

# Beyond Incomplete Spanning: Convenience Yields and Exchange Rate Disconnect\*

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## Abstract

We introduce safe asset demand for dollar-denominated bonds into a tractable incomplete-market equilibrium model of exchange rates and interest rates. The convenience yield enters as a stochastic wedge in the Euler equation for exchange rate determination. Our model makes progress on three exchange rate puzzles. (1) The model can rationalize the low pass-through of SDF shocks to exchange rates and hence low exchange rate volatility. (2) It helps address but does not fully resolve the exchange rate disconnect puzzle. (3) The model generates an unconditional log currency expected return on the dollar that is in line with the data. Our model also identifies a novel safe-asset convenience yield channel by which quantitative easing and the Fed's dollar swap lines impact the dollar exchange rate.

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# 1 Introduction

We introduce safe asset demand for dollar bonds into an otherwise standard, two-country, incomplete markets exchange economy. The model makes progress on outstanding exchange rate puzzles that complete market models cannot address. The analysis also uncovers a novel convenience yield mechanism through which quantitative easing affects the dollar exchange rate.

We focus on safe asset demand for dollar-denominated assets for three reasons. First, there is strong empirical evidence connecting movements in the dollar exchange rate and measures of the convenience (safety/liquidity) services that foreign investors attach to U.S. dollar safe bonds (see recent work by [Avdjiev, Du, Koch, and Shin, 2019](#); [Jiang, Krishnamurthy, and Lustig, 2021a](#); [Engel and Wu, 2021](#)). Second, recent work on market segmentation and intermediation has found that Euler equation wedges can help to address exchange rate puzzles. In particular, in the models developed by [Gabaix and Maggiori \(2015\)](#) and [Itskhoki and Mukhin \(2021\)](#), the exchange rate is determined by the Euler equation of a specialized FX intermediary. Intermediation frictions give rise to a wedge in the Euler equation of standard investors who do not operate in foreign exchange markets and/or foreign bond markets, and this research shows that the wedge can resolve exchange rate puzzles.<sup>1</sup> Convenience yields also enter as a stochastic wedge in the foreign investors' Euler equation when pricing U.S. dollar bonds relative to foreign bonds. These convenience yields can be inferred from the Covered Interest Rate Parity (C.I.P.) violations in government bond markets to discipline the Euler equation wedges. Third, there is a well-documented channel by which quantitative easing (QE) affects exchange rate (see [Neely, 2015](#); [Krishnamurthy and Lustig, 2019](#)). Since QE affects convenience yields on Treasury bonds, as outlined in [Krishnamurthy and Vissing-Jorgensen \(2011\)](#), the safe-asset demand ingredient may shed light on the QE-exchange rate connection.

We report four key findings in a calibrated version of the model. First, the wedges introduced by convenience yields mitigate the pass-through of shocks from the stochastic discount factors (SDF) to exchange rates. As a result, the model-implied exchange rates are not as volatile as in the complete markets model. Second, the covariance between shocks to the USD convenience yield and the SDFs substantially reduces the counter-cyclicality of exchange rates. Third, the model generates an unconditional log currency expected return that is in line with the data. Fourth, we uncover a connection between quantitative easing and exchange rates via convenience yields, distinct from the portfolio channel studied in [Gourinchas, Ray, and Vayanos \(2019\)](#); [Greenwood, Hanson, Stein, and Sunderam \(2020\)](#). We also present empirical evidence supporting the model's QE channel.

We consider a two-country exchange economy. Investors in both countries derive convenience utility on their holdings of U.S. Treasuries. Markets are not segmented. All investors can buy U.S. Treasuries and foreign bonds. We characterize the exchange rate processes that satisfy the

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<sup>1</sup>[Alvarez, Atkeson, and Kehoe \(2002\)](#); [Dou and Verdelhan \(2015\)](#); [Chien, Lustig, and Naknoi \(2020\)](#); [Sandulescu, Trojani, and Vedolin \(2020\)](#) develop models with segmented asset markets to address exchange rate puzzles. These models also generate an Euler equation wedge.

four first-order-conditions for the investors (2 investors  $\times$  2 bonds). Following the preference-free approach to FX markets (Backus, Foresi, and Telmer, 2001; Lustig, Roussanov, and Verdelhan, 2011; Lustig and Verdelhan, 2019), we posit a pair of U.S. (dollar) and foreign log SDFs, in the tradition of Hansen and Jagannathan (1991), and a stochastic convenience yield difference process. This approach allows us to derive a tractable expression for the exchange rate as a function of the histories of U.S. and foreign SDF shocks and USD convenience yield shocks.<sup>2</sup>

First, we make progress on the exchange rate volatility puzzle.<sup>3</sup> Our model's equilibrium exchange rates in logs are less volatile than the difference of U.S. and foreign log SDFs. This result, which is a convenience-yield variant of the result derived in Lustig and Verdelhan (2019), helps to resolve the volatility of exchange rates vis-à-vis stock prices (Brandt, Cochrane, and Santa-Clara, 2006). In our closed-form characterization, the covariance between the SDF shocks and the exchange rate is tightly connected to the covariance between the exchange rate and the convenience yield. In the complete-markets benchmark model without convenience yields, exchange rates have to fully close the gap between the pricing kernels, absorbing all of the residual shocks. In our model with convenience yields, the convenience yields can partially act as shock absorbers too. To calibrate the model, we match the comovement of convenience yields and exchange rates reported by Jiang, Krishnamurthy, and Lustig (2021a). This calibrated model then matches the volatility of exchange rates in the data. The convenience yields allow us to disentangle the volatility of the exchange rate from that of the SDFs.

Second, we make progress on the exchange rate disconnect puzzle. Convenience yield shocks impact exchange rates in our model. The equilibrium exchange rate reflects expected future interest rate spreads, currency risk premia and USD convenience yields. The dollar appreciates when dollar bonds carry a higher convenience yield. In the case of a foreign flight to the safety of U.S. Treasuries, the convenience yield shock has a higher covariance with the foreign SDF than the U.S. SDF. This being the case, the convenience yield channel counteracts the standard complete markets channel, through which the foreign currency appreciates as foreign investors experience higher than average marginal utility growth in a foreign flight to safety episode. As a result, the dollar can actually appreciate against the foreign currency. We explore this countervailing force in a calibrated version of our model. The model generates an a-cyclical exchange rate, but cannot deliver a pro-cyclical exchange rate in line with the data (Backus and Smith, 1993; Kollmann, 1995). We make progress on the Backus and Smith (1993) puzzle but we do not fully resolve it.

Third, our model generates sizable deviations from U.I.P. Foreign investors earns negative excess return on dollars because the dollar has a positive convenience yield and because it endogenously appreciates when the foreign SDF is high, thereby providing a hedge. Our model delivers a realistic unconditional log currency risk premium while matching the volatility of exchange rates. In stark contrast, Lustig and Verdelhan (2019) show that these moments cannot be

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<sup>2</sup>The long-run expected exchange rate level is well defined, which allows a Froot and Ramadorai (2005)-type representation. We also derive the risk premium implied by the model.

<sup>3</sup>In earlier work, Colacito and Croce (2011) show that correlated long-run risks to consumption help to account for the exchange rate disconnect.

matched jointly in a generic incomplete-market model without convenience yields or other Euler equation wedges, as progress on the risk premium puzzle immediately worsens the other exchange rate puzzles.

The baseline version of model does not feature time-varying prices or quantities of risk. We leave this out to keep the model tractable. As a result, our model does not generate time-variation in the conditional risk premium on foreign currencies, needed to replicate the failure of U.I.P. in the time series, first documented by [Hansen and Hodrick \(1980\)](#); [Fama \(1984\)](#). However, our model does generate time-variation in expected excess returns on long positions in foreign bonds, simply because of the variation in convenience yields. To keep the analysis tractable, we impose stationarity on the exchange rate process.

Fourth, the model sheds light on the connection between QE and exchange rates. In a large class of models with stationary exchange rates, the current exchange rate in deviation from its mean depends only on the spread in the long rates:

$$s_t - \bar{s} = \lim_{T \rightarrow \infty} (T - t)(r_t^{T-t} - r_t^{*,T-t}) \quad (1)$$

There is no long-run exchange rate risk. As a result, the USD bond and the domestic bond are perfect substitutes over longer holding periods for a long-horizon investor.

Recently, [Gourinchas et al. \(2019\)](#); [Greenwood et al. \(2020\)](#) develop equilibrium models of the joint pricing of bonds and currencies to elucidate the workings of QE. In their models, the exchange rate is stationary, so that a version of (1) applies. QE in their model affects the risk premium on long term bonds and thus long yields and the dollar exchange rate.

In our model, the exchange rate expression includes a novel convenience yield term:

$$s_t - \bar{s} = \lim_{T \rightarrow \infty} (T - t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du. \quad (2)$$

Holding fixed the differences in the long rates, the dollar appreciates when the future convenience yields  $\tilde{\lambda}^f$  foreign investors earn on U.S. Treasurys increase.

Our paper is the first to examine this distinct convenience yield channel of QE in FX markets, separate from the bond risk premium channel. When the Fed buys U.S. Treasurys, and reserves are more desirable as safe assets, then the convenience yield on dollar-denominated safe assets declines, and the dollar depreciates. If reserves are poor substitutes, then the convenience yields increase and the dollar appreciates. There is both empirical and theoretical support for the proposition that shifts in the supply of safe assets induced by QE changes the convenience yield on safe bonds ([Krishnamurthy and Vissing-Jorgensen, 2011](#)).

In closely related work, [Dou and Verdelhan \(2015\)](#); [Itskhoki and Mukhin \(2021\)](#); [Chien, Lustig, and Naknoi \(2020\)](#) develop international macro models with segmented markets to attack the exchange rate disconnect puzzle. Their models sever the equilibrium exchange rate from its macro-fundamentals by introducing market segmentation and deliver a pro-cyclical exchange rate based on the model's assumed patterns in the arbitrageur's portfolio. For example, [Chien et al. \(2020\)](#)

consider a model in which only small pool of investors arbitrage between domestic and foreign securities. As a result, the real exchange rate is disconnected from the differences in aggregate consumption growth between U.S. and foreign. Our model does not rely on market segmentation.

The rest of the paper proceeds as follows. Section 2 presents our model of exchange rate determination with convenience yields. Section 3 calibrates the model and examines the its implications for a common set of exchange rate puzzles. Section 4 discusses how quantitative easing impacts currency markets through the lens of our model and provides empirical support. Proofs of all propositions are in the Appendix.

## 2 Model

We develop an incomplete-market model of exchange rates in continuous time. Our economy has an infinite horizon. We fix a probability space  $(\Omega, \mathcal{F}, P)$  and a given filtration  $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$  satisfying the usual conditions. We assume that all stochastic processes are adapted to this filtration. There are two countries, U.S. (the U.S.) and foreign. Let  $s_t$  denote the log real exchange rate. A higher  $s_t$  means a stronger U.S. currency (USD). The key input into the model is that the USD is special in that U.S. and foreign investors earn a convenience yield on USD bonds.

### 2.1 Investor Preferences

The U.S. plays a unique role in the international financial system as the world’s provider of dollar-denominated safe assets, as analyzed by Caballero, Farhi, and Gourinchas (2008), Caballero and Krishnamurthy (2009), Maggiori (2017), He, Krishnamurthy, and Milbradt (2018). We formalize this by assuming that U.S. and foreign investors derive utility from their holdings of the U.S. country’s risk-free bond (U.S. Treasuries). Let  $c_t$  denote the U.S. households’ consumption, and let  $q_{H,t}$  denote the U.S. households’ holding in the U.S. risk-free bond. We assume that the processes  $c_t, q_{H,t}$  are integrable. The investors’ utility is derived over consumption and the dollar value of U.S. bond holdings:

$$u(c_t, q_{H,t}) = w(c_t) + v(q_{H,t}; \theta_t), \tag{3}$$

where  $\theta_t$  is a time-varying demand shifter for U.S. bonds. We assume that the utility is increasing in the consumption and the holding in the U.S. bonds, i.e.  $w'(c_t) > 0$  and  $v'(q_{H,t}; \theta_t) > 0$ , and the marginal utility for holding U.S. bonds is decreasing in quantity, i.e.,  $v''(q) < 0$ . In this way, the U.S. risk-free bond carries a convenience yield, which captures its non-pecuniary benefits to U.S. and foreign investors, which is decreasing in the quantity held. We also assume that the exponentially discounted utility functions  $w(\cdot)$  and  $v(\cdot; \theta)$  are integrable.

The expected lifetime utility for U.S. investors is

$$\mathbb{E}_0 \left[ \int_{t=0}^{\infty} e^{-\rho t} u(c_t, q_{H,t}) dt \right] = \mathbb{E}_0 \left[ \int_{t=0}^{\infty} e^{-\rho t} (w(c_t) + v(q_{H,t}; \theta_t)) dt \right]. \quad (4)$$

Similarly, for the foreign investors, we assume their expected lifetime utility is

$$\mathbb{E}_0 \left[ \int_{t=0}^{\infty} e^{-\rho t} u(c_t^*, q_{H,t}^*) dt \right] = \mathbb{E}_0 \left[ \int_{t=0}^{\infty} e^{-\rho t} (w(c_t^*) + v(q_{H,t}^* \exp(s_t); \theta_t^*)) dt \right], \quad (5)$$

where  $c_t^*$  denotes their aggregate consumption,  $\theta_t^*$  denotes their time-varying demand shifter for U.S. bonds, and let  $q_{H,t}^*$  denote their holding in the U.S. risk-free bond.

## 2.2 A Quartet of Euler Equations

Markets are not segmented. We assume that all investors can trade both U.S. and foreign risk-free bonds. The model analysis is centered around four Euler equations, 2 for the U.S. investors and 2 for the foreign ones. The asset markets for other risky claims, such as equity and long-term debt, may or may not be open, which allows us to model a general form of market incompleteness.

The U.S. [instantaneous] risk-free bond has a constant price  $P_t = 1$  and an interest rate  $r_t$ , and the foreign risk-free bond has a constant price  $P_t^* = 1$  and an interest rate  $r_t^*$ . These interest rates are determined in equilibrium from the Euler equations.

We start by characterizing the optimality conditions for U.S. households.

**Lemma 1.** *The first-order conditions for the U.S. investor with respect to holdings in the U.S. and foreign risk-free bonds are given by<sup>4</sup>*

$$0 = \mathbb{E}_t \left[ \frac{dM_t}{M_t} \right] + r_t + \frac{v'(q_{H,t}; \theta_t)}{w'(c_t)} \quad (6)$$

$$0 = \mathbb{E}_t \left[ \frac{d(M_t \exp(-s_t))}{M_t \exp(-s_t)} \right] + r_t^* \quad (7)$$

where,

$$M_t = e^{-\rho t} w'(c_t)$$

is the SDF of the U.S. investor.

To interpret this result, we note that the discrete-time counterparts to these equations are given by the following expressions:

$$1 - \frac{v'(q_{H,t}; \theta_t)}{w'(c_t)} = \mathbb{E}_t \left[ \frac{M_{t+1}}{M_t} \exp(r_t) \right] \quad (8)$$

$$1 = \mathbb{E}_t \left[ \frac{M_{t+1}}{M_t} \exp(r_t^* - \Delta s_{t+1}) \right] \quad (9)$$

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<sup>4</sup>With some abuse of notation we use the notation  $\mathbb{E}_t[dX_t]$  to represent the infinitesimal generator of a stochastic process  $X_t$ . The formal notation, which is adopted in the proof, is  $\mathcal{A}[X_t]$ .

Given  $v'(q_{H,t}; \theta_t) > 0$  and  $w'(c_t) > 0$ , the left-hand side of Eq.(8) is less than 1. U.S. investors are willing to accept a lower risk-neutral expected return in exchange for holding the U.S. risk-free bond, whereas the risk-neutral expected return on the foreign risk-free bond is exactly 1. We refer to the gap  $\frac{v'(q_{H,t}; \theta_t)}{w'(c_t)}$  as the convenience yield that U.S. households attach to dollar bonds.

Similarly, the foreign households' asset pricing conditions for the foreign bond and for U.S. Treasuries, respectively, are given by:

$$0 = \mathbb{E}_t \left[ \frac{dM_t^*}{M_t^*} \right] + r_t^* \quad (10)$$

$$0 = \mathbb{E}_t \left[ \frac{d(M_t^* \exp(s_t))}{M_t^* \exp(s_t)} \right] + r_t + \frac{v'(q_{H,t}^* \exp(s_t); \theta_t^*)}{w'(c_t^*)} \quad (11)$$

Let us define  $\tilde{\lambda}_t^h = \frac{v'(q_{H,t}; \theta_t)}{w'(c_t)}$  as the convenience yield earned U.S. investors on their dollar bond holdings. Likewise, define  $\tilde{\lambda}_t^f = \frac{v'(q_{H,t}^* \exp(s_t); \theta_t^*)}{w'(c_t^*)}$  as the foreign investors' convenience yield on dollar bonds. Then, we can rewrite pricing conditions (6) and (11) as:

$$0 = \mathbb{E}_t \left[ \frac{dM_t}{M_t} \right] + r_t + \tilde{\lambda}_t^h \quad (12)$$

$$0 = \mathbb{E}_t \left[ \frac{d(M_t^* \exp(s_t))}{M_t^* \exp(s_t)} \right] + r_t + \tilde{\lambda}_t^f \quad (13)$$

These Euler equations, together with (7) and (10) must be satisfied in equilibrium. We will search for interest rate and exchange rate processes that satisfy the four Euler equations.

### 2.3 Pricing Kernel Dynamics

We posit a pair of U.S. (dollar) and foreign log SDFs. In addition, we assume that foreign investors receive a stochastic convenience yield on their holding of U.S. (dollar) bonds. We then use the U.S. and foreign Euler equation for each bond to derive a closed-form expression for the exchange rate.

Let  $m_t = \log(M_t)$  and  $m_t^* = \log(M_t^*)$  denote the log SDFs. We posit that the log SDFs have the following dynamics:

$$dm_t = -\mu dt - \sigma dZ_t, \quad (14)$$

$$dm_t^* = \phi s_t dt - \sigma dZ_t^*, \quad (15)$$

We work with the dynamics of the SDF in our derivations, rather than the dynamics of the underlying consumption processes of U.S. and foreign households. We take these dynamics as primitives rather than solving a macroeconomic model to derive these dynamics.

Here,  $\{Z_t, Z_t^*\}$  are standard Brownian motion processes. The Brownian increments  $dZ_t$  and  $dZ_t^*$  represent shocks to the marginal utilities of each agent, which capture business cycle shocks to consumption. The dynamics for the foreign SDF describe risk-free rate dynamics in the foreign country engineered to keep the real exchange rate stationary. The SDF dynamics describe an

implicit monetary policy rule required for stationarity as in [Engel and West \(2005\)](#).

**Assumption 1.** *We assume that the mean-reversion parameter  $\phi > 0$  is strictly positive.*

Assumption 1 implies that the foreign real interest rate is decreasing in the level of the U.S. real exchange rate. In particular, if markets are complete, the log of the real exchange rate  $s_t^{cm}$  equals the difference in the log of the SDFs:

$$s_t^{cm} = m_t - m_t^*, \quad (16)$$

$$ds_t^{cm} = (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t), \quad (17)$$

which is a simple stationary process.

We assume that the convenience yields derived by U.S. and foreign investors can be different. As we show later, the difference in these yields is all that matters for exchange rate dynamics. We denote the difference between the two convenience yields as  $\tilde{\lambda}_t = \tilde{\lambda}_t^f - \tilde{\lambda}_t^h$ , and parameterize the difference as follows:

$$\tilde{\lambda}_t = \ell \frac{\exp(\lambda_t)}{\exp(\lambda_t) + 1}, \quad (18)$$

which is bounded between 0 and  $\ell$ . The variable  $\lambda_t$  satisfies

$$d\lambda_t = -\theta\lambda_t dt + \nu dX_t, \quad (19)$$

where  $dX_t$  is a standard Brownian motion on  $(\Omega, \mathcal{F}, \mathcal{P})$  adapted to the filtration  $\mathbb{F}$ .

Finally, with slight abuse of notation, let  $[dX_t, dY_t]$  denote the instantaneous conditional covariance between two diffusion processes  $X_t$  and  $Y_t$ . Formally, it is defined as  $[dX_t, dY_t] = d[X, Y]_t / dt$  where  $[X, Y]_t$  is the standard quadratic covariation between processes  $X_t$  and  $Y_t$ . We assume that the convenience yield shock and the SDF shocks can be pairwise correlated<sup>5</sup>:

$$[dZ_t, dX_t] = \rho, \quad [dZ_t^*, dX_t] = \rho^*, \quad [dZ_t, dZ_t^*] = \zeta.$$

We assume that the correlation of SDF (which loads negatively on  $dZ$  or  $dZ^*$  shocks) and convenience yield innovations is positive ( $\rho, \rho^* \leq 0$ ), so that the convenience yield tends to increase when the marginal utilities as represented by the pricing kernels rise. In high marginal utility states, there is an increased desire by foreign investors to own dollar bonds as in a “flight-to-Treasuries”. [Jiang et al. \(2021a\)](#) present empirical evidence on this point. When global volatility in financial markets increases, the convenience yield on U.S. Treasuries tends to increase.<sup>6</sup>

<sup>5</sup>As the correlation matrix of  $dX_t, dZ_t, dZ_t^*$  needs to be positive semidefinite, these correlation parameters need to satisfy  $\det = 1 + 2\rho\rho^*\zeta - \rho^2 - \rho^{*2} - \zeta^2 \geq 0$ .

<sup>6</sup>In [Jiang et al. \(2021a\)](#) use the U.S. Treasury basis to measure the convenience yield. We define the basis as the difference between the yield on a cash position in U.S. Treasuries  $y_t^{\$}$  and the synthetic dollar yield constructed from a



## 2.4 Equilibrium Exchange Rate Dynamics

Without loss of generality, we write the real exchange rate as satisfying the following stochastic differential equation,

$$ds_t = \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t \nu dX_t, \quad (21)$$

where  $\alpha_t$ ,  $\beta_t$ , and  $\gamma_t$  are  $\mathbb{F}$ -adapted stochastic processes.  $\beta_t$  governs the distance from complete markets. When  $\beta_t \equiv 1$ , and  $\gamma_t \equiv 0$ , we are back in the benchmark complete markets case: the change in the log exchange rate equals the difference in the log pricing kernels.

Our objective is to present a solution to (21) that satisfies the four pricing conditions (6), (7), (10) and (11). In our incomplete market setting, there are many candidate solutions. We restrict attention to a class of these solutions that we are able to characterize and (as we explain) has economically sensible properties. We assume that the loading of the exchange rate on the aggregate shocks is time-invariant:

**Assumption 2.**  $\beta_t \equiv \beta$  is constant.

**Proposition 1.** Under Assumption 2, there is a class of solutions indexed by constant  $k$  so that,

$$\beta_t = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sigma^2(1-\zeta) - 2k}{\sigma^2(1-\zeta)}}, \quad (22)$$

$$\gamma_t = \frac{(\rho^* - \rho)\sigma(1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2\sigma^2(1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}. \quad (23)$$

The log of the real exchange rate satisfies:

$$ds_t = \left( -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) \right) dt + \gamma_t\nu dX_t + \beta\sigma(dZ_t^* - dZ_t), \quad (24)$$

which loads on both the SDF shocks  $dZ$  and  $dZ^*$  and the convenience yield shock  $dX$ .

We explain this result in the next section, presenting the details of the proof in the appendix. For each  $k$ , there are two solutions for  $\beta$ . One root is between 1/2 and 1, and the other is between 0 and 1/2. We will calibrate  $\beta$  based on regression results. As for  $\gamma_t$ , note that  $(\rho^* - \rho)\sigma(1 - 2\beta)$  can

cash position in a foreign government bond, which earns a yield  $y_t^*$  in foreign currency, that is hedged back into dollars:

$$x_t^{Treas} \equiv y_t^{\$} + (f_t^1 - s_t) - y_t^*. \quad (20)$$

Here  $s_t$  denotes the log of the nominal exchange rate in units of foreign currency per dollar,  $f_t^1$  denotes the log of the forward exchange rate, and  $x_t^{Treas}$  measures the violation of the CIP constructed from U.S. Treasury and foreign government bond yields. A negative U.S. Treasury basis means that U.S. Treasuries are expensive relative to their foreign counterparts. The convenience yield can be inferred from the CIP deviations  $x_t^{Treas}$  in government bond markets:  $(1 - \beta^{basis})(\lambda_t^f - \lambda_t^h) = -x_t^{Treas}$ , where  $\beta^{basis}$  measures the fraction of the convenience yield earned on a synthetic Treasury constructed from a foreign currency bond. See [Jiang et al. \(2021a\)](#) for more details.

be either positive or negative. We pick the root of  $\gamma_t$  with the positive sign:

$$\gamma_t = \frac{(\rho^* - \rho)\sigma(1 - 2\beta) + \sqrt{(\rho^* - \rho)^2\sigma^2(1 - 2\beta)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu},$$

so that for  $k > \tilde{\lambda}_t$ , we can guarantee  $\gamma_t > 0$ . We focus on solutions with  $\gamma_t > 0$  to arrive at the natural result that the exchange rate appreciates when the foreign convenience yield for dollar bonds rises. Finally, note that unlike  $\beta_t$ ,  $\gamma_t$  is not constant and varies with the convenience yield,  $\tilde{\lambda}_t$ .

The SDFs are highly volatile. When  $\beta = 1$  and  $\gamma_t = 0$ , we are back in the workhorse complete markets case. This exchange rate process is the only one that satisfies not only the foreign and domestic bond investor Euler equations, but also all of the other Euler equations when a rich menu of assets is traded across borders. As soon as we deviate from the complete markets case ( $0 < \beta < 1, \gamma > 0$ ), the exchange rate responds to the convenience yield shocks and the exchange rates no longer absorb all of the shocks to the SDFs. The convenience yields do some of the shock absorption. These candidate exchange rate processes satisfy the bond investors' Euler equations.

Furthermore, we can solve the stochastic differential equation (24) to find a closed-form expression for the log of the real exchange rate.

**Proposition 2.** *The real exchange rate  $s_t$  can be expressed as*

$$s_t = f(\lambda_t) + H_t + \beta s_t^{cm}. \quad (25)$$

The first term  $f(\lambda_t)$  is a function of the current convenience yield  $\lambda_t$ . Let  $b = (\rho^* - \rho)\sigma(1 - 2\beta)$ , then

$$\begin{aligned} f(\lambda) &= \frac{1}{2\nu} \left\{ -\sqrt{b^2 + 4k} \log \left( 2e^{\lambda/2} \left( \cosh \left( \frac{\lambda}{2} \right) \left( \sqrt{b^2 + 4k} \sqrt{b^2 + 4k - 2\ell \tanh \left( \frac{\lambda}{2} \right) - 2\ell + b^2 + 4k - \ell} \right) - \ell \sinh \left( \frac{\lambda}{2} \right) \right) \right) \right. \\ &+ \sqrt{b^2 + 4k - 4\ell} \log \left( 2e^{\lambda/2} \left( \cosh \left( \frac{\lambda}{2} \right) \left( \sqrt{b^2 + 4k - 4\ell} \sqrt{b^2 + 4k - 2\ell \tanh \left( \frac{\lambda}{2} \right) - 2\ell + b^2 + 4k - 3\ell} \right) - \ell \sinh \left( \frac{\lambda}{2} \right) \right) \right) \\ &+ \lambda \left( \sqrt{b^2 + 4k + b} \right) \}. \end{aligned}$$

The second term  $H_t$  captures the history of past convenience yields:

$$\begin{aligned} H_t &= \exp(-\phi t) H_0 + \int_0^t \exp(-\phi(t-u)) h(\lambda_u) du, \\ h(\lambda_t) &= -\frac{1}{2} \tilde{\lambda}_t - \phi f - (1 - \beta)\mu + \frac{1}{2} \sigma \gamma_t \nu (\rho + \rho^*) + f' \theta \lambda_t - \frac{1}{2} f'' \nu^2. \end{aligned}$$

The third term is the real exchange rate  $s_t^{cm}$  under complete markets scaled by  $\beta$ , where

$$ds_t^{cm} = (-\mu - \phi s_t^{cm}) dt + \sigma (dZ_t^* - dZ_t). \quad (26)$$

The proof is in the appendix. This proposition shows that the real exchange rate level is determined by not only the relative pricing kernels, as summarized by the real exchange rate  $s_t^{cm}$  under

complete markets, but also the current convenience yield and the history of the convenience yields  $\lambda_t$ .

We also note that since the exchange rate's long-run expectation  $\lim_{T \rightarrow \infty} \mathbb{E}_t[s_T]$  is well-defined:

**Proposition 3.** *When the markets are incomplete, the exchange rate's long-run expectation  $\lim_{T \rightarrow \infty} \mathbb{E}_t[s_T]$  is*

$$\bar{s} \equiv \lim_{T \rightarrow \infty} \mathbb{E}_t[s_T] = \frac{1}{\phi} \left( -\frac{1}{2} \lim_{T \rightarrow \infty} \mathbb{E}_0[\tilde{\lambda}_T] - \mu + \frac{1}{2} \sigma \lim_{T \rightarrow \infty} \mathbb{E}_0[\gamma_T] \nu(\rho + \rho^*) \right).$$

In comparison, the complete-market counterpart is

$$\bar{s}^{cm} = \lim_{T \rightarrow \infty} \mathbb{E}_t[s_T^{cm}] = -\frac{\mu}{\phi},$$

which does not have the “convenience yield” term  $-\frac{1}{2} \lim_{T \rightarrow \infty} \mathbb{E}_0[\tilde{\lambda}_t]$  and the “risk premium” term  $\frac{1}{2} \sigma \lim_{T \rightarrow \infty} \mathbb{E}_0[\gamma_t] \nu(\rho + \rho^*)$ .

## 2.5 Exchange Rate Accounting

With this long-run expectation, the exchange rate level  $s_t$  has a forward-looking representation:

$$s_t = \bar{s} - \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T ds_u, \quad (27)$$

where the long-run expectation of the exchange rate  $\bar{s}$  is derived in Appendix A.3. Following [Froot and Ramadorai \(2005\)](#); [Jiang et al. \(2021a\)](#), we can further decompose the exchange rate level in the following way.

**Corollary 1.**

$$s_t = \bar{s} + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T (r_u - r_u^*) du + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du - \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T rp_u du. \quad (28)$$

On the right-hand side,  $\lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T (r_u - r_u^*) du$  captures expected future short rate differences,  $\lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du$  captures expected future convenience yields earned by the foreign investors, and  $-\lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T rp_u du$  captures expected future currency risk premia from the foreign perspective plus a Jensen's term,

$$rp_u = \left( \frac{1}{2} \tilde{\lambda}_u + \frac{1}{2} \sigma \gamma_u \nu(\rho + \rho^*) \right) = - \left( [dm_t^*, ds_t] + \frac{1}{2} [ds_t, ds_t] \right).$$

This decomposition in equation (28) is the equivalent of a Campbell-Shiller decomposition for exchange rates. The exchange rate level today reflects future interest rate differences (cash flows), future convenience yields, minus future risk premia (discount rates). This expression is forward-looking, which complements the backward-looking expression for the exchange rate level in equation (25). A version of this decomposition without convenience yields was derived by [Campbell](#)

and Clarida (1987); Froot and Ramadorai (2005). As we explain in section ?? of the Appendix, this expression without the convenience yields holds in large class of models. The dollar appreciates when future U.S. short rates increase and dollar currency risk premia decline. Jiang et al. (2021a) derive a version of this decomposition that allows for convenience yields. When foreign investors expect to earn larger convenience yields on USD bonds, the dollar appreciates in spot markets.

Alternatively, we can define a USD bond yield without convenience yield,  $r_t^{x^cy} = r_t + \tilde{\lambda}_t^h$ . Then, the exchange rate decomposition becomes

$$s_t = \bar{s} + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T (r_u^{x^cy} - r_u^*) du + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du - \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T r p_u du,$$

in which case the first term becomes the interest rate differential of USD and foreign bonds without convenience yields, and the second term becomes the differential in the convenience yields from foreign and U.S. investors' perspectives.

We use  $r_t^T$  to denote yield on a  $T$ -period zero coupon bond.  $rx_t^T$  to denote the conditional risk premium on  $T$ -period zero coupon bond. The USD long bond yields can be restated as the sum of the local currency bond risk premia and future risk-free rates:  $(T-t)r_t^{T-t} = \int_t^T rx_u^{T-t} du + \mathbb{E}_t \int_t^T (r_u - r_u^*) du$ . As a result, we can rewrite the exchange rate decomposition in equation (28) as follows:

$$\begin{aligned} s_t - \bar{s} &= \lim_{T \rightarrow \infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T (rx_u^{*,T-t-u} - rx_u^{T-t-u}) du \\ &- \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T r p_u du + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du. \end{aligned} \quad (29)$$

When the exchange rate is stationary, the long-term USD bond and foreign bond have to carry the same risk premium in the limit (Backus, Boyarchenko, and Chernov, 2018; Lustig, Stathopoulos, and Verdelhan, 2019). There is no difference in riskiness between holding a U.S. and a foreign bond over long holding periods. In this case the sum of currency risk premia are exactly offset by the sum of local currency bond risk premium differentials between the two countries. We obtain that the long-run U.I.P. is exactly captured by the future convenience yields:

$$s_t - \bar{s} = \lim_{T \rightarrow \infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du. \quad (30)$$

## 2.6 Characterizing the Candidate Exchange Rate Processes

This section explains why there are multiple solutions to the model. Consider the pair of Euler equations for the U.S. investor in the foreign bond (equation (7)) and the foreign investor in the U.S. bond (equation (11)). We rewrite these equations to derive expressions for the currency risk premia on long positions in USD and foreign currency, respectively:

$$r_t - r_t^* + \mathbb{E}_t[ds_t] = - \left( \frac{1}{2} [ds_t, ds_t] - [\sigma dZ_t^*, ds_t] \right) - \tilde{\lambda}_t^f,$$

$$r_t^* - r_t - \mathbb{E}_t[ds_t] = - \left( \frac{1}{2} [ds_t, ds_t] + [-\sigma dZ_t, -ds_t] \right) + \tilde{\lambda}_t^h.$$

These equations can be interpreted as follows. The expected log excess return on long positions in U.S. bonds harvested by the foreign investor, given by the interest rate difference plus the expected rate of appreciation of the U.S. currency, equals the log currency risk premium minus the convenience yield. The expression for the expected log excess return for the U.S. investor is similar: it equals the log currency risk premium plus the U.S. investors' convenience yields on holding U.S. bonds. The sum of these two Euler equations produces the following condition:

$$-\tilde{\lambda}_t = \left( \frac{1}{2} [ds_t, ds_t] - \sigma [dZ_t^*, ds_t] \right) + \left( \frac{1}{2} [ds_t, ds_t] + \sigma [dZ_t, ds_t] \right). \quad (31)$$

The two terms in parentheses are respectively the log currency risk premium for the U.S. investor going long foreign bonds and the foreign investor going long the U.S. bond.

First, we consider the case where there are no convenience yields;  $\tilde{\lambda}_t$  is always zero. In this symmetric case, these risk premia have to sum to zero. If one investor is earning a risk premium, the other investor must be paying the risk premium. In the case without convenience yields, consider the exchange rate process:

$$ds_t = \alpha_t dt + \beta \sigma (dZ_t^* - dZ_t).$$

That is, the only uncertainty is driven by the two Brownian motions driving the SDFs. Substituting into (31), we have that,

$$0 = \beta^2 \sigma^2 - \beta \sigma^2.$$

This equation has two solutions:  $\beta = 0$  and  $\beta = 1$ . The  $\beta = 1$  case corresponds to the complete-market model. The exchange rate is volatile, and the volatility carries a risk premium that compensates for the volatility. The  $\beta = 0$  case is also solution to all of the asset pricing equations. The exchange rate is non-stochastic and there is no risk premium in the model. All Euler equations are satisfied with a purely deterministic exchange rate (note that  $\alpha_t$  will not equal zero). We can think of this case as corresponding to an autarchic allocation: each agent holds their own U.S. bonds and the exchange adjusts deterministically to enforce uncovered interest parity.

Next, we consider a version of our model with stochastic convenience yields. With convenience yields, the foreign investor's demand for dollar bonds necessarily reduces the foreign investor's pecuniary return to going long dollar bonds relative to foreign bonds (i.e., the non-pecuniary convenience yield partially offsets this reduced pecuniary return). But this means that the U.S. investor can earn an excess return by going long foreign bonds relative to dollar bonds. If the exchange rate is non-stochastic, this cannot be an equilibrium since the excess return to the U.S. investor offers an infinite Sharpe ratio. Thus the exchange rate must be stochastic, but as we show next, there is still a family of solutions that arises. We substitute in the exchange rate process

from (21) into (31) to give,

$$-\tilde{\lambda}_t = \gamma_t^2 v^2 + 2\beta_t^2 \sigma^2 + 2\gamma_t v \beta_t (\rho^* - \rho) \sigma - 2\beta_t \sigma^2 - (\rho^* - \rho) \sigma \gamma_t v. \quad (32)$$

Under our Assumption 2, we take  $\beta_t$  as constant and look for solutions for  $\beta$  that satisfy:

$$0 = 2\beta^2 \sigma^2 - 2\beta \sigma^2 + k, \quad (33)$$

for constant  $k$ . Likewise, we look for solutions for  $\gamma_t$  that satisfy:

$$k - \tilde{\lambda}_t = \gamma_t^2 v^2 + 2\gamma_t v \beta (\rho^* - \rho) \sigma - (\rho^* - \rho) \sigma \gamma_t v. \quad (34)$$

Then  $k$  indexes a family of solutions with varying pass-through from the convenience yield and SDF shocks to the exchange rate. Mathematically, the zero volatility case is no longer a solution because if  $k = 0$ , the solution for  $\gamma_t$  is imaginary. This latter point can be seen by inspecting (23). We note that  $k$  indexes the solution for both  $\beta$  and  $\gamma$ . A key property of these solutions is that higher  $\beta$  goes together with higher  $\gamma$ . The next section builds on this observation.

Lastly, we point out that there exist additional sets of equilibrium exchange rates. In Appendix B, we consider an exchange rate process that depends not only on the SDF differential ( $dZ_t^* - dZ_t$ ) and the convenience yield shock  $dX_t$ , but also on an additional Euler equation wedge  $dW_t$  that is orthogonal to the convenience yield shock:

$$ds_t = \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t v dX_t + \omega_t dW_t. \quad (35)$$

In other words, we may consider the entire  $(\gamma_t v dX_t + \omega_t dW_t)$  term as the incomplete-market wedge. In our benchmark case, this wedge is assumed to be perfectly correlated with the convenience yield shock. In this more general setting, it could be partially correlated. The new term  $dW_t$  induces additional variance in the exchange rate movements.

### 3 Quantitative Implications of Convenience Yields for Exchange Rates

This section discusses (1) the comovement between dollar exchange rate and flight-to-safety as in Jiang et al. (2021a), (2) the partial SDF-FX pass-through and the Brandt et al. (2006) puzzle, (3) the Backus-Smith puzzle, (4) currency risk premium in log and in level, and (5) the Froot-Ramadorai decomposition of exchange rate level. Our model provides a quantitative account of these patterns FX dynamics driven by our convenience model of exchange rates. We begin by explaining our calibration choices.

### 3.1 Calibration Choices

Since the Great Financial Crisis, sizable deviations from Covered Interest Parity have opened up in LIBOR markets (Du, Tepper, and Verdelhan, 2018b), but even before the GFC, there were large, persistent deviations from CIP in government bond markets (see Du, Im, and Schreger, 2018a; Jiang et al., 2021a; Du and Schreger, 2021). U.S. Treasurys are always expensive relative to synthetic Treasurys constructed from foreign bonds. Following Jiang et al. (2021a), we infer the convenience yields earned on U.S. Treasurys by foreign investors from these C.I.P. deviations to discipline the Euler equation wedges.<sup>7</sup>

We calibrate the model at the annual frequency, and report our parameter values in Table 1. This set of parameter values implies that the convenience yield  $\tilde{\lambda}_t$  process has an unconditional mean of 2.5% and an unconditional standard deviation of 1.6% per annum. In Jiang et al. (2021a), we directly measure the U.S. Treasury basis. The convenience yield can be inferred from the CIP deviations  $x_t^{Treas}$  in government bond markets:

$$(1 - \beta^{basis})(\lambda_t^f - \lambda_t^h) = -x_t^{Treas},$$

where  $\beta^{basis}$  measures the fraction of the convenience yield earned on a synthetic U.S. Treasury constructed from a foreign currency denominated bond. Jiang et al. (2021a) estimate the constant of proportionality  $\beta^{basis}$  to be 0.9 so that the mean Treasury basis of 0.22 gives a mean convenience yield of  $0.22/(1 - 0.9) = 2.2\%$ . The standard deviation of the Treasury basis of 0.23 implies a standard deviation of the convenience yield of  $0.23/(1 - 0.9) = 2.3\%$ . Moreover, the mean-reversion parameter  $\theta = 3$  implies that the convenience yield shocks have a half-life of  $\log(2)/\theta = 0.23$  years. In the data, we estimate an AR(1) model of the Treasury basis and find the estimated model to have a half-life of 0.24 years.

The pricing kernel volatility  $\sigma$  is calibrated to 50% per annum, which implies that the maximal annual Sharpe ratio permitted by either country's pricing kernel is roughly 0.5 as well. We further assume that the correlation between the U.S. SDF shock and the convenience yield shock is  $\rho = 0$ , and the correlation between the foreign SDF shock and the convenience yield shock is  $\rho^* = -0.50$ . This assumption implies that the foreign agents' marginal utility goes up when the convenience yield increases. Moreover, we set the correlation between U.S. and foreign SDF shocks to 0.32, which is the average correlation between the U.S. consumption growth and other G10 countries' consumption growth, using annual data from 1970 to 2018. Alternatively, we could calibrate this parameter using the average correlation between the change in the U.S. stock log price-to-dividend ratio and other G10 countries', which yields a slightly higher value of 0.48.

The adjustment in interest rate in response to the exchange rate level is governed by the pa-

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<sup>7</sup>Foreign investors earn convenience yields of around 200 basis points, significantly larger than the CIP deviations. Using a demand-system-based approach, Koijen and Yogo (2020) report similar estimates of the convenience yields. In related work, Augustin, Chernov, Schmid, and Song (2020) study CIP deviations and convenience yields in a no-arbitrage framework. Van Binsbergen, Diamond, and Grotteria (2021) infer the true risk-free rates and the implied convenience yields from option prices. Jiang, Richmond, and Zhang (2021b) study the implications of the flight-to-U.S. safety for portfolio imbalances and international capital flows.

parameter  $\phi$ , which we set to 0.135. This parameter value implies that the half life of the variation in a shock to the real exchange rate is  $\log(2)/\phi = 5.13$  years. In the data, we estimate an AR(1) model of the log dollar index and find the estimated model to have a half-life of 5.18 years.

Note  $k$  can take values between  $[\frac{\ell - (\rho^* - \rho)^2 \sigma^2 / 4}{1 - (\rho^* - \rho)^2 / (2(1 - \zeta))}, \frac{\sigma^2(1 - \zeta)}{2}]$ . Equivalently, the equilibria in this system can be indexed by the value of  $\beta$ , which is bounded by  $[0.14, 0.86]$ . If  $\beta$  is outside of this range,  $\gamma_t$  will have imaginary roots when  $\lambda_t$  is large.

### 3.2 Choosing Parameter $k$

We write the the innovation in the exchange rate in terms of the underlying economic shocks:

$$ds_t - \mathbb{E}_t[ds_t] = \beta(dm_t^* - dm_t - \mathbb{E}_t[dm_t^* - dm_t]) + \gamma_t \frac{\ell}{\tilde{\lambda}_t(\ell - \tilde{\lambda}_t)} (d\tilde{\lambda}_t - \mathbb{E}_t[d\tilde{\lambda}_t]).$$

The first term on the right-hand side is the exchange rate's exposure to the pricing kernel differential's shock. The second term is the exchange rate's exposure to the convenience yield shock.

We note that  $k$  indexes a family of solutions for the pricing kernel exposure  $\beta$  and the convenience yield exposure  $\gamma_t$ . Let us start with the calibration choices above, but with  $\rho = \rho^* = 0$ . In the left panel of Figure 1, we plot  $\beta$  against  $\gamma_t$  evaluated at  $\lambda_t = 0$  for different values of  $\beta$ . This plot is generated by varying  $k$  over its domain. We see that the convenience yield loading  $\gamma_t$  is positively associated with the SDF loading  $\beta$  when  $\beta < 0.5$ , and is negatively associated with the SDF loading  $\beta$  when  $\beta > 0.5$ . When  $\rho = \rho^* = 0$ , our equations can be simplified to

$$\beta = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{\sigma^2(1 - \zeta)}}, \text{ and } \gamma_t = \frac{\sqrt{4(k - \tilde{\lambda}_t)}}{2\nu};$$

when  $\beta$  takes the smaller root (i.e. less than 0.5), both  $\beta$  and  $\gamma_t$  are increasing in  $k$ . For  $\beta > 0.5$ , the

Table 1: Parameter Choices

This table reports the parameter values we use to calibrate the model.

Symbol	Interpretation	Value	Calibration Target
<i>Panel A: SDF and FX</i>			
$\mu$	SDF drift	0	symmetry
$\sigma$	SDF shock volatility	0.5	max Sharpe ratio
$\zeta$	SDF shock correlation	0.32	consumption growth correlation
$\phi$	exchange rate persistence	0.135	half life of exchange rate shock
$\rho$	U.S. SDF loading on convenience shock	0	
$\rho^*$	foreign SDF loading on convenience shock	-0.5	
<i>Panel B: Convenience Yield</i>			
$\ell$	convenience yield level	5%	average Treasury basis
$\nu$	convenience yield volatility	5	Treasury basis volatility
$\theta$	convenience yield persistence	3	half life of Treasury basis shock



$\beta$  is decreasing in  $k$ , while  $\gamma_t$  is increasing in  $k$ .

In the panel on the right, we report the case with  $\rho^* = -0.5$  which corresponds to our principal calibration. Over most of the range of  $\beta$ , the convenience yield loading  $\gamma_t$  is positively associated with the SDF loading  $\beta$ . Algebraically, from equation (34), when  $\rho - \rho^* = 0.5$ , the term on the right-hand side contributes to the relation, thus strengthening the relation between  $\gamma$  and  $\beta$ . In our calibration exercise, for each value of  $k$ , we compute  $\gamma_t$  and select the smaller root of  $\beta$ , and then simulate the model. The greater root of  $\beta$  will generate much greater exchange rate volatility that is counterfactual.

### 3.3 Exchange Rate and Flight-to-Safety

Jiang et al. (2021a) show that the dollar’s real exchange rate is increasing in the convenience yield that foreign investors assign to the dollar risk-free bond. Specifically, when the U.S. Treasury’s convenience yield increases by one standard deviation (0.23% as measured by Treasury basis), the dollar appreciates by 2.35%. In the post-2008 sample, the one-standard deviation shock leads to a dollar appreciation of 3.28%. We target this regression coefficient to pin down  $\gamma_t$  and then via the logic of the model, also pin down  $\beta$ .

We discretize the model by a time increment of  $\Delta t = 0.001$  period and simulate 50,000 periods. Table 2 presents regression results from the simulated sample. The top panel reports results for the case with flight to quality by foreign investors. The bottom panel reports results for the case without flight to quality. We pick 6 different values of  $k$ , ranging from the minimum to the maximum possible values. Then, we run the regression of the exchange rate movement  $\Delta s_t$  on the change in the convenience yield  $\Delta \tilde{\lambda}_t$ , and report the regression coefficient in Column (3). In our preferred case in the second row of the top panel, with a low value of  $\beta = 0.18$ , this coefficient is 1.16. In comparison, the aforementioned empirical result in Jiang et al. (2021a) suggests that the slope coefficient should be between 1.02 and 1.49.

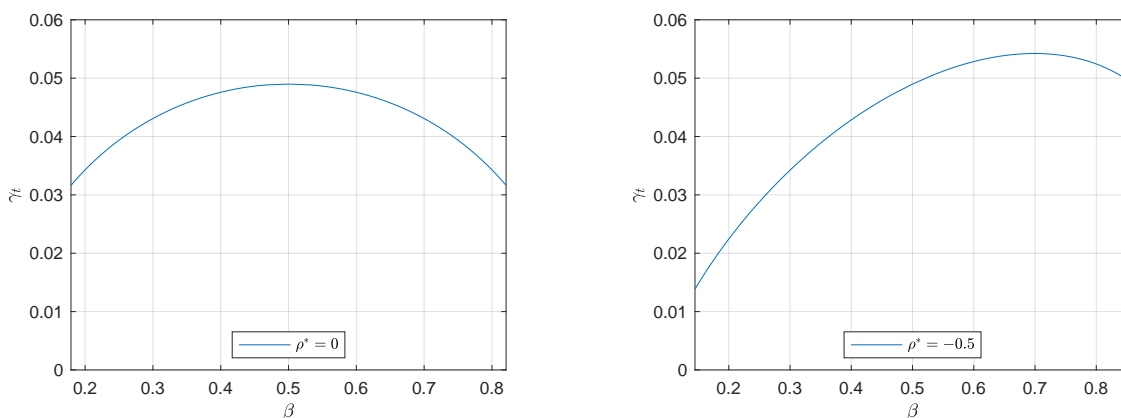


Figure 1: FX Loadings on the SDF and the Convenience Yield Shocks

The lower panel of the Table reports the results for the case of  $\rho^* = 0$ . In the version of the model without flight-to-quality, the model generates too high a regression coefficient on the convenience yield innovation. As we discuss in the next sections, this parameterization also generates too high an exchange rate volatility, as shown in column (4) of the table.

### 3.4 Partial SDF-FX Pass-through and FX Volatility

Under complete markets, the real exchange rate follows

$$ds_t^{cm} = \alpha_t^{cm} dt + \beta_t^{cm} \sigma(dZ_t^* - dZ_t) + \gamma_t^{cm} \nu dX_t = (-\mu - \phi s_t^{cm}) dt + \sigma(dZ_t^* - dZ_t), \quad (36)$$

which does not load on the convenience yield shock  $dX$ , i.e.  $\gamma_t^{cm} = 0$ , and moves one-to-one with the shocks to the pricing kernels, i.e.  $\beta_t^{cm} = 1$ .

Table 2: Simulation Results

Columns (1) and (2) report the parameter values. (3) reports the slope coefficient in regression of  $\Delta s_t$  on  $\Delta \tilde{\lambda}_t$ . (4) reports annual FX volatility. (5) reports the slope coefficient in regression of  $\Delta s$  on  $\Delta m - \Delta m^*$ . (6) reports the annual expected log excess return on long position in the U.S. dollar. The regressions are run at quarterly frequency. Our simulation is based on a long sample of  $T = 50,000 \times 1000$  subperiods.

<i>Panel A: <math>\rho^* = -0.5</math></i>						
(1)	(2)	(3)	(4)	(5)	(6)	
$k$	$\beta$	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp. Log Return (%)	
0.04	0.14	0.58	9.74	0.10	-1.97	
<b>0.05</b>	<b>0.18</b>	<b>1.16</b>	<b>11.56</b>	<b>0.11</b>	<b>-2.40</b>	
0.06	0.22	1.64	14.10	0.13	-2.77	
0.07	0.28	2.06	17.07	0.16	-3.15	
0.08	0.34	2.45	20.72	0.20	-3.55	
0.09	0.50	2.80	28.70	0.32	-4.24	
<i>Panel B: <math>\rho^* = 0</math></i>						
(1)	(2)	(3)	(4)	(5)	(6)	
$k$	$\beta$	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp. Log Return (%)	
0.05	0.18	3.59	18.66	0.18	-1.32	
0.06	0.21	4.16	21.47	0.21	-1.32	
0.06	0.25	4.65	24.27	0.25	-1.33	
0.07	0.30	5.07	27.22	0.29	-1.35	
0.08	0.36	5.46	30.65	0.35	-1.36	
0.09	0.50	5.83	37.69	0.50	-1.40	
<i>Panel C: Comparisons</i>						
(1)	(2)	(3)	(4)	(5)	(6)	
$k$	$\beta$	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp. Log Return (%)	
Data	-	1.02—1.49	10.00	< 0	-1.89	
1	0	-6.03	57.83	1.00	-0.23	

In contrast, under incomplete markets with a convenience yield, the real exchange rate follows

$$ds_t = \left( -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t v(\rho + \rho^*) \right) dt + \gamma_t v dX_t + \beta_t \sigma (dZ_t^* - dZ_t), \quad (37)$$

which loads on the convenience yield shock  $dX$  while having only a partial pass-through governed by  $0 < \beta < 1$  from the SDF shocks to the real exchange rate movement  $ds_t$ .

[Lustig and Verdelhan \(2019\)](#) provide a related result. They show that incomplete markets introduce a wedge in the exchange rate movement and this wedge is always negatively correlated with the SDF differential, which as a result partially offset the exchange rate movements induced by the SDF differential and lead to a less volatile exchange rate movement. In our model, we interpret this wedge as a convenience yield, and furthermore, we calibrate the relation between convenience yields and exchange rates based on the empirical analysis in [Jiang et al. \(2021a\)](#). This approach allows us to go further than [Lustig and Verdelhan \(2019\)](#) and nail down the extent of incomplete pass-through.

From [Table 2](#) we see that in our preferred calibration,  $\beta$  equals 0.11. The SDF volatility is 50%, but the exchange rate volatility is only 10%. Higher values of  $k$  lead to higher values of  $\beta$  and higher exchange rate volatility. This partial SDF-FX pass-through result helps resolve the volatility puzzle of [Brandt et al. \(2006\)](#); the complete markets  $dm - dm^*$  is more volatile than  $ds$ . In particular, the conditional variance of the exchange rate movement is

$$[ds_t, ds_t] = \gamma_t^2 v^2 + 2(1 - \zeta)\beta^2 \sigma^2 + 2\gamma_t v \beta \sigma (\rho^* - \rho), \quad (38)$$

whereas under complete markets, it is

$$[ds_t^{cm}, ds_t^{cm}] = 2(1 - \zeta)\sigma^2 \quad (39)$$

The reduced pass-through in our model is due to both a  $\beta$  that is much smaller than one, and  $\rho^* - \rho = -0.5$ , which reduces the volatility in [equation \(38\)](#).

### 3.5 Backus-Smith Puzzle

The exchange rate movement  $ds$  is exposed to both the SDF shock and the convenience yield shock. In relation to the Backus-Smith puzzle, we calculate the slope coefficient in a projection of the exchange rate changes the relative log SDF differential:

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \beta + \frac{\gamma_t v (\rho^* - \rho)}{2\sigma(1 - \zeta)}. \quad (40)$$

From Table 2 we see that this coefficient is 0.1 in our model. For comparison, under complete markets,  $\beta = 1$  and  $\gamma_t = 0$ , and therefore

$$\frac{[ds_t^{cm}, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = 1. \quad (41)$$

Under incomplete markets, as  $\beta$  is below one, the first term in (40) shrinks the covariance  $[ds_t, dm_t - dm_t^*]$  towards 0. This is the channel due to market incompleteness that was highlighted by [Lustig and Verdelhan \(2019\)](#). The second term results from the correlation between the SDF shock and the convenience yield shock. If  $\rho^* < \rho$ , i.e. the foreign country's pricing kernel is more exposed to the convenience yield shock than the U.S. country, this term is negative, which further reduces the slope coefficient in equation (40).

While our parameterization generates a Backus-Smith coefficient near zero, it does not generate a negative coefficient. The exchange rate is still counter-cyclical: the model generates an appreciation of the foreign currency when the foreign investors experience higher marginal utility growth than the U.S. investors<sup>8</sup>.

### 3.6 Currency Risk Premium

The expected log excess return on going long U.S. government bonds relative to foreign government bonds is given by:

$$\pi_t^f = \mathbb{E}_t[d \log \exp(r_t + s_t - r_t^*)] = \mathbb{E}_t[ds_t] + r_t - r_t^* \quad (42)$$

$$= -\frac{1}{2}\tilde{\lambda}_t + \frac{1}{2}\sigma\gamma_tv(\rho + \rho^*) - \tilde{\lambda}_t^h \quad (43)$$

$$= -\frac{1}{2}(\tilde{\lambda}_t^f + \tilde{\lambda}_t^h) + \frac{1}{2}\sigma\gamma_tv(\rho + \rho^*) \quad (44)$$

The first term captures the dollar's convenience yield earned by foreign investors. The foreign investors derive non-pecuniary benefits from holding the dollar bond, and therefore require a lower expected return to hold dollar government bonds. The second term captures the dollar's currency risk premium. As the convenience yield shock is correlated with the SDF shocks, the magnitude of the risk premium depends on the correlations  $\rho$  and  $\rho^*$ .

This term is  $-\frac{1}{2}\tilde{\lambda}_t$  instead of  $-\tilde{\lambda}_t$  because the other half of the convenience yield is in the Jensen's term  $\frac{1}{2}[ds_t, ds_t]$ . In levels, the expected return on a long position in USD, in excess of the foreign risk-free rate, from the perspective of the foreign investor, loads on the convenience yield  $\tilde{\lambda}_t$  with a coefficient of  $-1$ :

$$\Pi_t^f = \frac{\mathbb{E}_t[d \exp(r_t + s_t - r_t^*)]}{\exp(r_t + s_t - r_t^*)} = \mathbb{E}_t[ds_t] + \frac{1}{2}[ds_t, ds_t] + r_t - r_t^*$$

<sup>8</sup>In fact, Appendix A.4 shows that the Backus-Smith coefficient remains non-negative in our setting for arbitrary parameter values. That said, the incomplete-market wedge can generate negative Backus-Smith coefficients for some country pairs in a multi-currency world. See [Jiang \(2021\)](#) for a detailed discussion.

$$\begin{aligned}
&= -\tilde{\lambda}_t^f + \beta\sigma^2(1 - \zeta) + \sigma\gamma_t\nu\rho^* \\
&= -\tilde{\lambda}_t^f - [dm^*, ds_t]
\end{aligned}$$

This expected return in levels declines one-for-one with the dollar convenience yield. The expected return also declines as  $\rho^*$  declines, since higher foreign marginal utility growth coincides on average with higher convenience yields and smaller depreciation of the USD. On the other hand, the level of the expected foreign currency return, from the perspective of the U.S. investor, only reflects the covariance between the U.S. investor's SDF and the exchange rate movement:

$$\begin{aligned}
\Pi_t^h &= \frac{\mathbb{E}_t [d \exp(-s_t - r_t + r_t^*)]}{\exp(-s_t - r_t + r_t^*)} = -\mathbb{E}_t[ds_t] + \frac{1}{2}[ds_t, ds_t] - r_t + r_t^* \\
&= \tilde{\lambda}_t^h + \beta\sigma^2(1 - \zeta) - \sigma\gamma_t\nu\rho \\
&= \tilde{\lambda}_t^h + [dm_t, ds_t]
\end{aligned}$$

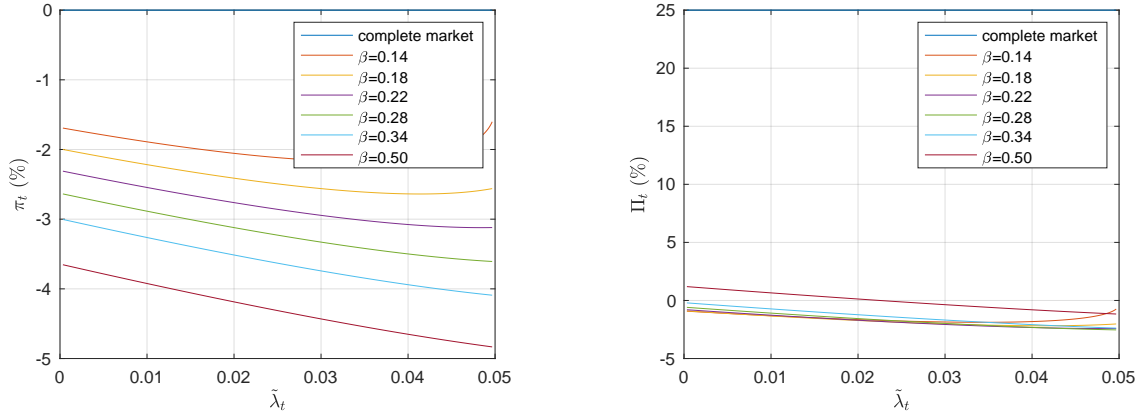
This risk premium of the dollar is solely driven by the combination of market incompleteness and the cyclicity of the convenience yield. For comparison, if markets are complete, since the U.S. and the foreign SDFs have the same volatilities, the log currency risk premium on USD is zero and the risk premium in levels equals the variance of the SDF (Bansal, 1997; Backus et al., 2001).

$$\begin{aligned}
\pi_t^{cm,f} &= 0, \\
\Pi_t^{cm,f} = \tilde{\Pi}_t^{cm,h} &= (1 - \zeta)\sigma^2.
\end{aligned} \tag{45} \tag{46}$$

In this case, the log currency risk premium is too small relative to the data whereas the level of currency risk premium is too large. The complete markets part of the model could be extended to generate non-zero complete markets currency risk premia by introducing asymmetries and time variation in the quantity and price of risk, as in the work of Verdelhan (2010); Colacito and Croce (2011); Farhi and Gabaix (2016).

In Table 2 we report that the expected log return in the model is  $-2.44\%$ . For comparison, in Jiang et al. (2021a) we compute the returns for a foreign investor to owning the entire U.S. Treasury bond index relative to their U.S. government bond index, over a sample from 1980 to 2019. We report that the dollar Treasury return is  $1.89\%$  lower than the foreign bond return, which is close to the model-implied estimate of  $2.44\%$ . According to equation (44), given an average convenience yield of  $\mathbb{E}[\tilde{\lambda}_t] = 1.9\%$ , our model indicates that about  $\frac{1}{2}\mathbb{E}[\tilde{\lambda}_t] = 0.95\%$  in the expected log return is attributable to the convenience yield, and the remaining  $2.44\% - 0.95\% = 1.49\%$  is attributable to the dollar's log risk premium.

Figure 2 further plot the currency risk premium as a function of  $\beta$  and for different values of  $\tilde{\lambda}_t$ . We plot both the level and the log expected return. As expected from equation (44), fixing  $\beta$ , the dollar's expected return declines with the current convenience yield  $\tilde{\lambda}_t$ .



Panel (a) Expected Log Excess Return  $\pi_t^f$

Panel (b) Expected Excess Return Level  $\Pi_t^f$

Figure 2: The Dollar's Expected Excess Return

**Forward Premium Puzzle** The SDFs have constant volatility in this model. The standard approach to introducing time variation in the conditional currency risk premium is to introduce time-varying volatility in the SDFs, which in turn can result from either changes in the quantities of risk or changes in the prices of risk. We have left out these features in order to derive a closed-form solution for the exchange rate dynamics. A more general model will be able to generate realistic variation in both the convenience yields and in the conditional currency risk premia.

That said, we note that while the dollar's risk premium is decreasing in the average convenience yield<sup>9</sup>,

$$\pi_t^f = -\frac{1}{2}(\tilde{\lambda}_t^f + \tilde{\lambda}_t^h) + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*),$$

whereas the interest rate differential is decreasing in the U.S. investors' convenience yield

$$r_t - r_t^* = \mu + \phi s_t - \tilde{\lambda}_t^h.$$

It is natural to expect that the U.S. and foreign investors' convenience yields on the U.S. bonds are positively correlated. Then, when the demand for the safe U.S. bonds goes up, the U.S. interest rate goes down while the U.S. dollar has a lower expected return. In this way, the convenience yield also generates the forward risk premium via our incomplete-market mechanism, without requiring a time-varying currency risk premium.

<sup>9</sup>In fact,  $\gamma_t$  is also a function of  $\tilde{\lambda}_t$ . So the relationship between  $\pi_t^f$  and convenience yields is more nuanced. Under our calibration,  $\gamma_t$  is decreasing in  $\tilde{\lambda}_t$ , so the dollar exchange rate's loading on the convenience yield shock is lower when the convenience yield is higher. Since  $\rho + \rho^* < 0$ , the risk premium component in the dollar's expected log excess return,  $\frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*)$ , is increasing in  $\tilde{\lambda}_t$ . However, this effect is dwarfed by the convenience yield component, so the dollar's expected log excess return is still decreasing in  $\tilde{\lambda}_t$  in Figure 2.

## 4 Quantitative Easing

Quantitative easing (QE) policies—that is, large scale purchases of long-term bonds matched by increases in bank reserves—have been shown to affect exchange rates (Neely, 2015). In this section, we show how our model can shed light on this connection.

### 4.1 Segmented Bond Markets and the Bond Risk Premium Channel

Gourinchas et al. (2019) and Greenwood et al. (2020) bring an equilibrium model of the term structure with market segmentation along the lines of Vayanos and Vila (2021) to bear on FX markets. These authors explore the impact of downward sloping demand curves for Treasuries. A decrease in the net U.S. supply of long bonds, as would occur under a QE purchase by the Fed, causes U.S. arbitrageurs to lower their required bond risk premium on long USD bonds. As a result, policy makers can control long rates.

In the settings of Gourinchas et al. (2019); Greenwood et al. (2020), the exchange rate is stationary. As a result, there is no difference in riskiness between holding a U.S. and a foreign bond over long holding periods. Then, they have to carry the same risk premium in the limit (Backus et al., 2018; Lustig et al., 2019). In the forward looking exchange rate expression in (28), the sum of currency risk premia are exactly offset by the sum of local currency bond risk premium differentials between the two countries. When the exchange rate is stationary, the exchange rate reflects differences in long yields:

$$s_t - \bar{s} = \lim_{T \rightarrow \infty} (T - t)(r_t^{T-t} - r_t^{*,T-t}). \quad (47)$$

Inside this model, the Fed is able to lower long yields via a QE purchase and thereby cause the USD to depreciate. Note that the modeled QE channel is symmetric; purchases by the ECB, BoJ, or BoE also affect bond risk premia, long yields, and the exchange rate via (47).

### 4.2 Convenience Yield Channel

Our work identifies a novel convenience yield channel through which large scale asset purchases affect exchange rates. The dollar appreciates when future U.S. Treasury convenience yields  $\lambda$  increase, holding constant the long yields. To explain the channel, let us consider the first-order condition for the foreign investor into U.S. bonds in (10):

$$0 = \mathbb{E}_t \left[ \frac{dM_t^*}{M_t^*} \right] + r_t + \overbrace{\frac{v'(q_{H,t}^* \exp(s_t); \theta_t)}{w'(c_t)}}^{\lambda_t^f}$$

Suppose also that the U.S. investor derives no convenience services on its holding of dollar bonds (i.e.  $v'(\cdot)$  for the U.S. investor is zero).<sup>10</sup> In this case, changes in the supply of dollar safe assets will lead to changes in  $q_{H,t}^*$  and alter the convenience yield,  $\lambda_t^f$ . Then, since,

$$s_t - \bar{s} = \lim_{T \rightarrow \infty} (T - t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T \rightarrow \infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u^f du \quad (48)$$

the exchange rate will also change.

The convenience yield channel creates a distinct role for flows/quantities. Shifts in the supply of dollar safe assets, as happens via QE will change convenience yields and exchange rates. The convenience yield channel is outlined in [Krishnamurthy and Vissing-Jorgensen \(2011\)](#). A swap of mortgage-backed securities for reserves likely increases the supply of safe assets, since reserves are a more convenient asset than mortgage-backed securities. A swap of Treasuries for reserves may increase or decrease the supply of safe assets depending on whether banks pass on the reserve expansion by expanding deposits, and the relative convenience of these deposits and Treasuries. Thus, convenience yields can either rise or fall with QE. Our theory of exchange rate connects the convenience yield with exchange rates. That is, QE that increases the convenience yield on dollar bonds should be expected to appreciate the dollar, while QE that decreases the convenience yield should be expected to depreciate the dollar. Thus our channel does not imply that QE always depreciates the dollar, as do the theories of [Gourinchas et al. \(2019\)](#) and [Greenwood et al. \(2020\)](#). Note also that the convenience yield channel assigns a special role to the U.S., to the extent that the U.S. is the world's safe asset supplier.

Figure 3 presents evidence linking changes in convenience yields around QE-event dates and changes in the dollar exchange rate. The dollar exchange rate is measured as the equal-weighted G-10 cross. The basis is the 1-year U.S. Treasury against an equal-weighted currency-hedged 1-year G-10 government bond. The data is from [Krishnamurthy and Lustig \(2019\)](#). As we show theoretically in [Jiang et al. \(2021a\)](#), the basis is proportional to the convenience yield on U.S. Treasury bonds relative to foreign bonds.

We note two key patterns in this figure that our consistent with our model. First, the dollar appreciates in some of these events, while it depreciates in others. Second, despite this fact, there is a clear association in both the sign and magnitude of the change in the dollar lines with changes in the basis. Table 3 presents this evidence in a regression. We regress the 2-day (Panel A) and 3-day (Panel B) change in the exchange rate against the change in the basis, controlling for the change in the relative interest rates in U.S. and foreign, which can control for shifts in the stance of monetary policy. At both horizons and measuring the basis using different maturity bonds, there is a strong relation between QE-induced changes in the basis and the dollar. Focusing on the 1-year basis in Panel A, we see that a 10 basis point change in the Treasury basis leads to a 1.66%

<sup>10</sup>In the case, where both U.S. and foreign households derive convenience yields, we need to model how each investor's portfolio will shift in response to changes in the convenience yield to map the quantity shift into changes in the convenience yield. But other than in pathological cases, the sign of a change in supply on convenience yields will be the same as in the case discussed above.



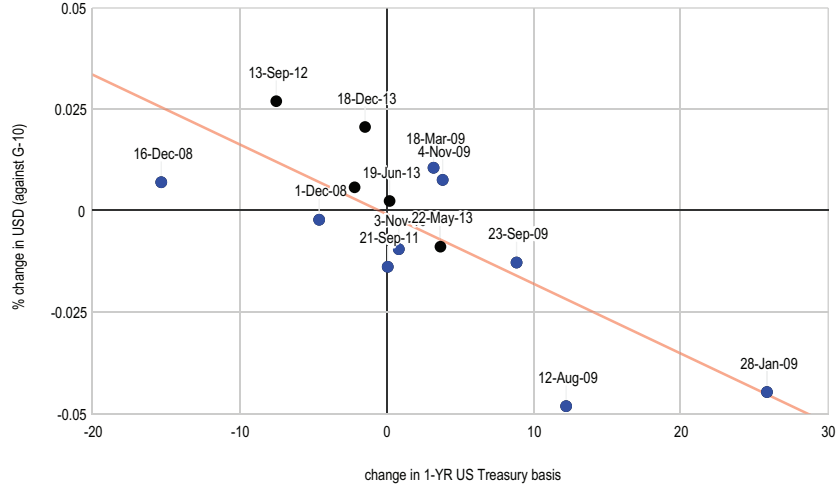


Figure 3: G-10 Dollar appreciation against change in basis around QE event dates. Sample of 14 QE event dates. 2-day window after QE-event dates. We include the event day and define the change in the basis ( $\Delta$  Basis) and the change in the dollar from the close of trading on the day prior to the event day to the close of trading 2-days later.

appreciation in the dollar. From the results in [Jiang et al. \(2021a\)](#), a 10 basis point change in the basis is equal to 1% change in the convenience yield.

Last consider the impact of the Fed’s dollar swap lines through the lens of our model. While our model predicts that the association between QE and exchange rate changes should not always have the same sign, its deeper prediction is between changes in the supply of safe dollar assets and exchange rates. The Fed’s dollar swap lines increase the supply of safe dollar assets abroad, which lowers the convenience yield that foreign investors impute on safe dollar assets. Through our expression of dollar exchange rate determination, this action supports foreign exchange rates. There is empirical evidence for this channel. [Baba and Packer \(2009\)](#); [Aizenman, Ito, and Pasricha \(2022\)](#) present evidence that the dollar swap lines between central banks reduce the dollar-foreign currency basis (which can measure the convenience yield) and depreciate the dollar.

### 4.3 QE Evaluation in the Model

Next, we turn to our model to see how well it can capture these patterns. We do not explicitly model the relation between the convenience yield  $\lambda$  and the quantity of safe assets. Instead, we focus directly on inducing a shock to  $\lambda$  and tracing out the impact of this shock on the exchange rate. We discretize the model by a time increment of  $\Delta t = 0.0025$  and start the model at  $t = 0$ . For initial values, we set  $s_0 = s_0^{cm} = \bar{s}$  and  $\lambda_0 = 0$ , and set  $H_0$  to satisfy

$$s_0 = f(\lambda_0) + H_0 + \beta s_0^{cm}. \quad (49)$$

We simulate  $dX_t$ ,  $dZ_t$  and  $dZ_t^*$  under the normal distribution with mean zero and standard

Table 3: QE, Basis, and Exchange Rate

Regression of changes in dollar (G-10) on QE-induced changes in U.S. Treasury basis and changes in yields. We include 14 QE event dates. We include the event day and define the change in the basis ( $\Delta$  Basis) and the change in the dollar from the close of trading on the day prior to the event day to the close of trading  $x$  days later.  $\Delta y$ -diff is the change in the 1-year interest rate differential between the U.S. and the G-10 average.

		3M	1Y	2Y	3Y	5Y	7Y	10Y
Panel A: 2-day window								
$\Delta$ Basis	coeff	-0.247	-0.166	-0.240	-0.225	-0.170	-0.189	-0.152
	s.e.	0.057	0.028	0.035	0.037	0.034	0.047	0.050
$\Delta y$ -diff	coeff	20.012	31.381	17.501	16.338	12.568	12.857	11.231
	s.e.	9.066	8.031	3.092	2.951	2.610	2.624	3.195
	$R^2$	0.637	0.828	0.837	0.800	0.751	0.697	0.563
Panel B: 3-day window								
$\Delta$ Basis	coeff	-0.219	-0.188	-0.175	-0.183	-0.135	-0.106	-0.083
	s.e.	0.051	0.027	0.036	0.037	0.036	0.037	0.043
$\Delta y$ -diff	coeff	15.319	22.568	15.494	13.861	12.186	12.068	11.944
	s.e.	7.054	6.307	3.227	2.541	2.064	2.253	2.685
	$R^2$	0.624	0.811	0.745	0.779	0.778	0.724	0.643

deviation  $\sqrt{\Delta t}$ . For the first quarter, i.e., periods  $(0, 0.25]$ , we introduce a positive impulse that raises all realizations of the shocks  $dX_t$  by one standard deviation. This impulse simulates a positive convenience yield shock in the first quarter. Then, we average across 100,000 simulated paths of the shocks  $(dX_t, dZ_t, dZ_t^*)$ . In this way, we estimate the average response following a positive convenience yield shock at date 0. We also simulate a benchmark case in which we draw from the normal distribution with mean 0 for the entire period  $t \in (0, T]$ . As expected, the average responses of exchange rate and convenience yield are close to zero in this benchmark case. We report the difference between the average responses in the case of a convenience yield shock and the benchmark case.

Figure 4 reports the result. In the top-left panel, we shock the convenience yield  $\lambda_t$  and then let the internal dynamics of mean reversion gradually bring the convenience yield to zero over the next 10 quarters. We can think of this shock as an announcement by the central bank to purchase assets at date 0, and then slowly unwind these purchases over the next 10 quarters.

The top-left panel of the figure graphs the instantaneous convenience yield over this path. The top-right panel plots the average convenience yield between time 0 and time  $t$  ( $= \frac{1}{t} \int_{k=0}^t \tilde{\lambda}_k dt$ ). This panel gives an expectations-hypothesis-type heuristic of how different maturity bases will react to this shock. We see that the largest response is in the short maturity bases with the effects dying out for longer maturity bases. At the one year point, the convenience yield rises by about 0.35% (given a 1:10 ratio between Treasury basis and convenience yield, this implies a widening in

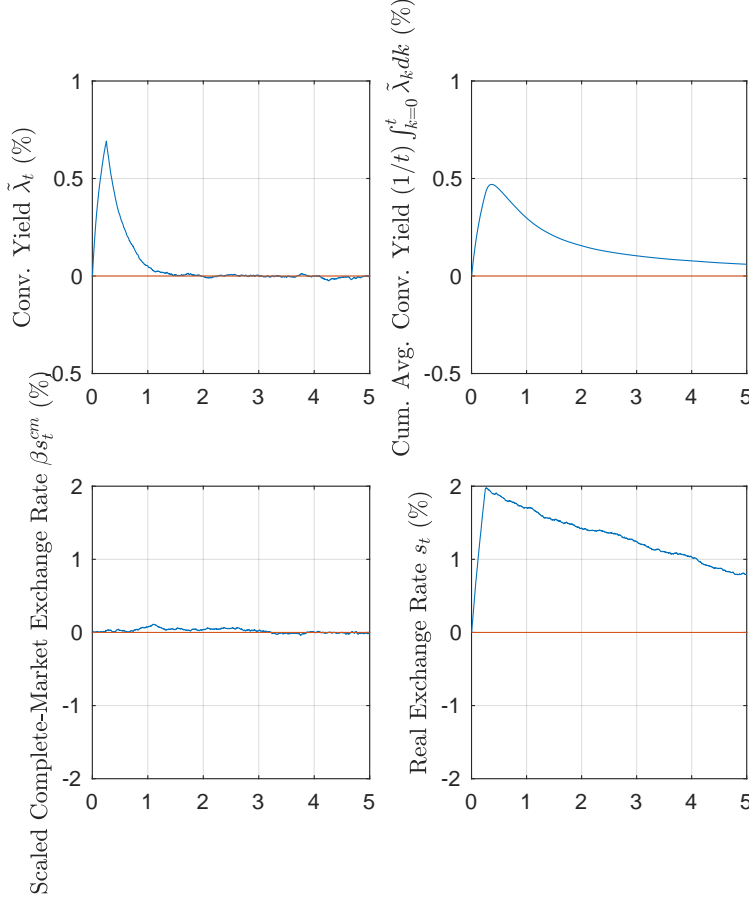


Figure 4: Impulse Response to a Convenience Yield Shock.

We report the average difference between simulations in which the convenience yield  $\lambda_t$  jumps up by 1 standard deviation in period  $(0, 0.25]$  and simulations in which all shocks have zero means. The cumulative average convenience yield in the top right panel is the expected average convenience yield for the next  $t$  periods at time 0.

the Treasury basis of 3.5 basis points). The bottom-left panel plots the complete markets exchange rate averaged across simulation paths. The last panel plots the exchange rate from the model. On impact, the exchange jumps by 1.7%, before gradually reverting to its long-run level. Thus quantitatively, our model generates a regression coefficient on the 1-year basis of  $-0.5$ , which is of the same magnitude but greater than the empirical estimates in Table 3.

The effect of this QE experiment unwinds gradually over the next several years. We note that the behavior in term  $H_t$  representing the cumulative convenience yields is also interesting. Since

$$H_t = \exp(-\phi t)H_0 + \int_0^t \exp(-\phi(t-u))h(\lambda_u)du, \quad (50)$$

it aggregates influences of past convenience yields with an exponential decay. As a result, the half life of the response in real exchange rate is longer than the half life of the response in the spot convenience yield.

## 5 Conclusion

Our paper delivers a fully specified no-arbitrage model of exchange rates, interest rates and convenience yields. We show how convenience yields interact with incomplete markets and make progress on the exchange rate puzzles. Moreover, in our model, the U.S. central bank can directly affect the dollar exchange rate, not by changing bond currency risk premia, but by changing the convenience yields on dollar-denominated government bonds. We refer to this as the convenience yield channel. Our paper is the first to embed this convenience yield channel in a no-arbitrage model of exchange rates. This channel is complementary to the bond risk premium channel in models with bond market segmentation. The convenience yield channel imputes a unique role to the U.S. central bank in affecting the dollar exchange rate without changing U.S. interest rates, because it can change the convenience yields earned by foreign investors.

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## A Appendix: Proofs

### A.1 Proof of Proposition 1

Recall that the real pricing kernels are

$$dM_t = M_t(-\mu + \frac{1}{2}\sigma^2)dt - M_t\sigma dZ_t \quad (\text{A.1})$$

$$dM_t^* = M_t^*(\phi s_t + \frac{1}{2}\sigma^2)dt - M_t^*\sigma dZ_t^* \quad (\text{A.2})$$

Rewrite the first FOC into the integral and reduced-form

$$0 = \mathcal{A}\left[\int dM_t + M_t r_t dt + M_t \tilde{\lambda}_t^h dt\right] \quad (\text{A.3})$$

$$r_t = \mu - \frac{1}{2}\sigma^2 - \tilde{\lambda}_t^h \quad (\text{A.4})$$

The third FOC becomes

$$0 = \mathcal{A}\left[\int dM_t^* + M_t^* r_t^* dt\right] \quad (\text{A.5})$$

$$r_t^* = -\phi s_t - \frac{1}{2}\sigma^2 \quad (\text{A.6})$$

Notice

$$d\exp(s_t) = \exp(s_t)ds_t + \frac{1}{2}\exp(s_t)[ds_t, ds_t]dt \quad (\text{A.7})$$

$$d\exp(-s_t) = -\exp(-s_t)ds_t + \frac{1}{2}\exp(-s_t)[ds_t, ds_t]dt \quad (\text{A.8})$$

The second FOC becomes

$$0 = \mathcal{A}\left[\int d(M_t \exp(-s_t)) + r_t^* M_t \exp(-s_t) dt\right] \quad (\text{A.9})$$

$$= \mathcal{A}\left[\int (\exp(-s_t)dM_t + M_t d\exp(-s_t) + [dM_t, d\exp(-s_t)])dt + M_t \exp(-s_t)r_t^* dt\right]$$

$$= -\mu + \frac{1}{2}\sigma^2 - \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t, -ds_t] + r_t^* \quad (\text{A.10})$$

The fourth FOC becomes

$$0 = \mathcal{A}\left[\int M_t^* \exp(s_t)\tilde{\lambda}_t^f dt + M_t^* \exp(s_t)r_t dt + d(M_t^* \exp(s_t))\right] \quad (\text{A.11})$$

$$= M_t^* \exp(s_t)(\tilde{\lambda}_t^f + r_t) + \mathcal{A}\left[\int \exp(s_t)dM_t^* + M_t^* d\exp(s_t) + [dM_t^*, d\exp(s_t)]dt\right]$$

$$= \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \quad (\text{A.12})$$

The sum of the third and the fourth FOC is

$$\tilde{\lambda}_t^h - \tilde{\lambda}_t^f = [ds_t, ds_t] - \sigma[dZ_t^* - dZ_t, ds_t], \quad (\text{A.13})$$

which is

$$-\tilde{\lambda}_t = [ds_t, ds_t] - \sigma[dZ_t^* - dZ_t, ds_t] \quad (\text{A.14})$$

Plug in the conjecture

$$ds_t = \alpha_t dt + \gamma_t \nu dX_t + \beta_t \sigma (dZ_t^* - dZ_t), \quad (\text{A.15})$$

then

$$-\tilde{\lambda}_t = \gamma_t^2 \nu^2 + 2\beta_t^2 \sigma^2 (1 - \zeta) + 2\gamma_t \nu \beta_t (\rho^* - \rho) \sigma - 2\beta_t \sigma^2 (1 - \zeta) - (\rho^* - \rho) \sigma \gamma_t \nu \quad (\text{A.16})$$

Suppose for a certain constant  $k$ ,

$$-k = 2(1 - \zeta) \beta_t^2 \sigma^2 - 2(1 - \zeta) \beta_t \sigma^2 \quad (\text{A.17})$$

$$k - \tilde{\lambda}_t = \gamma_t^2 \nu^2 + 2\gamma_t \nu \beta_t (\rho^* - \rho) \sigma - (\rho^* - \rho) \sigma \gamma_t \nu \quad (\text{A.18})$$

The solutions are

$$\beta_t = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{\sigma^2(1 - \zeta)}}, \quad (\text{A.19})$$

$$\gamma_t = \frac{(\rho^* - \rho) \sigma (1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}. \quad (\text{A.20})$$

which has real roots for all possible values of  $\lambda_t$  if and only if

$$k \leq \frac{\sigma^2(1 - \zeta)}{2} \quad (\text{A.21})$$

and

$$k \geq \frac{\ell - (\rho^* - \rho)^2 \sigma^2 / 4}{1 - (\rho^* - \rho)^2 / (2(1 - \zeta))} \quad (\text{A.22})$$

When the upper bound of  $k$  is obtained,  $\beta_t = 1/2$ . When the lower bound of  $k$  is obtained,

$$\beta_t = \frac{1}{2} \pm \frac{1}{2\sigma} \sqrt{\frac{\sigma^2(1 - \zeta) - 2\ell}{(1 - \zeta) - (\rho^* - \rho)^2 / 2}} \quad (\text{A.23})$$

which bounds the range of possible value of  $\beta_t$ .

Lastly, we also solve  $\alpha_t$  from

$$-\alpha_t = \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \quad (\text{A.24})$$

$$= \tilde{\lambda}_t + \phi s_t + \mu + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \quad (\text{A.25})$$

$$= \frac{1}{2}\tilde{\lambda}_t + \phi s_t + \mu - \frac{1}{2}\sigma[dZ_t^*, ds_t] - \frac{1}{2}\sigma[dZ_t, ds_t] \quad (\text{A.26})$$

$$\alpha_t = -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma(\gamma_t v \rho^* + \beta_t \sigma - \beta_t \sigma \zeta) + \frac{1}{2}\sigma(\gamma_t v \rho - \beta_t \sigma + \beta_t \sigma \zeta) \quad (\text{A.27})$$

$$= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t v(\rho + \rho^*) \quad (\text{A.28})$$

## A.2 Proof of Proposition 2

Recall the definition of the real exchange rate under complete markets, we have

$$d(s_t - \beta s_t^{cm}) = \left( -\frac{1}{2}\tilde{\lambda}_t - \phi(s_t - \beta s_t^{cm}) - (1 - \beta)\mu + \frac{1}{2}\sigma\gamma_t v(\rho + \rho^*) \right) dt + \gamma_t v dX_t \quad (\text{A.29})$$

We conjecture

$$s_t - \beta s_t^{cm} = f(\lambda_t) + H_t \quad (\text{A.30})$$

$$H_t = \exp(-\phi t)H_0 + \int_0^t \exp(-\phi(t-u))h(\lambda_u)du \quad (\text{A.31})$$

which implies

$$dH_t = \left( -\phi \exp(\phi(-t))H_0 + h(\lambda_t) - \phi \int_0^t \exp(\phi(u-t))h(\lambda_u)du \right) dt \quad (\text{A.32})$$

$$= (-\phi \exp(\phi(-t))H_0 + h(\lambda_t) - \phi(H_t - \exp(\phi(-t))H_0)) dt \quad (\text{A.33})$$

$$= (h(\lambda_t) - \phi H_t) dt \quad (\text{A.34})$$

We note

$$d(s_t - \beta s_t^{cm}) = f' d\lambda_t + \frac{1}{2}f''[d\lambda_t, d\lambda_t]^2 dt + dH_t \quad (\text{A.35})$$

$$= f'(-\theta\lambda_t dt + v dX_t) + \frac{1}{2}f''v^2 dt + (h(\lambda_t) - \phi H_t) dt \quad (\text{A.36})$$

and this has to match equation (A.29).

Matching  $dX_t$  term,

$$f' = \gamma_t = \frac{b + \sqrt{b^2 + 4(k - \tilde{\lambda}_t)}}{2v} \quad (\text{A.37})$$

where  $b = (\rho^* - \rho)\sigma(1 - 2\beta_t)$ . Then,

$$\begin{aligned}
f(\lambda) &= \frac{1}{2\nu} \left\{ -\sqrt{b^2 + 4k} \log \left( 2e^{\lambda/2} \left( \cosh \left( \frac{\lambda}{2} \right) \left( \sqrt{b^2 + 4k} \sqrt{b^2 + 4k - 2\ell \tanh \left( \frac{\lambda}{2} \right) - 2\ell + b^2 + 4k - \ell} \right) - \ell \sinh \left( \frac{\lambda}{2} \right) \right) \right) \right. \\
&+ \left. \sqrt{b^2 + 4k - 4\ell} \log \left( 2e^{\lambda/2} \left( \cosh \left( \frac{\lambda}{2} \right) \left( \sqrt{b^2 + 4k - 4\ell} \sqrt{b^2 + 4k - 2\ell \tanh \left( \frac{\lambda}{2} \right) - 2\ell + b^2 + 4k - 3\ell} \right) - \ell \sinh \left( \frac{\lambda}{2} \right) \right) \right) \right. \\
&+ \left. \lambda \left( \sqrt{b^2 + 4k} + b \right) \right\} \quad (\text{A.38})
\end{aligned}$$

and

$$f''(\lambda) = \frac{\frac{\ell e^{2\lambda}}{(e^\lambda + 1)^2} - \frac{\ell e^\lambda}{e^\lambda + 1}}{\nu \sqrt{b^2 + 4 \left( k - \frac{\ell e^\lambda}{e^\lambda + 1} \right)}} \quad (\text{A.39})$$

Matching  $dt$  term,

$$h(\lambda_t) = -\frac{1}{2}\tilde{\lambda}_t - \phi f - (1 - \beta)\mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) + f'\theta\lambda_t - \frac{1}{2}f''\nu^2 \quad (\text{A.40})$$

Since  $\gamma_t$  is also a function of  $\lambda_t$ , we confirm the conjecture that  $h(\lambda_t)$  is a function only of  $\lambda_t$ . So

$$s_t = f(\lambda_t) + H_t + \beta s_t^{cm} \quad (\text{A.41})$$

### A.3 Long-Term Expectation of Log Exchange Rate

Since,

$$ds_t = \left( -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) \right) dt + \gamma_t\nu dX_t + \beta\sigma(dZ_t^* - dZ_t), \quad (\text{A.42})$$

then,

$$d(e^{\phi t} s_t) = e^{\phi t} \left( -\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) \right) dt + e^{\phi t} \gamma_t \nu dX_t + e^{\phi t} \beta \sigma (dZ_t^* - dZ_t) \quad (\text{A.43})$$

The solution of the above Stochastic Differential Equation is:

$$s_T = e^{-\phi T} s_0 + \int_0^T e^{\phi(t-T)} \left( -\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) \right) dt + \int_0^T e^{\phi(t-T)} \gamma_t \nu dX_t + \int_0^T e^{\phi(t-T)} \beta \sigma (dZ_t^* - dZ_t) \quad (\text{A.44})$$

Recall that

$$\gamma_t = \frac{(\rho^* - \rho)\sigma(1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2\sigma^2(1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu},$$

$$|(1 - 2\beta_t)| = \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{\sigma^2(1 - \zeta)}},$$

then  $\gamma_t$  is bounded,

$$|\gamma_t| \leq \frac{|\rho^* - \rho| \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{(1 - \zeta)}} + \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{(1 - \zeta)}} (\rho^* - \rho)^2 + 4k}{2\nu}.$$

Hence, for  $s_T$ , the integrands in the stochastic integrals are all  $\mathcal{H}^2$ , and the stochastic integrals are martingales with expectation 0. Then,

$$\begin{aligned} \lim_{T \rightarrow \infty} \mathbb{E}_0[s_T] &= \lim_{T \rightarrow \infty} e^{-\phi T} s_0 + \lim_{T \rightarrow \infty} \mathbb{E}_0 \left[ \int_0^T e^{\phi(t-T)} \left( -\frac{1}{2} \tilde{\lambda}_t - \mu + \frac{1}{2} \sigma \gamma_t \nu (\rho + \rho^*) \right) dt \right] \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{\phi(t-T)} \left( -\frac{1}{2} \mathbb{E}_0[\tilde{\lambda}_t] - \mu + \frac{1}{2} \sigma \mathbb{E}_0[\gamma_t] \nu (\rho + \rho^*) \right) dt \\ &= \frac{1}{\phi} \left( -\frac{1}{2} \lim_{T \rightarrow \infty} \mathbb{E}_0[\tilde{\lambda}_T] - \mu + \frac{1}{2} \sigma \lim_{T \rightarrow \infty} \mathbb{E}_0[\gamma_T] \nu (\rho + \rho^*) \right). \end{aligned}$$

#### A.4 Backus-Smith Puzzle

In relation to the Backus-Smith puzzle, we calculate the slope coefficient in a projection of the exchange rate changes the relative log SDF differential:

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \beta + \frac{\gamma_t \nu (\rho^* - \rho)}{2\sigma(1 - \zeta)}. \quad (\text{A.45})$$

We plug the solutions of  $\gamma_t$  into the Backus-Smith coefficient. As argued in Section 2.4, we pick the positive  $\gamma_t$  solution.

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \beta + \frac{(\rho^* - \rho)}{2\sigma(1 - \zeta)} \frac{(\rho^* - \rho)\sigma(1 - 2\beta) + \sqrt{(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta)^2 + 4(k - \tilde{\lambda}_t)}}{2}$$

Since the greater root of  $\beta$  will generate much greater exchange rate volatility that is counterfactual, we pick the smaller root of  $\beta$ , which is between 0 and  $\frac{1}{2}$ . In addition, since when  $(\rho^* - \rho) > 0$ , this coefficient is guaranteed to be positive, we want  $(\rho^* - \rho)$  to be negative. If so, the coefficient is increasing in  $\tilde{\lambda}_t$ ; we therefore pick the lowest possible  $\tilde{\lambda}_t = 0$ . Also note that, in this case, the Backus-Smith coefficient is increasing in  $\beta$  when  $\beta \leq \frac{1}{2}$ .<sup>11</sup>

<sup>11</sup>When plugging in the positive  $\gamma_t$  solution, the derivative of the Backus-Smith coefficient in  $\beta$  is:

$$1 - \frac{(\rho^* - \rho)^2}{2(1 - \zeta)} + \frac{(\rho^* - \rho)^2}{2(1 - \zeta)} \frac{(1 - 2\beta_t)}{\sqrt{(1 - 2\beta)^2 + \frac{4}{(\rho^* - \rho)^2 \sigma^2} (k - \tilde{\lambda}_t)}}$$

Given the positive semi-definiteness of the correlation matrix, we have  $0 \leq \frac{(\rho^* - \rho)^2}{2(1 - \zeta)} \leq 1$ , and hence the Backus-Smith coefficient is increasing in  $\beta$  when  $\beta \leq \frac{1}{2}$ .

Then, the Backus-Smith coefficient is

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \frac{1}{2} - \frac{1}{2\sigma} \sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}} + \frac{(\rho^* - \rho)^2 \sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}} + (\rho^* - \rho) \sqrt{(\rho^* - \rho)^2 (\sigma^2 - \frac{2k}{(1-\zeta)})}}{4\sigma(1-\zeta)}$$

The equivalent condition for a negative Backus-Smith coefficient is

$$-(2(1-\zeta) - (\rho^* - \rho)^2) \sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}} + (\rho^* - \rho) \sqrt{(\rho^* - \rho)^2 (\sigma^2 - \frac{2k}{(1-\zeta)})} + 4k + 2\sigma(1-\zeta) < 0$$

Define  $q = \rho^* - \rho$ , and we restrict  $q$  to be negative. Denote the LHS of the inequality as a function of  $k, q, \zeta$ , and  $\sigma$ .

$$p(k, q, \zeta, \sigma) = -(2(1-\zeta) - q^2) \sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}} + q \sqrt{q^2 (\sigma^2 - \frac{2k}{(1-\zeta)})} + 4k + 2\sigma(1-\zeta)$$

Take derivative of  $p(k, q, \sigma, \zeta)$  over  $k$ ,

$$\frac{\partial p(k, q, \zeta, \sigma)}{\partial k} = \left(2 - \frac{q^2}{1-\zeta}\right) \left( \frac{1}{\sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}}} - \frac{1}{\sqrt{(\sigma^2 - \frac{2k}{(1-\zeta)}) + \frac{4k}{q^2}}} \right)$$

Recall that the positive semi-definiteness of the correlation matrix guarantees that  $2 - \frac{q^2}{1-\zeta} \geq 0$ . Also, we need  $k > 0$  to make sure that  $\gamma_t$  has real solutions in zero volatility case. Therefore, we have  $\frac{1}{\sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}}} - \frac{1}{\sqrt{(\sigma^2 - \frac{2k}{(1-\zeta)}) + \frac{4k}{q^2}}} \geq 0$ , and hence

$$\frac{\partial p(k, q, \zeta, \sigma)}{\partial k} \geq 0.$$

That is,  $p(k, q, \zeta, \sigma)$  is increasing in  $k$ , and hence the Backus-Smith coefficient is increasing in  $k$  as well. We then plug in the smallest value of  $k$  ( $k = 0$ ) into the formula of  $p(k, q, \zeta, \sigma)$ ,

$$\begin{aligned} p(k, q, \zeta, \sigma)|_{k=0} &= -(2(1-\zeta) - q^2) \sqrt{\sigma^2 - \frac{2k}{(1-\zeta)}} + q \sqrt{q^2 (\sigma^2 - \frac{2k}{(1-\zeta)})} + 4k + 2\sigma(1-\zeta) \\ &= -\sigma(2(1-\zeta) - q^2) - q^2\sigma + 2\sigma(1-\zeta) \\ &= 0 \end{aligned}$$

That is, the minimum  $p(k, q, \zeta, \sigma)$  is zero regardless of the values of  $q, \zeta$ , and  $\sigma$ . Hence, the corresponding minimum of the Backus-Smith coefficient is zero regardless of the values of  $(\rho^* - \rho), \zeta$ , and  $\sigma$ .

We further validate the above analytical conclusion by numerically searching for negative Backus-Smith coefficients. Recall that a Hermitian matrix is positive-semidefinite if and only if

all of its principal minors are nonnegative. Since the correlation parameters  $\rho, \rho^*, \zeta \in [-1, 1]$ , the positive semi-definiteness of the correlation matrix  $R$  requires:

$$\det(R) = 1 + 2\rho\rho^*\zeta - \rho^2 - \rho^{*2} - \zeta^2 \geq 0.$$

We solve for the lower bound of  $(\rho^* - \rho)$  through an optimization problem. Recall we assume that the correlation of SDF (which loads negatively on  $dZ$  or  $dZ^*$  shocks) and convenience yield innovations is positive ( $\rho, \rho^* \leq 0$ ), so that the convenience yield tends to increase when the marginal utilities as represented by the pricing kernels rise. Hence, we further impose the constraints  $\rho, \rho^* \leq 0$  in the following optimization problem:

$$\begin{aligned} V(\zeta) &= \min_{\rho^*, \rho} (\rho^* - \rho) & (A.46) \\ \text{subject to:} & \quad \rho^2 + \rho^{*2} + \zeta^2 - 2\rho\rho^*\zeta - 1 \leq 0 \\ & \quad -1 \leq \rho \leq 0 \\ & \quad -1 \leq \rho^* \leq 0 \end{aligned}$$

We solve the optimization problem numerically using MATLAB optimization solver and get the minimum  $(\rho^* - \rho)$  denoted as  $V(\zeta)$ . Due to the symmetry between  $(\rho^* - \rho)$  and  $(\rho - \rho^*)$ , the feasible set of  $(\rho^* - \rho)$  is set to be  $[V(\zeta), -V(\zeta)]$  in the numerical search. Then, we traverse over following grids to search for negative Backus-Smith coefficients

$$(k_i, (\rho^* - \rho)_i) \in \left[ \max\left(\frac{\ell - (\rho^* - \rho)^2 \sigma^2 / 4}{1 - (\rho^* - \rho)^2 / (2(1 - \zeta))}, 0\right), \frac{\sigma^2(1 - \zeta)}{2} \right] \times [V(\zeta), -V(\zeta)]$$

with given SDF correlations.

We use heatmaps to illustrate the results of the numeric search. The numerical search results with  $\zeta = 0.25$  is shown in Figure 5.

Figure 5: Backus-Smith Coefficient. We report the regression coefficient in equation (40) across different values of  $\beta$  and  $\rho^* - \rho$ .

## B Appendix: Extension with Incomplete-Market Wedge

### B.1 Extension of Proposition 1

We extend our model to the case where there is an incomplete-market wedge. We write the real exchange as satisfying the following stochastic differential equation,

$$ds_t = \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t \nu dX_t + \omega_t dW_t, \quad (\text{B.1})$$

where  $\alpha_t, \beta_t,$  and  $\gamma_t$  are  $\mathbb{F}$ -adapted stochastic processes.  $\beta_t$  governs the distance from complete markets. When  $\beta_t \equiv 1, \omega_t = 0,$  and  $\gamma_t \equiv 0,$  we are back in the benchmark complete markets case.  $W_t$  is the incomplete-market shock with<sup>12</sup>:

$$[dZ_t, dW_t] = 0, \quad [dZ_t^*, dW_t] = \chi, \quad [dX_t, dW_t] = 0.$$

Recall that from the four FOC conditions, we have

$$\tilde{\lambda}_t \triangleq \tilde{\lambda}_t^h - \tilde{\lambda}_t^f = [ds_t, ds_t] - \sigma [dZ_t^* - dZ_t, ds_t], \quad (\text{B.2})$$

Plug in the conjecture  $ds_t = \alpha_t dt + \gamma_t \nu dX_t + \beta_t \sigma (dZ_t^* - dZ_t) + \omega_t dW_t,$  then

$$\begin{aligned} -\tilde{\lambda}_t &= \gamma_t^2 \nu^2 + 2\beta_t^2 \sigma^2 (1 - \zeta) + 2\gamma_t \nu \beta_t (\rho^* - \rho) \sigma + \omega_t^2 + (2\beta_t - 1) \sigma \omega_t \chi \\ &- 2\beta_t \sigma^2 (1 - \zeta) - (\rho^* - \rho) \sigma \gamma_t \nu \end{aligned} \quad (\text{B.3})$$

We have different sets of solutions of  $\beta_t$  and  $\gamma_t$  depending on how much incomplete-market shock we want to load on the  $\beta_t$  and  $\gamma_t$  terms. We suppose that the incompleteness shock terms only load on the  $\gamma_t$  terms. Then, suppose for a certain constant  $k,$

$$-k = 2(1 - \zeta) \beta_t^2 \sigma^2 - 2(1 - \zeta) \beta_t \sigma^2 \quad (\text{B.4})$$

$$k - \tilde{\lambda}_t = \gamma_t^2 \nu^2 + 2\gamma_t \nu \beta_t (\rho^* - \rho) \sigma - (\rho^* - \rho) \sigma \gamma_t \nu + \omega_t^2 + (2\beta_t - 1) \sigma \omega_t \chi \quad (\text{B.5})$$

The solution is

$$\beta_t = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sigma^2 (1 - \zeta) - 2k}{\sigma^2 (1 - \zeta)}}, \quad (\text{B.6})$$

$$\gamma_t = \frac{(\rho^* - \rho) \sigma (1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t - \omega_t^2 + (1 - 2\beta_t) \sigma \omega_t \chi)}}{2\nu} \quad (\text{B.7})$$

which are real for all possible values of  $\lambda_t$  if and only if

$$k \leq \frac{\sigma^2 (1 - \zeta)}{2} \quad (\text{B.8})$$

<sup>12</sup>Regularity conditions of the positive semi-definiteness of the correlation matrix is discussed in section B.3



and

$$(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t - \omega_t^2 + (1 - 2\beta_t)\sigma\omega_t\chi) \geq 0 \quad (\text{B.9})$$

The second condition is equivalent to

$$\begin{aligned} & (\rho^* - \rho)^2 \frac{\sigma^2(1 - \zeta) - 2k}{(1 - \zeta)} + 4(k - \tilde{\lambda}_t - \omega_t^2) \geq 4(2\beta_t - 1)\sigma\omega_t\chi \\ \text{i.e.} \quad & (\rho^* - \rho)^2 \frac{\sigma^2(1 - \zeta) - 2k}{(1 - \zeta)} + 4(k - \tilde{\lambda}_t - \omega_t^2) \geq \pm 4\sigma\omega_t\chi \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{\sigma^2(1 - \zeta)}} \end{aligned} \quad (\text{B.10})$$

Lastly, we also solve  $\alpha_t$  from

$$-\alpha_t = \tilde{\lambda}_t^f + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \quad (\text{B.11})$$

$$= \tilde{\lambda}_t + \phi s_t + \mu + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \quad (\text{B.12})$$

$$= \frac{1}{2}\tilde{\lambda}_t + \phi s_t + \mu - \frac{1}{2}\sigma[dZ_t^*, ds_t] - \frac{1}{2}\sigma[dZ_t, ds_t] \quad (\text{B.13})$$

$$\alpha_t = -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma(\gamma_t \nu \rho^* + (1 - \zeta)\beta_t \sigma + \omega_t \chi) + \frac{1}{2}\sigma(\gamma_t \nu \rho - (1 - \zeta)\beta_t \sigma) \quad (\text{B.14})$$

$$= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t \nu (\rho + \rho^*) + \frac{1}{2}\omega_t \chi \sigma \quad (\text{B.15})$$

So, we can extend Proposition 1 to the more general case with incomplete-market shock.

**Proposition 1.** *Under Assumption 2, there is a class of solutions indexed by constant  $k$  so that,*

$$\begin{aligned} \beta_t &= \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{\sigma^2(1 - \zeta) - 2k}{\sigma^2(1 - \zeta)}}, \\ \gamma_t &= \frac{(\rho^* - \rho)\sigma(1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t - \omega_t^2 + (1 - 2\beta_t)\sigma\omega_t\chi)}}{2\nu} \end{aligned}$$

The log of the real exchange rate satisfies:

$$ds_t = \left( 2\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t \nu (\rho + \rho^*) + \frac{1}{2}\omega_t \chi \sigma \right) dt + \gamma_t \nu dX_t + \beta\sigma(dZ_t^* - dZ_t) + \omega_t dW_t,$$

which loads on the SDF shocks  $dZ$  and  $dZ^*$ , the convenience yield shock  $dX$ , and the incomplete-market shock  $dW_t$ .

## B.2 Backus Smith Puzzle

In the above extension, the Backus-Smith coefficient is then

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]}$$

$$\begin{aligned}
&= \frac{\sigma\gamma_t v(\rho^* - \rho) + 2\beta_t \sigma^2(1 - \zeta) + \omega_t \sigma \chi}{2\sigma^2(1 - \zeta)} \\
&= \beta_t + \frac{\gamma_t v(\rho^* - \rho)}{2\sigma(1 - \zeta)} + \frac{\omega_t \chi}{2\sigma(1 - \zeta)}
\end{aligned}$$

We plug the solutions of  $\gamma_t$  into the Backus-Smith coefficient. As argued in Section 2.4, we pick the positive  $\gamma_t$  solution.

$$\begin{aligned}
&\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} \\
&= \beta_t + \frac{\omega_t \chi}{2\sigma(1 - \zeta)} + \frac{(\rho^* - \rho)^2 \sigma(1 - 2\beta_t) + (\rho^* - \rho) \sqrt{(\rho^* - \rho)^2 \sigma^2(1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t - \omega_t^2 + (1 - 2\beta_t)\sigma\omega_t\chi)}}{4\sigma(1 - \zeta)}
\end{aligned}$$

Since the greater root of  $\beta$  will generate much greater exchange rate volatility that is counterfactual, we pick the smaller root of  $\beta$ , which is between 0 and  $\frac{1}{2}$ . In addition, when  $\chi \geq 0$  and  $(\rho^* - \rho) > 0$ , this coefficient is guaranteed to be positive; when  $\chi \leq 0$ , picking  $(\rho^* - \rho) < 0$  is more likely to generate a negative Backus-Smith coefficient. So, we want  $(\rho^* - \rho)$  to be negative. If so, the coefficient is increasing in  $\tilde{\lambda}_t$ ; we therefore pick the lowest possible  $\tilde{\lambda}_t = 0$ . The Backus-Smith coefficient is not monotone in either  $k$  or  $(\rho^* - \rho)$ . So a numeric search is needed. We solve for the lower bound of  $(\rho^* - \rho)$  and  $k$  through optimization problems.

First, consider the optimization problem for the lower bound of  $(\rho^* - \rho)$ . The positive semi-definiteness of the correlation matrix introduces optimization constraints discussed in section B.3. Recall we assume that the correlation of SDF (which loads negatively on  $dZ$  or  $dZ^*$  shocks) and convenience yield innovations is positive ( $\rho, \rho^* \leq 0$ ). Hence, we further impose the constraints  $\rho \leq 0$  and  $\rho^* \leq 0$  in the following optimization problem:

$$\begin{aligned}
V(\zeta, \chi) &= \min_{\rho^*, \rho} (\rho^* - \rho) && \text{(B.16)} \\
\text{subject to:} & && \\
&-1 - 2\rho\rho^*\zeta + \rho^2 + \rho^{*2} + \zeta^2 \leq 0 \\
&\rho^{*2} + \chi^2 - 1 \leq 0, \\
&-1 + \rho^2 + \rho^{*2} + \zeta^2 - 2\rho\rho^*\zeta - \chi^2(-1 + \rho^2) \leq 0, \\
&-1 \leq \rho \leq 0, \\
&-1 \leq \rho^* \leq 0,
\end{aligned}$$

where  $1 - \zeta^2 - \chi^2 \geq 0$ . We solve the optimization problem numerically using MATLAB optimization solver and get the minimum  $(\rho^* - \rho)$  denoted as  $V(\zeta, \chi)$ . Due to the symmetry between  $(\rho^* - \rho)$  and  $(\rho - \rho^*)$ , the feasible set of  $(\rho^* - \rho)$  is set to be  $[V(\zeta, \chi), -V(\zeta, \chi)]$  in the numerical search.

Then, given values of  $(\rho^* - \rho)$  and other parameters, we nail down the lower bound of  $k$  by solving the corresponding optimization problem. Note that the optimization constraints are different when we pick different  $\beta$  solutions. When we pick the smaller  $\beta$  solution, the optimization

problem is

$$\begin{aligned}
\underline{k}(\zeta, \rho, \rho^*, \chi, \omega_t, \sigma) &= \min k & (B.17) \\
\text{Subject to:} & k \geq \omega_t^2 \\
& k \leq \frac{\sigma^2(1-\zeta)}{2} \\
& (\rho^* - \rho)^2 \frac{\sigma^2(1-\zeta) - 2k}{(1-\zeta)} + 4(k - \omega_t^2) \geq -4\sigma\omega_t\chi \sqrt{\frac{\sigma^2(1-\zeta) - 2k}{\sigma^2(1-\zeta)}}
\end{aligned}$$

While when we pick the larger  $\beta_t$  solution, the optimization problem is

$$\begin{aligned}
\underline{k}(\zeta, \rho, \rho^*, \chi, \omega_t, \sigma) &= \min k & (B.18) \\
\text{Subject to:} & k \leq \frac{\sigma^2(1-\zeta)}{2} \\
& k \geq \omega_t^2 \\
& (\rho^* - \rho)^2 \frac{\sigma^2(1-\zeta) - 2k}{(1-\zeta)} + 4(k - \omega_t^2) \geq 4\sigma\omega_t\chi \sqrt{\frac{\sigma^2(1-\zeta) - 2k}{\sigma^2(1-\zeta)}}
\end{aligned}$$

We then numerically solve these optimization problems using MATLAB optimization solver and denote the lower bound of  $k$  as  $\underline{k}(\zeta, \rho, \rho^*, \chi, \omega_t, \sigma)$ .

Then, we search the following grids for negative Backus-Smith coefficients<sup>13</sup>

$$(k_i, (\rho^* - \rho)_i) \in [\underline{k}(\zeta, \rho, \rho^*, \chi, \omega_t, \sigma), \frac{\sigma^2(1-\zeta)}{2}] \times [V(\zeta, \chi), -V(\zeta, \chi)]$$

We use heatmaps to illustrate the numeric search results. The numeric search results with  $\zeta = 0.25, \chi = 0.25, \omega_t = 0.1$  is shown in Figure 6.

Figure 6: Backus-Smith Coefficient. We report the regression coefficient in equation (B.16) across different values of  $\beta$  and  $\rho^* - \rho$ .

<sup>13</sup>In the numeric search, be careful when setting the parameter values.

- First, we need to make sure that  $\omega_t^2 \leq \frac{\sigma^2(1-\zeta)}{2}$ , so that the feasible set of  $k$  is not empty.
- Make sure  $1 - \zeta^2 - \chi^2 \geq 0$ , which is necessary for the positive semi-definiteness of the correlation matrix.

### B.3 Auxiliary

Denote  $R$  as the correlation matrix of  $X_t, Z_t, Z_t^*$ , and  $W_t$ .

$$R = \begin{pmatrix} 1 & \rho & \rho^* & 0 \\ \rho & 1 & \zeta & 0 \\ \rho^* & \zeta & 1 & \chi \\ 0 & 0 & \chi & 1 \end{pmatrix}$$

The positive semi-definiteness of  $R$  imposes additional regularity constraints on the values of  $\rho$ ,  $\rho^*$ ,  $\zeta$ , and  $\chi$ . Recall that a Hermitian matrix is positive-semidefinite if and only if all of its principal minors are nonnegative. Hence, the positive semi-definiteness of  $R$  requires:

$$\det(PM_1) = 1 + 2\rho\rho^*\zeta - \rho^2 - \rho^{*2} - \zeta^2 \geq 0, \quad (\text{B.19})$$

$$\det(PM_2) = 1 - \rho^2 \geq 0, \quad (\text{B.20})$$

$$\det(PM_3) = 1 - \rho^{*2} - \chi^2 \geq 0, \quad (\text{B.21})$$

$$\det(PM_4) = 1 - \zeta^2 - \chi^2 \geq 0, \quad (\text{B.22})$$

$$\det(R) = 1 - \rho^2 - \rho^{*2} - \zeta^2 + 2\rho\rho^*\zeta + \chi^2(-1 + \rho^2) \geq 0. \quad (\text{B.23})$$

where  $PM_1, PM_2, PM_3, PM_4$  are the third-order principal minors of  $R$ ,

$$PM_1 = \begin{pmatrix} 1 & \rho & \rho^* \\ \rho & 1 & \zeta \\ \rho^* & \zeta & 1 \end{pmatrix}; PM_2 = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; PM_3 = \begin{pmatrix} 1 & \rho^* & 0 \\ \rho^* & 1 & \chi \\ 0 & \chi & 1 \end{pmatrix}; PM_4 = \begin{pmatrix} 1 & \zeta & 0 \\ \zeta & 1 & \chi \\ 0 & \chi & 1 \end{pmatrix}.$$