# A Long and a Short Leg Make For a Wobbly Equilibrium

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**Abstract**

We provide evidence that the online discussion on the WSB subreddit had a substantial negative impact on the profitability of shorting strategies across a number of stocks — even those that were neither heavily discussed on the subreddit, nor experienced an unusual increase in retail buying volume. We provide a model to explain how fears among short sellers can become self-fulfilling. In the model, the market for shares and the lending market must clear jointly. Despite standard assumptions, the model features multiple equilibria and "run-type" behavior by shorting agents. More broadly, the model provides a tractable framework to interpret several empirical observations on the relation between short interest, lending fees, and expected returns.

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The events surrounding Gamestop and the WSB subreddit caught the attention of the popular press. Somewhat less noticed was the broader "ripple effect" of these events on short-selling strategies. We show that, as the price of Gamestop skyrocketed, several other stocks with high short interest also experienced high returns, leading to losses on short positions and a subsequent decline in short interest. These outcomes occurred despite the fact that these stocks were substantially larger than Gamestop, were not particularly discussed on the WSB subreddit, and did not experience an extraordinary rise in retail purchases.

A natural explanation is that the events surrounding Gamestop spread fear among short sellers. In this paper we develop a model that helps explain how such fears can become self-reinforcing and go as far as to cause a collapse in share lending and short-selling. In the model the stock market and the lending market have to clear jointly. Remarkably, even under a set of standard, "neoclassical" assumptions (Walrasian markets, price-taking investors, etc.) the model features multiple equilibria: In the baseline version of the model, a high short-interest, low Sharpe-ratio equilibrium with high expected gains for short sellers coexists with two other equilibria featuring smaller, respectively zero, short interest with short-sellers experiencing lower gains. This multiplicity provides a stark way of conveying the intuition that shorting strategies may be exposed to behaviors akin to a "run."

While motivated by recent events, the model provides a broader framework to interpret some findings of the empirical literature. For example, the model can help explain the observation that a supply shift in the market for loanable shares impacts lending fees and short interest but at the same time has a muted or even ambiguous effect on expected returns.

In the remainder of the introduction we provide a more detailed summary of the paper's empirical findings and the assumptions and results of the model.

The motivating facts for our analysis can be summarized as follows. January 2021 was an unprecedented month for a strategy that goes long stocks with high short interest and finances this position by shorting the broad market portfolio. (A bet against the "shorts").

Also during that month, the online discussion on the WSB subreddit saw explosive growth, largely centered around Gamestop. Consistent with articles in the popular press linking this online discussion with retail purchases, we show that high-frequency fluctuations of Gamestop mentions on the WSB subreddit exhibited a very high correlation (at hourly intervals) with retail purchases of Gamestop. This strong high-frequency correlation suggests that the WSB subreddit was an effective vehicle in coordinating retail purchases for this particular stock. Interestingly, several other high-short-interest stocks, that were barely mentioned on the WSB subreddit, had substantially higher stock market capitalization, and did not experience any unusual increase in retail purchases, also experienced dramatic declines in short interest over that same period. Indeed, the performance of shorting strategies continues to be strikingly poor, even if we remove stocks that attracted some attention on the WSB subreddit, and if we only focus on large market-capitalization stocks. For stocks with these properties, it appears implausible that they were the target of a coordinated "short squeeze."

The goal of our model is to explain why shorting strategies may be particularly susceptible to self-propagating fears that manifest themselves as multiple equilibria. The model features investors with heterogeneous beliefs. One group of investors holds optimistic and one group of investors holds rational beliefs about the expected return of a positive-supply, risky stock.[1] This difference of opinion between the traders prompts trade among the investors, with the rational investors having an incentive to short the stock whenever the expected excess return becomes negative. If the rational investors want to short the stock, they must borrow it from the optimistic investors. A lending fee clears the lending market. To sharpen our results, we keep the rest of the model structure standard, in the sense that all investors are competitive (thus precluding market manipulation) and market clearing in both the stock market and

[1] Motivated by the empirical fact that stocks with high short interest tend to have low subsequent returns, we assume that the comparatively more pessimistic investors are actually rational, but this is not an essential assumption for our results.

the lending market is Walrasian.

As is known, the presence of lending fees modifies the returns experienced by the rational and the optimistic investors. The equilibrium risk compensation (the "Sharpe ratio") is impacted both by the magnitude of the lending fee and the fraction of the shares that are shorted (the short interest). All else equal, higher short interest acts essentially as a subsidy for long positions. This basic property of the model is responsible for equilibrium multiplicity. To show this, suppose that — for whatever reason — some rational investors decide to refrain from shorting. The resulting reduction in lending income diminishes the implicit subsidy to the long positions. Since the effective return on long positions declines, the equilibrium Sharpe ratio needs to rise in order to keep investors with long positions content with their holdings. But this rise in the Sharpe ratio reinforces the incentive of short sellers to abandon their positions, which further raises the Sharpe ratio until the market settles on a new equilibrium with (possibly zero) short interest. An additional implication of this reasoning is that even marginal changes in the supply of loanable shares (a "short squeeze") can be quite effective in terms of causing lending income to decline, raise the Sharpe ratio, prompt short sellers to abandon their positions, and ignite the self-propagating circle that can lead to a discontinuous change in equilibrium.

The paper is organized as follows. After reviewing related literature, section 1 presents the empirical evidence that motivates the model. Section 2 presents the baseline model. Section 3 contains the analysis of the model and describes the equilibria of the model. Section 4 discusses equilibrium properties. In that section we also discuss some broader implications of the model for the empirical literature on short selling, placing some emphasis on the ambiguous relation between short interest and lending fees on the one hand and the Sharpe ratio on the other. Section 5 presents an extension of the model to an economy with multiple risky assets. In that section we also discuss the quantitative implications of the model. Section 6 discusses a version of the model in which high short interest is associated with high lending fees. Section 7 concludes. Proofs, detailed data descriptions, various model extensions, and additional results are contained in the appendix.

## Related Literature

Our work relates to several strands of the asset-pricing literature. The most closely related one considers the joint determination of lending fees, short interest, and returns. In particular, D'Avolio (2002), Duffie et al. (2002), Vayanos and Weill (2008), and Banerjee and Graveline (2013) consider explicit frictions to lending and borrowing shares, which translate into non-zero lending fees that in turn impact expected returns.[2] Our model is closer in spirit to D'Avolio (2002)[3] and Banerjee and Graveline (2013), which also feature instantaneous clearing of lending and spot markets — rather than explicit search frictions. For our purposes, a key feature of all these papers is the uniqueness of the equilibrium. Put simply, a higher short interest must necessarily be accompanied by a lower effective return for the short sellers — despite the higher lending fee — and simultaneously a higher effective return for the long investors. The only way to achieve such a joint outcome is to increase the proportion of one's long portfolio that is lent along with the lending fee. Given the hard constraints on lending imposed in these papers — specifically, up to a given exogenous proportion of shares are loanable without frictions, but no more can be lent — positive fees only obtain for a fixed level of short interest, guaranteeing at most one equilibrium with non-zero fees. In addition, our model allows for a different class of dynamics, driven by the endogenous fluctuations in wealth of the different types of agents.

A large body of work studies the empirical relation between short interest and stock returns. Desai et al. (2002), Diether et al. (2009), Asquith et al. (2005), and Dechow et

[2] Such frictions also motivated the empirical studies of Geczy et al. (2002), Lamont (2012), Jones and Lamont (2002), Kaplan et al. (2013), and Asquith et al. (2005) among others.

[3] More precisely, to a working-paper version of this study, which contains a theoretical model that did not make it in the published article. Unfortunately, we have been unable to find a copy.

al. (2001) study the cross-sectional relation and find that stocks with higher short interest underperform those with lower short interest. Later work by Cohen et al. (2007) and Boehmer et al. (2008) uses proprietary data on quantities lent as well as shorting fees and find consistent results. Drechsler and Drechsler (2014) documents that asset pricing anomalies concentrate in stocks with high shorting fees, while Lamont and Stein (2004) studies the information content in aggregate short interest and finds that short interest declined as stock market valuations rose in the late 90's.

In our model, the motive for shorting is generated by differences in beliefs among the two types of agents. This feature traces back to seminal papers such as Williams (1956), Harrison and Kreps (1978), and Miller (1977).

Our paper also relates to a sizable theoretical literature analyzing multiple equilibria in asset pricing and macroeconomics. Multiple equilibria can arise through a number of mechanisms, chief among them a) bubbles (or money) in OLG economies, b) increasing returns to scale and production externalities, and c) portfolio constraints.[4] To our knowledge, ours is the first paper in which multiple equilibria are due to shorting fees that can make a long position sufficiently attractive to sustain a higher level of the short interest.

Finally, as in canonical asset pricing models with heterogeneous agents — see, e.g., Dumas (1989) and Gârleanu and Panageas (2015) — the relative wealth shares of agents is an important state variable in our model. Rather than attempting to summarize the large body of work in this area, we refer readers to Panageas (2020), which discusses both theoretical and empirical contributions in this area of asset pricing research.

[4] We refer the reader to the survey by Benhabib and Farmer (1999), which lists and discusses the different mechanisms that lead to multiple equilibria and indeterminacies. Gârleanu and Panageas (2021) discusses the asset-pricing implications of models with multiple equilibria and "sunspot shocks".

# 1 Empirical Motivation

We motivate our theoretical model by first documenting some empirical facts. Specifically, we show that: a) January 2021 was the worst month for shorting strategies across all years for which data are available (48 years); b) these remarkably bad returns coincided with an exponential growth in discussion on the WSB subreddit, focused primarily on Gamestop and a few other stocks with high short interest; c) the events of January 2021 impacted shorting strategies across the board — even for stocks substantially larger than Gamestop and not particularly heavily discussed on the WSB subreddit.

In terms of data, we combine standard academic data-sets with social media posts collected from the WallstreetBets subreddit (WSB), a subdomain of the Reddit website. (For a detailed description of the data collection process, see Appendix D.) Reddit is a large online website featuring specialized communities in which users post messages and other users can comment on these posts in message-board fashion. Users on the WSB subreddit actively discuss financial news, investments, and individual securities with one another. We plot the daily submissions to WSB in Figure 1 on a logarithmic scale. Though the subreddit has existed for a number of years (it was created in 2012), daily activity on WSB grew exponentially in late 2020, starting around November and continuing to increase through January 2021.

Concurrently with the rise in this online discussion, the returns of shorting strategies collapsed. In Figure 2, we plot the returns to an equal-weighted portfolio that "bets against the shorts." The portfolio is long the top decile of Russell 3000 stocks, ranked by short interest, and short the broad market. To put the recent returns of shorting strategies in historical perspective, we plot a histogram of the monthly returns of this strategy for as long as data are readily available (since 1973). Stock return data are from CRSP and short interest data are from the SEC. Figure 2 depicts these returns and shows that the January

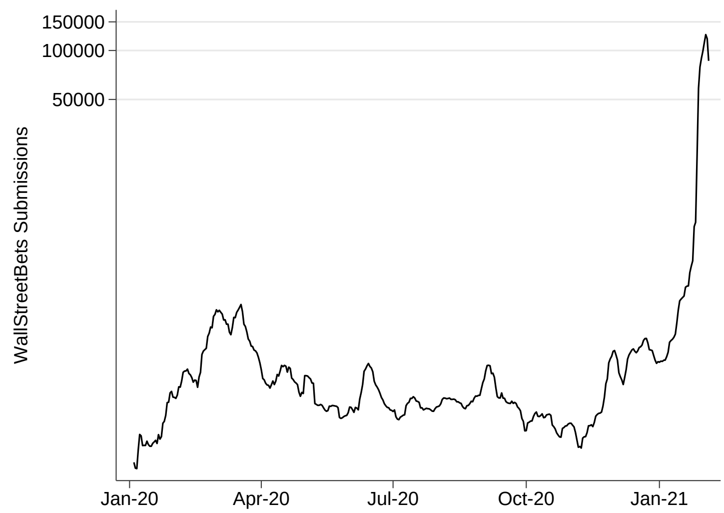


Figure 1: 7-Day Moving Average of Daily Submissions to the WallstreetBets subreddit, (January 1, 2020 - February 7, 2021). The vertical axis is on a logarithmic scale.

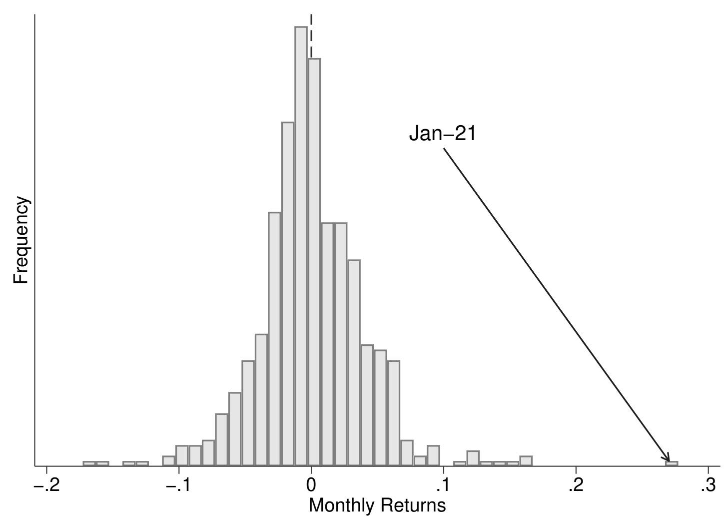


Figure 2: Histogram of Monthly Returns, 1973-2021. Equal weighted returns on a portfolio long highly shorted stocks and short the market. The arrow indicates the portfolio return in the month of January 2021.

2021 return is unprecedented historically (approximately six standard deviations larger than the historical mean). For details of the portfolio construction, see Appendix D.

The popular press attributed the unprecedented bad returns on shorting strategies to the belligerent posts on the WSB subreddit targeting the hedge funds that pursue these shorting

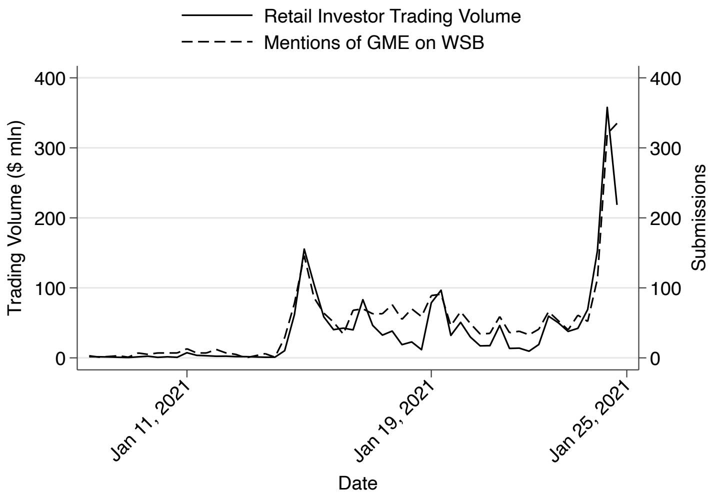


Figure 3: Retail Trading Volume in GME (January 7 - January 25, 2021). Hourly trading volume in GME, measured using the methodology of Boehmer et al. (2020), plotted against hourly mentions of the GME ticker on the WallStreetBets subreddit.

strategies. There is high-frequency evidence that this was indeed the case for Gamestop. One advantage of using high-frequency data on WSB mentions is that we are able to identify a strong "real-time" link between mentions of Gamestop on WSB and retail trading volume, which we measure in TAQ data with the methodology of Boehmer et al. (2020). Figure 3 plots the two series at hourly frequency. Over the three trading weeks shown, Gamestop mentions on WSB and retail trading purchases exhibit strikingly strong comovement (0.94 correlation). Figure 17 in Appendix E plots analogous time series of purchase volume and WSB discussion for other popular tickers over this period.

While high short-interest stocks had a higher likelihood of being the subject of discussion on the WSB subreddit, it is also important to note that the vast majority of the discussion focused on a very limited number of stocks. Our textual analysis enables the creation of high-frequency time series of ticker-mentions by aggregating mentions within a time interval. As can be seen in Figure 4, the vast majority of mentions centered around GME. From December 1, 2020 to February 1, 2021, six stocks account for about 80% of the discussion. That number

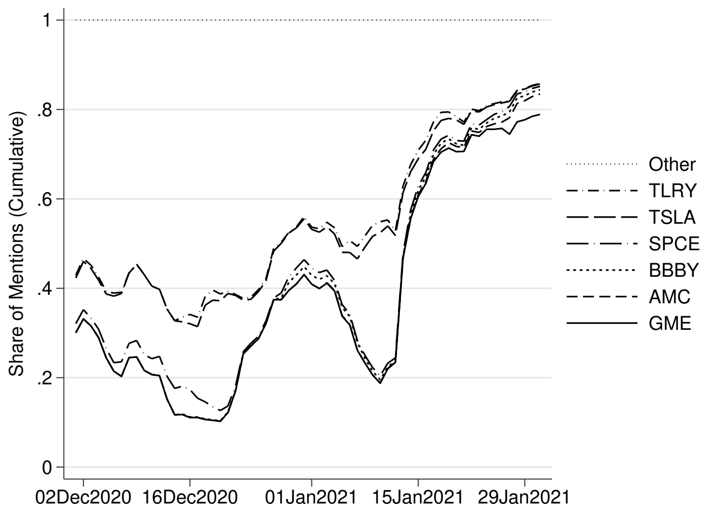


Figure 4: Cumulative fraction of discussion, (December 1, 2020 – January 31, 2021). Relative shares for ticker $s$ are computed as:

$$m_{st} = \frac{\text{Mentions}_{st}}{\sum_{s' \in S} \text{Mentions}_{s't}}$$

"Other" consists of all other tickers mentioned on WSB.

peaks to over 80% in the week of January 18–25, primarily driven by the rise in Gamestop mentions.

Despite the focus of the discussion on only a few stocks, the events surrounding Gamestop had a substantial impact on short positions — even in stocks that were not particularly discussed on the WSB subreddit. As two final pieces of motivating evidence, we show that a) shorting activity exhibited a broad retreat over the two weeks between January 15 and January 29, and b) losses on short positions extended to the larger universe of highly shorted stocks, not just those discussed on WSB.

In Figure 5, we show a bin scatter of short interest on January 15, 2021 against the subsequent decline in short interest, as well as a quadratic curve of best fit. Whereas short interest is typically highly persistent at the individual stock level, there was a systematic decline in short interest over this period. Furthermore, this decline is concentrated among

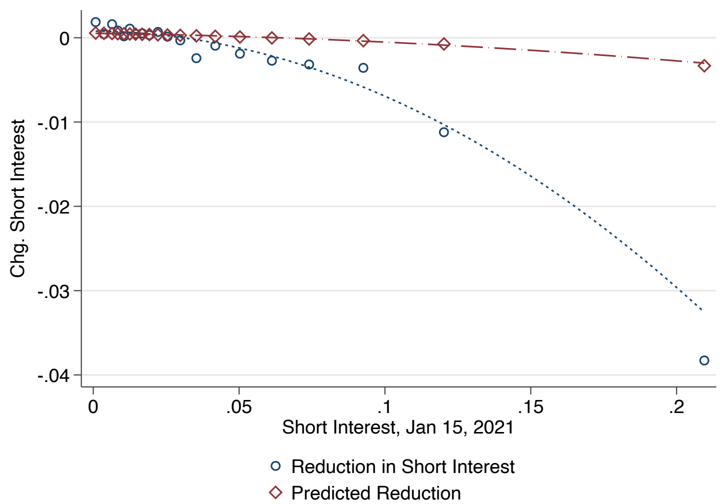


Figure 5: Change in Short Interest, January 15–January 29, 2021. Stocks are binned into one of twenty groups based on short interest as of January 15, 2021. The average percentage point reduction in short interest from January 15 to January 29, 2021 for each bin is plotted on the y-axis. Predicted reduction is computed based on an AR(1) fitted on five-percent bins of short interest fitted on historical changes in shorting data. The dashed and dotted lines represent best-fit quadratic curves.

those stocks with high short interest on January 15, 2021. Figure 5 also shows that the decline in short interest amongst stocks with high short interest is not just a manifestation of mean reversion. The change in short interest that would be predicted by a simple AR1 model would be associated with much smaller declines in short interest compared to the one observed in the data. In Figure 6 we put the January 2021 reduction in aggregate short interest into historical context. We compute the monthly reduction in short interest for stocks in our betting against the shorts trading strategy and show that the steep decline in January 2021 was a significant negative outlier. In Appendix E, we present analogous histograms controlling for the lagged level of short interest and date fixed effects .

While short sellers certainly lost money as the stock price of GME increased, the historically unparalleled losses in January 2021 on short selling were the result of broad declines across the strategy. The returns we show in Figure 2 correspond to monthly returns on

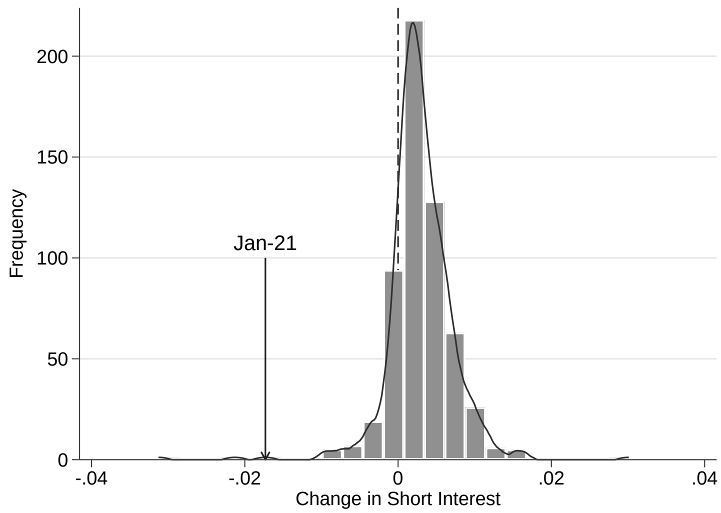


Figure 6: Monthly Change in Short Interest, January 1973-January 2021. Histogram of equal weighted reduction in short interest for stocks in the betting against the shorts long portfolio. Kernel density in black.

an equal-weighted portfolio, of which GME and other reddit stocks comprise only a small fraction. In the first column of Figure 7, we repeat that histogram, but also show that the unusual returns hold for: a) a value-weighted portfolio; b) the residuals of a regression of the equal-weighted "bet against the shorts" strategy regressed on the Fama-French factors and c) the residuals of a regression of the value-weighted "bet against the shorts" strategy regressed on the Fama-French factors. Fama-French factor portfolio returns are taken from Ken French's website. The second column repeats this exercise while explicitly excluding popular reddit tickers from the set of portfolio holdings. Finally, in the third column we restrict our sample to S&P 500 constituents in which there is high short interest. Regardless of the exact choices underlying the construction of the long-short portfolio, January 2021 was among the worst, if not the worst, months for shorting strategies (i.e., the portfolio that bets against the shorts had an unprecedentedly strong performance).

The systematic reduction in short interest in January 2021 is not the result of retail traders' security purchases. In Figure 8, we plot the abnormal retail purchase volume against

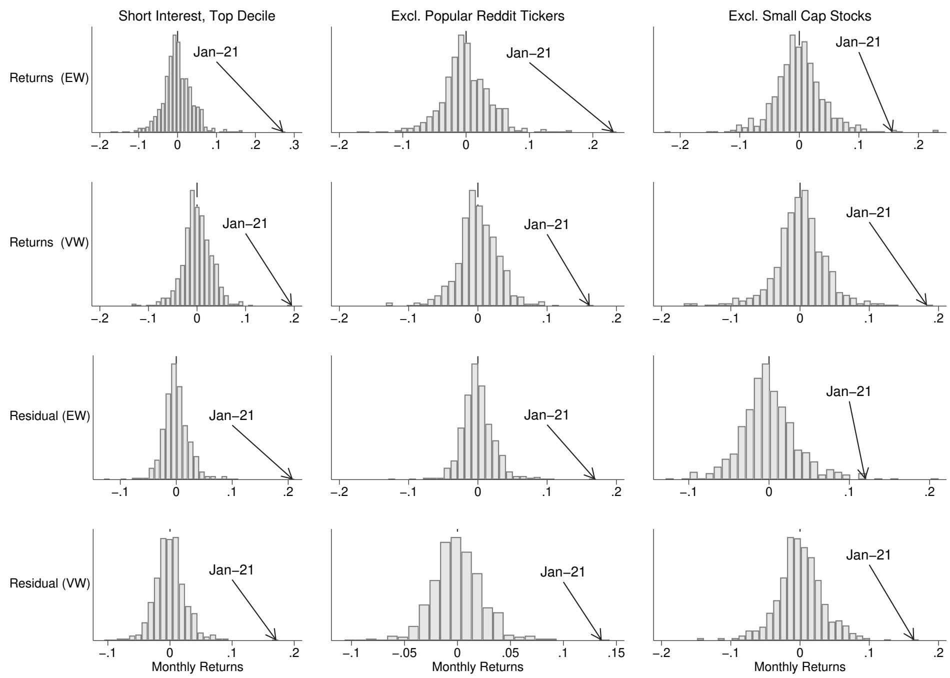


Figure 7: Monthly returns, 1973-2021. Histograms show monthly returns to a trading strategy long highly shorted stocks and short the market index. The first column shows returns to holding the top decile of stocks, sorted on short interest. The second column excludes popular stocks discussed on Reddit. The third column excludes small market capitalization stocks. Returns during the month of January 2021 are indicated with an arrow. The first row shows a histogram of unadjusted returns to the long-short strategy that equal weights stocks in the portfolio. The second row shows the same for a value-weighted portfolio. The third row shows the monthly returns to the equal-weighted portfolio, net of exposures to the 3 Fama-French factors. The fourth row shows the adjusted returns for the value-weighted portfolio.

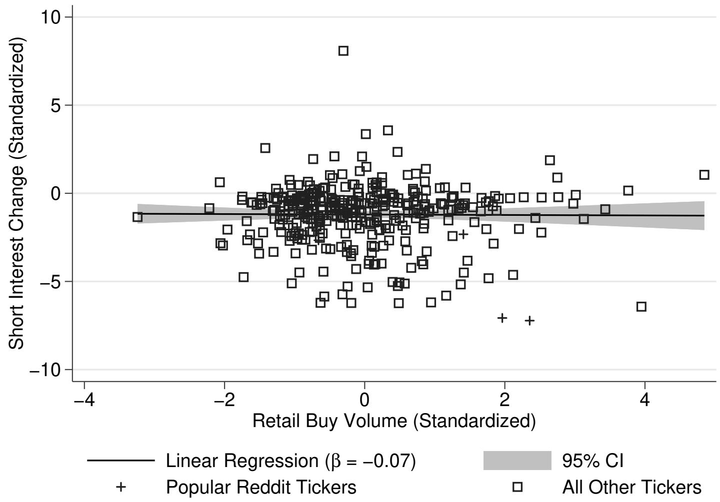


Figure 8: Cross-sectional Relationship between Retail Volume and Short Interest, January 2021. Abnormal changes in both retail purchase volume and change in short interest were calculated as standardized $z$-scores using TAQ and SEC data from January 2015 through January 2021. Each month, we calculate retail purchase volume and short interest; standardized values are demeaned and divided by the sample standard deviation on this period. Tickers which were popular discussion topics on WSB and which are also in the top decile of short interest are indicated with "+", while all other tickers are indicated with "□".

the abnormal change in short interest for all stocks in the top decile of short interest in January 2021. We can see that, while there is a negative slope coefficient, it is economically small and statistically indistinguishable from zero. Abnormal changes in both retail purchase volume and change in short interest were calculated as standardized $z$-scores using TAQ and SEC data from January 2015 through January 2021. Each month, we calculate retail purchase volume and short interest; standardized values are demeaned and divided by the sample standard deviation over this sample period.

Taken together, these panels show that the losses on shorting were broad-based and systematic, extending even to stocks not heavily discussed on reddit as well as to stocks with much larger market caps than GME. Table 1 makes this point in a regression framework, in which we show the magnitude of the portfolio return in January 2021 and test whether it is statistically different from the average return on the strategy over the full 48 year sample.

| | Highly Shorted Stocks | Excl. Popular Reddit Stocks | Excl. Small Stocks |
|---|---|---|---|
| $r^{EW}$ | 0.270 | 0.232 | 0.156 |
| | (6.728) | (5.773) | (3.491) |
| $r^{VW}$ | 0.194 | 0.161 | 0.183 |
| | (6.287) | (5.246) | (4.716) |
| $\alpha^{EW}$ | 0.207 | 0.169 | 0.120 |
| | (8.458) | (6.894) | (3.218) |
| $\alpha^{VW}$ | 0.170 | 0.136 | 0.164 |
| | (6.686) | (5.431) | (4.781) |

Table 1: Portfolio returns, January 2021. Monthly return to a long-short portfolio buying highly shorted stocks and shorting the market index. Here, $r$ denotes raw returns, and $\alpha$ residuals after controlling for Fama-French 3-factor exposure. $EW$ denotes equal weighted portfolio returns and $VW$ denotes value weighted portfolio returns. $t$-statistics on a January 2021 indicator variable are shown in parentheses. Standard errors are the larger of OLS standard errors and White standard errors.

In summary, the month of January 2021 was associated with historically bad returns for short sellers even for stocks that were not the focus of discussion on the WSB subreddit. This suggests that the events surrounding Gamestop were a game-changer for short sellers across the board. In the next section we develop a model to better understand why short selling is a strategy that is likely to be particularly exposed to changes in market fears and sentiments – to the point where any event that makes short sellers fearful could make the shorting market unravel.

# 2 Model

## 2.1 Heterogeneous agents

Time is continuous and infinite for tractability. To obtain a stationary wealth distribution, we follow Gârleanu and Panageas (2015) and assume that investors continuously arrive ("births") and depart ("deaths") from the economy. Per unit of time a mass $\pi$ of investors arrives, and a mass $\pi$ departs. By the law of large numbers, the population of agents who were born at time $s \leq t$ is $\pi e^{-\pi(t-s)}$, while the total population is constant and equal to $\int_{-\infty}^{t} \pi e^{-\pi(t-s)} ds = 1$.

To introduce trade in equities, we assume that investors have heterogeneous beliefs. For simplicity, a fraction $\nu \in (0,1)$ of investors perceive the correct data-generating process. We refer to them as rational investors ("$R$" investors). The remaining fraction are overly optimistic (we model this optimism shortly), and we refer to these investors as "$I$" investors.

For tractability, both investors have logarithmic utilities and their expected discounted utility from consumption is

$$V_t^i \equiv E_t^{(i)} \int_t^{\infty} e^{-(\rho+\pi)(u-t)} \log\left(c_{u,t}^i\right) du \tag{1}$$

for $i \in \{I, R\}$, with $\rho$ a discount factor and $c_{u,t}^i$ the time-$u$ consumption of an agent of type $i$ born at time $t \leq u$. The notation $E_t^{(i)}$ reflects the different investor beliefs. Because of death, the effective discount rate is $\rho + \pi$.

Before proceeding we comment on some of our assumptions. While it is crucial for our results that investors have heterogeneous beliefs (so as to introduce a motivation for trading), the assumption that one group has correct beliefs helps mostly to save notation and can be easily relaxed. Similarly, the overlapping-generations structure is just a technical device to ensure that no investor type disappears in the long run.[5]

## 2.2 Endowments

In order to support their consumption over their lives, we assume that the arriving investors at time $t$ are equally endowed with shares of new "trees," which are created at time $t$. Letting

[5]In particular, the lack of inter-generational risk sharing, which is a feature of some of these models, is not driving any of the results in this paper.

$s \leq t$ denote the time of creation of a tree, we specify its time-$t$ dividends as

$$D_{t,s} = \delta e^{-\delta(t-s)} D_t, \tag{2}$$

where $\delta > 0$ captures depreciation and $D_t$ follows a geometric Brownian motion with mean $\mu_D$ and volatility $\sigma_D > 0$,

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t, \tag{3}$$

with $B_t$ a standard Brownian motion. Accordingly, the time-$t$ total endowment of this economy is the sum of the endowment produced by all trees born up to to time $t$,

$$\int_{-\infty}^{t} D_{t,s} ds = \left( \int_{-\infty}^{t} \delta e^{-\delta(t-s)} ds \right) \times D_t = D_t.$$

The assumption that investors are endowed with shares of newly arriving trees follows Gârleanu et al. (2012) and Panageas (2020). We adopt this assumption rather than introducing labor income (as in Gârleanu and Panageas, 2015 or Gârleanu and Panageas, 2020), because –for the purposes of this paper– labor income would just complicate matters without providing any novel insights.

## 2.3 Beliefs

The irrational investors are optimistic and believe that the aggregate endowment grows at the rate $\mu^I > \mu_D$. Irrational investors hold this optimistic view over their life-time and do not learn ("dogmatic beliefs"). Introducing learning would be a distraction for the purposes of this paper and therefore we omit it.

Letting

$$\eta \equiv \frac{\mu^I - \mu_D}{\sigma_D},$$

define $Z_t^I$ as

$$Z_t^I \equiv e^{-\frac{(\eta)^2}{2}t + \eta B_t}. \tag{4}$$

The process $Z_t^I$ constitutes the likelihood ratio between the correct probability measure and the probability measure perceived by the irrational investors. An implication of Girsanov's theorem is that we can write an irrational agent's maximization objective as

$$E_t^{(I)} \int_t^\infty e^{-(\rho+\pi)(u-t)} \log\left(c_{u,t}^I\right) du = E_t \int_t^\infty e^{-(\rho+\pi)(u-t)} \left(\frac{Z_u^I}{Z_t^I}\right) \log c_{u,t}^I du. \tag{5}$$

## 2.4 Dynamic budget constraint and short-selling frictions

As in Gârleanu et al. (2012) and Panageas (2020) the arriving investors support their lifetime consumption by selling their firms into the stock market. These firms become part of the market index (the "market portfolio"). Given our assumptions, the flow of arriving companies $P_{t,t}$ over total market capitalization $P_t$ is $\frac{P_{t,t}}{P_t} = \delta$.

Letting $P_t$ denote aggregate stock market capitalization, the instantaneous return of the market portfolio is

$$\begin{aligned} dR_t &= \underbrace{\frac{dP_t}{P_t}}_{\text{Change in aggregate market cap.}} - \underbrace{\frac{P_{t,t}}{P_t}dt}_{\text{cost to purchase the new firms}} + \underbrace{\frac{D_t}{P_t}dt}_{\text{dividend yield}} \\ &= \frac{dP_t + D_t dt}{P_t} - \delta dt. \end{aligned}$$

The definition of $dR_t$ reflects the fact that existing investors need to pay arriving investors to purchase the new firms, and hence the increase in stock market capitalization, $\frac{dP_t}{P_t}$, needs to be reduced by the payments that existing investors need to make to new investors, $\frac{P_{t,t}}{P_t}$.[6]

Aside from investing in (positive supply) shares of the market portfolio and (zero net supply) risk-free assets, we follow Blanchard (1985) in assuming that each investor annuitizes her entire wealth (since there are no bequest motives) by pledging it to an insurance company upon death in exchange for receiving an income stream while alive. Assuming a perfectly competitive insurance market, the income stream is $\pi W_{t,s}$ per unit of time $dt$, allowing the insurance company to break even.

The main departure from a frictionless market is that if investors want to short stocks, they have to pay a lending fee, $f_t$. Specifically, letting $W^i_{t,s}$ denote the time-$t$ wealth of an investor of type $i$ who was born at time $s \leq t$ and $w^i_{t,s}$ the fraction of wealth invested in stocks, the dynamic budget constraint is

$$dW^i_{t,s} = W^i_{t,s}\left(r_t + \pi + n_t + w^i_{t,s}\left(\mu_t - r_t + \lambda^i_{t.s}\right) - \frac{c^i_{t,s}}{W^i_{t,s}}\right)dt + w^i_{t,s}W^i_{t,s}\sigma_t dB_t, \tag{6}$$

where $\mu_t$ and $\sigma_t$ are the equilibrium expected return and volatility (respectively) of a stock investment and $r_t$ is the equilibrium interest rate.

The terms $n_t$ and $\lambda^i_{t,s}$ are the two terms that capture the presence of shorting frictions.

[6] For a more detailed derivation, start from $P_t = \int_{-\infty}^t P_{t,s}ds$. Time-differentiating $\frac{dP_t}{P_t}$, using Leibniz's rule, and adding $\frac{D_t}{P_t} = \frac{\int D_{t,s}ds}{P_t}$ we obtain

$$\begin{aligned}\frac{dP_t}{P_t} + \frac{D_t}{P_t}dt &= \frac{\int_{-\infty}^t \left(dP_{t,s} + D_{t,s}\right)ds}{P_t} + \frac{P_{t,t}}{P_t}dt = \int_{-\infty}^t \left(\frac{P_{t,s}}{P_t}\right)\left(\frac{dP_{t,s} + D_{t,s}}{P_{t,\,s}}\right)ds + \frac{P_{t,t}}{P_t}dt \\ &= \int_{-\infty}^t w_{t,s}\left(\frac{dP_{t,s} + D_{t,s}}{P_{t,\,s}}\right)ds + \frac{P_{t,t}}{P_t}dt = dR_t + \frac{P_{t,t}}{P_t}dt = dR_t + \delta dt,\end{aligned}$$

where $w_{t,s}$ are market-capitalization weights and the equality $dR_t = \int_{-\infty}^t w_{t,s}\left(\frac{dP_{t,s}+D_{t,s}dt}{P_{t,\,s}}\right)ds$ follows from the fact that the market portfolio is a self-financing strategy.

The term $\lambda^i_{t,s}$ is defined as

$$\lambda^i_{t.s} \equiv f_t \times \left(1_{\left\{w^i_{t,s}<0\right\}} + \tau y_t 1_{\left\{w^i_{t,s}\geq 0\right\}}\right), \tag{7}$$

where $y_t$ is the fraction of a long portfolio that is loaned out by the representative "brokerage house" and $\tau$ is the fraction of the lending fees that accrues to the investor. (We discuss the optimizing choice of $y_t$ by the brokerage house and the term $n_t$ shortly.) Equation (7) reflects that an investor with a short position $w^i_{t,s} < 0$ has to pay a multiple $f_t$ of the value of her entire short position, $|w^i_{t,s}|W^i_{t,s}$, so that the post-fee rate of return per dollar shorted is $-(\mu_t - r_t + f_t)\,dt - \sigma_t dB_t$. Similarly, an investor holding a positive position, $w^i_{t,s} > 0$, obtains a rate of return equal to $(\mu_t - r_t + \tau y_t f_t)\,dt + \sigma_t dB_t$ on her stock investments.

Market clearing in the lending market requires

$$y_t \overline{W}^+_t = \overline{W}^-_t, \tag{8}$$

where $\overline{W}^-_t$ is the value of the aggregate short interest and $\overline{W}^+_t$ that of the aggregate long position,

$$\overline{W}^-_t \equiv \sum_{i\in\{I,R\}} \int_{-\infty}^{t} |w^i_{t,s}|W^i_{t,s} 1_{w^i_{t,s}<0} ds, \text{ and } \overline{W}^+_t \equiv \sum_{i\in\{I,R\}} \int_{-\infty}^{t} w^i_{t,s} W^i_{t,s} 1_{w^i_{t,s}>0} ds.$$

To model the choice $y_t$ as an optimizing choice, in Appendix A we introduce competitive firms ("brokerage houses") who obtain an investor's long portfolio and choose the fraction of shares that they supply to the lending market, $y_t$. Placing shares on the lending market allows brokerage houses to collect lending revenue equal to the prevailing lending fee times the value of shares loaned. However, lending shares is a costly activity, which requires a resource ("human capital ") owned by households. Brokerage houses pay a fraction $\tau$ of the lending fees to (long-portfolio) investors in order to obtain their permission to place their shares on the lending market. The remaining $1-\tau$ fraction of lending revenue is used by the competitive broker to compensate households for providing their human capital. Appendix A shows that the optimizing choice of $y_t$ by the brokerage firms results in an upward-sloping relation $f_t = l\left(y_t\right)$ with $l'\left(\cdot\right) \geq 0$. We refer to $l(y_t)$ as the supply curve for lendable shares.

The appendix also shows that compensating the household sector (as renumeration for services rendered) results in the following expression for $n_t$ in equation (6),

$$n_t = \frac{\left(1-\tau\right) f_t W_t^-}{W_t}. \tag{9}$$

Using (8), (9) and aggregating across all households we obain

$$\begin{aligned} f_t \overline{W}_t^- &= \left(1-\tau\right) f_t \overline{W}_t^- + \tau f_t \overline{W}_t^- \\ &= n_t W_t + \tau f_t y_t \overline{W}_t^+. \end{aligned} \tag{10}$$

The left hand side of (10) reflects the aggregate lending fees $f_t \overline{W}_t^-$. The right-hand side reflects the ultimate division of lending income between the households (who obtain a fraction $1-\tau$ of lending, irrespective of their portfolio) and long investors, who obtain a fraction $\tau$ of the lending income.

Equation (10) shows that share lending does not result in any loss of aggregate resources: All payments made by investors with short positions are received either by investors with long positions or by brokerage firms, who rebate them to the household sector. This shows the dual role played by brokerage firms. On the one hand, their optimizing choices provide a micro-foundation for an upward sloping supply curve $f_t = l(y_t)$. On the other hand, they help capture the notion that only a fraction $\tau$ of lending fees is received by long investors, with the remainder of the lending fees being rebated to the households in a manner that does not impact their portfolio choice.

## 2.5 Equilibrium

Equilibrium in the lending market requires that the lending fee be such that the supply of loanabe shares $y_t\overline{W}_t^+ = l^{-1}(f_t)\overline{W}_t^+$ is equal to the demanded short interest, $\overline{W}_t^-$ (equation (8)).

The rest of the equilibrium definition is standard. We require that investors $I$ and $R$ maximize (1) over $c_{t,s}^i$ and $w_{t,s}^i$ subject to the budget constraint (6), and $\mu_t$, $r_t$, and $\sigma_t$ are such that the bond market clears, $\sum_{i\in\{I,R\}}\int_{-\infty}^t \nu^i\left(1-w_{t,s}^i\right)W_{t,s}^i ds = 0$, the stock market clears, $\sum_{i\in\{I,R\}}\int_{-\infty}^t \nu^i w_{t,s}^i W_{t,s}^i ds = P_t$, where $P_t$ is aggregate stock market capitalization, and the goods market clears, $\sum_{i\in\{I,R\}}\int_{-\infty}^t \nu^i c_{t,s}^i ds = D_t$. By Walras' Law, market clearing of the bond market implies stock market clearing and vice versa, and accordingly the asset-market clearing requirements can be written equivalently as $W_t = \sum_{i\in\{I,R\}}\int_{-\infty}^t \nu^i W_{t,s}^i ds = P_t$.

For future reference, we note that the clearing of the stock market requires that $y_t < 1$:

$$y_t = \frac{\overline{W}_t^-}{W_t^+} = \frac{\overline{W}_t^-}{P_t + \overline{W}_t^-} < 1.$$

# 3 Analysis

We analyze the model in two steps. First, we consider a special parametric case that allows us to characterize all equilibrium quantities in closed form. The special case we analyze is the "elastic supply" case, that is, the limiting case where the supply of loanable shares is horizontal at some level $l(y_t) = \beta$. (As we explain in Appendix A, this special case corresponds to a linear specification for the disutility of lending out shares.) In Section 6 we repeat the analysis for increasing functions $l(y_t)$ and show how the key results readily extend to this more general case.

## 3.1 Optimal portfolio and consumption

For a log investor the wealth-to-consumption ratio is constant and equal to

$$\frac{c_{t,s}^i}{W_{t,s}^i} = \rho + \pi. \tag{11}$$

Another convenient property of logarithmic utility is that the portfolio is myopic and maximizes the logarithmic growth rate of an investor's wealth,

$$w_{t,s}^i = \arg\max_{w_{t,s}^i} \left\{ w_{t,s}^i \left( \mu_t + \eta \sigma_t 1_{\{i=I\}} - r_t + \lambda_{t,s}^i \right) - \frac{1}{2} \left( w_{t,s}^i \sigma_t \right)^2 \right\},$$

where $1_{\{i=I\}}$ is an indicator function taking the value one when $i = I$ and zero otherwise. Letting

$$\widehat{\mu}_t^i \equiv \mu_t + \eta \sigma_t 1_{\{i=I\}}$$

denote the expected return on the stock as perceived by investor $i \in \{I, R\}$, the optimal portfolio is

$$w_{t,s}^i = \begin{cases} \frac{\widehat{\mu}_t^i - r_t + f_t}{\sigma_t^2} & \text{if } \widehat{\mu}_t^i - r_t + f_t < 0 \\ \frac{\widehat{\mu}_t^i - r_t + \tau f_t y_t}{\sigma_t^2} & \text{if } \widehat{\mu}_t^i - r_t + f_t y_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

By inspection, the optimal portfolios do not depend on the cohort $s$, only on the type of investor $i \in \{I, R\}$ . Therefore, from now on we drop the subscript $s$ and write $w_t^R$ and $w_t^I$.

One straightforward implication of equation (12) is that if investor $R$ is actively shorting ($w_t^R < 0$) then it must be the case that the excess rate of return per dollar shorted is positive, even after netting out the fee $f_t$. Indeed, evaluating (12) with $i = R$, assuming that $w_t^R < 0$

and re-arranging leads to $-(\mu_t - r - f_t) = -(\widehat{\mu}_t^R - r - f_t) = -w_t^R\sigma_t^2 > 0$. The term $-w_t^R\sigma_t^2$, which corresponds to the absolute value of the covariance of the stock's return with the short seller's portfolio, is the risk compensation to the short seller for taking a short position.

## 3.2 Equilibrium

It is useful to start by defining the consumption share of each type of agent $i \in \{I, R\}$:

$$x_t^i \equiv \frac{\nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} c_{t,s}^i ds}{D_t}.$$

Using (11), the wealth-weight $\omega_t^i$ and the consumption weight $x_t^i$ coincide:

$$\omega_t^i \equiv \frac{\nu^i \int_{-\infty}^t \pi e^{-\pi(t-s)} W_{t,s}^i ds}{W_t} = \frac{(\rho+\pi)^{-1} x_t^i}{\sum_{i\in\{I,R,S\}} (\rho+\pi)^{-1} x_t^i} = x_t^i, \tag{13}$$

where $W_t$ is aggregate wealth.

With these definitions in hand, the goods market and stock market clearing requirements imply

$$\begin{aligned} D_t &= \sum_{i\in\{I,R\}} \int_{-\infty}^t \nu^i c_{t,s}^i ds = (\rho+\pi) \sum_{i\in\{I,R\}} \int_{-\infty}^t \nu^i W_{t,s}^i ds \\ &= (\rho+\pi) W_t = (\rho+\pi) P_t. \end{aligned} \tag{14}$$

Taking logarithms gives $d\log D_t = d\log P_t$ and therefore the stock market volatility equals $\sigma = \sigma_D$. The implication that the volatility of the stock is constant and equal to the volatility of dividend growth is convenient for obtaining closed-form solutions. In Section 5 we discuss extensions of the model that allow for a time-varying price-dividend ratio and volatility.

Applying Itô's Lemma to (13) and using (6) and (14) yields

$$d\omega_t^i = \mu_t^i dt + \sigma_t^i dB_t,$$

where

$$\sigma_t^i = \omega_t^i \left(w_t^i - 1\right) \sigma_D, \tag{15}$$

$$\mu_t^i = \omega_t^i \left(-\mu_D + \sigma_D^2 - \pi + r_t - \rho + w_t^i \left(\mu_t - r_t + s_t^i\right) - w_t^i \sigma_D^2\right) + \nu^i \delta. \tag{16}$$

The market clearing requirement $\sum_{i\in\{I,R\}} \omega_t^i = 1$ implies that $\sum_{i\in\{I,R\}} d\omega_t^i = 0$ and therefore $\sum_{i\in\{I,R\}} \sigma_t^i = 0$ and $\sum_{i\in\{I,R\}} \mu_t^i = 0$. As mentioned earlier, in an effort to obtain a tractable solution, we assume that the supply of loanable shares is perfectly elastic at the rate $\beta$:

**Assumption 1** $l(y) = \beta > 0$.

We maintain this assumption until Section 6 in order to develop intuition. In Section 6 we generalize the results to an upward-sloping supply function $l(\cdot)$, so that the lending fees increase with short interest.

In preparation for the description of the equilibrium, we start with the following definition and assumptions on the parameters.

**Definition 1** *Let*

$$\omega^{(1)} \equiv \frac{\eta - \sigma_D - \frac{\beta}{\sigma_D}}{\eta - \frac{\beta}{\sigma_D}}. \tag{17}$$

*and*

$$F(\omega) \equiv \left(\sigma_D - \omega\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right)\right)^2 - 4\tau\frac{\omega^2}{1-\omega}\frac{\beta}{\sigma_D}\left(\sigma_D + (1-\omega)\left(\frac{\beta}{\sigma_D} - \eta\right)\right). \tag{18}$$

The following assumption collects all the requirements we impose on the parameters.

**Assumption 2** *Assume that $\eta$, $\beta$, $\sigma_D$, and $\tau$ are such that*

$$(1+\tau)\frac{\beta}{\sigma_D} > \eta > \frac{\beta}{\sigma_D} \tag{19}$$

$$\omega^{(1)} > \frac{\sigma_D}{(1+\tau)\frac{\beta}{\sigma_D} - \eta} > 0 \tag{20}$$

*and $F(\omega)$ has a unique root in the interval $(0,1)$, denoted by $\omega^{(2)}$.*

The following proposition asserts that the set of parameters $\eta$, $\beta$, $\sigma_D$, and $\tau$ that satisfy Assumption 2 is non-empty.

**Proposition 1** *There is an open set of positive values $\eta$, $\beta$, $\sigma_D$, and $\tau$ that jointly satisfy Assumption 2.*

The next proposition describes the equilibria in our economy.

**Proposition 2** *Suppose that Assumption 2 holds. Then $\omega^{(2)} > \omega^{(1)}$ and the equilibria in this economy can be described as follows.*

*i) If $\omega_t^R \in (\omega^{(2)}, 1]$ there is no short-selling in equilibrium. The equilibrium is unique and the Sharpe ratio $\kappa_t \equiv \frac{\mu_t - r_t}{\sigma_D}$ is given by*

$$\kappa_t = \begin{cases} \sigma_D - \left(1-\omega_t^R\right)\eta & \text{if } \omega_t^R > 1 - \frac{\sigma_D}{\eta} \\ \frac{\sigma_D}{1-\omega_t^R} - \eta & \text{if } \omega_t^R \in (\omega^{(2)}, 1 - \frac{\sigma_D}{\eta}] \end{cases}. \tag{21}$$

*ii) If $\omega_t^R \in [\omega^{(1)}, \omega^{(2)}]$, then there are three equilibria. The first equilibrium continues to be given by (21) and involves no short-selling. The second and third equilibria involve shorting and the ratio of shorted-to-loanable shares $y_t$ corresponds to the two roots $y^+$ and $y^-$ of the* *quadratic equation*

$$y\left(\eta+\frac{\sigma_D}{\omega_t^R}-\frac{\beta}{\sigma_D}\left(1-\tau y\right)\right)-\left(\eta-\frac{\sigma_D}{1-\omega_t^R}-\frac{\beta}{\sigma_D}\left(1-\tau y\right)\right)=0, \tag{22}$$

*which has two real roots* $y^+$ *and* $y^-$ *in the interval* $(0,1)$*. The Sharpe ratio in the equilibrium associated with* $y^+$ *(resp.* $y^-$*) is*

$$\kappa_t^{\pm}=\sigma_D-\left(1-\omega_t^R\right)\eta-\frac{\omega_t^R}{\sigma_D}\beta\left(1+\tau y^{\pm}\frac{1-\omega_t^R}{\omega_t^R}\right). \tag{23}$$

*iii) If* $\omega_t\in[0,\omega^{(1)})$*, then the equilibrium is unique and involves shorting. In this case only the larger of the two roots (*$y^+$*) of equation (22) lies in the interval* $(0,1)$*, and the unique equilibrium Sharpe ratio is given by* $\kappa^+$*.*

*In all three cases the interest rate is given by*

$$r_t=\rho+\pi+\mu_D-\delta-\kappa_t\sigma_D. \tag{24}$$

*Additionally, because* $\kappa_t$*,* $r_t$*, and* $y_t$ *are functions of* $\omega_t^R$*, equations (12), (15), and (16) imply that* $\mu_t^R$ *and* $\sigma_t^R$ *are functions of* $\omega_t^R$ *and hence the equilibrium is Markov in* $\omega_t^R$*.*

Figure 9 illustrates Proposition 2. The figure plots $\kappa\left(\omega_t^R\right)$, the Sharpe ratio, as a function of the wealth share of rational agents.

As a benchmark, the line labeled "Zero shorting cost" depicts $\sigma_D-\left(1-\omega_t^R\right)\eta$, i.e., the Sharpe ratio that would obtain in this economy in the absence of any shorting frictions ($\beta=0$). The curve "No shorting Equil." depicts the Sharpe ratio in the equilibrium that involves no shorting and for the values of $\omega_t^R$ that such an equilibrium exists. Similarly for the curves "Medium shorting" and "High shorting," which depict equilibria with shorting, assuming that the value of $\omega_t^R$ permits such equilibria.

The figure shows that when $\omega_t^R$ is larger than $1-\frac{\sigma_D}{\eta}$ the lines "Zero shorting cost" and

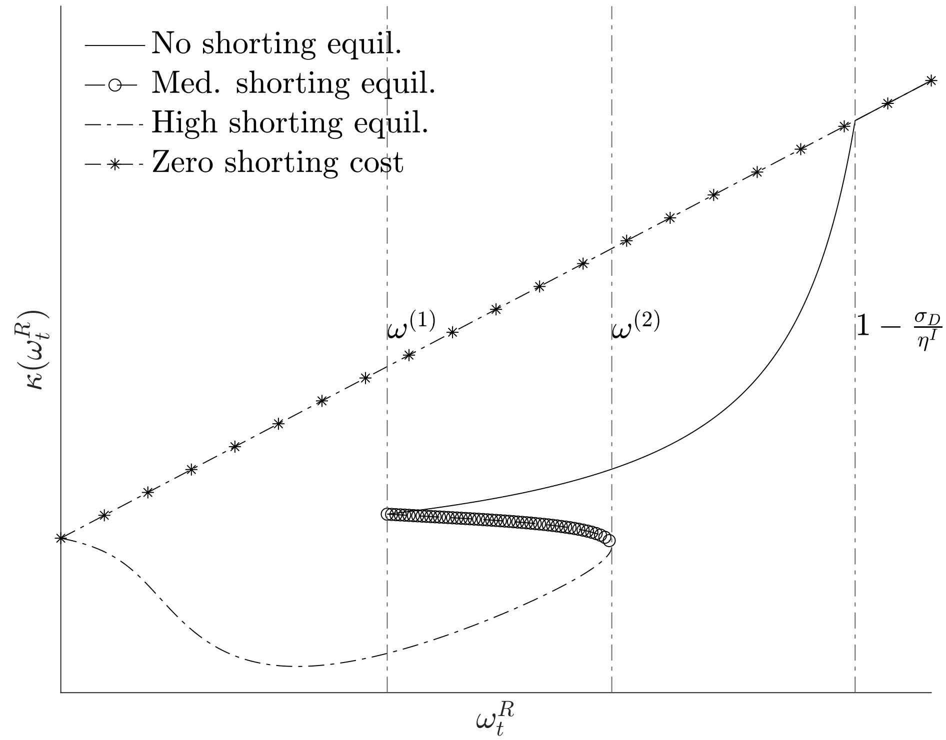


Figure 9: An illustration of Proposition 2

"No shorting Equil." coincide, reflecting that all investors invest strictly positive amounts in the stock market in this region.

When $\omega_t^R$ becomes smaller than $1 - \frac{\sigma_D}{\eta}$ (but larger than $\omega^{(2)}$), the rational investor puts zero weight on stocks, but the shorting fee $\beta$ deters her from actively short-selling. Since only the irrational investor is marginal in financial markets, the lines "Zero shorting cost" and "No shorting Equil." deviate from each other when $\omega_t^R < 1 - \frac{\sigma_D}{\eta}$ . In this region the magnitude of the lending fee, $\beta$, does not impact the Sharpe ratio directly, (except for deterring the $R$ investors from actively shorting).

If $\omega_t^R$ becomes smaller than $\omega^{(2)}$ (but larger than $\omega^{(1)}$) the economy exhibits three equilibria. In the first equilibrium, there is still no shorting. In the second and third, there is

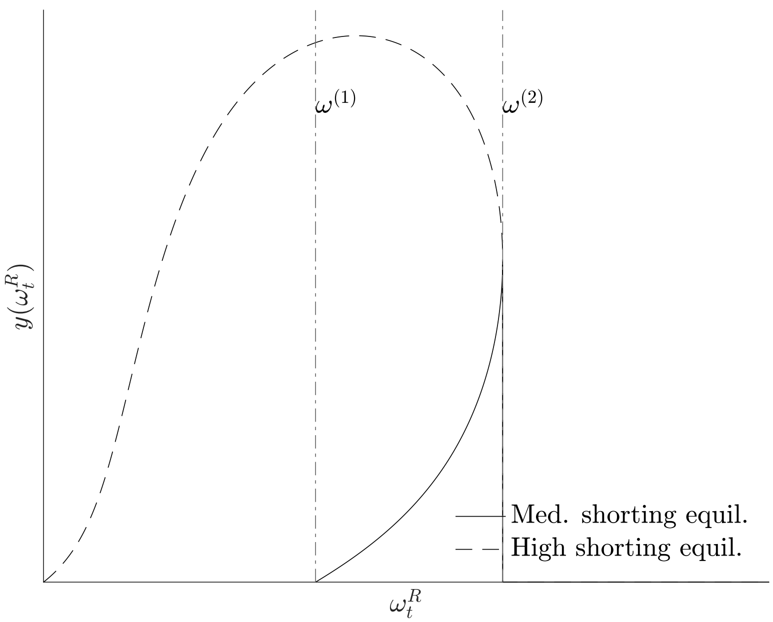


Figure 10: The ratio of shorted-to-loanable shares, $y_t$ in the equilibria of model as a function of $\omega_t^R$.

active shorting by the rational investor. Across these three equilibria, the higher the extent of shorting, the lower the Sharpe ratio. This is illustrated in Figure 10. If $\omega_t^R$ becomes smaller than $\omega^{(1)}$, then the equilibrium becomes unique and involves shorting.[7]

Several features of Figure 9 are noteworthy. First, the Sharpe ratio is always lower than it would be in the absence of lending fees, even if investor $R$ is not actively shorting shares, but is only investing in bonds.

Second, the presence of a region where multiple equilibria co-exist is not a very common feature of asset pricing models, especially when there is only one good and one positive-

[7] To see why an equilibrium without shorting can no longer be an equilibrium when $\omega_t^R < \omega^{(1)}$, assume otherwise. Indeed assume that the $R$ investor holds zero stocks and is not marginal in the stock market $\left(w_t^R = 0\right)$. The market clearing requirement, $\omega_t^R w_t^R + \left(1 - \omega_t^R\right) w_t^I = 1$, along with $w_t^I = \frac{\kappa_t + \eta}{\sigma_D}$ implies that the Sharpe ratio would be $\kappa_t = \frac{\sigma_D}{1 - \omega_t^R} - \eta$. Under this supposition, it would therefore be the case that $\mu_t - r + \beta = \sigma_D \left(\kappa_t + \frac{\beta}{\sigma^D}\right) = \sigma_D \left(\frac{\sigma_D}{1 - \omega_t^R} - \eta + \frac{\beta}{\sigma^D}\right) < 0$, where the inequality follows from $\omega_t^R < \omega^{(1)}$. Because $\mu_t - r + \beta < 0$, equation (12) implies that the $R$ investor would want to short the market, contradicting the assumption that she is optimally holding zero stocks.

supply asset. To better understand the source of this multiplicity, it is useful to provide a concise derivation of the key statements in Proposition 1.

Specifically, suppose that we consider equilibria that involve active shorting ($w_t^R < 0$). In such equilibria, the optimal portfolio holdings can be expressed as

$$w_t^R = \frac{\kappa_t + \frac{\beta}{\sigma_D}}{\sigma_D} \tag{25}$$

$$w_t^I = \frac{\kappa_t + \eta + \frac{\beta}{\sigma_D}\tau y_t}{\sigma_D}, \tag{26}$$

while asset market clearing requires

$$\omega_t^R w_t^R + \left(1 - \omega_t^R\right) w_t^R = 1. \tag{27}$$

Combining equations (25)–(27) leads to

$$\kappa_t = \sigma_D - \left(1 - \omega_t^R\right)\eta - \frac{\omega_t^R}{\sigma_D}\beta\left(1 + \tau y_t \frac{1 - \omega_t^R}{\omega_t^R}\right), \tag{28}$$

which is equation (23) of Proposition 1. Note that the partial derivative of $\kappa_t$ with respect to $y_t$ is negative. This is intuitive: A higher value of $y_t$ increases the effective rate of return to (long-portfolio) stock holders ($I$ investors). The increased appetite by $I$ investors to hold long positions lowers the Sharpe ratio. (Phrased differently, the absolute value of the Sharpe ratio increases, since the Sharpe ratio is negative when $w_t^R < 0$.)

This lowering of the Sharpe ratio strengthens the short-sellers' appetite to borrow the stock and short it. In equilibrium, the increased shorting demand raises the ratio of shorted-to-loanable shares, $y_t$, increasing the effective return to $I$ investors, which further reduces the Sharpe ratio, etc.

These self-reinforcing effects are the root cause of the multiple equilibria. The easiest way to see this is by completing the computation of the Sharpe ratio, which requires us to determine the value of $y_t$ that clears the lending market. Indeed, in any equilibrium involving $w_t^R < 0$ and $w_t^I > 0$ we must have

$$y_t = \frac{W_t^-}{W_t^+} = \frac{-w_t^R W_t^R}{w_t^I W_t^I} = -\frac{w_t^R}{w_t^I} \times \frac{\omega_t^R}{1-\omega_t^R}. \tag{29}$$

Using (25) to compute the ratio $\frac{w_t^R}{w_t^I}$ gives

$$\begin{aligned} y_t &= -\frac{\kappa_t + \frac{\beta}{\sigma_D}}{\kappa_t + \eta^I + \frac{\beta}{\sigma_D}\tau y_t} \times \frac{\omega_t^R}{1-\omega_t^R} \\ &= \frac{\left(1-\omega_t^R\right)\eta - \sigma_D - \frac{\left(1-\omega_t^R\right)\beta}{\sigma_D}\left(1-\tau y_t\right)}{\sigma_D + \omega_t^R \eta - \frac{\omega_t^R \beta}{\sigma_D}\left(1-\tau y_t\right)} \times \frac{\omega_t^R}{1-\omega_t^R}, \end{aligned} \tag{30}$$

where the last line follows from (28) after collecting terms and simplifying. Rearranging (30) gives the quadratic equation (22), which is the key equation of Proposition 1. The rest of the proposition is devoted to studying this quadratic equation and confirming that its roots correspond to valid equilibria.

While equation (30) is particularly simple to analyze, the multiplicity of equilibria does not hinge on assuming that the supply curve $l\left(y_t\right)$ is constant at the level $\beta$, as we show in Section 6.

The intuition behind the multiplicity of equilibria is contained in equation (30). For a given wealth distribution and belief discrepancy, a higher $y_t$ makes long investors content with holding the same positive position at a lower equilibrium Sharpe ratio. This negative relation between the Sharpe ratio and $y_t$ is responsible for equilibrium multiplicity: For instance, if something prompts rational investors to abandon their short positions, the resulting reduction in lending income requires a higher Sharpe ratio to clear the market. But this rise in the Sharpe ratio reinforces the incentive of short sellers to abandon their positions, which lowers lending income, and further raises the Sharpe ratio, etc., until the market settles on a new equilibrium with (possibly zero) short interest.

# 4 Properties of the Equilibria

In Section 4.1 we show that equilibria with high shorting are beneficial for $R$ investors. This implies that the worst possible outcome for $R$ investors is for markets to coordinate on the equilibrium that deters them from short selling. Sections 4.2 and 4.3 discuss some broader implications of the model that are unrelated to coordination, but help further illustrate the model's key intuitions. Specifically, we perform comparative statics with respect to changes in $\beta$, which capture shifts in the supply of loanable shares. Section 4.2 shows how marginal changes in the supply curve can lead to discontinuous drops in short interest. Section 4.3 shows that exogenous shifts in the supply of loanable shares may impact lending fees and short interest but have a muted impact on equilibrium expected returns.

## 4.1 Dynamics of the wealth shares

The three equilibria we identified above have different implications for the dynamics of the wealth shares of $R$ investors. The next proposition shows that both the drift rate $\mu_t^R\left(\omega_t^R\right)$ of the wealth share of type $R$ investors and the expected logarithmic growth rate of their wealth are higher in equilibria that feature higher short interest $y_t$.

**Proposition 3** *For a fixed wealth share of the R-agents, $\omega_t^R$, consider two equilibria A and B with the following properties:*

1. $w_t^R \leq 0$ *in both equilibria A and B.*

2. $y_t^B > y_t^A$ *(and accordingly $\kappa_t^B < \kappa_t^A$).*

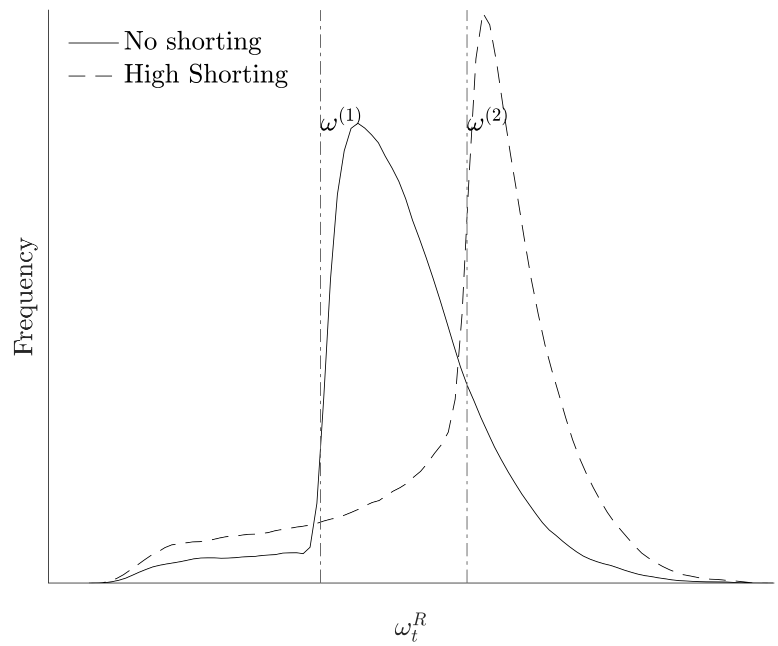


Figure 11: An illustration of Proposition 3. The figure depicts the stationary distribution of $\omega_t^R$ for the "high shorting" and the "no shorting" equilibrium of Figure 9.

*Then the drift of investor R's wealth share, $\mu_t^R\left(\omega_t^R\right)$, satisfies*

$$\mu_t^{B,R}\left(\omega_t^R\right) > \mu_t^{A,R}\left(\omega_t^R\right),$$

*where $\mu_t^{A,R}\left(\omega_t^R\right)$ is the drift of investor R's wealth in equilibrium A and $\mu_t^{B,R}\left(\omega_t^R\right)$ is the respective drift in equilibrium B.*

*In addition, the drift of the logarithmic growth rate of investor R, defined as*

$$g_t^R \equiv r_t + \max_{w^R \leq 0}\left\{w_t^R\left(\kappa_t\sigma_D + \beta\right) - \frac{1}{2}\left(w_t^R\sigma_D\right)^2\right\} - \left(\rho + \pi\right), \tag{31}$$

*is higher in equilibrium B than in equilibrium A, i.e., $g_t^{B,R}\left(\omega_t^R\right) > g_t^{A,R}\left(\omega_t^R\right)$.*

Figure 11 provides an illustration of Proposition 3. The figure shows the stationary distribution of $\omega_t^R$ in the equilibrium associated with no shorting for values $\omega_t^R \in (\omega^{(1)}, \omega^{(2)})$ and in the equilibrium associated with the highest shorting, $y^+\left(\omega_t^R\right)$, for $\omega_t^R \in (\omega^{(1)}, \omega^{(2)})$. The figure shows that the distribution of $\omega_t^R$ has a higher stationary mean in the second equilibrium rather than in the first equilibrium. This is consistent with Proposition 3, which asserts a higher (logarithmic) growth rate for the wealth of $R$ investors in the second equilibrium.

The comparatively higher probability mass of larger values of $\omega_t^R$ in the equilibrium that features shorting implies that there are two competing effects on the stationary mean of the Sharpe ratio $\kappa_t$. On the one hand – for a fixed $\omega_t^R$ – the Sharpe ratio is lower in equilibria featuring comparatively higher short selling. On the other hand, low values of $\omega_t^R$ become less likely in equilibria with comparatively more shorting activity. The first effect tends to lower the stationary mean of the Sharpe ratio in equilibria with comparatively higher shorting, the second effect tends to raise it. The overall effect on the stationary value of the Sharpe ratio is ambiguous. We revisit this issue in section 5, when we discuss extensions of the model that allow the price-dividend ratio to differ across equilibria and across different values of $\omega_t^R$.

## 4.2 The instability of short interest

Besides the sensitivity that emanates from demand-side coordination, our model also implies that small shifts in the supply of loanable shares can lead to discontinuous changes in equilibrium short interest.

**Lemma 1**

$$\frac{d\omega^{(2)}}{d\beta} < 0.$$

Lemma 1 states that an increase in $\beta$ lowers the range of values $\omega_t^R$ that are associated

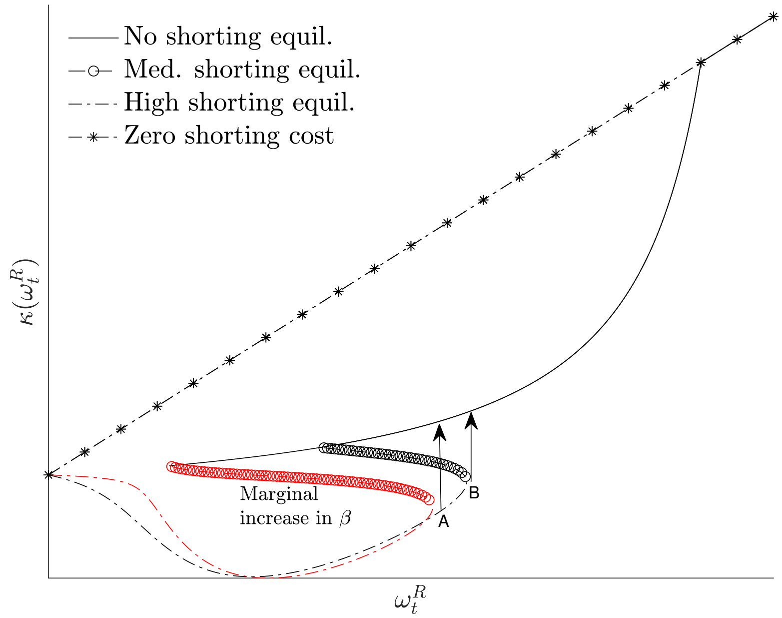


Figure 12: Points $A$ and $B$ are no longer equilibrium points when $\beta$ increases.

with multiple equilibria. By implication, if, say, a given company can take some action to to induce an upward shift in the supply curve for its loanable shares (i.e., an increase in $\beta$), this can lead to a discontinuous change of short interest from a positive value to 0 if $\omega_t^R$ is close to $\omega^{(2)}$.

Figure 12 illustrates the effects of a change in $\beta$ (an upward shift in the supply for loanable shares) on the equilibrium values $\omega_t^R$. The figure shows that points such as $A$ and $B$, which used to correspond to equilibria associated with positive short interest stop being eqilibura if $\beta$ increases, even marginally.

## 4.3 The ambiguous relation between Sharpe ratio and short interest

It would seem natural to expect that an increase in the supply of loanable shares (a downward shift of the loan supply curve) should raise the Sharpe ratio, since this should incentivize short sellers to short the stock and thus lower the absolute value of the (negative) Sharpe ratio. Remarkably, this is not the case in this model. Depending on the equilibrium, there is no unambiguous relation between the Sharpe ratio and shorting costs. This may be one of the reasons why the empirical literature finds that randomized increases in loanable shares affect short interest and lending rates but not excess returns.

The following proposition illustrates the novel implications of the model by focusing on the case of small $\omega_t^R$.

**Proposition 4** *Assume that the equilibrium involves a positive amount of short interest. In the equilibrium associated with $y^+$ (which is the unique equilibrium if $\omega_t^R < \omega^{(1)}$), it holds that, for sufficiently small $\omega_t^R$,*

$$\frac{d\kappa}{d\beta} > 0. \tag{32}$$

*In the equilibrium associated with $y^-$, for any value of $\omega_t^R$,*

$$\frac{d\kappa}{d\beta} < 0. \tag{33}$$

Equation (32) in Proposition 4 appears counterintuitive. The explanation is that decreasing $\beta$ has two opposing effects. Inspection of equation (28) shows that a decline in $\beta$ has the direct effect of raising $\kappa_t$; however, since $y_t$ is endogenous, the decline in $\beta$ also increases $y_t$, which — for a given $\beta$ — has the effect of lowering $\kappa_t$. Therefore, it is possible that a decline in $\beta$ (say, because of an exogenous change in the cost of supplying shares) lowers the fee $f_t$

and increases the short interest $y_t$, but leaves the expected return on the stock unchanged. This is consistent with the empirical findings of Kaplan et al. (2013).

Figure 12 illustrates that an increase in $\beta$ could either raise or lower $\kappa_t\left(\omega_t^R\right)$, depending on the equilibrium and on whether $\omega_t^R$ is large or small.

# 5 Multiple Risky Assets and Time-Varying Price-Dividend Ratio

In the baseline model, the price-dividend ratio and the volatility of the stock market are both constant. This is an implication of a) logarithmic utility over intermediate consumption (which implies a constant wealth-to-consumption ratio) and b) a single asset in positive net supply. As is typical of models with similar setups, fluctuations in the interest rate offset the fluctuations of the risk premium, thus rendering the overall discount rate — and by implication the price-dividend ratio[8] — constant.

We next present a version of the model that features multiple risky assets and a time-varying price-dividend ratio. After extending Proposition 2 to allow for multiple risky assets — a result of independent interest — we consider a limiting case that permits simple computations. Specifically, we study the limit in which there is a "small" stock subject to shorting costs and a "large" stock that can be shorted costlessly. In that limit, only the endowment of the large stock matters for the interest rate and thus the price-dividend ratio of the small stock is time-varying and reflects variations in its risk premium.

## 5.1 Multiple risky assets

In this section we introduce an additional Lucas tree (stock "2") to our baseline model, which is not subject to any trading frictions, and comprises a potentially large part of the total

[8]Note also that the expected dividend growth is constant.

market capitalization. We continue to assume that borrowing stock 1, which now comprises only (a possibly small) part of the market capitalization, requires lending fees, as in the baseline model.

We make one more convenient and realistic assumption. Specifically, while all investors hold interior portfolios in stock 2 and the risk-free asset, only a fraction of investors pays any "attention" to stock 1. The remaining fraction of investors simply optimize their portfolio over the risk-free asset and stock 2 and assign zero weight to stock 1. This assumption is in the spirit of Robert Merton's "limited recognition hypothesis," the idea that only a fraction of investors actively trade in some smaller stocks.

Because stock 1 is no longer the only positive-supply asset, consumption-market clearing no longer implies a constant price-to-dividend ratio for stock 1. This prevents a full analytical solution of the model. However, we can still provide an analytic "CAPM-style" formula,[9] which constitutes a natural extension of the results of Proposition 2.

To start, we assume that equilibrium returns on stocks 1 and 2 follow diffusion processes of the form

$$dR_{1,t} = \mu_{1,t}dt + \sigma_{1,t}dB_{1,t} + b_t\sigma_{2,t}dB_{2,t} \tag{34}$$

$$dR_{2,t} = \mu_{2,t}dt + \sigma_{2,t}dB_{2,t}, \tag{35}$$

where $B_{1,t}$ and $B_{2,t}$ are independent Brownian motions, $\mu_{1,t}$ and $\mu_{2,t}$ are the expected excess returns of the two stocks, and

$$\sigma_t \equiv \begin{bmatrix} \sigma_{1,t} & b_t\sigma_{2,t} \\ 0 & \sigma_{2,t} \end{bmatrix}$$

[9]By "CAPM-style" formula we mean that the formula provides a connection between expected excess returns and the covariance matrix of returns.

is a matrix capturing the loadings of the two stocks to the two Brownian motions. We assume that investors $I$ believe that Brownian motion 1 follows the dynamics[10] $dB_{1,t} + \eta dt$, while no investor has any belief distortions pertaining to Brownian motion 2. We let $m_{j,t}$ for $j \in \{1, 2\}$ denote the market capitalization of the two stocks.

Letting $\widehat{\omega}_t$ denote the wealth share of the investors who actively participate in the market for stock 1, the market clearing condition is

$$\widehat{\omega}_t \sum_{i \in \{I,R\}} \frac{\omega_t^i}{\widehat{\omega}_t} \overrightarrow{w}_t^i + (1 - \widehat{\omega}_t) \begin{bmatrix} 0 \\ \widehat{w}_{2,t} \end{bmatrix} = m_t, \tag{36}$$

where $\widehat{w}_{2,t} = \frac{\mu_{2,t} - r_t}{\sigma_{2,t}^2}$ is the optimal portfolio holdings of stock 2 by investors who don't participate in stock 1, and $\overrightarrow{w}_t^i$ is the vector of portfolio holdings of an investor $i \in \{I, R\}$ that is active in the market for stock 1. The market clearning condition (36) leads to the following result.

**Proposition 5** *Define $\kappa_{1,t} = \frac{(\mu_{1,t} - r) - b_t(\mu_{2,t} - r)}{\sigma_{1,t}}$ as the Sharpe ratio of a portfolio that invests 1 unit in asset 1 and shorts $b_t$ units of asset 2. Let $\widetilde{m}_{1,t} \equiv \frac{m_{1,t}}{\widehat{\omega}_t}$ and $\widetilde{\omega}_t^i \equiv \frac{\omega_t^i}{\widehat{\omega}_t}$ for $i \in \{I, R\}$.*

*Then, in an equilibrium with shorting in asset 1 ($y_t > 0$), $y_t$ corresponds to the root(s) $y^{\pm}$ of the quadratic equation*

$$0 = y_t \left( \eta + \frac{\widetilde{m}_{1,t}}{\widetilde{\omega}_t^R} \sigma_{1,t} - \frac{\beta}{\sigma_{1,t}} (1 - \tau y_t) \right) - \left( \eta - \frac{\widetilde{m}_{1,t}}{1 - \widetilde{\omega}_t^R} \sigma_{1,t} - \frac{\beta}{\sigma_{1,t}} (1 - \tau y_t) \right) \tag{37}$$

[10]More formally, the Radon-Nikodym derivative of the true probability measure to the subjective one is given by

$$Z_t^I \equiv e^{-\frac{\eta^2}{2} t + \eta B_{1,t}}.$$

*that lie(s) in the interval* $[0,1)$, *and the Sharpe ratio is given by*

$$\kappa_{1,t} = \widetilde{m}_{1,t}\sigma_{1,t} - \left(1-\widetilde{\omega}_t^R\right)\eta - \widetilde{\omega}_t^R \frac{\beta}{\sigma_{1,t}}\left(1 + \frac{1-\widetilde{\omega}_t^R}{\widetilde{\omega}_t^R}\tau y_t^{\pm}\right). \tag{38}$$

*Similarly, in an equilibrium without shorting in stock 1, we have* $\kappa_{1,t} = \sigma_{1,t} - \left(1-\omega_t^R\right)\eta$ *if investor R holds an interior position in asset* 1 *and* $\kappa_{1,t} = \frac{\sigma_{1,t}}{1-\omega_t^R} - \eta$ *otherwise.*

*The excess return to asset 2 is given by the conventional CAPM relationship*

$$\mu_{2,t} - r_t = [0,1]\,\sigma_t\sigma_t' m_t. \tag{39}$$

Equations (38) and (37) specialize to (23) and (22), respectively, when $\widetilde{m}_{1,t} = 1$, $\sigma_{1,t} = \sigma_D$, and $\widetilde{\omega}_t^R = \omega_t^R$. In this sense, Proposition 5 is a natural extension of Proposition 2, except that the Sharpe ratio in Proposition 5 pertains to a portfolio that invests one dollar in stock 1 and shorts $b_t$ units of asset 2 (so as to "hedge out" the exposure of the portfolio to the second Brownian shock).

As in Proposition 2, the excess return on asset 1 can be decomposed into a risk premium, a (wealth-weighted) belief distortion, and a component that reflects the impact of shorting costs. Specifically, equation (38) implies that in an equilibrium with active shorting, the expected return of stock 1 is

$$\mu_{1,t} - r_t = \underbrace{b_t\left(\mu_{2,t} - r_t\right) + \widetilde{m}_{1,t}\sigma_{1,t}^2}_{\text{risk compensation}} - \underbrace{\left(1-\widetilde{\omega}_t^R\right)\eta\sigma_{1,t}}_{\text{wealth-weighted optimism}} - \underbrace{\beta\left(\widetilde{\omega}_t^R + \left(1-\widetilde{\omega}_t^R\right)\tau y_t\right)}_{\text{impact of shorting costs}}. \tag{40}$$

All else equal, a higher level of $y_t$ lowers $\mu_{1,t} - r$, consistent with the empirical finding that short interest negatively predicts returns. Also, higher values of the lending fee $\beta$ lower equilibrium expected excess returns.

## 5.2 A limiting economy with a small and a large stock

Next we consider a limiting, multi-stock economy, where trees of type 1 are small compared to trees of type 2 and also the fraction of investors that pay attention to trees of type 1 is small.

Specifically, suppose that there are two types of trees, namely "small" trees (type-1 trees) and "large" trees (type-2 trees). Type-2 trees have dividends similar to the baseline model, namely $D_{2,t,s} = \phi_2 \delta_2 D_{2,t} e^{-\delta_2 (t-s)}$, where $\phi_2 > 0$, $\delta_2 > 0$, and $D_{2,t}$ follows a geometric Brownian motion, $\frac{dD_{2,t}}{D_{2,t}} = \mu_{2,D} dt + \sigma_{2,D} dB_{2,t}$. Type-1 trees produce dividends

$$D_{1,t,s} = \phi_1 \delta_1 D_{2,s} e^{-\delta_1 (t-s) + \sigma_{1,D}(B_{1,t} - B_{1,s})},$$

with $\phi_1 > 0$ and $\delta_1 > 0$. With the above dividend specifications, the dividend share of type-1 trees is

$$\frac{D_{1,t}}{D_{2,t}} = \frac{\int_{-\infty}^{t} D_{1,t,s} ds}{\int_{-\infty}^{t} D_{2,t,s} ds} = \phi_1 z_t, \tag{41}$$

where

$$z_t \equiv \delta_1 \int_{-\infty}^{t} \left( \frac{D_{2,s}}{D_{2,t}} \right) e^{-\delta_1 (t-s) + \sigma_{1,D}(B_{1,t} - B_{1,s})} ds \tag{42}$$

is a stationary process.

The above assumptions imply that the dividend shares of type-1 and type-2 trees are stationary fractions of aggregate consumption $D_{1,t} + D_{2,t}$, while the dividends of a given tree belonging to a fixed cohort $s$ follow a geometric Brownian motion.[11] Moreover, when type-1

[11] Ito's Lemma implies that

$$\frac{dD_{1,t,s}}{D_{1,t,s}} = \left( \frac{\sigma_{1,D}^2}{2} - \delta_1 \right) dt + \sigma_{1,D} dB_{1,t} \text{ and } \frac{dD_{2,t,s}}{D_{2,t,s}} = (\mu_{2,D} - \delta_2)\, dt + \sigma_{2,D} dB_{2,t}$$

trees are small compared to type-2 trees $\left(\frac{\phi_1}{\phi_2} \approx 0\right)$, aggregate consumption is approximately equal to the aggregate dividends of the type-2 trees, and therefore aggregate consumption follows a geometric Brownian motion. The implication is that the interest rate and the risk premium for type-2 trees both converge to constants as the ratio $\frac{\phi_1}{\phi_2} \to 0$ goes to zero.

In Appendix B we also specify entry and exit by investors into the market for stock 1 to ensure that a) per unit of time $dt$ a mass $\delta\nu^i dt$ arrives in each group $i \in \{I, R\}$ and b) the aggregate wealth share of investors who pay attention to stock 1 at each point in time is proportional to the stock market capitalization of stock 1 ($\widetilde{m}_t = \widetilde{m}$ is constant). We refer to that appendix for the precise mathematical details. Here we simply note that with these specifications, the price-dividend ratio of stock 1 is the solution to an ordinary differential equation, as the next proposition shows.

**Proposition 6** *Using the expressions for $w_t^i$, $\kappa_{1,t}$ (with $b = 0$) and $y_t$ from Proposition 5, the wealth shares $\widetilde{\omega}_t^i$ follow the diffusion process*

$$d\widetilde{\omega}_t^i = \mu_t^i dt + \sigma_t^i dB_{1,t}, \tag{43}$$

*where*

$$\begin{aligned}
\mu_t^i &= \widetilde{\omega}_t^i \left( \left(w_{1,t}^i - \widetilde{m}\right) \sigma_{1,t} \left(\kappa_t - \sigma_{1,t}\widetilde{m}\right) + w_{1,t}^i \beta + \frac{y_t \widetilde{m}}{1 - y_t} \beta \left(1 - \tau\right) \right) + \delta \left(\nu^i - \widetilde{\omega}_t^i\right), \\
\sigma_t^i &= \widetilde{\omega}_t^i \left(w_{1,t}^i - \widetilde{m}\right) \sigma_{1,t},
\end{aligned}$$

*where $\sigma_{1,t}$ is the volatility of the stock 1 and is given by $\sigma_{1,t} = \frac{p'\left(\omega_t^R\right)}{p\left(\omega_t^R\right)} \sigma_t^R + \sigma_D$, and the price-dividend ratio $p_t = p(\widetilde{\omega}_t^R)$ is the solution of the ordinary differential equation*

$$\frac{1}{2} \frac{\partial^2 p}{\left(\partial \omega^R\right)^2} \left(\sigma_t^R\right)^2 + \frac{\partial p}{\partial \omega^R} \left(\mu_t^R + \left(\sigma_t^R - \kappa_{1,t}\right) \sigma_D\right) - p \times \left(r + \delta_1 + \kappa_{1,t} \sigma_{D,1}\right) + 1 = 0. \tag{44}$$

The expressions for $\mu_t^i$ and $\sigma_t^i$ in Proposition 6 coincide with (16) and (15) when $\widetilde{m} = 1$ and $\sigma_{1,t} = \sigma_D$.[12] Moreover, with the dividend growth of type-1 and type-2 stocks being independent, so are their stock-price processes (in the limit where stock-1 becomes small) and the expressions for $y_t$, $w_{1,t}^i$, and $\kappa_{1,t}$ in Proposition 6 (when $\widetilde{m} = 1$ and $\sigma_{1,t} = \sigma_D$) coincide with the respective expressions of the baseline model. In short, if one dropped the goods-market clearing requirement from the baseline model and instead postulated a constant interest rate, the resulting expression for the price-to-dividend ratio would be given by (44) (with $\widetilde{m} = 1$).

Equation (44) is a non-linear ordinary differential equation, which has to be solved numerically.[13] Figure 13 presents the solution of this ordinary differential equation. For this numerical exercise we assume that the volatility of dividends is $\sigma_{1,D} = 12\%$. The disagreement is $\eta = 0.5$. To translate this disagreement into more meaningful units, it is useful to perform the following back-of-the envelope exercise: If a type-$I$ agent were to behave like an econometrician and reject her null hypothesis that $\eta = 0.5$ in favor of the hypothesis that $\eta = 0$, based on empirical observations, it would take on average $\left(\frac{1.65}{\eta}\right)^2 \approx 11$ years to reject the hypothesis that $\eta = 0.5$ (in favor of $\eta = 0$) at the 95% level.[14] The values of $\nu$ and $\delta$ control the stationary mean and the degree of mean reversion of $\widetilde{\omega}_t^R$. We choose $\nu = 0.1$ to focus on a market where rational investors are the minority and consider $\delta = 0.2$ in our

[12]To see this substitute the expression for the equilibrium interest rate (24) to (16) .

[13]The reason why (44) is a non-linear (rather than linear) ordinary differential equation is that $\mu_t^i, \sigma_t^i$ depend on $\sigma_{1,t}$, which in turn depends on $p(\omega_t^R)$ and $p'(\omega_t^R)$. For this reason we use an iterative numerical procedure to solve (44). We start with $\sigma_t = \sigma_{1,D}$ as an "initial guess". We then exploit the fact that for a given function $\sigma_{1,t}(\omega_t^R)$, (44) is a linear ODE to compute $p(\omega_t^R)$ and $p'(\omega_t^R)$. With these estimates we compute a new $\sigma_{1,t}$ and iterate to convergence.

[14]In the model it is known that the Brownian motion $B_{1,t}$ has a volatility equal to one, but agent $I$ thinks it has a drift of $\eta$, when the true drift is zero. In a normal experiment where the true mean is zero and the variance is one (and known), the empirical mean after observing $T$ years of data is normally distributed with a mean of zero and a standard deviation equal to $T^{-0.5}$. Suppose that someone adopts a rule of rejecting the null that $\eta = 0.5$ in favor of the alternative that $\eta = 0$ whenever $\sqrt{T}(0.5 - \overline{X}) > 1.65$ to reflect the one-sided nature of the hypothesis. Under the truth, the expected value of $E(\overline{X})$ is zero, and therefore the expected value of $\sqrt{T}(0.5 - \overline{X})$ is just $0.5\sqrt{T}$. This quantity exceeds 1.65 when $\sqrt{T} > \frac{1.65}{0.5}$. For the above calculations it is immaterial whether the data are observed discretely or continuously (see, e.g., Merton (1980).)

baseline. We also assume that the sum of interest rate and depreciation $r + \delta_1$ for stock 1 is 0.08. (The appendix presents results for several different combinations of $\delta$ and $\delta_1$.)[15] We choose a value of $\tau = 0.8$ based on the industry practice of rebating about 80% to the mutual funds or ETFs that provide their shares for lending.[16] Finally, we consider several values of the lending fee $\beta$. For our baseline case, we choose a large value ($\beta = \eta\sigma_D = 0.06$) because a) we want to focus on stocks that are "on special" and, more importantly, b) this choice implies $\omega_1 = 0$, and thus, if there exists an equilibrium with positive shorting (for a given $\omega_t^R$), it always co-exists with an equilibrium without shorting.

Figure 13 depicts the price-dividend ratio as a function of $\widetilde{\omega}_t^R$. The line "shorting" depicts the log-pd ratio under the assumption that the market always coordinates on the equilibrium with the highest value of $y_t$, while the "no shorting" line depicts the log-price dividend ratio assuming that the market always coordinates on the equilibrium with the lowest value of $y_t$ (which is zero when $\beta = \sigma_D\eta$.). The grey area in the left plot depicts the 99% interval of values of $\widetilde{\omega}_t^R$ in the stationary distribution. The right plot depicts the difference between the "shorting" and the "no shorting" log-pd ratio, as well as the stationary density of $\widetilde{\omega}_t^R$. The vertical line labeled depicts $\omega^{(2)}$, i.e., the highest value of $\widetilde{\omega}_t^R$ above which the equilibrium value of $y_t$ is zero and the equilibrium is unique.

As is evident from the figure, the price-dividend ratio in the no-shorting equilibrium is higher than the price-dividend ratio in the shorting equilibrium. This is remarkable, because — for a fixed $\widetilde{\omega}_t^R$ — the Sharpe ratio in the no-shorting equilibrium is higher than in the equilibrium that features shorting. The reason is that in the equilibrium with shorting the process $\widetilde{\omega}_t^R$ tends to grow faster when the shorting market is active (i.e, when $\widetilde{\omega}_t^R$ is low). Quantitatively, the gap between the shorting and the no-shorting equilibrium is non-trivial.

[15] Note that in the baseline model $\delta$ and $\delta_1$ had to be the same number. With multiple stocks, $\delta$ and $\delta_1$ can be separated, with $\delta$ determining the speed of mean reversion in $\widetilde{\omega}_t^R$, and $\delta_1$ capturing the depreciation of stocks of type 1.

[16] Source: "Unlocking the potential of your portfolios: iShares Security Lending." Blackrock, 2021. Available at `https://www.ishares.com`.

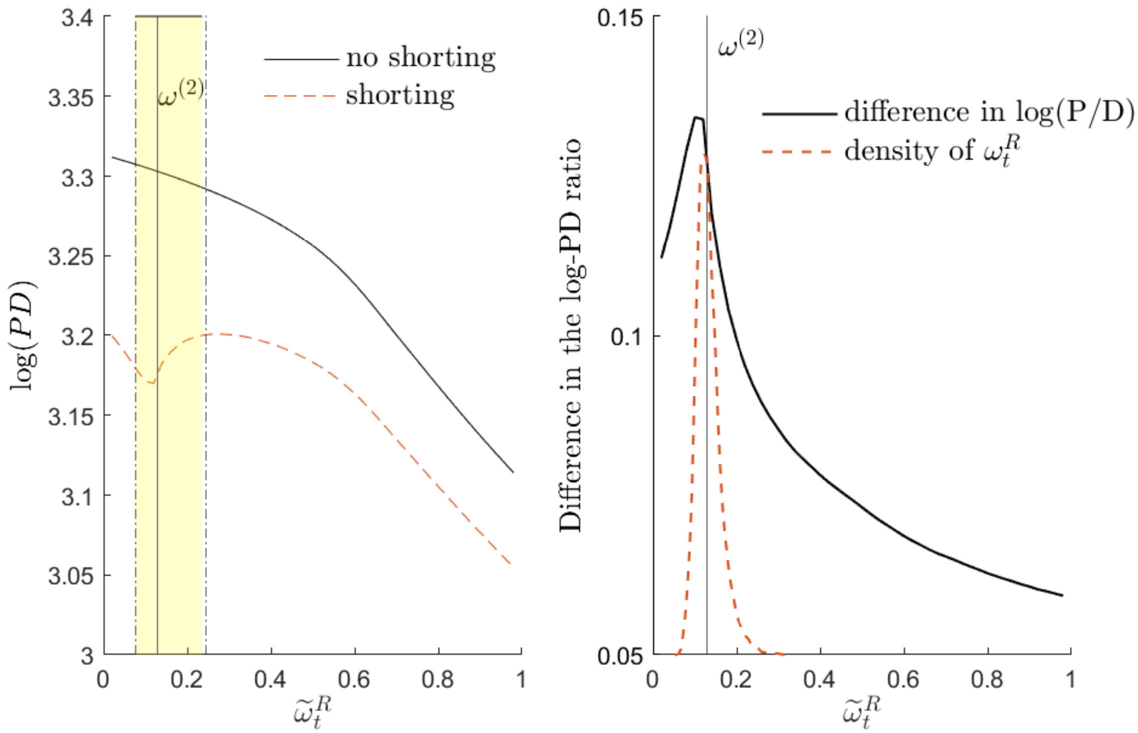


Figure 13: This figure repeats the quantitative exercise of Figure 13 using the same parameters ($\sigma_{D,1} = 0.12, \eta = 0.5, \tau = 0.8, \delta = 0.2, \nu = 0.1, \beta = \eta\sigma_{D,1}$). We choose $\delta_1$ so that $r + \delta_1 = 0.08$. Figure 18 in the appendix reports results for various alternative specifications.

In our base calibration the price jumps by approximately 13% (when there is a permanent and unexpected transition to the no-shorting equilibrium).

Figure 14 illustrates another property of the model, namely that the stock price is not monotone in the level of the shorting fee $\beta$. The figure shows that, for any fixed $\widetilde{\omega}_t^R$, there is no monotone relation between the stock price and the level of the fee. This observation is not surprising in light of the discussion in Section 4.3.

# 6 General supply curve for shorting

The baseline version of the model assumes an elastic supply of loanable shares, so that the lending fee is constant. The results generalize readily to the case where the supply of loanable

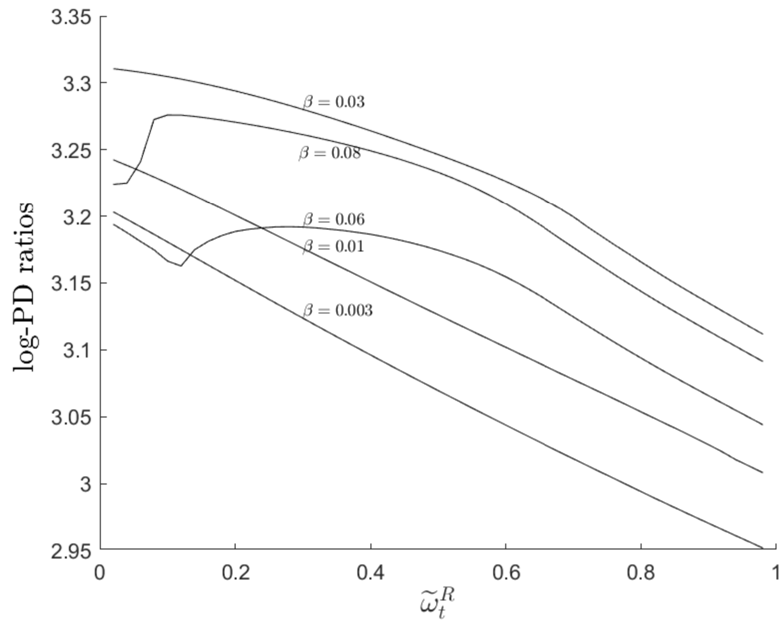


Figure 14: Log price-dividend ratio for various levels of $\beta$.

shares is increasing in $y_t$ so that $f_t = l(y_t)$, where $l'(y_t) > 0$.

We obtain the following proposition.

**Proposition 7** *Consider the model of Section 2, but without Assumption 1. Define*

$$z(y_t) \equiv f_t(1 - \tau y_t) = l(y_t)(1 - \tau y_t). \tag{45}$$

*Assume that* $\frac{\sigma_D}{1-\omega_t^R} - \eta < 0$ *and that there exists* $y_t^* \in (0,1)$ *such that*

$$y_t^* = \frac{\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y_t^*)}{\eta + \frac{\sigma_D}{\omega_t^R} - \frac{1}{\sigma_D} z(y_t^*)}, \tag{46}$$

*and* $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y^*) > 0$. *Moreover, if* $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(0) < 0$, *then there exist at least two values of* $y_t^{(j)}$, $j = \{1,2\}$ *satisfying both (46) and* $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(y^{(i)}\right) > 0$ *and three equilibria co-exist. In the first equilibrium, the R investor holds a zero portfolio, and the Sharpe ratio is* $\kappa_t = \frac{\sigma_D}{1-\omega_t^R} - \eta$. *There also exist another two equilibria, with the R investors*

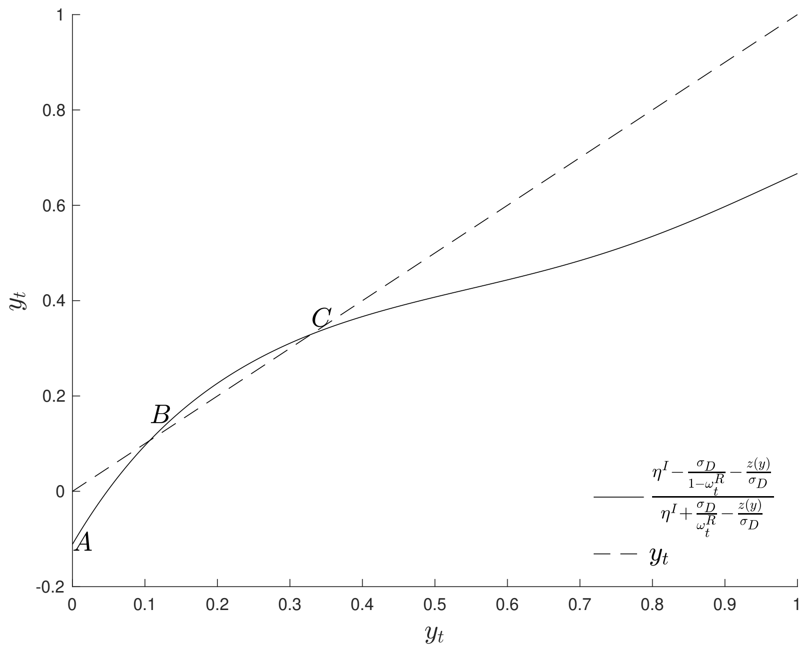


Figure 15: An illustration of Proposition 7.

*holding negative portfolios, and the Sharpe ratio given by*

$$\kappa_t = \sigma_D - \left(1 - \omega_t^R\right)\eta - \frac{\omega_t^R}{\sigma_D} l\left(y_t^{(j)}\right)\left[1 + \tau y_t^{(j)} \frac{1 - \omega_t^R}{\omega_t^R}\right]. \tag{47}$$

*In all equilibria the interest rate is given by (24), and the lending fee is given by* $l\left(y_t^{(j)}\right)$.

**Remark 8** *In the special case* $l(y_t) = \beta$, *equations (46) and (47) become identical to (22) and (23), respectively.*

Figure 15 illustrates Proposition 7. For this particular numerical example we choose $l(y_t) = \beta(1 + 2y_t^3)$, keeping all parameters (including $\beta$) as in Figure 9. The figure plots the left hand side (dotted line) of equation (46) and the right hand side (solid line). Points $B$ and $C$ correspond to the two fixed points. Point $A$ in the figure illustrates the assumption $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(0) < 0$. This inequality implies that there is a third equilibrium in which

$R$ investors are deterred by the presence of the lending fee and the shorting market is inactive. The fees in the three equilibria differ, with the lending fee being lowest ($l(0) = \beta$) in equilibrium $A$, in which the shorting market is inactive, and highest in equilibrium $C$, in which $y$ is highest.

# 7 Conclusion

The main finding of this paper is that shorting can be fickle. We utilized a time-honored device in economic theory, namely the presence of multiple equilibria to illustrate this point.

In the model shorting exhibits "run-type" patters. Any event that makes some short sellers abandon their short positions ignites a self-propagating circle: Less shorting also implies less lending income for investors with long positions, who now need to be compensated with a higher Sharpe ratio, which in turn further prompts short sellers to abandon their strategies. (Going in the opposite direction, high shorting activity acts as a subsidy to long positions, thus lowering the equilibrium Sharpe ratio and attracting further short selling). Thus, for the same fundamentals, there can be multiple equilibria — a manifestation of the self-reinforcing nature of shorting decisions.

The paper can consequently provide an explanation of how events affecting a single stock of small market capitalization can spiral quickly across many shorting strategies — even among stocks that did not see an unusual increase in retail trading purchases. The model also implies that (possibly small) shifts in the supply of loanable shares can have a disproportionately large impact on the level of short interest and the Sharpe ratio, by shifting the equilibrium.

While motivated by recent events, the analysis of the paper has broader implications. For instance, the model helps explains why shifts in the supply of loanable shares need not have a clear impact on the equilibrium Sharpe ratio, or why higher short interest and higher lending fees negatively predict subsequent returns.

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# Appendix

## A The Supply of Loanable Shares

In this section we provide the micro-foundations for the supply curve $f_t = l(y_t)$.

To start, we extend the model by introducing competitive firms that we label "brokers." Brokers facilitate share lending operations. A broker that facilitates the lending of $S_t$ shares obtains revenue equal to $f_t S_t P_t$, where $f_t$ is the market lending rate and $P_t$ is the value of the stock. The broker only retains a fraction $(1-\tau)$ of that income, with the remaining income being rebated to (long-position) stock investors as the outcome of a bargaining game to obtain their permission to lend out their shares.

Facilitating the share lending also requires a resource ("human capital ") that is owned by households. With these assumptions, a firm's problem is

$$\max_{S_t \geq 0}(1-\tau) f_t S_t P_t - q_t H(S_t), \tag{48}$$

where $q_t$ is the prevailing market compensation ("wage") per unit of resource employed and $H(S_t)$ are the resource units that are required to process $S_t$ shares. (We assume $H'(S_t) \geq 0$ and $H''(S_t) \geq 0$.) Solving the broker's optimization problem leads to the resource demand function

$$H'(S_t) = \frac{(1-\tau) f_t P_t}{q_t}. \tag{49}$$

To model the supply of the household-controlled resource, we assume that each household incurs a disutility equal to $\frac{1}{\rho+\pi} d\left(\frac{h_t^i}{\chi_t^i}\right)$ from providing $h_t^i$ units of human capital where $d'(\cdot) \geq 0, d''(\cdot) \geq 0$ and $\chi_t^i \equiv \frac{W_t^i}{W_t}\overline{H}$. The term $\chi_t^i$ ensures that households with higher relative wealth, $\frac{W_t^i}{W_t}$, are de-facto endowed with a better ability to provide efficiency units of human capital (say due to their higher education). But fundamentally, the assumption that wealthier households obtain a higher fraction of the human-capital compensation is for technical reasons, since this assumption safeguards that a household's value function continues to be logarithmic in wealth. (Otherwise, one would have to introduce the present value of human capital compensation as a separate state variable in each household's problem, which would undermine the tractability of the model.)

Given logarithmic utilities, a household's supply decision is characterized by the first-order condition

$$\frac{1}{\rho+\pi}\frac{1}{\chi_t^i}d'\left(\frac{h_t^i}{\chi_t^i}\right)=\frac{1}{(\rho+\pi)\,W_t^i}\times q_t, \tag{50}$$

where the right-hand side is the marginal disutility of labor, whereas the right hand side is the marginal utility of consumption, $\frac{1}{(\rho+\pi)W_t^i}$, times the wage $q_t$. Using the definition of $\chi_t^i$ and simplifying gives

$$h_t^i=d'^{-1}\left(\frac{\overline{H}}{W_t}q_t\right)\chi_t^i, \tag{51}$$

where $d'^{-1}(\cdot)$ is the inverse function of $d'(\cdot)$. Aggregating (51) across all households $j$ (across all cohorts and agent types) and noting that $\int_j x_t^j dj=1$ leads to $H_t\equiv\int_j h_t^j dj=\overline{H}d'^{-1}\left(\overline{H}\frac{q_t}{W_t}\right)$ or equivalently

$$q_t=\frac{W_t}{\overline{H}}d'\left(\frac{H_t}{\overline{H}}\right). \tag{52}$$

Substituting (52) into (49), noting that in equilibrium $P_t=W_t$ and also that the supply $H_t$ equals the demand $H(S_t)$ leads to

$$f_t=\frac{1}{(1-\tau)\,\overline{H}}H'(S_t)\,d'\left(\frac{H(S_t)}{\overline{H}}\right). \tag{53}$$

The right-hand side of (53) is non-decreasing in $S_t$ (given the assumed monotonicity and convexity of $d(\cdot)$ and $H(\cdot)$). In turn, the equilibrium short interest equals $S_t = \frac{y_t}{1-y_t}$.[17] Accordingly, $S_t$ is a monotone function of $y_t$ and therefore $f_t$ can be expressed as a non-decreasing function of the equilibrium value of $y_t$. We write

$$f_t = l\left(y_t\right) \text{ with } l'\left(\cdot\right) \geq 0.$$

Assuming that $H\left(S_t\right) = \phi S_t$ for some constant $\phi > 0$, the broker makes no profits and therefore $\left(1-\tau\right) f_t S_t P_t = H_t q_t$. In turn, combining (51) with (52) implies

$$\frac{h_t^i q_t}{H_t q_t} = \frac{W_t^i}{W_t}. \tag{54}$$

Accordingly, the wealth evolution of household $i$ is

$$\begin{aligned} \frac{dW_t^i}{W_t^i} &= \left(\left(r_t + \pi + w_{t,s}^i\left(\mu_t - r_t + s_{t.s}^i\right)\right) - \frac{c_t^i}{W_t^i} + \frac{h_t^i q_t}{W_t^i}\right) dt + w_t^i \sigma_t dB_t \\ &= \left(\left(r_t + \pi + w_{t,s}^i\left(\mu_t - r_t + s_{t.s}^i\right)\right) - \frac{c_t^i}{W_t^i} + \frac{\left(1-\tau\right) f_t S_t P_t}{W_t}\right) dt + w_t^i \sigma_t dB_t \\ &= \left(\left(r_t + \pi + w_{t,s}^i\left(\mu_t - r_t + s_{t.s}^i\right)\right) - \frac{c_t^i}{W_t^i} + \left(1-\tau\right) f_t \frac{y_t}{1-y_t}\right) dt + w_t^i \sigma_t dB_t, \end{aligned} \tag{55}$$

where the second line follows from combining (54) with $\left(1-\tau\right) f_t S_t P_t = H_t q_t$ and the last line follows from $P_t = W_t$ and $S_t = \frac{y_t}{1-y_t}$.

We note that, if we were to assume a non-linear $H\left(S_t\right)$, then we would additionally have to assume the broker profits to be distributed to the households in proportion to their wealth.

[17] The amount of shorted shares $S_t$ is equal to $S_t = \omega_t^R \left|w_t^R\right|$. The definition $y_t = \frac{\omega_t^R \left|w_t^R\right|}{\omega_t^I w_t^I}$ along with the stock market clearing requirement $\omega_t^I w_t^I - \omega_t^R \left|w_t^R\right| = 1$ implies that

$$S_t = \omega_t^R \left|w_t^R\right| = \frac{y_t}{1-y_t}.$$

# B The Price-Dividend Ratio of a Small Stock

This section provides the details of the entry-and-exit process for the model of section 5.2 and proves Proposition 6.

We assume that the fraction of investors that pay attention to stock 1, which we denote by $\widehat{\omega}_t$, is small: $\widehat{\omega}_t = \omega_t^R + \omega_t^I \approx 0$. Further, in the interest of tractability, we allow for entry and exit into the market for stock 1 to not be fully driven by entry and exit into the economy (via births and deaths). Specifically, letting $W_t^i$ denote the (aggregate) wealth of type-$i$ investors (where $i \in \{I, R\}$), we assume

$$dW_t^i = dW_t^{i,\text{part}} + \delta\left(\nu_i\left(W_t^I + W_t^R\right) - W_t^i\right)dt + \omega_t^i\left(dL_t - dN_t\right), \tag{56}$$

where $dW_t^{i,\text{part}}$ is the wealth growth of an investor of type $i \in \{I, R)$ who is already participating in the market[18] and the last two terms on the right-hand side of equation (56) capture entry and exit into market 1, i.e., how investors from the broad economy gain and lose interest in stock 1. The term $\delta\left(\nu_i\left(W_t^I + W_t^R\right) - W_t^i\right)dt$ reflects that investors enter and exit the market for stock 1 at a rate $\delta$ per unit of time $dt$ for exogenous reasons. Similar to the baseline model, this term affects the composition, but not the sum of $W_t^I + W_t^R$, since $\sum_{i\in\{I,R\}}\delta\left(\nu_i\left(W_t^I + W_t^R\right) - W_t^i\right) = 0$. The third term on the right-hand side of (56) affects the sum of $W_t^I + W_t^R$, but not the the shares $\omega_t^i = \frac{W_t^i}{W_t^I + W_t^R}$. Specifically, we assume that $dL_t$ and $dN_t$ are two singular, increasing processes that "control" $W_t^I + W_t^R$ so that the ratio of stock market capitalization in market 1 to the total wealth of investors participating in market one, $\widetilde{m}_t = \frac{M_{1,t}}{W_t^I + W_t^R}$ stays constant across time ($\widetilde{m}_t = \widetilde{m}$). This assumption ensures that in

[18]For completeness, $dW_t^{i,\text{part}} = W_t^{i,\text{part}}\mu_W^i dt + W_t^{i,\text{part}}\overrightarrow{\sigma}_W^i dW_t$, where

$$\mu_W^i = r_t + \pi + n_t + \overrightarrow{w}_{t,s}^i\left(\overrightarrow{\mu}_t - r_t 1_{\{2\times1)} + \lambda_{t.s}^i\begin{bmatrix}1\\0\end{bmatrix}\right) - \frac{c_{t,s}^i}{W_{t,s}^i}$$

and $\overrightarrow{\sigma}_W^i = \overrightarrow{w}^{i\prime}\sigma_t$.

the absence of any differences of opinion ($\eta = 0$), the excess return, the price-dividend ratio, and the volatility of stock 1 would all be constant. Thus, we can attribute any fluctuations in the price-dividend ratios exclusively to the forces we wish to highlight — namely, the differences in opinion and the shorting frictions — while eliminating the fluctuations that would emerge from limited participation in the market for stock 1. In addition, these assumptions on $dL_t$ and $dN_t$ are convenient from a technical standpoint, since they allow us to a express the price-dividend ratio for stock 1 as a function of $\widetilde{\omega}_t^R$, rather than a function of three state variables $\left(\widetilde{\omega}_t^R, \widetilde{m}_t, z_t\right)$, which would require the solution of a non-linear, three-dimensional partial differential equation.

Using the market clearing condition $\sum_{i \in \{I,R\}} \widetilde{\omega}_t^i w_t^{i,1} = \widetilde{m}$, and applying Ito's Lemma to $\widetilde{\omega}_t^i = \frac{W_t^i}{W_t^I + W_t^R}$ leads to

$$d\widetilde{\omega}_t^i = \mu_t^i dt + \sigma_t^i dB_{1,t}, \tag{57}$$

where

$$\begin{aligned} \mu_t^i &= \widetilde{\omega}_t^i \left[\left(w_{1,t}^i - \widetilde{m}\right) \sigma_{1,t} \left(\kappa_t - \sigma_{1,t}\widetilde{m}\right) + w_{1,t}^i f_t + \widetilde{n}_t\right] + \delta\left(\nu^i - \widetilde{\omega}_t^i\right), \\ \sigma_t^i &= \widetilde{\omega}_t^i \left(w_{1,t}^i - \widetilde{m}\right) \sigma_{1,t}, \end{aligned}$$

and[19]

$$\widetilde{n}_t \equiv - \sum_{i \in \{I,R\}} w_{1,t}^i \widetilde{\omega}_t^i \lambda_{t.s}^i = \frac{y_t \widetilde{m}}{1 - y_t} f_t \left(1 - \tau\right).$$

[19]Using $\sum_{i \in \{I,R\}} w_{t,s}^i \widetilde{\omega}_t^i = \widetilde{m}_t$, the definition $y_t = -\frac{\left(w_{t,s}^R \widetilde{\omega}_t^R\right) 1_{\left\{w_{t,s}^R < 0\right\}}}{w_{t,s}^I \widetilde{\omega}_t^I}$ and the definition of $\lambda_{t.s}^i$ leads to

$$-\sum_{i \in \{I,R\}} w_{t,s}^i \widetilde{\omega}_t^i \lambda_{t.s}^i = \frac{y_t \widetilde{m}}{1 - y_t} f_t \left(1 - \tau\right).$$

Since $\frac{\phi_1}{\phi_2} \approx 0$, the aggregate endowment follows a geometric Brownian motion in the limit, and the interest rate is constant $r_t = r$. Accordingly, price of a stock of type 1 follows the dynamics

$$\frac{dP_{1,t,s} + D_{1,t,s}dt}{P_{1,t,s}} = (r + \kappa_{1,t}\sigma_{1,t})dt + \sigma_t dB_{1,t}. \tag{58}$$

Applying Ito's Lemma to the product $P_{t,s} = p\left(\omega_t^R\right) D_{1,t,s}$ also implies that

$$\frac{dP_{1,t,s}}{P_{1,t,s}} = \frac{dp_t}{p_t} + \frac{dD_{1,t,s}}{D_{1,t,s}} + \frac{p'\left(\omega_t^R\right)}{p\left(\omega_t^R\right)}\sigma_t^R \sigma_D dt. \tag{59}$$

Combining (58) with (59) and using $\sigma_{1,t} = \frac{p'\left(\omega_t^R\right)}{p\left(\omega_t^R\right)}\sigma_t^R + \sigma_D$ and Ito's Lemma to compute the drift of $\frac{dp_t}{p_t}$ leads to

$$\frac{1}{2}\frac{\partial^2 p}{\left(\partial \omega^R\right)^2}\left(\sigma_t^R\right)^2 + \frac{\partial p}{\partial \omega^R}\left(\mu_t^R + \sigma_t^R \sigma_D\right) - p \times (r + \delta_1 + \kappa_{1,t}\sigma_{1,t}) + 1 = 0, \tag{60}$$

which in turn leads to (44) after substituting $\sigma_{1,t} = \frac{p'\left(\omega_t^R\right)}{p\left(\omega_t^R\right)}\sigma_t^R + \sigma_D$.

## C Proofs

**Proof of Proposition 1.** Fix a value $\eta$ and consider a sequence of parameters such that $\sigma_D \to 0$ and along that sequence $\beta$ is set according to

$$\beta = \sigma_D \left(\eta - \xi \sigma_D\right), \tag{61}$$

for some $\xi > 1$. Note that since $\sigma_D \to 0, \eta - \xi\sigma_D$ is positive for sufficiently small $\sigma_D$, and hence $\beta$ is positive.

We show next that as $\sigma_D$ gets close to zero, Assumption 2 is satisfied.

Re-arranging (61) gives

$$\frac{\eta}{\frac{\beta}{\sigma_D}} = \frac{1}{1 - \xi \frac{\sigma_D}{\eta}}. \tag{62}$$

For sufficiently small $\sigma_D$ we obtain

$$1 + \tau > \frac{1}{1 - \xi \frac{\sigma_D}{\eta}} > 1. \tag{63}$$

Combining (62) and (63) yields (19).

Turning to (20), we note that the definition of $\omega^{(1)}$ along with (61) implies

$$\omega^{(1)} = 1 - \frac{\sigma_D}{\xi \sigma_D} = \frac{\xi - 1}{\xi} > 0,$$

while also

$$\lim_{\sigma_D \to 0} \frac{\sigma_D}{(1+\tau)\frac{\beta}{\sigma_D} - \eta} = \lim_{\sigma_D \to 0} \frac{\sigma_D}{(1+\tau)\left(\eta - \xi\sigma_D\right) - \eta} = 0.$$

Therefore, for sufficiently small $\sigma_D$, the left-hand side of (20) converges to $\frac{\xi-1}{\xi} > 0$, while the right-hand side converges to zero, and therefore the inequality holds.

We conclude the proof by showing that $F(\omega)$ has a unique root in the interval $(0,1)$. To this end, it is useful to introduce the definitions

$$
\begin{aligned}
A(\omega) &\equiv \tau \frac{\omega}{\sigma_D}\beta, && (64)\\
B(\omega) &\equiv \sigma_D - \omega\left((1+\tau)\frac{\beta}{\sigma_D} - \eta\right), && (65)\\
C(\omega) &\equiv \frac{\omega}{1-\omega}\left(\sigma_D + (1-\omega)\left(\frac{\beta}{\sigma_D} - \eta^I\right)\right). && (66)
\end{aligned}
$$

With these definitions, $F(\omega)$ can be written as $F(\omega) = B^2(\omega) - 4A(\omega)C(\omega)$. We start by observing that $C\left(\omega^{(1)}\right) = 0$ for any parametric choice (since the definition of $\omega^{(1)}$ in equation (17) implies $\sigma_D + \left(1-\omega^{(1)}\right)\left(\frac{\beta}{\sigma_D} - \eta\right) = 0$). Also the inequality (20) implies that $B\left(\omega^{(1)}\right) \neq 0$, which implies that $B^2\left(\omega^{(1)}\right) > 0$. Accordingly, $F\left(\omega^{(1)}\right) > 0$. Also $B(1) < \infty$, while $C(1) = \infty$. By continuity, there exists at least one value $\omega^{(2)} \in (\omega^{(1)}, 1)$ such that $F\left(\omega^{(2)}\right) = 0$.

To show that this value is unique, consider any value $\omega^{(2)} \in \left(\omega^{(1)}, 1\right)$ such that $F(\omega^{(2)}) = 0$. We next show that $F'(\omega^{(1)}) < 0$.

To this end, note that

$$
\begin{aligned}
F'(\omega) &= 2B(\omega)B'(\omega) - 4\left[A'(\omega)C(\omega) + A(\omega)C'(\omega)\right]\\
&= 2B^2(\omega)\frac{B'(\omega)}{B(\omega)} - 4A(\omega)C(\omega)\left(\frac{A'(\omega)}{A(\omega)} + \frac{C'(\omega)}{C(\omega)}\right).
\end{aligned}
$$

Since $\omega^{(2)}$ is a root of $F(\omega)$ it follows that $B^2\left(\omega^{(2)}\right) = 4A\left(\omega^{(2)}\right)C\left(\omega^{(2)}\right)$. Therefore,

$$
F'\left(\omega^{(2)}\right) = B^2\left(\omega^{(2)}\right)\left(2\frac{B'\left(\omega^{(2)}\right)}{B\left(\omega^{(2)}\right)} - \frac{A'\left(\omega^{(2)}\right)}{A\left(\omega^{(2)}\right)} - \frac{C'\left(\omega^{(2)}\right)}{C\left(\omega^{(2)}\right)}\right). \tag{67}
$$

The sign of $F'\left(\omega^{(2)}\right)$ is the same as the sign of the expression inside the parentheses in equation (67). We have

$$\frac{A'\left(\omega^{(2)}\right)}{A\left(\omega^{(2)}\right)}=\frac{1}{\omega^{(2)}},\frac{B'\left(\omega^{(2)}\right)}{B\left(\omega^{(2)}\right)}=-\frac{\left((1+\tau)\frac{\beta}{\sigma_D}-\eta\right)}{\sigma_D-\omega^{(2)}\left((1+\tau)\frac{\beta}{\sigma_D}-\eta\right)}$$

and

$$\frac{C'\left(\omega^{(2)}\right)}{C\left(\omega^{(2)}\right)}=\frac{1}{\omega^{(2)}\left(1-\omega^{(2)}\right)}+\frac{\eta-\frac{\beta}{\sigma_D}}{\sigma_D+\left(1-\omega^{(2)}\right)\left(\frac{\beta}{\sigma_D}-\eta^I\right)}.$$

Combining terms gives

$$2\frac{B'\left(\omega^{(2)}\right)}{B\left(\omega^{(2)}\right)}-\frac{A'\left(\omega^{(2)}\right)}{A\left(\omega^{(2)}\right)}-\frac{C'\left(\omega^{(2)}\right)}{C\left(\omega^{(2)}\right)}=$$

$$-\frac{2\left((1+\tau)\frac{\beta}{\sigma_D}-\eta^I\right)}{\sigma_D-\omega^{(2)}\left((1+\tau)\frac{\beta}{\sigma_D}-\eta\right)}-\frac{1}{\omega^{(2)}}-\frac{1}{\omega^{(2)}\left(1-\omega^{(2)}\right)}-\frac{\eta-\frac{\beta}{\sigma_D}}{\sigma_D+\left(1-\omega^{(2)}\right)\left(\frac{\beta}{\sigma_D}-\eta\right)}. \quad (68)$$

For future reference, we note that using $\omega^{(2)}>\omega^{(1)}$ along with (19) and the definition of $\omega^{(1)}$ implies that

$$\sigma_D+\left(1-\omega^{(2)}\right)\left(\frac{\beta}{\sigma_D}-\eta\right)>\sigma_D+\left(1-\omega^{(1)}\right)\left(\frac{\beta}{\sigma_D}-\eta\right)=0. \quad (69)$$

Using (61) we can write the right-hand side of (68) as

$$-\frac{2\left((1+\tau)\left(\eta-\xi\sigma_D\right)-\eta\right)}{\sigma_D-\omega^{(2)}\left((1+\tau)\left(\eta-\xi\sigma_D\right)-\eta\right)}-\frac{1}{\omega^{(2)}}-\frac{1}{\omega^{(2)}\left(1-\omega^{(2)}\right)}-\frac{\xi}{1-\xi\left(1-\omega^{(2)}\right)}. \quad (70)$$

Taking the limit as $\sigma_D$ approaches zero, the expression (70) converges to

$$-\frac{1}{1-\omega^{(2)}} - \frac{\xi}{1-\xi\left(1-\omega^{(2)}\right)} < 0,$$

where the inequality follows from (69) along with (61).[20]

The fact that the derivative $F'\left(\omega^{(2)}\right) < 0$ for any root of the equation $F\left(\omega^{(2)}\right) = 0$ in the interval $\left(\omega^{(1)}, 1\right)$ implies that the root $\omega^{(2)}$ must be unique. ■

**Proof of Proposition 2.** In preparation for the proof of Proposition 2, we state and prove an auxiliary result.

**Lemma 2** *The following statements hold for the quadratic equation (22).*

1. $\omega^{(1)} < \omega^{(2)}$ *and the discriminant of (22) is non-negative for all* $\omega_t^R \leq \omega^{(2)}$.
2. *When* $\omega^{(1)} \leq \omega_t^R \leq \omega^{(2)}$, *the two roots of the equation are both in the interval* $[0, 1)$.
3. *For* $\omega_t^R \in [0, \omega^{(1)})$, *only the larger root of (22) is in the interval* $(0, 1)$.
4. *If* $y$ *is a root of (22), then* $\left(1-\omega_t^R\right)\eta - \sigma_D - \frac{1-\omega_t^R}{\sigma_D}\beta\left(1-\tau y\right) > 0$.

**Proof of Lemma 2.** We start with part 1. Using the definitions (64)–(66), equation (22) can be written in the familiar form

$$A\left(\omega_t^R\right)y^2 + B\left(\omega_t^R\right)y + C\left(\omega_t^R\right) = 0,$$

and the discriminant of this quadratic equation is given by $F\left(\omega_t^R\right)$, as defined in equation (18).

For $\omega_t^R \leq \omega^{(1)}$, $C\left(\omega_t^R\right) < 0$ and the discriminant, $B^2\left(\omega_t^R\right) - 4A\left(\omega_t^R\right)C\left(\omega_t^R\right)$, is positive. The assumption that $\omega^{(2)}$ is the unique root of $F(\omega)$ along with the facts that $F\left(\omega^{(1)}\right) =$

[20]Equation (61) implies that $\frac{\beta}{\sigma_D} = \eta - \xi\sigma_D$, and therefore $\sigma_D + \left(1-\omega^{(2)}\right)\left(\frac{\beta}{\sigma_D} - \eta\right) = \sigma_D\left[1 - \left(1-\omega^{(2)}\right)\xi\right] < 0$, where the inequality follows from (69).

$B^2\left(\omega^{(1)}\right) > 0$ and $F(1) = -\infty$ imply that $\omega^{(1)} < \omega^{(2)}$.[21] The uniqueness of the root $\omega^{(2)}$ also implies that $F\left(\omega_t^R\right) = B^2\left(\omega_t^R\right) - 4A\left(\omega_t^R\right)C\left(\omega_t^R\right) \geq 0$ for all $\omega_t^R \leq \omega^{(2)}$.

We now turn to part 2. To economize on notation we write $A$ rather $A\left(\omega_t^R\right)$ and similarly for $B$ and $C$. Fix a given $\omega_t^R$ and let $g(y) = Ay^2 + By + C$.We have that $g(1) = A + B + C = \frac{\sigma_D}{1-\omega_t^R} > 0$ and $g'(1) = 2A + B = \sigma_D + \omega_t^R\left(\eta^I - (1-\tau)\frac{\beta}{\sigma_D}\right) > 0$, where the inequality follows from (19). Since $A > 0$, it follows that all roots of $g(y)$ must be smaller than one. Also, the fact that $\omega_t^R \geq \omega^{(1)}$ implies that $g(0) = C > 0$, while assumptions (19) and (20) together with the fact that $\omega_t^R \geq \omega^{(1)}$ imply that $g'(0) = B < 0$.

The facts that i) $g(y)$ is a convex, quadratic function of $y$, ii) $g(1) > 0, g(0) > 0$ and $g'(1) > 0, g'(0) < 0$ and iii) $B^2 - 4AC > 0$ for $\omega_t^R \in [\omega^{(1)}, \omega^{(2)})$ imply that there are two roots in $(0,1)$.

For part 3, we note that, when $\omega_t^R < \omega^{(1)}$, $g(0) = C < 0$, while $g(1) = A + B + C = \frac{\sigma_D}{1-\omega_t^R} > 0$. Therefore there exists one and only one root in $(0,1)$.

Finally, let $y \in (0,1)$ denote a root of the quadratic equation (22). Accordingly,

$$
\begin{aligned}
\left(1-\omega_t^R\right)\eta - \sigma_D - \left(1-\omega_t^R\right)\frac{\beta}{\sigma_D}(1-\tau y) &= \frac{1-\omega_t^R}{\omega_t^R}y\left(\sigma_D + \omega_t^R\eta^I - \omega_t^R\frac{\beta}{\sigma_D}(1-\tau y)\right) \\
&= \frac{1-\omega_t^R}{\omega_t^R}y\left[\sigma_D + \omega_t^R\left(\eta - \frac{\beta}{\sigma_D}\right) + \omega_t^R\frac{\beta}{\sigma_D}\tau y\right] \\
&> 0
\end{aligned}
$$

where the last inequality follows from (19). This proves property 4. ■

We now continue with the proof of the proposition. We provide expressions for $r_t$ and $\kappa_t$ that apply in any equilibrium where $w_t^R \neq 0$. Since $\sum_i \omega_t^i = 1$, it follows that $\sum_i \sigma_t^i = 0$ and $\sum_i \mu_t^i = 0$. Using (15) and $\sum_i \sigma_t^i = 0$ implies that $\sum_i \omega_t^i w_t^i = 1$. Combining $\sum_i \omega_t^i w_t^i = 1$ with

[21] Assumption (20) implies that $B\left(\omega^{(1)}\right) \neq 0$ and therefore $B^2\left(\omega^{(1)}\right) > 0$.

(12) along with the definition $y_t = \frac{\overline{W}_t^-}{\overline{W}_t^+}$ gives

$$\kappa_t + \left(1-\omega^R\right)\eta + \left(\omega_t^R \frac{1}{\sigma_D}\beta + \left(1-\omega_t^R\right)\tau y_t \frac{1}{\sigma_D}\beta\right) 1_{\left\{w_t^R<0\right\}} = \sigma^D. \tag{71}$$

Similarly, using (16) along with $\sum_i \mu_t^i = 0$ and $\sum_i \omega_t^i \left(n_t + w_t^i s_t^i\right) = 0$ gives (24).

We next describe the equilibria for the three subintervals of $\omega_t^R$ described in the statement of the proposition.

i) In this case, $\omega_t^R > \omega^{(2)}$. The equilibrium prescribes non-negative portfolios for both investors. If $\omega_t^R > 1 - \frac{\sigma_D}{\eta}$, equation (71) implies that $\kappa_t > 0$ and (12) implies that both investors hold positive portfolios and the shorting market is inactive. If $\omega_t^R \in [\omega^{(1)}, 1 - \frac{\sigma_D}{\eta})$, then there is an equilibrium that involves no shorting and a zero portfolio for investor $R$. Under this supposition the market clearing requirement becomes $\left(1-\omega_t^R\right) w_t^I = 1$, which together with $y_t = 0$ leads to (21). To confirm that this is indeed an equilibrium, note that

$$\begin{aligned}
\kappa_t + \frac{\beta}{\sigma_D} &= \frac{\sigma_D}{1-\omega_t^R} - \eta + \frac{\beta}{\sigma_D} \\
&> \frac{\sigma_D}{1-\omega^{(1)}} - \eta + \frac{\beta}{\sigma_D} \\
&= 0.
\end{aligned} \tag{72}$$

The first line follows from (21), the second line follows from $\omega_t^R > \omega^{(1)}$ and the third line follows from the definition of $\omega^{(1)}$. Since $\kappa_t + \frac{\beta}{\sigma_D} > 0$, investor $R$ does not choose a negative portfolio. And since $\kappa_t < 0$ for $\omega_t^R \in [\omega^{(1)}, 1 - \frac{\sigma_D}{\eta})$, the investor chooses a zero portfolio.

ii) In this case, $\omega^{(1)} < \omega_t^R < \omega^{(2)}$. Since $\omega_t^R > \omega^{(1)}$, equation (72) implies that the no-shorting equilibrium continues to be an equilibrium. However, we have two more equilibria. To compute them, we guess (and verify shortly) that $w_t^R < 0$. Using (12) and (71) gives

$$\begin{aligned} y_t &= \frac{\overline{W}_t^-}{\overline{W}_t^+} = \frac{-\omega_t^R w_{t,s}^R}{\left(1-\omega_t^R\right) w_{t,s}^I} = \frac{\omega_t^R}{1-\omega_t^R} \frac{-\left[\kappa_t + \frac{1}{\sigma_D}\beta\right]}{\kappa_t + \eta_t + \frac{1}{\sigma_D}\beta\tau y_t} \\ &= \frac{\omega_t^R}{1-\omega_t^R} \frac{\left(1-\omega_t^R\right)\eta - \sigma_D - \frac{1-\omega_t^R}{\sigma_D}\beta\left(1-\tau y_t\right)}{\sigma_D + \omega_t^R \eta - \frac{\omega_t^R}{\sigma_D}\beta\left(1-\tau y_t\right)}. \end{aligned}$$

Re-arranging leads to (22). Statement 1 of Lemma 2 implies that when $\omega_t \in (\omega^{(1)}, \omega^{(2)})$, then (22) has two roots between $(0,1)$. Under the supposition that $w_t^R < 0$, (71) leads to (23). In turn

$$\begin{aligned} \kappa_t^{\pm} + \frac{\beta}{\sigma_D} &= \sigma_D - \left(1-\omega_t^R\right)\eta - \frac{\omega_t^R}{\sigma_D}\beta\left[1 + \tau y^{\pm}\frac{1-\omega_t^R}{\omega_t^R}\right] + \frac{\beta}{\sigma_D} \\ &= \sigma_D - \left(1-\omega_t^R\right)\left(\eta + \frac{\beta}{\sigma_D}\left(1-\tau y_t^{\pm}\right)\right) < 0, \end{aligned} \tag{73}$$

where the last inequality follows from statement 4 of Lemma 2. Combining this observation with (12) confirms that $w_t^R < 0$. Note that in the second and third equilibria we have that

$$\kappa_t^{\pm} + \eta_t + \frac{1}{\sigma_D}\beta\tau y_t^{\pm} = \sigma_D + \omega_t^R \eta - \frac{\beta\omega_t^R}{\sigma_D}\left(1-\tau y_t^{\pm}\right) > 0,$$

where the last inequality follows from (73) along with the fact that $y^{\pm}$ satisfy the equation (22). This implies that $w_t^I > 0$.

iii) In this case, $\omega_t^R < \omega^{(1)}$. Statement 3 of Lemma 2 implies that the quadratic equation (22) has only one solution in $(0,1)$. This shows that there can only be one equilibrium with shorting. Moreover, this is the unique equilibrium If $w_t^R$ were zero and the Sharpe ratio were $\frac{\sigma_D}{1-\omega_t^R} - \eta^I$, then the argument given in (72) would imply that $\frac{\sigma_D}{1-\omega_t^R} - \eta + \frac{\beta}{\sigma_D} < 0$ and investor $R$ would want to deviate from the equilibrium prescription and choose a negative portfolio.

■

**Proof of Proposition 3.** Case I: Suppose that $w_t^R = 0$ in equilibrium $A$ and $w_t^R < 0$ in equilibrium $B$. Let $g_t^{R,j}$ denote the logarithmic growth rate of investor $R$ in equilibrium $j \in \{A, B\}$. We have

$$
\begin{aligned}
g_t^{R,B} - g_t^{R,A} &= -\left(\kappa^B - \kappa^A\right)\sigma_D + \max_{w_t \leq 0}\left\{w_t\left(\kappa^B\sigma_D + \beta\right) - \frac{1}{2}w_t^2\sigma_t^2\right\} \\
&> \left(\kappa^A - \kappa^B\right)\sigma_D \geq 0,
\end{aligned}
$$

where the first inequality follows from the fact that $w_t^R = 0$ is suboptimal for investor $R$ in equilibrium $B$ (by assumption). Similarly, using (16) gives

$$
\begin{aligned}
\mu_t^{R,B} - \mu_t^{R,A} &= \omega_t^R\left(\left(\kappa^A - \kappa^B\right)\sigma_D + w_t^B\sigma_D\left(\kappa^B + \frac{\beta}{\sigma_D} - \sigma_D\right)\right) \\
&= \omega_t^R\left[\left(\kappa^A - \kappa^B\right)\sigma_D + \left(1-\omega_t^R\right)w_t^B\sigma_D\left(\frac{\beta}{\sigma_D}(1-y) - \eta\right)\right] \\
&= \omega_t^R\left[\left(\kappa^A - \kappa^B\right)\sigma_D + \left(1-\omega_t^R\right)\left|w_t^B\right|\sigma_D\left(\eta - \frac{\beta}{\sigma_D}(1-y)\right)\right] \\
&> 0,
\end{aligned}
$$

where the first equality follows from (23), the second equality from $w_t^B < 0$ and the inequality from assumption (19) along with $y < 1$.

Case II: In this case the portfolio choice of investor $R$ is interior in both equilibria. Using the fact that in any interior equilibrium the optimal value of $w_t$ satisfies

$$
w_t\left(\kappa^B\sigma_D + \beta\right) - \frac{1}{2}w_t^2\sigma_t^2 = \frac{1}{2}w_t^2\sigma_t^2,
$$

we obtain

$$\begin{aligned}
g_t^{R,B} - g_t^{R,A} &= -\left(\kappa^B - \kappa^A\right)\sigma_D + \frac{\sigma_D^2}{2}\left[\left(w_t^{R,B}\right)^2 - \left(w_t^{R,A}\right)^2\right] \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D + \left(w_t^{R,B} + w_t^{R,A}\right)\frac{\sigma_D^2}{2}\left[\left(w_t^{R,B} - w_t^{R,A}\right)\right] \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D + \left(w_t^{R,B} + w_t^{R,A}\right)\frac{\sigma_D}{2}\left(\kappa^B - \kappa^A\right) \\
&= \left(\kappa^A - \kappa^B\right)\sigma_D\left[1 + \left|w_t^{R,B} + w_t^{R,A}\right|\frac{\sigma_D}{2}\right] \\
&> 0.
\end{aligned}$$

Using (16) gives

$$\begin{aligned}
\mu_t^{R,B} - \mu_t^{R,A} &= \omega_t^R\left(\left(\kappa^A - \kappa^B\right)\sigma_D + w_t^{R,B}\sigma_D\left(\kappa^B + \frac{\beta}{\sigma_D} - \sigma_D\right) - w_t^{R,A}\sigma_D\left(\kappa^A + \frac{\beta}{\sigma_D} - \sigma_D\right)\right) \\
&= \omega_t^R\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[w_t^{R,B}\left(w_t^{R,B} - 1\right) - w_t^{R,A}\left(w_t^{R,A} - 1\right)\right]\right) \\
&= \omega_t^R\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[\left(w_t^{R,B} - \frac{1}{2}\right)^2 - \left(w_t^{R,A} - \frac{1}{2}\right)^2\right]\right) \\
&= \omega_t^R\left(\left(\kappa^A - \kappa^B\right)\sigma_D + \sigma_D^2\left[\left(\left|w_t^{R,B}\right| + \frac{1}{2}\right)^2 - \left(\left|w_t^{R,A}\right| + \frac{1}{2}\right)^2\right]\right) \\
&> 0,
\end{aligned}$$

where the last inequality follows from $w_t^{R,B} < w_t^{R,A} < 0$ (since $0 > \kappa^A > \kappa^B$) and therefore $\left|w_t^{R,B}\right| > \left|w_t^{R,A}\right|$. ■

**Proof of Lemma 1.** The implicit function theorem states that

$$\frac{d\omega^{(2)}}{d\beta} = -\frac{F_\beta}{F_\omega}.$$

Since $\lim_{\omega\to\infty} F(\omega) = -\infty$ and the root $F\left(\omega^{(2)}\right) = 0$ is unique (by assumption), it follows that $F_\omega(\omega^{(2)}) < 0$. So it suffices to prove that $F_\beta(\omega^{(2)}) < 0$.

Differentiating $F$ with respect to $\beta$, multiplying the resulting expression by $\beta$ and evaluating at $\omega^{(2)}$ (which implies that $F\left(\omega^{(2)}\right)=0$) gives

$$\beta F_\beta=-2\omega^{(2)}\frac{\beta}{\sigma_D}\left(1+\tau\right)\left(\sigma_D-\omega^{(2)}\left(\left(1+\tau\right)\frac{\beta}{\sigma_D}-\eta\right)\right)-\left(\sigma_D-\omega^{(2)}\left(\left(1+\tau\right)\frac{\beta}{\sigma_D}-\eta\right)\right)^2-4\tau\left(\omega^{(2)}\right)^2\frac{\beta^2}{\sigma_D^2}$$

Completing the square gives

$$\begin{aligned}\beta F_\beta &= -\left(\sigma_D-\omega^{(2)}\left(\left(1+\tau\right)\frac{\beta}{\sigma_D}-\eta^I\right)+\omega^{(2)}\frac{\beta}{\sigma_D}\left(1+\tau\right)\right)^2+\left(\omega^{(2)}\right)^2\frac{\beta^2}{\sigma_D^2}\left(1-\tau\right)^2\\ &= -\left(\sigma_D+\omega^{(2)}\eta\right)^2+\left(\omega^{(2)}\right)^2\frac{\beta^2}{\sigma_D^2}\left(1-\tau\right)^2\\ &= -\left(\sigma_D+\omega^{(2)}\eta+\omega^{(2)}\frac{\beta}{\sigma_D}\left(1-\tau\right)\right)\left(\sigma_D+\omega^{(2)}\eta-\omega^{(2)}\frac{\beta}{\sigma_D}\left(1-\tau\right)\right)\\ &< 0,\end{aligned}$$

where the last inequality follows from the assumption $\eta\geq\frac{\beta}{\sigma_D}$. ■

**Proof of Proposition 4.** Differentiating $\kappa_t$ with respect to $\beta$ (in an equilibrium where $y>0$), we obtain

$$\frac{d\kappa_t}{d\beta}=-\frac{\omega_t^R}{\sigma_D}\left\{1+\frac{1-\omega_t^R}{\omega_t^R}\tau y_t\left[1+\frac{\beta}{y_t}\frac{dy_t}{d\beta}\right]\right\}. \tag{74}$$

In turn, the implicit function theorem applied to (22) gives

$$\frac{dy_t}{d\beta}=-\frac{\frac{\omega_t^R}{\sigma_D}\left(1-\tau y\right)\left(1-y\right)}{\sigma_D+\omega_t^R\eta-\frac{\omega_t^R}{\sigma_D}\beta\left(1+\tau-2\tau y\right)}=-\frac{\frac{\omega_t^R}{\sigma_D}\left(1-\tau y\right)\left(1-y\right)}{g'\left(y\right)},$$

where $g\left(y\right)$ is defined in Proposition 1. Since $g'\left(y^-\right)<0$ and $g'\left(y^+\right)>0$, we have $\frac{dy^-}{d\beta}>0$ and $\frac{dy^+}{d\beta}<0$. Combining $\frac{dy^-}{d\beta}>0$ with (74) implies $\frac{d\kappa_t}{d\beta}<0$ in the equilibrium associated with $y^-$. For the equilibrium associated with $y^+$ we have

$$1+\frac{\beta}{y^+}\frac{dy^+}{d\beta}=\frac{\left(\sigma_D+\omega_t^R\eta-\frac{\omega_t^R}{\sigma_D}\beta\left(1+\tau-2\tau y^+\right)\right)y^+-\frac{\beta\omega_t^R}{\sigma_D}\left(1-\tau y^+\right)\left(1-y^+\right)}{\left(\sigma_D+\omega_t^R\eta-\frac{\omega_t^R}{\sigma_D}\beta\left(1+\tau-2\tau y^+\right)\right)y^+}. \tag{75}$$

We are interested in the behavior of (75) as $\omega_t^R$ approaches zero. Letting $x \equiv \frac{y^+}{\omega_t^R}$, dividing both sides of (22) by $\omega_t^R$ and re-arranging terms yields

$$x\left(\sigma_D + \omega_t^R \eta\right) + \frac{\beta}{\sigma_D}\left(1 - \omega_t^R x\right)\left(1 - \tau \omega_t^R x\right) = \frac{1}{1 - \omega_t^R}\left(\left(1 - \omega_t^R\right)\eta^I - \sigma_D\right).$$

Taking limits as $\omega_t^R$ approaches zero, implies

$$\lim_{\omega_t^R \to 0} x = \frac{\eta - \sigma_D - \frac{\beta}{\sigma_D}}{\sigma_D}. \tag{76}$$

Using (76), as $\omega_t^R$ approaches zero we obtain

$$\begin{aligned}
\lim_{\omega_t^R \to 0}\left\{1 + \frac{1 - \omega_t^R}{\omega_t^R}\tau y_t\left[1 + \frac{\beta}{y_t}\frac{dy_t}{d\beta}\right]\right\} &= 1 + \tau \lim_{\omega_t^R \to 0}\{x_t\} \times \lim_{\omega_t^R \to 0}\left\{1 + \frac{\beta}{y_t}\frac{dy_t}{d\beta}\right\} \\
&= \tau \lim_{\omega_t^R \to 0}\{x_t\} \times \left\{1 - \frac{\frac{\beta}{\sigma_D}}{\sigma_D \lim_{\omega_t^R \to 0}\{x_t\}}\right\} \\
&= \tau\left[\frac{\eta - \sigma_D - 2\frac{\beta}{\sigma_D}}{\sigma_D}\right] \qquad (77) \\
&< 0, \qquad (78)
\end{aligned}$$

where we used (19) to derive the last inequality. Combining (78) with (74) implies that, for small $\omega_t^R$, $\frac{d\kappa\left(y^+\right)}{d\beta} > 0$. ■

**Proof of Proposition 5.** The proof essentially repeats the steps from the one-risky asset case, so we provide only a sketch, focusing on the elements that differ. We define

$$\overrightarrow{\beta} = \begin{bmatrix} \beta \\ 0 \end{bmatrix}, \overrightarrow{\eta} = \begin{bmatrix} \eta \\ 0 \end{bmatrix}.$$

We consider first an equilibrium with $y_t > 0$. Investor $R$'s and $I$'s optimal portfolios are given by

$$\overrightarrow{w}_t^R = (\sigma_t \sigma_t')^{-1} \left( \overrightarrow{\mu}_t - r_t 1_{2\times 1} + \overrightarrow{\beta} \right), \tag{79}$$

$$\overrightarrow{w}_t^I = (\sigma_t \sigma_t')^{-1} \left( \overrightarrow{\mu}_t - r_t 1_{2\times 1} + \sigma_{1,t} \overrightarrow{\eta} + \tau y_t \overrightarrow{\beta} \right). \tag{80}$$

Using (79) inside (36) yields

$$\begin{aligned} (\sigma_t \sigma_t') \overrightarrow{m}_t = (1 - \widehat{\omega}_t) \left[ \widetilde{\omega}_t^R \left( \overrightarrow{\mu}_t - r 1_N + \overrightarrow{\beta} \right) + \left(1 - \widetilde{\omega}_t^R\right) \left( \overrightarrow{\mu}_t - r 1_N + \sigma_1 \overrightarrow{\eta} + \tau y_t \overrightarrow{\beta} \right) \right] \\ + \widehat{\omega}_t (\sigma_t \sigma_t') \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix}, \end{aligned} \tag{81}$$

where we introduced the short-hand notation $\widetilde{\omega}_t^R = \frac{\omega_t^R}{1-\widehat{\omega}_t}$ and $\widetilde{\omega}_t^I = \frac{\omega_t^I}{1-\widehat{\omega}_t}$. (Since $\omega_t^R + \omega_t^I + \widehat{\omega}_t = 1$, it follows that $\widetilde{\omega}_t^R + \widetilde{\omega}_t^I = 1$.)

Next we use the row selection vector $[0, 1]$ to pre-multiply both sides of (81). Noting that $[0, 1] \overrightarrow{\beta} = [0, 1] \overrightarrow{\eta} = 0$, and also

$$(\sigma_t \sigma_t') \begin{bmatrix} 0 \\ \frac{\mu_{2,t} - r}{\sigma_{2,t}^2} \end{bmatrix} = \begin{bmatrix} b_t (\mu_{2,t} - r) \\ \mu_{2,t} - r \end{bmatrix}, \tag{82}$$

leads to (39). We next note that

$$\begin{aligned} [1, -b_t] \sigma \sigma' \begin{bmatrix} m_{1,t} \\ m_{2,t} \end{bmatrix} &= [1, -b_t] \begin{bmatrix} \left(\sigma_{1,t}^2 + b_t^2 \sigma_{2,t}^2\right) m_{1,t} + m_{2,t} b_t \sigma_{2,t}^2 \\ m_{1,t} b_t \sigma_{2,t}^2 + m_{2,t} \sigma_{2,t}^2 \end{bmatrix} \\ &= \sigma_{1,t}^2 m_{1,t}. \end{aligned} \tag{83}$$

Pre-multiplying both sides of (81) with the row vector $[1, -b_t]$, using (82) and (83) and the definition of $\kappa_{1,t}$ re-arranging yields

$$\kappa_{1,t} = \widetilde{m}_{1,t}\sigma_{1,t} - \left(1-\widetilde{\omega}_t^R\right)\eta - \widetilde{\omega}_t^R \frac{\beta}{\sigma_{1,t}}\left(1+\frac{1-\widetilde{\omega}_t^R}{\widetilde{\omega}_t^R}y_t\right). \tag{84}$$

Using the definition of $\kappa_{1,t}$ inside (79) gives

$$w_{1,t}^R = \frac{\kappa_{1,t}}{\sigma_1,t} + \frac{\beta}{\sigma_{1,t}^2}, \text{ and } w_{1,t}^I = \frac{\kappa_{1,t}+\eta}{\sigma_1} + \frac{\tau y_t \beta}{\sigma_{1,t}^2}, \tag{85}$$

where we used the notation $w_t^{1,R}, w_t^{1,I}$ to denote the first element of $w_t^R$ and $w_t^I$ respectively.

Using the market clearing condition $y_t = \frac{\omega_t^R w_{1,t}^R}{\omega_t^I w_{1,t}^I} = \frac{\widetilde{\omega}_t^R w_{1,t}^R}{\widetilde{\omega}_t^I w_{1,t}^I} = \frac{\widetilde{\omega}_t^R w_{1,t}^R}{\left(1-\widetilde{\omega}_t^R\right) w_{1,t}^I}$ leads to (85) leads to (37).

If agent $R$ chooses not to short then the market clearing condition becomes

$$\omega_t^I w_t^I + \left(\widehat{\omega}_t + \omega_t^R\right)\begin{bmatrix} 0 \\ \widehat{w}_t \end{bmatrix} = m_t. \tag{86}$$

Substituting (80), pre-mutiplying by $(\sigma_t\sigma_t')$ gives

$$(\sigma_t\sigma_t')\overrightarrow{m}_t = \omega_t^I\left(\overrightarrow{\mu}_t - r1_N + \sigma_1\overrightarrow{\eta}\right) + \left(\widehat{\omega}_t + \omega_t^R\right)(\sigma_t\sigma_t')\begin{bmatrix} 0 \\ \frac{\mu_{2,t}-r}{\sigma_{2,t}^2} \end{bmatrix}. \tag{87}$$

Pre-multpiplying (87) by the row $[1, -b_t]$ and using (82) and (83) gives

$$\sigma_{1,t}^2 m_{1,t} = \omega_t^I \sigma_{1,t}\left(\kappa_{1,t}+\eta\right).$$

Divding both sides by $\omega_t^I\sigma_{1,t}$ yields $\kappa_{1,t} = \sigma_{1,t}\frac{m_{1,t}}{\omega_t^I} - \eta = \sigma_{1,t}\frac{\widetilde{m}_{1,t}}{\widetilde{\omega}_t^I} - \eta = \sigma_{1,t}\frac{\widetilde{m}_{1,t}}{1-\widetilde{\omega}_t^R} - \eta$. Finally, when both agents hold positive portfolios, the optimal portfolios are $\overrightarrow{w}_t^R = (\sigma_t\sigma_t')^{-1}\left(\overrightarrow{\mu}_t - r_t 1_{2\times 1}\right)$, $\overrightarrow{w}_t^I = (\sigma_t\sigma_t')^{-1}\left(\overrightarrow{\mu}_t - r_t 1_{2\times 1} + \sigma_{1,t}\overrightarrow{\eta}\right)$. Repeating the arguments in equations (79)-(84), $\kappa_{1,t}$

becomes $\kappa_{1,t} = \widetilde{m}_{1,t}\sigma_{1,t} - \left(1 - \widetilde{\omega}_t^R\right)\eta$. ■

**Proof of Proposition 6.** The proof of this Proposition is contained in Appendix B. ■

**Proof of Proposition 7.** Since this proof is essentially identical to the proof of Proposition 2, we only provide a sketch. Combining (12) with $\sum_i \omega_t^i w_t^i = 1$ implies that in any equilibrium with $w_t^R < 0$ and $w_t^I > 0$ the Sharpe ratio is

$$\kappa_t + \left(1 - \omega^R\right)\eta + \omega_t^R \frac{1}{\sigma_D} f_t + \left(1 - \omega_t^R\right)\tau y_t \frac{1}{\sigma_D} f_t = \sigma_D. \tag{88}$$

Re-arranging (88) and using $f_t = l\left(y_t\right)$ gives (47). Substituting (47) back into the investors' optimal portfolios (12) and the fact that $y_t = -\frac{\omega_t^R w_t^R}{\left(1-\omega_t^R\right) w_t^I}$ leads to (46).

We next study the roots of (46). Let Let $Z\left(y\right) \equiv \frac{\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z(y)}{\eta + \frac{\sigma_D}{\omega_t^R} - \frac{1}{\sigma_D} z(y)}$, so that equation (46) can be expressed as $y = Z\left(y\right)$. The assumption of the proposition is that there exists at least one $y$ such that $y = Z\left(y\right)$. Let $y^{(*)}$ be the largest root of (46). We consider two cases i) $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(y\right) > 0$ for all $y \in [y^{(*)}, 1]$ and ii) $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(\overline{y}\right) = 0$ for some $\overline{y} \in [y^{(*)}, 1]$. In case i) it must be that $Z'\left(y^{(*)}\right) \leq 1$, since $y^{(*)} = Z\left(y^{(*)}\right)$ and $1 > Z\left(1\right)$. In case ii) it must also be that $Z'\left(y^{(*)}\right) \leq 1$ since $\overline{y} > Z\left(\overline{y}\right) = 0$. The assumption that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(0\right) < 0$ implies that there exists some interval $\left[\underline{y}, y^{(*)}\right]$ such that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(y\right) > 0$ for all $y \in \left[\underline{y}, y^{(*)}\right]$. Since $Z'\left(y^{(*)}\right) \leq 1$ it must be the case that there exists at least one more root in the interval $y \in \left[\underline{y}, y^{(*)}\right]$.

Finally, to confirm that a no-shorting equilibrium is also an equilibrium, $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} z\left(0\right) < 0$ implies that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} l\left(0\right) < 0$. If the Sharpe ratio is given by $\kappa_t = \frac{\sigma_D}{1-\omega_t^R} - \eta < 0$, the assumption that $\eta - \frac{\sigma_D}{1-\omega_t^R} - \frac{1}{\sigma_D} l\left(0\right) < 0$ implies that $\kappa_t + \frac{1}{\sigma_D} l\left(0\right) > 0$. Accordingly, investor $R$ does not wish to short when the fee is $f_t = l\left(0\right)$ and the lending market clears with $y = 0$ at the lending fee $l\left(0\right)$. Moreover, $w_t^I = \frac{\kappa_t + \eta}{\sigma_D} = \frac{1}{1-\omega_t^R}$. Therefore $\omega_t^R \times 0 + \left(1 - \omega_t^R\right) \times w_t^I = 1$ and the stock market clears. ■

# D Additional Data Discussion

## D.1 Methodology

### D.1.1 Measuring ticker discussion on WallstreetBets

Our measure of ticker mentions on WallstreetBets is constructed as follows. We use the PushshiftAPI to collect all submissions posted on WallstreetBets subreddit from January 1, 2020 through February 7, 2021 (Baumgartner et al., 2020). For each submission, we observe the title text, the body of the submission, the author of the submission, and the time of the submission.

In order to identify which tickers are discussed in the submission, we take advantage of the fact that users often tag tickers with a leading $ (i.e. $TSLA or $AAPL). This practice is entirely voluntary and is therefore insufficient for identifying all mentions of a ticker. We use regular expressions to identify all words tagged in this way and match those words to CRSP tickers that were traded on the NYSE, AMEX, and NASDAQ exchanges in 2020. This gives us a set $S$ of roughly 4,000 tickers that are mentioned on WSB between January 2020 and February 2021.

We then identify all cases in which these tickers are mentioned in submissions, irrespective of whether they are prefixed with a dollar sign. To address the possibility of falsely identifying tickers, we require that, if the ticker is a common word in the written English language, it must be prefaced by a dollar sign. For example, AT&T's ticker T is also a common word in written English, and thus we require that the text "$T" appear in a submission for it to be considered as mentioned AT&T. We consider a ticker as being mentioned in a submission if it appears in either the title or the body of the submission. We identify common word-stems based on the Google Trillion Word Corpus (Michel et al., 2011). In a robustness check, we account for the downward bias this restriction introduces by scaling common-word tickers by an in-sample estimated adjustment factor. This adjustment leaves the relative ranking of ticker mentions largely unchanged. We estimate the adjustment factor by comparing the frequency of tagged ticker mentions versus untagged ticker mentions for the set of tickers which do not commonly appear in written English.

**Revised submissions and comments.** Authors of Reddit comments have the ability to edit their comments even after the comment has been posted. The PushshiftAPI records the comment text as of a certain day, and does not update to reflect potential revised comments. The same constraint applies to the content body of submissions. Titles of submissions cannot be revised and thus do not have this measurement problem.

**Missed tickers** Tickers that, for whatever reason, are never tagged with a leading dollar sign will be omitted from our dataset. Similarly, we under-count the occurrences of tickers that are common words, owing to requiring they appear with a leading "$" We attempt to correct for this by scaling the observed counts for common word tickers. For AAPL and GME, which are not common word tickers, the ticker appears with the leading "$" roughly 20% of the time. We can thus simply multiply our observed frequencies by a factor of five to adjust for the more stringent matching procedure. As can be seen in Figures 16a and 16b, the adjustment does not have a significant impact on the relative popularity of the top tickers.

In some cases, users may choose to refer to the company by its name, rather than by its ticker. We do not attempt to identify mentions of companies by name.

## D.2 Measuring retail trading

We adopt the methodology of Boehmer et al. (2020) to identify retail trades in the TAQ data. We briefly summarize the methodology here and refer readers to the paper for details.

The intuition behind the methodology is the knowledge that retail trades are often executed by wholesalers or via broker internalization, rather than on the major trading exchanges. These trades appear in the TAQ consolidated tape data under the exchange code "D". These trades are given a small price improvement on the order of tenths of a penny as a means to induce brokers to route orders to the wholesaler. Similarly, brokers which internalize retail trades offer a subpenny price improvement in order to comply with Regulation 606T. Importantly, institutional trades are rarely, if ever, internalized or directed to wholesalers and their trades are usually in round penny prices, with the notable exception of midpoint trades.

The methodology of Boehmer et al. (2020) uses these institutional details to identify retail trades in the TAQ consolidated tape data. Trades flagged with exchange code "D" and with a subpenny amount in the set $(0, 0.40) \bigcup (0.60, 1.00)$ are identified as retail trades. Splitting these trades further, retail trades with subpenny amounts between zero- and forty-hundredths of a penny are labeled as "sell orders", whereas subpenny amounts between sixty- and one hundred-hundredths are considered "buy orders". The midpoint trades are excluded to avoid mis-classifying institutional trades executed at midpoints as retail trades.

### D.2.1 Challenges

**Derivatives** The TAQ data only contains trades of equities. Options offer another way to benefit for investors to benefit from increases in the price of stock. As an added advantage for retail investors, options offer embedded leverage greater than what might otherwise be available through their broker. The Boehmer et al. (2020) methodology relies on institutional details to identify off-exchange retail trades, and thus cannot reliably identify replication trades by market makers.

## D.3 Betting against the shorts portfolio

As is standard in the literature, we restrict attention to common shares of COMPUSTAT firms which trade on the NYSE, AMEX, and NASDAQ exchanges. We further exclude companies for whom no share class has a price exceeding \$1. The strategy equally weights each firm in the top decile, shorts the market index, and reconstitutes 8 trading days following the disclosure date, which is the first opportunity following the public dissemination of the short interest data.

# E Additional Figures and Tables

## E.1 Analyzing Reddit Discussion

Consistent with the views expressed in the press, there is evidence that stocks with comparatively higher short interest were more likely to be discussed on the WSB subreddit. To substantiate this claim, we analyze the text of all WSB posts and identify all mentions of individual stocks. In Table 2, we investigate which stock characteristics make a stock a likely topic for discussion on the WSB subreddit over the two week period from January 15 to January 25, 2021. Consistent with the popular press, short interest as of January 15, 2021 is strongly correlated with these mentions. We show this using two specifications. In columns (1)–(3), we use ticker-mention percentile as the y-variable. The most discussed ticker, GME, takes a value of 1, while a stock that is never discussed takes a value of 0. By using the ranked ticker-mentions, rather than the raw counts, we avoid over-weighting right-tail observations in the data.

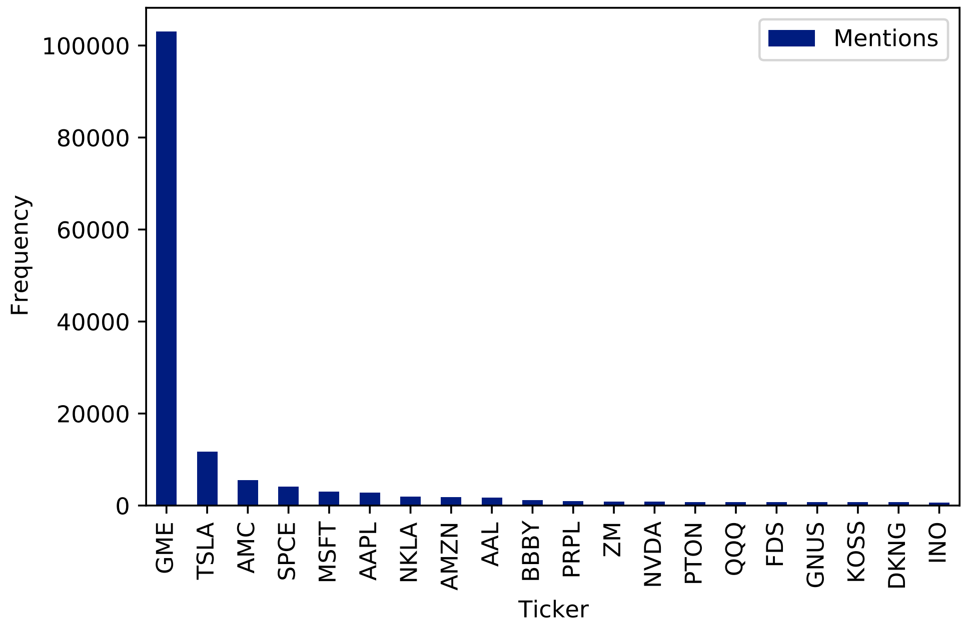
Since several stocks are not discussed at all, we estimate a Tobit model to account for the left-censoring of the y-variable at 0. In column (1), we first include only a concise set of explanatory variables: short interest and index inclusion for both the Russell 3000 and the S&P 500. The reported coefficient on short interest is for a standardized version of the variable, and has the interpretation that a one standard-deviation increase in short interest is associated with a 34 percent increase in the distribution of ticker-mentions. The coefficient on short interest is economically large and significant even when we include financial ratios (column (2)) and factor betas (column (3)). Both financial ratios and factor betas are calculated as of December 2020. A one standard deviation increase in short interest leads to a 24 to 34 percent increase in the distribution of ticker mentions. Short interest is measured as shares shorted divided by total shares outstanding, value-weighted across share classes of a given firm. In columns (4)–(6), we repeat the analysis with a different y-variable: an indicator for whether the ticker was discussed at all on WSB. This linear probability model, which we estimate using OLS, uses only the extensive margin of ticker discussion on WSB, whereas the Tobit model uses both the extensive and intensive margins of ticker-mentions. We find that a one standard deviation increase in short interest is associated with a 8 to 11 percent higher probability of being discussed on WSB. As can be seen from the count of uncensored observations in our Tobit model, roughly 13 percent of tickers are ever mentioned on WSB, so this coefficient is large in magnitude.

Table 2: Characteristics of highly-discussed tickers

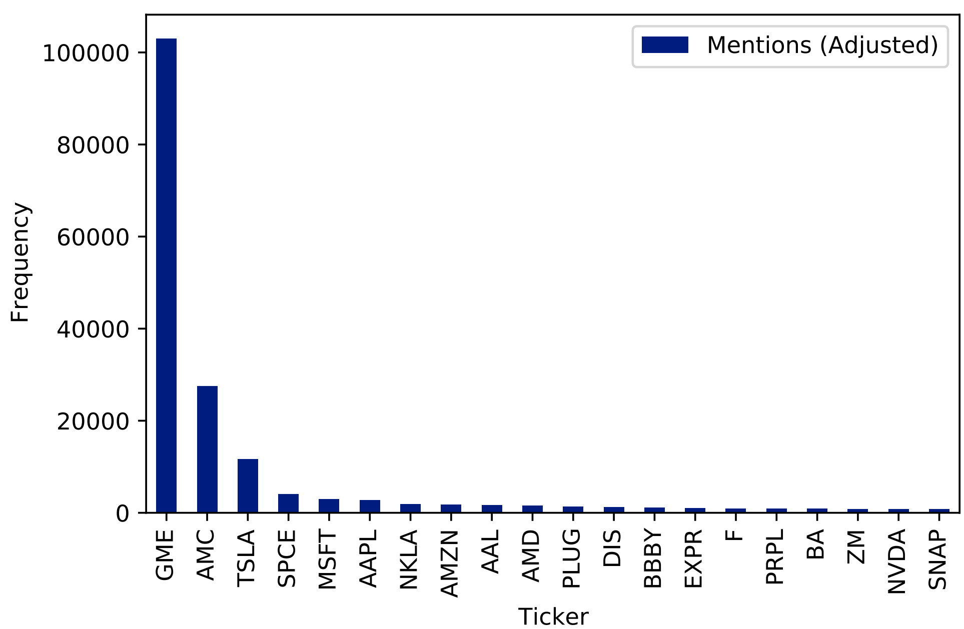
| | Tobit | | | OLS | | |
|---|---|---|---|---|---|---|
| | Rank | Rank | Rank | Indicator | Indicator | Indicator |
| Short Interest | 0.335*** | 0.309*** | 0.241*** | 0.107*** | 0.099*** | 0.081*** |
| | (0.023) | (0.024) | (0.024) | (0.006) | (0.006) | (0.007) |
| Russell Constituent | -0.482*** | -0.502*** | -0.197** | -0.113*** | -0.115*** | -0.044** |
| | (0.065) | (0.071) | (0.086) | (0.014) | (0.015) | (0.020) |
| SPX Constituent | 0.905*** | 0.977*** | 1.024*** | 0.244*** | 0.242*** | 0.249*** |
| | (0.077) | (0.086) | (0.091) | (0.018) | (0.019) | (0.020) |
| Book-Market | | -0.171*** | -0.067* | | -0.022*** | -0.012* |
| | | (0.042) | (0.037) | | (0.006) | (0.007) |
| Debt-Assets | | 0.028 | 0.107** | | 0.011 | 0.029*** |
| | | (0.040) | (0.042) | | (0.009) | (0.009) |
| Dividend Yield | | -0.002 | 0.026 | | 0.002 | 0.008 |
| | | (0.047) | (0.045) | | (0.009) | (0.009) |
| Dividend Yield Missing | | 0.282** | 0.144 | | 0.042* | 0.008 |
| | | (0.112) | (0.111) | | (0.022) | (0.023) |
| $\beta_{MKT}$ | | | 0.091*** | | | 0.015** |
| | | | (0.029) | | | (0.007) |
| $\beta_{SMB}$ | | | -0.019 | | | -0.003 |
| | | | (0.032) | | | (0.007) |
| $\beta_{HML}$ | | | -0.213*** | | | -0.046*** |
| | | | (0.038) | | | (0.009) |
| $\beta_{UMD}$ | | | 0.167*** | | | 0.037*** |
| | | | (0.035) | | | (0.008) |
| Idio. Vol. | | | 0.229*** | | | 0.058*** |
| | | | (0.035) | | | (0.008) |
| $N_{unc}$ | 428 | 390 | 390 | | | |
| $N$ | 3100 | 2947 | 2947 | 3100 | 2947 | 2947 |
| $R^2$ | 0.118 | 0.132 | 0.167 | 0.125 | 0.128 | 0.154 |

Columns (1)-(3) report results estimated using a left-censored tobit model on ticker discussion percentile as the left hand side variable. Columns (4)-(6) report results estimated using OLS on a dummy variable indicating whether a ticker was discussed on the WallstreetBets sub-reddit. Both Rank and Indicator variables are based on tickers mention between January 15 and January 25. $\beta$ and volatility co-variates are calculated as of January 15, 2021. Financial ratios are calculated as of most recent reporting date available on WRDS.

*Note:* Coefficients reported are for continuous co-variates standardized to have mean zero and unit standard deviation. $N_{unc}$ denotes number of uncensored observations in the tobit specification. $R^2$ for columns (1)-(3) denotes McFadden's pseudo R-squared. * p <0.1, ** p <0.05, *** p <0.01. Standard errors in parentheses.

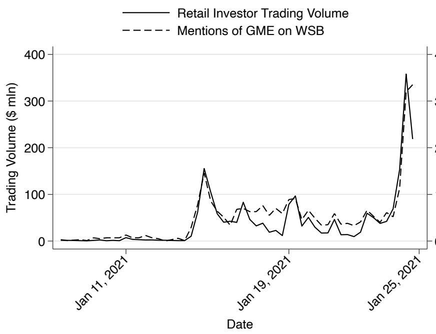



(a) Submissions mentioning each Ticker

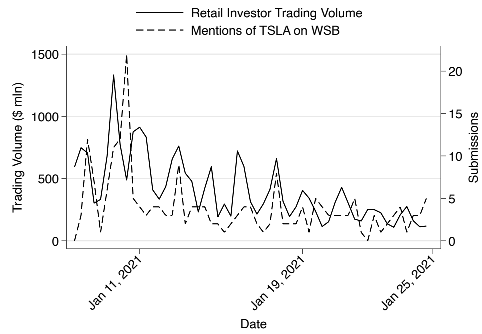



(b) Submissions mentioning each Ticker, adjusted for word-ticker overlap

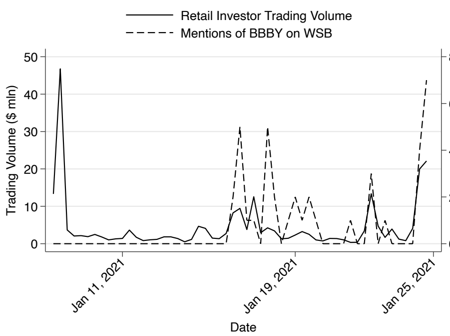
Figure 16: Popular Tickers on WallstreetBets (January 1, 2020 - February 7, 2021).



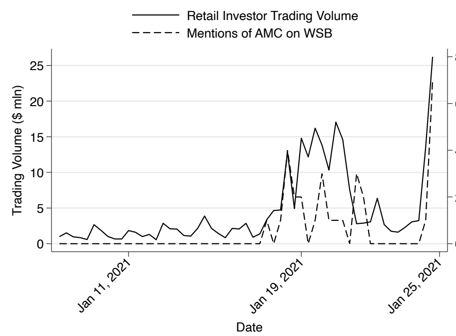

(a) Gamestop (GME)



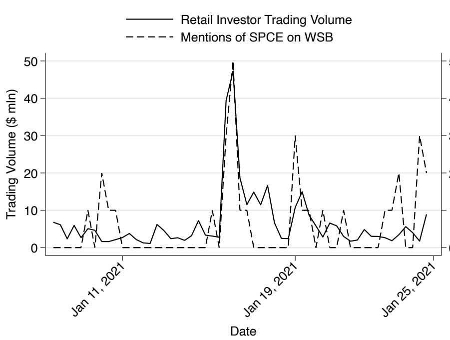

(b) Telsa (TSLA)



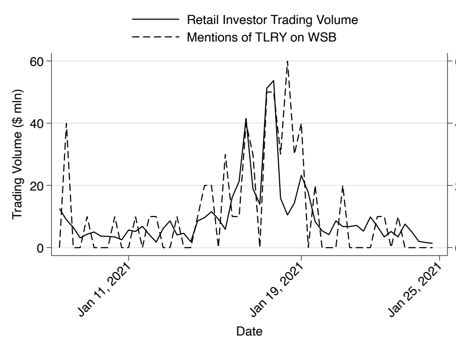

(c) Bed, Bath and Beyond (BBBY)




(d) AMC (AMC)

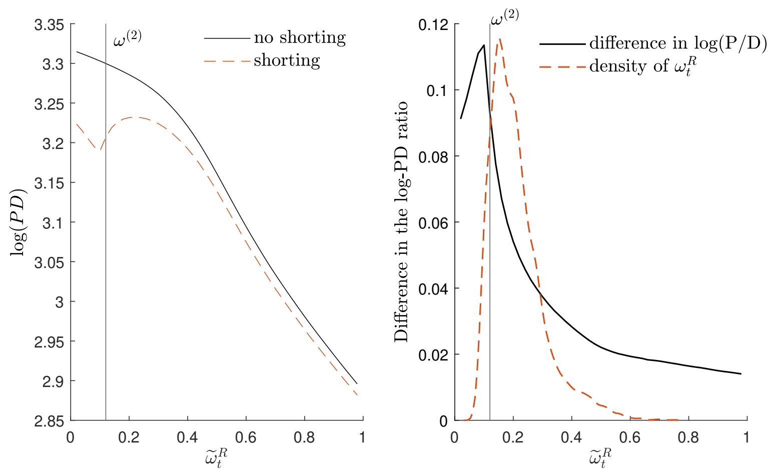



(e) Virgin Galactic (SPCE)

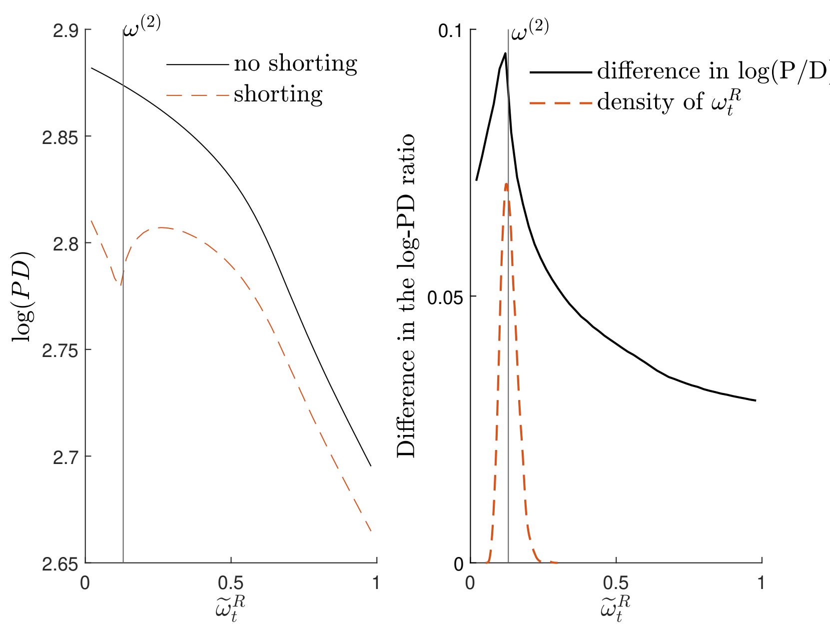



(f) Tilray (TLRY)

Figure 17: Retail Trading Volume and Reddit Discussion (January 7 - January 25, 2021).




(a) $\delta = 0.1$




(b) $r + \delta_1 = 0.10$

Figure 18: This figure repeats the same quantitative exercise as Figure 13 but with modified $r + \delta_1$ and $\delta$.