_t \geq \xi_{def}$, where ξ_{def} is defined by $H(\cdot, \xi_{def}, S = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by equations (28)-(29).

Between capital calls and distributions, $H(t, \xi, S = D)$ is characterized by

$$0 = \max_{c,\theta} \left[\frac{[c(1-\xi)]^{1-\gamma}}{1-\gamma} - \delta H + H_t + A_0(\xi, c, \theta, S = D)H + A_1(\xi, c, \theta, S = D)H_\xi \right. \\ \left. + A_2(\xi, c, \theta, S = D)H_{\xi\xi} \right] \tag{60}$$

where A_0, A_1 and A_2 are given by equations (31)-(33).

The investor strategically sells their PE stakes on the secondary market at $t \in [0, 1/\lambda_D)$ if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H(t, \xi_{sec\_mkt}, S = D) = \left(\frac{1+\alpha\xi_{sec\_mkt}}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\cdot, \xi', S = C)$. The smooth pasting and super contact conditions are given by equations (35)-(36).

A.7 Solution in Economy 0 - 2 funds

In Economy 0 of the 2-fund problem, the Hamilton-Jacobi-Bellman (HJB) equation is as follows:

$$\begin{aligned}
\delta F = \max_{\{\theta, c\}} \Big\{ & U(c) + \left(r + (\mu - r)\theta - c - \mathbb{1}_{S^1=C} f \frac{X^1}{W} - \mathbb{1}_{S^2=C} f \frac{X^2}{W} \right) W F_W \\
& + \frac{1}{2}\theta^2\sigma^2 W^2 F_{WW} + \nu X^1 F_{X^1} + \nu X^2 F_{X^2} + \frac{1}{2}\psi^2 (X^1)^2 F_{X^1X^1} + \frac{1}{2}\psi^2 (X^2)^2 F_{X^2X^2} \\
& + \theta\sigma\psi\rho_L W (X^1 F_{WX^1} + X^2 F_{WX^2}) + \psi^2 \rho_{PE} X^1 X^2 F_{X^1X^2} \\
& + \lambda_C \left[\left(F^{1+C} - F\right) \mathbb{1}_{S^1=C} + \left(F^{2+C} - F\right) \mathbb{1}_{S^2=C} \right] \\
& + \lambda_D \left[\left(F^{1+D} - F\right) \mathbb{1}_{S^1=D} + \left(F^{2+D} - F\right) \mathbb{1}_{S^2=D} \right] \Big\}
\end{aligned} \tag{61}$$

with

$$\begin{aligned}
&F^{1+C}(W, X^1, X^2, S^1 = C, S^2) = F(W - X^1, X^1, X^2, S^1 = D, S^2), \\
&F^{2+C}(W, X^1, X^2, S^1, S^2 = C) = F(W - X^2, X^1, X^2, S^1, S^2 = D), \\
&F^{1+D}(W, X^1, X^2, S^1 = D, S^2) = \max_{X^1} F(W + X^1, X^1, X^2, S^1 = C, S^2), \\
&F^{2+D}(W, X^1, X^2, S^1, S^2 = D) = \max_{X^2} F(W + X^2, X^1, X^2, S^1, S^2 = C).
\end{aligned}$$

Similarly to the one-fund case, the value function can be decomposed as follows:

$$F(W, X^1, X^2, S^1, S^2) = TW^{1-\gamma} H\left(\xi^1, \xi^2, S^1, S^2\right), \tag{62}$$

with $TW \equiv W + X^1 \mathbb{1}_{S^1=D} + X^2 \mathbb{1}_{S^2=D}$. We give below the reduced HJB equation solved by H.

Proposition 8 (Economy 0, 2 funds) *The investor's value function can be written as in (62), where $H(\xi^1, \xi^2, S^1, S^2)$ exists and is finite, continuous, and concave for $(\xi^1, \xi^2) \in [0, 1)^2$. At time 0, the investor selects $(\xi^1, \xi^2)^* \equiv \arg\max_{\xi^1, \xi^2} H(\xi^1, \xi^2, S^1 = C, S^2 = C)$,* *which exists. Whenever the investor can commit capital to fund* $i \in \{1,2\}$*, she selects* $\xi^{i*} \equiv \arg\max_{\xi^i} H(\xi^i, \xi^{\neq i}, S^i = C, S^{\neq i})$*, which exists.*

When both funds are in their commitment periods, $H(\xi^1, \xi^2, S^1 = C, S^2 = C)$ *is characterized by*

$$
\begin{aligned}
0 = \max_{c,\theta} \Big[& \frac{c^{1-\gamma}}{1-\gamma} - \delta H + A_0(c, \theta, S^1 = C, S^2 = C)H + A_1(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C)H_{\xi^1} \\
& + A_2(\xi^1, c, \theta, S^1 = C, S^2 = C)H_{\xi^1\xi^1} + A_3(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C)H_{\xi^2} \\
& + A_4(\xi^2, c, \theta, S^1 = C, S^2 = C)H_{\xi^2\xi^2} + A_5(\xi^2, c, \theta, S^1 = C, S^2 = C)H_{\xi^1\xi^2} \\
& + \lambda_C \left(H^{1+C} + H^{2+C} - 2H\right) \Big]
\end{aligned} \tag{63}
$$

where

$$
\begin{aligned}
A_0(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C) =&\ (1-\gamma)(r + \theta(\mu - r) - c - f\xi^1 - f\xi^2) + \frac{\gamma}{2}(\gamma - 1)\sigma^2\theta^2 \\
A_1(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C) =&\ \left[-(r + \theta(\mu - r) - c - f\xi^1 - f\xi^2) + \sigma^2\theta^2\gamma\right]\xi^1 \\
A_2(\xi^1, c, \theta, S^1 = C, S^2 = C) =&\ \frac{1}{2}\sigma^2\theta^2(\xi^1)^2 \\
A_3(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C) =&\ \left[-(r + \theta(\mu - r) - c - f\xi^1 - f\xi^2) + \sigma^2\theta^2\gamma\right]\xi^2 \\
A_4(\xi^2, c, \theta, S^1 = C, S^2 = C) =&\ \frac{1}{2}\sigma^2\theta^2(\xi^2)^2 \\
A_5(\xi^1, \xi^2, c, \theta, S^1 = C, S^2 = C) =&\ \sigma^2\theta^2\xi^1\xi^2 \\
H^{1+C}(\xi^1, \xi^2, S^1 = C, S^2 = C) =&\ H(\xi^1, \xi^2, S^1 = D, S^2 = C) \\
H^{2+C}(\xi^1, \xi^2, S^1 = C, S^2 = C) =&\ H(\xi^1, \xi^2, S^1 = C, S^2 = D).
\end{aligned}
$$

The investor strategically defaults at time t *if* $\xi^1_t \geq \xi^1_{def}(\xi^2_t)$ *or* $\xi^2_t \geq \xi^2_{def}(\xi^1_t)$*, where* $\xi^1_{def}(\xi^2)$ *solves* $H(\xi^1_{def}, \xi^2, S^1 = C, S^2 = C) = H^{Merton}$ *and* $\xi^2_{def}(\xi^1)$ *solves* $H(\xi^1, \xi^2_{def}, S^1 = C, S^2 = C) = H^{Merton}$*. The smooth pasting and super contact conditions are given by*

$$\lim_{\xi^1 \to \xi^1_{def}} H_{\xi^1}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0 \tag{64}$$

$$\lim_{\xi^2 \to \xi^2_{def}} H_{\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0 \tag{65}$$

$$\lim_{\xi^1 \to \xi^1_{def}} H_{\xi^1\xi^1}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0. \tag{66}$$

$$\lim_{\xi^2 \to \xi^2_{def}} H_{\xi^2\xi^2}(\xi^1, \xi^2, S^1 = C, S^2 = C) = 0. \tag{67}$$

When one fund is in its commitment period and the other is in its holding period (w.l.o.g., *assume that $S^1 = C$ and $S^2 = D$), $H(\xi^1, \xi^2, S^1 = C, S^2 = D)$ is characterized by*

$$0 = \max_{c,\theta}\Big[\frac{[c(1-\xi^2)]^{1-\gamma}}{1-\gamma} - \delta H + A_0(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D)H \\ + A_1(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D)H_{\xi^1} + A_2(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D)H_{\xi^1\xi^1} \\ + A_3(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D)H_{\xi^2} + A_4(\xi^2,c,\theta,S^1=C,S^2=D)H_{\xi^2\xi^2} \\ + A_5(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D)H_{\xi^1\xi^2} + \lambda_C\left(H^{1+C} - H\right) + \lambda_D\left(H^{2+D} - H\right)\Big] \tag{68}$$

where

$$\begin{aligned} &A_0(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D) = \ (1-\xi^2)(1-\gamma)(r+\theta(\mu-r)-c-f\xi^1) + \xi^2(1-\gamma)\nu \\ &\quad + \frac{\gamma}{2}(\gamma-1)[(\xi^2)^2\psi^2 + \sigma^2\theta^2(1-\xi^2)^2 + 2\xi^2(1-\xi^2)\rho_L\psi\sigma\theta] \\ &A_1(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D) = -\xi^1(1-\xi^2)(r+\theta(\mu-r)-c-f\xi^1) - \nu\xi^1\xi^2 \\ &\quad + \gamma\xi^1\left[\sigma^2\theta^2(1-\xi^2)^2 + \xi^2(1-\xi^2)2\rho_L\psi\sigma\theta + \psi^2(\xi_2)^2\right] \\ &A_2(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D) = \ \frac{1}{2}(\xi^1)^2\left[2\rho_L\sigma\theta\psi(\xi^2)^2(1-\xi^2)^2 + \sigma^2\theta^2(1-\xi^2)^2 + \psi^2(\xi^2)^2\right] \\ &A_3(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D) = \ [-(r+\theta(\mu-r)-c-f\xi^1) + \sigma^2\theta^2\gamma(1-\xi^2) - \psi\gamma\xi^2 \\ &\quad - \theta\sigma\psi\rho_L\gamma(1-2\xi^2) + \nu\xi^2] + \xi^2(1-\xi^2) \\ &A_4(\xi^2,c,\theta,S^1=C,S^2=D) = \ \frac{1}{2}(\xi^2)^2(1-\xi^2)2[\sigma^2\theta^2 + \psi^2 - 2\theta\sigma\psi\rho_L] \\ &A_5(\xi^1,\xi^2,c,\theta,S^1=C,S^2=D) = \ \xi^1\xi^2(1-\xi^2)[\theta^2\sigma^2(1-\xi^2) - \psi^2(\xi^2) - \theta\sigma\psi\rho_L(1-2\xi^2)] \\ &H^{1+C}(\xi^1,\xi^2,S^1=C,S^2=D) = H(\xi^1,\xi^2,S^1=D,S^2=D) \\ &H^{2+D}(\xi^1,\xi^2,S^1=C,S^2=D) = \max_{\xi^2} H(\xi^1,\xi^2,S^1=C,S^2=C). \end{aligned}$$

The investor strategically defaults at time t if $\xi^1_t \geq \xi^1_{def}(\xi^2_t)$, where $\xi^1_{def}(\xi^2)$ solves $H(\xi^1_{def}, \xi^2, S^1 = C, S^2 = C) = H^{Merton}$. The smooth pasting and super contact conditions are given by equations (64) and (66).

The investor strategically sells their PE stakes on the secondary market at t if $\xi^2_t = \xi^2_{sec\_mkt}$, where $\xi^2_{sec\_mkt}$ solves $H(\xi^1, \xi^2_{sec\_mkt}, S^1 = C, S^2 = D) = \left(\frac{1+\alpha\xi^2_{sec\_mkt}}{1+\xi^2_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi^1, \xi', S^1 = C, S^2 = C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^2\to\xi^2_{sec\_mkt}} H_{\xi^2}(\xi^1,\xi^2,S^1=C,S^2=D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^2_{sec\_mkt})^{2-\gamma}}\max_{\xi'} H(\xi^1,\xi',S^1=C,S^2=C) \tag{69}$$

$$\lim_{\xi^2\to\xi^2_{sec\_mkt}} H_{\xi^2\xi^2}(\xi^1,\xi^2,S^1=C,S^2=D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^2_{sec\_mkt})^{3-3\gamma}}\left[\frac{-\gamma\alpha(1+\xi^2_{sec\_mkt})}{1+\alpha\xi^2_{sec\_mkt}}-2+\gamma\right] \max_{\xi'} H(\xi^1,\xi',S^1=C,S^2=C). \tag{70}$$

Finally, when both funds are in their holding period, $H(\xi^1,\xi^2,S^1=D,S^2=D)$ is characterized by

$$\begin{aligned}0=\max_{c,\theta}\Big[&\frac{[c(1-\xi^1-\xi^2)]^{1-\gamma}}{1-\gamma}-\delta H+A_0(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H\\&+A_1(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H_{\xi^1}+A_2(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H_{\xi^1\xi^1}\\&+A_3(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H_{\xi^2}+A_4(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H_{\xi^2\xi^2}\\&+A_5(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)H_{\xi^1\xi^2}+\lambda_D\left(H^{1+D}+H^{2+D}-2H\right)\Big]\end{aligned} \tag{71}$$

where

$$\begin{aligned}A_0(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=\ &(1-\xi^1-\xi^2)(1-\gamma)(r+\theta(\mu-r)-c)+(\xi^1+\xi^2)(1-\gamma)\nu\\&+\frac{\gamma}{2}(\gamma-1)[((\xi^1)^2+(\xi^2)^2)\psi^2+\sigma^2\theta^2(1-\xi^1-\xi^2)^2+2\xi^1\xi^2\rho_{PE}\psi^2+2(1-\xi^1-\xi^2)\rho_L\psi\sigma\theta(\xi^1+\xi^2)]\\A_1(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=&-\xi^1(1-\xi^1-\xi^2)(r+\theta(\mu-r)-c)+\nu\xi^1(1-\xi^1-\xi^2)\\&+\gamma\xi^1\sigma^2\theta^2(1-\xi^1-\xi^2)^2+\psi^2(\xi^1)^2(1-\gamma)(1-\xi^1)+\psi^2(\xi^2)^2\gamma\xi^1-\psi^2\rho_{PE}\xi^1\xi^2\gamma(1-2\xi^1)\\&+\rho_L\psi\sigma\theta(1-\xi^1-\xi^2)\xi^1[(2-\gamma)(1-\xi^1)+\gamma\xi^2]\\A_2(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=\ &\frac{1}{2}(\xi^1)^2\left[2\rho_L\sigma\theta\psi(1-\xi^1+\xi^2)(1-\xi^1-\xi^2)+\sigma^2\theta^2(1-\xi^1-\xi^2)^2\right.\\&\left.+\psi^2[(\xi^2)^2-(1-\xi^1)^2-2\rho_{PE}\xi^2(1-\xi^1)]\right]\\A_3(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=\ &\left[-(r+\theta(\mu-r)-c)+\sigma^2\theta^2\gamma\xi^2(1-\xi^1-\xi^2)-\theta\sigma\psi\rho_L\gamma(1-\xi^1-\xi^2)+\nu\right]\\A_4(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=\ &\frac{1}{2}(\xi^2)^2(1-\xi^1-\xi^2)^2\sigma^2\theta^2+\frac{1}{2}(\xi^2)^2\psi^2[(\xi^1)^2+(1-\xi^2)^2\\&-2\rho_{PE}\xi^1(1-\xi^2)]-\theta\sigma\psi\rho_L(1-\xi^1-\xi^2)^2(\xi^2)^2\\A_5(\xi^1,\xi^2,c,\theta,S^1=D,S^2=D)=\ &\xi^1\xi^2(1-\xi^1-\xi^2)[\theta^2\sigma^2(1-\xi^1-\xi^2)-\psi^2(\xi^1)(1-\xi^2)\\&+\theta\sigma\psi\rho_L2\xi^2]+\psi^2\rho_{PE}\xi^1\xi^2[(1-\xi^1)(1-\xi^2)+\xi^1\xi^2]\\H^{1+D}(\xi^1,\xi^2,S^1=D,S^2=D)=&\max_{\xi^1}H(\xi^1,\xi^2,S^1=C,S^2=D)\\H^{2+D}(\xi^1,\xi^2,S^1=D,S^2=D)=&\max_{\xi^2}H(\xi^1,\xi^2,S^1=D,S^2=C).\end{aligned}$$

The investor strategically sells their stakes in fund 1 on the secondary market at t if $\xi_t^1 = \xi^1_{sec\_mkt}$, where $\xi^1_{sec\_mkt}$ solves $H(\xi^1_{sec\_mkt}, \xi^2, S^1 = C, S^2 = D) = \left(\frac{1+\alpha\xi^1_{sec\_mkt}+\xi^2}{1+\xi^1_{sec\_mkt}+\xi^2}\right)^{1-\gamma} \max_{\xi'} H(\xi', \xi^2, S^1 = C, S^2 = D)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^1 \to \xi^1_{sec\_mkt}} H_{\xi^1}(\xi^1, \xi^2, S^1 = D, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^1_{sec\_mkt}+\xi^2)^{-\gamma}}{(1+\xi^1_{sec\_mkt}+\xi^2)^{2-\gamma}} \max_{\xi'} H(\xi', \xi^2, S^1 = C, S^2 = D) \quad (72)$$

$$\lim_{\xi^1 \to \xi^1_{sec\_mkt}} H_{\xi^1\xi^1}(\xi^1, \xi^2, S^1 = D, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\alpha\xi^1_{sec\_mkt}+\xi^2)^{-\gamma}}{(1+\xi^1_{sec\_mkt}+\xi^2)^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi^1_{sec\_mkt}+\xi^2)}{1+\alpha\xi^1_{sec\_mkt}+\xi^2} - 2 + \gamma\right] \max_{\xi'} H(\xi', \xi^2, S^1 = C, S^2 = D). \quad (73)$$

The investor strategically sells their stakes in fund 2 on the secondary market at t if $\xi_t^2 = \xi^2_{sec\_mkt}$, where $\xi^2_{sec\_mkt}$ solves $H(\xi^1, \xi^2_{sec\_mkt}, S^1 = C, S^2 = D) = \left(\frac{1+\xi^1+\alpha\xi^2_{sec\_mkt}}{1+\xi^1+\xi^2_{sec\_mkt}}\right)^{1-\gamma} \max_{\xi'} H(\xi^1, \xi', S^1 = D, S^2 = C)$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2}(\xi^1, \xi^2, S^1 = D, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\xi^1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^1+\xi^2_{sec\_mkt})^{2-\gamma}} \max_{\xi'} H(\xi^1, \xi', S^1 = D, S^2 = C) \quad (74)$$

$$\lim_{\xi^2 \to \xi^2_{sec\_mkt}} H_{\xi^2\xi^2}(\xi^1, \xi^2, S^1 = D, S^2 = D) = (1-\gamma)(\alpha-1)\frac{(1+\xi^1+\alpha\xi^2_{sec\_mkt})^{-\gamma}}{(1+\xi^1+\xi^2_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi^1+\xi^2_{sec\_mkt})}{1+\xi^1+\alpha\xi^2_{sec\_mkt}} - 2 + \gamma\right] \max_{\xi'} H(\xi^1, \xi', S^1 = D, S^2 = C). \quad (75)$$

In Economies 2 and 4, the solution to the two-fund problem is derived as in the one-fund problem, see Sections A.3 and A.5.

A.8 Solution in Economy 0 - Infinity of funds

The investor solves problem (6). The function F can be written as (17). We give below the reduced HJB equation solved by H^∞.

We give below the reduced HJB equation solved by H^∞.

Proposition 9 (Economy 0, infinity of funds) *The investor's value function can be written as in (17), where $H^\infty(\pi, \xi)$ exists and is finite, continuous, and concave for $\xi \in [0, 1)$*

and $\pi \in [0, 1-\xi)$.

The investor chooses dJ_t *such that* $\frac{dJ_t}{W_t+X_t} = \max(0, \pi^*(\xi_t) - \pi_t)$, *where* $\pi^*(\xi)$ *is characterized by the value matching and super-contact conditions* $H^\infty_\pi(\pi^*, \xi) = H^\infty_{\pi\pi}(\pi^*, \xi) = 0$.

On $\pi \in [\pi^*(\xi), 1-\xi)$ *and* $\xi \in [0, 1)$, $H^\infty(\pi, \xi)$ *is characterized by*

$$0 = \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma}(1-\xi)^{1-\gamma} - \delta H + A(\pi, \xi, c, \theta)H + B(\pi, \xi, c, \theta)H_\xi \right. \tag{76}$$

$$\left. + C(\pi, \xi, c, \theta)\frac{1}{2}H_{\xi\xi} + D(\pi, \xi, c, \theta)H_\pi + E(\pi, \xi, c, \theta)\frac{1}{2}H_{\pi\pi} + F(\pi, \xi, c, \theta)H_{\xi\pi} \right]$$

where

$$\begin{aligned}
A(\pi, \xi, c, \theta) &= (1-\gamma)\left((1-\xi)(r + \theta(\mu - r) - c) + \xi\lambda_D - \pi\lambda_C - \pi f\right) + (1-\gamma)\left(\xi\nu + \pi\lambda_C + \pi f - \xi\lambda_D\right) \\
&\quad + \frac{\gamma}{2}(\gamma - 1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\
B(\pi, \xi, c, \theta) &= -\xi(1-\xi)(r + \theta(\mu - r) - c) + (1-\xi)\left(\xi\nu + \pi\lambda_C + \pi f - \xi\lambda_D\right) \\
&\quad + \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) \\
C(\pi, \xi, c, \theta) &= \xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right) \\
D(\pi, \xi, c, \theta) &= -\pi\left((1-\xi)(r + \theta(\mu - r) - c) - \xi\lambda_D + \pi\lambda_C + \pi f\right) - \pi\left(\lambda_C + \pi\lambda_C + \pi f - \xi\lambda_D + \xi\nu\right) \\
&\quad + \frac{\gamma}{2}\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\
E(\pi, \xi, c, \theta) &= \pi^2\xi^2\psi^2 + 2\xi\pi^2(1-\xi)\rho_L\sigma\theta\psi + \pi^2(1-\xi)^2\sigma^2\theta^2 \\
F(\pi, \xi, c, \theta) &= -\xi(1-\xi)\pi\left(\xi\psi^2 + (1-2\xi)\rho_L\sigma\theta\psi - (1-\xi)\sigma^2\theta^2\right).
\end{aligned}$$

The investor strategically defaults at time t if $\xi_t \geq \xi_{def}(\pi_t)$, where $\xi_{def}(\pi)$ solves $H^\infty(\pi, \xi_{def}) = H^{Merton}$. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{def}} H^\infty_\xi(\pi, \xi) = 0 \tag{77}$$

$$\lim_{\xi \to \xi_{def}} H^\infty_{\xi\xi}(\pi, \xi) = 0. \tag{78}$$

The investor strategically sells stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H^\infty(\pi, \xi_{sec\_mkt}) = \max_{\xi' \leq \xi_{sec\_mkt}} \left(\frac{1+\alpha(\xi_{sec\_mkt} - \xi') + \xi'}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H^\infty(\pi, \xi')$. Let us denote by $\xi' = \xi_m$ the value of ξ' at which the maximum is reached. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} \quad H^{\infty}_{\xi}(\pi, \xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}}(1+\xi_m)H(\pi,\xi_m) \tag{79}$$

$$\lim_{\xi \to \xi_{sec\_mkt}} \quad H_{\xi\xi}(\pi, \xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}} \left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m} - 2 + \gamma\right] H(\pi,\xi_m). \tag{80}$$

A.9 Solution in Economy 2 - Infinity of funds

The following proposition characterizes the solution to the investor's problem:

Proposition 10 (Economy 2, infinity of funds) *The investor's value function can be written as $F^{\infty}(W, X^{\infty}, Y^{\infty}) = (W + X^{\infty})^{1-\gamma} H^{\infty}(\xi)$, where $H^{\infty}(\xi)$ exists and is finite, continuous, and concave for $\xi \in [0,1)$.*

The investor chooses dJ_t such that $\frac{dJ_t}{W_t+X_t} = \max(0, \xi^*_t - \xi_t)$, where ξ^* is characterized by the value matching and super-contact conditions $H^{\infty}_{\xi}(\xi^*) = H^{\infty}_{\xi\xi}(\xi^*) = 0$.

On $\xi \in [0,1)$, $H^{\infty}(\xi)$ is characterized by

$$0 = \max_{c,\theta}\left[\frac{1}{1-\gamma}c^{1-\gamma}(1-\xi)^{1-\gamma} - \delta H + A(\xi,c,\theta)H + B(\xi,c,\theta)H_{\xi} + C(\xi,c,\theta)\frac{1}{2}H_{\xi\xi}\right] \tag{81}$$

where

$$\begin{aligned}
A(\xi,c,\theta) &= (1-\gamma)\left((1-\xi)(r+\theta(\mu-r)-c)+\xi\lambda_D\right) + (1-\gamma)\left(\xi\nu - \xi\lambda_D\right) \\
&\quad + \frac{\gamma}{2}(\gamma-1)\left(\xi^2\psi^2 + \sigma^2\theta^2(1-\xi)^2 + 2\xi(1-\xi)\rho_L\psi\sigma\theta\right) \\
B(\xi,c,\theta) &= -\xi(1-\xi)(r+\theta(\mu-r)-c) + (1-\xi)\left(\xi\nu - \xi\lambda_D\right) \\
&\quad + \gamma\left(-\psi^2\xi^2(1-\xi) + \sigma^2\theta^2(1-\xi)^2\xi - \xi(1-\xi)(1-2\xi)\rho_L\psi\sigma\theta\right) \\
C(\xi,c,\theta) &= \xi^2(1-\xi)^2\left(\psi^2 - 2\rho_L\sigma\theta\psi + \sigma^2\theta^2\right).
\end{aligned}$$

The investor strategically sells stakes on the secondary market at t if $\xi_t = \xi_{sec\_mkt}$, where ξ_{sec_mkt} solves $H^{\infty}(\xi_{sec\_mkt}) = \max_{\xi' \le \xi_{sec\_mkt}} \left(\frac{1+\alpha(\xi_{sec\_mkt}-\xi')+\xi'}{1+\xi_{sec\_mkt}}\right)^{1-\gamma} H^{\infty}(\xi')$. Let us denote by $\xi' = \xi_m$ the value of ξ' at which the maximum is reached. The smooth pasting and super contact conditions are given by

$$\lim_{\xi \to \xi_{sec\_mkt}} H_\xi^\infty(\xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{2-\gamma}}(1+\xi_m)H(\xi_m) \quad (82)$$

$$\lim_{\xi \to \xi_{sec\_mkt}} H_{\xi\xi}(\xi) = (1-\gamma)(\alpha-1)\frac{(1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m)^{-\gamma}}{(1+\xi_{sec\_mkt})^{3-3\gamma}}$$
$$\left[\frac{-\gamma\alpha(1+\xi_{sec\_mkt})}{1+\alpha(\xi_{sec\_mkt}-\xi_m)+\xi_m} - 2 + \gamma\right] H(\xi_m). \quad (83)$$

B Numerical Methods

After changing variables: $\xi = e^Z$ when $S = C$ and $\xi = \frac{1}{1+e^{-Z}}$ when $S = D$, the value function (10) can be rewritten as

$$F(W, X, S) = W^{1-\gamma}G(Z, S), \quad (84)$$

where $Z = \log(X/W)$, and G solves the following transformed reduced HJB equation:

$$0 = \max_{c,\theta}\left\{\frac{c^{1-\gamma}}{1-\gamma} + B_0(c,\theta,Z)G + B_1(c,\theta,Z)G_Z + B_2(c,\theta)G_{ZZ} + \lambda_S\left(G^{+S} - G\right)\right\}, \quad (85)$$

where $\lambda_S = \lambda_C$ (resp. $\lambda_S = \lambda_D$) when $S = C$ (resp. $S = D$). The functions B_0, B_1 and B_2 and G^{+S} are defined as follows:

$$\begin{cases} B_0(c,\theta,Z) = -\delta + (1-\gamma)\left(r + (\mu - r)\theta - c - fe^Z\right) - \frac{1}{2}\gamma(1-\gamma)\theta^2\sigma^2 \\ B_1(c,\theta,Z) = -\left(r + (\mu-r)\theta - c - fe^Z\right) + \theta^2\sigma^2(\gamma - \frac{1}{2}) + \nu - \frac{1}{2}\psi^2 + \theta\sigma\psi\rho_L(1-\gamma) \\ B_2(c,\theta) = \frac{1}{2}\theta^2\sigma^2 + \frac{1}{2}\psi^2 - \theta\sigma\psi\rho_L \\ G^{+S} = G(Z, S = D) \text{ if } S = C \text{ and } G^{+S} = G(Z, S = C) \text{ if } S = D. \end{cases} \quad (86)$$

When $S = C$, the parameters ρ_L, ψ and ν are set equal to zero.

We use the Markov Chain Approximation (MCA) of Kushner and Dupuis (2001) to evaluate equations (85), for $S = C$ and $S = D$. The method used applies an MCA to continuous-time continuous state stochastic control problems, by renormalizing finite differences forms as proper Markov chain transition probabilities. These transition probabilities arise when deriving finite difference versions of the HJB equation. In our case, the ODEs (PDEs with more than one fund) are linked: $G(Z, S = C)$ depends on $G(Z, S = D)$ and vice versa.

We present below the main steps of the algorithm, for one private equity fund and with power utility.

We define a grid z_i of values taken by Z, ranging from -10 to 5 with intervals of $h = \frac{1}{100}$. For each state S, we discretize the value function G on the grid of z_i: $g_i = G(z_i)$. g_i satisfies an equation of the form:

$$0 = \max_{c,\theta} \left[\tilde{U}(z_i, c, \theta) + B_0(z_i, c, \theta) g_i + B_1^+(z_i, c, \theta) \frac{g_{i+1} - g_i}{h} + B_1^-(z_i, c, \theta) \frac{g_i - g_{i-1}}{h} + \right.$$
$$\left. B_2(z_i, c, \theta) \frac{g_{i+1} + g_{i-1} - 2g_i}{h^2} \right],$$

where $\tilde{U}(z_i, c, \theta)$ includes the utility term as well as the jump term, $B_1^+(z_i, c, \theta)$ (resp. $B_1^-(z_i, c, \theta)$ is the positive (resp. negative) part of $B_1(z_i, c, \theta)$.

The optimal allocation and consumption policy, as functions of f_i, are obtained from the first order condition.

We rearrange the terms to define transition probabilities and the time step of a Markov chain:

$$\begin{cases} p_u(z_i, c, \theta) = \dfrac{\frac{1}{h} B_1^+(z_i, c, \theta) + \frac{1}{h^2} B_2(z_i, c, \theta)}{-B_0(z_i, c, \theta) + \frac{1}{h} B_1^+(z_i, c, \theta) - \frac{1}{h} B_1^-(z_i, c, \theta) + \frac{2}{h^2} B_2(z_i, c, \theta)} \\ \\ p_d(z_i, c, \theta) = \dfrac{-\frac{1}{h} B_1^-(z_i, c, \theta) + \frac{1}{h^2} B_2(z_i, c, \theta)}{-B_0(z_i, c, \theta) + \frac{1}{h} B_1^+(z_i, c, \theta) - \frac{1}{h} B_1^-(z_i, c, \theta) + \frac{2}{h^2} B_2(z_i, c, \theta)} \\ \\ \delta(z_i, c, \theta) = \dfrac{1}{-B_0(z_i, c, \theta) + \frac{1}{h} B_1^+(z_i, c, \theta) - \frac{1}{h} B_1^-(z_i, c, \theta) + \frac{2}{h^2} B_2(z_i, c, \theta)} \end{cases} \tag{87}$$

Conditional on the allocation and consumption policy, the value function g_i is then given by:

$$g_i = p_u(z_i, c, \theta) g_{i+1} + p_d(z_i, c, \theta) g_{i-1} + \tilde{U}(z_i, c, \theta) \delta(z_i, c, \theta) \tag{88}$$

We observe that g_i depends on $\tilde{U}(z_i, c, \theta)$, which itself depends on the jump term $G^{+S} - G$ in (85), hence on g_i.

The algorithm then consists of two steps: i) the estimation of the value function g_i, for the two states, conditional on the liquid asset portfolio policy, the consumption policy, and the previous round's estimate of g_i used in $\tilde{U}(z_i, c, \theta)$ and ii) the estimation of the optimal allocation and consumption policy given the value function. These two steps are iterated until convergence. We deviate from strictly alternating between (i) and (ii) by running (i) multiple times in succession with updated values for g_i in order to speed up convergence. Denote the value function at iteration k by g_i^k. We stop the procedure when the sum of the absolute value of all innovations is below 10^{-6}: $\sum_i \left| g_i^{k+1} - g_i^k \right| < 10^{-6}$.

To include strategic default and the secondary market, g is set in step (i) equal to the maximum of its derived value and the value the agent would obtain by defaulting in state $S = C$, or selling their stake on the secondary market in state $S = D$. Thus, the smooth pasting and super-contact conditions emerge from optimality and are checked; they are not imposed directly.

For two private equity funds, we use grid intervals for z_i of $\frac{1}{50}$ and a total tolerance of

10^{-1}. Because $N = 2$ require two state variables, we are solving linked-PDEs on $751 \times 751 = 564,001$ points, rather than linked-ODEs on 1501 points. Recall that our tolerances of 10^{-6} and 10^{-1} are the sum of absolute deviations, not the average of absolute deviations.

For probability distributions we use Monte Carlo methods: we use $dt = \frac{1}{100}$ ($dt = \frac{1}{50}$ for $N = 2, \infty$) and simulate wealth shocks using $dZ_t = \epsilon \sim N(0, \sqrt{dt})$. We create a single times series lasting for $1,000,000$ years, taking the evolution of wealth from the budget equations and optimal allocations and consumption from the HJB equation.

Our standard parameters are shown in Table 1.

C Sensitivity of Results to Preferences

A drawback of power utility is that it ties the degree of consumption smoothing of the investor across states (governed by her risk aversion) to the degree of smoothing over time (governed by the elasticity of intertemporal substitution, a.k.a. EIS). In practice, the reciprocal relation that is imposed by power utility on the risk aversion and EIS coefficients does not necessarily hold. To disentangle the two effects, recursive preferences were introduced by Epstein and Zin (1989) and Epstein and Zin (1991) in a discrete-time setup, and extended by Duffie and Epstein (1992) in continuous-time. We consider a variation of our baseline problem using recursive utility. The investor's value function is given by:

$$F^{EZ}(W_t, X_t, S_t) = \max_{\{\theta, X, c\}} \mathrm{E}_t \left[\int_t^{\infty} f\left(C_u, F^{EZ}(W_u, X_u, S_u)\right) du \right], \tag{89}$$

where the time aggregator f is defined as in Duffie and Epstein (1992):

$$f(C, J) = \frac{\delta}{1-\zeta} \left(\frac{C^{1-\zeta}}{((1-\gamma)J)^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)J \right). \tag{90}$$

The maximization is subject to the budget constraints (3) and (5). ζ denotes the inverse of the elasticity of intertemporal substitution. When $\zeta = \gamma$, the problem reduces to our baseline problem with CRRA preferences.

If the investor has Epstein-Zin preferences and solves Problem (89), the function H^{EZ} solves analogous ODEs to those describer in Section A.1.

Proposition 11 (Baseline Epstein-Zin, one fund) *The investor's value function can be written as in (10). Between private equity commitments and capital calls, $H^{EZ}(\xi, S = C)$ is characterized by*

$$\begin{aligned} 0 = \max_{c,\theta} \Bigg[& \frac{\delta}{1-\zeta} \left(\frac{c^{1-\zeta}}{((1-\gamma)H^{EZ})^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)H^{EZ} \right) + A_0(c, \theta, S = C)H^{EZ} \\ & + A_1(\xi, c, \theta, S = C)H^{EZ}_{\xi} + A_2(\xi, c, \theta, S = C)H^{EZ}_{\xi\xi} + \lambda_C \left(H^{EZ,+C} - H^{EZ}\right) \Bigg] \end{aligned} \tag{91}$$

Between capital calls and distributions, $H(\xi, S = D)$ is characterized by

$$0 = \max_{c,\theta} \left[\frac{\delta}{1-\zeta} \left(\frac{[c(1-\zeta)]^{1-\zeta}}{((1-\gamma)H^{EZ})^{\frac{\gamma-\zeta}{1-\gamma}}} - (1-\gamma)H^{EZ} \right) + A_0(\xi, c, \theta, S = D)H^{EZ} \right.$$
$$\left. + A_1(\xi, c, \theta, S = D)H^{EZ}_{\xi} + A_2(\xi, c, \theta, S = D)\frac{1}{2}H^{EZ}_{\xi\xi} + \lambda_D(\max_{\xi} H^{EZ,+D} - H^{EZ}) \right] \tag{92}$$

The functions A_0, A_1 and A_2 are defined as in Online Appendix A.1.

Figure A1 shows the investor's consumption, as a function of the PE allocation and for two values of the EIS: 2 and 4 (reduces to power utility). The thin dashed (respectively, plain) line represents the consumption policy of an investor with Epstein-Zin utility during the commitment period (resp. holding period), to be compared to the wide dashed (resp. plain) line which represents the consumption policy of an investor with power utility. As a lower EIS captures a weaker desire to smooth consumption over time, consumption is higher when using Epstein-Zin utility with $\zeta = 2$ than when using power utility ($\zeta = \gamma = 4$), for all PE allocations. Interestingly, the choice of the utility function does not impact the allocation in liquid asset, similarly to Ang et al. (2014). It only affects consumption, and hence the allocation in the risk-free asset.

Figure A1: Consumption under Recursive Preferences

This figure represents the optimal consumption of an investor as a function of their PE allocation. The consumption of an investor with power utility is the thicker dashed line during the commitment period and the thicker plain line during the holding period. The consumption of an investor with Epstein-Zin utility and EIS = 6 is the thinner upper dashed line during the commitment period and the thinner upper plain line during the holding period. With EIS=2, it is, respectively the thinner lower dashed and plain lines. Default is represented as a circle, sale on the secondary market as a square.

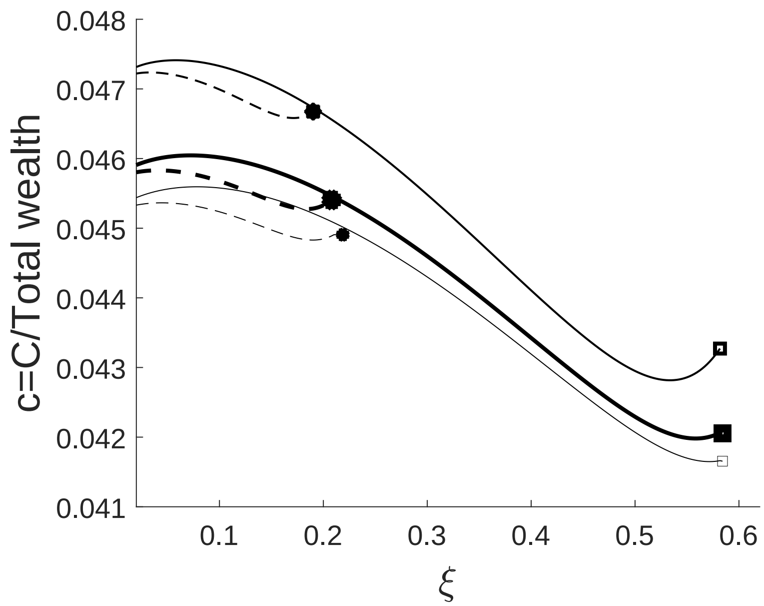


Table A1 reports the optimal PE commitments as well as the welfare costs and return premiums of commitment-quantity risk for investors with $\zeta = 2, 4$ and 6. $\gamma = 4$ throughout. All our findings still hold with Epstein-Zin utility.

Table A1: PE allocations, welfare costs and premiums with recursive utility

This table displays the results of the calibration of our one-fund model described in Section 3 for an investor with Epstein-Zin utility and an EIS varying from 2 (first line) to 6 (last line). The optimal commitment to PE is reported in column 2. The welfare cost and the return premium of commitment-quantity risk (Economy 2 relative to Economy 0) are in columns 3 and 4. Parameters are set as listed in Table 1. The case of EIS=4 corresponds to an investor with power utility and reproduces the results presented in Sections 4.2 and 5.

	PE allocation	Welfare cost	Return premium
ζ=2	5.13%	1.17%	1.06%
ζ=4	5.23%	1.21%	1.05%
ζ=6	5.23%	1.22%	1.03%