Sufficient Statistics for Nonlinear Tax Systems with Preference Heterogeneity

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October, 2021

Abstract

This paper provides general and empirically-implementable sufficient statistics formulas for optimal nonlinear tax systems in the presence of preference heterogeneity. We study unrestricted tax systems on income and savings (or other commodities) that implement the optimal direct-revelation mechanism, as well as simpler tax systems that impose common restrictions like separability between earnings and savings taxes. We characterize the optimum using familiar elasticity concepts and a sufficient statistic for across-income preference heterogeneity: the difference between the cross-sectional variation of savings with income, and the causal effect of income on savings. The Atkinson-Stiglitz Theorem is a knife-edge case corresponding to zero difference, and a number of other key results in optimal tax theory are subsumed as special cases. Our formulas also apply to other sources of across-income heterogeneity, including heterogeneity in rates of return on savings, inheritances, and the ability to shift income between tax bases. We provide tractable extensions of these results that include multidimensional heterogeneity, additional efficiency rationales for taxing heterogeneous asset returns, and corrective motives to encourage more saving. Applying these formulas in a calibrated model of the U.S. economy, we find that the optimal savings tax is positive and progressive.

*We are grateful to Afras Sial for excellent research assistance. This project was supported through a Quartet Pilot Research award and was funded by the Center for Health Incentives and Behavioral Economics at the University of Pennsylvania, by The Research Council of Norway 315765, and by the Alfred P. Sloan Foundation. Ferey gratefully acknowledges the financial support of Labex ECODEC at CREST, and of the Deutsche Forschungsgemeinschaft through CRC TRR 190 at LMU Munich. The content is solely the responsibility of the authors and does not necessarily represent the official views of the University of Pennsylvania. We thank Pierre Boyer, Philippe Choné, Ashley Craig, Laurence Jacquet, Dirk Krueger, Etienne Lehmann, Jonas Loebbing, Jean-Baptiste Michau, Andreas Peichl, Dominik Sachs, Emmanuel Saez, Florian Scheuer, Stefanie Stantcheva, Aleh Tsyvinski, Nicolas Werquin, participants at NBER Macro Public Finance, NBER Public Economics, IIPF, LAGV, NTA, and audiences at CREST - Ecole Polytechnique and LMU Munich. Ferey: LMU Munich (antoine.ferey@econ.lmu.de). Lockwood: Wharton & NBER (ben.lockwood@wharton.upenn.edu). Taubinsky: UC Berkeley & NBER (dmitry.taubinsky@berkeley.edu).

1 Introduction

Taxes on capital income, estates, inheritances, and certain categories of consumption are a widespread feature of modern tax systems. Yet there is considerable debate, both among economists and in policy circles, about their optimal design. The celebrated theorem of Atkinson and Stiglitz (1976) is often interpreted to suggest that such taxes should be eliminated: the theorem states that if preferences are homogeneous and weakly separable, then differential taxes on commodities—including on future consumption in the form of savings—are suboptimal, and welfare is maximized when redistribution is carried out solely through an income tax. However, as was appreciated by contemporaneous work (Mirrlees, 1976) and emphasized by the authors themselves (Stiglitz, 2018), the assumptions underpinning the Atkinson-Stiglitz Theorem are strong, and the theorem does not apply in settings where earnings ability co-varies with commodity preferences, or with other attributes that affect saving levels, such as heterogeneous inheritances, rates of return, or income-shifting abilities.

As a result, an active literature has developed to demonstrate that non-zero commodity and capital income taxes may be optimal when the Atkinson-Stiglitz assumptions are relaxed. Yet general, elasticity-based "sufficient statistics" formulas for optimal nonlinear commodity and savings taxes, of the kind common in the optimal income tax literature (e.g., Saez, 2001), have remained elusive. Existing results have instead studied settings with restrictions to a small number of discrete "types" or on functional forms of the utility and tax functions, or they have focused on qualitative insights.[1]

In this paper, we derive generally-applicable, sufficient statistics formulas for optimal linear and nonlinear commodity taxes in a setting where preferences or other consumer attributes, such as inheritances or rates of return, vary with income-earning ability. We study a general version of standard models where consumers with heterogeneous earning abilities and tastes choose labor supply and a consumption and savings bundle that exhausts their after-tax income.[2] Our formulas nest prior results in this setting, as well as the Atkinson-Stiglitz Theorem itself, as special cases. For concreteness in what follows, we describe results in terms of taxes on savings, although they also apply to other commodities.

We organize the paper around the following key contributions.

The first is a set of results about the optimal unrestricted, nonlinear tax system on income and savings. We begin with the question of implementation: can the optimal allocation be implemented by a smooth (i.e., differentiable) tax on income and savings? A smooth tax system allows for *double deviations*, where individuals can jointly alter their income and savings to reach bundles not chosen

[1] Of particular note, Saez (2002) used a model like the one in this paper to answer the qualitative question of when a "small" *linear* commodity (savings) tax can increase welfare in the presence of preference heterogeneity, but left to future work the task of deriving an expression for the optimal tax, writing "It would of course be extremely useful to obtain optimal commodity tax formulas" in such a framework.

[2] See, e.g., Atkinson and Stiglitz (1976); Saez (2002); Farhi and Werning (2010); Diamond and Spinnewijn (2011); Golosov et al. (2013); Piketty and Saez (2013); Scheuer and Wolitzky (2016); Saez and Stantcheva (2018); Allcott et al. (2019)

by another type, which can simply be disallowed under the optimal mechanism. This introduces a complication not present in the standard income taxation model of Mirrlees (1971) and related work: in the presence of preference heterogeneity—and in contrast to the Atkinson-Stiglitz setting with homogeneous preferences and weak separability—double deviations are in general the most attractive direction of adjustment. Nevertheless, we show that under mild regularity assumptions, it is possible to construct a smooth tax system, dependent only on income and savings, that implements the optimal direct-revelation mechanism.

We then present new elasticity-based formulas for the optimal nonlinear tax on savings and income. We show that these formulas can be written entirely in terms of welfare weights and empirically measurable statistics, including a key sufficient statistic for preference heterogeneity: the difference between the cross-sectional variation of savings s with earnings z, denoted $s'(z)$, and the causal effect of income changes on savings, which we denote $s'_{inc}(z)$. The residual, $s'_{pref}(z) := s'(z) - s'_{inc}(z)$, is a sufficient statistic for (local) preference heterogeneity.[3] Intuitively, the total derivative of s with respect to z is the sum of two partial derivatives: (i) the causal income effect s'_{inc}, holding preferences constant and (ii) the degree to which higher-ability types prefer more s, holding earnings constant.

The s'_{pref} statistic can be estimated from existing empirical data and from behavioral responses to policy reforms, avoiding the need to explicitly measure or model the relationship between unobserved preferences and ability. The condition for optimal savings tax rates take a form resembling the optimal earnings tax condition in Saez (2001), with earnings z replaced by $s'_{pref}(z)$, and with the elasticity of taxable income replaced by the elasticity of savings with respect to the savings tax rate. This sufficient statistics formulation provides an immediate generalization of the Atkinson-Stiglitz Theorem, as it implies that the optimal savings tax rate is everywhere zero when $s'_{pref}(z) = 0$ for all earnings levels z.

This characterization of optimal nonlinear tax systems spans a variety of other structural models that depart from the Atkinson-Stiglitz setting, including heterogeneous endowments or inheritances, differential rates of return on investments, and the ability to engage in income shifting (Slemrod, 1995). In each case, the difference between the cross-sectional profile of savings and the causal income effect on savings, $s'(z) - s'_{inc}(z)$, is the key sufficient statistic for heterogeneity. Consequently, our formula can be viewed as both a synthesis of prior work that qualitatively studied these extensions in isolation, and a key step forward that allows empirically-grounded quantification of optimal tax rates when multiple of these forces are at play.[4] For simplicity, we still refer to $s'_{pref}(z)$ as a measure of preference heterogeneity, but we emphasize that it is a sufficient statistic for many other forms of across-income heterogeneity.

Our second contribution is a characterization of what we call "simple tax systems." We docu-

[3]To our knowledge, this statistic was first employed in Allcott et al. (2019), in a setting restricted to a separable linear commodity tax, which of course cannot implement the optimal mechanism.

[4]See Gahvari and Micheletto (2016) and Gerritsen et al. (2020) on heterogeneous rates of return; Boadway et al. (2000) and Cremer et al. (2003) on heterogeneous endowments; and Christiansen and Tuomala (2008) on income shifting.

ment that across a large number of countries, the tax system consists of a nonlinear tax on income, accompanied by taxes on savings vehicles that can be classified as one of three types: (i) a separable linear (SL) savings tax; (ii) a separable nonlinear (SN) savings tax; or (iii) a system with a linear earnings-dependent (LED) savings tax, which allows, for example, lower-income people to have their savings taxed at a lower linear rate, as is the case for long-term capital gains in the U.S. We show that the optimal tax policy within each of these classes of simple systems can be expressed using the same sufficient statistics that appear in our formulas for the optimal smooth unrestricted tax system. We also present formulas for Pareto efficiency conditions that can be used to test of whether existing tax systems are consistent with the nonlinear income tax, without additional assumptions about social marginal welfare weights. These simple systems cannot always implement the optimal mechanism, but we characterize sufficient conditions for when the SN and LED systems can.

We provide further generality in four tractable extensions of our baseline results. First, we consider multidimensional heterogeneity. We show that our results characterizing the optimal SL, SN, and LED systems can be extended tractably to multidimensional heterogeneity and to a potentially suboptimal income tax. In this more general setting, the causal effect of income on savings, together with the cross-sectional profile of savings across the income distribution, remain sufficient statistics for characterizing the optimal savings tax. Second, we consider many dimensions of consumption. In this case, the additional necessary sufficient statistics are cross-price elasticities, which allow us to compute *tax diversion ratios*—the fiscal spillovers to taxes collected on goods $j \neq i$ relative to the reduction in taxes collected on good i when the price of good i is increased. Third, we consider situations where the government wants to alter or correct individual behavior. This analysis generalizes the Farhi and Werning (2010) setting, in which parents underweight the wellbeing of their descendents, to allow for heterogeneous preferences, and also covers the case where individuals under-save due to behavioral biases, as in Moser and Olea de Souza e Silva (2019). Fourth, we study settings in which there is an additional efficiency rationale for taxing savings, because the government can collect savings taxes either before or after returns are earned, and therefore can arbitrage heterogeneous private rates of return by shifting tax collections onto post-returns savings for high earners. This extension relates to independent work by Gerritsen et al. (2020), who study the special case where all across-income heterogeneity is from heterogeneity in rates of return, and characterize the optimal separable nonlinear savings tax in terms of model primitives.

In the final part of our paper, we apply our sufficient statistics formulas to study the optimal tax treatment of savings in the U.S. We calibrate the distribution of savings across the income distribution using the Distributional National Accounts micro-files of Piketty et al. (2018). This evidence suggests that savings are approximately constant at low incomes but increase convexly at higher incomes, so that the cross-sectional slope $s'(z)$ is increasing with income. To calibrate the causal income effect on savings, we draw on two sources. The first is estimates from Fagereng et al. (2019) of the medium-run marginal propensity to save out of windfall income. The second is a new probability-based survey representing the United States adult population, conducted on the AmeriSpeak panel. The survey asked people about their savings behavior in response to a possible raise. The two sources are consistent in suggesting similar magnitudes for $s'_{inc}(z)$, with little variation across incomes. Together, these findings yield a positive and increasing value of the residual $s'(z) - s'_{inc}(z) = s'_{pref}(z)$, our sufficient statistic for heterogeneity, across most of the income distribution. Combined with our formulas, this implies a positive and progressive optimal tax on savings. Our baseline estimates of optimal savings tax rates are somewhat higher than those currently in place in the U.S., although as in other work, they are sensitive to the elasticity of savings with respect to tax rates, about which there is still substantial uncertainty.

Our paper contributes most directly to the literature studying optimal commodity and savings taxation in the presence of correlated preference heterogeneity. Saez (2002) considers the special case of a separable linear commodity tax and derives conditions under which its optimal value is non-zero, but does not provide a formula for the magnitude. Golosov et al. (2013) derive conditions characterizing the optimal mechanism in a model like the one we study, but formulate their results in terms of first-order conditions on structural primitives rather than empirically-estimable sufficient statistics. Their empirical estimates suggest substantially less across-income heterogeneity than ours do, resulting in much lower optimal savings tax rates. This difference could be because they study heterogeneity in time discounting only, rather than the broader set of forces that can contribute to $s'_{pref}(z)$ and that we allow in our general characterization.[5] Saez and Stantcheva (2018) study nonlinear capital taxation in a setting without income effects, which corresponds to the special case of our model where $s'_{inc}(z) = 0$ and $s'_{pref}(z) = s'(z)$. They consider multidimensional heterogeneity when tax systems are restricted to be either separable linear or separable nonlinear, so their results can be viewed as a special case of our extension characterizing optimal simple tax systems with multidimensional heterogeneity. Allcott et al. (2019) derive a sufficient statistics formula for the optimal separable linear commodity tax in the presence of correlated preference heterogeneity.[6] We build on these insights by developing methods to characterize and implement the optimal mechanism using an unrestricted smooth nonlinear tax system, by studying other more restricted but still nonlinear tax systems that are commonly used in the tax-treatment of savings, and by incorporating forms of across-income heterogeneity that are not just preference-based.

As we note above, our sufficient statistics strategy for quantifying preference heterogeneity spans several other structural departures from the Atkinson-Stiglitz setting, which have been studied independently. Gahvari and Micheletto (2016) and Gerritsen et al. (2020) study heterogeneous rates

[5]The lower measured heterogeneity in Golosov et al. (2013) could also be driven by attenuation bias. They measure preference heterogeneity by regressing a structural estimate of time preferences on a plausibly noisy proxy of earnings ability (performance on the Armed Forces Qualication Test), which may be biased toward zero due to a noisy right-hand-side variable.

[6]The application of separable linear savings taxes in the presence of multidimensional heterogeneity is also considered in Piketty and Saez (2013), Diamond and Spinnewijn (2011), and Gauthier and Henriet (2018). Piketty and Saez (2013) derive sufficient statistics formulas but make the additional restriction of a linear income tax. Diamond and Spinnewijn (2011) and Gauthier and Henriet (2018) allow for a nonlinear income tax but assume a finite number of possible earnings levels, and derive results in terms of model primitives. Jacquet and Lehmann (2021) provide a generalization to a separable sum of many one-dimensional tax schedules.

of return, Boadway et al. (2000) and Cremer et al. (2003) study heterogeneous endowments, and Christiansen and Tuomala (2008) study income shifting. Our methods provide a unified treatment not only of these different sources of across-income heterogeneity, but also a unified approach that can address—using the same set of sufficient statistics—both the growing literature on simpler tax systems with multidimensional heterogeneity and the smaller literature on optimal mechanisms with unidimensional heterogeneity.

The literature on dynamic taxation (see overviews by Golosov and Tsyvinski, 2006; Stantcheva, 2020) typically assumes homogeneous preferences, but derives a theoretically robust role for capital taxation via the inverse Euler equation (e.g., Golosov et al., 2003; Farhi and Werning, 2013). This literature tends to find optimal savings "wedges" of only several percentage points (see, e.g., Golosov and Tsyvinski, 2015; Golosov et al., 2016; Farhi and Werning, 2013)—substantially lower than those suggested by our baseline calibrations at the same assumed values of elasticities. This suggests that preference heterogeneity may play a quantitatively larger role in determining optimal savings tax policy than do the social insurance motives analyzed in the dynamic taxation literature. Our work is complementary in relaxing the assumption of homogeneous savings preferences, but limiting to a more static framework. Extensions of our approach to measuring and incorporating preference heterogeneity could be fruitfully applied to more dynamic models.

The rest of this paper proceeds as follows. Section 2 presents our model and assumptions. Section 3 shows that smooth tax systems can implement the optimal mechanism, and provides sufficient statistics for optimal smooth tax systems. Section 4 studies simple tax systems. Section 5 presents extensions to our results. Section 6 applies our formulas to characterize optimal savings tax rates in the United States. Section 7 concludes. All proofs are gathered in the Appendix.

2 Model and Assumptions

Agents There is a population of heterogeneous agents who differ in earnings ability and preferences for s, with their types denoted by θ. We begin with the common assumption that $\theta \in \Theta \subset \mathbb{R}$, where Θ is compact; Section 5.1 considers multidimensional heterogeneity. We assume that θ has a continuously differentiable cumulative distribution function $F(\theta)$.

Agents choose earnings z, and a consumption bundle (c, s), and derive utility $U(c, s, z; \theta)$. One application is where c is period-1 consumption and s is the realized savings in period 2, as in Saez (2002), Golosov et al. (2013), and many others. A second application is where c is period-1 consumption by the parents, while s is the wealth bequeathed to their children and consumed in period 2, as in Farhi and Werning (2010). A third application is where c is numeraire consumption and s is another dimension of commodity consumption that could be tax nonlinearly, such as the energy efficiency.

We assume a linear production technology where with marginal rate of transformation p between s and c. In the savings and inheritance interpretations of the model, $p = 1/R$, where R is the rate of return in a linear savings technology between the two periods.

Throughout the paper, we assume that:

Assumption 1. *$U(c, s, z; \theta)$ is twice continuously differentiable, increasing and weakly concave in c and s, and decreasing and strictly concave in z. The first derivatives U_c' and U_s' are bounded.*

For example, a frequently-used functional form (e.g. Saez, 2002; Golosov et al., 2013) involves additively separable utility and heterogeneity in agents' productivity w and discount factor δ:

$$U(c, s, z; \theta) = u(c) + \delta(\theta) u(s) - k(z/w(\theta)), \tag{1}$$

with $u(.)$ the utility from consumption and $k(z/w)$ the disutility from work. We say that there is correlated preference heterogeneity when the discount factor $\delta(\theta)$ covaries with productivity $w(\theta)$.

More generally, we say that there is correlated preference heterogeneity when marginal rates of substitution between c and s vary with earnings ability. We define the marginal rate of substitution

$$\mathcal{S}(c, s, z; \theta) := \frac{U_s'(c, s, z; \theta)}{U_c'(c, s, z; \theta)}, \tag{2}$$

and we use the shorthand $S_\theta'(c, s, z; \theta_0) := \frac{\partial}{\partial \theta} \mathcal{S}_\theta(c, s, z; \theta)|_{\theta=\theta_0}$. We define correlated preference heterogeneity as follows:

Definition 1. *There is correlated preference heterogeneity if some agents prefer different consumption-savings bundles conditional on having the same earnings level; i.e.,*

$$\exists \theta_0, \forall (c, s, z), \; S_\theta'(c, s, z; \theta_0) \neq 0 \tag{3}$$

In the formulation in (1), $S_\theta'(c, s, z; \theta) > 0$ whenever $\delta'(\theta) > 0$.

We similarly let $\mathcal{Z}$ denote the marginal rate of substitution between consumption c and earnings z,

$$\mathcal{Z}(c, s, z; \theta) := \frac{U_z'(c, s, z; \theta)}{U_c'(c, s, z; \theta)}. \tag{4}$$

Government An agent's type θ is private information and cannot be observed by the government; only the distribution of types, $F(\theta)$, is known. The government must design a tax and transfer system that only depends on the observable variables (c, s, z).

Without any restrictions on the form of the optimal tax system, the resulting optimal allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ must solve the following program:

$$\max \int_\theta \alpha(\theta) U(c(\theta), s(\theta), z(\theta); \theta) \, dF(\theta) \tag{5}$$

subject to the resource constraint

$$\int_\theta [z(\theta) - ps(\theta) - c(\theta)] \, dF(\theta) \geq E \tag{6}$$

and incentive compatibility constraints

$$\forall \left(\theta, \theta'\right) \in \Theta^2, \; U\left(c(\theta), s\left(\theta\right), z\left(\theta\right); \theta\right) \geq U(c(\theta'), s(\theta'), z(\theta'); \theta). \tag{7}$$

We refer to an allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ that maximizes (5) subject to (6) and (7) as the *optimal incentive-compatible allocation.*

3 Optimal Smooth Tax Systems

In this section, we provide two key results about *smooth tax systems*, by which we mean twice continuously differentiable tax functions $\mathcal{T}(s, z)$.[7] First, we show that the optimal incentive-compatible allocation, characterized by the program in (5)-(7) is implementable by a smooth tax system under intuitive regularity conditions. Second, we leverage our first result to derive a sufficient statistics characterization of optimal smooth tax systems. We assume that all individuals can trade c for s at rate p because, for example, they are part of a competitive market that sets the relative price of s to p.

We maintain the following assumptions throughout the rest of our analysis.

Assumption 2. *In the optimal incentive-compatible allocation* $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ *,* c*,* s*, and* z *are smooth functions of* θ*. Any type* θ *strictly prefers its allocation* $(c(\theta), s(\theta), z(\theta))$ *to the allocation* $(c(\theta'), s(\theta'), z(\theta'))$ *of another type* $\theta' \neq \theta$

Assumption 3. *Along the path of* $\{c, s, z\}$ *offered in the optimal incentive-compatible allocation* $\mathcal{A}$*, cand* s *are smooth functions of* z*, with* c *increasing, and the following extended Spence-Mirrlees condition holds:*

$$\mathcal{S}'_\theta(c, s, z; \theta)\frac{ds}{dz} + \mathcal{Z}'_\theta(c, s, z; \theta) > 0. \tag{8}$$

Assumption 2 is a standard assumption that is also required to apply standard optimal control methods to characterize the optimal allocation.

The main component of Assumption 3 is the extended Spence-Mirrlees condition, which generalizes the standard assumption, first stated in Mirrlees (1971), that $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$. If $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$ and s is increasing in z, this condition states that the relationship between earnings ability and preferences for s isn't too *negative.* If $\mathcal{Z}'_\theta(c, s, z; \theta) > 0$ and s is decreasing in z, this condition states that the relationship between earnings ability and preferences for s isn't too *positive.* In the savings applications we consider, where evidence suggests positive correlated preference heterogeneity $\mathcal{S}'_\theta(c, s, z; \theta) > 0$, this assumption would be violated only if the optimal mechanism featured a savings allocation that is *decreasing* with earnings. We have not been able to find examples of such unintuitive mechanisms numerically.[8]

[7]Expressing the tax more generally as $\mathcal{T}(c, s, z)$ is redundant. Given such a tax function, any choice of s and z implies a consumption value given by $\mathcal{C}(s, z) := \max\{c | c = z - s - \mathcal{T}(c, s, z)\}$; thus, one can re-express tax burden as $\tilde{\mathcal{T}}(s, z) = \mathcal{T}(\mathcal{C}(s, z), s, z)$.

[8]Note that the assumption that c is increasing could alternatively be characterized not as an assumption about

One consequence of Assumption 3 is that $z(\theta)$ is strictly increasing in θ (Appendix Lemma A1). This allows us to define the function $\vartheta(z)$, which is the type assigned to an earnings level z from the optimal incentive-compatible allocation.[9]

3.1 Implementability with Smooth Tax Systems

Definition 2. *We say that an allocation* $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ *is implementable with a tax system* $\mathcal{T}$ *if*

1. $\mathcal{T}$ *satisfies type-specific feasibility:* $c(\theta) + ps(\theta) + \mathcal{T}(s(\theta), z(\theta)) = z(\theta)$ *for all* $\theta \in \Theta$, *and*

2. $\mathcal{T}$ *satisfies individual optimization:* $(c(\theta), s(\theta), z(\theta))$ *maximizes* $U(c, s, z; \theta)$ *for all* $\theta \in \Theta$, *subject to the constraint* $c + ps + \mathcal{T}(s, z) \leq z$.

Our first result shows that the optimal incentive-compatible allocation is implementable by some smooth tax system.

Theorem 1. *Suppose that assumptions 2 and 3 hold. Then the optimal incentive-compatible allocation is implementable by a smooth tax system. In this smooth tax system, agents' choices are interior (first-order conditions hold), and their local optima are strict (strict second-order conditions).*

Although it is clear that the optimal incentive-compatible allocation $\{(c(\theta), s(\theta), z(\theta))\}_\theta$ can always be implemented by *some* two-dimensional tax system—for example, by defining $\mathcal{T}(s(\theta), z(\theta)) = z(\theta) - c(\theta) - s(\theta)$ for $\theta \in \Theta$ and letting $\mathcal{T}(s, z) \to \infty$ for $(c, s, z) \notin \{(c(\theta), s(\theta), z(\theta))\}_\theta$—such a tax system is not guaranteed to be smooth. A smooth tax system allows agents to independently adjust s and z locally to points not chosen by any other type in the optimal allocation, and thus the set of possible deviations is much larger than when optimal mechanism can simply disallow certain allocations.

Starting from any given allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta\}_\theta$, a smooth tax system can implement the allocation only by satisfying the first-order conditions

$$\mathcal{T}'_s\left(s\left(z\left(\theta\right)\right), z\left(\theta\right)\right) = \mathcal{S}\left(c(\theta), s(\theta), z(\theta); \theta\right) - 1 \tag{9}$$

$$\mathcal{T}'_z\left(s\left(z\left(\theta\right)\right), z\left(\theta\right)\right) = \mathcal{Z}\left(c(\theta), s(\theta), z(\theta); \theta\right) + 1. \tag{10}$$

In the presence of preference heterogeneity, individuals' incentives to deviate from their assigned allocation $(c(\theta), s(\theta), z(\theta))$ are higher under a smooth tax system than under an optimal incentive-compatible mechanism. For example, suppose that higher types θ have a higher taste for savings. If they deviate downward to some other earnings level $z(\theta') < z(\theta)$, then under the optimal mechanism

the nature of the optimal mechanism, but rather as a modest assumption on the space of allowable mechanisms. Since the space of increasing functions is compact, an optimal mechanism within this space is guaranteed to exist.

[9] Incentive compatibility implies $\mathcal{S}'_\theta(c(\theta), s(\theta), z(\theta); \theta)s'(\theta) + \mathcal{Z}'_\theta\left(c(\theta), s(\theta), z(\theta); \theta\right) z'(\theta) \geq 0$ for any type θ. Absent preference heterogeneity, $\mathcal{S}'_\theta = 0$, which reduces to the standard Spence-Mirrlees condition $\mathcal{Z}'_\theta > 0$, which is known to imply that earnings z increase with type in any incentive compatible allocation.

they will be forced to choose savings level $s(\theta')$. Under a smooth tax system, however, the deviating type θ will choose a higher savings level $s' > s(\theta')$ at earnings level $z(\theta')$, making this *double deviation* more appealing.

Tax implementation results that involve multidimensional consumption bundles and multidimensional tax systems typically avoid the difficulties associated with double deviations by ruling out the type of heterogeneity that we consider here. Thus, to our knowledge, our proof of Theorem 1 involves a different and more subtle mathematical approach. The proof, contained in Appendix B.2, proceeds in three steps. The first step is to construct a sequence of tax systems $\mathcal{T}_k$ such that each element in the sequence satisfies type-specific feasibility and the first-order conditions above. The sequence is ordered such that each successive element is more convex around the bundles $(s(\theta), z(\theta))$ offered in the optimal mechanism.

In the second step of the proof, we show that for each type θ there exists k sufficiently large such that for this type the second-order conditions hold at the point $(c(\theta), s(\theta), z(\theta))$. In other words, for each type there is a sufficiently large k such that $(c(\theta), s(\theta), z(\theta))$ is a *local* optimum under the tax system $\mathcal{T}_k$. This step requires auxiliary Lemmas B1 and B2, which characterize individuals' budget constraints and second derivatives of indirect utility functions for any tax system $\mathcal{T}$ that preserves only the first-order conditions of the optimal mechanism.

In the third step, we show that there exists a sufficiently large k such that $(c(\theta), s(\theta), z(\theta))$ is a *global* optimum for *all* types θ under $\mathcal{T}_k$. We complete this step via a proof by contradiction. Under the assumption that such a k does not exist, there exists an infinite sequence of values k and types θ_k such that type θ_k prefers to deviate from $(c(\theta_k), s(\theta_k), z(\theta_k))$ under $\mathcal{T}_k$. Because the type space is compact, the Bolzano-Weirstrass Theorem allows us to extract a convergent subsequence of types θ_j who all prefer to deviate from the allocation assigned to them under the optimal mechanism. We show that this implies a contradiction because, roughly speaking, the limit type of this sequence, $\hat{\theta}$, must then prefer to deviate from $(c(\hat{\theta}), s(\hat{\theta}), z(\hat{\theta}))$ to some other allocation $(c(\theta'), s(\theta'), z(\theta'))$ offered in the optimal mechanism.

Theorem 1 is an "existence result," and our proof of the theorem does not offer insight into the structure of an optimal tax system. However, because agents' choices are shown to satisfy first-order and second-order conditions in a smooth tax system, we can use variational methods to characterize optimal tax systems. We now proceed by deriving optimal tax formulas expressed in terms of empirically-estimable sufficient statistics that transparently highlight the key economic forces governing the optimal tax system.

3.2 Sufficient Statistics for Smooth Tax Systems

3.2.1 Definitions

To define the sufficient statistics, it is helpful to write agents' optimization problem under a tax system $\mathcal{T}(s, z)$ as

$$\max_{z} \left\{ \max_{c,s} U(c, s, z; \theta) \text{ s.t. } c \leq z - s - \mathcal{T}(s, z) \right\}, \tag{11}$$

where the inner problem represents the optimal choices of consumption $c(z;\theta)$ and savings $s(z;\theta)$ for a given earnings level z, and the outer problem represents the optimal choice of earnings $z(\theta)$ taking into account endogenous consumption and savings choices.

Earnings responses to tax reforms are captured through ζ_z^c, the compensated elasticity of labor income with respect to the marginal labor income tax rate, and η_z, the income effect parameter. Formally, for each level of earnings $z(\theta)$ chosen by a type θ, we define

$$\zeta_z^c(z(\theta)) := -\frac{1-\mathcal{T}_z'(s(\theta),z(\theta))}{z(\theta)}\frac{\partial z(\theta)}{\partial \mathcal{T}_z'(s(\theta),z(\theta))}$$
$$\eta_z(z(\theta)) := -(1-\mathcal{T}_z'(s(\theta),z(\theta)))\frac{\partial z(\theta)}{\partial \mathcal{T}(s(\theta),z(\theta))}$$

where $\mathcal{T}(s(\theta),z(\theta))$ is the tax liability and $\mathcal{T}_z'(s(\theta),z(\theta))$ is the marginal labor income tax rate. Since the earnings choice takes into account endogenous consumption and savings choices, these elasticity concepts take into account the full sequence of adjustments due to changes in consumption and savings choices, as well as those due to any nonlinearities in the tax system.[10]

Savings responses to tax reforms are captured through $\zeta_{s|z}^c$, the compensated elasticity of savings with respect to the marginal savings tax rate, $\eta_{s|z}$, the income effect parameter, and s_{inc}', the causal effect on consumption of s from a marginal change in gross pre-tax income z. These are formally defined as follows:

$$\zeta_{s|z}^c(z(\theta)) := -\frac{1+\mathcal{T}_s'(s(z;\theta),z)}{s(z;\theta)}\frac{\partial s(z;\theta)}{\partial \mathcal{T}_s'(s(z;\theta),z)}\Big|_{z=z(\theta)}$$
$$\eta_{s|z}(z(\theta)) := -\left(1+\mathcal{T}_s'(s(z;\theta),z)\right)\frac{\partial s(z;\theta)}{\partial \mathcal{T}(s(z;\theta),z)}\Big|_{z=z(\theta)}$$
$$s_{inc}'(z(\theta)) := \frac{\partial s(z;\theta)}{\partial z}\Big|_{z=z(\theta)}$$

where $\mathcal{T}_s'(s(z;\theta),z)$ is the marginal savings tax rate of an agent of type θ who earns labor income z. These elasticity concepts are conditional on z. They measure responses of consumption and savings to tax reforms and nonlinearities in the tax system, holding labor income z fixed fixed at $z(\theta)$. For all elasticity concepts, we use the "bar" notation, as in $\overline{\zeta_{s|z}^c}$, to denote a population elasticity.

For convenience, we define the elasticity of s with respect to one plus the marginal tax rate, rather than with respect to $p+\mathcal{T}_s'$, which may be natural in applications where s represents savings. However, defining the elasticity with respect to $p+\mathcal{T}_s'$ may be paricularly natural in applications where s is a commodity sold at after-tax price of $q=p+\mathcal{T}_s'$. With this alternative definition of the elasticities, the key results in Propositions 2 and 2 can be obtained by simply multiplying $\zeta_{s|z}^c$ by $(p+\mathcal{T}_s')/(1+\mathcal{T}_s')$.[11]

[10] This corresponds to the type of circular adjustment process described in e.g. Jacquet and Lehmann (2020).

[11] In this case, the only change in Theorem 2 is that the left-hand-side in equation (16) becomes $\frac{\mathcal{T}_s'(s(z),z)}{p+\mathcal{T}_s'(s(z),z)}$, and analogously for Proposition 2 .

To quantify correlated preference heterogeneity, we decompose the cross-sectional profile of s consumption. Intuitively, $s'(z)$, the cross-sectional change in s with respect to z, reflects both the causal income effect and the degree to which preferences are changing with earnings z. We thus define our measure of correlated preference heterogeneity, $s'_{pref}(z)$, as the difference between the cross-sectional variation of s along the earnings distribution and the causal income effect $s'_{inc}(z)$:[12]

$$s'_{pref}(z(\theta)) := s'(z(\theta)) - s'_{inc}(z(\theta))$$

To illustrate how $s'_{pref}(z)$ relates to model parameters, suppose that $U = \ln c + \delta(\theta) \ln s - \psi(z/\theta)$, where s is period-2 consumption, δ is the discount factor, and $\mathcal{T}$ is a *simple* tax system (see Section 4) that is separable in s and z. Then a z-earner chooses

$$s(z) = \frac{1}{p} \frac{\delta(z)}{1 + \delta(z)} (z - \mathcal{T}) \tag{12}$$

where $\delta(z)$ is used to denote the discount factor of the z-earner. Cross-sectional heterogeneity in s is given by

$$s'(z) = \underbrace{\frac{1}{p} \frac{\delta(z)}{1 + \delta(z)} (1 - \mathcal{T}'_z)}_{s'_{inc}(z)} + \underbrace{\frac{1}{p} (z - \mathcal{T}) \frac{d}{dz} \left(\frac{\delta(z)}{1 + \delta(z)} \right)}_{s'_{pref}(z)} \tag{13}$$

The causal income effect s'_{inc} is obtained by differentiating (12) with respect to z while holding the discount factor $\delta(z)$ constant. The local preference heterogeneity term $s'_{pref}(z)$ is obtained by differentiating (12) with respect to z while holding after-tax income $z - \mathcal{T}$ constant. The total derivative of s with respect to z that makes up the cross-sectional profile is the sum of the two partial derivatives. The key insight that facilitates measurement is that $s'(z)$ and $s'_{inc}(z)$ are commonly-measured statistics that can be used to indirectly estimate $s'_{pref}(z)$, which is often more tractable than a direct estimate of how time preferences $\delta(\theta)$ vary across the income distribution.

3.2.2 Measurement

Because $s'(z)$ is the cross-sectional variation of savings along the income distribution, it can be directly observed from survey or administrative data featuring both earnings and savings. The result below shows how $s'_{inc}(z)$ can be measured in the data.

Proposition 1. *The sufficient statistic $s'_{inc}(z)$ can be measured as follows.*

- *If preferences are weakly separable and the tax system is separable in s and z, $s'_{inc}(z) = \frac{1 - T'_z(z)}{1 + T'_s(s(z))} \eta_{s|z}(z)$*
- *If wage rates w and hours h are observable, $s'_{inc}(z) = s(z) \frac{\xi^s_w(z)}{w(z) + h(z) \xi^h_w(z)}$*

[12] Put another way, let $\vartheta(z)$ denote the type θ that earns z. Then $s'(z)$ is a total derivative given by $\frac{d}{dz} s(z, \vartheta(z)) = \frac{\partial}{\partial z} s(z, \theta)|_{\theta = \vartheta(z)} + \frac{\partial}{\partial \theta} s(z, \theta)|_{\theta = \vartheta(z)}$, where the first term measures causal income effects and corresponds to $s'_{inc}(z)$, and the second term measures local preference heterogeneity and corresponds to $s'_{pref}(z)$.

- *If we savings and earnings changes in response to tax reforms are observable,* $s'_{inc}(z) = \frac{s(z)}{z}\frac{\chi_s^c(z)}{\zeta_z^c(z)}$,

where $\xi_w^s(z)$ *is the elasticity of savings with respect to the wage rate,* $\xi_w^h(z)$ *is the elasticity of hours with respect to the wage rate, and* $\chi_s^c(z)$ *is the elasticity of savings with respect the marginal net of tax rate on labor income.*

If individuals' preferences are weakly separable as in (1) above, and if the tax system is separable in s and z, then $s'_{inc}(z)$ is proportional to the income effect parameter. If individuals' preferences are not weakly separable but wage rates w and hours h are observable, $s'_{inc}(z)$ can be related to the elasticity of savings with respect to the wage rate and to the elasticity of hours with respect to the wage rate. If both earnings and savings changes in response to a change in the marginal tax rate z are observable, $s'_{inc}(z)$ can be recovered from the ratio of savings responses to labor income responses. Intuitively, responses to this type of tax reform are induced by changes in labor income.

3.3 Sufficient Statistics Characterization of Optimal Smooth Tax Systems

A key result result used to derive our sufficient statistics formula is an equivalence result for tax reforms affecting marginal tax rates on s versus z. The result is a generalization of Lemma 1 in Saez (2002) to arbitrarily nonlinear smooth tax systems.

Lemma 1. *A small increase* $d\tau$ *in the marginal savings tax rate faced by agent* θ *at earnings* z *induces the same earnings change as a small increase* $s'_{inc}(z)\,d\tau$ *in the marginal earnings tax rate.*

Lemma 1 relates the labor supply distortions induced by increasing taxes on s to the labor supply distortions induced by earnings taxes. The intuition behind the lemma is as follows. An individual who reduces earnings by dz reduces s by $s'_{inc}dz$. Thus, when taxes on s increase by $d\tau$, reducing earnings by dz reduces tax liability by an additional $s'_{inc}d\tau dz$. By comparison, when earnings taxes are increased by $d\tau$, a dz reduction in earnings reduces tax liability by an additional$d\tau dz$. Thus, the dis-incentive to work generated by a $d\tau$ increase in the tax on s is equal to the product of s'_{inc} and the dis-incentive to work generated by a $d\tau$ increase in the earnings tax.

To encode the policymaker's redistributive objective, we follow the literature and define $\hat{g}(z)$ as the social marginal welfare weights augmented with the fiscal impact of income effects. These capture the social value of marginally increasing the disposable income of agents earning z. Formally,

$$\begin{aligned}\hat{g}(z(\theta)) := \frac{\alpha(\theta)}{\lambda}U'_c(c(\theta), s(\theta), z(\theta); \theta) + \mathcal{T}'_z(s(\theta), z(\theta))\frac{\eta_z(z(\theta))}{1-\mathcal{T}'_z(s(\theta), z(\theta))} \\ + \mathcal{T}'_s(s(\theta), z(\theta))\left(\frac{\eta_{s|z}(z(\theta))}{1+\mathcal{T}'_s(s(\theta), z(\theta))} + s'_{inc}(z(\theta))\frac{\eta_z(z(\theta))}{1-\mathcal{T}'_z(s(\theta), z(\theta))}\right),\end{aligned} \tag{14}$$

where the last term captures the fact that income effects on earnings proportional to $\eta_z(z(\theta))$ induce changes in savings proportional to $s'_{inc}(z(\theta))$ affecting savings tax revenues. We use $g(z(\theta))$ to denote the first component, $\frac{\alpha(\theta)}{\lambda}U'_c(c(\theta), s(\theta), z(\theta); \theta)$.

In the results that follow, we use $H(s,z)$ and $h(s,z)$ to denote the cumulative and density functions over (s,z), we let h_s denote the marginal density over s, and we let h_z denote the marginal density over z.

Theorem 2. *Suppose that assumptions 1, 2, and 3 hold. At each bundle (c,s,z) chosen by a type θ, an optimal smooth tax system satisfies the following conditions on marginal tax rates on z and s, respectively:*

$$\frac{\mathcal{T}_z'(s,z)}{1-\mathcal{T}_z'(s,z)} = \frac{1}{z\,\zeta_z^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{\mathcal{T}_s'(s,z)}{1-\mathcal{T}_z'(s,z)} \tag{15}$$

$$\frac{\mathcal{T}_s'(s,z)}{1+\mathcal{T}_s'(s,z)} = s_{pref}'(z)\frac{1}{s\,\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) \tag{16}$$

Any Pareto efficient smooth tax system satisfies

$$\frac{\mathcal{T}_s'(s,z)}{1+\mathcal{T}_s'(s,z)} = s_{pref}'(z)\frac{z\,\zeta_z^c(z)}{s\,\zeta_{s|z}^c(z)}\frac{\mathcal{T}_z'(s,z)+s_{inc}'(z)\mathcal{T}_s'(s,z)}{1-\mathcal{T}_z'(s,z)} \tag{17}$$

Formula (15) constitutes a familiar condition analogous to Saez (2001), with one modification. Because there is a non-zero savings tax rate, the formula also accounts for the distortionary effects on savings caused by distortions to earnings levels.

Formula (16) is one of our key results about optimal marginal tax rates on s. Optimal tax rates on s satisfy a condition that is remarkably similar to the standard "ABC" formula for optimal income tax rates, as presented in equation (15). When $s_{pref}' > 0$, the magnitude of the optimal tax rate at point (s,z) is decreasing in the elasticity of s with respect to the tax rate, increasing in the strength of redistributive motives, and decreasing in the relative density of individuals at point (s,z).

Formula (16) also gives a transparent generalization of the Atkinson-Stiglitz Theorem. When the sufficient statistic for preference heterogeneity, s_{pref}', is equal to zero, the condition implies that the optimal tax on s must equal zero as well. When the statistic is $s_{pref}' > 0$, implying that higher earners have a higher taste for s, the condition implies that the optimal tax rate on s must be positive.

We can combine conditions (15) and(16) to derive the Pareto efficiency condition in (17). Because the condition in (17) does not feature social marginal welfare weights, it is an efficiency condition that must hold for any tax system that is not Pareto dominated. Intuitively, it quantifies the efficient balance between taxing s and taxing z, given the measure s_{pref}' of how tastes for s vary with earnings ability. The larger is the association between tastes for s and earnings ability, the more efficient it is to tax s instead of z.

An implication of the Pareto efficiency condition in (17) is that in the absence of preference heterogeneity, positive savings tax rates are Pareto dominated, providing an extension of the Atkinson-Stiglitz Theorem for nonlinear tax systems. On the other hand, any Pareto efficient tax system must feature nonzero savings tax rates in the presence of preference heterogeneity.

3.4 Other Determinants of Tax Elasticities Captured by the Sufficient Statistics Formulas

In practice, the elasticities and measures of preference heterogeneity that we utilize may be shaped by other economic considerations, such as differences in prices of s across individuals (Gahvari and Micheletto, 2016), *income shifting* (Slemrod, 1995; Christiansen and Tuomala, 2008), and heterogeneity in endowments (Boadway et al., 2000; Cremer et al., 2003). A key feature of our sufficient statistics characterization of optimal tax schedules, including the characterization of "simple" tax systems in Section 4 below, is that they apply in the presence of these other determinants of taxpayer behavior. The intuition is in the spirit of the Feldstein (1999) result that the elasticity of taxable income is a sufficient statistic for efficiency losses irrespective of whether it is due to real labor supply responses or costly avoidance behavior.[13] While these other mechanisms have previously been studied in isolation in specialized models that qualitatively address the robustness of the Atkinson-Stiglitz Theorem, our sufficient statistics techniques allow us to account for these considerations in a quantitative and general manner.

Heterogeneous Prices. Suppose that individuals face prices $p(e, z, s, \theta)$ that might depend on effort e to seek out lower prices, types θ, earnings z, and the level of s. For example, higher ability types might better able to seek out lower prices or higher returns on investment, higher income z might generate beneficial network effects that expose individuals to better opportunities, and higher levels of s might allow individuals to lock in better prices or interest rates. Individuals' utility function is now $U(c, s, z, e; \theta)$ and their budget constraint is

$$c + p(e, z, s, \theta)s \leq z - \mathcal{T}(s, z).$$

This economy is then equivalent to an economy where $p \equiv 1$ and where individuals maximize the utility function

$$\tilde{U}(c, s, z; \theta) = \max_e U\left(c + (1-p)s, s, z, e; \theta\right) \tag{18}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This is because with a price of $p = p'$ instead of $p = 1$, individuals receive $(1 - p')s$ more consumption c at a given choice s. The feasibility constraint $\int_\theta \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is independent of the price p because the tax is deducted directly from an individual's earnings z.[14] Thus, it is without loss of generality to assume that $p \equiv 1$, which is a normalization we adopt in our proofs.

Type-dependent heterogeneity in prices will generally lead to $s'_{pref}(z) \neq 0$ and thus deviations

[13]Chetty (2009) suggests limitations to Feldstein's (1999) results due to some avoidance behaviors generating new types of fiscal externalities, or due to behavioral biases. Variations of these considerations are relevant in our setting as well, as explored in Sections 5.3 and 5.4, respectively.

[14]For example, if s is units of soda purchased, the tax applies to individuals' earnings directly because it is units of dollars. However, if the tax had a two-part structure where the individual must pay T_1 to in units of c to the government and T_2 in units of s to the government then the equivalence above would not apply. We explore this in Section 5.4, extending the Gerritsen et al. (2020) analysis of capital income taxation with heterogeneous rates of return.

from the Atkinson-Stiglitz result. For example, suppose that types θ see prices $p(\theta)$. Then heterogeneity in prices functions much like heterogeneity in discount factors. To illustrate, suppose again that $U = \ln c + \delta \ln s - \psi(z/\theta)$ and that $\mathcal{T}$ is separable. Then, as in the example with heterogeneous discount rates, $s'_{pref}(z) = \delta\,(z - \mathcal{T})\,\frac{d}{dz}\left(\frac{1}{p(z)}\right)$, where $p(z)$ is the price seen by z-earners.

Heterogeneity in prices that is due to scale effects, however, does not contribute to s'_{pref} . For example, when $U = \ln c + \delta \ln s - \psi(z/\theta)$, allowing scale effects where p is an increasing function of s (but not θ) increases s'_{inc} but leaves $s'_{pref} \equiv 0$ when δ is homogeneous.

Income Shifting. Slemrod (1995) argues that some tax system may provide incentives for individuals to "convert ordinary income into preferentially taxed capital gains," "convert corporations into pass-through legal entities such as partnerships, or retain labor compensation within the corporation." Generalizing the two-type model in Christiansen and Tuomala (2008), suppose that individuals can shift χ units of their labor income to capital income or savings, so that the observable pre-tax labor income that they are taxed on is $z = \tilde{z} - \chi$, where $\tilde{z}$ is actual earned income. Individuals choose z, c, s, χ to maximize $U(c, s, z, \chi; \theta)$ subject to the budget constraint $c + p = s \leq z - \mathcal{T}(s, z) - m(\chi; \theta)$. This formulation allows income shifting to both enter the utility function directly—e.g., due to effort or psychic costs—as well as through the budget constraint because it may require financial resources m. This formulation also allows for the natural possibility individuals with higher earnings ability may be better able to engage in income shifting. The budget constraint can be rewritten as

$$\begin{aligned} c &= \tilde{z} - \mathcal{T}(s, z) - s - m(\chi) \\ &= z + \chi - \mathcal{T}(s, z) - s - m(\chi) \end{aligned}$$

Thus, we can alternatively rewrite this as an optimization problem where individuals maximize

$$\tilde{U}(c, s, z; \theta) = \max_{\chi} U\left(c + \chi - m(\chi, \theta), s, z + \chi, \chi; \theta\right)$$

subject to the budget constraint that $c + s \leq z - \mathcal{T}(s, z)$. The feasibility constraint $\int_\theta \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is independent of the degree to which agents engage in income shifting because the tax is deducted directly from an individual's observed earnings z.

Heterogeneous Endowments. Suppose that individuals have endowments $y_0(\theta)$, from inheritance or other sources, such that their budget constraint is given by $c + ps \leq y_0(\theta) + z - \mathcal{T}(s, z)$. This economy is then equivalent to an economy where $y_0 \equiv 0$ and where individuals maximize the utility function

$$\tilde{U}(c, s, z; \theta) = U\left(c + y_0(\theta), s, z; \theta\right) \tag{19}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This is because at a given choice s, individuals receive y_0 more consumption of c. Again, the feasibility constraint $\int_\theta \mathcal{T}\left(s\left(\theta\right), z\left(\theta\right)\right) dF\left(\theta\right) \geq E$ is independent of the endowments.

4 Optimal "Simple" Tax Systems

In practice, tax systems must be defined by policymakers and implemented by institutions that might face constraints on the degree of complexity. As a result, typical tax functions are not fully flexible. Although the details of the restrictions vary across institutions, most can be classified into a few common functional forms.

We consider two questions in this section. First, does the importance of preference heterogeneity, as measured by s'_{pref}, extend to simple tax systems? Second, when can simple tax systems implement the optimal allocation?

4.1 A Taxonomy of Common Simple Savings Tax Systems

Many governments tax both labor income earnings and savings / capital interest income. While these tax systems take a variety of forms, the details of which depend on specifics such as timing, many of these tax policies can be interpreted as a function of earnings and savings, analogous to our function $\mathcal{T}(s, z)$. In table 1 we propose three types of simple tax systems: separable linear (SL), separable nonlinear (SN), and linear income-dependent (LED). In Table 2 we show that many of the tax systems used in practice can be categorized as one of three simple types in Table 1. In Table 2 categorize the tax policies on each class of savings vehicle for 21 countries, most of which fit one of the three simple tax system types from Table 1. In cases where there is some ambiguity (such as the distinction between short-term and long-term capital gains in the United States) we provide supplementary information in Appendix D.[15]

Each of the three tax systems in Table 1 can for instance be found in the United States. Most property taxes, levied at the state and local level, take the form of separable linear taxes, with a flat tax rate, independent of one's labor earnings, applied to the assessed value of the total property. The estate tax takes the form of a separable nonlinear tax, as the tax rate rises progressively with the value of the estate, but it does not vary with labor income of the donor or the recipient. Finally, taxes on long-term capital gains and on qualified dividends both take the form of linear earnings-dependent taxes. In 2020, for example, an individual with \$50,000 in labor earnings faced a long-term capital gains tax rate of 15%, whereas an individual earning \$500,000 faced a rate of 20%.[16]

[15] We impose several simplifications for our interpretation of the tax codes. First, we treat ordinary income as consisting primarily of labor income (earnings), written as z in our notation. Second, we separately consider taxes on five broad categories of savings vehicles: wealth, capital gains, real property, private pensions, and inheritances. These categories may overlap—real property is a component of wealth, for example—but we use these groups to reflect the tax instruments that many governments use in practice.

[16] Given our application to taxation of savings, bequests, and capital interest income, where individuals consume c in period 1 and s in period 2, it may be more natural to formalize the tax function as imposing a period-1 labor income tax and a period-2 tax on gross savings. Let $1 + r = 1/p$ denote the rate of transformation between c and s, so that r can be interpreted as the interest rate. Now consider separable tax functions where the policymaker imposes a tax $T_1(z)$ in period 1, and imposes a tax $T_2(s_g, z)$ on gross pre-tax savings $s_g = (1 + r)(z - T_1(z) - c)$ in

4.2 Optimal Simple Tax Systems

We now present optimality conditions for SL, SN and LED tax systems. We focus on marginal tax rates on s in the body of the paper, and present conditions for marginal tax rates on z in Appendix (A.4). The preference heterogeneity statistic s'_{pref} remains the key sufficient statistic for the marginal tax rate on s. For SL systems, it is convenient to introduce the term $s_{pref}(z) = \int_{x=z_{min}}^{z} s'_{pref}(x)dx$, which integrates local preference heterogeneity to obtain a measure of total preference heterogeneity for with gross earnings of z or less.

Proposition 2. *Suppose that the optimal SL, SN, and LED systems are smooth and that at the optimum: (i) agents' optima are unique their first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, these systems satisfy the following optimality conditions :*

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\bar{s}\overline{\zeta^c_{s|z}}} \int_z s'_{pref}(z) \left[\int_{x \geq z} (1-\hat{g}(x)) dH_z(x)\right] dz \tag{20}$$

$$= -\frac{1}{\bar{s}\overline{\zeta^c_{s|z}}} Cov[s_{pref}(z), \hat{g}(z)] \tag{21}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{pref}(z) \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) \tag{22}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = \frac{1}{s\,\zeta^c_{s|z}(z)} \frac{1}{h_z(z)} s'_{pref}(z) \int_{x \geq z} (1-\hat{g}(x))\, dH_z(x) \tag{23}$$

Moreover, if a SL/SN/LED tax system is not Pareto dominated by another SL/SN/LED system, then it must satisfy the following conditions:

$$SL: \frac{\tau_s}{1+\tau_s} = \frac{1}{\bar{s}\overline{\zeta^c_{s|z}}} \int_z s'_{pref}(z)\, z\zeta^c_z(z)\, \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)}\, dH_z(z) \tag{24}$$

$$SN: \frac{T'_s(s)}{1+T'_s(s)} = s'_{pref}(z)\, \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)}\, \frac{T'_z(z) + s'_{inc}(z)T'_s(s)}{1-T'_z(z)} \tag{25}$$

$$LED: \frac{\tau_s(z)}{1+\tau_s(z)} = s'_{pref}(z)\, \frac{z\zeta^c_z(z)}{s\zeta^c_{s|z}(z)}\, \frac{T'_z(z) + \tau'_s(z)\, s + s'_{inc}(z)\tau_s(z)}{1-T'_z(z) - \tau'_s(z)\, s} \tag{26}$$

The optimal tax formulas and the Pareto efficiency conditions for SN and LED systems are

period 2, so that period-2 consumption is given by $s = s_g - T_s(s_g, z)$. In this formulation, a SL structure is one where $T_2(s_g, z) = \tau_s s_g$, a SN structure is one where $T_2(s_g, z)$ is a function of s_g only, and a LED structure is one where $T_2(s_g, z) = \tau_s(z)s_g$. As we show in Appendix (A.6), these formulations of two-period tax systems are equivalent to the ones we study in the body of the paper in two fundamental ways. First, the set of allocations implementable by these systems is identical, as there is a simple and natural translation between the "static" tax function $\mathcal{T}$ we consider and the two-period function. Second, if $T_1(z) + T_2(s_g, z)$ implements the same allocation as $\mathcal{T}(s, z)$, then $T_1(z) + T_2(s_g, z)$ will be SL/SN/LED if and only if $\mathcal{T}(s, z)$ is SL/SN/LED.

essentially identical to the conditions for fully flexible smooth tax systems derived in Theorem (2). This equivalence is guaranteed for conditions under which SN and LED systems can implement the optimal allocation, which we summarize in Section (4.3). However, there are also cases in which SN or LED systems cannot implement the same allocation as the optimal nonlinear tax system. In these cases, Proposition (3) shows that the formulas for the optimum are still similar, even if the resulting allocations are not.

The SL system is the most restrictive, and generically cannot implement the same allocation as the optimal nonlinear tax system. This is because the optimal nonlinear tax system does not generally feature constant marginal tax rates on s. Correspondingly, the optimal tax formula for the SL systems takes a different form. As shown in expression (20), the constant marginal tax rate τ_s for SL systems is in a certain sense an average of the z-earner specific marginal tax rates in expressions (22) and (23). Intuitively, the constant marginal tax rate is a population aggregate of the tax rates that would be optimal for individuals with different earnings levels. This is mirrored in the Pareto efficiency condition (24). Expression (20) is an alternative optimality condition, which was first derived by Allcott et al. (2019). This formulation has a familiar form resembling the Diamond (1975) "Many-person Ramsey tax rule." The expression is identical to the Diamond (1975) formula when $s_{pref}(z) \equiv s(z)$; i.e., when all consumption is driven by preference heterogeneity. This illustrates both the relevance of Ramsey taxation principles even in the presence of nonlinear taxation, as well as their limitations. The SL formula when $s_{pref}(z) \equiv 0$ reduces to the original statement of the Atkinson-Stiglitz theorem. More generally, it is helpful to note that even for arbitrarily nonlinear taxes on s, the optimal tax rule rate is always inversely proportional to the elasticity $\zeta^c_{s|z}(z)$, consistent with standard Ramsey principles, as long as $s'_{pref}(z) \neq 0$.

4.3 When can Simple Savings Tax Systems Implement the Optimum?

In Appendix A.2 we present sufficient conditions for the optimal allocation to be implementable by a SN or LED tax system. We assume that the optimal allocation satisfies the conditions in Theorem 1, implying that it can be implemented by *some* smooth tax system, so that the question of interest is whether SN or LED systems are among those that implement the optimum. We focus on SN and LED systems because the SL system is plainly too restrictive to implement the optimum. We summarize the results here.

We proceed in three steps to characterize the sufficient conditions. First, we define candidate SN and LED tax systems constructed to satisfy type-specific feasibility and individuals' first-order conditions at the optimal allocation, as in Section 3.1. This completely defines candidate tax systems in terms of agents' marginal rates of substitutions $\mathcal{S}$ and $\mathcal{Z}$, and the optimal incentive-compatible allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$.

Second, in Proposition A1 we present sufficient conditions under which these candidate SN and LED tax systems also satisfy the local second-order conditions, implying that each type's assigned allocation is a local optimum. The sufficient conditions for the SN tax system are relatively weak, and are satisfied by many common utility functions used in calibrations of savings and income taxation models, including the example in Equation (1). The sufficient conditions for the LED tax system are somewhat more restrictive, as they place a constraint on the extent of local preference heterogeneity. A high degree of local preference heterogeneity at income z implies a large positive savings tax rate for any individual choosing earnings z. This large increase in the savings tax liability must then be compensated by a sharp decrease in the earnings tax liability, which can create incentives for deviation.

Third, in Proposition A2 we derive sufficient conditions under which local optima are ensured to be global optima for SN and LED tax systems. These conditions involve separability of agents' preferences and further concavity requirements on preferences for consumption and savings. Together with Proposition A1, this produces a characterization of when SN and LED tax systems implement the optimal allocation.

5 Extensions

In this section we provide four key extensions of our results. First, we generalize our results to consider multidimensional heterogeneity and potentially suboptimal income taxes. Second, we generalize our results to many dimensions of consumption. Third, we extend our results to the possibility that the government wants people to save more than their perceived private optima, either because of the government's inter-generational social preferences or because of behavioral biases. Fourth, we consider the case where tax systems have a *multidimensional range* so that, for example, they are collected both in period 1 (in units of c) and in period 2 (in units of s).

5.1 Multidimensional Heterogeneity and Non-Optimal Earnings Tax

We begin by characterizing optimal simple tax systems with multidimensional heterogeneity. We show that s'_{inc} remains the key sufficient statistic. Because $s' = s'_{pref} + s'_{inc}$, this implies that s'_{pref} remains the key sufficient statistic for preference heterogeneity.

We focus on simple tax systems for two reasons: because the separability in these mechanisms allows us to study taxation of s in the presence of a suboptimal earnings tax, which may be a realistic possibility in practice, and because the optimal mechanism can feature atoms, discontinuities, and randomization in the presence of multidimensional heterogeneity, making it impossible to characterize in all but a few special cases.[17] Proposition 3 below generalizes Proposition 2 in two key ways. First, it allows for multidimensional heterogeneity, where types θ belong to a subset of $\mathbb{R}^n$ for $n \geq 2$, so that the support of the distribution of (s, z) is two-dimensional. Second, it characterizes optimal taxes on s even when the earnings tax $T_z(z)$ is not necessarily optimal. Proposition (A5) in the Appendix characterizes the optimal earnings tax schedule.

We use $\zeta_z^c(s, z)$ to define the compensated elasticity of s of an individual choosing the bundle (s, z). We utilize the notation $\hat{g}(s, z)$ and $s'_{inc}(s, z)$ analogously.

[17] In related work, Jacquet and Lehmann (2021) consider a sum of many one-dimensional tax schedules for similar reasons.

Proposition 3. *Consider smooth simple tax systems with (potentially suboptimal) smooth earnings tax schedule $T_z(z)$ and optimally-set marginal tax rates on s. Assume that agents' first- and second-order conditions hold in these tax systems, and that there is no bunching. Then, at each bundle (c^0, s^0, z^0) chosen some type θ, the marginal tax rates on s in SL/SN/LED systems must satisfy the following optimality conditions:*

$$SL: \frac{\tau_s}{1+\tau_s} \int_z \left\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z) \Big| z \right] \right\} dH_z(z) = \int_z \left\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s \Big| z \right] \right.$$
$$\left. - \mathbb{E}\left[\frac{T_z'(z) + s'_{inc}(s,z)\,\tau_s}{1-T_z'(z)} z\zeta^c_z(s,z)\, s'_{inc}(s,z) \Big| z \right] \right\} dH_z(z) \tag{27}$$

$$SN: \frac{T_s'(s^0)}{1+T_s'(s^0)} \int_z \left\{ s^0 \zeta^c_{s|z}(s^0,z) \right\} h(s^0,z)\, dz = \int_z \left\{ \mathbb{E}\left[1-\hat{g}(s,z) \Big| z, s \geq s^0 \right] \right\} dH_z(z)$$
$$- \int_z \left\{ \frac{T_z'(z) + s'_{inc}(s^0,z)\, T_s'(s^0)}{1-T_z'(z)} z\zeta^c_z(s^0,z) s'_{inc}(s^0,z) \right\} h(s^0,z)\, dz \tag{28}$$

$$LED: \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta^c_z(s,z) s \Big| z^0 \right] h_z(z^0) = \int_{z \geq z^0} \mathbb{E}\left[(1-\hat{g}(s,z))\, s \Big| z \right] dH_z(z)$$
$$- \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta^c_{s|z}(s,z) \Big| z \right] + \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z)\, s + s'_{inc}(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta^c_z(s,z) s'_{inc}(s,z) \Big| z \right] \right\} dH_z(z) \tag{29}$$

The main difference between multidimensional and unidimensional heterogeneity is that with unidimensional heterogeneity, the expectation operators $\mathbb{E}[\cdot|z]$ are not needed. In expression (27), $\hat{g}(s,z)$, $\zeta^c_z(s,z)$, and $s'_{inc}(s,z)$ are functions of z only with unidimensional heterogeneity, which implies that the operator $\mathbb{E}[\cdot|z]$ is not needed with unidimensional heterogeneity. In expression (28), the term $\mathbb{E}\left[1-\hat{g}(s,z)\Big|z, s \geq s^0\right]$ reduces to $\mathbb{E}\left[1-\hat{g}(z)\Big|z \geq z^0\right]$ in the unidimensional case, and the functions s'_{inc} and h can be written as functions of z only. Analogous simplifications apply to expression (29).

When all terms inside the expectation operators $\mathbb{E}[\cdot|z]$ in Proposition A5 are independent of each other, the expectation can be applied to each separate statistic separately, and thus the unidimensional formulas are similar to the multidimensional formulas provided that all statistics are interpreted as averages conditional on z. For example, the first term in the integral in expression (27) can be written as $\left(1-\overline{\hat{g}(z)}\right)\overline{s(z)}$, where the "bar" notation denotes income-conditional averages. The second term in the integral can be written as

$$\frac{T_z'(z)\,\overline{s'_{inc}(z)} + \tau_s \overline{s'_{inc}(s,z)^2}}{1-T_z'(z)} z\overline{\zeta^c_z(z)}$$

The main new effect is the square of s'_{inc} inside the integral. Because $\int (s'_{inc})^2 dH > \left(\int s'_{inc} dH\right)^2$ and because the square enters into the optimal tax expression negatively, this implies that ignoring multidimensional heterogeneity can lead to over-estimates of optimal marginal tax rates on s. The formulas in (28) and (29) also involve squares of s'_{inc}, and thus also imply that multidimensional heterogeneity can lower the optimal tax rate on s through $(s'_{inc})^2$. We quantify the importance of this insight in our empirical application in Section 6. More generally, positive covariances between pairs of statistics inside the expectation operator will tend to lower the optimal tax rate on s, while negative covariances will tend to increase it.

5.2 Multiple Goods

We now extend our analysis to a setting where agents consume $n+1$ goods, c and $\boldsymbol{s} = (s_1, s_2, ..., s_n)$. We consider a tax system $\mathcal{T}(\boldsymbol{s}, z) = \mathcal{T}(s_1, s_2, ..., s_n, z)$, where we retain the normalization that the numeraire c is untaxed. We normalize $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ to measure consumption in units of the numeraire. An individual of type θ then maximizes $U(c, \boldsymbol{s}, z; \theta)$ subject to the budget constraint $c + \sum_{i=1}^{n} s_i \leq z - \mathcal{T}(\boldsymbol{s}, z)$.

We denote own-price elasticities of goods by $\zeta^c_{s_i|z}(z)$, and we define cross-substitution elasticities by $\xi^c_{s_{j,i}|z}(z) := -\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}{s_j(z;\theta)} \frac{\partial s_j(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta),z)}\Big|_{\theta=\vartheta(z)}$, where $s_j(z;\theta)$ denotes type θ consumption of good j when earning labor income z. We denote causal income effects on good s_j by $s'_{j,inc}(z) := \frac{\partial s_j(z;\theta)}{\partial z}\Big|_{\theta=\vartheta(z)}$. We continue using $\hat{g}(z)$ to denote the social marginal welfare effect of increase a z-earner's consumption of c by one unit; the formula for $\hat{g}$ in this more general setting is in Appendix B.9.

For the result below, as well as for Propositions 5, 6, and the supplementary results in Appendices A.3-A.5, we assume that

Assumption 4. *The tax systems under consideration are such that at the optimum: (i) they are smooth (ii) agents' optima are unique their first-order and second-order conditions strictly hold, (iii) there is no bunching, (iv) c and s are smooth functions of z, and (v) when SN system is studied, s is strictly monotonic (increasing or decreasing) in z.*

Proposition 4. *With n taxed goods $s_1, ..., s_n$, for each good i and at each bundle $(c, \boldsymbol{s}, z)$ chosen by a type θ, an optimal smooth tax system satisfies*

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)}{1 + \mathcal{T}'_{s_i}(\boldsymbol{s}, z)} = s'_{i,pref}(z) \frac{1}{s_i \zeta^c_{s_i|z}(z)} \frac{1}{h_z(z)} \int_{x \geq z} \left[1 - \hat{g}(x)\right] dH_z(x) + \underbrace{\sum_{j \neq i} \frac{\mathcal{T}'_{s_j}(\boldsymbol{s}, z)}{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)} \frac{s_j \xi^c_{s_{j,i}|z}(z)}{s_i \zeta^c_{s_i|z}(z)}}_{\text{Tax diversion ratio}}. \tag{30}$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)}{1 + \mathcal{T}'_{s_i}(\boldsymbol{s}, z)} = s'_{i,pref}(z) \frac{z \zeta^c_z(z)}{s_i \zeta^c_{s_i|z}(z)} \frac{\mathcal{T}'_z(\boldsymbol{s}, z) + \sum_{j=1}^{n} s'_{j,inc}(z) \mathcal{T}'_{s_j}(\boldsymbol{s}, z)}{1 - \mathcal{T}'_z(\boldsymbol{s}, z)} + \underbrace{\sum_{j \neq i} \frac{\mathcal{T}'_{s_j}(\boldsymbol{s}, z)}{\mathcal{T}'_{s_i}(\boldsymbol{s}, z)} \frac{s_j \xi^c_{s_{j,i}|z}(z)}{s_i \zeta^c_{s_i|z}(z)}}_{\text{Tax diversion ratio}}. \tag{31}$$

Proposition 4 features all of the familiar terms of Theorem 2, and includes a novel term that captures the tax implications of substitution effects. Intuitively, substituting from s_i to higher-taxed goods generates positive fiscal externalities that motivate higher marginal tax rates on s_i, while substitution to lower-taxed goods generates negative fiscal externalities that motivate lower marginal tax rates on s_i. These effects are summarized by what we call the tax diversion ratio—the impact on taxes collected on goods $j \neq i$ relative to the impact on taxes collected on good i when the price of good i is increased. The higher is the diversion ratio, the more favorable are the fiscal externalities associated with substitution away from good i, and thus the higher is the optimal tax rate on good i.

5.3 Optimal Taxation when the Government Wants Agents to Save More

Our framework can be interpreted as a bequest model in which parents work and consume in the first period, and leave a bequest to their heirs in the second period. Under this interpretation, our baseline model makes the implicit assumption that the government's preferred level of bequests corresponds to parents' choices. Farhi and Werning (2010) consider a model where the weight that parents attach to the wellbeing of future generations may be too low from a normative perspective. This misalignment introduces a policy motive to encourage bequests, which we consider in this extension.

Following Farhi and Werning (2010), we assume additively separable preferences given by

$$U\big(c, s, z; \theta\big) = u\left(c; \theta\right) - k\left(z; \theta\right) + \beta v\left(s; \theta\right), \tag{32}$$

where $u(c;\theta)$ is the utility parents derive from consumption c, $k(z;\theta)$ is the disutility parents incur to obtain earnings z, $v(s;\theta)$ is the utility heirs derive from a bequest s, and β is the weight parents attach to the wellbeing of their heirs. As in Farhi and Werning (2010), the government maximizes

$$\int_\theta \big[U\left(c\left(\theta\right), s\left(\theta\right), z\left(\theta\right); \theta\right) + \nu v\left(s\left(\theta\right); \theta\right)\big] dF\left(\theta\right), \tag{33}$$

where ν parametrizes the degree of misalignment. Farhi and Werning (2010) microfound ν as the Lagrange multiplier associated with a constraint that the future generation attains a required level of well-being $\int_\theta \nu v\left(s\left(\theta\right); \theta\right) dF\left(\theta\right) \geq \underline{V}$.

The formal model above can be interpreted more generally beyond the bequest application, and can be used to analyze behavioral biases as a motivation for encouraging savings. In particular, suppose that $v(x;\theta) = \delta(\theta)u(x)$, where δ is the "exponential discount factor" and β is "present focus," as in Laibson (1997). If the government utilizes the "long-run criterion" for welfare, then the degree of misalignment is given by $\nu = (1-\beta)$.[18] More generally, β may be heterogeneous, so that misalignment is type-dependent and given by $\nu(\theta) = (1-\beta(\theta))$. For example, Lockwood (2020) summarizes evidence suggesting that individuals with higher earnings ability have lower degrees of present focus.

[18]See Bernheim and Taubinsky (2018) for a detailed discussion of such a criterion, as well as alternative normative approaches to studying the implications of present focus.

Below, we characterize optimal taxation with heterogeneous misalignment, where $\beta(z)$ and $\nu(z)$ denote the parameters corresponding to a z-earner. This generalizes the result in Farhi and Werning (2010) by (i) allowing heterogeneity in preferences for s and (ii) by allowing heterogeneity in the misalignment parameter ν.

Proposition 5. *At each bundle (c, s, z) chosen by a type θ, an optimal smooth tax system satisfies the following marginal tax rate condition*

$$\frac{\mathcal{T}_s'(s,z)}{1+\mathcal{T}_s'(s,z)} = s_{pref}'(z)\frac{1}{s\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x) - \frac{\nu(z)}{\beta(z)}g(z). \tag{34}$$

Any Pareto-efficient smooth tax system satisfies

$$\frac{\mathcal{T}_s'(s,z)}{1+\mathcal{T}_s'(s,z)} = s_{pref}'(z)\frac{z\zeta_z^c(z)}{s\zeta_{s|z}^c(z)}\left[\frac{\mathcal{T}_z'(s,z)+s_{inc}'(z)\mathcal{T}_s'(s,z)}{1-\mathcal{T}_z'(s,z)} + s_{inc}'(z)\frac{\nu(z)}{\beta(z)}g(z)\right] - \frac{\nu(z)}{\beta(z)}g(z) \tag{35}$$

The main result is an intuitive generalization of Proposition 2, where the key new term is a form of Pigovian correction, given by $\frac{\nu(z)}{\beta(z)}g(z)$. As equation (34) shows, the presence of misalignment motivates the government to lower the tax rate on s. The degree by which the government lowers the tax rate depends on the relative degree of misalignment (relative to the discount factor β), and on the social marginal welfare weight. Because social marginal welfare weights decline with z, equation (34) gives the "progressive estate taxation" result of Farhi and Werning (2010)—i.e., savings subsidies that decline with income—under the special assumptions that (i) $s_{pref}' \equiv 0$ and (ii) $\beta(z) \equiv \beta \in \mathbb{R}$, $\nu(z) \equiv \nu \in \mathbb{R}$. This core result of Farhi and Werning (2010) extends the standard Pigovian taxation logic to optimal screening of distortions with a nonlinear tax.

More generally, Proposition 5 provides a simple formula for balancing the "corrective motives" studied by Farhi and Werning (2010) with the additional motives to tax s in the presence of correlated preference heterogeneity studied in this paper. This extends the Allcott et al. (2019) results for linear commodity taxes with biased consumers to study optimal screening of biases with a nonlinear tax. If $s_{pref}'(z) > 0$ and $\nu(z)/\beta(z)$ and $g(z)$ are decreasing with z, Proposition 5 suggests a progressive tax on s that can feature subsidies at low incomes and taxes at high incomes.

5.4 Multi-Dimensional Tax Range with Heterogeneous Rates of Return

Thus far we have considered tax functions where the tax is always paid in units of the numeraire commodity c. In some applications it is also natural to consider tax systems with *multidimensional range*, which include taxes paid in units of c and also taxes paid in units of s. This is natural, for example, if c and s correspond to period 1 and period 2 consumption, respectively, and taxes must be paid in both periods. The additional richness in the range does not alter the optimal tax implications when the rates of transformation p are homogeneous. However, we shall show that there are new economic effects with heterogeneity in p.

Formally, suppose that the government uses a two-part tax structure, where a person must pay a tax $T_1(z)$ in units of c and a tax $T_2(s,z)$ in units of s. For instance, in a two-period model where s is savings, T_1 would be the earnings tax levied in period 1 and paid in period 1 dollars, T_2 would be the savings tax levied in period 2 and paid in period 2 dollars, and $p = 1/(1+r)$ is a function of the rate of return r. For concreteness, we refer to T_1 as period-1 taxes and to T_2 as period-2 taxes, but we stress that the nature of the extension is not about dynamics per se, but rather that T_2 is collected in units of s.

Following Gahvari and Micheletto (2016), we consider heterogeneous $p(z,\theta)$ which are a function of gross earnings and type. For example, wealthier individuals have access to better rates of return on savings or prices of commodities. Alternatively, higher earnings ability may be associated with a better ability to obtain high rates of return or to find better prices. Denoting by $\vartheta(z)$ the type θ of individuals who choose earnings z, we slightly abuse notation to define $p(z) := p(z,\vartheta(z))$

Individuals maximize $U(c,s,z;\theta)$ subject to the budget constraint $c + p(z,\theta)s \leq z - T_1(z) - p(z,\theta)T_2(s,z)$. The government, as before, maximizes a weighted average of utilities,

$$\int_z \left\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z),z)\big), s(z), z; \vartheta(z)\Big) \right\} dH_z(z),$$

subject to the constraints

$$\int_z T_1(z) dH_z(z) \geq E_1$$
$$\int_z T_2(s(z),z) dH_z(z) \geq E_2,$$

which generate marginal values of public funds λ_1 and λ_2 . In Appendix B.11.3 we define $\hat{g}(z)$, analogously to before, as the impact of a unit of after-tax income on z-earners, normalized by the marginal value of public funds λ_1.

Heterogeneity in p generates two novel considerations in this model. First, for individuals with relatively low $p(z)$, it is efficient for the government to decrease T_1 and increase T_2. This efficiency effect is present irrespective of the mechanism for the cross-sectional variation of p with z, and leads to a novel deviation from the Atkinson-Stiglitz theorem explored by Gerritsen et al. (2020) and generalized in our results below.

Second, lump-sum changes in T_2 trigger novel substitution effects. This is because a lump-sum increase dT in T_2 has the same effect on an agent's utility as a $p(z)dT$ increase in T_1, and thus changes behavior as much as a marginal tax rate change of $\frac{\partial p}{\partial z} dT$ in T_1 (see Appendix B.11.2). We denote by $\varphi(z) := -\left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(z)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{z\zeta_z^c(z)}{1-T_1'(z)}\frac{\partial p}{\partial z}$ the fiscal impacts of this substitution effect at earnings z. The impact of a uniform lump sum decrease in T_2 is $\overline{\hat{g}p - \varphi}$, where the "bar" notation is used to denote a population average across all earnings levels. Thus, $\lambda_2/\lambda_1 = \overline{\hat{g}p - \varphi}$, as we formally show in Appendix B.11.3.

Taking these new considerations into account, we characterize optimal taxes on s for SN tax systems where $T_2(s,z) = T_2(z)$, and for LED tax systems where $T_2(s,z) = \tau_s(z)\, s$. We supplement the results in Proposition 6 with derivations of the earnings tax schedule $T_1(z)$ at the optimum

(Appendix B.11.4).

Proposition 6. *With heterogeneous prices, at each bundle (c, s, z) chosen by a type θ, an optimal SN two-part tax system $\{T_1(z), T_2(s)\}$ satisfies*

$$\frac{\lambda_2/\lambda_1 T_2'(s)}{1+p(z)T_2'(s)} = \frac{1}{s\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\left\{s'_{pref}(z)\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x) + \frac{s'(z)}{p(z)}\left(\Psi(z)+\int_{x\geq z}\left[\varphi(x)-\overline{\varphi}\right]dH_z(x)\right)\right\} \tag{36}$$

where

$$\Psi(z) := \Big(1-H_z(z)\Big)\int_{x\leq z}\hat{g}(x)\Big(p(x)-p(z)\Big)dH_z(x) + H_z(z)\int_{x\geq z}\hat{g}(x)\Big(p(z)-p(x)\Big)dH_z(x),$$

An optimal LED two-part tax system $\{T_1(z), \tau_s(z)s\}$ satisfies

$$\frac{\lambda_2/\lambda_1 \tau_s(z)}{1+p(z)\tau_s(z)} = \frac{1}{s(z)\zeta_{s|z}^c(z)}\frac{1}{h_z(z)}\left\{s'_{pref}(z)\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x) + \frac{p'(z)}{p(z)}s(z)\int_{x\geq z}\left[1-\hat{g}(x)\right]dH_z(x)\right\}$$
$$+\frac{1}{\zeta_{s|z}^c(z)}\frac{1}{p(z)}\left\{\overline{\hat{g}p}-\overline{\hat{g}}p(z)+\varphi(z)-\overline{\varphi}\right\} \tag{37}$$

Proposition 6 shows that the sufficient statistic $s'_{pref}(z)$ remains critical for optimal marginal tax rates on s, although it now additionally captures correlations between p and earnings ability. On the left-hand side of (36) and (37), the presence of $p(z)$ in the denominator is because an agent's marginal tax rate on s, translated to units of c, is $p(z)\frac{\partial T_2}{\partial s}$. The presence of λ_2/λ_1 in the numerator of the left-hand side is because fiscal externalities generated by substitution away from s must be weighted by the "period 2" marginal value of public funds.

Proposition 6 also introduces novel efficiency terms that to lead to taxes on s even when $s'_{pref} \equiv 0$. In the SN formula, there are two additional efficiency effects. These terms are both positive and thus push toward taxing s when higher earners (i) face lower prices p (e.g., higher rates of returns on savings) and choose higher levels s and (ii) exhibit larger substitution effects φ. The first term, proportional to $\Psi(z)$, captures the efficiency effects of increasing period-2 taxes. This term is unambiguously positive when p decreases cross-sectionally with z, and captures the intuition that with a SN system, increasing marginal tax rates on s at any point $z > z_{\min}$ increases period-2 taxes on T_2 on individuals with below-average p. The second term, proportional to $\int_{x\geq z}\left[\varphi(x)-\overline{\varphi}\right]dH_z(x)$, captures the fact that increasing marginal tax rates on s motivates agents to increase labor supply when $\frac{\partial p}{\partial z} < 0$. The SN formula generalizes the result in Gerritsen et al. (2020) to incorporate other forms of correlated heterogeneity, and extends their somewhat-difficult-to-sign conditions to a formula employing measurable sufficient statistics with clear economic implications.[19]

The implications are different for LED tax systems. Assume again that p declines cross-sectionally with z, meaning that $p'(z) < 0$. The first novel term, proportional to $p'(z)/p(z)$, captures the fact that $s'_{pref}(z)$ is no longer a sufficient statistic for how marginal tax rate changes

[19] Gerritsen et al. (2020) sign the implications for tax rates in the special cases where returns are either purely scale-dependent or purely type-dependent, but not in the general case that involves both possibilities.

in T_2 versus T_1 affect behavior. As shown in Appendix Lemma B4, higher earners are less responsive to marginal changes in T_2 when $p(z)$ declines with income, since period-2 consumption is "cheaper" for them than period-1 consumption. The second term, proportional to $\overline{\hat{g}p} - \bar{\hat{g}}p(z) + \varphi(z) - \overline{\varphi}$, is also negative for sufficiently low values of z, as in this case both $\overline{\hat{g}p} - \bar{\hat{g}}p(z)$ and $\varphi(z) - \overline{\varphi}$ are negative. However, this term is positive for sufficiently high values of z. Thus, when $s'_{pref}(z) \equiv 0$, the optimal LED system features subsides on s for lower-income individuals and taxes on s for higher-income individuals.

The contrast in implications for SN versus LED tax systems—everywhere-positive tax rates in the former, subsidies followed by taxes in the latter—highlights that the new efficiency considerations from heterogeneous rates of return depend on the types of restrictions imposed on the tax system. The reason for this dependence is because positive tax rates on s are a consequence of a missing instrument problem. In a fully flexible tax system, the efficiency gains of taxing a person in period 2 instead of period 1 could be obtained without increasing marginal tax rates on s, but less-flexible tax systems can only generate this shifting of the tax burden by altering marginal tax rates on s, and the optimal means of doing this depend on the particulars of the tax system.

6 Empirical Application

We now apply our formulas to the question of savings taxes in the United States. We first calibrate the relevant sufficient statistics from micro data and empirical studies, devoting particular attention to the calibration of the sufficient statistic $s'_{pref}(z)$. We then use the Pareto efficiency conditions derived in Proposition 2 to compute the SL, SN and LED savings tax schedules that would be consistent with the status quo income tax schedule. This allows us to study the welfare-improving reforms that could be made to the existing tax system, taking as given the distributional preferences already embedded in the existing income tax. As is typical for calculations based on sufficient statistics formulas, these results are approximations, as they do not account for changes in the underlying distributions and sufficient statistics that might arise if the savings tax were reformed. These results suggest that preference heterogeneity leads to a (mostly) positive and progressive schedule of savings tax rates, which range from approximately 0% at the bottom of the income distribution up to between 15% and 20% at the top in our baseline calibration.

Calibration We calibrate a model of the U.S. economy that can be interpreted through the lens of the joint savings and income tax function $\mathcal{T}(s, z)$, and the three tax systems described in Table 1. We calibrate a two-period model economy with a fine grid of incomes, where the first period corresponds to work-life and the second to retirement. We assume these periods are separated by 20 years, and we assume a constant (and, in our baseline, homogeneous) annual rate of return of 3.8% before taxes, drawing from Fagereng et al. (2020).We calibrate a version of the economy with unidimensional heterogeneity (i.e., a single level of savings at each income) and a version with multidimensional heterogeneity, reporting results for each below.

Note that because our model builds on standard models of commodity taxation, it implicitly assumes that z and $\mathcal{T}(s, z)$ are measured in the same units as consumption, which in a dynamic setting corresponds to "period-1" dollars. In practice, savings taxes are typically imposed after returns, and thus measured in "period-2" dollars. We accordingly translate all tax rates into units of period-2 dollars when reporting results. Appendix C.1 discusses details of our calibration and this conversion.

To calibrate the earnings and savings distributions—and thus the across-income savings profile $s(z)$—we use the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65). Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the nth percentile is constructed by averaging earnings at the nth percentile across those aged 20 to 44 and those aged 45 to 64. Our measure of s is the average value of net personal wealth, *hweal*, projected forward to age 65 based on the value observed at each income percentile in the 45-64 age bucket.[20] We normalize both labor earnings and retirement savings by the number of years worked. See Appendix C.1 for further details.

Figure 1 plots our estimate of gross (i.e., after returns and before tax) savings per year worked, across the income distribution. This does not include Social Security, which we model as lump-sum forced savings that are received during retirement. The figure shows that savings upon retirement are approximately zero at low incomes, but increase substantially with income. We convert this to net-of-tax savings using a calibration of savings tax rates across the income distribution in the U.S., derived by computing the weighted average of savings tax rates across incomes using the asset composition reported in Bricker et al. (2019); see Appendix C.3 for details. The convex shape of the savings profile in Figure A2b, which persists after accounting for taxes, indicates that the cross-sectional slope $s'(z)$ rises with income, as shown by the solid blue line in Figure 2.

To calibrate the causal income effect on savings, $s'_{inc}(z)$, and thus our measure of local preference heterogeneity $s'_{pref}(z) = s'(z) - s'_{inc}(z)$, we draw from two sources. The first is Fagereng et al. (2019), who estimate the marginal propensity to consume (MPC) out of windfall income across the earnings distribution using information on lottery prizes linked with administrative data in Norway. Lottery consumption is widespread in Norway—over 70% of adults from all income groups participated in 2012—and administrative records of asset and wealth holdings allow for direct measures of savings and consumption responses to lottery winnings. They find that individuals' consumption peaks during the winning year and gradually reverts to their previous value afterwards. Over a 5-year horizon, they estimate that winners consume close to 90% of the prize (see their Figure 2, aggregate consumption response) which translates into an MPC of 0.9, and a marginal propensity to save of

[20] This measure of wealth includes housing assets, business assets, and financial assets, net of liabilities, as well as pension and life insurance assets. It does not include Social Security. We add an adjustment for Social Security, which is treated as forced savings, in our income and savings distributions. To account for taxes on these savings, we compute an average total tax rate on savings at each income level, by weighting asset-specific tax rates (e.g., for capital gains on financial and business assets) by the share of such assets held at each point in the income distribution, as reported in Bricker et al. (2019). See Appendix C.3 for details.

0.1. They do not find significant heterogeneity across incomes in this MPC. We convert this MPC into a response of net retirement savings to changes in pre-tax labor income using our calibrated schedules of income and savings tax rates as described in Appendix C.1.2.

The second is a new probability-based survey representing the U.S. adult population, conducted on the AmeriSpeak panel in the spring of 2021. In the survey we asked respondents to report what portion of a hypothetical raise they would save. The relative advantages of this survey is that it is about the U.S. population and that it asks directly about pre-tax income. Appendix C.1.2 summarizes the results from this survey, which suggest a short-run MPC close to that reported in Fagereng et al. (2019), with little variation across incomes. We translate this into a long-run MPC using the response profile of Fagereng et al. (2019). Figure 2 reports these two schedules of $s'_{inc}(z)$. There is a substantial difference between $s'(z)$ and s'_{inc}, which is positive across most of the income distribution and rises with income.[21] This is the key force that pushes toward a progressive and mostly positive schedule of optimal savings taxes.

We assume a constant compensated earnings elasticity of $\bar{\zeta}_z^c = 0.33$, drawn from the meta-analysis of Chetty (2012). The value of the savings elasticity $\zeta_{s|z}^c$ is harder to obtain, in part because different savings vehicles may trigger different responses to taxes and because different time horizons can lead to different results. We assume it is constant, and we report results for a broad range of values spanning $\zeta_{s|z}^c = 0.7$ to $\zeta_{s|z}^c = 3$, with a baseline of $\zeta_{s|z}^c = 1$, which approximately aligns with the baseline calibration considered in Golosov et al. (2013), in which the intertemporal elasticity of substitution is set to one. Appendix C.1.3 discusses this conversion, as well as other potential sources of evidence on the size of the savings elasticity, including recent work by Jakobsen et al. (2020) and by Agersnap and Zidar (2020)

By way of comparison, Golosov et al. (2013) estimate preference heterogeneity by estimating differences in discount factors across ability levels. They infer discount factors from a simple optimization model applied to survey data on individuals' household income paths and net worth, while they use survey respondents' results to the Armed Forces Qualification Test (AFQT) to as a proxy for ability. In contrast to our findings, their estimation strategy finds very little measured preference heterogeneity, amounting to less than 1% of the cross sectional variation in savings (see Appendix C.2 for a discussion). This discrepancy could be driven by attenuation bias due to measurement error in their proxy for ability—an issue we avoid by computing preference heterogeneity directly as a difference of two statistics rather than from regression analysis. It could also be driven by their use of a narrower measure of across-income heterogeneity based only on time preferences, as opposed to all of the possible forms of across-income heterogeneity that our statistic accounts for.

[21] Our measure of $s'_{pref}(z)$ appears to be slightly negative at low incomes, which in our simulations gives rise to slight savings subsidies at low incomes. However we note that this could be driven by limitations in our ability to precisely measure $s'_{inc}(z)$ specifically at low incomes. This emphasizes the value of additional empirical research on this statistic.

Results Figure 3 reports the schedule of marginal tax rates for SL, SN and LED tax systems that satisfy the Pareto efficiency formulas in Proposition 2, taking the existing U.S. income tax schedule and income distribution as given. In each case, we translate the tax into a marginal tax rate on gross savings at retirement, measured in period 2 dollars.[22] Each panel reports results for a different value of the savings elasticity. For SL tax systems, the linear savings tax rate τ_s is by definition constant across earnings levels. For LED tax systems, the linear savings tax rate $\tau_s(z)$ is earnings dependent and we thus report the linear savings tax rate at each earnings level. For SN tax systems, the nonlinear savings tax schedule $T_s(s)$ depends on the value of savings s, and not on earnings z. But to make the SN system visually comparable to the other systems, we plot the marginal savings tax rate faced at the margin by each earner, accounting for their level of saving.

In each panel, marginal savings tax rates are positive, and the nonlinear tax schedules are progressive, with marginal rates increasing with income. The magnitudes depend on the value of the savings elasticity parameter. In the baseline case of $\zeta^c_{s|z} = 1$, savings tax rates in SN and LED tax systems average approximately 0% for annual incomes below $50,000, then steadily increase up to nearly 20% for annual incomes around $200,000, remaining stable thereafter. The savings tax rate in a SL tax system is approximately 6%. Changing the savings elasticity parameter scales the efficient savings tax rates without affecting the overall pattern: preference heterogeneity calls for positive and progressive savings tax rates. At the lower elasticity values, our estimates of optimal tax rates are substantially higher than the prevailing savings tax rates in the U.S., which are also shown in Figure 1. We draw on Bricker et al. (2019) to compute the current U.S. savings tax rates, and the details of our methodology are in Appendix C.3.

Figure 4 considers two key extensions considered from Section 5. The first is multidimensional heterogeneity. The second is heterogeneous rates of return when there is additional efficiency rationale to tax high-return individuals because taxes on s are collected in period 2. To maintain comparability with our baseline results, all other parameters, including elasticity parameters and welfare weights, are held fixed at the values from our baseline calibrations, in order to isolate the effect of these extensions.

In the case of multidimensional heterogeneity, we use the same measure of gross savings described in Appendix C.1, but instead allow for four different level of savings at each level of income, each representing a quartile of the income-conditional savings distribution. As in Saez (2002), we assume that social marginal welfare weights $g(z)$ depend only on earnings z, and we calculate the implied weights under unidimensional heterogeneity using the inverse optimum approach summarized in Appendix C.4. Further details of this calibration are described in Appendix C.5.

In the case of heterogeneous rates of return, we follow Gerritsen et al. (2020) in assuming that that rates of return rise by 1.4% from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return. Further details are in Appendix C.6.

[22] Appendix A.6 presents a proof that the simplicity structure of a SL, SN, or LED tax system is preserved under such a translation.

Consistent with the intuition described in Section 5.1, the top two panels of Figure 4 show that incorporating multidimensional heterogeneity reduces the magnitude of optimal tax ratesin SL and LED systems (top left panel) and in SN systems (top right panel). The effect is particularly pronounced for SN systems. At the same time, marginal savings tax rates are still progressive, and are above status quo savings tax rates across high incomes in our baseline specification.

The bottom two panels show that the presence of heterogeneous rates of returns tends to raise optimal savings tax rates, reflecting the efficiency effects highlighted in Proposition 6. The bottom right panel shows that tax rates in the SN system are higher at all levels of income, consistent with our discussion of the formula for SN systems in Proposition 6. On the other hand, recall that the formula for LED systems implied lower savings tax rates at low incomes and higher tax rates higher incomes. Consistent with this, the bottom left panel shows that relative to the baseline, the optimal savings tax rates with heterogeneous rates of return are even more progressive. For example, substantial savings subsidies are optimal for incomes below about \$40,000, whereas savings taxes are substantially higher higher incomes.

7 Conclusion

This paper characterizes optimal smooth tax systems on earnings and savings (or other dimensions of consumption) in the presence of across-income heterogeneity in preferences. We first prove that with unidimensional heterogeneity, the optimal allocation can be implemented by a smooth tax on earnings and savings. We then derive formulas which characterize the optimal smooth tax system, expressed using familiar empirical statistics, as well as a key sufficient statistic for preference heterogeneity, $s'_{pref}(z)$. This statistic can be estimated from empirical data, and can also accommodate other dimensions of heterogeneity such as rates of return. We then consider a set of "simple" separable tax systems that are widely used in practice. We derive intuitive sufficient statistics formulas for these separable tax systems, which we generalize to several extensions including multidimensional heterogeneity and multiple goods. Finally, we apply our theoretical formulas to the U.S. savings tax, where results suggest the savings tax rates that would be consistent with existing income tax rates is positive and progressive. Together, these results provide a practical and general method for quantifying optimal tax systems for saving, inheritances, and other commodities in the presence of correlated preference heterogeneity.

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Tables & Figures

Table 1: Types of simple tax systems

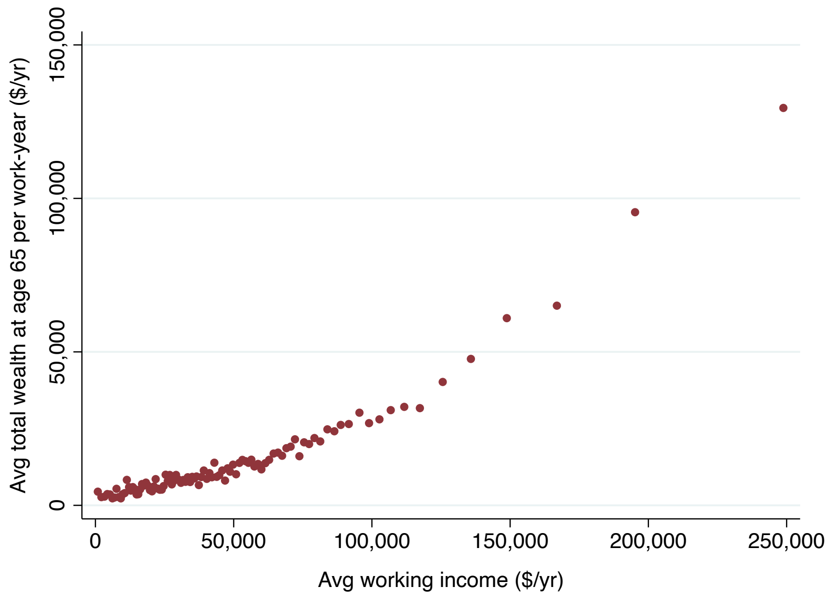
Type of simple tax system	$\mathcal{T}(s,z)$	$\mathcal{T}'_s(s,z)$	$\mathcal{T}'_z(s,z)$
SL: separable linear	$\tau_s\, s + T_z(z)$	τ_s	$T'_z(z)$
SN: separable nonlinear	$T_s(s) + T_z(z)$	$T'_s(s)$	$T'_z(z)$
LED: linear earnings-dependent	$\tau_s(z)\, s + T_z(z)$	$\tau_s(z)$	$T'_z(z) + \tau'_s(z)\, s$

Table 2: Tax systems applied to different savings vehicles, by country.

Country	Wealth	Capital Gains	Property	Pensions	Inheritance
Australia	–	Other	SL, SN	SL	–
Austria	–	Other	SL, SN	SN	–
Canada	–	Other	SL	SN	–
Denmark	–	SN	SL, SN	SL, SN	SN
France	–	Other	Other	SL, SN	SN
Germany	–	Other	SL	SN	SN
Ireland	–	SN	SL, SN	SN	SN
Israel	–	Other	Other	SN	–
Italy	SL, SN	SL	SL	SL	SL, SN
Japan	–	SL, SN	SN	SN	SN
Netherlands	SN	SL	SL, SN	SN	SN
New Zealand	–	Other	SN	SL, LED	–
Norway	SN	SL	SL	SN	–
Portugal	–	SL	Other	SN	SL
Singapore	–	Other	SN	SN	–
South Korea	–	SN	SN	SN	SN
Spain	SN	SN	SL, SN	SN	SN
Switzerland	SN	SN	SL, SN	SN	SN
Taiwan	–	SL, SN	SL, SN	SN	SN
United Kingdom	–	Other	SN	SN	SN
United States	–	LED	SL	SN	SN

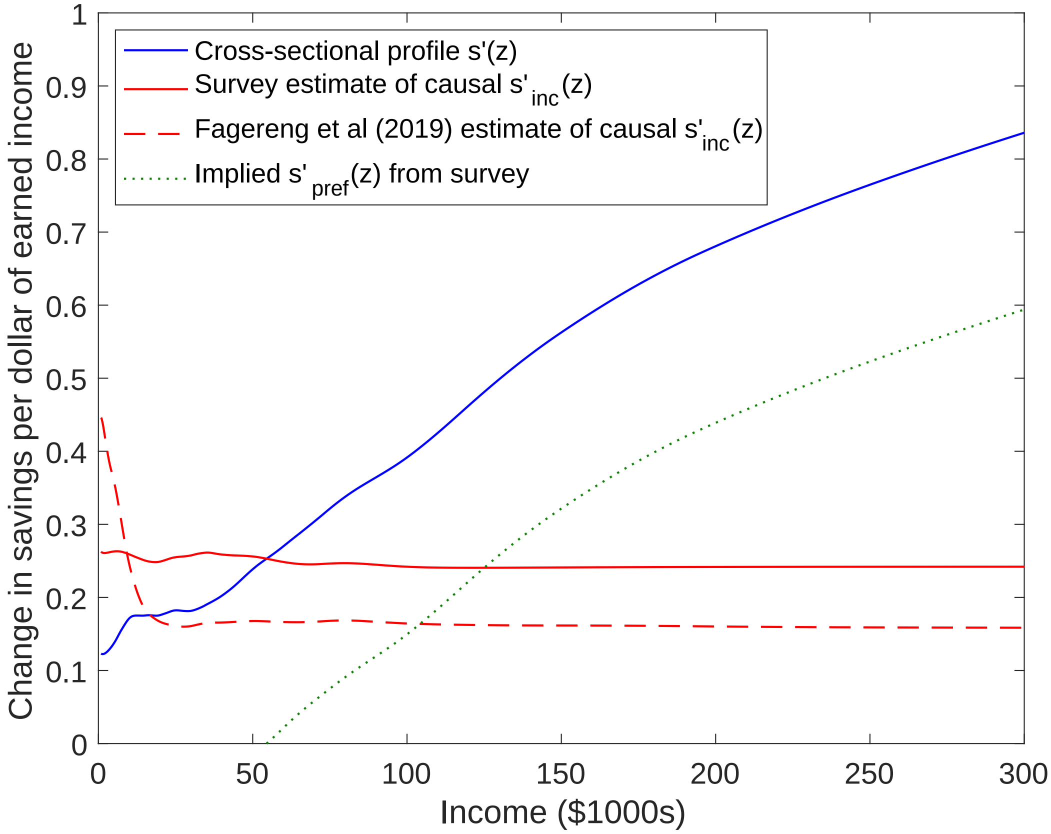
Notes: This table classifies tax systems applied to different savings vehicles across countries in 2020 according to the types in Table 1.

Figure 1: Savings Rates by Income in the United States



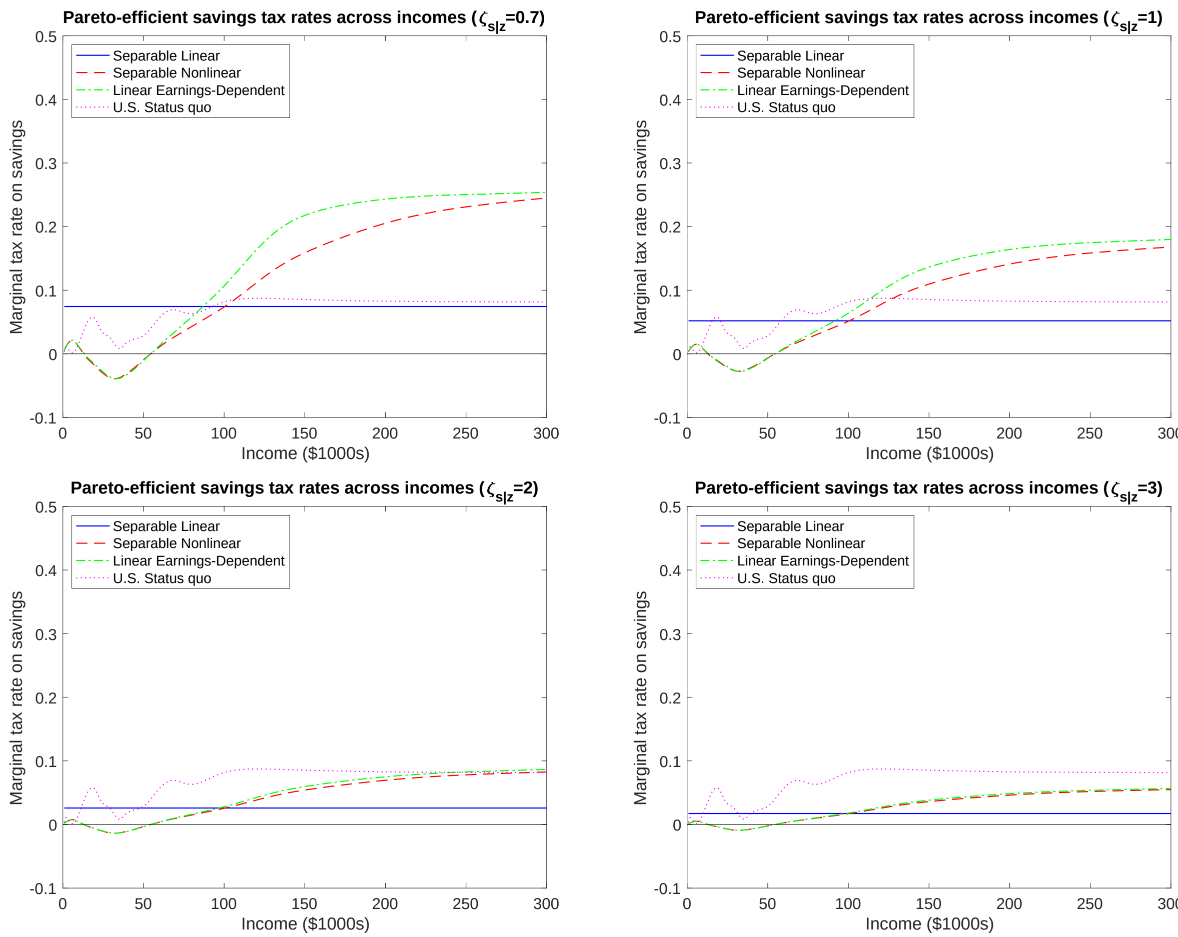
Notes: The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (*plinc*) and net personal wealth (*hweal*) at the individual level, as well as the age category (20 to 44 years old, 45 to 64, and above 65) to impute gross savings at the time of retirement, which we normalize by the number of work years. See Appendix C.1 for further details.

Figure 2: Savings Rates by Income in the United States



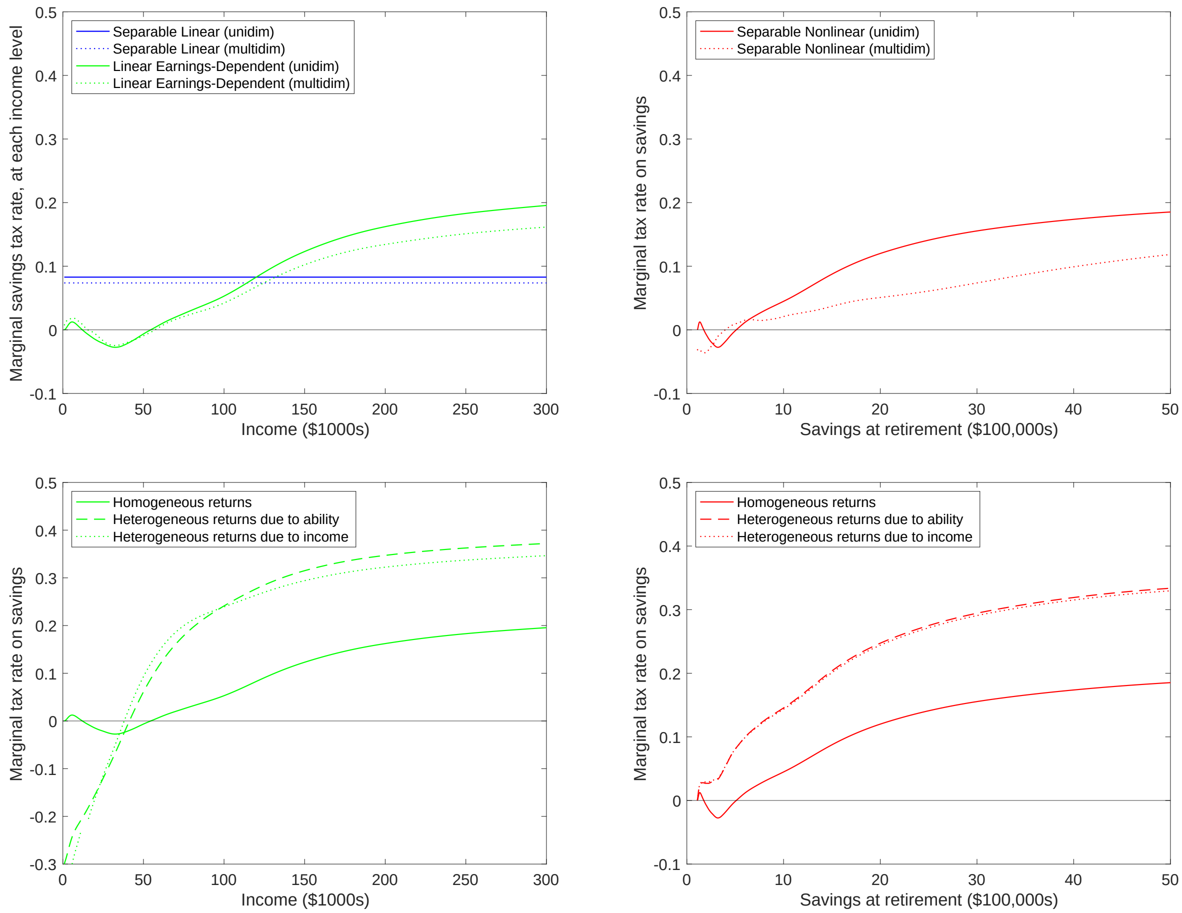
Notes: This figure reports the slope of the cross-sectional profile of savings $s'(z)$ (blue), as well as our calibrations of $s'_{inc}(z)$ based on causal income effects, derived from Fagereng et al. (2019) and from a new nationally representative survey. See Section 6 and Appendix C.1 for details.

Figure 3: Savings Tax Rates Implied by Pareto Efficiency Formulas



Notes: This figure presents the marginal savings tax rates values that satisfy the Pareto efficiency formulas in Proposition 2, plotted against the earnings level to which they apply.

Figure 4: Effects of Multidimensional Heterogeneity and Heterogeneous Returns



Notes: This figure plots the marginal savings tax rate schedules which are optimal, according to the first-order condition formulas presented in the text, for two extensions discussed in Section 5: multidimensional heterogeneity, and heterogeneous returns. All plots use the same set of social welfare weights, calibrated to rationalize the status quo income tax in the unidimensional model. See Section 6 and Appendices C.5 and C.6 for details.

Online Appendix

Sufficient Statistics for Nonlinear Tax Systems With Preference Heterogeneity

Antoine Ferey, Benjamin B. Lockwood, and Dmitry Taubinsky

Table of Contents

A Supplementary Theoretical Results

A.1 Monotonicity with Preference Heterogeneity

Lemma A1. *Under Assumption 1, 2, and 3, earnings z are strictly increasing with type θ in the optimal incentive-compatible allocation $\mathcal{A}$.*

A.2 Implementation Results for Simple Tax Systems

We proceed in three steps to provide sufficient conditions under which some SN and LED tax systems decentralize the optimal incentive-compatible allocation.

First, we define candidate SN and LED tax systems that satisfy type-specific feasibility and agents' first-order conditions at the optimal allocation. Second, in Proposition A1 we present sufficient conditions under which these SN and LED tax systems also satisfy agents' second-order conditions at the optimal allocation, implying local optimality. Third, in Proposition A2, we present sufficient conditions under which local optima are ensured to be global optima, implying that the candidate SN and LED systems are indeed implementing the optimal allocation.

Step 1: Defining candidate tax systems. We first define a candidate SN tax system $\mathcal{T}(s,z) = T_s(s)+T_z(z)$, with the nonlinear functions T_s and T_z defined across all savings and earnings bundles of the optimal allocation $\mathcal{A} = \left(c^*(\theta), s^*(\theta), z^*(\theta)\right)_\theta$ as follows:

$$T_s(s^*(\theta)) := \int_{\vartheta=\theta_{min}}^{\theta} \left(U_s'(\vartheta)/U_c'(\vartheta) - 1\right) s^{*\prime}(\vartheta) d\vartheta, \tag{38}$$

$$T_z(z^*(\theta)) := z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) + \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta \tag{39}$$

where θ_{min} denotes the lowest earning type of the compact type space Θ, and the derivatives are evaluated at the bundle assigned in the optimal allocation (e.g., $U_s'(\vartheta) = U_s'(c^*(\vartheta), s^*(\vartheta), z^*(\vartheta); \vartheta)$). Under this tax system, the optimal allocation satisfies by definition each type's first-order conditions for individual optimization given in Equations (9) and (10). By Lemma B1, this tax system thus satisfies type-specific feasibility.

We similarly define a candidate LED tax system $\mathcal{T}(s,z) = \tau_s(z) \cdot s + T_z(z)$ as follows:

$$\tau_s(z^*(\theta)) := U_s'(\theta)/U_c'(\theta) - 1, \tag{40}$$

$$\begin{aligned} T_z(z^*(\theta)) :=& z^*(\theta_{min}) - s^*(\theta_{min}) - c^*(\theta_{min}) \\ &+ \int_{\vartheta=\theta_{min}}^{\theta} \left(U_z'(\vartheta)/U_c'(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta - s^*(z) \cdot \left(\tau_s(z) - \tau_s(z^*(\theta_{min}))\right). \end{aligned} \tag{41}$$

This tax system also satisfies local first-order conditions for individual optimization and type-specific feasibility.

Step 2: Local maxima. We can now derive sufficient conditions under which the above candidate SN and LED tax systems satisfy the second-order conditions for individual optimization, implying that under these conditions assigned bundles are local optima . These conditions can be simply stated in terms of the marginal rates of substitution between consumption and, respectively, savings $\mathcal{S}(c, s, z; \theta)$ and earnings $\mathcal{Z}(c, s, z; \theta)$. These marginal rates of substitutions are smooth functions of c, s, z, and θ by the smoothness of the allocation and the utility function, and sufficient conditions for local second-order conditions are given by the following proposition.

Proposition A1. *Suppose that an allocation satisfies the conditions in Theorem 1. Under the SN tax system defined by Equations* (38) *and* (39)*, each agent's assigned choice of savings and earnings is a local optimum if the following conditions hold at each point in the allocation:*

$$\mathcal{S}_c' \geq 0,\ \mathcal{S}_z' \geq 0,\ \mathcal{S}_\theta' \geq 0 \tag{42}$$

and

$$\mathcal{Z}_c' \leq 0,\ \mathcal{Z}_s' \geq 0,\ \mathcal{Z}_\theta' \geq 0. \tag{43}$$

Under the LED tax system defined by Equations (40) *and* (41)*, each agent's assigned choice of savings and earnings is a local optimum if the utility function is additively separable in consumption, savings, and earnings (*$U_{cs}'' = 0$*,* $U_{cz}'' = 0$*, and* $U_{sz}'' = 0$*), and additionally the following conditions hold at each point in the allocation:*

$$\mathcal{S}_\theta' \geq 0,\ \mathcal{S}_\theta' \leq \frac{z^{*\prime}(\theta)}{s^{*\prime}(\theta)} \mathcal{Z}_\theta',\ \mathcal{S}_\theta' \leq s^{*\prime}(\theta) \left(\mathcal{S} \cdot \mathcal{S}_c' - \mathcal{S}_s'\right). \tag{44}$$

The sufficiency conditions (42) and (43) are quite weak; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in Equation (1). Conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$ are single crossing conditions for savings and earnings, while other conditions intuitively relate to the concavity of preferences.

The sufficiency conditions for LED systems are more restrictive. Beyond the single crossing conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{Z}_\theta' \geq 0$, they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. In words, the preference for savings must not increase "too quickly" across types, or else the second-order condition for earnings may be violated. The intuition for this result can be seen from the definition of the potentially optimal LED system. If the marginal rate of substitution for saving, $\mathcal{S}$, increases very quickly with income at some point in the allocation, then the savings tax rate $\tau_s(z)$ must rise very quickly with z at that point, by Equation (40). Since the savings tax rate $\tau_s(z)$ applies to total savings (including inframarginal savings), this increase in $\tau_s(z)$ must be offset by a sharp decrease in $T_z(z)$ at the same point in the distribution, by Equation (41). Yet a sufficiently steep decrease in $T_z(z)$ will cause the second-order condition for earnings choice—holding fixed savings choice—to be violated.

Step 3: Global maxima. Having presented conditions under which the bundle $\left(c^*(\theta), s^*(\theta), z^*(\theta)\right)$ assigned to type θ is a local optimum under the candidate SN and LED tax systems, we now present a set of regularity conditions ensuring that these local optima are also global optima.

Proposition A2. *Assume single crossing conditions for earnings and savings ($\mathcal{Z}'_\theta \geq 0$ and $\mathcal{S}'_\theta \geq 0$), that preferences are weakly separable ($U''_{cz} = 0$ and $U''_{sz} = 0$), and that commodities c and s are weak complements ($U''_{cs} \geq 0$). If $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, local optima correspond to global optima when*

1. *$\mathcal{T}$ is a SN system, and we have that for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{-T''_{ss}(s)}{1+T'_s(s)}$.*

2. *$\mathcal{T}$ is a LED system, and we have that*

   (a) for all $s < s^*(\theta)$ and θ, $\frac{-U''_{cc}(c(s,\theta),s,z^*(\theta);\theta)}{U'_c(c(s,\theta),s,z^*(\theta);\theta)} > \frac{1}{1+\tau_s(z^*(\theta))} \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,

   (b) for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$.

where $c(s,\theta) := z^*(\theta) - s - \mathcal{T}(s, z^*(\theta))$

In essence, global optimality is ensured under the following assumptions. First, higher types θ derive higher gains from working and allocating those gains to consumption or savings — generalized single-crossing conditions. Second, additive separability of consumption and savings from labor. Third, the utility function U is sufficiently concave in consumption and savings.

For the case of *SN* tax systems, condition 1 imposes a particular concavity requirement with respect to savings. For the case of LED tax systems, condition 2 imposes particular concavity requirements with respect to both consumption and savings. Notably, these concavity conditions need only be checked when earnings are fixed at each type's assigned earnings level $z^*(\theta)$.

We can naturally apply this result to the candidate SN tax system defined in Equations (38) and (39), and to the candidate LED tax system defined in Equations (40) and (41). Because these candidate tax systems are defined in terms of agents' preferences and optimal allocations, we can then express conditions 1 and 2 fully in terms of agents' preferences and optimal allocations.

A.3 Optimal Tax Rates on s in Simple Tax Systems

We present optimal savings tax formulas for simple tax systems, which characterize the optimal savings tax schedule for *any* given earnings tax schedule—including a potentially suboptimal one. These formulas are derived assuming unidimensional heterogeneity and are similar to those presented in Proposition 3, where heterogeneity is allowed to be multidimensional.

Proposition A3. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, and assume that under each simple tax system (i) agents first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z.. At each bundle (c, s, z) chosen by a type θ,* *SL, SN, and LED systems satisfy the following optimality conditions on the savings tax rates:*

$$SL:\frac{\tau_s}{1+\tau_s}\int_{x=z_{min}}^{z_{max}} s(x)\zeta^c_{s|z}(x)dH_z(x)=\int_{x=z_{min}}^{z_{max}}\left\{s\left(x\right)\left(1-\hat{g}(x)\right)-\frac{T'_z(x)+s'_{inc}(x)\,\tau_s}{1-T'_z\left(x\right)}x\zeta^c_z(x)\,s'_{inc}(x)\right\}dH_z(x) \quad (45)$$

$$SN:\frac{T'_s(s)}{1+T'_s(s)}s\zeta^c_{s|z}(z)h_z(z)=s'\left(z\right)\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x)-\frac{T'_z(z)+s'_{inc}(z)T'_s(s)}{1-T'_z(z)}z\zeta^c_z(z)s'_{inc}(z)h_z(z) \quad (46)$$

$$LED:\frac{T'_z\left(z\right)+\tau'_s\left(z\right)s+s'_{inc}(z)\tau_s(z)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}z\zeta^c_z(z)\,s\,h_z(z)+\int_{x\geq z}\frac{\tau_s(x)}{1+\tau_s(x)}s(x)\zeta^c_{s|z}(x)\,dH_z(x) \quad (47)$$
$$=\int_{x\geq z}\left\{\left(1-\hat{g}(x)\right)s\left(x\right)-\frac{T'_z\left(x\right)+\tau'_s\left(x\right)s\left(x\right)+s'_{inc}(x)\tau_s(x)}{1-T'_z\left(x\right)-\tau'_s\left(x\right)s\left(x\right)}x\zeta^c_z(x)s'_{inc}(x)\right\}dH_z(x)$$

These optimal savings tax formulas are all different, reflecting differences between the savings tax instruments that we consider. Yet, they share common elements. First, the preference heterogeneity term $s'_{pref}\left(z\right)$ no longer appears in the formulas. The intuition is that outside of the full optimum, it may still be desirable to tax savings in the absence of preference heterogeneity, implying that optimality may clash with Pareto-efficiency when the earnings tax is suboptimal. Second, $s'_{inc}\left(z\right)$ is a key sufficient statistic for optimal savings tax schedules. Indeed, by Lemma 1, a larger $s'_{inc}\left(z\right)$ means that savings tax reforms impose higher distortions on earnings and thus generally calls for lower savings tax rate.

A.4 Optimal Earnings Tax Rates in Simple Tax Systems

We now present optimal earnings tax formulas for simple tax systems.

Proposition A4. *Consider given (and potentially suboptimal) SL, SN and LED savings tax schedules, and assume that under each simple tax system (i) agents first-order and second-order conditions strictly hold, (ii) there is no bunching, (iii) c and s are smooth functions of z, and (iv) in the SN system s is strictly monotonic (increasing or decreasing) in z. At each bundle (c, s, z) chosen by a type θ, SL, SN, and LED systems satisfy the following optimality conditions on the earnings tax rates:*

$$SL:\frac{T'_z\left(z\right)}{1-T'_z\left(z\right)}=\frac{1}{z\zeta^c_z(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x)-s'_{inc}(z)\frac{\tau_s}{1-T'_z\left(z\right)} \quad (48)$$

$$SN:\frac{T'_z\left(z\right)}{1-T'_z\left(z\right)}=\frac{1}{z\zeta^c_z(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x)-s'_{inc}(z)\frac{T'_s\left(s\right)}{1-T'_z\left(z\right)} \quad (49)$$

$$LED:\frac{T'_z\left(z\right)+\tau'_s\left(z\right)s}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s}=\frac{1}{z\zeta^c_z(z)}\frac{1}{h_z(z)}\int_{x\geq z}\left(1-\hat{g}(x)\right)dH_z(x)-s'_{inc}(z)\frac{\tau_s\left(z\right)}{1-T'_z\left(z\right)-\tau'_s\left(z\right)s} \quad (50)$$

These conditions pinning down the optimal schedule of marginal earnings tax rates are a direct application of Equation (15) presented in Theorem 2 for smooth tax systems. While formulas for SL and SN tax systems look almost identical to the general condition, the formula for LED tax system looks a bit different. This difference only reflects the fact that for a LED tax system the marginal earnings tax rate is given by $\mathcal{T}'_z(s,z) = T'_z(z) + \tau'_s(z)\, s(z)$, accounting for the earnings-dependent nature of savings taxes.

A.5 Optimal Earnings tax Rates in Simple Tax Systems with Multidimensional Heterogeneity

Proposition A5. *Consider given (and potentially suboptimal) SL, SN, and LED savings tax schedule, and assume that under each simple tax system agents first-order and second-order conditions strictly hold. Then, at each bundle (c^0, s^0, z^0) chosen by a type θ^0, marginal tax rates on z in SL/SN/LED systems must satisfy the following optimality conditions:*

$$SL: \frac{T'_z(z^0)}{1-T'_z(z^0)}\mathbb{E}\left[\zeta_z^c(s,z)\middle|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{\tau_s}{1-T'_z(z)}\zeta_z^c(s,z)\middle|z^0\right] \tag{51}$$

$$SN: \frac{T'_z(z^0)}{1-T'_z(z^0)}\mathbb{E}\left[\zeta_z^c(s,z)\middle|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{T'_s(s)}{1-T'_z(z)}\zeta_z^c(s,z)\middle|z^0\right] \tag{52}$$

$$LED: \mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)\,s}{1-T'_z(z)-\tau'_s(z)\,s}\zeta_z^c(s,z)\middle|z^0\right] = \frac{1}{z^0 h_z(z^0)}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\right\} dH_z(z) - \mathbb{E}\left[s'_{inc}(s,z)\frac{\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\,s}\zeta_z^c(s,z)\middle|z^0\right] \tag{53}$$

These conditions are similar to those presented above for optimal marginal earnings tax rates under unidimensional heterogeneity (Proposition A4). Indeed, Lemma 1 still applies such that the previous derivations carry over when adding an expectation with respect to savings. Proofs are thus omitted.

A.6 Equivalences with Tax Systems Involving Gross Period-2 Savings

Suppose that there are two periods, and set $1+r = 1/p$. In period 1 the individual earns z and consumes c, and pays income taxes $T_1(z)$. In period 2 the individual realizes *gross pre-tax savings* $s_g = (z-c-T_1(z))(1+r)$, and pays income taxes $T_2(s_g,z)$. The realized savings s are given by $s_g - T_2(s_g,z)$. The total taxes paid in "period 1 dollars" is given by $T_1(z)+T_2(s_g,z)/(1+r)$. The individual maximizes $U(c,s,z)$ subject to the constraint

$$s \leq (z-c-T_1(z))(1+r) - T_2(s_g,z)$$
$$\Leftrightarrow c + \frac{s}{1+r} \leq z - T_1(z) - \frac{T_2((z-c-T_1(z))(1+r),z)}{1+r}$$

In our baseline formulation with period-1 tax function $\mathcal{T}(s, z)$, individuals choose s and z to maximize $U(z - s - \mathcal{T}(s, z), s, z; \theta)$. To convert from the formulation with period-2 taxes to our baseline formulation, define a function $\tilde{s}_g(s, z)$ implicitly to satisfy the equation

$$\tilde{s}_g - T_2(\tilde{s}_g, z) = s$$

Note that $\tilde{s}_g$ is generally uniquely defined if we have a system with monotonic realized savings s. Then the equivalence in tax schedules is given by

$$\mathcal{T}'_s(s, z) = \frac{1}{1+r} \frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g} \frac{\partial}{\partial s} \tilde{s}_g \tag{54}$$

and $\mathcal{T}'_z = T'_z$. Equation (54) simply computes how a marginal change in s changes the tax burden in terms of period-1 units of money, and the division by $1 + r$ is to convert to period-1 units. Now differentiating the definition of $\tilde{s}_g$ gives

$$\frac{\partial}{\partial s} \tilde{s}_g - \frac{\partial}{\partial s_g} T_2(s_g, z) \frac{\partial}{\partial s} \tilde{s}_g = 1$$

and thus

$$\frac{\partial}{\partial s} \tilde{s}_g = \frac{1}{1 - \frac{\partial}{\partial s_g} T_2(s_g, z)}$$

from which it follows that

$$\mathcal{T}'_s(s, z) = \frac{1}{1+r} \frac{\frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g}}{1 - \frac{\partial}{\partial s_g} T_2(s_g, z)|_{s_g = \tilde{s}_g}} \tag{55}$$

We can also start with a schedule $\mathcal{T}$ and converts it to the 2-period tax schedule. Now if s is the realized savings, we can define gross savings in period 2 as $s_g = s + \mathcal{T}(z, s)(1 + r) - \mathcal{T}(z, 0)$, and we define the function $\tilde{s}(s_g, z)$ to satisfy

$$s_g = \tilde{s} + (1 + r) \left(\mathcal{T}(\tilde{s}, z) - \mathcal{T}(0, z) \right)$$

Then

$$\begin{aligned} \frac{\partial}{\partial s_g} T_2(s_g, z) &= (1 + r) \mathcal{T}'_s(\tilde{s}, z) \frac{\partial}{\partial s_g} \tilde{s} \\ &= \frac{(1 + r) \mathcal{T}'_s(\tilde{s}, z)}{1 + (1 + r) \mathcal{T}'_s(\tilde{s}, z)} \end{aligned} \tag{56}$$

A.6.1 Separable tax systems (SN).

Now if T_2 is a function of s_g only (a separable tax system), then s_g will be a function of s only, and thus $\mathcal{T}'_s$ will only depend on s. Conversely, note that if $\mathcal{T}$ is a separable system, so that $\mathcal{T}'_s$ does not depend on z, then (56) implies that $\frac{\partial}{\partial s_g} T_2(s_g, z)$ does not depend on z either. Thus, separability is a property preserved under these transformations.

Now if we start with a separable $\mathcal{T}$, then T_2 is given by

$$T_2'(s_g) = (1+r)\frac{\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}{1+\frac{\partial}{\partial s}\mathcal{T}_s'(s))|_{s=\tilde{s}}}$$

where $\tilde{s}$ is the value that solves $s_g = \tilde{s} + \mathcal{T}(\tilde{s})$.

A.6.2 Linear tax systems (LED and SL).

If $T_2 = s_g\tau(z)$, a linear earnings-dependent system, then $s = s_g(1-\tau(z))$ and $s_g = \frac{s}{1-\tau(z)}$. Moreover, $\frac{\partial}{\partial s}s_g = \frac{1}{1-\tau(z)}$, and thus we have that

$$\mathcal{T}_s' = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$$

which again implies that we have a linear earnings dependent system with rate $\tilde{\tau}(z) = \frac{1}{1+r}\frac{\tau(z)}{1-\tau(z)}$.

Conversely, if we start with a LED system $\mathcal{T}$ with $\mathcal{T}_s' = \tau(z)$, then

$$\frac{\partial}{\partial s_g}T_2(s_g, z) = (1+r)\frac{\tau(z)}{1+\tau(z)}$$

When the tax rates τ are not functions of z, the calculations above show that the conversions preserve not just linearity, but also independence of z.

B Proofs

B.1 Proof of Lemma A1 (Monotonicity with Preference Heterogeneity)

By Assumption 3, any (c, s) offered in the optimal incentive-compatible allocation can be written as functions of z: $c(z)$ and $s(z)$. By Assumption 2, c, s, z are differentiable in θ which implies that c and s are differentiable in z as well. The total derivative of agent θ utility $U(c(z), s(z), z; \theta)$ with respect to earnings z is

$$\begin{aligned}\frac{dU(c(z), s(z), z;\theta)}{dz} &= U_c'(c(z), s(z), z;\theta)\,c'(z) + U_s'(c(z), s(z), z;\theta)\,s'(z) + U_z'(c(z), s(z), z;\theta) \\ &= U_c'(c(z), s(z), z;\theta)\left(c'(z) + \mathcal{S}(c(z), s(z), z;\theta)\,s'(z) + \mathcal{Z}(c(z), s(z), z;\theta)\right)\end{aligned}$$

The first order condition for an agent θ choosing earnings z therefore implies[23]

$$c'(z) + \mathcal{S}(c(z), s(z), z;\theta)\,s'(z) + \mathcal{Z}(c(z), s(z), z;\theta) = 0$$

[23] The first-order condition has to be satisfied for any type θ in the interior of the compact space of types Θ since we assume smooth allocations (Assumption 2) and smooth preferences (Assumption 1).

As a result, the extended Spence-Mirrlees condition implies that each type θ chooses a different earnings level z since it implies

$$\mathcal{S}'_{\theta}\left(c(z), s(z), z; \theta\right) s'(z) + \mathcal{Z}'_{\theta}\left(c(z), s(z), z; \theta\right) > 0,$$

meaning that the first-order condition cannot hold for two different θ.

Moreover, the extended Spence-Mirrlees condition also implies that type $\theta_2 > \theta_1$ chooses earnings $z_2 > z_1$. To prove this result, we proceed by contradiction. Assume (without loss of generality) that there is an open set $(\theta_1, \theta_2) \in \Theta$ where z is decreasing with θ, and denote $z_1 := z(\theta_1)$ and $z_2 := z(\theta_2)$ where $z_2 < z_1$. Then,

$$\begin{aligned}
&U\left(c(z_2), s(z_2), z_2; \theta_2\right) - U\left(c(z_1), s(z_1), z_1; \theta_2\right) \\
&= \int_{z=z_1}^{z_2} \frac{dU\left(c(z), s(z), z; \theta_2\right)}{dz} dz \\
&= -\int_{z=z_2}^{z_1} U'_c\left(c(z), s(z), z; \theta_2\right)\left(c'(z) + \mathcal{S}\left(c(z), s(z), z; \theta_2\right) s'(z) + \mathcal{Z}\left(c(z), s(z), z; \theta_2\right)\right) dz
\end{aligned}$$

But for each $\theta \in (\theta_1, \theta_2)$ choosing earnings $z \in [z_2, z_1]$ the first-order condition is

$$c'(z) + \mathcal{S}\left(c(z), s(z), z; \theta\right) s'(z) + \mathcal{Z}\left(c(z), s(z), z; \theta\right) = 0$$

such that the extended Spence-Mirrlees condition implies that with $\theta_2 > \theta$

$$c'(z) + \mathcal{S}\left(c(z), s(z), z; \theta_2\right) s'(z) + \mathcal{Z}\left(c(z), s(z), z; \theta_2\right) > 0$$

Since $U'_c > 0$, this means that the integral above is positive and thus that

$$U\left(c(z_2), s(z_2), z_2; \theta_2\right) < U\left(c(z_1), s(z_1), z_1; \theta_2\right).$$

This is a contradiction with the assumption that type $\theta_2 > \theta_1$ chooses earnings $z_2 < z_1$, and concludes the proof.

B.2 Proof of Theorem 1 (Implementation with a Smooth Tax System)

In the appendix, we adopt the notation that agent's allocations in the optimal mechanism are labeled with a "star"; i.e., $(c^*(\theta), s^*(\theta), z^*(\theta))$. We construct a smooth tax system that implements the optimal incentive-compatible allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ by introducing penalties for deviations away from these allocations. This proof relies on Lemma B1 and Lemma B2, which we derive at the end of this subsection.

With unidimensional heterogeneity, type θ belongs to the compact space $\Theta = [\theta_{min}, \theta_{max}]$. Moreover, there is always a mapping $s^*(z)$ that denotes the savings level associated with earnings level $z = z^*(\theta)$ at the optimal incentive-compatible allocation.[24] Let $z_{max} := z^*(\theta_{max})$ and $z_{min} := z^*(\theta_{min})$ denote the maximal and minimal, respectively, earnings levels in the allocation. Let $s_{max} := \max_z s^*(z)$ and $s_{min} := \min_z s^*(z)$ denote the maximal and minimal savings levels.

Step 1: Defining the smooth tax system. We start from a separable and smooth tax system $T_s(s) + T_z(z)$ that satisfies type-specific feasibility and agents' first-order conditions at the optimal incentive-compatible allocation. We then add quadratic penalty terms parametrized by k for deviations from this allocation. For a given deviation, this allows to make the penalty arbitrarily large and enables us to make agents problem locally concave around the optimal incentive-compatible allocation. The other terms that we add are there to guarantee the smoothness of the penalized tax system $\mathcal{T}(s, z; k)$ at the boundaries of the set of optimal allocations.

Formally, $\mathcal{T}_k = \mathcal{T}(s, z; k)$ is defined by:

1. $T_s(s_{min}) = 0$ and $T_z(z_{min}) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\forall z \in [\, z_{min}\, ;\, z_{max}\,]\,, T_z'(z) = \mathcal{Z}(c^*(\theta_z), s^*(\theta_z), z^*(\theta_z); \theta_z) + 1$ with θ_z such that $z = z^*(\theta_z)$
3. $\forall s \in [\, s_{min}\, ;\, s_{max}\,]\,, T_s'(s) = \mathcal{S}(c^*(\theta_s), s^*(\theta_s), z^*(\theta_s); \theta_s) - 1$ with θ_s such that $s = s^*(\theta_s)$
4. $$\mathcal{T}(s, z; k) = \begin{cases} T_s(s) + T_z(z) + k(s - s^*(z))^2 \\ T_s(s_{min}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{min})(s - s_{min}) \\ T_s(s_{max}) + T_z(z) + k(s - s^*(z))^2 + T_s'(s_{max})(s - s_{max}) \\ T_s(s) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 + T_z'(z_{min})(z - z_{min}) \\ T_s(s_{min}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 + T_z'(z_{min})(z - z_{min}) + T_s'(s_{min})(s - s_{min}) \\ T_s(s_{max}) + T_z(z_{min}) + k(s - s_{min})^2 + k(z - z_{min})^2 + T_z'(z_{min})(z - z_{min}) + T_s'(s_{max})(s - s_{max}) \\ T_s(s) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 + T_z'(z_{max})(z - z_{max}) \\ T_s(s_{max}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 + T_z'(z_{max})(z - z_{max}) + T_s'(s_{max})(s - s_{max}) \\ T_s(s_{min}) + T_z(z_{max}) + k(s - s_{max})^2 + k(z - z_{max})^2 + T_z'(z_{max})(z - z_{max}) + T_s'(s_{min})(s - s_{min}) \end{cases}$$

Assumptions 1 and 2 guarantee that the separable tax system $T_s(s) + T_z(z)$ is smooth i.e. a twice continuously differentiable function. Our construction of the penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ guarantees that it inherits this smoothness property.

Step 2: Local maxima for sufficiently large k. For a given agent θ, we show that the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum under the tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$ for sufficiently large k. To do so, we first establish that type-specific feasibility is satisfied together with the first-order conditions of agent θ maximization problem. We then show that for sufficiently large k, second-order conditions are also satisfied implying that the intended bundle is a local maximum.

[24] We consider (without loss of generality) the case in which s is strictly increasing, the proof is similar if s is strictly decreasing instead.

The previous definition of the tax system implies

$$\mathcal{T}'_z(s^*(\theta), z^*(\theta); k) = T'_z(z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$$
$$\mathcal{T}'_s(s^*(\theta), z^*(\theta); k) = T'_s(s^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$$

meaning type-specific feasibility is satisfied by Lemma B1 (see below).

Now, defining

$$V(s, z; \theta, k) := U(z - s - \mathcal{T}(s, z; k), s, z; \theta), \tag{57}$$

first-order conditions of agent θ choice of savings s and earnings z are

$$V'_s(s, z; \theta, k) = -(1 + \mathcal{T}'_s(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_s(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$
$$V'_z(s, z; \theta, k) = (1 - \mathcal{T}'_z(s, z; k))U'_c(z - s - \mathcal{T}(s, z; k), s, z; \theta) + U'_z(z - s - \mathcal{T}(s, z; k), s, z; \theta) = 0$$

and they are by construction satisfied at $(s^*(\theta), z^*(\theta))$ for each type θ.

Using Lemma B2 (see below), second-order conditions at $(s^*(\theta), z^*(\theta))$ are

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \leq 0 \tag{58}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \leq 0 \tag{59}$$

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta + \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg] \leq 0 \tag{60}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Here, U, $\mathcal{S}$, and $\mathcal{Z}$ are smooth functions implying that their derivatives are continuous functions over compact spaces, and are thus bounded. Now, by definition of $\mathcal{T}_k = \mathcal{T}(s, z; k)$ we have $\mathcal{T}''_{sz} = -2ks^{*\prime}(z)$ which is negative for any $k \geq 0$ and increasing in magnitude with k.

Noting $U'_c \geq 0$ and $s^{*\prime}(z) \geq 0$, it implies that V''_{ss} and V''_{zz} must be negative for sufficiently large k thanks to the last term since other terms are bounded and do not depend on k. By the same reasoning, under the extended Spence-Mirrlees single-crossing assumption that $\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta \geq 0$, we also have that $(V''_{sz})^2 - V''_{ss}V''_{zz}$ must be negative for sufficiently large k.

This shows that for a given agent θ, there exists a k_0 such that for all $k \geq k_0$ the allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum to agent θ maximization problem under the smooth penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$.

Step 3: Global maxima for sufficiently large k. Let $s_{\mathcal{T}_k}(\theta)$ and $z_{\mathcal{T}_k}(\theta)$ denote the level of savings and earnings, respectively, that a type θ chooses given a smooth penalized tax system $\mathcal{T}_k$. To prove implementability of optimal incentive-compatible allocations, we show that there exists a sufficiently large k such that for all θ, $s_{\mathcal{T}_k}(\theta) = s^*(\theta)$ and $z_{\mathcal{T}_k}(\theta) = z^*(\theta)$.

Let's proceed by contradiction, and suppose that it is not the case. Then, there exists an infinite sequence of types θ_k, choosing savings $s_{\mathcal{T}_k}(\theta_k) \neq s^*(\theta_k)$ and earnings $z_{\mathcal{T}_k}(\theta_k) \neq z^*(\theta_k)$ under tax system $\mathcal{T}_k$, and enjoying utility gains from this "deviation" to allocation $(s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$.

First, the fact that we impose quadratic penalties for earnings choices outside of $[\, z_{min}\, ;\, z_{max}\,]$ guarantees that for any $\varepsilon > 0$, there exists k_1, such that for all $k \geq k_1$ and for all types θ, agents' earnings choices belong to the compact set $[\, z_{min} - \varepsilon\, ;\, z_{max} + \varepsilon\,]$. Indeed, starting from a given earnings level $z > z_{max} + \varepsilon$, the utility change associated with an earnings change is $[(1 - \mathcal{T}_z')\, U_c' + U_z']\, dz$. By construction, the marginal earnings tax rate in this region is $\mathcal{T}_z' = 2k(z - z_{max}) + T_z'(z_{max})$. Noting that $U_c' > 0$, $U_z' < 0$, and that the type space is compact, we can make for all individuals the utility change from a decrease in earnings arbitrarily positive for sufficiently large k. This shows that all individuals choose earnings $z \leq z_{max} + \varepsilon$ for sufficiently large k. Symetrically, we can show that all individuals choose earnings $z \geq z_{min} - \varepsilon$ for sufficiently large k.

Second, since earnings shape agents' disposable income, the fact that earnings belong to a compact set for sufficiently large k implies that agents' allocation choices also belong to a compact set. Indeed, for sufficiently large k agents' allocation choices must belong to the set of (c, s, z) such that $c \geq 0$, $s \geq 0$, $z \in [\, z_{min} - \varepsilon\, ;\, z_{max} + \varepsilon\,]$, and $c + s \leq z - \mathcal{T}\,(s, z; k)$ where the tax function is smooth and finite. These constraints make the space of allocations compact.

As a result, the infinite sequence $(\theta_k, s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$ belongs to a compact space of allocations and types, it is thus bounded. By the Bolzano–Weierstrass theorem, this means that there exists a convergent subsequence $\left(\theta_j, s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{\theta}, \hat{s}, \hat{z})$. Since all types θ_j belong to $[\, \theta_{min}\, ;\, \theta_{max}\,]$, we know that the limit type $\hat{\theta}$ must belong to $[\, \theta_{min}\, ;\, \theta_{max}\,]$. Now, from the previous paragraph, agents' earnings choices have to be arbitrarily close to $[\, z_{min}\, ;\, z_{max}\,]$ as the penalty grows. This implies that the limit $\hat{z}$ must belong to $[\, z_{min}\, ;\, z_{max}\,]$.

Next, we establish that the limit $\hat{s}$ must be such that $\hat{s} = s^*(\hat{z})$. Fix an earnings level $z \in [\, z_{min}\, ;\, z_{max}\,]$, starting from a savings level $s \neq s^*(z)$ the utility change associated with a savings change is $[-(1 + \mathcal{T}_s')\, U_c' + U_s']\, ds$. Assuming without loss of generality that s belongs to $[\, s_{min}\, ;\, s_{max}\,]$, the marginal savings tax rate in this region is $\mathcal{T}_s' = T_s'(s) + 2k(s - s^*(z))$. Noting that $U_c' > 0$, and that U_s' is bounded, we can make the utility gains from a savings change towards $s^*(z)$ arbitrarily large for sufficiently large k. As a result, for any $\varepsilon > 0$, there exists k_2 such that for all $k \geq k_2$, agent $\hat{\theta}$ chooses savings $s \in [\, s^*(z) - \varepsilon\, ;\, s^*(z) + \varepsilon\,]$ for a fixed z.[25] Since agent $\hat{\theta}$ savings choice can be made arbitrarily close to $s^*(z)$ for sufficiently large k, we must have at the limit $s = s^*(z)$. Now, because earnings z converge to $\hat{z}$ and the function $s^*(z)$ is by assumption continuous, we must have at the limit $\hat{s} = s^*(\hat{z})$.

We have thus established that the limit $(\hat{\theta}, \hat{s}, \hat{z})$ is such that $\hat{\theta} \in [\, \theta_{min}\, ;\, \theta_{max}\,]$, $\hat{z} \in [\, z_{min}\, ;\, z_{max}\,]$, and $\hat{s} = s^*(\hat{z})$. This means that the limit allocation $(\hat{c}, \hat{s}, \hat{z})$ belongs to the set of optimal incentive-compatible allocations. Given our continuity and monotonicity assumptions, this implies that it

[25] A way to see this is that the marginal rate of substitution between consumption and savings $\mathcal{S}$ is continuous on a compact space and thus bounded, whereas the marginal tax rate which is parametrized by k can be made arbitrarily large. As a result, agents' first-order condition can never hold for sufficiently large k, while we can similarly exclude corner solutions for sufficiently large k.

is the optimal allocation of some type θ and it has to be by definition that of agent $\hat{\theta}$. Hence, $(\hat{c}, \hat{s}, \hat{z}) = (c^*(\hat{\theta}), s^*(\hat{\theta}), z^*(\hat{\theta}))$.

To complete the proof and find a contradiction, fix a value $k^\dagger$ that is large enough such that second-order conditions hold for type $\hat{\theta}$ at allocation $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ under tax system $\mathcal{T}_{k^\dagger}$ – this $k^\dagger$ exists by step 2. This implies that there exists an open set N containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ such that $V(s, z; \hat{\theta}, k^\dagger)$ is strictly concave over $(s, z) \in N$.

Now, consider types θ^j with $j \geq k^\dagger$. Since these agents belong to the previously defined subsequence, they prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_j$. Because penalties are increasingly large and $j \geq k^\dagger$, this implies that agents θ^j also prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_{k^\dagger}$.

Yet, by continuity, the function $V\left(s, z; \theta_j, k^\dagger\right)$ gets arbitrarily close to the function $V(s, z; \hat{\theta}, k^\dagger)$ for sufficiently large j since $\theta_j \to \hat{\theta}$. For the same reason, $(s^*(\theta_j), z^*(\theta_j)) \to (s^*(\hat{\theta}), z^*(\hat{\theta}))$. Moreover, by definition $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{s}, \hat{z})$. As a result, for any open set $N' \subsetneq N$ containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$, there exists a $j^\dagger \geq k^\dagger$ such that $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ and such that both $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ and $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ belong to the set N'.

Since $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ it has a unique optimum on N'. By definition of $\mathcal{T}_{k^\dagger}$, agent $\theta_{j^\dagger}$ first-order conditions are satisfied at $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$. Hence, $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ is agent $\theta_{j^\dagger}$ maximum under the tax system $\mathcal{T}_{k^\dagger}$. This contradicts the fact established above that agent $\theta_{j^\dagger}$ prefers $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ to allocation $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ under tax system $\mathcal{T}_{k^\dagger}$, which completes the proof.

Lemma for type-specific feasibility.

Lemma B1. *A smooth tax system $\mathcal{T}$ satisfies type-specific feasibility over the compact type space $[\,\theta_{min}\,;\,\theta_{max}\,]$ if it satisfies the following conditions*

1. $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = z^*(\theta_{min}) - c^*(\theta_{min}) - s^*(\theta_{min})$
2. $\mathcal{T}'_z(s^*(\theta), z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$
3. $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$

Proof. Consider the type-specific feasible tax system $T^*_\theta(\theta) = z^*(\theta) - s^*(\theta) - c^*(\theta)$, and note that the lemma amounts to showing that $T^*_\theta(\theta) = \mathcal{T}(s^*(\theta), z^*(\theta))$ for all θ. To that end, note that the first-order condition for truthful reporting of θ under the optimal mechanism implies

$$U'_c \cdot (z'(\theta) - s'(\theta) - T^{*\prime}_\theta(\theta)) + U'_s \cdot s'(\theta) + U'_z \cdot z'(\theta) = 0,$$

with derivatives evaluated at the optimal allocation. This can be rearranged as

$$\begin{aligned} T^{*\prime}_\theta(\theta) &= \left(\frac{U'_s}{U'_c} - 1\right) s'(\theta) + \left(\frac{U'_z}{U'_c} + 1\right) z'(\theta) \\ &= \mathcal{T}'_s(s^*(\theta)) s^{*\prime}(\theta) + \mathcal{T}'_z(z^*(\theta)) z^{*\prime}(\theta). \end{aligned}$$

It thus follows that

$$\begin{aligned}\mathcal{T}(s^*(\theta), z^*(\theta)) - \mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) &= \int_{\vartheta=\theta_{min}}^{\vartheta=\theta} \left(\mathcal{T}'_s(s^*(\vartheta))s^{*\prime}(\vartheta) + \mathcal{T}'_z(z^*(\vartheta))z^{*\prime}(\vartheta)\right) d\vartheta \\ &= T^*_\theta(\theta) - T^*_\theta(\theta_{min})\end{aligned}$$

Since $\mathcal{T}(s^*(\theta_{min}), z^*(\theta_{min})) = T^*_\theta(\theta_{min})$, this implies that $\mathcal{T}(s^*(\theta), z^*(\theta)) = T^*_\theta(\theta)$ for all θ. The smooth tax system $\mathcal{T}$ therefore satisfies type-specific feasibility. □

Lemma on second-order conditions.

Lemma B2. *Consider a smooth tax system $\mathcal{T}$ satisfying the conditions in Lemma B1 and define*

$$V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta). \tag{61}$$

When evaluated at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$, we show that

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \tag{62}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \tag{63}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg]\end{aligned} \tag{64}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Proof. First-order derivatives are

$$\begin{aligned}V'_s(s, z; \theta) &= -(1 + \mathcal{T}'_s(s, z))U'_c(z - s - \mathcal{T}(s, z), s, z; \theta) + U'_s(z - s - \mathcal{T}(s, z), s, z; \theta) \\ V'_z(s, z; \theta) &= (1 - \mathcal{T}'_z(s, z))U'_c(z - s - \mathcal{T}(s, z), s, z; \theta) + U'_z(z - s - \mathcal{T}(s, z), s, z; \theta)\end{aligned}$$

Second-order derivatives are

$$V''_{ss}(s, z; \theta) = -\mathcal{T}''_{ss}U'_c - (1 + \mathcal{T}'_s)\left(-(1 + \mathcal{T}'_s)U''_{cc} + U''_{cs}\right) - (1 + \mathcal{T}'_s)U''_{cs} + U''_{ss} \tag{65}$$

$$V''_{zz}(s, z; \theta) = -\mathcal{T}''_{zz}U'_c + (1 - \mathcal{T}'_z)\left((1 - \mathcal{T}'_z)U''_{cc} + U''_{cz}\right) + (1 - \mathcal{T}'_z)U''_{cz} + U''_{zz} \tag{66}$$

To obtain the first result of the Lemma, we compute $\mathcal{T}''_{ss}$ by differentiating both sides of $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$ with respect to θ

$$\begin{aligned}\mathcal{T}''_{ss}s^{*\prime}(\theta) + \mathcal{T}''_{sz}z^{*\prime}(\theta) &= \frac{d}{d\theta}\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &= \mathcal{S}'_c c^{*\prime}(\theta) + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta\end{aligned}$$

plugging in $c^{*\prime}(\theta) = (1-\mathcal{T}_z')\, z^{*\prime}(\theta) - (1+\mathcal{T}_s')\, s^{*\prime}(\theta)$ and denoting $s^{*\prime}(z^*) := s^{*\prime}(\theta)/z^{*\prime}(\theta)$, the previous expression can be rearranged as

$$\mathcal{T}_{ss}'' = \mathcal{S}_c' \frac{1-\mathcal{T}_z'}{s^{*\prime}(z^*)} - \mathcal{S}_c'(1+\mathcal{T}_s') + \mathcal{S}_s' + \frac{\mathcal{S}_z'}{s^{*\prime}(z^*)} + \frac{\mathcal{S}_\theta'}{s^{*\prime}(\theta)} - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z^*)}. \tag{67}$$

Moreover, we differentiate Equation (2) to express the derivative of $\mathcal{S}$ with respect to c as

$$\begin{aligned}
\mathcal{S}_c'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U_c' U_{sc}'' - U_s' U_{cc}''}{(U_c')^2} \\
&= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'} U_{cc}'' + U_{sc}''\right) \\
&= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s') U_{cc}'' + U_{sc}''\right)
\end{aligned} \tag{68}$$

and the derivative of $\mathcal{S}$ with respect to s as

$$\begin{aligned}
\mathcal{S}_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U_c' U_{ss}'' - U_s' U_{cs}''}{(U_c')^2} \\
&= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'} U_{cs}'' + U_{ss}''\right) \\
&= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s') U_{cs}'' + U_{ss}''\right).
\end{aligned} \tag{69}$$

Substituting equations (67), (68) and (69) into (65), we have

$$\begin{aligned}
V_{ss}''\left(s^*(\theta), z^*(\theta); \theta\right) &= -U_c' \cdot \left(\mathcal{S}_c' \frac{1-\mathcal{T}_z'}{s^{*\prime}(z)} - \mathcal{S}_c'(1+\mathcal{T}_s') + \mathcal{S}_s' + \frac{\mathcal{S}_z'}{s^{*\prime}(z)} + \frac{\mathcal{S}_\theta'}{s^{*\prime}(\theta)} - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) - (1+\mathcal{T}_s') U_s' \mathcal{S}_c' + U_c' \mathcal{S}_s' \\
&= -U_c' \cdot \left(\frac{1-T_z'}{s^{*\prime}(z)} \mathcal{S}_c' + \frac{1}{s^{*\prime}(z)} \mathcal{S}_z' + \frac{1}{s^{*\prime}(\theta)} \mathcal{S}_\theta' - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) \\
&= \frac{U_z'}{s^{*\prime}(z)} \mathcal{S}_c' - \frac{U_c'}{s^{*\prime}(z^*)} \mathcal{S}_z' - \frac{U_c'}{s^{*\prime}(\theta)} \mathcal{S}_\theta' + \frac{U_c'}{s^{*\prime}(z^*)} \mathcal{T}_{sz}''.
\end{aligned} \tag{70}$$

where we have used $U_z' = -U_c'\,(1-\mathcal{T}_z')$ in the last line.

Similarly, we can obtain the second result of the Lemma by writing $\mathcal{T}_{zz}''$ as

$$\mathcal{T}_{zz}'' = \mathcal{Z}_c'\left(1-\mathcal{T}_z'\right) - \mathcal{Z}_c'\left(1+\mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{Z}_s' s^{*\prime}(z^*) + \mathcal{Z}_z' + \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} - \mathcal{T}_{sz}'' s^{*\prime}(z^*). \tag{71}$$

Using

$$\mathcal{Z}_c' = \frac{1}{U_c'}\left(U_{cz}'' + \left(1-\mathcal{T}_z'\right) U_{cc}''\right)$$

as well as

$$\mathcal{Z}_z' = \frac{1}{U_c'}\left(U_{zz}'' + \left(1-\mathcal{T}_z'\right) U_{cz}''\right)$$

we get

$$V''_{zz}\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - U'_c \frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} + U'_c \mathcal{T}''_{sz} s^{*\prime}(z^*). \tag{72}$$

Finally, to obtain the third result of the Lemma, we must compute $(V''_{sz})^2 - V''_{ss}V''_{zz}$. Note that the first-order condition $V'_s(s^*(\theta), z^*(\theta); \theta) = 0$ holds at every θ by construction. Differentiating with respect to θ we get

$$\frac{d}{d\theta}V'_s(s^*(\theta), z^*(\theta); \theta) = V''_{ss}s^{*\prime}(\theta) + V''_{sz}z^{*\prime}(\theta) + V''_{s\theta} = 0 \tag{73}$$

which we can rearrange as

$$-V''_{sz} = V''_{ss}s^{*\prime}(z^*) + \frac{V''_{s\theta}}{z^{*\prime}(\theta)}. \tag{74}$$

Similarly, by totally differentiating the first-order condition $V'_z(s^*(\theta), z^*(\theta); \theta) = 0$ and rearranging we find

$$-V''_{sz} = \frac{V''_{zz}}{s^{*\prime}(z^*)} + \frac{V''_{z\theta}}{s^{*\prime}(\theta)}. \tag{75}$$

Writing $(V''_{sz})^2$ as the product of the right-hand sides of Equations (74) and (75) yields

$$\begin{aligned}\left(V''_{sz}\right)^2 =& \left(V''_{ss}s^{*\prime}(z) + \frac{V''_{s\theta}}{z^{*\prime}(\theta)}\right)\left(\frac{V''_{zz}}{s^{*\prime}(z)} + \frac{V''_{z\theta}}{s^{*\prime}(\theta)}\right) \\ =& V''_{ss}V''_{zz} + \frac{1}{z^{*\prime}(\theta)}V''_{ss}V''_{z\theta} + \frac{1}{s^{*\prime}(\theta)}V''_{zz}V''_{s\theta} + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}V''_{s\theta}V''_{z\theta}\end{aligned} \tag{76}$$

Now from the definition $V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta)$, we can compute

$$\begin{aligned}V''_{s\theta}(s, z; \theta) &= -\left(1 + \mathcal{T}'_s(s, z)\right)U''_{c\theta} + U''_{s\theta} \\ V''_{z\theta}(s, z; \theta) &= \left(1 - \mathcal{T}'_z(s, z)\right)U''_{c\theta} + U''_{z\theta}\end{aligned}$$

and use the fact that at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ we have

$$\begin{aligned}\mathcal{S}'_\theta &= \frac{1}{U'_c}\left(U''_{s\theta} - \left(1 + \mathcal{T}'_s\right)U''_{c\theta}\right) \\ \mathcal{Z}'_\theta &= \frac{1}{U'_c}\left(U''_{z\theta} + \left(1 - \mathcal{T}'_z\right)U''_{c\theta}\right)\end{aligned}$$

to obtain

$$V''_{s\theta}\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = U'_c\mathcal{S}'_\theta \tag{77}$$

$$V''_{z\theta}\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = U'_c\mathcal{Z}'_\theta. \tag{78}$$

Substituting these into Equation (76) and rearranging, we have

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{1}{z^{*\prime}(\theta)} V''_{ss} U'_c \mathcal{Z}'_\theta + \frac{1}{s^{*\prime}(\theta)} V''_{zz} U'_c \mathcal{S}'_\theta + \frac{1}{s^{*\prime}(\theta) z^{*\prime}(\theta)} \left(U'_c\right)^2 \mathcal{S}'_\theta \mathcal{Z}'_\theta. \tag{79}$$

Expanding V''_{ss} from Equation (70), and V''_{zz} from Equation (72) yields after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)} \Bigg[\left(U'_z \mathcal{S}'_c - U'_c \mathcal{S}'_z\right) \mathcal{Z}'_\theta + \left(U'_s \mathcal{Z}'_c - U'_c \mathcal{Z}'_s - U'_c \frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)} \right) s^{*\prime}(z^*) \mathcal{S}'_\theta + \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*) \mathcal{S}'_\theta \right) U'_c \mathcal{T}''_{sz} \Bigg],$$

which gives the third result of the Lemma above. □

B.3 Proof of Proposition A1 & A2 (Implementation with Simple Tax Systems)

B.3.1 Proof of Proposition A1

SN tax system. The sufficient conditions for local optimality under the candidate SN tax system follow directly from Lemma B2 which computes agents' SOCs at the optimal incentive-compatible allocation under a general tax system $\mathcal{T}(s, z)$. Indeed, agents' SOCs are satisfied if Equations (62), (63), and (64) are negative under the SN tax system. Since the cross-partial derivative $\mathcal{T}''_{sz}$ is zero for a SN tax system, it is easy to verify that conditions (42) and (43) on the derivatives of $\mathcal{S}$ and $\mathcal{Z}$, combined with monotonicity ($s^{*\prime}(\theta) > 0$, $s^{*\prime}(z) > 0$) and Assumption 1 on the derivatives of U, jointly imply that each of these three equations is the sum of negative terms. This implies that agents' SOCs are satisfied at the optimal incentive-compatible allocation under the candidate SN tax system.

LED tax system. To derive sufficient conditions for local optimality under the candidate LED tax system, we begin from results obtained in the derivations of Lemma B2 which computes agents' SOCs at the optimal incentive-compatible allocation. We consider the requirements $V''_{ss} < 0$, $V''_{zz} < 0$, and $V''_{ss}V''_{zz} > (V''_{sz})^2$ in turn.

First, to show that V''_{ss} is negative, note that under a LED tax system, $\mathcal{T}''_{ss} = 0$. Therefore, using the fact that under the candidate LED tax system we have $1 + \mathcal{T}'_s = \frac{U'_s}{U'_c}$ at the optimal incentive-compatible allocation, the general expression for V''_{ss} given in Equation (65) reduces to

$$V''_{ss}(s^*(\theta), z^*(\theta); \theta) = \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} - 2\frac{U'_s}{U'_c} U''_{cs} + U''_{ss}$$

Therefore when utility is additively separable in c and s (implying $U''_{cs} = 0$), the concavity of preferences ($U''_{cc} \leq 0$ and $U''_{ss} \leq 0$) guarantees that this expression is negative.

Second, to show that V''_{zz} is negative, note that under the candidate LED tax system defined in

Equations (40) and (41) we have

$$\mathcal{T}''_{sz}(s,z) = \tau'_s(z).$$

We can thus find an expression for $\tau'_s(z)$ at any point in the allocation in question by totally differentiating Equation (40) with respect to θ:

$$\begin{aligned}
\tau'_s\left(z^*(\theta)\right) z^{*\prime}(\theta) &= \frac{d}{d\theta}\Big[\mathcal{S}\left(c^*(\theta), s^*(\theta), z^*(\theta);\theta\right)\Big] \\
&= \frac{d}{d\theta}\Big[\mathcal{S}\left(z^*(\theta) - s^*(\theta) - \mathcal{T}\left(s^*(\theta), z^*(\theta)\right), s^*(\theta), z^*(\theta);\theta\right)\Big] \\
&= \mathcal{S}'_c \cdot \left[(1-\mathcal{T}'_z)z^{*\prime}(\theta) - \left(1+\mathcal{T}'_s\right)s^{*\prime}(\theta)\right] + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,
\end{aligned}$$

which yields

$$\tau'_s\left(z^*(\theta)\right) = \mathcal{S}'_c \cdot \left(1-\mathcal{T}'_z\right) - \mathcal{S}'_c \cdot \left(1+\mathcal{T}'_s\right)s^{*\prime}(z^*) + \mathcal{S}'_s \cdot s^{*\prime}(z^*) + \mathcal{S}'_z + \frac{\mathcal{S}'_\theta}{z^{*\prime}(\theta)}.$$

Substituting this into the expression for V''_{zz} in (72), we have

$$\begin{aligned}
V''_{zz}\left(s^*\left(\theta\right), z^*\left(\theta\right);\theta\right) = {}& U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - U'_c \frac{\mathcal{Z}'_\theta}{z^{*\prime}(\theta)} \\
&+ U'_c s^{*\prime}(z^*)\left[\mathcal{S}'_c \cdot \left(1-\mathcal{T}'_z\right) - \mathcal{S}'_c \cdot \left(1+\mathcal{T}'_s\right)s^{*\prime}(z^*) + \mathcal{S}'_s \cdot s^{*\prime}(z^*) + \mathcal{S}'_z + \frac{\mathcal{S}'_\theta}{z^{*\prime}(\theta)}\right].
\end{aligned} \tag{80}$$

Now employing the assumption that utility is separable in c, s, and z, (implying both $U''_{cz} = 0$ and $U''_{cs} = 0$) we have

$$\begin{aligned}
U'_s\mathcal{Z}'_c + U'_c\mathcal{S}'_c(1-\mathcal{T}'_z) &= U'_s\mathcal{Z}'_c - U'_z\mathcal{S}'_c \\
&= U'_s\frac{U'_cU''_{cz} - U'_zU''_{cc}}{(U'_c)^2} - U'_z\frac{U'_cU''_{cs} - U'_sU''_{cc}}{(U'_c)^2} \\
&= 0.
\end{aligned}$$

Substituting this result into Equation (80), and noting that $\mathcal{Z}'_s = \mathcal{S}'_z = 0$ by separability, yields

$$V''_{zz}(s^*(\theta), z^*(\theta);\theta) = \left(s^{*\prime}(z^*)\right)^2\left[U'_c\mathcal{S}'_s - U'_s\mathcal{S}'_c\cdot\right] - \frac{U'_c}{z^{*\prime}(\theta)}\left[\mathcal{Z}'_\theta - s^{*\prime}(z^*)\mathcal{S}'_\theta\right] \tag{81}$$

Again employing separability, we have

$$U'_c\mathcal{S}'_s - U'_s\mathcal{S}'_c = U'_c\frac{U'_cU''_{ss} - U'_sU''_{cs}}{(U'_c)^2} - U'_s\frac{U'_cU''_{cs} - U'_sU''_{cc}}{(U'_c)^2} = U''_{ss} + \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} \leq 0,$$

implying that the first term on the right-hand side of Equation (81) is negative. The condition

$\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta' \geq 0$ from (44) in the Proposition then implies Equation (81) (and thus V_{zz}'') is negative.

Third, to show $V_{ss}''V_{zz}'' > (V_{sz}'')^2$, we proceed from Equation (64) in Lemma B2:

$$\begin{aligned}
&\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' \\
&= \frac{U_c'}{s^{*\prime}(\theta)}\left[\left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\mathcal{Z}_\theta' + \left(U_s'\mathcal{Z}_c' - U_c'\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}_\theta' + \left(\mathcal{Z}_\theta' + s^{*\prime}(z^*)\mathcal{S}_\theta'\right)U_c'\mathcal{T}_{sz}''\right] \\
&= \left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta'U_c'\mathcal{T}_{sz}'' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta'\left(U_s's^{*\prime}(z^*)\mathcal{Z}_c' - U_c's^{*\prime}(z^*)\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c's^{*\prime}(z^*)\mathcal{T}_{sz}''\right).
\end{aligned}$$

Recognizing that the last bracket term is exactly the expression for V_{zz}'' given in Lemma B2 this gives

$$\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' = \left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta'U_c'\mathcal{T}_{sz}'' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta'V_{zz}''$$

using the previous expression derived for $\mathcal{T}_{sz}'' = \tau_s'$, and the fact that separability ensures $\mathcal{S}_z' = 0$, we obtain after simplification

$$\left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' = -\frac{(U_c')^2}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\mathcal{Z}_\theta'\left[s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}_c' - \mathcal{S}_s'\right) - \mathcal{S}_\theta'\right] + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta'V_{zz}''.$$

We have already shown that V_{zz}'' is negative. Thus the conditions $\mathcal{S}_\theta' \geq 0$ and $\mathcal{S}_\theta' \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}_c' - \mathcal{S}_s'\right)$ from (44) in the Proposition imply that both terms on the right-hand side are negative, implying that all second-order conditions hold.

B.3.2 Proof of Proposition A2

We begin with a more general statement, and then derive Proposition A2 as a corollary. For a fixed type θ, let $c(z,\theta)$ and $s(z,\theta)$ be its preferred consumption and savings choices at earnings z, given the budget constraint induced by $\mathcal{T}(s,z)$

Lemma B3. *Suppose that $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, where $z^*(\theta)$ is increasing. Individuals' local optima correspond to their global optima when*

1. *$\mathcal{Z} = \frac{U_z'(c,s,z;\theta)}{U_c'(c,s,z;\theta)}$ and $\mathcal{S} = \frac{U_s'(c,s,z;\theta)}{U_c'(c,s,z;\theta)}$ are strictly increasing in θ for all (c,s,z)*
2. *For any two types θ and θ', we cannot have both*

$$\begin{aligned}
&U_c'\Big(c^*(\theta), s^*(\theta), z^*(\theta); \theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U_z'\Big(c^*(\theta), s^*(\theta), z; \theta\Big) \\
&< U_c'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta\Big)\sigma_c\Big(s\left(z^*(\theta), \theta'\right), z^*(\theta)\Big) + U_z'\Big(c\left(z^*(\theta), \theta'\right), s\left(z^*(\theta), \theta'\right), z^*(\theta); \theta\Big)
\end{aligned} \tag{82}$$

and

$$U_s'\Big(c^*(\theta),s^*(\theta),z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta),z^*(\theta)\Big)+U_z'\Big(c^*(\theta),s^*(\theta),z;\theta\Big)\Big) \tag{83}$$
$$<U_s'\Big(c\left(z^*(\theta),\theta'\right),s\left(z^*(\theta),\theta'\right),z^*(\theta);\theta\Big)\sigma_s\Big(s\left(z^*(\theta),\theta'\right),z^*(\theta)\Big)+U_z'\Big(c\left(z^*(\theta),\theta'\right),s\left(z^*(\theta),\theta'\right),z^*(\theta);\theta$$

where $\sigma_c(s,z) := 1-\mathcal{T}_z'(s,z)$ *and* $\sigma_s(s,z) := \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Condition 1 corresponds to single crossing assumptions for earnings and savings. Condition 2 is a requirement that if type θ preserves its assigned earnings level $z^*(\theta)$, but chooses some other consumption level s (corresponding to a level that some other type θ' would choose if forced to choose earnings level $z^*(\theta)$), then at this alternative consumption bundle agent θ cannot have both higher marginal utility from increasing its savings through one more unit of work *and* increasing its consumption through one more unit of work. Generally, this condition will hold as long as U is sufficiently concave in consumption and savings when type θ chooses earnings level $z^*(\theta)$.

Proof. To prove agents' local optima are global optima, we want to show that for any given agent θ^*, utility decreases when moving from allocation $(c^*(\theta^*),s^*(\theta^*),z^*(\theta^*))$ to allocation $(c(z,\theta^*),s(z,\theta^*),z)$.

The first step is to compute agent θ^* utility change. The envelope theorem applied to savings choices $s(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\\&=U_c'\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\sigma_c\left(s(z,\theta^*),z\right)+U_z'\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\end{aligned}$$

where $\sigma_c(s,z) = 1-\mathcal{T}_z'(s,z)$. Note that, as established by Milgrom and Segal (2002), these equalities hold as long as U is differentiable in z (holding s and c fixed)—differentiability of $c(z,\theta^*)$ or $s(z,\theta^*)$ is actually not required.

Similarly, the envelope theorem applied to consumption choices $c(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\\&=U_s'\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\sigma_s\left(s(z,\theta^*),z\right)+U_z'\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)\end{aligned} \tag{84}$$

where $\sigma_s(s,z) = \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Therefore, agent's θ^* utility change when moving from allocation $(c^*(\theta^*),s^*(\theta^*),z^*(\theta^*))$ to allocation $(c(z,\theta^*),s(z,\theta^*),z)$ is

$$\begin{aligned}&U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)-U\left(c(z^*(\theta^*),\theta^*),s(z^*(\theta^*),\theta^*),z^*(\theta^*);\theta^*\right)\\&=\int_{x=z^*(\theta^*)}^{x=z}\Big[\min\left\{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_c\left(s(x,\theta^*),x\right),U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_s\left(s(x,\theta^*),x\right)\right\}\\&\qquad+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\Big]dx\end{aligned} \tag{85}$$

where the min operator is introduced without loss of generality given the fact that both terms are equal.

The second step is to show that under our assumptions, agent θ^* utility change (85) is negative. To do so, let θ_x be the type that chooses earnings x. Then, by definition, agent θ_x utility is maximal at earnings x implying both

$$U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$
$$U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0$$

such that

$$\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right)\right\}$$
$$+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) = 0 \tag{86}$$

Now, by condition 2, we either have[26]

$$U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)$$
$$\geq U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)$$

or

$$U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)$$
$$\geq U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)$$

implying that

$$\max\left\{U_c'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_c\left(s^*(\theta_x), x\right), U_s'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right)\sigma_s\left(s^*(\theta_x), x\right)\right\}$$
$$+ U_z'\left(c^*(\theta_x), s^*(\theta_x), x; \theta_x\right) \tag{87}$$
$$\geq \min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}$$
$$+ U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)$$

But since the maximum is zero, this minimum has to be negative. Hence, we have either

$$U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)\sigma_c\left(s(x,\theta^*), x\right) + U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right) \leq 0$$
$$\iff \frac{U_z'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)}{U_c'\left(c(x,\theta^*), s(x,\theta^*), x; \theta_x\right)} \leq -\sigma_c\left(s(x,\theta^*), x\right)$$

[26]Not having $\{a < c \quad and \quad b < c\}$ means having $\{a \geq c \quad \text{or} \quad b \geq d\}$ which implies $\max(a,b) \geq \min(c,d)$

or

$$\begin{aligned}&U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_s\left(s(x,\theta^*),x\right)+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\leq 0\\ \iff &\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}{U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}\leq-\sigma_c\left(s(x,\theta^*),x\right)\end{aligned}$$

Suppose that $z > z^*(\theta^*)$ such that $x > z^*(\theta^*)$; the case $z < z^*(\theta^*)$ follows identically. For any $x > z^*(\theta^*)$, the monotonicity of the earnings function means that $\theta_x > \theta^*$. Then, by the single-crossing conditions for $\mathcal{Z} = \frac{U_z'}{U_c'}$ and $\mathcal{S} = \frac{U_s'}{U_c'}$, this means that we have either[27]

$$\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}\leq-\sigma_c\left(s(x,\theta^*),x\right)$$

or

$$\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}{U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}\leq-\sigma_c\left(s(x,\theta^*),x\right)$$

implying that for any $x > z^*(\theta^*)$,

$$\begin{aligned}&\min\left\{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_c\left(s(x,\theta^*),x\right),U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_s\left(s(x,\theta^*),x\right)\right\}\\&+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\leq 0.\end{aligned}\tag{88}$$

As a result, the right hand-side of Equation (85) is an integral of negative terms, which shows that

$$U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right)-U\left(c^*(\theta^*),s^*(\theta^*),z^*(\theta^*);\theta^*\right)\leq 0\tag{89}$$

The case with $z < z^*(\theta^*)$ follows identically, proving Lemma B3. □

Proof of Proposition A2

We now derive Proposition A2 as a consequence of Lemma B3 by deriving assumptions under which condition 2 is met for SN and LED tax systems.

SN systems First, suppose that $s < s^*(\theta)$, then $c > c^*(\theta)$. Noting that $\sigma_c = 1 - T_z'(z^*(\theta))$ is not a function of s, we can use $U_{cc}'' \leq 0$ and $U_{cs}'' \geq 0$ to obtain

$$U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)\geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right).$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we obtain

$$\begin{aligned}&U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right)+U_z'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\\&\geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right)+U_z'\left(c,s,z^*(\theta);\theta\right).\end{aligned}$$

[27]Note that having both $\mathcal{Z}$ and $\mathcal{S}$ increasing in θ also implies that $\frac{\mathcal{Z}}{\mathcal{S}} = \frac{U_z'}{U_s'}$ is increasing in θ.

Conversely, suppose that $s > s^*(\theta)$, then $c < c^*(\theta)$. We have

$$\frac{d}{ds}\left[\frac{U_s'(z - T_z(z) - s - T_s(s), s, z^*(\theta);\theta)}{1 + T_s'(s)}\right]$$
$$= -U_{cs}'' + \frac{1}{(1 + T_s'(s))}\left[U_{ss}'' - U_s'\frac{T_{ss}''(s)}{1 + T_s'(s)}\right].$$

The condition that $\frac{U_{ss}''(c(s,\theta),s,z^*(\theta);\theta)}{U_s'(c(s,\theta),s,z^*(\theta);\theta)} < \frac{T_{ss}''(s)}{1+T_s'(s)}$, together with $U_{cs}'' > 0$, implies that $\frac{U_s'(c(s,\theta),s,z^*(\theta);\theta)}{1+T_s'(s)}$ is decreasing in s and thus that

$$\frac{U_s'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)}{1 + T_s'(s^*(\theta))} \geq \frac{U_s'(c, s, z^*(\theta);\theta)}{1 + T_s'(s)}.$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, and that $\mathcal{T}_s' = T_z'(z)$ is independant of s, we obtain

$$U_s'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)\,\sigma_s(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)$$
$$\geq U_s'(c, s, z^*(\theta);\theta)\,\sigma_s(s, z^*(\theta)) + U_z'(c, s, z^*(\theta);\theta).$$

LED systems First, consider a type θ' choosing earnings $z = z^*(\theta) > z^*(\theta')$. We have

$$\frac{d}{ds}\left[U_c'(z - s - \tau_s(z^*(\theta))s - T_z(z^*(\theta)), s, z^*(\theta);\theta)\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))s\right)\right]$$
$$= U_{cs}''\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))s\right) - U_{cc}''(1 + \tau_s(z^*(\theta)))\left(1 - T_z'(z^*(\theta)) - \tau_s'(z^*(\theta))s\right) - U_c'\tau_s'(z^*(\theta)).$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z^*(\theta) > z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') < s^*(\theta)$. In this case, condition *(a)* of the proposition implies that the remaining terms are negative such that

$$U_c'(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta);\theta)\,\sigma_c(s, z^*(\theta))$$

is increasing in s for $s < s^*(\theta)$, where $\sigma_c(s, z^*(\theta)) = 1 - T_z'(z) - \tau_s'(z)s$. As a result,

$$U_c'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)\,\sigma_c(s^*(\theta), z^*(\theta))$$
$$\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta);\theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$U_c'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)\,\sigma_c(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta);\theta)$$
$$\geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta);\theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta);\theta\right).$$

Second consider a type θ' choosing $z = z^*(\theta) < z^*(\theta')$. We have

$$\frac{d}{ds}\left[U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right) \frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1+\tau_s(z)}\right]$$
$$= -U_{cs}''\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) + U_{ss}''\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1+\tau_s(z)} + U_s'\frac{\tau_s'\left(z^*(\theta)\right)}{1+\tau_s(z)}$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z = z^*(\theta) < z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') > s^*(\theta)$. Hence, condition *(b)* of the proposition implies that the remaining terms are negative such that

$$U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right)\sigma_s\left(s, z^*(\theta)\right)$$

is decreasing in s for $s > s^*(z)$, where $\sigma_s\left(s, z^*(\theta)\right) = \frac{1-T_z'(z)-\tau_s'(z)s}{1+\tau_s(z)}$. This ensures that

$$U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right)$$
$$\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)$$
$$\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right).$$

B.4 Proof of Proposition 1 (Measurement of Causal Income Effects)

We here derive the different expressions of the sufficient statistic $s_{inc}'(z)$ can be measured empirically.

Case 1. If agents preferences are weakly separable between the utility of consumption $u(.)$ and the disutility to work $k(.)$, agent θ problem writes

$$\max_{c,s,z}\; u\left(c, s; \theta\right) - k\left(z/w(\theta)\right) \;\; s.t.\; c \leq z - s - \mathcal{T}\left(s, z\right)$$

meaning that conditional on earnings z, savings $s\left(z;\theta\right)$ is defined as the solution to

$$-\left(1 + \mathcal{T}_s'\left(s, z\right)\right)u_c'\left(z - s - \mathcal{T}\left(s, z\right), s; \theta\right) + u_s'\left(z - s - \mathcal{T}\left(s, z\right), s; \theta\right) = 0.$$

Differentiating this equation with respect to savings s and earnings z yields

$$\frac{\partial s}{\partial z} = -\frac{\left[-\mathcal{T}_{sz}''u_c' - \left(1+\mathcal{T}_s'\right)\left(1-\mathcal{T}_z'\right)u_{cc}'' + \left(1-\mathcal{T}_z'\right)u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c' + \left(1+\mathcal{T}_s'\right)^2 u_{cc}'' - 2\left(1+\mathcal{T}_s'\left(s,z\right)\right)u_{cs}'' + u_{ss}''\right]}.$$

Differentiating this equation with respect to savings s and disposable income y yields

$$\frac{\partial s}{\partial y} = -\frac{\left[-\left(1+\mathcal{T}_s'\right)u_{cc}''+u_{cs}''\right]}{\left[-\mathcal{T}_{ss}''u_c'+\left(1+\mathcal{T}_s'\right)^2 u_{cc}''-2\left(1+\mathcal{T}_s'\left(s,z\right)\right)u_{cs}''+u_{ss}''\right]}.$$

Hence, if $\mathcal{T}_{sz}''=0$, we get

$$s_{inc}'\left(z\right) := \frac{\partial s\left(z;\theta\right)}{\partial z} = \left(1-\mathcal{T}_z'\right)\frac{\partial s}{\partial y} = \left(1-\mathcal{T}_z'\right)\frac{\eta_{s|z}\left(z(\theta)\right)}{1+\mathcal{T}_s'},$$

where the last equality follows from the definition of $\eta_{s|z}\left(z(\theta)\right)$. The intuition behind this result is that with separable preferences, savings s depend on earnings z only through disposable income $y = z - s - \mathcal{T}\left(s,z\right)$.

Case 2. If agents wage rates w and hours h are observable, and earnings z are given by $z = w \cdot h$, we can infer s_{inc}' from changes in wages through

$$\begin{aligned}
\frac{\partial s}{\partial w} &= \frac{\partial s\left(w\cdot h;\theta\right)}{\partial w} = \frac{\partial s\left(z;\theta\right)}{\partial z}\left(1+\frac{\partial h}{\partial w}\right)\\
\iff \frac{\partial s\left(z;\theta\right)}{\partial z} &= \frac{\frac{\partial s}{\partial w}}{1+\frac{\partial h}{\partial w}} = s\frac{\frac{w}{s}\frac{\partial s}{\partial w}}{w+h\frac{w}{h}\frac{\partial h}{\partial w}}\\
\iff s_{inc}'\left(z\right) &= s\left(z\right)\frac{\xi_w^s\left(z\right)}{w\left(z\right)+h\left(z\right)\xi_w^h\left(z\right)}
\end{aligned}$$

where $\xi_w^s\left(z\right) \equiv \frac{w(z)}{s(z)}\frac{\partial s(z)}{\partial w(z)}$ is individuals' elasticity of savings with respect to their wage rate, and $\xi_w^h\left(z\right) \equiv \frac{w(z)}{h(z)}\frac{\partial h(z)}{\partial w(z)}$ is individuals' elasticity of hours with respect to their wage rate.

Case 3. Otherwise, if we can measure the elasticity of savings s and earnings z upon a compensated change in the marginal earnings tax rate $\mathcal{T}_z'$, respectively denoted $\chi_s^c := -\frac{1-\mathcal{T}_z'}{s}\frac{\partial s}{\partial \mathcal{T}_z'}$ and $\zeta_z^c := -\frac{1-\mathcal{T}_z'}{z}\frac{\partial z}{\partial \mathcal{T}_z'}$, we then have

$$\begin{aligned}
\frac{\partial s}{\partial \mathcal{T}_z'} &= \frac{\partial s\left(z;\theta\right)}{\partial z}\frac{\partial z}{\mathcal{T}_z'}\\
\iff \left(-\frac{s}{1-\mathcal{T}_z'}\chi_s^c\right) &= s_{inc}'\left(z\right)\left(-\frac{z}{1-\mathcal{T}_z'}\zeta_z^c\right)\\
\iff s_{inc}'\left(z\right) &= \frac{s\left(z\right)}{z}\frac{\chi_s^c\left(z\right)}{\zeta_z^c\left(z\right)}.
\end{aligned}$$

B.5 Proof of Lemma 1 (Earnings Responses to Taxes on s)

Throughout the paper, we characterize earnings responses to (different) savings tax reforms using generalizations of Lemma 1 in Saez (2002). The robust insight in all cases is that a $d\tau$ increase in the marginal tax rate on s induces the same earnings changes (through substitution effects) as a

$s'_{inc}(z)d\tau$ increase in earnings tax rate. This is what appears in the body of the text as Lemma 1.

In our proofs we use a version that pertains reforms that have an LED, SL, or SN structure. For example, a reform with LED structure adds a linear tax rate $d\tau_s dz$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0 + dz\right]$. Note that the reform itself has an LED structure, but it can applied to any nonlinear tax system, not just one with an LED structure. The results below allow for multidimensional heterogeneity.

Let

$$V\left(\mathcal{T}(.,z), z; \theta\right) = \max_s\, U\left(z - s - \mathcal{T}(s,z), s, z; \theta\right)$$

be agent θ's indirect utility function at earnings z.

LED reform. Consider a tax reform $d\mathcal{T}_s$ that consists in adding a linear tax rate $d\tau_s dz$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0, z^0 + dz\right]$, that is:[28]

$$d\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ d\tau_s\left(z - z^0\right)s & if\ z \in [z^0, z^0 + dz] \\ d\tau_s\, dz\, s & if\ z \geq z^0 + dz \end{cases}$$

We now construct for each type θ a tax reform $d\mathcal{T}_z^\theta$ that affects marginal earnings tax rates, and induces the same earnings response as the initial perturbation $d\mathcal{T}_s$. We define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z) + d\mathcal{T}_s(.,z), z; \theta) = V(\mathcal{T}(.,z) + d\mathcal{T}_z^\theta(.,z), z; \theta)$$

Then, by construction, the perturbation $d\mathcal{T}_z^\theta$ induces the same earnings response dz as the initial perturbation $d\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with earnings $z(\theta) \in [z^0, z^0 + dz]$, this implies

$$\begin{aligned} U'_c\, d\tau_s\left(z - z^0\right)s\left(z;\theta\right) &= U'_c\, d\mathcal{T}_z^\theta(z) \\ \iff d\mathcal{T}_z^\theta(z) &= d\tau_s\left(z - z^0\right)s\left(z;\theta\right) \end{aligned}$$

Differentiating both sides with respect to z and letting $dz \to 0$, this implies that in the phase-in region, the reform induces the same earnings change as a small increase $s'_{inc}\left(z\right)d\tau_s$ in the marginal earnings tax rate.

[28] This reform, which is natural to consider for LED tax systems, allows us to derive a sufficient statistics characterization of the optimal smooth tax system (Theorem 2) without the requirement that $s(z)$ is monotonic. In contrast, if we were to rely on an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we would need further assumptions.

For types θ with earnings $z(\theta) \geq z^0 + dz$, this implies

$$U_c' \, d\tau_s \, dz \, s\,(z;\theta) = U_c' \, d\mathcal{T}_z^{\theta}(z)$$
$$\iff d\mathcal{T}_z^{\theta}(z) = d\tau_s \, dz \, s\,(z;\theta)$$

That is, above the phase-in region, the reform induces the same earnings changes as a $d\tau_s dz\, s\,(z)$ increase in tax liability combined with a $d\tau_s dz\, s_{inc}'\,(z)$ increase in the marginal earnings tax rate.

SL reform. Consider a tax reform $d\mathcal{T}_s$ that consists in adding a linear tax rate $d\tau_s$ on s for all individuals. This is a special case of a LED reform. As a result, we directly obtain that this reform induces the same earnings changes as a $d\tau_s\, s\,(z)$ increase in tax liability combined with a $d\tau_s\, s_{inc}'\,(z)$ increase in the marginal earnings tax rate.

SN reform. Consider a tax reform $d\mathcal{T}_s$ that consists in a small increase $d\tau_s$ in the marginal tax rate on s in a bandwidth $\left[s^0, s^0 + ds\right]$, with $d\tau_s$ much smaller than ds:

$$d\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ s \leq s^0 \\ d\tau_s(s - s^0) & if\ s \in [s^0, s^0 + ds] \\ d\tau_s ds & if\ s \geq s^0 + ds \end{cases}$$

We now construct for each type θ a perturbation of the earnings tax $d\mathcal{T}_z^{\theta}$ that induces the same earnings response as the initial perturbation $d\mathcal{T}_s$. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z) + d\mathcal{T}_s(.,z), z; \theta) = V(\mathcal{T}(.,z) + d\mathcal{T}_z^{\theta}(.,z), z; \theta)$$

Then, by construction, the perturbation $d\mathcal{T}_z^{\theta}$ induces the same earnings response dz as the initial perturbation $d\mathcal{T}_s$. Moreover, both tax reforms must induce the same utility change for type θ. To compute these utility changes, we make use of the envelope theorem.

For types θ with $s(z;\theta) \in [s^0, s^0 + ds]$, this implies

$$U_c' d\tau_s \left(s\,(z;\theta) - s^0\right) = U_c' d\mathcal{T}_z^{\theta}(z)$$
$$\iff d\mathcal{T}_z^{\theta}(z) = \left(s\,(z;\theta) - s^0\right) d\tau_s.$$

Differentiating both sides with respect to z and letting $ds \to 0$, this implies that a small increase $d\tau_s$ in the marginal tax rate on s induces the same earnings change as a small increase $s_{inc}'\,(z)\, d\tau_s$ in the marginal earnings tax rate.

For types θ with $s(z;\theta) \geq s^0 + ds$, this implies

$$U_c' \, d\tau_s \, ds = U_c' \, d\mathcal{T}_z^\theta(z)$$
$$\iff d\mathcal{T}_z^\theta(z) = d\tau_s \, ds.$$

Thus, a $d\tau_s \, ds$ lump-sum (savings) tax increase induces the same earnings change as a $d\tau_s \, ds$ lump-sum (earnings) tax increase.

B.6 Proof of Theorem 2 (Optimal Smooth Tax Systems)

When $z(\theta)$ is a strictly increasing function, we can define its inverse by $\vartheta(z)$. This allows us to define consumption of good c as $c(z) := c(z;\vartheta(z))$, consumption of good s as $s(z) := s(z;\vartheta(z))$, and the planner's weights as $\alpha(z) := \alpha(\vartheta(z))$.

In this notation, the problem of the government is to maximize the Lagrangian

$$\mathcal{L} = \int_z \Big[\alpha(z)\, U\Big(c(z), s(z), z; \vartheta(z)\Big) + \lambda\Big(\mathcal{T}(s(z), z) - E\Big)\Big] \, dH_z(z), \tag{90}$$

where λ is the social marginal value of public funds, and the tax function implicitly enters agents' utility through $c(z) = z - s(z) - \mathcal{T}(s(z), z)$.

B.6.1 Optimality Condition for Marginal Tax Rates on z

Reform. We consider a small reform at earnings level $z^0 = z(\theta^0)$ that consists in a small increase $d\tau_z$ of the marginal tax rate on earnings in a small bandwidth dz. Formally,

$$d\mathcal{T}(s,z) = \begin{cases} 0 & if \ z \leq z^0 \\ d\tau_z(z - z^0) & if \ z \in [z^0, z^0 + dz] \\ d\tau_z dz & if \ z \geq z^0 + dz \end{cases}$$

We characterize the impact of this reform on the government's objective function $\mathcal{L}$ as $dz \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*:

$$\int_{z \geq z^0} \Big(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\Big) \, d\tau_z dz \, dH_z(z)$$

- *behavioral effects from changes in z:*[29]

$$-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)d\tau_z\,dzh_z(z^0)$$
$$-\int_{z\geq z^0}\mathcal{T}_z'\left(s\left(z\right),z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}d\tau_z dz\,dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}_s'\left(s(z^0),z^0\right)s_{inc}'(z^0)\left[\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)d\tau_z\right]dzh_z(z^0)$$
$$-\int_{z\geq z^0}\mathcal{T}_s'\left(s(z),z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right]d\tau_z dz\,dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz}=\int_{z\geq z^0}\left(1-\hat{g}(z)\right)\,d\tau_z dH_z(z)$$
$$-\left(\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)d\tau_z\,h_z(z^0) \quad (91)$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}\eta_{s|z}(z).$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal schedule of marginal tax rates on z. Indeed, at the optimum the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, meaning that at each earnings z^0 the optimal marginal earnings tax rate satisfies

$$\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}=\frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0h_z(z^0)}\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} \quad (92)$$

which is the optimality condition (15) presented in Theorem 2.

B.6.2 Optimality Condition for Marginal Tax Rates on s

Reform. We consider a small reform $d\mathcal{T}_s$ that consists in adding a linear tax rate $d\tau_s dz$ on s for all individuals with earnings z above z^0, and phased-in over the earnings bandwidth $\left[z^0,z^0+dz\right]$,

[29] Note that by definition elasticity concepts include all circularities and adjustments induced by tax reforms such that the impact on z and s is given by

$$\begin{cases}dz=-\frac{z}{1-\mathcal{T}_z'}\zeta_z^c(z)d\mathcal{T}_z'(s,z)-\frac{\eta_z(z)}{1-\mathcal{T}_z'}d\mathcal{T}(s,z)\\ ds=-\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'}d\mathcal{T}(s,z)+s_{inc}'(z)dz\end{cases}$$

that is:[30]

$$d\mathcal{T}_s(s,z) = \begin{cases} 0 & if\ z \leq z^0 \\ d\tau_s\,(z - z^0)\, s & if\ z \in [z^0, z^0 + dz] \\ d\tau_s\, dz\, s & if\ z \geq z^0 + dz \end{cases}$$

Let $s^0 = s(z^0)$. We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $dz \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \geq z^0} \left(1 - \frac{\alpha(z)}{\lambda} U'_c\left(c(z), s(z), z; \theta(z)\right)\right) d\tau_s dz\, s\,(z)\ dH_z(z) \tag{93}$$

- *behavioral effects from changes* in z:[31]

$$\begin{aligned} &- \mathcal{T}'_z\left(s^0, z^0\right) \left[\frac{z^0 \zeta^c_z(z^0)}{1 - \mathcal{T}'_z\left(s^0, z^0\right)}\, d\tau_s s^0\right] h_z(z^0) dz \\ &- \int_{z \geq z^0} \mathcal{T}'_z\left(s\left(z\right), z\right) \left[\frac{z \zeta^c_z(z) s'_{inc}(z)}{1 - \mathcal{T}'_z\left(s\left(z\right), z\right)} + \frac{\eta_z(z) s(z)}{1 - \mathcal{T}'_z\left(s\left(z\right), z\right)}\right] d\tau_s dz\, dH_z(z) \end{aligned} \tag{94}$$

- *behavioral effects from changes* in s:

$$\begin{aligned} &- \mathcal{T}'_s\left(s^0, z^0\right) s'_{inc}(z^0) \left[\frac{z^0 \zeta^c_z(z^0)}{1 - \mathcal{T}'_z\left(s^0, z^0\right)}\, d\tau_s\, s^0\right] h_z(z^0) dz \\ &- \int_{z \geq z^0} \mathcal{T}'_s\left(s(z), z\right) \left[\frac{\zeta^c_{s|z}(z) + \eta_{s|z}(z)}{1 + \mathcal{T}'_s\left(s(z), z\right)}\, s(z) + s'_{inc}(z) \left[\frac{z \zeta^c_z(z) s'_{inc}(z)}{1 - \mathcal{T}'_z\left(s(z), z\right)} + \frac{\eta_z(z) s(z)}{1 - \mathcal{T}'_z\left(s(z), z\right)}\right]\right] d\tau_s dz\, dH_z(z) \end{aligned} \tag{95}$$

Summing over these different effects yields the total impact of the reform

$$\begin{aligned} &\frac{1}{\lambda} \frac{d\mathcal{L}}{d\tau_s dz} \\ &= - \frac{\mathcal{T}'_z\left(s^0, z^0\right) + s'_{inc}(z^0) \mathcal{T}'_s\left(s^0, z^0\right)}{1 - \mathcal{T}'_z\left(s^0, z^0\right)} z^0 \zeta^c_z(z^0)\, s^0\, h_z(z^0) \\ &+ \int_{z \geq z^0} \left\{\left(1 - \hat{g}(z)\right) s\left(z\right) - \frac{\mathcal{T}'_z\left(s\left(z\right), z\right) + s'_{inc}(z) \mathcal{T}'_s\left(s(z), z\right)}{1 - \mathcal{T}'_z\left(s\left(z\right), z\right)} z \zeta^c_z(z) s'_{inc}(z) - \frac{\mathcal{T}'_s\left(s(z), z\right)}{1 + \mathcal{T}'_s\left(s(z), z\right)} s(z) \zeta^c_{s|z}(z)\right\} dH_z(z) \end{aligned} \tag{96}$$

[30] This reform allows us to derive a sufficient statistics characterization of the optimal smooth tax system (Theorem 2) without the requirement that $s(z)$ is monotonic. In contrast, if we were to rely on an increase in the marginal savings tax rates over a certain bandwidth of savings, which is natural to consider for SN tax systems, we would need further assumptions.

[31] Applying Lemma 1, changes in z and s at earnings z^0 and above earnings z^0 are respectively

$$\begin{cases} dz = -\frac{z^0 \zeta^c_z(z^0)}{1 - \mathcal{T}'_z}\, d\tau_s\, s^0 \\ ds = s'_{inc}(z^0) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = -\frac{z \zeta^c_z(z)}{1 - \mathcal{T}'_z}\, d\tau_s dz\, s'_{inc}(z) - \frac{\eta_z(z)}{1 - \mathcal{T}'_z}\, d\tau_s dz\, s\,(z) \\ ds = -\frac{s(z) \zeta^c_{s|z}(z)}{1 + \mathcal{T}'_s}\, d\tau_s dz - \frac{\eta_s(z)}{1 + \mathcal{T}'_s} d\tau_s dz\ s\,(z) + s'_{inc}(z) dz \end{cases}$$

where $\hat{g}(z)$ is the social marginal welfare weight augmented with income effects, given by

$$\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\vartheta(z)\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s_{inc}'(z)\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}\eta_{s|z}(z).$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum the reform should have a zero impact on the government objective, $d\mathcal{L}=0$, which implies that at each $s^0=s\left(z^0\right)$ and earnings z^0, the optimal marginal tax rate on s satisfies

$$\begin{aligned}&\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_z'\left(s^0,z^0\right)}{1-\mathcal{T}_s'\left(s^0,z^0\right)}z^0\zeta_z^c(z^0)\,s^0\,h_z(z^0) \qquad (97)\\ &=\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\frac{\mathcal{T}_z'\left(s(z),z\right)+s_{inc}'(z)\mathcal{T}_z'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s(z),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z).\end{aligned}$$

Using the formula for the optimal schedule of marginal earnings tax rates (92) to replace the term on the left-hand side, this formula can be rearranged as

$$\begin{aligned}&s(z^0)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z) \qquad (98)\\ &=\int_{z\geq z^0}\left\{\left(1-\hat{g}(z)\right)s(z)-\frac{\mathcal{T}_z'\left(s(z),z\right)+s_{inc}'(z)\mathcal{T}_z'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s(z),z\right)}z\zeta_z^c(z)s_{inc}'(z)-\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s(z),z\right)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z).\end{aligned}$$

Differentiating both sides with respect to z^0 yields

$$\begin{aligned}&s'(z^0)\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z)-s^0(1-\hat{g}(z^0))h_z(z^0)-\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s^0\zeta_{s|z}^c(z^0)h_z(z^0)\\ &=-\left(1-\hat{g}(z^0)\right)s^0h_z(z^0)+s_{inc}'(z^0)\frac{\mathcal{T}_z'\left(s^0,z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s^0,z^0\right)}{1-\mathcal{T}_z'\left(s^0,z^0\right)}\zeta_z^c(z^0)z^0h_z(z^0)\end{aligned}$$

where both $s^0(1-\hat{g}(z^0))h_z(z^0)$ terms cancel out. Using (92) again, the last term is equal to $s_{inc}'(z^0)\int_{z\geq z^0}(1-\hat{g}(z))\,dH_z(z)$ such that we finally obtain

$$\frac{\mathcal{T}_s'\left(s^0,z^0\right)}{1+\mathcal{T}_s'\left(s^0,z^0\right)}s(z^0)\zeta_{s|z}^c(z^0)h_z(z^0)=\underbrace{\left[s'(z^0)-s_{inc}'(z^0)\right]}_{s_{pref}'(z^0)}\int_{z\geq z^0}\left(1-\hat{g}(z)\right)dH_z(z), \qquad (99)$$

which is the optimality condition (16) presented in Theorem 2.

B.6.3 Pareto-efficiency Condition

We can combine formulas for optimal marginal tax rates on z and on s to obtain a characterization of Pareto-efficiency. Indeed, leveraging the above optimal formula for marginal tax rates on s written in terms of $s_{pref}'(z^0)$, and replacing the integral term by its value from the optimal formula for marginal earnings tax rates (92) yields

$$\frac{\mathcal{T}_s'(s^0,z^0)}{1+T_s'(s^0,z^0)} s^0 \zeta_{s|z}^c(z^0) = s_{pref}'(z^0) \frac{\mathcal{T}_z'(z^0) + s_{inc}'(z^0)\mathcal{T}_s'(s^0,z^0)}{1-\mathcal{T}_z'(s^0,z^0)} z^0 \zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (17) presented in Theorem (2).

B.7 Proof of Propositions 2, A3, and A4 (Optimal Simple Tax Systems)

The derivation of optimal earnings tax formulas for simple tax systems parallels that of general smooth tax systems and the optimal formula for marginal earnings tax rates formula, (15), continues to hold. This proves Proposition A4.

Moreover, the particular linear reforms of considered in the sufficient statistics characterization of optimal marginal tax rates on s for general smooth tax systems $\mathcal{T}(s,z)$ are also available for LED tax systems. As a result, the derivation of optimal marginal tax rates on s in LED tax systems is identical to the derivation for general smooth tax systems, and the optimality formula (16) continues to hold. This, in turn, implies that the Pareto-efficiency condition (17) also holds, thereby proving all sufficient statistics characterizations for LED tax systems.

In contrast, LED reforms of tax rates on s are not available under SL and SN tax systems, and we below derive sufficient statistics characterization of optimal tax rates on s and Pareto-efficiency conditions in SL and SN tax systems.

B.7.1 SL tax system

SL tax reform. When the government uses a linear tax on s such that $\mathcal{T}(s,z) = \tau_s\, s + T_z(z)$, we consider a small reform of the linear tax rate τ_s that consists in a small increase $d\tau_s$. For an individual with earnings z, this reform increases tax liability by $d\tau_s\, s(z)$ and increases the marginal tax rate on s by $d\tau_s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_z \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \vartheta(z)\right)\right) d\tau_s\, s\,(z)\; dH_z(z) \tag{100}$$

- *behavioral effects from changes in z:*[32]

$$-\int_z T_z'(z) \left[\frac{z\zeta_z^c(z)}{1-T_z'(z)}\, d\tau_s\, s_{inc}'(z) + \frac{\eta_z(z)}{1-T_z'(z)}\, d\tau_s\, s\,(z)\right] dH_z(z) \tag{102}$$

[32] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z\zeta_z^c(z)}{1-T_z'(z)}\, d\tau_s\, s_{inc}'(z) - \frac{\eta_z(z)}{1-T_z'(z)}\, d\tau_s\, s\,(z) \\ ds = -\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\, d\tau_s - \frac{\eta_{s|z}(z)}{1+\tau_s} d\tau_s\; s\,(z) + s_{inc}'(z) dz \end{cases} \tag{101}$$

- *behavioral effects from changes in* s:

$$-\int_z \tau_s \left[\frac{s(z)\zeta^c_{s|z}(z)}{1+\tau_s} d\tau_s + \frac{\eta_{s|z}(z)}{1+\tau_s} d\tau_s\, s\left(z\right)\right] dH_z(z)$$
$$-\int_z \tau_s s'_{inc}(z) \left[\frac{z\zeta^c_z(z)}{1-T'_z} d\tau_s\, s'_{inc}(z) + \frac{\eta_z(z)}{1-T'_z} d\tau_s\, s\left(z\right)\right] dH_z(z) \quad (103)$$

Summing over these different effects yields the total impact of the reform

$$\frac{d\mathcal{L}}{\lambda} = \int_z \left\{ \left(1-\hat{g}(z)\right) s\left(z\right) - \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z\left(z\right)} z\zeta^c_z(z)\, s'_{inc}(z) - \frac{\tau_s}{1+\tau_s} s(z)\zeta^c_{s|z}(z) \right\} d\tau_s\, dH_z(z), \quad (104)$$

with social marginal welfare weights augmented with the fiscal impact of income effects given by

$$\hat{g}(z) = \frac{\alpha(z)}{\lambda} U'_c\left(c(z), s(z), z; \theta(z)\right) + \frac{T'_z(z)}{1-T'_z\left(z\right)}\eta_z(z) + \tau_s \left[\frac{\eta_{s|z}(z)}{1+\tau_s} + s'_{inc}(z)\frac{\eta_z(z)}{1-T'_z}\right].$$

Optimal linear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal linear tax rate τ_s. Indeed, at the optimum the reform should have a zero impact on the government objective meaning that the optimal τ_s satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z) dH_z(z) = \int_z \left\{ \left(1-\hat{g}(z)\right)\, s\left(z\right) - \frac{T'_z(z) + \tau_s\, s'_{inc}(z)}{1-T'_z\left(z\right)} z\zeta^c_z(z)\, s'_{inc}(z) \right\} dH_z(z). \quad (105)$$

This is Equation (48) in Proposition A3, and it holds for any (potentially suboptimal) nonlinear earnings tax schedule $T_z\left(z\right)$.

Now, let assume that the earnings tax schedule is optimal. Equation (92) applied to SL tax systemsthen implies that at each earnings z,

$$\frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} = \frac{1}{\zeta^c_z(z)} \frac{1}{z h_z(z)} \int_{x\geq z} \left(1-\hat{g}(x)\right) dH_z(x) \quad (106)$$

such that plugging in this expression to replace the last term, we obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z) dH_z(z) = \int_z \left\{ s\left(z\right)\left(1-\hat{g}(z)\right) \right\} dH_z(z) - \int_z \left\{ s'_{inc}(z) \int_{x\geq z} \left(1-\hat{g}(x)\right) h_z(x) dx \right\} dz \quad (107)$$

Defining $s_{inc}(z) \equiv \int_{x=0}^{z} s'_{inc}\left(x\right) dx$, we can now integrate the last term to reexpress it as[33]

$$\int_z \left\{ s'_{inc}(z) \int_{x\geq z} \left(1-\hat{g}(x)\right) h_z(x) dx \right\} dz = \int_z \left\{ s_{inc}(z) (1-\hat{g}(z))\, h_z(z) \right\} dz \quad (108)$$

[33]Define $\phi\left(z\right) = \int_{x=0}^{z} s'_{inc}\left(x\right) dx$ such that $\phi'\left(z\right) = s'_{inc}\left(z\right)$ and $\psi\left(z\right) = \int_{x=z}^{z_{max}} \left(1-\hat{g}(x)\right) h_z(x) dx$ such that $\psi'\left(z\right) = -\left(1-\hat{g}(z)\right) h_z(z)$, and apply

$$\int_{x=z}^{z_{max}} \left[\phi'\left(x\right)\psi\left(x\right)\right] dx = \left[\phi\left(z_{max}\right)\psi\left(z_{max}\right) - \phi\left(z\right)\psi\left(z\right)\right] - \int_{x=z}^{z_{max}} \left[\phi\left(x\right)\psi'\left(x\right)\right] dx$$

to obtain

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z)dH_z(z) = \int_z \left\{ \left[s\left(z\right) - s_{inc}(z)\right]\left(1-\hat{g}(z)\right)\right\} dH_z(z). \tag{109}$$

Note that here $\int_z \frac{s(z)}{1+\tau_s}\zeta^c_{s|z}(z)dH_z(z)$ is the aggregate population responses to a change in τ_s. Defining $\overline{\zeta^c_{s|z}}$ as the aggregate elasticity of $\overline{s} := \int_z s(z)dH_z(z)$, we can rewrite this term as $\frac{\overline{s}}{1+\tau_s}\overline{\zeta^c_{s|z}}$ such that

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}}\int_z s_{pref}\left(z\right)\left(1-\hat{g}(z)\right) dH_z(z). \tag{110}$$

This is Equation (20) in Proposition 2. Integrating by part the right-hand side, this formula is also equivalent to

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\overline{s}\overline{\zeta^c_{s|z}}}\int_z \left[s'_{pref}\left(z\right)\int_{x\geq z}\left(1-\hat{g}(x)\right) dH_z(x)\right] dz. \tag{111}$$

Pareto-efficiency for SL tax systems. To characterize Pareto-efficiency, we combine tax reforms in a way that anihilates all lump-sum changes in tax liability, thereby offsetting all utility changes.

We start with a small reform of the linear tax rate τ_s that consists in small increase $d\tau_s$. At the bottom of the earnings distribution ($z = z_{min}$), the mechanical effect of the reform is an increase in tax liability by $s\left(z_{min}\right)d\tau_s$. We thus adjust the earnings tax liability through a downward lump-sum shift by $s\left(z_{min}\right)d\tau_s$ at all earnings levels. This joint reform has the following impact on the government objective

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}}^{z_{max}}\left\{\left[1-\hat{g}(z)\right]\left[s\left(z\right)-s\left(z_{min}\right)\right] - \frac{T'_z(z)+\tau_s\, s'_{inc}(z)}{1-T'_z} z\zeta^c_z(z)\, s'_{inc}(z) - \tau_s\frac{s(z)}{1+T'_s}\zeta^c_{s|z}(z)\right\} d\tau_s\, dH_z(z) \tag{112}$$

meaning that the lump-sum change in tax liability is nil at earnings $z = z_{min}$, but not at earnings $z \geq z_{min}$.

To cancel out lump-sum changes in tax liability at all earnings levels, we construct a sequence of earnings tax reforms. We discretize the range of earnings $[z_{min}, z_{max}]$ into N bins and consider reforms in the small earnings bandwidths $dz = \frac{\Delta z}{N}$ where $\Delta z = z_{max} - z_{min}$. We proceed by induction to derive a general formula:

- First, consider a decrease in the marginal earnings tax rate by $d\tau_z = s'\left(z_{min}\right)d\tau_s$ over the bandwidth $[z_{min}, z_{min} + dz]$. In this bandwidth, this additional reform (i) cancels out lump-sum changes in tax liability to a first-order approximation since $\left[s\left(z_{min}+dz\right) - s\left(z_{min}\right)\right]d\tau_s \approx s'\left(z_{min}\right)dzd\tau_s$, and (ii) induces earnings responses through the change in marginal tax rates. Moreover, it also decreases the lump-sum tax liability on all individuals with earnings $z \geq$

$z_{min} + dz$ by $s'(z_{min})\, dz d\tau_s$. The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}+dz}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - s'(z_{min})\, dz \right] \right\} d\tau_s\, dH_z(z)$$
$$- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z)$$
$$+ \int_{z=z_{min}}^{z_{min}+dz} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min})\, d\tau_s \right) dH_z(z) \quad (113)$$

- Second, consider a decrease in the marginal earnings tax rate by $d\tau_z = s'(z_{min} + dz)\, d\tau_s$ over the bandwidth $[z_{min} + dz, z_{min} + 2dz]$. This again cancels out lump-sum changes in this bandwidth up to a first-order approximation since $[s(z_{min} + 2dz) - s(z_{min}) - s'(z_{min})\, dz] \approx s'(z_{min} + dz)\, dz$. The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}+2dz}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - s'(z_{min})\, dz - s'(z_{min}+dz)\, dz \right] \right\} d\tau_s\, dH_z(z)$$
$$- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z)$$
$$+ \int_{z=z_{min}}^{z_{min}+dz} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min})\, d\tau_s \right) dH_z(z)$$
$$+ \int_{z=z_{min}+dz}^{z_{min}+2dz} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) \left(s'(z_{min}+dz)\, d\tau_s \right) dH_z(z) \quad (114)$$

- Iterating over to step k, in which we consider a decrease in the marginal earnings tax rate by $d\tau_z = s'\left(z_{min} + (k-1)\frac{\Delta z}{N}\right) d\tau_s$ over the bandwidth $[z_{min} + (k-1)\frac{\Delta z}{N}, z_{min} + k\frac{\Delta z}{N}]$. The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=z_{min}+k\frac{\Delta z}{N}}^{z_{max}} \left\{ [1-\hat{g}(z)] \left[s(z) - s(z_{min}) - \frac{\Delta z}{N} \left[\sum_{p=0}^{k-1} s'\left(z_{min} + p\frac{\Delta z}{N} \right) \right] \right] \right\} d\tau_s\, dH_z(z)$$
$$- \int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z)$$
$$+ \sum_{p=0}^{k-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N} \right) d\tau_s\, dH_z(z) \quad (115)$$

- Pushing the iteration forward until $k = N$, the first integral disappears (integration over an empty set) such that the total impact of this sequence of reforms is given by

$$\frac{d\mathcal{L}}{\lambda} = -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z) + \sum_{p=0}^{N-1} \int_{z=z_{min}+p\frac{\Delta z}{N}}^{z_{min}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(z_{min} + p\frac{\Delta z}{N}\right) d\tau_s\, dH_z(z) \quad (116)$$

Let's now compute the last term at the limit $N \to \infty$. Denoting $z^p := z_{min} + p\frac{\Delta z}{N}$, we have

$$\begin{aligned} &\sum_{p=0}^{N-1} \int_{z=z^p}^{z^p+\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'(z^p)\, d\tau_s\, dH_z(z) \\ &\approx \sum_{p=0}^{N-1} \frac{T_z'(z^p) + s_{inc}'(z^p)\tau_s}{1-T_z'(z^p)} (z^p)\, \zeta_z^c(z^p)\, s'(z^p)\, d\tau_s h_z(z^p) \frac{\Delta z}{N} \\ &\underset{N\to\infty}{\longrightarrow} \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, d\tau_s\, h_z(z)\, dz \end{aligned} \quad (117)$$

where the last line follows from the (Riemann) definition of the integral in terms of Riemann sums. Hence, the total impact of this sequence of reforms is at the limit given by

$$\frac{d\mathcal{L}}{\lambda} = -\int_{z=z_{min}}^{z_{max}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z) \right\} d\tau_s\, h_z(z) dz + \int_{z=z_{min}}^{z_{max}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, d\tau_s\, h_z(z)\, dz \quad (118)$$

By construction, the sequence of reforms we have constructed does not affect agents' utility, and only affects tax revenue through the expression above. When the impact of this reform is non-zero, the type of sequence of reforms we consider delivers a Pareto-improvement over the existing tax system. Characterizing a Pareto-efficient SL tax system as one that cannot be reformed in a Pareto-improving way yields the following Pareto-efficiency formula

$$\frac{\tau_s}{1+\tau_s} \int_z s(z)\zeta_{s|z}^c(z) h_z(z) dz = \int_z \underbrace{\left[s'(z) - s_{inc}'(z)\right]}_{s_{pref}'(z)} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, h_z(z)\, dz, \quad (119)$$

which is Equation (24) in Proposition 2.

B.7.2 SN tax systems

SN tax reform. When the government uses a SN tax system such that $\mathcal{T}(s,z) = T_s(s) + T_z(z)$, we consider a small reform of the tax on s at $s^0 = s(\theta^0)$ that consists in a small increase $d\tau_s$ of the marginal tax rate on s in a small bandwidth ds. Formally,

$$d\mathcal{T}(s,z) = \begin{cases} 0 & if\ s \leq s^0 \\ d\tau_s(s-s^0) & if\ s \in [s^0, s^0+ds] \\ d\tau_s ds & if\ s \geq s^0 + ds \end{cases}$$

Assuming there exists a strictly increasing mapping between z and s, we denote z^0 the earnings level such that $s^0 = s(z^0)$.[34] We characterize the impact of this reform on the government objective function $\mathcal{L}$ as $ds \to 0$. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \geq z^0} \left(1 - \frac{\alpha(z)}{\lambda} U'_c\left(c(z), s(z), z; \vartheta(z)\right)\right) d\tau_s ds\, dH_z(z)$$

- *behavioral effects from changes in z:*[35]

$$-\mathcal{T}'_z(s^0,z^0)\left[\frac{z^0}{1-\mathcal{T}'_z(s^0,z^0)}\zeta^c_z(z^0)\, s'_{inc}(z) d\tau_s\right] ds\, \frac{h_z(z^0)}{s'(z^0)} - \int_{z \geq z^0} \mathcal{T}'_z\left(s,z\right) \frac{\eta_z(z)}{1-\mathcal{T}'_z\left(s,z\right)} d\tau_s ds\, dH_z(z)$$

- *behavioral effects from changes in s:*

$$-\mathcal{T}'_s\left(s^0,z^0\right)\left[\frac{s^0}{1+\mathcal{T}'_s\left(s^0,z^0\right)}\zeta^c_{s|z}(z^0) d\tau_s + s'_{inc}(z^0)\frac{z^0}{1-\mathcal{T}'_z\left(s^0,z^0\right)}\zeta^c_z(z^0)\, s'_{inc}(z^0) d\tau_s\right] ds\, \frac{h_z(z^0)}{s'(z^0)}$$
$$- \int_{z \geq z^0} \mathcal{T}'_s\left(s,z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}'_s\left(s,z\right)} + s'_{inc}(z)\frac{\eta_z(z)}{1-\mathcal{T}'_z\left(s,z\right)}\right] d\tau_s ds\, dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{ds} = s'\left(z^0\right)\int_{z \geq z^0}\left(1-\hat{g}(z)\right) d\tau_s\, dH_z(z) - \left\{\mathcal{T}'_s\left(s^0,z^0\right)\frac{s^0}{1+\mathcal{T}'_s\left(s^0,z^0\right)}\zeta^c_{s|z}(z^0)\right.$$
$$\left. + \left[\mathcal{T}'_z(s^0,z^0) + s'_{inc}(z^0)\mathcal{T}'_s(s^0,z^0)\right]\frac{z^0}{1-\mathcal{T}'_z\left(s^0,z^0\right)}\zeta^c_z(z^0) s'_{inc}(z^0)\right\} d\tau_s\, h_z(z^0) \qquad (121)$$

Optimal nonlinear tax rate on s. A direct implication of this result is a sufficient statistics characterization of the optimal marginal tax rates on s. Indeed, at the optimum the reform should

[34] Our sufficient statistic characterization fundamentally relies on monotonicity of the function $s(z)$. Hence, it is also valid if we assume a strictly decreasing mapping $s(z)$. Moreover, it can be extended to weakly monotonic $s(z)$ (i.e. non-decreasing or non-increasing) with slight modifications.

[35] Applying Lemma 1, changes in z and s are here given by

$$\begin{cases} dz = -\frac{z}{1-T'_z}\zeta^c_z(z)\delta T^{\theta\prime}_z - \frac{\eta_z(z)}{1-T'_z}\delta T^\theta_z \\ ds = -\frac{s(z)}{1+T'_s}\zeta^c_{s|z}(z)\delta T'_s - \frac{\eta_{s|z}(z)}{1+T'_s}\delta T_s + s'_{inc}(z)dz \end{cases} \qquad (120)$$

where T^θ_z is a $s'_{inc}(z)d\tau_s$ increase in the marginal earnings tax rate when $s \in [s^0, s^0+ds]$, and a $d\tau_s ds$ increase in tax liability when $s \geq s^0 + ds$. Moreover, the mass of individuals in the bandwitdh is $ds\, h_s(s(z^0)) = ds\, \frac{h_z(z^0)}{s'(z^0)}$.

have a zero impact on the government objective, $d\mathcal{L} = 0$, which implies that at each $s^0 = s\left(z^0\right)$ the optimal marginal tax rate on s satisfies

$$\frac{\mathcal{T}_s'(s^0, z^0)}{1 + \mathcal{T}_s'(s^0, z^0)} s^0 \zeta_{s|z}^c(z^0) h_z(z^0)$$
$$= s'\left(z^0\right) \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z) - s_{inc}'(z^0) \frac{\mathcal{T}_z'(s^0, z^0) + s_{inc}'(z^0)\mathcal{T}_s'(s^0, z^0)}{1 - \mathcal{T}_z'(s^0, z^0)} z^0 \zeta_z^c(z^0) h_z(z^0) \tag{122}$$

which is Equation (46) in Proposition A3, recognizing that $\mathcal{T}_z'(s,z) = T_z'(z)$ and $\mathcal{T}_s'(s,z) = T_s'(s)$. This characterization holds for any (potentially optimal) nonlinear earnings tax schedule $T_z(z)$.

Now, let further assume that the earnings tax schedule is optimal. Equation (92) applied to SN tax systems then implies that at each earnings z^0,

$$\frac{T_z'(z^0) + s_{inc}'(z^0) T_s'(s(z^0))}{1 - T_z'(z^0)} = \frac{1}{\zeta_z^c(z^0)} \frac{1}{z^0 h_z(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z). \tag{123}$$

Using this expression to substitute the last term in the formula for optimal marginal tax rates on s yields

$$\frac{T_s'(s^0)}{1 + T_s'(s^0)} s^0 \zeta_{s|z}^c(z^0) h_z(z^0) = \underbrace{\left[s'(z^0) - s_{inc}'(z^0)\right]}_{s_{pref}'(z^0)} \int_{z \geq z^0} (1 - \hat{g}(z))\, dH_z(z)$$

which is Equation (22) in Proposition 2.

Pareto-efficiency for SN tax systems. We can combine formulas for optimal marginal tax rates on s and z to obtain a characterization of Pareto-efficiency. Indeed, leveraging the previous optimal formula for marginal tax rates on s written in terms of $s_{pref}'(z^0)$, and replacing the integral term by its value given from the optimal formula for marginal earnings tax rates (97) yields

$$\frac{\mathcal{T}_s'(s^0, z^0)}{1 + T_s'(s^0, z^0)} s^0 \zeta_{s|z}^c(z^0) = s_{pref}'(z^0) \frac{\mathcal{T}_z'(z^0) + s_{inc}'(z^0)\mathcal{T}_s'(s^0, z^0)}{1 - \mathcal{T}_z'(s^0, z^0)} z^0 \zeta_z^c(z^0)$$

which is the Pareto-efficiency condition (25) presented in Proposition 2, recognizing that $\mathcal{T}_z'(s,z) = T_z'(z)$ and $\mathcal{T}_s'(s,z) = T_s'(s)$.

B.8 Proof of Proposition 3 (Simple Tax systems and Multidimensional Heterogeneity)

We characterize in Proposition 3 optimal savings tax formulas for each type of simple tax system in the presence of multidimensional heterogeneity. These formulas take the actual earnings tax schedule as given, be they optimally set or not, and extend the results derived in the unidimensional case. Crucially, we are able to provide similar characterizations because Lemma 1 still holds in the presence of multidimensional heterogeneity.

B.8.1 Separable linear (SL) tax system

Consider a reform that consists in a $d\tau_s$ increase in the linear savings tax rate τ_s. For all agents, this triggers an increase in tax liability by $s\, d\tau_s$ and an increase in the marginal savings tax rate by $d\tau_s$ – which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s'_{inc} d\tau_s$.

This reform has the following effects on the government objective:

- mechanical effects

$$\int_z \int_s \Big[(1 - g(s,z))\, s\, d\tau_s \Big] h(s,z)\, ds dz \tag{124}$$

$$= \int_z \mathbb{E}\left[(1 - g(s,z))\, s | z \right] \, d\tau_s h_z(z)\, dz \tag{125}$$

- earnings responses[36]

$$\int_z T'_z(z) \left\{ \int_s \left(-\frac{z}{1 - T'_z(z)} \zeta^c_z(s,z)\, s'_{inc}(s,z)\, d\tau_s - \frac{\eta_z(s,z)}{1 - T'_z(z)} s\, d\tau_s \right) h(s,z)\, ds \right\} dz \tag{126}$$

$$= - \int_z \frac{T'_z(z)}{1 - T'_z(z)} \left\{ \mathbb{E}\left[z \zeta^c_z(s,z) s'_{inc}(s,z) + \eta_z(s,z) s | z \right] \right\} d\tau_s h_z(z)\, dz \tag{127}$$

- savings responses[37]

$$\tau_s \int_z \int_s \Bigg\{ -\frac{s}{1+\tau_s} \zeta^c_{s|z}(s,z)\, d\tau_s - \frac{\eta_{s|z}(s,z)}{1+\tau_s} s\, d\tau_s \tag{128}$$
$$+ s'_{inc}(s,z) \left(-\frac{z}{1 - T'_z(z)} \zeta^c_z(s,z)\, s'_{inc}(s,z)\, d\tau_s - \frac{\eta_z(s,z)}{1 - T'_z(z)} s\, d\tau_s \right) \Bigg\} h(s,z)\, ds dz$$

$$= -\tau_s \int_z \Bigg\{ \frac{1}{1+\tau_s} \mathbb{E}\left[s \zeta^c_{s|z}(s,z) + \eta_{s|z}(s,z)\, s | z \right] \tag{129}$$
$$+ \frac{1}{1 - T'_z(z)} \left(\mathbb{E}\left[z \zeta^c_z(s,z)\ \left(s'_{inc}(s,z) \right)^2 + \eta_z(s,z)\, s\, s'_{inc}(s,z)\, | z \right] \right) \Bigg\} d\tau_s h_z(z)\, dz$$

[36]We have

$$dz = \frac{\partial z}{\partial T'_z} \delta T'_z + \frac{\partial z}{\partial T_z} \delta T_z = -\frac{z}{1 - T'_z} \zeta^c_z\, \delta T'_z - \frac{\eta_z}{1 - T'_z} \delta T_z$$

[37]We have

$$ds = \frac{\partial s}{\partial T'_s} \delta T'_s + \frac{\partial s}{\partial T_s} \delta T_s + \frac{ds(z;\theta)}{dz} dz = -\frac{s}{1 + T'_s} \zeta^c_{s|z}\, \delta T'_s - \frac{\eta_{s|z}}{1 + T'_s} \delta T_s + s'_{inc} dz$$

Such that the total impact of the reform on the government objective is

$$\frac{d\mathcal{L}}{d\tau_s} = \int_z \mathbb{E}\left[(1-g(s,z))\, s|z\right]\, h_z(z)\, dz \tag{130}$$

$$-\int_z \frac{T_z'(z)}{1-T_z'(z)} \left\{ \mathbb{E}\left[z\zeta_z^c(s,z) s_{inc}'(s,z) + \eta_z(s,z)\, s|z\right] \right\} h_z(z)\, dz \tag{131}$$

$$-\tau_s \int_z \left\{ \frac{1}{1+\tau_s} \mathbb{E}\left[s\zeta_{s|z}^c(s,z) + \eta_{s|z}(s,z)\, s|z\right] \right.$$

$$\left. + \frac{1}{1-T_z'(z)} \left(\mathbb{E}\left[z\zeta_z^c(s,z)\, \left(s_{inc}'(s,z)\right)^2 + \eta_z(s,z)\, s\, s_{inc}'(s,z)\, |z\right] \right) \right\} h_z(z)\, dz \tag{132}$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T_z'(z)}{1-T_z'(z)} \eta_z(s,z) + \frac{\tau_s}{1+\tau_s} \eta_{s|z}(s,z) + \frac{\tau_s}{1-T_z'(z)} \eta_z(s,z) s_{inc}'(s,z) \tag{133}$$

we finally get

$$\frac{d\mathcal{L}}{d\tau_s} = \int_z \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z\right]\, h_z(z)\, dz - \int_z \frac{T_z'(z)}{1-T_z'(z)} \left\{ \mathbb{E}\left[z\zeta_z^c(s,z) s_{inc}'(s,z)\, |z\right] \right\} h_z(z)\, dz \tag{134}$$

$$-\tau_s \int_z \left\{ \frac{1}{1+\tau_s} \mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right] + \frac{1}{1-T_z'(z)} \left(\mathbb{E}\left[z\zeta_z^c(s,z)\, \left(s_{inc}'(s,z)\right)^2 |z\right] \right) \right\} h_z(z)\, dz$$

The optimal savings linear tax rate τ_s thus satisfies

$$\frac{\tau_s}{1+\tau_s} \int_z \left\{ \mathbb{E}\left[s\zeta_{s|z}^c(s,z)|z\right] \right\} dH_z(z) \tag{135}$$

$$= \int_z \left\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z\right] - \mathbb{E}\left[\frac{T_z'(z) + s_{inc}'(s,z)\,\tau_s}{1-T_z'(z)} z\zeta_z^c(s,z) s_{inc}'(s,z)\, |z\right] \right\} dH_z(z)$$

B.8.2 Separable nonlinear (SN) tax system

Consider a reform that consists in a small $\delta\tau_s$ increase in the marginal savings tax rate across the savings bandwidth $\left[s^0, s^0+\delta s\right]$.[38] For all agents with savings above s^0, this triggers a $\delta s\, \delta\tau_s$ increase in tax liability. For agents at s^0, this triggers a $\delta\tau_s$ increase in the marginal savings tax rate – which by Lemma 1 produces earnings responses equivalent to a $s_{inc}'\delta\tau_s$ increase in the marginal earnings tax rate.

This reform has the following effects on the government objective:

- mechanical effects

$$\int_{s\geq s^0} \int_z \left\{ (1-g(s,z))\, \delta s \delta\tau_s \right\} h(s,z)\, dsdz \tag{136}$$

$$= \int_z \left\{ \mathbb{E}\left[1-g(s,z)\, |z, s\geq s^0\right] \right\} \delta s \delta\tau_s h_z(z)\, dz$$

[38] To avoid any ambiguity in this multidimensional context, we here use d for integration and δ for attributes of the reform we consider.

- earnings responses[39]

$$-\int_z T_z'(z)\left\{\frac{z}{1-T_z'}\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)\delta\tau_s\right\}\delta s\, h\left(s^0,z\right)dz \tag{137}$$
$$-\int_{s\geq s^0}\int_z T_z'(z)\left\{\frac{\eta_z(s,z)}{1-T_z'(z)}\delta\tau_s\delta s\right\}h(s,z)\,dsdz$$
$$=-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)s|z,s\geq s^0\right]\right\}\delta\tau_s\delta s\, h_z(z)\,dz$$

- savings responses[40]

$$-T_s'\left(s^0\right)\int_z\left\{\frac{s^0}{1+T_s'\left(s^0\right)}\zeta_{s|z}^c(s^0,z)\,\delta\tau_s+s_{inc}'\left(s^0,z\right)\frac{z}{1-T_z'(z)}\zeta_z^c(s^0,z)\,s_{inc}'\left(s^0,z\right)\delta\tau_s\right\}\delta s\, h\left(s^0,z\right)dz \tag{138}$$
$$-\int_{s\geq s^0}\int_z\left\{T_s'(s)\left(\frac{\eta_{s|z}(s,z)}{1+T_s'(s)}\delta s\delta\tau_s+s_{inc}'(s,z)\frac{\eta_z(s,z)}{1-T_z'(z)}\delta s\delta\tau_s\right)\right\}h(s,z)\,dsdz$$
$$=-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}\delta\tau_s\delta s\, h_z(z)\,dz \tag{139}$$
$$-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}\delta s\delta\tau_s h_z(z)\,dz$$

Such that the total impact of the reform on the government objective is

$$\frac{d\mathcal{L}}{\delta s\delta\tau_s}=\int_z\left\{\mathbb{E}\left[1-g(s,z)\,|z,s\geq s^0\right]\right\}dH_z(z) \tag{140}$$
$$-\int_z\frac{T_z'(z)}{1-T_z'}\left\{z\zeta_z^c\left(s^0,z\right)s_{inc}'\left(s^0,z\right)+\mathbb{E}\left[\eta_z(s,z)|z,s\geq s^0\right]\right\}dH_z(z)$$
$$-\int_z\left\{\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c(s^0,z)+\frac{T_s'\left(s^0\right)}{1-T_z'(z)}s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z)\right\}dH_z(z)$$
$$-\int_z\left\{\mathbb{E}\left[\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^0\right]+\mathbb{E}\left[s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z)\middle|z,s\geq s^0\right]\right\}dH_z(z)$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z)=g(s,z)+\frac{T_z'(z)}{1-T_z'(z)}\eta_z(s,z)+\frac{T_s'(s)}{1+T_s'(s)}\eta_{s|z}(s,z)+s_{inc}'(s,z)\frac{T_s'(s)}{1-T_z'(z)}\eta_z(s,z) \tag{141}$$

[39]We have
$$dz=\frac{\partial z}{\partial T_z'}\delta T_z'+\frac{\partial z}{\partial T_z}\delta T_z=-\frac{z}{1-T_z'}\zeta_z^c(z)\,\delta T_z'-\frac{\eta_z(z)}{1-T_z'}\delta T_z$$

[40]We have
$$ds=\frac{\partial s}{\partial T_s'}\delta T_s'+\frac{\partial s}{\partial T_s}\delta T_s+\frac{ds(z;\theta)}{dz}dz=-\frac{s}{1+T_s'}\zeta_{s|z}^c\,\delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\delta T_s+s_{inc}'(z)dz$$

we finally get

$$\frac{d\mathcal{L}}{\delta s \delta \tau_s} = \int_z \Big\{ \mathbb{E}\left[1-\hat{g}\left(s,z\right)|z, s\geq s^0\right] \Big\} dH_z\left(z\right) - \int_z \left\{ \frac{T_z'\left(z\right)}{1-T_z'} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right) \right\} dH_z\left(z\right) \quad (142)$$
$$- \int_z \left\{ \frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} s^0 \zeta_{s|z}^c(s^0,z) + \frac{T_s'\left(s^0\right)}{1-T_z'\left(z\right)} s_{inc}'\left(s^0,z\right)^2 z\zeta_z^c(s^0,z) \right\} dH_z\left(z\right)$$

The optimal marginal savings tax rate $T_s'(.)$ thus satisfies at each savings s^0,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)} \int_z \left\{ s^0 \zeta_{s|z}^c(s^0,z) \right\} dH_z\left(z\right) \quad (143)$$
$$= \int_z \Big\{ \mathbb{E}\left[1-\hat{g}\left(s,z\right)|z, s\geq s^0\right] \Big\} dH_z\left(z\right) - \int_z \left\{ \frac{T_z'\left(z\right) + s_{inc}'\left(s^0,z\right) T_s'\left(s^0\right)}{1-T_z'\left(z\right)} z\zeta_z^c\left(s^0,z\right) s_{inc}'\left(s^0,z\right) \right\} dH_z\left(z\right)$$

which is equation (52) in Proposition (A4).

B.8.3 Linear earnings-dependent (LED) tax system

Consider a reform that consists in a $\delta\tau_s\delta z$ increase in the linear savings tax rate phased-in across the earnings bandwidth $\left[z^0, z^0+\delta z\right]$.[41]

For all agents with earnings above z^0, this triggers an increase in the linear savings tax rate by $\delta\tau_s\delta z$ meaning that the marginal savings tax rate increases by the same magnitude – which triggers by Lemma 1 earnings responses equivalent to those induced by a $s_{inc}'\delta\tau_s\delta z$ increase in the marginal earnings tax rate – and that agents' tax liability increases by $s\,\delta\tau_s\delta z$.

For agents at z^0, the only direct effect of the reform is to induce earnings responses which by Lemma 1 are equivalent to an increase in the marginal earnings tax rate given by $s\,\delta\tau_s$ (as in the unidimensional case).

This reform has the following effects on the government objective:

- mechanical effects[42]

$$\int_{z\geq z^0} \int_s \Big\{ \left(1-g\left(s,z\right)\right) \delta z \delta\tau_s s \Big\} h\left(s,z\right) ds dz \quad (144)$$
$$= \int_{z\geq z^0} \Big\{ E_s\Big[\left(1-g\left(s,z\right)\right) s \Big| z \Big] \delta z \delta \tau_s \Big\} h_z\left(z\right) dz$$

[41] To avoid any ambiguity, we here use d for integration and δ for attributes of the reform we consider.

[42] We use

$$h\left(s,z\right) = h\left(s|z\right) h_z\left(z\right)$$

- earnings responses[43]

$$
\begin{aligned}
&-\int_s \left(T_z'\left(z^0\right)+\tau_s'\left(z^0\right)s\right)\left\{\frac{z^0}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s}\zeta_z^c(s,z^0)\,s\,\delta\tau_s\right\}\delta z\, h\left(s,z^0\right)ds \qquad (145)\\
&-\int_{z\geq z^0}\int_s \left(T_z'\left(z\right)+\tau_s'\left(z\right)s\right)\left\{\frac{z}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\zeta_z^c(s,z)s_{inc}'\left(s,z\right)\delta z\delta\tau_s+\frac{\eta_z(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\,s\,\delta z\delta\tau_s\right\}h\left(s,\right.\\
&=-\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s\middle|z=z^0\right]\delta z\delta\tau_s\, h_z\left(z^0\right) \qquad (146)\\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{T_z'\left(z\right)+\tau_s'\left(z\right)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\Big(z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)+s\,\eta_z(s,z)\Big)\middle|z\right]\right\}\delta z\delta\tau_s h_z\left(z\right)dz
\end{aligned}
$$

- savings responses[44]

$$
\begin{aligned}
&-\tau_s\left(z^0\right)\int_s s_{inc}'\left(s,z^0\right)\left\{\frac{z^0}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)s}\zeta_z^c(s,z^0)\,s\,\delta\tau_s\right\}\delta z\, h\left(s,z^0\right)ds \qquad (147)\\
&-\int_{z\geq z^0}\int_s \tau_s\left(z\right)\left\{\frac{s}{1+\tau_s\left(z\right)}\zeta_{s|z}^c(s,z)\,\delta z\delta\tau_s+s_{inc}'\left(s,z\right)\frac{z}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\zeta_z^c(s,z)\,s_{inc}'\left(s,z\right)\delta z\delta\tau_s\right\}h\left(s,z\right)d\\
&-\int_{z\geq z^0}\int_s\left\{\tau_s\left(z\right)\left(\frac{\eta_{s|z}(s,z)}{1+\tau_s\left(z\right)}\,s\,\delta z\delta\tau_s+s_{inc}'\left(s,z\right)\frac{\eta_z(s,z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\,s\,\delta z\delta\tau_s\right)\right\}h\left(s,z\right)dsdz\\
&=-\tau_s\left(z^0\right)\mathbb{E}\left[s_{inc}'\left(s,z\right)\frac{z\zeta_z^c(s,z)s}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}\middle|z=z^0\right]\delta z\delta\tau_s\, h_z\left(z^0\right) \qquad (148)\\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}z\zeta_z^c(s,z)s_{inc}'\left(s,z\right)^2\right]\right\}\delta z\delta\tau_s\, h_z\left(z\right)dz\\
&-\int_{z\geq z^0}\left\{\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{\tau_s\left(z\right)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s}s\eta_z(s,z)s_{inc}'\left(s,z\right)\middle|z\right]\right\}\delta z\delta\tau_s\, h_z\left(z\right)dz
\end{aligned}
$$

Such that the total impact of the reform on the government objective is

[43] We have

$$dz=\frac{\partial z}{\partial \mathcal{T}_z'}\delta\mathcal{T}_z'+\frac{\partial z}{\partial \mathcal{T}_z}\delta\mathcal{T}_z=-\frac{z}{1-\mathcal{T}_z'}\zeta_z^c\,\delta\mathcal{T}_z'-\frac{\eta_z}{1-\mathcal{T}_z'}\delta\mathcal{T}_z$$

[44] We have

$$ds=\frac{\partial s}{\partial T_s'}\delta T_s'+\frac{\partial s}{\partial T_s}\delta T_s+\frac{ds(z;\theta)}{dz}dz=-\frac{s}{1+T_s'}\zeta_{s|z}^c\,\delta T_s'-\frac{\eta_{s|z}}{1+T_s'}\delta T_s+s_{inc}'(z)dz$$

$$
\begin{aligned}
\frac{d\mathcal{L}}{\delta s \delta \tau_s} &= \int_{z \geq z^0} \left\{ E_s\left[(1-g(s,z))\, s \middle| z \right] \right\} h_z(z)\, dz \qquad (149)\\
&- \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) \\
&- \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s}{1-T_z'(z)-\tau_s'(z)\, s} \left(z\zeta_z^c(s,z) s_{inc}'(s,z) + s\, \eta_z(s,z) \right) \middle| z \right] \right\} h_z(z)\, dz \\
&- \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s \middle| z = z^0 \right] \delta z \delta \tau_s\, h_z\left(z^0\right) \\
&- \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta_{s|z}^c(s,z) \middle| z \right] + \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s_{inc}'(s,z) \middle| z \right] \right\} h_z(z)\, dz \qquad (150)\\
&- \int_{z \geq z^0} \left\{ \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\eta_{s|z}(s,z) \middle| z \right] + \mathbb{E}\left[\frac{s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} s\eta_z(s,z) \middle| z \right] \right\} h_z(z)\, dz
\end{aligned}
$$

Redefining augmented social marginal welfare weights as

$$
\hat{g}(s,z) = g(s,z) + \frac{T_z'(z)+\tau_s'(z)\, s + s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} \eta_z(s,z) + \frac{\tau_s(z)}{1+\tau_s(z)} \eta_{s|z}(s,z) \qquad (151)
$$

we finally get

$$
\begin{aligned}
\frac{d\mathcal{L}}{\delta s \delta \tau_s} &= \int_{z \geq z^0} \mathbb{E}\left[(1-\hat{g}(s,z))\, s \middle| z \right] h_z(z)\, dz \qquad (152)\\
&- \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) \\
&- \int_{z \geq z^0} \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s_{inc}'(s,z) \Big) \middle| z \right] h_z(z)\, dz \\
&- \int_{z \geq z^0} \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta_{s|z}^c(s,z) \middle| z \right] h_z(z)\, dz
\end{aligned}
$$

The optimal linear earnings-dependent savings tax rate $\tau_s(.)$ satisfies at each earnings z^0,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s \middle| z = z^0 \right] h_z\left(z^0\right) + \int_{z \geq z^0} \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s\zeta_{s|z}^c(s,z) \middle| z \right] h_z(z)\, dz \\
&= \int_{z \geq z^0} \mathbb{E}\left[(1-\hat{g}(s,z))\, s \middle| z \right] h_z(z)\, dz - \int_{z \geq z^0} \mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\, s + s_{inc}'(s,z)\, \tau_s(z)}{1-T_z'(z)-\tau_s'(z)\, s} z\zeta_z^c(s,z) s_{inc}'(s,z) \middle| z \right] h_z(z)\, dz \qquad (153)
\end{aligned}
$$

which is equation (53) in Proposition (A4).

B.9 Proof of Proposition 4 (Many Goods)

B.9.1 Setting and definitions

The problem of the government is to maximize the following Lagrangian

$$\mathcal{L} = \int_z \left\{ \alpha(z)\, U\Big(z - \mathcal{T}(\boldsymbol{s}(z), z) - \sum_{i=1}^{n} s_i(z), \boldsymbol{s}(z), z; \vartheta(z) \Big) + \lambda \mathcal{T}(\boldsymbol{s}(z), z) - E \right\} dH_z(z). \tag{154}$$

where we use the fact that $z(\theta)$ is a bijective mapping to denote $\vartheta(z)$ its inverse to denote Pareto-weights $\alpha(z) := \alpha(\vartheta(z))$ and the vector of n consumption goods as $\boldsymbol{s}(z) := \boldsymbol{s}(z; \vartheta(z))$.

Note that when characterizing optimal marginal tax rates on good i, we use the existence of a mapping between s_i consumption and earnings z, such that we implicitly assume that consumption patterns are strictly monotonic. That is, for each good i, we assume that $s_i'(z) > 0$ for normal goods, and that $s_i'(z) < 0$ for inferior goods.

In this setting, we are going to express optimal tax formulas of the following elasticity concepts that measure consumption responses of s_i and s_j to changes in $\mathcal{T}'_{s_i}$

$$\zeta^c_{s_i|z}(z(\theta)) := -\frac{1 + \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}{s_i(z;\theta)} \frac{\partial s_i(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}\Big|_{z=z(\theta)} \tag{155}$$

$$\xi^c_{s_{j,i}|z}(z(\theta)) := \frac{\mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}{s_j(z;\theta)} \frac{\partial s_j(z;\theta)}{\partial \mathcal{T}'_{s_i}(\boldsymbol{s}(z;\theta), z)}\Big|_{z=z(\theta)} \tag{156}$$

and in terms of the following statistics

$$s'_{i,inc}(z(\theta)) := \frac{\partial s_i(z;\theta)}{\partial z}\Big|_{z=z(\theta)} \tag{157}$$

$$\hat{g}(z(\theta)) := \left[\alpha(z)\frac{U_c'(z)}{\lambda} - \left(\mathcal{T}'_z(\boldsymbol{s}(z), z) + \sum_{i=1}^{n} s'_{i,inc}(z)\mathcal{T}'_{s_i}(\boldsymbol{s}(z), z) \right) \frac{\partial z(.)}{\partial \mathcal{T}} - \sum_{i=1}^{n} \mathcal{T}'_{s_i}(\boldsymbol{s}(z), z)\frac{\partial s_i(.)}{\partial \mathcal{T}} \right]\Big|_{z=z(\theta)}. \tag{158}$$

B.9.2 Optimal marginal tax rates on earnings z

We consider a small reform at earnings level z^0 that consists in a small increase $d\tau_z$ of the marginal earnings tax rate $\mathcal{T}'_z$ in a small bandwidth dz. The impact of this reform on the Lagrangian as $dz \to 0$ is

$$\begin{aligned} \frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_z dz} &= \int_{x=z^0}^{z_{max}} \Big(1 - \alpha(x)\frac{U_c'(x)}{\lambda}\Big) dH_z(x) \\ &+ \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0) + \int_{x=z_0}^{z_{max}} \mathcal{T}'_z(\boldsymbol{s}(x), x)\, \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} dH_z(x) \\ &+ \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) s'_{i,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z^0) + \int_{x=z_0}^{z_{max}} \sum_{i=1}^{n} \mathcal{T}'_{s_i}(\boldsymbol{s}(x), x) \left[\frac{\partial s_i(.)}{\partial \mathcal{T}}\Big|_{z=x} + s'_{i,inc}(x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} \right] d \end{aligned} \tag{159}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(z)$, we obtain

$$-\left[\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{i=1}^{n} \mathcal{T}'_{s_i}\left(\boldsymbol{s}(z^0), z^0\right) s'_{i,inc}(z^0)\right] \frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} h_z(z_0) = \int_{x=z_0}^{z_{max}} \Big(1-\hat{g}(x)\Big) dH_z(x) \quad (160)$$

B.9.3 Optimal marginal tax rates on good i

Intuitively, an increase in the marginal tax rate on good i induces (a) substitution effects away from s_i, (b) labor supply distortions on earnings z, and, new to this setting, (c) cross-effects on the consumption of goods s_{-i}.

We consider a small reform at $s_i^0 = s_i(z^0)$ that consists in a small increase $d\tau_s$ of the marginal tax rate $\mathcal{T}'_{s_i}$ in a small bandwidth ds. The impact of this reform on the Lagrangian as $ds \to 0$ is

$$\begin{aligned}
\frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_z dz} = & \int_{x=z^0}^{z_{max}} \Big(1 - \alpha(x)\frac{U'_c(x)}{\lambda}\Big) dH_z(x) \quad (161)\\
& + \mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial z(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} \frac{h_z(z^0)}{s'_i(z^0)} + \int_{x=z^0}^{z_{max}} \mathcal{T}'_z\left(\boldsymbol{s}(x), x\right) \frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x} dH_z(z)\\
& + \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) \left[\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} + s'_{j,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0}\right] \frac{h_z(z^0)}{s'_i(z^0)}\\
& + \int_{x=z^0}^{z_{max}} \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(x), x\right) \left[\frac{\partial s_j(.)}{\partial \mathcal{T}}\Big|_{z=x} + s'_{j,inc}(x)\frac{\partial z(.)}{\partial \mathcal{T}}\Big|_{z=x}\right] dH_z(z)
\end{aligned}$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}$ capture cross-effects for all $j \neq i$.

We characterize optimal taxes through $d\mathcal{L} = 0$. Plugging in social marginal welfare weights augmented with the fiscal impacts of income effects $\hat{g}(x)$, and using the fact that by Lemma 1 (which still holds in this setting) $\frac{\partial z(.)}{\partial \mathcal{T}'_{s_i}} = s'_{i,inc}\frac{\partial z(.)}{\partial \mathcal{T}'_z}$, we obtain

$$\begin{aligned}
& -\left[\left(\mathcal{T}'_z\left(\boldsymbol{s}(z^0), z^0\right) + \sum_{j=1}^{n} s'_{j,inc}(z^0)\mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right)\right) s'_{i,inc}(z^0)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0} + \sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(\boldsymbol{s}(z^0), z^0\right) \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0}\right] h_z(z^0)\\
& = s'_i(z^0) \int_{x=z^0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) dH_z(x) \quad (162)
\end{aligned}$$

B.9.4 Deriving Proposition 4

For any good i, we combine the optimality condition for marginal tax rates on earnings z with the one for i to obtain

$$-\sum_{j=1}^{n} \mathcal{T}'_{s_j}\left(z^0\right) \frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = \frac{1}{h_z(z^0)}\Big(s'_i(z^0) - s'_{i,inc}(z^0)\Big) \int_{x=z^0}^{z_{max}} \Big(1 - \hat{g}(x)\Big) dH_z(x). \quad (163)$$

Isolating the term relative to $\mathcal{T}'_{s_i}\left(z^0\right)$ on the left-hand side yields the following optimal tax formula in terms of $s'_{i,pref}$

$$-\mathcal{T}'_{s_i}\left(z^0\right)\frac{\partial s_i(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = \frac{1}{h_z(z^0)}s'_{i,pref}(z^0)\int_{x=z^0}^{z_{max}}\Big(1-\hat{g}(x)\Big)dH_z(x)+\sum_{j\neq i}\mathcal{T}'_{s_j}\left(z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} \quad (164)$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}$ capture cross-effects for all $j \neq i$.

We can rewrite this optimality condition in terms of the compensated elasticity $\zeta^c_{s_i|z}$ and the cross elasticity $\xi^c_{s_{j,i}|z}$ to finally obtain

$$\begin{aligned}\frac{\mathcal{T}'_{s_i}(z^0)}{1+\mathcal{T}'_{s_i}(z^0)} &= \frac{1}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}\frac{1}{h_z(z^0)}s'_{i,pref}(z^0)\int_{z=z^0}^{z_{max}}\Big(1-\hat{g}(z)\Big)dH_z(z) \\ &+\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}(z^0)}{\mathcal{T}'_{s_i}(z^0)}\frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}\end{aligned} \quad (165)$$

which is the first condition stated in Proposition 4.

To derive the second condition stated in Proposition 4, we again combine the optimality condition for marginal tax rates on earnings z with the one for i eliminating this time the integral terms. This yields

$$\sum_{j=1}^{n}\mathcal{T}'_{s_j}\left(z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = \Big(s'_i(z^0)-s'_{i,inc}(z^0)\Big)\left(\mathcal{T}'_z(z^0)+\sum_{j=1}^{n}s'_{j,inc}(z^0)\mathcal{T}'_{s_j}(z^0)\right)\frac{\partial z(.)}{\partial \mathcal{T}'_z}\Big|_{z=z^0}. \quad (166)$$

Isolating the term relative to $\mathcal{T}'_{s_i}\left(z^0\right)$ on the left-hand side yields the following Pareto-efficiency condition in terms of $s'_{i,pref}$

$$\mathcal{T}'_{s_i}\left(z^0\right)\frac{\partial s_i(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} = s'_{i,pref}(z^0)\left(\mathcal{T}'_z(z^0)+\sum_{j=1}^{n}s'_{j,inc}(z^0)\mathcal{T}'_{s_j}(z^0)\right)\frac{\partial z(.)}{\partial \mathcal{T}'_z}-\sum_{j\neq i}\mathcal{T}'_{s_j}\left(z^0\right)\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}\Big|_{z=z^0} \quad (167)$$

where $\frac{\partial s_j(.)}{\partial \mathcal{T}'_{s_i}}$ capture cross-effects for all $j \neq i$.

We can rewrite this optimality condition in terms of the compensated elasticity $\zeta^c_{s_i|z}$ and the cross elasticity $\xi^c_{s_{j,i}|z}$ to finally obtain

$$\begin{aligned}\frac{\mathcal{T}'_{s_i}(z^0)}{1+\mathcal{T}'_{s_i}(z^0)} &= s'_{i,pref}(z^0)\frac{\mathcal{T}'_z(z^0)+\sum_{j=1}^{n}\mathcal{T}'_{s_j}(z^0)s'_{j,inc}(z^0)}{1-\mathcal{T}'_z(z^0)}\frac{z^0\zeta^c_z(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)} \\ &+\sum_{j\neq i}\frac{\mathcal{T}'_{s_j}(z^0)}{\mathcal{T}'_{s_i}(z^0)}\frac{s_j(z^0)\xi^c_{s_{j,i}|z}(z^0)}{s_i(z^0)\zeta^c_{s_i|z}(z^0)}\end{aligned} \quad (168)$$

This completes the proof of Proposition 4.

B.10 Proof of Proposition 5 (Bequest Taxation and Behavioral Agents)

We here provide a sufficient statistics characterization of a smooth tax system $\mathcal{T}(s,z)$ under the following additively separable representation of agents' preferences

$$U\big(c,s,z;\theta\big) = u(c;\theta) - k(z;\theta) + \beta(\theta)\, v(s;\theta),$$

and for a utilitarian government that maximizes

$$\int_\theta \Big[U(c(\theta), s(\theta), z(\theta);\theta) + \nu(\theta)\, v(s(\theta);\theta)\Big] dF(\theta), \tag{169}$$

where $\nu(\theta)$ parametrizes the degree of misalignment on the valuation of s.

Using the mapping between types θ and earnings z, the Lagrangian of the problem writes

$$\mathcal{L} = \int_z \Big[U(c(z), s(z), z; \vartheta(z)) + \nu(z)\, v(s(z);\vartheta(z)) + \lambda(\mathcal{T}(s,z) - E)\Big] dH_z(z), \tag{170}$$

As before, we derive optimal tax formulas by considering reforms of marginal tax rates on z and s. Thanks to the additively separable representation of preferences, there are no income effects on labor supply choices. As a result, the only substantial change is that savings changes now lead to changes in social welfare proportional to the degree of misalignment.

Optimal marginal tax rates on z. A small reform at earnings z^0 that consists in a small increase $d\tau_z$ of the marginal earnings tax rate in a small bandwidth dz has the following effect as $dz \to 0$,

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_z dz} = &\int_{z^0}^{z_{max}} (1-\hat{g}(z))\, dH_z(z)\\ &- \left(\mathcal{T}_z'\left(s\left(z^0\right), z^0\right) + s_{inc}'(z^0)\left(\mathcal{T}_s'\left(s(z^0), z^0\right) + \nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\right)\right)\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right), z^0\right)}\zeta_z^c(z^0)\, h_z(z^0).\end{aligned}$$

In this context, social marginal welfare weights augmented with income effects $\hat{g}(z)$ are equal to

$$\hat{g}(z) = \frac{u'(c(z))}{\lambda} + \left(\mathcal{T}_s'(s(z),z) + \nu(z)\frac{v'(s(z))}{\lambda}\right)\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'(s(z),z)}$$

and we can use agents' first-order condition for s, $(1+\mathcal{T}_s')\, u'(c) = \beta v'(s)$, to express the misalignment wedge in terms of the social marginal welfare weights $g(z) := \frac{u'(c(z))}{\lambda}$ as

$$\nu(z)\frac{v'(s(z))}{\lambda} = \frac{\nu(z)}{\beta(z)} g(z)\left(1+\mathcal{T}_s'\right).$$

The optimal schedule of marginal earnings tax rates is thus characterized by

$$\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} = \frac{1}{\zeta_z^c(z^0)}\frac{1}{z^0 h_z(z^0)}\int_{z=z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z) \\ - s_{inc}'(z^0)\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)} - s_{inc}'(z^0)\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\frac{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}$$

Optimal marginal tax rates on s. A small reform at $s^0 = s(z^0)$ that consists in a small increase $d\tau_s$ of the marginal tax rate on s in a small bandwidth ds has the following effect as $ds \to 0$,

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_s ds} = s'\left(z^0\right)\int_{z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z) \\ - \left\{\left(\mathcal{T}_s'\left(s(z^0),z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\right)\frac{s(z^0)}{1+\mathcal{T}_s'\left(s\left(z^0\right),z^0\right)}\zeta_{s|z}^c(z^0)+\right. \\ \left.\left[\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\left(\mathcal{T}_s'\left(s(z^0),z^0\right)+\nu\left(z^0\right)\frac{v'\left(s\left(z^0\right)\right)}{\lambda}\right)\right]\frac{z^0}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)s_{inc}'(z^0)\right\}h_z(z^0)$$

Replacing the misalignment wedge by its expression in terms of social marginal welfare weights $g(z)$, we obtain that the optimal schedule of marginal tax rates on s is characterized by

$$\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s\left(z^0\right),z^0\right)}\zeta_{s|z}^c(z^0)s(z^0)h_z(z^0) \\ = s'\left(z^0\right)\int_{z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z) - s_{inc}'(z^0)\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)z^0h_z(z^0) \\ -\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\left[\zeta_{s|z}^c(z^0)s(z^0)+\left(s_{inc}'(z^0)\right)^2\frac{1+\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}\zeta_z^c(z^0)z^0\right]h_z(z^0) \tag{171}$$

Plugging in the formula for optimal marginal earnings tax rates, we finally obtain

$$\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s\left(z^0\right),z^0\right)} = s_{pref}'\left(z^0\right)\frac{1}{\zeta_{s|z}^c(z^0)}\frac{1}{s(z^0)h_z(z^0)}\int_{z=z^0}^{z_{max}}\left(1-\hat{g}(z)\right)dH_z(z) - \frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)$$

which is the first optimality condition in Proposition (5).

Pareto-efficiency condition. Combining optimal formulas for earnings and savings (or bequests) taxes to eliminate the integral term yields the following Pareto-efficiency condition

$$\frac{\mathcal{T}_s'\left(s(z^0),z^0\right)}{1+\mathcal{T}_s'\left(s\left(z^0\right),z^0\right)}+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z) \\ = s_{pref}'\left(z^0\right)\frac{\zeta_z^c(z^0)z^0}{\zeta_{s|z}^c(z^0)s(z^0)}\left[\frac{\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)+s_{inc}'(z^0)\mathcal{T}_s'\left(s(z^0),z^0\right)}{1-\mathcal{T}_z'\left(s\left(z^0\right),z^0\right)}+s_{inc}'(z^0)\frac{\nu\left(z^0\right)}{\beta\left(z^0\right)}g(z^0)\right]$$

which is the second optimality condition in Proposition (5).

B.11 Proof of Proposition 6 (Multi-Dimensional Tax Range with Heterogeneous Prices)

B.11.1 Setting

We consider heterogeneous marginal rates of transformation or "prices" $p(z,\theta)$ between c and s, and a two-part tax structure, where a person must pay a tax $T_1(z)$ in units of c and a tax $T_2(s,z)$ in units of s. We here consider simple tax systems of the SN type, where the tax on s is nonlinear but independent of earnings z such that $T_2(s,z) = T_2(s)$, and of the LED type, where the tax on s is linear but earnings-dependent such that $T_2(s,z) = \tau_s(z)\, s$.

In this setting, we can write agent θ problem as

$$\max_{c,s,z} U(c,s,z;\theta) \text{ s.t. } c + p(z,\theta)s \leq z - T_1(z) - p(z,\theta)T_2(s,z) \tag{172}$$

$$\iff \max_{z} \left\{ \max_{s} U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big) \right\}. \tag{173}$$

where the inner problem leads to consumption choices $c\,(z;\theta)$ and $s\,(z;\theta)$, while the outter problem leads to an earnings choice $z\,(\theta)$. Maintaining the assumption that $z\,(\theta)$ is a bijective mapping, we again denote $\vartheta\,(z)$ its inverse, which indicates the type θ with earnings z. This allows us to define $s\,(z) := s\,(z;\vartheta\,(z))$, $p(z) := p(z(\vartheta(z));\vartheta(z))$ and to formulate the problem in terms of observable earnings z.[45]

Let λ_1 and λ_2 be the marginal values of public funds associated with the following resource constraints:

$$\int_z T_1(z) dH_z(z) \geq E_1 \tag{174}$$

$$\int_z T_2(s(z),z) dH_z(z) \geq E_2 \tag{175}$$

The problem of the government is to maximize the following Lagrangian

$$\mathcal{L} = \int_z \Bigg\{ \alpha(z)\, U\Big(z - T_1(z) - p(z)\big(s(z) + T_2(s(z),z)\big), s(z), z; \vartheta(z)\Big) + \lambda_1 T_1(z) + \lambda_2 T_2(s(z),z) - E_1 - E_2 \Bigg\} dH_z(z) \tag{176}$$

B.11.2 Adapting Lemma 1

Lemma B4. *For an agent $\theta = \vartheta(z)$, we have that:*
(1a) a small increase $d\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)d\tau_z$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.
(1b) a small increase $d\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a

[45] When taking derivatives the presence of these two arguments is implicit in the notation. For instance, a total derivative corresponds to $\frac{dp}{dz} := \frac{\partial p}{\partial z} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial z}$, whereas a partial derivative $\frac{\partial p}{\partial z}$ represents variation in only the first argument.

small increase $p(z)s'_{inc}(z)d\tau_s$ in the marginal tax rate $\frac{\partial T_1}{\partial z}$.
(2) a small increase dT in the T_2 tax liability faced by agent $\theta = \vartheta(z)$ generates the same earnings change as a small increase $p(z)dT$ in the T_1 tax liability.

Proof. We first derive an abstract characterization, that we then apply to different tax reforms.

Let agent θ indirect utility function at earnings z be

$$V\left(T_1(z), T_2(., z), z; \theta\right) := \max_s \; U\Big(z - T_1(z) - p(z,\theta)\left(s + T_2(s,z)\right), s, z; \theta\Big). \tag{177}$$

Consider a small reform $dT_2(s,z)$ of T_2, and let construct for each type θ a perturbation $dT_1^\theta(z)$ of T_1 that induces the same earnings response as the initial perturbation. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(T_1(z) + dT_1^\theta(z), T_2(., z), z; \theta) = V(T_1(z), T_2(., z) + dT_2(., z), z; \theta) \tag{178}$$

then, by construction, the perturbation $dT_1^\theta(z)$ induces the same earnings response dz as the initial perturbation $dT_2(., z)$. Moreover, both tax reforms must induce the same utility change for agent θ. Applying the envelope theorem yields

$$-U'_c(z;\theta) \cdot dT_1^\theta(z) = -U'_c(z;\theta)\, p(z,\theta) \cdot dT_2\left(s(z;\theta), z\right) \tag{179}$$

such that finally, the perturbation $dT_1^\theta(z)$ is

$$dT_1^\theta(z) = p(z,\theta) \cdot dT_2\left(s(z;\theta), z\right). \tag{180}$$

and we can now apply this abstract characterization to different tax reforms.

(1a) Consider a small increase $d\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ over a small bandwidth of income $\left[z^0, z^0 + dz\right]$. Then, for any agent θ such that $z(\theta) \in \left[z^0, z^0 + dz\right]$, we have $dT_2\left(s(z;\theta), z\right) = d\tau_z\left(z - z^0\right)$ such that $dT_1^\theta(z) = p(z,\theta)d\tau_z\left(z - z^0\right)$ and differentiating with respect to z we get

$$\left(dT_1^\theta(z)\right)' = \frac{\partial p(z,\theta)}{\partial z} d\tau_z\left(z - z^0\right) + p(z,\theta)d\tau_z \tag{181}$$

At the limit $dz \to 0$ such that $z \to z^0$, a small increase $d\tau_z$ in the marginal tax rate $\frac{\partial T_2}{\partial z}$ generates the same earnings change as a small increase $p(z)d\tau_z$ in the marginal tax rate $T_1'(z)$.

(1b) Consider a small increase $d\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ over a small bandwidth of savings $\left[s^0, s^0 + ds\right]$. Then, for any agent θ such that $s(\theta) \in \left[s^0, s^0 + ds\right]$, we have $dT_2\left(s(z;\theta), z\right) = d\tau_s\left(s(z;\theta) - s^0\right)$ such that $dT_1^\theta(z) = p(z,\theta)d\tau_z\left(s(z;\theta) - s^0\right)$ and differentiating with respect to z we get

$$\left(dT_1^\theta(z)\right)' = \frac{\partial p(z,\theta)}{\partial z} d\tau_z\left(s(z;\theta) - s^0\right) + p(z,\theta)d\tau_z s'_{inc}(z) \tag{182}$$

At the limit $ds \to 0$ such that $s \to s^0$, a small increase $d\tau_s$ in the marginal tax rate $\frac{\partial T_2}{\partial s}$ generates the same earnings change as a small increase $p(z)s'_{inc}(z)d\tau_z$ in the marginal tax rate $T_1'(z)$.

(2) Consider a small lump-sum increase dT in the T_2 tax liability for an agent θ who earns z, we then have $dT_1^{\theta}(z) = p(z,\theta)dT$ such that the equivalent reform is no longer a lump-sum increase. Hence, a small increase dT in the T_2 tax liability faced by an agent $\vartheta(z)$ generates the same earnings change as a small increase $p(z)dT$ in the T_1 tax liability. □

B.11.3 Marginal values of public funds

An important prerequisite to derive optimality conditions is to pin down the marginal values of public funds λ_1 and λ_2. At the optimum, λ_1 and λ_2 are pinned down by optimally setting the tax level T_1 and T_2. Characterizing the impact of lump-sum changes in tax liabilities yields the following two equations that can be solved for λ_1 and λ_2:

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)U_c'(x) + \lambda_1 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'\lambda_2 \frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} + \lambda_2 \frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \right\} dH_z(x) = 0 \tag{183}$$

$$\int_{x=z_{min}}^{z_{max}} \left\{ -\alpha(x)p(x)U_c'(x) + \lambda_2 + \left(\lambda_1 T_1'(x) + \lambda_2 \frac{\partial T_2}{\partial z} + s_{inc}'\lambda_2 \frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_2} + \lambda_2 \frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_2} \right\} dH_z(x) = 0 \tag{184}$$

where $z(.)$ and $s(.)$ denote with a slight abuse of notations the earnings and savings choices and all partial derivatives are evaluated at earnings x.

Renormalizing these equations by λ_1, we can use the fact that by Lemma 2, $\frac{\partial z(.)}{\partial T_2} = \frac{\partial z(.)}{\partial T_1}p(z) + \frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z}$ and that $\frac{\partial s(.)}{\partial T_2} = \frac{\partial s(.)}{\partial T_1}p(z)$ to obtain

$$\int_{x=z_{min}}^{z_{max}} \left\{ 1 - \left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1}\right]\right\} dH_z(z) = 0 \tag{185}$$

$$\begin{aligned}\int_{x=z_{min}}^{z_{max}} \Bigg\{ \frac{\lambda_2}{\lambda_1} - p(x)\left[\alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1}\right] \\ + \left(T_1'(z) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z}\Bigg\} dH_z(x) = 0\end{aligned} \tag{186}$$

At any given earnings x, defining social marginal welfare weights augmented with the fiscal impact of income effects $\hat{g}(x)$, and the fiscal impacts of the novel substitution effects $\varphi(x)$ as respectively

$$\hat{g}(x) := \alpha(x)\frac{U_c'(x)}{\lambda_1} - \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1} - \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\frac{\partial s(.)}{\partial T_1} \tag{187}$$

$$\varphi(x) := \left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z} + s_{inc}'(x)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\right)\frac{\partial z(.)}{\partial T_1'}\frac{\partial p}{\partial z} \tag{188}$$

where all partial derivatives are evaluated at x, we finally obtain

$$\bar{\hat{g}} := \int_{x=z_{min}}^{z_{max}} \hat{g}(x)\, dH_z(x) = 1 \tag{189}$$

$$\overline{\hat{g}p - \varphi} := \int_{x=z_{min}}^{z_{max}} \Big(\hat{g}(x)p(x) - \varphi(x)\Big)\, dH_z(x) = \frac{\lambda_2}{\lambda_1} \tag{190}$$

B.11.4 Optimal tax rates on z

We consider a small reform at earnings level z^0 that consists in a small increase $d\tau_z$ of the marginal earnings tax rate $T_1'(z)$ in a small bandwidth dz. The impact on the Lagrangian is as $dz \to 0$,

$$\begin{aligned}\frac{d\mathcal{L}}{d\tau_z dz} = &\int_{x\geq z^0} \Big(\lambda_1 - \alpha(x)U_c'(x)\Big) dH_z(x) \\ &+ \left[\lambda_1 T_1'\left(z^0\right) + \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x\geq z^0} \left[\lambda_1 T_1'\left(x\right) + \lambda_2 \frac{\partial T_2}{\partial z}\right] \frac{\partial z(.)}{\partial T_1} dH_z(x) \\ &+ \lambda_2 \frac{\partial T_2}{\partial s}\Big|_{z=z^0} s_{inc}'(z^0) \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} h_z(z^0) \\ &+ \int_{x\geq z^0} \lambda_2 \frac{\partial T_2}{\partial s} \left[\frac{\partial s(.)}{\partial T_1} + s_{inc}'(x) \frac{\partial z(.)}{\partial T_1}\right] dH_z(x). \end{aligned} \tag{191}$$

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing everything by λ_1, plugging in social marginal welfare weights augmented with income effects $\hat{g}(x)$, we obtain the following optimality condition for marginal earnings tax rates at each earnings z^0

$$-\left[T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0} + s_{inc}'(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right] \frac{\partial z(.)}{\partial T_1'\left(z^0\right)} = \frac{1}{h_z(z^0)} \int_{x\geq z^0} \Big[1 - \hat{g}(x)\Big]\, dH_z(x) \tag{192}$$

B.11.5 Optimal tax rates on s

SN tax system. We consider a small reform at $s^0 = s(z^0)$ that consists in a small increase $d\tau_s$ of $\frac{\partial T_2}{\partial s}$, the marginal tax rate on s, in a small bandwidth ds. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $ds \to 0$

$$\begin{aligned}\frac{d\mathcal{L}}{d\tau_s ds} &= \int_{x\geq z^0}\Big(\lambda_2-\alpha(x)p(x)U_c'(x)\Big)\,dH_z(x) \qquad (193)\\
&+\left[\lambda_1 T_1'(z^0)+\lambda_2\frac{\partial T_2}{\partial z}\Big|_{z=z^0}\right]\frac{\partial z(.)}{\partial T_1'(z^0)}s_{inc}'(z^0)p(z^0)\,\frac{h_z(z^0)}{s'(z^0)}\\
&+\int_{x\geq z^0}\left[\lambda_1 T_1'(x)+\lambda_2\frac{\partial T_2}{\partial z}\right]\left(\frac{\partial z(.)}{\partial T_1}p(x)+\frac{\partial z(.)}{\partial T_1'(x)}\frac{\partial p}{\partial z}\right)\,dH_z(x)\\
&+\lambda_2\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\left[\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z=z^0}\right)}+s_{inc}'(z^0)\frac{\partial z(.)}{\partial T_1'(z^0)}s_{inc}'(z^0)p(z^0)\right]\,\frac{h_z(z^0)}{s'(z^0)}\\
&+\int_{x\geq z^0}\lambda_2\frac{\partial T_2}{\partial s}\left[\frac{\partial s(.)}{\partial T_2}+s_{inc}'(x)\left(\frac{\partial z(.)}{\partial T_1}p(x)+\frac{\partial z(.)}{\partial T_1'(x)}\frac{\partial p}{\partial z}\right)\right]\,dH_z(x)\end{aligned}$$

We characterize optimal taxes through $d\mathcal{L}=0$. Renormalizing by λ_1 and using $\frac{\partial s(.)}{\partial T_2}=\frac{\partial s(.)}{\partial T_1}p(x)$, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for marginal tax rates on s at each savings $s^0=s(z^0)$

$$\begin{aligned}&-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}h_z(z^0)=s'(z^0)\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\right\}dH_z(x) \qquad (194)\\
&+\left[T_1'(z^0)+\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial z}\Big|_{z^0}+s_{inc}'(z^0)\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\right]\frac{\partial z(.)}{\partial T_1'(z^0)}s_{inc}'(z^0)p(z^0)h_z(z^0)\end{aligned}$$

LED tax system. We consider a small reform at $s^0=s(z^0)$ that consists in a small increase $d\tau_s$ of the linear savings tax rate $\tau_s(z)$ phased in over the earnings bandwidth $[z^0,z^0+dz]$. Using Lemma 2, we characterize the impact of the reform on the Lagrangian as $dz\to 0$

$$\begin{aligned}\frac{d\mathcal{L}}{d\tau_s dz} &= \int_{x\geq z^0}\Big(\lambda_2-\alpha(x)p(x)U_c'(x)\Big)s(x)\,dH_z(x) \qquad (195)\\
&+\big(\lambda_1 T_1'(z^0)+\lambda_2\tau_s'(z^0)s(z^0)\big)\,\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\,h_z(z^0)\\
&+\int_{x\geq z^0}\big(\lambda_1 T_1'(x)+\lambda_2\tau_s'(z^0)s(z^0)\big)\left[\frac{\partial z(.)}{\partial T_1}p(x)s(x)+\frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x)+p(x)s_{inc}'(x)\right)\right]\,dH_z(x)\\
&+\lambda_2\tau_s(z^0)s_{inc}'(z^0)\left[\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\right]\,h_z(z^0)\\
&+\int_{x\geq z^0}\lambda_2\tau_s(x)\left[\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}+\frac{\partial s(.)}{\partial T_1}p(x)s(x)+s_{inc}'(x)\left(\frac{\partial z(.)}{\partial T_1}p(x)s(x)+\frac{\partial z(.)}{\partial T_1'(x)}\left(\frac{\partial p}{\partial z}s(x)+p(x)s_{inc}'(x)\right)\right)\right]\,dH_z\end{aligned}$$

since the reform triggers for individuals at z^0 changes in earnings z equivalent to those induced by a $p(z)\,d\tau_s\,s(z)$ increase in $T_1'(z^0)$, and for individuals above z^0 an increase in tax liability equivalent to a $p(z)\,d\tau_s\,dz\,s(z)$ increase in T_1 and a change in marginal earnings tax rates equivalent to a

$\left(\frac{\partial p}{\partial z}\, s(z) + p(z)\, s'_{inc}(z)\right) d\tau_s\, dz$ increase in $T_1'(z)$, in addition to the $d\tau_s dz$ increase in the linear tax rate on s.

We characterize optimal taxes through $d\mathcal{L} = 0$. Renormalizing by λ_1, we can plug in $\hat{g}(x)$ and $\varphi(x)$ to obtain the following optimality condition for linear earnings-dependent tax rates on s at each earnings z^0

$$\begin{aligned}
&-\left(T_1'(z^0) + \frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0) + \frac{\lambda_2}{\lambda_1}s'_{inc}(z^0)\tau_s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s(z^0)\, h_z(z^0) \\
&= \int_{x\geq z^0}\left\{\left(\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right)s(x) + \frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}\right\} dH_z(x) \qquad (196) \\
&+ \int_{x\geq z^0}\left(T_1'(x) + \frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x) + \frac{\lambda_2}{\lambda_1}s'_{inc}(x)\tau_s(x)\right)\frac{\partial z(.)}{\partial T_1'(x)}p(x)s'_{inc}(x)\, dH_z(x)
\end{aligned}$$

B.11.6 Deriving Proposition 6

SN tax system. A two-part SN tax system $\{T_1(z), T_2(s)\}$ thus satisfies two optimality conditions: the optimality condition (192) for $T_1'(z)$ and the optimality condition (194) for $T_2'(s)$. Combining these two conditions, we get that at each earnings z^0 the optimal SN tax system satisfies

$$-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} = \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\} dH_z(x) - p(z^0)\frac{s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1 - \hat{g}(x)\Big]\, dH_z(x) \qquad (197)$$

Adding and subtracting $p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x=z^0}^{z_{max}}\Big[1 - \hat{g}(x)\Big]\, dH_z(x)$ yields

$$\begin{aligned}
-\frac{\lambda_2}{\lambda_1}\frac{\partial T_2}{\partial s}\Big|_{z^0}\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)} &= p(z^0)\frac{s'(z^0) - s'_{inc}(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1 - \hat{g}(x)\Big]\, dH_z(x) \qquad (198) \\
&+ \frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\left\{\frac{\lambda_2}{\lambda_1} - \hat{g}(x)p(x) + \varphi(x)\right\} dH_z(x) - p(z^0)\frac{s'(z^0)}{h_z(z^0)}\int_{x\geq z^0}\Big[1 - \hat{g}(x)\Big]\, dH_z(x)
\end{aligned}$$

Defining $\zeta^c_{s|z}(z) = -\frac{1+p\frac{\partial T_2}{\partial s}\big|_{z^0}}{s}\frac{\partial s(.)}{p\,\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)}$ such that $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)} = -\frac{p\,s}{1+p\frac{\partial T_2}{\partial s}\big|_{z^0}}\zeta^c_{s|z}(z)$, we get[46]

$$\frac{\overline{\hat{g}p - \varphi\frac{\partial T_2}{\partial s}\big|_{z^0}}}{1+p(z^0)\frac{\partial T_2}{\partial s}\big|_{z^0}} \tag{199}$$
$$= \frac{1}{s(z^0)\zeta^c_{s|z}(z^0)}\frac{1}{h_z(z^0)}\left\{s'_{pref}(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x) + \frac{s'(z^0)}{p(z^0)}\Big[\Psi(z^0)+\Phi(z^0)\Big]\right\}$$

where we use $\overline{\hat{g}p-\varphi} = \frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}(x)} = 1$ to obtain the additional terms

$$\begin{aligned}
\Psi(z^0) &:= \int_{z\geq z^0}\left[\overline{\hat{g}p} - \hat{g}(z)p(z)\right]dH_z(z) - p(z^0)\int_{z=z^0}^{z_{max}}\Big[\overline{\hat{g}} - \hat{g}(z)\Big]dH_z(z) \qquad (200)\\
&= \int_{z\geq z^0}\left[\int_{x=z_{min}}^{z_{max}}\hat{g}(x)p(x)dH_z(x) - \hat{g}(z)p(z)\right]dH_z(z) - p(z^0)\int_{z\geq z^0}\Big[\int_{x=z_{min}}^{z_{max}}\hat{g}(x)dH_z(x) - \hat{g}(z)\Big]dH_z(z)\\
&= \big(1-H_z(z^0)\big)\int_{x=z_{min}}^{z_{max}}\hat{g}(x)p(x)dH_z(x) - \int_{z\geq z^0}\hat{g}(z)p(z)dH_z(z)\\
&\qquad - p(z^0)\big(1-H_z(z^0)\big)\int_{x=z_{min}}^{z_{max}}\hat{g}(x)dH_z(x) - p(z^0)\int_{z\geq z^0}\hat{g}(z)dH_z(z)\\
&= \big(1-H_z(z^0)\big)\int_{x=z_{min}}^{z_{max}}\hat{g}(x)\Big(p(x)-p(z^0)\Big)dH_z(x) - \int_{x\geq z^0}\hat{g}(x)\Big(p(x)-p(z^0)\Big)dH_z(x)\\
&= \Big(1-H_z(z^0)\Big)\int_{x\leq z^0}\hat{g}(x)\Big(p(x)-p(z^0)\Big)dH_z(z) + H_z(z^0)\int_{x\geq z^0}\hat{g}(x)\Big(p(z^0)-p(x)\Big)dH_z(x)
\end{aligned} \tag{201}$$

$$\Phi(z^0) := \int_{x\geq z^0}\Big[\varphi(x) - \overline{\varphi(x)}\Big]\,dH_z(x) \tag{202}$$

which proves the optimal formula for SN tax systems in Proposition 6.

LED tax system. A two-part LED tax system $\{T_1(z), \tau_s(z)s\}$ thus satisfies two optimality conditions: the optimality condition (192) for $T_1'(z)$ and the optimality condition (196) for $\tau_s(z)$. Combining these two conditions, we get that at each earnings z^0 the optimal LED tax system

[46] With homogeneous p, a SN savings tax levied in period 1 dollar $T_s(s)$ is simply equal to $T_s(s) = pT_2(s)$. As a result, this elasticity definition ensures that $\zeta^c_{s|z}(z)$ coincides with the elasticity concept introduced before:

$$\zeta^c_{s|z}(z) = -\frac{1+T_s'(s)}{s}\frac{\partial s(.)}{\partial T_s'(s)} = -\frac{1+pT_2'(s)}{s}\frac{\partial s(.)}{p\partial T_2'(s)}$$

satisfies

$$
\begin{aligned}
&p(z^0)s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\\
&=\int_{x\geq z^0}\left\{\Big(\frac{\lambda_2}{\lambda_1}-\hat{g}(x)p(x)+\varphi(x)\Big)s(x)+\frac{\lambda_2}{\lambda_1}\tau_s(x)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_x\right)}\right\}\,dH_z(x)\\
&+\int_{x\geq z^0}\left(T_1'(x)+\frac{\lambda_2}{\lambda_1}\tau_s'(x)s(x)+\frac{\lambda_2}{\lambda_1}s_{inc}'(x)\tau_s(x)\right)\frac{\partial z(.)}{\partial T_1'(x)}p(x)s_{inc}'(x)\,dH_z(x)
\end{aligned}
$$

Differentiating with respect to z^0 yields

$$
\begin{aligned}
&\Big(p'(z^0)s(z^0)+p(z^0)s'(z^0)\Big)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)-p(z^0)s(z^0)\Big[1-\hat{g}(z^0)\Big]\,h_z(z^0)\\
&=-\left\{\Big(\frac{\lambda_2}{\lambda_1}-\hat{g}(z^0)p(z^0)+\varphi(z^0)\Big)s(z^0)+\frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}\right\}\,h_z(z^0)\\
&-\left(T_1'(z^0)+\frac{\lambda_2}{\lambda_1}\tau_s'(z^0)s(z^0)+\frac{\lambda_2}{\lambda_1}s_{inc}'(z^0)\tau_s(z^0)\right)\frac{\partial z(.)}{\partial T_1'(z^0)}p(z^0)s_{inc}'(z^0)\,h_z(z^0)
\end{aligned}
$$

Using the optimality condition (192) for $T_1'(z)$, the last term is equal to $p(z^0)s_{inc}'(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)$ at the optimum such that

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1}\tau_s(z^0)\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\Big|_{z^0}\right)}\,h_z(z^0)\\
&=p(z^0)s_{pref}'(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+p'(z^0)s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\\
&+\left\{\frac{\lambda_2}{\lambda_1}-\Big(\hat{g}(z^0)p(z^0)-\varphi(z^0)\Big)-p(z^0)\Big[1-\hat{g}(z^0)\Big]\right\}s(z^0)h_z(z^0).
\end{aligned}
$$

We can now plug in the elasticity $\frac{\partial s(.)}{\partial\left(\frac{\partial T_2}{\partial s}\big|_{z^0}\right)}=-\frac{p(z^0)s(z^0)}{1+p(z^0)\frac{\partial T_2}{\partial s}\big|_{z^0}}\zeta_{s|z}^c(z^0)$ with $\frac{\partial T_2}{\partial s}\big|_{z^0}=\tau_s(z^0)$ and use the fact that $\overline{\hat{g}p-\varphi}=\frac{\lambda_2}{\lambda_1}$ and $\overline{\hat{g}}=1$ to obtain

$$
\begin{aligned}
&\overline{\hat{g}p-\varphi}\frac{\tau_s(z^0)}{1+p(z^0)\tau_s(z^0)} \qquad (203)\\
&=\frac{1}{s(z^0)\zeta_{s|z}^c(z^0)}\frac{1}{h_z(z^0)}\left\{s_{pref}'(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)+\frac{p'(z^0)}{p(z^0)}s(z^0)\int_{x\geq z^0}\Big[1-\hat{g}(x)\Big]\,dH_z(x)\right\}\\
&+\frac{1}{p(z^0)}\frac{1}{\zeta_{s|z}^c(z^0)}\left\{\left[\overline{\hat{g}p}-p(z^0)\overline{\hat{g}}\right]-\left[\overline{\varphi}-\varphi(z^0)\right]\right\}
\end{aligned}
$$

which proves the optimal formula for LED tax systems in Proposition 6.

C Details on the Empirical Application

This appendix describes the details underlying the numerical results presented in Section 6. Those results report the savings tax rates that would be Pareto efficient — holding fixed the status quo income tax — within each class simple tax systems described in Table 1: SL, SN, and LED. Throughout this section, it will be necessary to translate between simple tax systems as expressed in Table 1, in which all taxes are measured in units of "period 1" (i.e., working life) dollars and are written as a function of net-of-tax savings s, vs. simple tax systems in which the savings component is measured in units of "period 2" dollars (at the time of retirement) and written as a function of gross savings at the time of retirement (before any taxes are paid) since this is the format often used to express savings taxes in practice. We draw this distinction using the notation of Appendix A.6, with s_1 and s_g denoting gross savings (before taxes) measured in period 1 and period 2 dollars, respectively, and $T_2(s_g, z)$ denoting the savings tax function in period 2. Appendix A.6 demonstrates that the simplicity structure of a tax system (SL, SN, and LED) is preserved when translating between $\mathcal{T}(s, z)$ and $T_1(z)$, $T_2(s_g, z)$. Consistent with the formulas expressed throughout the paper, all optimal taxes are computed in terms of $\mathcal{T}(s, z)$, but marginal tax rates are converted into $\frac{\partial T_2(s_g,z)}{\partial s_g}$ when plotted in figures.

Throughout this section, we assume that income effects on labor supply are negligible, so that $\eta_z \approx 0$, which simplifies the computation of $\hat{g}(z)$ from Equation (14) to

$$\hat{g}(z) = g(z) + \left(\frac{\mathcal{T}_s'}{1 + \mathcal{T}_s'}\right) \eta_{s|z}(z). \tag{204}$$

We rescale welfare weights $g(z)$ to ensure that $\int_z \hat{g}(z) dH_z(z) = 1$, so that if our starting weights are $g^0(z)$, we set $g(z) = \kappa g^0(z)$, where

$$\kappa = \frac{1 - \int_z \left(\frac{\mathcal{T}_s'}{1+\mathcal{T}_s'}\right) \eta_{s|z}(z) dH_z(z)}{\int_z g^0(z) dH_z(z)}. \tag{205}$$

We also assume that preferences are weakly separable so that as shown in Proposition 1,we have

$$\eta_{s|z}(z) = s'_{inc}(z) \frac{1 + T_s'(s(z))}{1 - T_z'(z)}. \tag{206}$$

We compute $\eta_{s|z}(z)$ from the status quo schedule of $s'_{inc}(z)$, and then assume that $\eta_{s|z}(z)$ is stable when savings tax schedules are recomputed using our optimal tax conditions.

C.1 Details of the Calibration of Income, Savings, and Elasticities

Here we describe the details of our calibration of a two-period version of the U.S. economy, with the first period corresponding to working life and the second to retirement. We assume these periods are separated by 20 years, with a risk-free annual rate of return of 3.8% per year between period 1 and period 2 (see Fagereng et al. (2020), Table 3).

C.1.1 Distribution of earnings $h(z)$, gross savings $s_g(z)$, and the nonlinear income tax schedule $T(z)$

The distribution of earnings and savings is calibrated for the U.S. from the Distributional National Accounts micro-files of Piketty et al. (2018), henceforth PSZ. We use individual measures of pretax labor income (*plinc*) and net personal wealth (*hweal*) as well as the age category (20 to 44 years old, 45 to 64, and above 65) and household information. We discretize the income distribution into percentiles by age group. Our measure of annualized earnings during work life z at the n-th percentile is constructed by averaging earnings at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. For married households, we use the average earnings of the couple and assign both members of the couple to the same percentile of income. This can be important for households in which only one member of the household is working. If one member of the household is above 65 years old, we only keep the younger spouse in the sample. We drop the bottom 2% of observations with non-positive labor income; these individuals have positive average income from other sources, suggesting they are not representative of the zero ability types which would correspond to $z = 0$ in our model. We construct a smoothed savings distribution by fitting the log of our discretized probability mass function to log income.[47]

Our measure of gross retirement savings per year worked, which we denote s_g in the notation of Appendix A.6, at the n-th percentile of income is constructed by projecting forward to age 65 the average wealth we observe in the 45 to 65 age category at each percentile, assuming that individuals in this category experience another 10 years of returns on average before retirement.[48] For married households, we take household wealth to be the average wealth of its members. This can be important for couples if assets are legally held by one household member. We then normalize the total wealth at retirement by the number of working years $(65 - 25 = 40)$ so that z and s_g are in comparable units measured per working year. This yields a monotonic distribution of earnings z and gross savings $s_g(z)$, and pins down the cross-sectional variation in savings $s'_g(z)$.

We construct the status quo income tax function using the PSZ measure *diinc* ("extended disposable income") as a measure of post-tax income $z - T_1(z)$, using the median value within each pre-tax income percentile, constructing disposable income y as a smooth spline fit of ln *diinc* to

[47] This fitted schedule, and the savings and post-tax income fitted schedules below, are constructed using the smoothing spline fit in Matlab. The density uses a smoothing parameter of 0.99, while the savings and post-tax income schedules use a parameter of 0.9. Measures of savings are noisy at low incomes, which also have outlier values of $\ln(z)$ after the logarithmic transformation used for our savings fit. To avoid having those percentiles generate a strong pull on the fit, we fit the log of savings to $\ln(z + k)$, where a larger k reduces the extent to which the low incomes are outliers. Our baseline uses $k = \$20,000$.

[48] To account for the remaining 10 years (on average) of saving among 45- to 65-year-olds before retirement, we first impute the amount of net-of-tax income saved by middle-aged workers at each income percentile, which are compounded until the time of retirement. We do this imputation by constructing a representative working agent in each income percentile in each age category: a "young" agent of age 35 (the 20 to 44 age category, for which we assume work begins at age 25), and a "middle-aged" agent of age 55 (the 45 to 64 age category). We assume wealth at middle-aged consists of the sum of 20 years worth of savings while young, with returns compounded for an average of $55 - 35 = 20$ years, and 10 years of saving while middle-aged, compounded for an average of 5 years. Assuming that individuals within each percentile save a the same share of post-tax income while young and middle-aged, this calculation pins down annual savings among middle-aged workers, which are assumed to continue until retirement at age 65.

ln *plinc* with smoothing parameter 0.9. We then calibrate the smooth marginal income tax rate schedule as $1 - \frac{dy}{dz}$. We treat social security as a fixed amount of forced savings, which are added to net-of-tax disposable savings to arrive at our total measure of net savings s. These forced savings are funded by a reduction in disposable income y (during working life) of an amount necessary to fund the fixed Social Security receipt upon retirement.[49]

C.1.2 Measures of s'_{inc}

A key input for our sufficient statistics is the marginal propensity to save out of earned income, $s'_{inc}(z) := \frac{\partial s(z)}{\partial z}\big|_{\theta=\theta(z)}$, which relates changes in the amount of net-of-tax savings at the time of retirement to changes in the amount of pre-tax earnings z. We here show how we draw from two sources of empirical data to calibrate our marginal propensities to consume (or save), translated into measures of $s'_{inc}(z)$. These results are plotted in Figure 2.

Norwegian estimates from Fagereng et al. (2019). Fagereng et al. (2019) estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. They find that individuals' consumption peaks during the winning year and gradually reverts to their previous value afterwards. Over a 5-year horizon, they estimate winners consume close to 90% of the tax-exempt lottery prize, which translates into an MPC of 0.9, and thus a marginal propensity to save of 0.1. Under the assumption that preferences are weakly separable with respect to the disutility of labor supply, this is also the marginal propensity to save out of net earned income from labor supply. (See Proposition 1.)

They find little evidence of variation in MPCs across income levels which implies

$$\frac{\partial c(z)}{\partial (z - T_1(z))} = 0.9$$

and thus

$$\frac{\partial s_1(z)}{\partial (z - T_1(z))} = 1 - \frac{\partial c(z)}{\partial (z - T_1(z))} = 0.1.$$

Using $\partial (z - T_1(z)) = (1 - T_1'(z))\, \partial z$ and $s = (s_1 - T_s(s))(1+r) \implies \frac{\partial s}{\partial s_1} = \frac{1}{\frac{1}{1+r} + T_s'(s)}$, the local causal effect of *pre-tax* income z on *net* savings s satisfies

$$s'_{inc}(z) = \frac{\partial s_1(z)}{\partial (z - T_1(z))} \cdot \frac{\partial s}{\partial s_1} \cdot \frac{\partial (z - T_1(z))}{\partial z} \tag{207}$$

$$= 0.1 \cdot \frac{1 - T_1'(z)}{\frac{1}{1+r} + T_s'(s(z))}. \tag{208}$$

[49] The amount is computed as follows, using the SSA Fact Sheet: Retired workers receive on average \$1,514 per month from social security, meaning $12 * 1,514 = \$18,168$ annually. Through the lens of our two-period model, these benefits are received over an average retirement length of 20 years, and stem from contributions paid over 40 working years. We therefore approximate this as forced savings at the time of retirement of \$90009,000 per working year. Discounting this amount at our assumed rate of return of 3.8% over a period of 20 years, we find an implied forced savings contribution of \$4,269, which is deducted from disposable income.

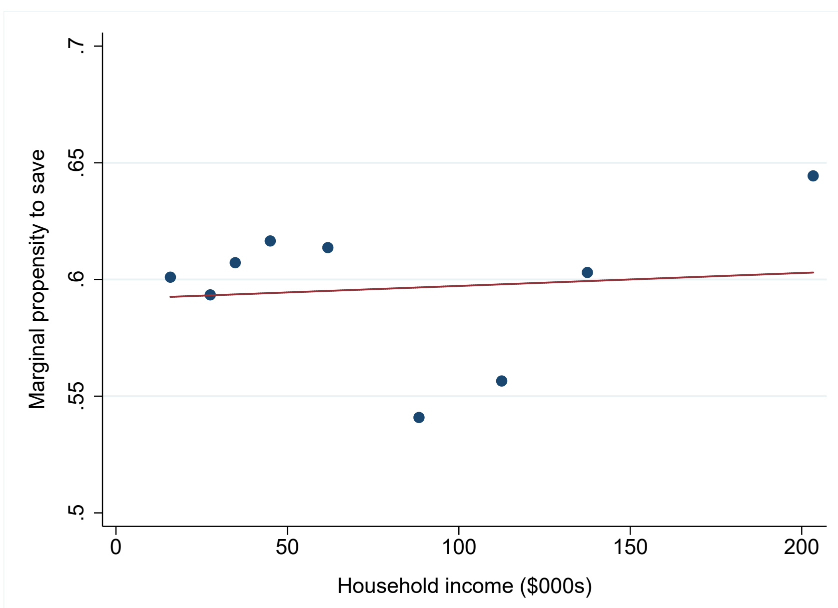
We can then use our calibrated U.S. tax schedule to obtain a profile of $s'_{inc}(z)$, under the key assumption that U.S. households have similar MPCs as Norwegian households. This profile is plotted in Figure 2.

U.S. estimates from a new Amerispeak survey In a survey experiment ran on a representative sample of US households, we ask the following question:

> Imagine that you or someone else in your household gets a raise such that over the next five years, your household's income is $1,000 higher each year than what you expected. How much of this would your household spend, and how much would your household save over each of the next five years? (For purposes of this question, consider paying off debt, such as reducing your mortgage, a form of saving.) If no one in your household is going to be employed for most of the next five years, please write "N/A."
>
> Spend an extra $ ______ per year
>
> Save an extra $ ______ per year

Answers to this question provide information about individuals' reported marginal propensity to consume (MPC) and marginal propensity to save (MPS) out of earned income. Since we also collect information on household income in the survey, we can observe marginal propensity to save across earnings levels, plotted in Figure A1. Respondents report on average a marginal propensity to save of 0.6 in the year of the raise. Moreover, marginal propensities to save seem strikingly constant across income levels which is consistent with the results of Fagereng et al. (2019).

Figure A1: Marginal propensity to save across household income (own survey)



Notes: Marginal propensities to save are computed from the answers to our survey question. They are computed as the ratio between the amount respondents report they would save and the amount of the raise.

Since we ask about consumption and spending within each year, we interpret these estimates as short-run responses. Yet, Fagereng et al. (2019) show that positive income shocks are followed by consumption responses that can last up to 5 years. We thus rely on the evidence that they provide to convert these 1-year MPS into 5-year MPS that we interpret as a total effect on savings before returns. To do so, we use the fact that they report a 1-year MPC of 0.52 and a 5-year MPC of 0.90 and accordingly compute our long run marginal propensity to save as

$$MPS_{5y} = MPS_{1y} \cdot \frac{1-0.90}{1-0.52} = 0.6 \cdot 0.208 = 0.125.$$

Since our question asked about a change in pre-tax income, we do not need to multiply by $1-T_1'(z)$ as in equation (208); we just divide by $\frac{1}{1+r} + T_s'(s(z))$ to reach our measure of $s_{inc}'(z)$. This results in an estimate somewhat higher than that obtained by Fagereng et al. (2019) for Norway, plotted in Figure 2. We use this as the measured of $s_{inc}'(z)$ for our simulations.

C.1.3 Savings elasticity

For purposes of calibration, we assume that the income-conditional compensated elasticity of savings is constant, $\zeta_{s|z}^c(z) = \bar{\zeta}_{s|z}^c$. This statistic is related to the elasticity of taxable wealth (e.g., Jakobsen et al. (2020)) and to the elasticity of capital gains realizations with respect to the capital gains tax (e.g., Agersnap and Zidar (2020)), although both statistics require some translation and additional assumptions, since they pertain to variations in gross (rather than net) savings. We instead follow Golosov et al. (2013) in drawing on the literature estimating the intertemporal elasticity of substitution (IES), and reporting results for a substantial range of values. To motivate this range of values, we describe here how we can translate from the IES to a compensated elasticity $\zeta_{s|z}^c$ in the case of a representative agent.

The IES is defined as the elasticity of the growth rate of consumption with respect to the net price of consumption. We assume consumption is smoothed during retirement, so that retirement consumption is proportional to the net stock of savings s, and thus the elasticity of the growth rate of consumption (with respect to a tax change) is the same as the elasticity of the ratio of s to work-life consumption c. We consider a change in the price of retirement consumption induced by a small reform to a SL system like the one described in Table 1 with a constant linear tax rate τ_s, in which case the net-of-tax price of retirement savings is $\frac{R}{1+R\tau_s}$. (This can be found using the relationship $(s_1 - \tau_s s)R = s$ and solving for $\frac{ds}{ds_1} = -\frac{ds}{dc}$.) We can therefore write

$$\begin{aligned}
IES &= \frac{d\ln(s/c)}{d\ln(\frac{R}{1+R\tau_s})} \\
&= -\frac{d\ln(s/c)}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{d\ln c}{d\ln(1+R\tau_s)} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{dc}{d\ln(1+R\tau_s)}\frac{1}{c} \\
&= -\frac{d\ln s}{d\ln(1+R\tau_s)} + \frac{ds}{d\ln(1+R\tau_s)}\frac{dc}{ds}\frac{1}{c}
\end{aligned}$$

Substituting for $\frac{dc}{ds} = \frac{1+R\tau_s}{R}$, we then obtain

$$\begin{aligned}
IES &= -\frac{d\ln s}{d\ln(1+R\tau_s)} - \frac{d\ln s}{d\ln(1+R\tau_s)}\frac{1+R\tau_s}{R}\frac{s}{c} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln s}{d\ln(1+R\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{d\ln(1+\tau_s)}{d\ln(1+R\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\left(\frac{d(1+R\tau_s)}{d\tau_s}\right)^{-1}\frac{1+R\tau_s}{1+\tau_s}\frac{d\ln s}{d\ln(1+\tau_s)} \\
&= -\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}\frac{d\ln s}{d\ln(1+\tau_s)}
\end{aligned}$$

$$\implies \quad \frac{d\ln s}{d\ln(1+\tau_s)} = -\frac{IES}{\left(1+\left(\frac{1+R\tau_s}{R}\right)\frac{s}{c}\right)\frac{1+R\tau_s}{R(1+\tau_s)}} \tag{209}$$

Using a value of $s/c = 0.72$ (the population average in our calibrated 2-period economy), and using the values $R = 2.1$ (from our real interest rate of 3.8% compounded over 20 years) and $\tau_s = 0.01$ (corresponding to a linear tax of 4% on capital gains, the approximate average in Figure A2b), we find[50]

$$\frac{d\ln s}{d\ln(1+\tau_s)} = -\frac{IES}{0.65}.$$

[50] A linear tax rate τ^{cg} on capital gains $(R-1)\,s_1$ leads to net savings $s = s_1(1+(R-1)(1-\tau^{cg}))$. Similarly, a period 1 linear tax τ_s on net savings s leads to net savings $s = (s_1 - \tau_s s)\,R \iff s = \frac{s_1 R}{1+\tau_s R}$. As a result,

$$\begin{aligned}
s_1(1+(R-1)(1-\tau^{cg})) &= \frac{s_1 R}{1+\tau_s R} \\
\iff 1+\tau_s R &= \frac{R}{1+(R-1)(1-\tau^{cg})} \\
\iff \tau_s &= \frac{1}{1+(R-1)(1-\tau^{cg})} - \frac{1}{R}.
\end{aligned} \tag{210}$$

Treating this as the population estimate of $\frac{d\ln\bar{s}}{d\ln(1+\tau_s)}$, we can then compute the value of the elasticity $\bar{\zeta}^c_{s|z}$ that is consistent with this estimate. From the proof of the optimal SL tax system (see Appendix B.7.1, equation (103)), the response of aggregate savings $\bar{s}$ to a change in the separable linear tax rate τ_s (measured in period 1 dollars, as distinct from $\tau_{s,2}$) can be written as follows:

$$\begin{aligned}
\frac{d\bar{s}}{d\tau_s} &= -\int_z \left\{ \frac{1}{1+\tau_s}\left(s(z)\bar{\zeta}^c_{s|z} + \eta_{s|z}(z)s(z)\right) + \frac{s'_{inc}(z)}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right\} dH_z(z) \\
\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{1} &= -\bar{\zeta}^c_{s|z}\bar{s} - \int_z \left\{ \eta_{s|z}(z)s(z) + s'_{inc}(z)\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right\} dH_z(z) \\
\underbrace{\frac{d\bar{s}}{d\tau_s}\frac{1+\tau_s}{\bar{s}}}_{\frac{d\ln\bar{s}}{d\ln(1+\tau_s)}} &= -\bar{\zeta}^c_{s|z} - \int_z \left\{ \eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + s(z)\,\eta_z(z)\right)\right\} dH_z(z) \\
\bar{\zeta}^c_{s|z} &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\eta_{s|z}(z)\frac{s(z)}{\bar{s}} + \frac{s'_{inc}(z)}{\bar{s}}\frac{1+\tau_s}{1-T'_z(z)}\left(z\,\zeta^c_z(z)s'_{inc}(z) + \eta_z(z)s(z)\right)\right]
\end{aligned}$$

This could be computed directly if we have an independent esitmate of the income-conditional income effect $\eta_{s|z}$. Here, we instead invoke our assumptions of weak separability and a separable tax system, implying $\eta_{s|z}(z) = s'_{inc}(z)\frac{1+T'_s(s(z))}{1-T'_z(z)}$ (see Proposition 1), and negligible income effects on earnings, to write

$$\begin{aligned}
\bar{\zeta}^c_{s|z} &= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)}\frac{s'_{inc}(z)}{\bar{s}}\left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z)\right)\right] \\
&= -\frac{d\ln\bar{s}}{d\ln(1+\tau_s)} - \frac{1}{\bar{s}}\cdot\mathbb{E}\left[\frac{1+T'_s(z)}{1-T'_z(z)}s'_{inc}(z)\left(s(z) + z\bar{\zeta}^c_z s'_{inc}(z)\right)\right] \qquad (211)
\end{aligned}$$

In our calibration, the value of the second term is 0.35, suggesting a translation of $\bar{\zeta}^c_{s|z} \approx IES/0.65 - 0.35$. Thus a value of $IES = 1$, the baseline in Golosov et al. (2013), suggests an elasticity of $\bar{\zeta}^c_{s|z} = 1.2$. We use a baseline value of $\bar{\zeta}^c_{s|z} = 1$. IES Values of 0.5 and 2 (the "low" and "high" values considered in Golosov et al. (2013)) suggest savings elasticities of $\bar{\zeta}^c_{s|z} = 0.4$ and $\bar{\zeta}^c_{s|z} = 2.7$. This is a wide range, and in particular values of savings elasticities below $\bar{\zeta}^c_{s|z} = 0.6$ suggest that consistency with the status quo income tax requires a savings tax that is extreme or non-convergent.[51] We report results for alternative values of $\bar{\zeta}^c_{s|z} = 0.7$, $\bar{\zeta}^c_{s|z} = 2$, and $\bar{\zeta}^c_{s|z} = 3$.

C.2 Comparison to Golosov et al. (2013)

In their baseline calibration, Golosov et al. (2013) assume individuals preferences are CRRA

$$U = \frac{\alpha(w)}{1+\alpha(w)}\ln c + \frac{1}{1+\alpha(w)}\ln s - \frac{1}{\sigma}(l)^\sigma$$

[51] Intuitively, as the savings elasticity becomes low, one's level of savings becomes a reliable signal of underyling ability, and more of the total redistribution in the tax system should be carried out through the savings tax, rather than the income tax. Thus for sufficiently low $\bar{\zeta}^c_{s|z}$, the status quo income tax cannot be Pareto efficient.

where l is the labor supply of an individual with hourly wage w. The risk aversion parameter is here set to $\gamma = 1$, the isoelastic disulity from labor effort is such that $\sigma = 3$, and the taste parameter is given by

$$\alpha(w) = 1.0526\,(w)^{-0.0036}$$

meaning that it varies from 1.0433 for individuals in the bottom quintile of the earnings distribution (mean hourly wage of \$12.35, in 1992 dollars) to 1.0406 for individuals in the top quintile of the earnings distribution (mean hourly wage of \$25.39, in 1992 dollars). In other words, this parameter is almost constant with income around an average of $\overline{\alpha} = 1.042$.

To translate this structural calibration into our sufficient statistics, we solve the individuals' problem, which is to maximize utility U subject to the budget constraint $c \leq wl - \frac{1}{R}s - \mathcal{T}(s, wl)$. This enables us to express our elasticity concepts in terms of primitives.

We obtain that the earnings elasticity is given by $\zeta_z^c = \frac{1}{\sigma} = 0.33$, and that the savings elasticity is given by $\zeta_{s|z}^c = \frac{\alpha(1+\mathcal{T}_s')}{(1+\alpha)(1/R+\mathcal{T}_s')}$. Using the values $R = 2.1 \approx (1.038)^{20}$ and $\tau_s \approx 0.02$ as in the preceding subsection, yields $\zeta_{s|z}^c = \frac{1.042*(1+0.12)}{(1+1.042)*(1/2.1+0.12)} = 0.52$.

Moreover, assuming $\mathcal{T}'' \approx 0$, we can decompose the variation of savings s across earnings z as

$$\underbrace{\frac{ds}{dz}}_{s'(z)} = \underbrace{\frac{1-\mathcal{T}_z'}{1/R + \alpha\left(1/R + \mathcal{T}_s'\right) + \mathcal{T}_s'}}_{s_{inc}'(z)} + \underbrace{\frac{-\left(1/R + \mathcal{T}_s'\right)}{1/R + \alpha\left(1/R + \mathcal{T}_s'\right) + \mathcal{T}_s'}\frac{d\alpha}{dz}s}_{s_{pref}'(z)}.$$

Given the separability assumptions, note that the causal income effect on savings s_{inc}' corresponds to $s_{inc}' = \frac{1-\mathcal{T}_z'}{1+\mathcal{T}_s'}\eta_{s|z}$. Assuming a constant marginal earnings tax rate of 30%, and a constant marginal savings tax rate of 2% yields $s_{inc}' = \frac{1-0.3}{1/2.1+1.042*(1/2.1+0.02)+0.02} = 0.69$.

To obtain an order of magnitude of the s_{pref}' term, we use the fact that Golosov et al. (2013) report in their simulation results that individuals with an annual income $z = \$100,000$ have an hourly wage $w = \$40$ while those with an annual income $z = \$150,000$ have an hourly wage $w = \$62.5$. Using these values, we can approximate $\frac{d\alpha}{dz} = \frac{\alpha(62.5)-\alpha(40)}{150,000-100,000} = \frac{1.0370-1.0387}{50,000} = -34*10^{-9}$. Assuming a constant s_{inc}', we can also infer that at an annual income of \$125,000 the annual amount of savings available for consumption in period 2 (including coumpounded interests) is approximately equal to $s = s_{inc}' * \$125,000 = 0.69 * 125,000 = \$86,250$. We finally obtain that $s_{pref}' = \frac{1/2.1+0.02}{1/2.1+1.042*(1/2.1+0.02)+0.02} * (34 * 10^{-9}) * 86,250 = 0.0014$.[52]

These values for s_{inc}' and s_{pref}' imply that in the calibration of Golosov et al. (2013), preference heterogeneity is minuscule as it only explains $\frac{s_{pref}'}{s_{pref}'+s_{inc}'} = \frac{0.0014}{0.69+0.0014} = 0.2\%$ of the variation in savings between individuals earning \$100,000 annually and those earning \$150,000.

[52] More specifically, we postulate $s_{pref}' \ll s_{inc}'$ to infer $s(z) = s_{inc}' \cdot z$ and then compute s_{pref}'. Since we obtain a value that verifies $s_{pref}' \ll s_{inc}'$, this reasoning is consistent and proves that $s_{pref}' \ll s_{inc}'$. Put differently, even if we assume $s_{pref}' \approx s_{inc}'$ which implies that $s(z) = 2s_{inc}' \cdot z$, we still obtain $s_{pref}' \ll s_{inc}'$.

C.3 Estimates of Savings Tax Rates in the United States

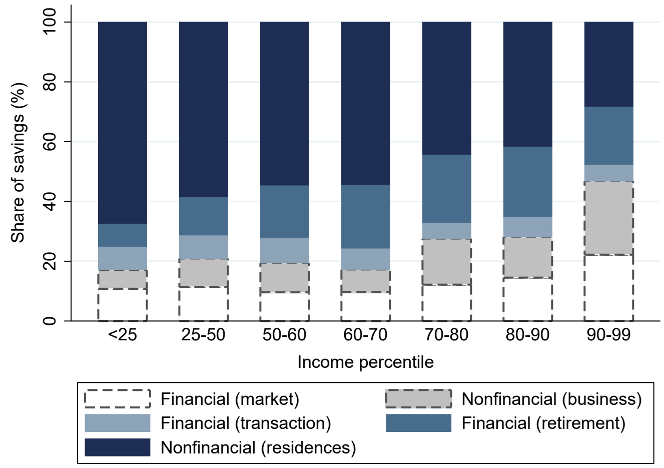
Bricker et al. (2019) provide a decomposition of saving types by asset ownership percentile; we summarize the analogous decomposition by income percentile in Figure A2 below. We use the share of savings of each type in Figure A2(a) to estimate the savings tax rates in Figure A2(b). We estimate that Financial (market) and Nonfinancial (business) savings are subject to a 15% tax, while the other types of savings are not subject to any additional savings taxes.[53] Thus, we estimate the savings tax rate for each income group as 15%×(share of Financial (market) savings + share of Nonfinancial (business) savings).

For the purposes of our numerical calibrations, we use the data underlying Bricker et al. (2019) to compute this rate at each percentile of the income distribution. We smooth this schedule of average rates using the spline fit procedure described above, and apply that average tax rate to the calibrated level of gross savings at each point in the income distribution, to reach a calibrated schedule of total savings taxes paid, measured in period 2 dollars. We then convert this to period 1 dollars using our assumed rate of return, and map this schedule to our calibrated schedule of total net savings in order to find a SN tax structure equivalent to the U.S. status quo, which we use in formulas that rely on status quo savings tax rates, such as the inverse otimum weight calculation below.

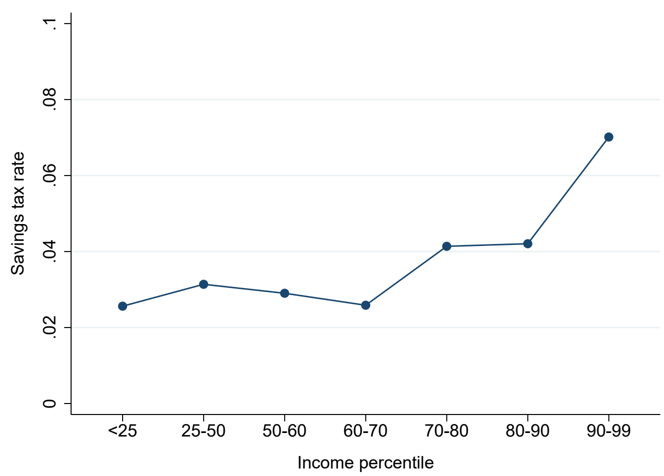
[53]The 15% estimated tax on Financial (market) savings and Nonfinancial (business) savings represent capital gains taxes. We assume that Financial (transaction) savings—which include checkings and savings accounts among other transaction accounts—and Financial (retirement) savings are tax exempt. We view property taxes on "Nonfinancial (residences)" savings as a tax that is incident on renters, and thus a component of imputed rent, which is paid regardless of whether the asset is owned by the user, so we also assume the tax rate on these savings is 0%.

Figure A2: Calibration of Savings Tax Rates Across Incomes in the U.S.

(a) Decomposition of Savings Types: Bricker et al. (2019)



(b) Calibrated Savings Tax Rates in the United States, by Income Percentile



Notes: This figure illustrates the calibration of savings tax rates in the U.S. across the income distribution. Panel (a) plots the composition of asset types in individuals' portfolios across the income distribution, reported by Bricker et al. (2019). Panel (b) plots the implied weighted average savings tax rate in each bin. See Appendix C.3 for details.

C.4 Inverse Optimum Approach

The inverse optimum approach aims at inferring the social marginal welfare weights (SMWW) consistent with actual tax policy (Bourguignon and Spadaro, 2012; Lockwood and Weinzierl, 2016). Indeed, assuming the actual policy is optimal, one can invert optimality conditions to obtain the social marginal welfare weights.

In the general case where we consider arbitrary smooth tax systems, we derive two optimality conditions gathered in Proposition 2. Proposition A4 characterizes the optimal schedule of marginal earnings tax rates for any given (and potentially suboptimal) schedule of marginal savings tax rates, whereas Proposition 3 characterizes the optimal schedule of marginal savings tax rates for any given (and potentially suboptimal) schedule of marginal earnings tax rates.

There are thus two methodological choices to be made here. The first choice is whether we assume that the actual schedule of marginal earnings tax rates is optimal, or whether we assume that the actual schedule of marginal savings tax rates is optimal—if both are optimal then the actual tax system should verify Pareto-efficiency, which does not appear to be the case for our baseline specification in Figure 3. The second choice is how to compute the actual tax schedules on earnings and savings, and in particular whether we impose a particular functional form on the actual tax system.

If we assume that the actual schedule of marginal earnings tax rates is optimal, that it only depends on earnings, and that the savings tax is of the separable nonlinear (SN) type, we can infer social marginal welfare weights at each earnings z through

$$\frac{T_z'(z)}{1-T_z'(z)} = \frac{1}{\zeta_z^c(z)\,z}\frac{1}{h_z(z)}\int_{x=z}^{z_{max}}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z)\frac{T_s'(s(z))}{1-T_z'(z)} \tag{212}$$

$$\iff \int_{x=z}^{z_{max}}(1-\hat{g}(x))\,dH_z(x) = \zeta_z^c(z)\,z\,h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)} \tag{213}$$

where the right-hand side term can be identified from the data.

Differentiating with respect to z yields the expression we use to implement this computation numerically,

$$\hat{g}(z) = 1 + \frac{1}{h_z(z)}\cdot\frac{d}{dz}\left[\zeta_z^c(z)\,z\,h_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}\right], \tag{214}$$

which can be further manipulated as follows:

$$\begin{aligned} -(1-\hat{g}(z))\,h_z(z) = {} & \frac{d\zeta_z^c(z)}{dz}zh_z(z)\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)} \\ & + \zeta_z^c(z)\left[h_z(z)+zh_z'(z)\right]\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)} \\ & + \zeta_z^c(z)zh_z(z)\frac{d}{dz}\left[\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)}\right] \end{aligned} \tag{215}$$

and further assuming that the compensated elasticity of earnings is approximately constant across incomes ($\frac{d\zeta_z^c(z)}{dz}\approx 0$) yields

$$\begin{aligned} \hat{g}(z) = {} & 1 + \zeta_z^c(z)\left[1+\frac{zh_z'(z)}{h_z(z)}\right]\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)} \\ & + \zeta_z^c(z)z\left[\frac{\left[1+s_{inc}'(z)T_s'(s(z))\right]T_z''(z)}{\left[1-T_z'(z)\right]^2} + \frac{s_{inc}''(z)T_s'(s(z^*))+s_{inc}'(z)s'(z)T_s''(s(z))}{1-T_z'(z)}\right] \end{aligned} \tag{216}$$

which gives augmented social marginal welfare weights as a function of the local Pareto parameter of the income distribution $1+\frac{zh_z'(z)}{h_z(z)}$ and other observables.

Using the fact that augmented social marginal welfare weights are defined as

$$\hat{g}(z) := g(z) + T_z'(z)\frac{\eta_z(z)}{1-T_z'(z)} + T_s'(s(z))\left(\frac{\eta_{s|z}(z)}{1+T_s'(s(z))} + s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'(z)}\right), \tag{217}$$

and assuming preferences are weakly separable, such that by Proposition 1 we have $s_{inc}'(z) = \frac{1-T_z'(z)}{1+T_s'(s(z))}\eta_{s|z}(z)$, we finally get

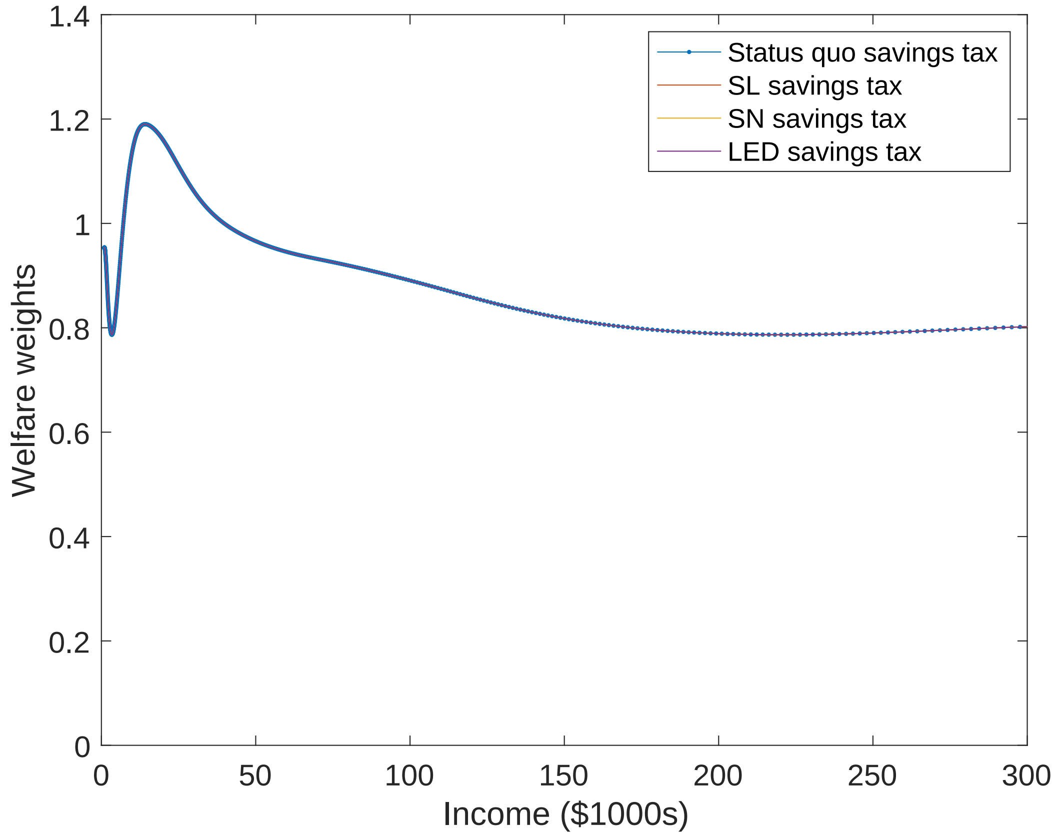
$$\begin{aligned} g(z) = {} & 1+\zeta_z^c(z)\left[1+\frac{zh_z'(z)}{h_z(z)}\right]\frac{T_z'(z)+s_{inc}'(z)T_s'(s(z))}{1-T_z'(z)} \\ & +\zeta_z^c(z)z\left[\frac{[1+s_{inc}'(z)T_s'(s(z))]\,T_z''(z)}{[1-T_z'(z)]^2}+\frac{s_{inc}''(z)T_s'(s(z))+s_{inc}'(z)s'(z)T_s''(s(z))}{1-T_z'(z)}\right] \\ & -T_z'(z)\frac{\eta_z(z)}{1-T_z'(z)} - s_{inc}'(z)\,T_s'(s(z))\frac{1+\eta_z(z)}{1-T_z'(z)}. \end{aligned} \tag{218}$$

Employing the assumption that labor supply income effects $\eta_z(z)$ are negligible, we can write the expression we use to numerically compute inverse optimum weights $g(z)$ from $\hat{g}(z)$:

$$g(z) = \hat{g}(z) - s_{inc}'(z)\left(\frac{T_s'(s(z))}{1-T_z'(z)}\right), \tag{219}$$

This expression depends on both the current schedule of marginal earnings tax rates, $T_z'(z)$, and savings tax rates, T_s'. We seek the schedule of weights that rationalizes the existing income tax schedule, and thus we set $T_z'(z)$ to reflect the observed schedule of income tax rates. Since the current schedule of savings tax rates is not Pareto efficient, there is a question of whether we compute weights assuming that the savings tax were counterfactually optimal, or whether we use the status quo (suboptimal) savings tax schedule. For consistency with the inverse "optimum" motivation, we use the former approach, but results look essentially identical if we instead use the suboptimal status quo schedule. Figure A3 plots our estimated profile of inverse optimum weights, both under the assumption that marginal savings tax rates are SN Pareto-efficient, and assuming the status quo schedule of savings tax rates.

Figure A3: Schedule of Inverse Optimum Social Welfare Weights in the U.S.



Notes: This figure plots the schedule of inverse optimum welfare weights that would rationalize the U.S. income tax schedule. Separate schedules are plotted using the status quo (Pareto-inefficient) savings tax, as well as under the assumption that the savings tax were (counterfactually) Pareto-efficient, with either a separable linear, separable nonlinear, or linear earnings-dependent structure.

C.5 Simulations of Optimal Savings Taxes with Multidimensional Heterogeneity

C.5.1 Overview and assumptions

Through numerical simulations, we assess how the presence of multidimensional heterogeneity affects optimal savings taxes in each of the simple tax system that we consider. To this end, we use optimal savings tax formulas derived in the presence of multidimensional heterogeneity (Proposition 3). Here our goal is to quantify the effect of adding to multidimensionality to our model while holding fixed other aspects of the model, such as welfare weights and behavioral elasticities.

To extend our calibrated 2-period model economy to a multidimensional setting, we retain the same discretized grid of incomes as in the unidimensional case, using the calibration described in Appendix C.1. At each income, we now allow for heterogeneous levels of savings. Specifically, using the same measure of gross savings described in Appendix C.1, we now use a calibration with four different level of savings at each level of income, each representing a quartile of the income-conditional savings distribution. Across the income distribution, we assume savings within each quartile are a constant ratio of the income-conditional average level of saving. These ratios are 15%, 40%, 70%, and 280% of the income-conditional average savings level; they are calibrated to reflect the average ratios across percentiles 50 to 100 in the PSZ data. (We calibrate these ratios excluding the bottom portion of the distribution, because the average level of saving itself is very low in the bottom half, resulting in noisily measured ratios.)

For this simulation, we assume marginal social welfare weights depend on income, but not on savings conditional on income i.e. $g(z)$. To isolate the effect of accounting for multidimensional heterogeneity without simultaneously altering distributional preferences, we hold welfare weights fixed at the values we find using the inverse optimum approach assuming unidimensional heterogeneity.

Maintaining our assumption that preferences are weakly separable, our empirical estimate of $s'_{inc}(z)$ from the data allows us to pin down the income effect parameter on savings $\eta_{s|z}(z)$ in the unidimensional case. We then infer the value of $s'_{inc}(s,z)$ in the multidimensional case by assuming that the income elasticity of savings is constant within earnings and equal to its unidimensional counterpart. Formally, this means that at a given earnings z, for any savings s we have $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, where $\overline{s(z)} := \mathbb{E}\left[s\middle|z\right]$ denotes the average savings level at earnings z. And similarly, $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}}s'_{inc}(z)$. Using these expressions, we can adapt Equation (205)—the welfare weight scaling factor necessary to ensure that $\hat{g}(s,z)$ integrates to one—to this setting:

$$\kappa = \frac{1 - \int_z \int_s \left(\frac{\mathcal{T}'_s}{1+\mathcal{T}'_s}\right) \eta_{s|z}(s,z) h(s,z) ds dz}{\int_z g^0(z) dH_z(z)}$$

Letting $\mathcal{T}'_s = \tau$ in the SL case and $\mathcal{T}'_s = \tau_s(z)$ in the LED case, we have

$$\kappa = \frac{1 - \int_z \frac{\tau_s(z)}{1+\tau_s(z)} \eta_{s|z}(z) dH_z(z)}{\int_z g^0(z) dH_z(z)}, \tag{220}$$

and with $\mathcal{T}'_s = T'_s(s)$ in the SN case such that

$$\kappa = \frac{1 - \int_s \frac{T'_s(s)}{1+T'_s(s)} \int_z \eta_{s|z}(s,z) h(s,z) dz ds}{\int_z g^0(z) dH_z(z)}. \tag{221}$$

C.5.2 Separable linear (SL) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 3) is

$$\begin{aligned} &\frac{\tau_s}{1+\tau_s} \int_z \left\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z) \middle| z \right] \right\} dH_z(z) \\ &= \int_z \left\{ \mathbb{E}\left[(1-\hat{g}(s,z)) s \middle| z \right] - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z) s'_{inc}(s,z) \middle| z \right] \right\} dH_z(z) \end{aligned} \tag{222}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s'_{inc}(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\frac{\tau_s}{1+\tau_s}\int_z\left\{\overline{s(z)}\zeta^c_{s|z}\right\}dH_z(z) \tag{223}$$
$$=\int_z\left\{\mathbb{E}\left[\left(1-g(z)-\tau_s\frac{\eta_{s|z}(z)}{1+\tau_s}\frac{s}{\overline{s(z)}}\right)s\Big|z\right]-\frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[T'_z(z)\frac{s}{\overline{s(z)}}+s'_{inc}(z)\tau_s\left(\frac{s}{\overline{s(z)}}\right)^2\Big|z\right]\right\}dH_z(z)$$

which after rearranging yields

$$\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z\overline{s(z)}dH_z(z) \tag{224}$$
$$=\int_z\left\{(1-g(z))\overline{s(z)}-\frac{\tau_s}{1+\tau_s}\frac{\eta_{s|z}(z)}{\overline{s(z)}}\mathbb{E}\left[s^2\Big|z\right]-\frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[\frac{T'_z(z)}{\overline{s(z)}}s+\frac{s'_{inc}(z)\tau_s}{\overline{s(z)}^2}s^2\Big|z\right]\right\}dH_z(z).$$

We can now use $\mathbb{E}\left[s^2\Big|z\right]=\mathbb{E}\left[\left(s-\overline{s(z)}\right)^2+2s\overline{s(z)}-\overline{s(z)}^2\Big|z\right]=\mathbb{V}(s|z)+\overline{s(z)}^2$ to obtain

$$\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z\overline{s(z)}dH_z(z) \tag{225}$$
$$=\int_z\left\{\left[1-g(z)-\frac{\tau_s\eta_{s|z}(z)}{1+\tau_s}\left(1+\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\overline{s(z)}-\frac{z\zeta^c_z s'_{inc}(z)}{1-T'_z(z)}\left[T'_z(z)+s'_{inc}(z)\tau_s\left(1+\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}\right)\right]\right\}dH_z(z)$$

which we can finally rewrite as

$$\frac{\tau_s}{1+\tau_s}\int_z\overline{s(z)}\zeta^c_{s|z}dH_z(z) \tag{226}$$
$$=\int_z\left\{\left(1-g(z)-\frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\right)\overline{s(z)}-\frac{T'_z(z)+s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right\}dH_z(z)$$
$$-\int_z\left\{\underbrace{\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}}_{\geq 0}\underbrace{\left(\frac{\tau_s}{1+\tau_s}\eta_{s|z}(z)\overline{s(z)}+\frac{s'_{inc}(z)\tau_s}{1-T'_z(z)}z\zeta^c_z s'_{inc}(z)\right)}_{\geq 0}\right\}dH_z(z).$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition A3) and the last line captures the effect of multidimensional heterogeneity through $\mathbb{V}(s|z)$. Multidimensional heterogeneity adds a corrective term which is unambiguously negative, it thus prescribes a lower linear savings tax rate.

C.5.3 Separable nonlinear (SN) tax system

At any given savings level s^0, the optimal savings tax formula with multidimensional heterogeneity (Proposition 3) is

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}\int_z\left\{s^0\zeta_{s|z}^c(s^0,z)\right\}h\left(s^0,z\right)dz \tag{227}$$
$$=\int_z\left\{\mathbb{E}\Big[1-\hat{g}\left(s,z\right)\Big|z,s\geq s^0\Big]\right\}h_z\left(z\right)dz-\int_z\left\{\frac{T_z'\left(z\right)+s_{inc}'\left(s^0,z\right)T_s'\left(s^0\right)}{1-T_z'\left(z\right)}z\zeta_z^c(s^0,z)s_{inc}'\left(s^0,z\right)\right\}h\left(s^0,z\right)dz.$$

Under the aforementioned assumptions, expanding $\hat{g}\left(s,z\right)$ and assuming η_z is negligible gives

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c\int_z h\left(s^0,z\right)dz \tag{228}$$
$$=\int_{s\geq s^0}\left\{\int_z\left[1-g(z)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}s\frac{\eta_{s|z}\left(z\right)}{s\left(z\right)}\right]h\left(s,z\right)dz\right\}ds-\int_z\left[\frac{T_z'\left(z\right)+s_{inc}'\left(s^0,z\right)T_s'\left(s^0\right)}{1-T_z'\left(z\right)}z\zeta_z^c s_{inc}'\left(s^0,z\right)\right]h\left(s^0,\right.$$

or equivalently, expressing this as a function of the savings density $h_s\left(s\right)=\int_z h\left(s,z\right)dz$,

$$\frac{T_s'\left(s^0\right)}{1+T_s'\left(s^0\right)}s^0\zeta_{s|z}^c h_s\left(s^0\right) \tag{229}$$
$$=\int_{s\geq s^0}\left\{\mathbb{E}\left[1-g(z)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}\left(s,z\right)\Big|s\right]\right\}h_s\left(s\right)ds-\mathbb{E}\left[\frac{T_z'\left(z\right)+s_{inc}'\left(s,z\right)T_s'\left(s\right)}{1-T_z'\left(z\right)}z\zeta_z^c s_{inc}'\left(s,z\right)\Big|s=s^0\right]h_s\left(s^0\right.$$

where expectations operator denote integration with respect to earnings conditional on savings.

For implementation, we assume that at each point in the income continuum, there are M different equal-sized saver-bins (e.g., bottom-, middle-, and top-third savers), indexed by $m=1,\ldots,M$. Thus we can write $s_m(z)$ as the savings map for saver-bin m at each income, with $s_m'(z)$ the cross-sectional savings profile within each saver-bin. Then the income density in each saver-bin is $h_{z,m}(z)=h(z)/M$, since the bins are equally sized conditional on income. The savings density among saver-bin m is therefore $h_{s,m}(s)=h_{z,m}(z)/s_m'(z)$, and we have $H(s)=\sum_{m=1}^M\int_{s=0}^\infty h_{s,m}(s)ds$, and $h_s(s)=\sum_{m=1}^M h_{s,m}(s)$. And the savings-conditional average of some $x(s,z)$ is $\mathbb{E}[x(s,z)|s]=\frac{\sum_{m=1}^M x(s_m,z)h_{s,m}(s)}{h_s(s)}$.

To better picture the link with the unidimensional formula (46), let also rewrite the latter as a function of the savings density $h_s\left(s\right)$ – implicitly defining $z\left(s\right)$ as the earnings level of individuals with savings s – this yields

$$\frac{T_s'(s^0)}{1+T_s'(s^0)}s^0\zeta_{s|z}^c h_s\left(s^0\right) \tag{230}$$
$$=\int_{s\geq s^0}\left\{1-g\left(z(s)\right)-\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}\left(z\left(s\right)\right)\right\}h_s\left(s\right)ds-\frac{T_z'(z\left(s^0\right))+s_{inc}'(z\left(s^0\right))T_s'(s^0)}{1-T_z'(z\left(s^0\right))}z\left(s^0\right)\zeta_z^c s_{inc}'(z\left(s^0\right))h_s\left(s^0\right.$$

While it is clear that the multidimensional formula extends the unidimensional formula, determining the impact of multidimensional heterogeneity on tax rates is analytically more difficult and we thus rely on numerical simulations.

C.5.4 Linear earnings dependent (LED) tax system

At earnings z^0, the optimal LED savings tax formula in the presence of multidimensional heterogeneity (Proposition 3) is

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]h_z(z)\,dz$$
$$=\int_{z\geq z^0}\mathbb{E}\Big[(1-\hat{g}(s,z))\,s\Big|z\Big]h_z(z)\,dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\middle|z\right]h_z(z)\,dz \tag{231}$$

which proves particularly cumbersome to use in numerical simulations, even under the aforementioned assumptions. To obtain an expression that is more easily implementable numerically, we further assume that the earnings tax is optimal (see Proposition A5) such that

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)\middle|z^0\right]h_z\left(z^0\right)=\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\Big\}h_z(z)\,dz \tag{232}$$

Now, observing that $s=\overline{s\left(z^0\right)}+s-\overline{s\left(z^0\right)}$, we can rewrite the first term of the optimal savings tax formula as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s\middle|z^0\right] \tag{233}$$
$$=\overline{s\left(z^0\right)}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\middle|z^0\right]+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\middle|z=z^0\right]$$

plugging this back into the optimal savings tax formula and using the optimal earnings tax formula, this implies that

$$\overline{s\left(z^0\right)}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\Big\}h_z(z)\,dz \tag{234}$$
$$+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\middle|z=z^0\right]h_z\left(z^0\right)+\int_{z\geq z^0}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\middle|z\right]h_z(z)\,dz$$
$$=\int_{z\geq z^0}\mathbb{E}\Big[(1-\hat{g}(s,z))\,s\Big|z\Big]h_z(z)\,dz-\int_{z\geq z^0}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s_{inc}'(s,z)\middle|z\right]h_z(z)\,dz$$

Differentiating with respect to z^0 then yields

$$\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\Big\{\mathbb{E}\left[1-\hat{g}(s,z)\,|z\right]\Big\}h_z(z)\,dz-\overline{s\left(z^0\right)}\,\mathbb{E}\left[1-\hat{g}(s,z)\,|z^0\right]h_z\left(z^0\right) \tag{235}$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c\left(s-\overline{s\left(z^0\right)}\right)\middle|z=z^0\right]h_z\left(z^0\right)\right)-\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c\middle|z^0\right]h_z\left(z^0\right)$$
$$=-\mathbb{E}\Big[(1-\hat{g}(s,z))\,s\Big|z^0\Big]h_z\left(z^0\right)+\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c s_{inc}'(s,z)\middle|z^0\right]h_z\left(z^0\right)$$

Rearranging gives

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta^c_{s|z}+\frac{T'_z(z)+\tau'_s(z)s+s'_{inc}(s,z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}z\zeta^c_z s'_{inc}(s,z)\Big|z^0\right]h_z(z^0) \quad (236)$$
$$=\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-\hat{g}(s,z)|z\right]\right\}h_z(z)\,dz-\mathbb{E}\Big[(\hat{g}(s,z))\left(s-\overline{s(z^0)}\right)\Big|z^0\Big]h_z(z^0)$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+s'_{inc}(s,z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}z\zeta^c_z\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)\right)$$

Now, with the aforementioned assumptions that $s'_{inc}(s,z)=\frac{s}{s(z)}s'_{inc}(z)$ as well as $\eta_{s|z}(s,z)=\frac{s}{s(z)}\eta_{s|z}(z)$ and $\hat{g}(s,z)=g(z)+\frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)$, we get

$$\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta^c_{s|z}+\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}z\zeta^c_z\frac{s}{s(z)}s'_{inc}(z)\Big|z^0\right]h_z(z^0) \quad (237)$$
$$=\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{\mathbb{E}\left[1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)\Big|z\right]\right\}h_z(z)\,dz-\mathbb{E}\left[\left(g(z)+\frac{\tau_s(z)}{1+\tau_s(z)}\frac{s}{s(z)}\eta_{s|z}(z)\right)\left(s-\right.\right.$$
$$+\frac{d}{dz^0}\left(\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}z\zeta^c_z\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)\right)$$

which simplifies to the following exact formula

$$\frac{\tau_s(z^0)}{1+\tau_s(z^0)}\overline{s(z^0)}\zeta^c_{s|z}h_z(z^0)+z^0\zeta^c_z\frac{s'_{inc}(z^0)}{\overline{s(z^0)}}\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}s\Big|z^0\right]h_z(z^0) \quad (238)$$
$$=\frac{d\left(\overline{s(z^0)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\}h_z(z)\,dz-\frac{\tau_s(z^0)}{1+\tau_s(z^0)}\eta_{s|z}(z^0)\frac{\mathbb{V}\left[s\big|z^0\right]}{\overline{s(z^0)}}h_z(z^0)$$
$$+\frac{d}{dz^0}\left(z^0\zeta^c_z\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)s+\frac{s}{s(z)}s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}\left(s-\overline{s(z^0)}\right)\Big|z=z^0\right]h_z(z^0)\right)$$

using $\mathbb{E}\left[s\left(s-\overline{s(z^0)}\right)\Big|z^0\right]=\mathbb{E}\left[s^2\Big|z^0\right]-\left(\overline{s(z^0)}\right)^2=\mathbb{V}\left[s\Big|z^0\right]$.

Since the marginal tax rate on earnings $T'_z(z)+\tau'_s(z)s$ features savings s, it is hard to further simplify this formula while retaining an exact characterization. To further simplify this expression, we disregard this dependence by setting $s=\overline{s(z^0)}$ in marginal earnings tax rates. We believe that these formulas are informative in that they converge to exact expressions as the linear earnings dependent savings tax rate tends to a simple linear savings tax rate—that is $\tau'_s(z)=0$ for all z. Moreover, although these approximations are not unbiased in that they provide an upper bound on the linear-earnings dependent savings tax rate, these upper bounds are tight as the approximation only amounts to assuming $\tau_s'\left(z^0\right)\mathbb{V}\left(s|z^0\right)$ is negligible.[54]

We thus use

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+\frac{s}{\overline{s(z)}}s_{inc}'(z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}s\Big|z^0\right]$$
$$\approx\overline{s\left(z^0\right)}\left[\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}+\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\left(1+\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}\right)\right]$$

as well as

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s+\frac{s}{\overline{s(z)}}s_{inc}'(z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}\left(s-\overline{s\left(z^0\right)}\right)\Big|z=z^0\right]$$
$$\approx\overline{s\left(z^0\right)}\left[\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}\right]$$

to finally obtain

$$\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\overline{s\left(z^0\right)}\zeta_{s|z}^c h_z\left(z^0\right)+s_{inc}'\left(z^0\right)\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}+s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}z^0\zeta_z^c h_z\left(z^0\right)$$
$$=\frac{d\left(\overline{s\left(z^0\right)}\right)}{dz^0}\int_{z\geq z^0}\left\{1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\right\}h_z(z)\,dz \tag{241}$$
$$-z^0\zeta_z^c s_{inc}'\left(z^0\right)\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}^2}h_z\left(z^0\right)-\frac{\tau_s\left(z^0\right)}{1+\tau_s\left(z^0\right)}\eta_{s|z}\left(z^0\right)\frac{\mathbb{V}\left(s\Big|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)$$
$$+\frac{d}{dz^0}\left(z^0\zeta_z^c\frac{s_{inc}'\left(z^0\right)\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\frac{\mathbb{V}\left(s|z^0\right)}{\overline{s\left(z^0\right)}}h_z\left(z^0\right)\right).$$

As an element of comparison, a similar derivation under unidimensional heterogeneity combining the optimal LED savings tax formula (Proposition A3) and the optimal earnings tax formula

[54] Indeed, because functions $X\mapsto\frac{X}{1-X}$ and $X\mapsto\frac{1}{1-X}$ are positive and increasing in X over $[0,1]$ the interval where optimal earnings marginal tax rates lie, we have that

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)s}{1-T_z'(z)-\tau_s'(z)s}s\Big|z^0\right]\geq\frac{T_z'\left(z^0\right)+\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\mathbb{E}\left[s\Big|z^0\right] \tag{239}$$
$$\mathbb{E}\left[\frac{\tau_s(z)}{1-T_z'(z)-\tau_s'(z)s}s^2\Big|z^0\right]\geq\frac{\tau_s\left(z^0\right)}{1-T_z'\left(z^0\right)-\tau_s'\left(z^0\right)\overline{s\left(z^0\right)}}\mathbb{E}\left[s^2\Big|z^0\right] \tag{240}$$

(Proposition A4) yields the following unidimensional analogue

$$\frac{\tau_s(z^0)}{1+\tau_s(z^0)} s(z^0)\zeta^c_{s|z}(z^0)\, h_z(z^0) + z^0\zeta^c_z(z^0)s'_{inc}(z^0)\frac{T'_z\left(z^0\right)+\tau'_s\left(z^0\right)s\left(z^0\right)+s'_{inc}(z^0)\tau_s(z^0)}{1-T'_z\left(z^0\right)-\tau'_s\left(z^0\right)s\left(z^0\right)}h_z(z^0) \tag{242}$$

$$= s'(z^0)\int_{z\geq z^0}\left(1-g(z)-\frac{\tau_s\left(z\right)}{1+\tau_s\left(z\right)}\eta_{s|z}\left(z\right)\right)dH_z(z).$$

Multidimensional heterogeneity thus add new terms related to $\mathbb{V}\left(s|z^0\right)$ that naturally wash out under unidimensional heterogeneity. The two terms on the third line of 241 are clearly negative and thus push for lower savings tax rates in the presence of multidimensional heterogeneity. The term on the fourth line cannot be signed unambiguously. In our calibratreion, it appears to be negative at low earnings but positive at high earnings. However, its order magnitude is so small (around 10^{-4}) that it does not meaningully affects the optimal LED savings tax rate and can thus be neglected.

C.6 Simulations of Optimal Savings Taxes with Heterogeneous Returns

The bottom two panels of Figure 4 display schedules of LED and SN savings tax rates computed under the assumption that individuals with different income levels differ in their private rates of return. We compute the tax schedules which satisfy the equations for the optimal tax conditions in Proposition 6. As in the case of multidimensional heterogeneity, we hold fixed the schedule of marginal social welfare weights $g(z)$ proportional to those which rationalize the status quo income tax in our baseline inverse optimum calculation. For comparability with Gerritsen et al. (2020), we assume that rates of return rise by 1.4% from the bottom to the top of the income distribution. We linearly interpolate this difference across income percentiles, centered on our 3.8% baseline rate of return.

Employing our maintained assumptions (stated at the beginning of this Appendix) of negligible labor supply income effects and weakly separable preferences, Equation (187) simplifies to

$$\hat{g}(x) := g(x) + \frac{\lambda_2}{\lambda_1}\frac{T'_2(s)}{1+pT'_2(s)}\eta_{s|z}(z) \tag{243}$$

for an SN system, or with $\tau_s(z)$ replacing $T'_2(s)$ for an LED system with $\tau_s(z)s$ levied in period-2 dollars. To ensure that $\hat{g}(z)$ still integrates to one, the rescaling factor in Equation (205) now becomes

$$\kappa = \frac{1-\int_z\left(\frac{\lambda_2}{\lambda_1}\frac{T'_2(s)}{1+pT'_2(s)}\right)\eta_{s|z}(z)dH_z(z)}{\int_z g^0(z)dH_z(z)}. \tag{244}$$

Similarly, Equation (188) simplifies to

$$\varphi(x) = -\left(T'_1(x)+s'_{inc}(x)\frac{\lambda_2}{\lambda_1}T'_2(s)\right)\left(\zeta^c_z(x)\frac{x}{1-T'_1(x)}\right)\frac{\partial p}{\partial z}. \tag{245}$$

D Details of Tax Systems by Country

In Table 2, we consider five categories of savings subject to various taxation regimes in different countries: (i) wealth, (ii) capital gains, (iii) property, (iv) pensions, and (v) inheritance, which are typically defined in tax codes as follows. First, wealth, which is free from taxation in most advanced economies, is defined as the aggregate value of certain classes of assets, such as real estate, stocks, and bank deposits. Next, capital gains consist of realized gains from financial and real estate investments, and include interest and dividend payments. Third, property consists of real estate holdings, such as land, private residences, and commercial properties. Fourth, for our purposes, pensions are defined as private retirement savings in dedicated accounts, excluding government transfers to retired individuals, such as Social Security in the United States. Lastly, inheritances—also known as estates—are the collections of assets bequeathed by deceased individuals to living individuals, often relatives.

For each country, we label the tax system applied to each category of savings with the types described in Table 1 or "Other," which encompasses all other tax systems. An additional common simple tax structure is a "composite" tax, in which savings and labor income are not distinguished for the purposes of taxation. Composite taxes are often applied to classes of income for which it is unclear whether the income should be considered capital income or labor income. For example, in a majority of the countries in Table 2, rental income—which requires some active participation from the recipient of the income—is subject to composite taxation.

In the subsections below, we have included additional details about the tax system in each country in Table 2. Note that we characterize tax systems that feature a flat tax on savings above an exempt amount as having a separable nonlinear tax system. In addition, when benefits are withdrawn from pension accounts, they are often subject to the same progressive tax rates as labor income. We characterize these tax systems as separable nonlinear rather than composite since benefits are generally received after retirement from the labor force when the taxpayer's income is primarily composed of savings.

Australia

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a composite tax applies. Gains from certain assets are exempt or discounted.
- **Property:** At the state level, land tax rates are progressive; primary residence land is typically exempt. At the local level, generally flat taxes are assessed on property but the taxes can be nonlinear as well, depending on the locality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. A component of pension benefits may be subject to taxation when withdrawn, in which case the lesser of a flax tax or the same progressive tax rates as apply to labor income is assessed.

- **Inheritance:** No inheritance tax.

Austria

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed, with the rate depending on the type of asse; taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt.
- **Property:** Either flat or progressive tax rates are assessed on property, depending on its intended use. Rates vary by municipality.
- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with discounts applicable to certain types of withdrawals.
- **Inheritance:** No inheritance tax.

Canada

- **Wealth:** No wealth tax.
- **Capital gains:** For most capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. For certain gains, such as interest income, no discount is applied. Lifetime exemptions up to a limit apply to gains from certain classes of assets.
- **Property:** Generally a flat tax is assessed on property, with rates varying by province and locality.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with exemptions applicable to certain types of withdrawals.
- **Inheritance:** No separate inheritance tax. A final year tax return is prepared for the deceased, including income for that year, that treats all assets as if they have just been sold and applies the relevant taxes (e.g., labor income and capital gains taxes) accordingly.

Denmark

- **Wealth:** No wealth tax.
- **Capital gains:** Progressive taxation with two tax brackets. Gains from certain classes of assets are exempt.

- **Property:** At the national level, property is subject to progressive taxation with two tax brackets. Pensioners under an income threshold can receive tax relief. Land taxes—assessed at the local level—are flat taxes, with rates varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income (excluding a labor market surtax), a flat tax, or are exempt from taxation, depending on the type of pension.
- **Inheritance:** Generally a flat tax is assessed on the inheritance above an exemption, with a higher tax rate for more distant relatives. Transfers to spouses and charities are exempt. Inheritances above a certain value are subject to additional taxes.

France

- **Wealth:** No wealth tax.
- **Capital gains:** Different rates—progressive and flat—apply to gains from different classes of assets. Certain low-income individuals are either exempt from taxes or can opt to apply their labor income tax rate, depending on the type of asset. High-income individuals are subject to a surtax. Gains from certain assets are exempt or discounted.
- **Property:** Residence taxes are assessed on property users, while property taxes on developed and undeveloped properties are assessed on owners. Rates are set at the local level and apply to the estimated rental value of the property. Exemptions, reductions, and surcharges may apply depending on the taxpayer's reference income and household composition, certain events, and property characteristics. Surcharges may also apply to higher-value properties. An additional property wealth tax applies at the national level; rates are progressive above an exemption.
- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits beyond an exemption are generally subject to the same progressive tax rates as labor income. A flat tax is assessed on certain types of withdrawals, and special rules apply to certain types of accounts.
- **Inheritance:** Either a flat tax or progressive tax rates are assessed on the inheritance above an exemption, with rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Transfers to spouses/civil partners are exempt. Certain shares are required to pass to the deceased's children.

Germany

- **Wealth:** No wealth tax.

- **Capital gains:** Generally a flat tax is assessed on gains above an exemption, but taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt or subject to special rules.
- **Property:** A flat tax is assessed on property, with rates depending on the class of property and subject to a multiplier, which varies by locality.
- **Pensions:** No tax on capital gains made within the pension account. A portion of pension benefits, which depends on the type of account, is subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions both depending on the relation of the recipient to the deceased. Pension entitlements are exempt.

Ireland

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains above an exemption, with the rate depending on the type of asset. Certain classes of individuals, such as farmers and entrepreneurs, qualify for lower rates and additional exemptions.
- **Property:** Progressive tax rates are assessed on residential properties, with local authorities able to vary the rates to a certain extent. A flat tax is assessed on commercial properties, with rates varying by locality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of withdrawal, pension benefits are either subject to the same progressive tax rates as labor income or different progressive tax rates beyond an exemption. A surtax is assessed on high-value accounts.
- **Inheritance:** A flat tax is assessed on inheritances above an exemption. Exemptions are associated with the recipient and apply to the sum of all inheritances bequeathed to the recipient from certain classes of relatives.

Israel

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on real gains (i.e., the inflationary component of gains is exempt). High-income individuals are subject to a surtax.
- **Property:** Generally the tax increases in the area of the property, with amounts depending on property characteristics and varying by municipality. Tax relief may apply to certain taxpayers, such as new immigrants and low-income individuals, depending on the municipality.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; certain taxpayers qualify for exemptions.
- **Inheritance:** No inheritance tax.

Italy

- **Wealth:** A flat tax is assessed on bank deposits and financial investments held abroad, with exemptions on bank deposits if the average annual account balance is below a certain threshold.
- **Capital gains:** Generally a flat tax is assessed on financial capital gains. For certain real estate capital gains, individuals can choose between separable or composite taxation, either applying a flat tax or their labor income tax rate.
- **Property:** Generally a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of asset. Pension benefits are also subject to flat taxes, with rates varying with the duration of the contribution period.
- **Inheritance:** A flat tax is assessed on inheritances, with higher rates for more distant relatives. Different amounts of the inheritance are exempt from taxation for certain close relatives.

Japan

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains from certain classes of assets, such as securities and real estate, with the rate depending on the type of asset. Progressive tax rates, composite taxation, exemptions, and discounts apply to gains from different classes of assets.
- **Property:** A flat tax is assessed on property above an exemption, with a lower rate or reduction applicable to certain types of property.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to progressive tax rates, with the rates depending on the type of withdrawal.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above a general exemption and an exemption that depends on the relation of the recipient to the deceased and their disability status. A surtax applies to more distant relatives. Certain shares are required to pass to certain relatives.

Netherlands

- **Wealth:** A progressive, fictitious estimated return from net assets not intended for daily use is taxed at a flat rate depending on the amount above the exemption.
- **Capital gains:** Gains from a company in which an individual has a substantial stake are subject to a flat tax. Most other capital gains are not subject to taxation.
- **Property:** At the municipal level, a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality. At the national level, progressive tax rates are assessed on the fictitious estimated rental values of primary residences, with substantial deductions applicable to the portion of the tax exceeding the mortgage interest deduction.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, though certain accounts with taxed contributions allow tax-free withdrawals.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Additional exemptions apply to certain classes of assets.

New Zealand

- **Wealth:** No wealth tax.
- **Capital gains:** Capital gains from financial assets are generally either subject to composite taxation or are exempt from taxation, depending on the type of gain. Special rules apply to certain classes of assets. Capital gains from real estate are generally subject to composite taxation. Depending on transaction characteristics, gains from the sale of commercial property may be subject to an additional tax, while gains from the sale of residential property may be exempt from taxation.
- **Property:** Generally a fixed fee plus a flat tax is assessed on property, with rates set at the municipal level. Low-income individuals qualify for rebates for owner-occupied residential property.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of account; for certain accounts, the rate depends on the taxpayer's labor income in prior years. Pension benefits are generally exempt from taxation.
- **Inheritance:** No inheritance tax.

Norway

- **Wealth:** A flat tax is assessed on wealth above an exemption, with the value of certain classes of assets, such as primary and secondary residences, discounted.
- **Capital gains:** A flat tax is assessed on gains from financial assets above the "risk-free" return (i.e., the counterfactual return on treasury bills of the same value). Gains from certain financial assets, such as dividends, are multiplied by a factor before the tax is assessed. A flat tax is assessed on real estate gains, with exemptions for certain types of property.
- **Property:** A flat tax is assessed on discounted property values, with rates varying by municipality and discounts varying by property type.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to a lower tax rate than labor income, and taxpayers with smaller benefits qualify for larger tax deductions.
- **Inheritance:** No inheritance tax.

Portugal

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on gains from financial assets, but for certain types of gains, such as interest, low-income individuals can opt to apply their labor income tax rate. For real estate capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. Certain classes of real estate are exempt.
- **Property:** Progressive tax rates are assessed on property, with exemptions for certain taxpayers. Rates and exemptions vary based on property characteristics, and an additional exemption applies to low-income individuals.
- **Pensions:** No tax on capital gains made within the pension account, except for dividends, which are generally subject to a flat tax. For different types of withdrawals above an exemption, capital gains are either subject to a flat tax or the same progressive tax rates as labor income when withdrawn. Depending on how contributions were initially taxed and the type of withdrawal, the non-capital gains component of benefits is exempt from taxation, or subject to a flat tax or the same progressive tax rates as labor income on the amount above an exemption.
- **Inheritance:** A flat tax is assessed on the inheritance, with a higher rate for real estate transfers. Transfers to spouses/civil partners, ascendants, and descendants are exempt (except for real estate transfers, which are subject to a low flat tax).

Singapore

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains are not subject to taxation. Depending on transaction characteristics, composite taxation may apply.
- **Property:** Progressive tax rates are assessed on the estimated rental value of the property, with rates varying by property type and occupancy status.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; benefits from contributions made before a certain year are exempt from taxation.
- **Inheritance:** No inheritance tax.

South Korea

- **Wealth:** No wealth tax.
- **Capital gains:** Various flat and progressive tax rates are assessed on gains above an exemption; rates and exemptions depend on the type of asset. Gains from certain classes of assets are entirely exempt. Dividends and interest are subject to flat taxation below a certain limit and composite taxation above that limit.
- **Property:** Progressive tax rates are assessed on property, with rates varying by property type.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond a progressive exemption (i.e, greater portions are exempt at smaller benefit levels) are generally subject to the same progressive tax rates as labor income; the exempt amount may also depend on the type of withdrawal and taxpayer characteristics.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above either a lump-sum or itemized deduction, which depends on the composition of the inheritance and relation of the recipient to the deceased. Transfers to spouses are exempt. The top tax rate increases for controlling shares in a company.

Spain

- **Wealth:** Progressive tax rates are assessed on net assets above an exemption, with an additional exemption for residences.
- **Capital gains:** Progressive tax rates are generally assessed on gains, with exemptions for elderly individuals under certain conditions and for certain real estate gains.

- **Property:** Generally a flat tax is assessed on property, with rates depending on the property type and varying by locality. Exemptions or discounts may apply depending on taxpayer and property characteristics, including taxpayer income.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Certain classes of assets, such as family businesses and art collections, are eligible for additional exemptions.

Switzerland

- **Wealth:** A flat tax is assessed on the net value of certain classes of assets and liabilities, with tax rates and exemptions varying by canton.
- **Capital gains:** Progressive tax rates are assessed on gains from real estate, with rates varying by canton. Most capital gains from financial assets are not subject to taxation. Dividends and interest are subject to composite taxation.
- **Property:** Generally a flat tax is imposed on property, with rates varying by canton; a minimum amount per property may apply. For owner-occupied properties not rented out, an estimated rental value is subject to composite taxation.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to either the same progressive tax rates as labor income or lower progressive tax rates, depending on the type of withdrawal.
- **Inheritance:** In most cantons, progressive tax rates are assessed on the inheritance and depend on the relation of the recipient to the deceased. Transfers to spouses and children are exempt in most cantons.

Taiwan

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains from financial assets are subject to composite taxation; taxpayers can opt for a flat tax to be assessed on dividends, and certain gains are exempt from taxation. A flat tax is assessed on gains from real estate, with the rate depending on the type of asset, and an exemption for primary residences.
- **Property:** Flat or progressive tax rates are assessed on land, depending on its intended use. A flat tax is generally assessed on buildings, with rates depending on their intended use. Certain classes of land and buildings are exempt or subject to reduced rates.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption—which depends on the duration of the contribution period—are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, which depends on the relation of the recipient to the deceased, their disability status, and their age.

United Kingdom

- **Wealth:** No wealth tax.
- **Capital gains:** Either flat or progressive tax rates are assessed on gains, with rates depending on the taxpayer's labor income tax bracket; higher rates generally apply to taxpayers in higher labor income tax brackets. Exemptions for part or all of the gain apply to certain types of assets, such as dividends and primary residences.
- **Property:** Progressive tax rates are assessed on property, with rates varying by locality. Exemptions or discounts may apply to certain taxpayers depending on characteristics, such as age.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption are subject to the same progressive tax rates as labor income. An additional flat tax may be imposed on accounts with a value exceeding a lifetime limit, with the tax rate depending on the type of withdrawal.
- **Inheritance:** A flat tax is assessed on the inheritance above an exemption, with larger exemptions for transfers to children. Transfers to spouses/civil partners, charities, and amateur sports clubs are exempt. The tax rate is reduced if a certain share is transferred to charity.

United States

- **Wealth:** No wealth tax.
- **Capital gains:** Gains from "short-term" assets (held for less than a year) are subject to composite taxation. Gains from "long-term" assets are subject to a flat tax, with higher rates for higher-income individuals. Dividends are also subject to either composite taxation or flat taxes that increase with labor income, depending on their source.
- **Property:** Generally a flat tax is assessed on property, with rates varying by state, county, and municipality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of account, benefits are generally either exempt from taxation or subject to the same progressive tax rates as labor income.

- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption. Transfers to spouses are generally exempt.