The Macroeconomic Effects of Corporate Tax Reforms*

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Abstract

We investigate the effects of corporate tax reforms on aggregate capital behavior. In a model of lumpy investment with fixed adjustment costs and partial irreversibility, we show that corporate taxes and investment frictions jointly determine three interconnected macroeconomic outcomes: (i) capital allocation, (ii) capital valuation, and (iii) capital dynamics. We validate the theory using firm-level investment data and corporate tax data from Chile. We discover that a higher corporate income tax rate correlates with higher capital misallocation, lower aggregate Tobin's q, and slower propagation of aggregate productivity shocks.

JEL: D30, D80, E20, E30

Keywords: corporate taxes, investment frictions, fixed adjustment costs, irreversibility, lumpiness, capital misallocation, Tobin's q, transitional dynamics, inaction, propagation.

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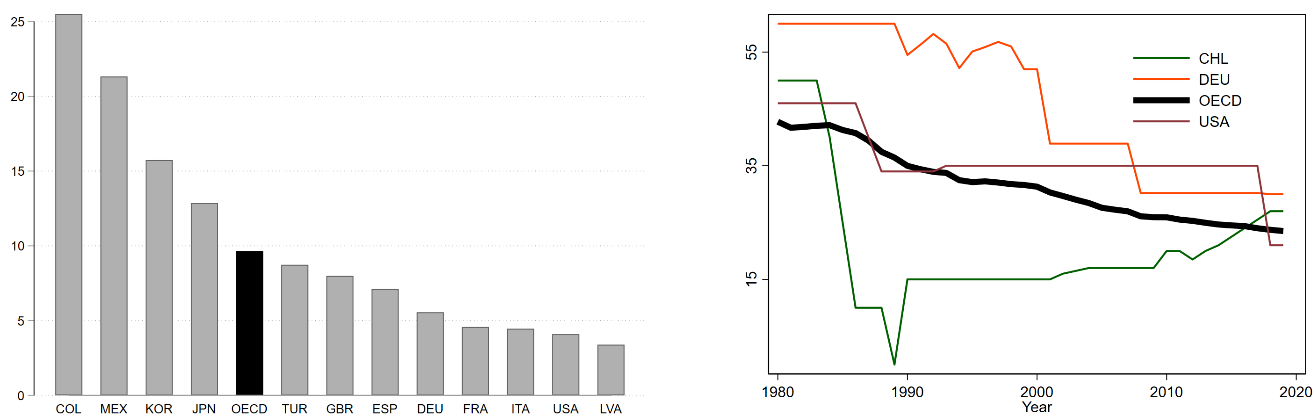
1 Introduction

Taxes paid by companies are a key source of tax revenues for governments around the globe. Among OECD countries, corporate income tax revenue in 2018 accounted for an average of 10% of total tax revenue, ranging from 3.4% in Latvia to 25% in Colombia (Panel A in Figure I). The importance of corporate taxation remains large, despite a generalized falling trend in tax rates over the last four decades (Panel B in Figure I); in particular, the median corporate income tax rate has decreased from 42% in 1980 to 25% in 2020. At the country level, corporate tax reforms happen infrequently and thus are very persistent. In the US, for instance, only two reforms in the corporate income tax rate have occurred in the last 40 years, in 1986 and 2018.

Figure I – Corporate Taxes in OECD Countries

(A) Corporate income tax revenue
(% of total tax revenue)

(B) Corporate income tax rate
(%, highest marginal rate)



Source: OECD Revenue Statistics Database. Corporate income tax revenue includes corporate income tax and capital gains tax revenue. Data for the largest OECD countries in terms of GDP and the countries with the lowest and the highest value in the sample.

Beyond their importance as a source of government revenue, corporate taxes have significant economic effects on private investment, a key driver of long-run growth and short-run fluctuations. Moreover, recent economic developments have revived the interest in corporate taxation. The most dramatic development is the COVID-19 pandemic, which has forced governments to increase their spending through economic relief plans and accumulate massive debts. The current policy debate on how governments will repay these new debts—together with tax competition for foreign direct investment, the exhaustion of monetary policy after years at the zero lower bound, and the secular increase in business profits—has put corporate tax reforms into the spotlight. In this paper, we ask: What are the macroeconomic effects of persistent corporate tax reforms?

We answer this question in four steps. First, we develop a parsimonious lumpy investment model with empirically relevant investment frictions, consisting of a fixed capital adjustment cost (Caballero and Engel, 1999) and a wedge between the purchase and sale prices of capital that makes investment partially irreversible (Abel and Eberly, 1996; Bertola and Caballero, 1994; Veracierto, 2002; Lanteri, 2018). Second, we formalize the relationships between three macroeconomic outcomes: (i) *capital allocation*, measured by the cross-sectional dispersion of the log marginal product of capital; (ii) *capital valuation*, measured by the capital-weighted marginal q; and (iii) *capital dynamics*, measured by the cumulative impulse response following an aggregate productivity shock. Third, following Auerbach (1979) and Summers (1981), we introduce a comprehensive corporate tax schedule and characterize the joint effects of taxes and investment frictions in shaping the three macroeconomic outcomes. In particular, we show that investment depends exclusively on investment frictions relative to *after-tax* profits. Lastly, using Chilean data, we provide suggestive evidence of the theory showing the low-frequency comovement of the time-series of corporate income taxes and the three macroeconomic outcomes and exploiting cross-sectoral data. We discover that a higher corporate income tax rate increases capital misallocation, decreases capital valuation, and slows down capital dynamics following an aggregate productivity shock.

Qualitative predictions. Consider a firm making investment choices under idiosyncratic productivity shocks, fixed capital adjustment costs, partial irreversibility, a constant interest rate, and a corporate tax schedule—including a corporate income tax, a personal income tax, a capital gains tax, and a geometric depreciation allowance. The firm chooses when and how much to invest to maximize its present value of profits minus the adjustment costs. We characterize the investment strategy as an inaction region for the capital-productivity ratio and two points where the capital-productivity ratio is reset after a positive or a negative adjustment. In this environment, we show that corporate taxation affects the optimal investment policy through three channels: the *after-tax* adjustment costs, the *after-tax* investment prices, and the *after-tax* discount factor.

Next, consider a continuum of ex-ante identical firms. The economy features a steady-state cross-sectional distribution of capital-productivity ratios, which is the outcome of the investment frictions and the corporate tax system. We establish the structural relationships that exist between three steady-state macroeconomic outcomes. The first macroeconomic outcome is *capital misallocation*, defined as the cross-sectional variance of the log marginal revenue product of capital (Hsieh and Klenow, 2009; Restuccia and Rogerson, 2013; Asker, Collard-Wexler and De Loecker, 2014; David and Venkateswaran, 2019). We show that misallocation is proportional to the variance of capital-productivity ratios. The second macroeconomic outcome is capital valuation, defined as the capital-weighted Tobin's marginal q. We show that aggregate q decreases monotonically with capital misallocation and with the aggregate capital stock, implying that aggregate Tobin's q is a sufficient statistic for aggregate investment and clarifying contradicting views in the literature (Tobin, 1969; Abel, 1979; Hayashi, 1982; Abel and Eberly, 1996; Caballero and Leahy, 1996; Philippon, 2009). The third macroeconomic outcome is *capital dynamics*, defined as the cumulative impulse response (CIR) of aggregate capital following an unanticipated, once-and-for-all, and small shock to all firms' productivity (Álvarez, Le Bihan and Lippi, 2016; Baley and Blanco, 2021). We show that, up to first order, the CIR increases with capital misallocation and irreversibility: economies with more pervasive frictions feature more persistent economic fluctuations.

Since capital misallocation is the primary link across the different macroeconomic outcomes, we characterize the channels through which corporate taxes and investment frictions affect misallocation. We focus on the corporate income tax rate. A higher corporate income tax decreases profitability and thus increases fixed costs relative to after-tax profits. As a result, capital misallocation increases, aggregate q may increase or decrease (depending on the tax effects on capital accumulation and its allocation across firms), and the CIR increases (the propagation of aggregate shocks slows down). These results put forward a new policy channel: Corporate tax policy can effectively change the size of investment frictions—technological constraints or market prices typically considered outside the control of a policymaker—and structurally change the behavior of aggregate capital.

Quantitative predictions. The challenge in analyzing the effects of corporate tax reforms on the distribution of capital-productivity ratios, which in turn determines the three macro outcomes, stems from the fact that this distribution is not directly observed. Yet, economists have available panel data with information on the actions of adjusting firms, namely, the size of discrete capital adjustments and the duration of completed inaction spells. We put forward a new estimation technique that maps the data into the cross-sectional moments of the capital-productivity distribution, establishing a connection between macro outcomes and microdata.

We calibrate our model to match the average level of taxes and the relevant investment moments during the last 40 years in Chile to assess investment frictions' magnitude. Using the calibrated model, we examine the elasticity of macroeconomic outcomes to the corporate income tax rate. Then, we provide suggestive evidence for the validity of our theory using low-frequency time-series correlations. We confirm that a higher corporate income tax rate associates with larger misallocation and slower propagation of productivity shocks, while it is associated with lower aggregate q. Finally, we exploit cross-sectoral data and confirm the theoretical prediction that sectors with initially higher investment frictions, manifested in higher misallocation, suffer a larger change in the macroeconomic outcomes after an increase in the corporate tax rate.

In summary, our framework constitutes a laboratory for assessing their interaction of corporate taxation with empirically-relevant investment frictions and for examining the macroeconomic effects of persistent corporate tax reforms.

Contributions to the literature. The impact of corporate tax reforms is a widely studied topic. A vast and active literature uses detailed firm-level data to estimate the short-run investment response to tax reforms, including investment tax credits (Hall and Jorgenson, 1967; Lerche, 2019); bonus depreciation (Hassett and Hubbard, 2002; House and Shapiro, 2008; Zwick and Mahon, 2017; Maffini, Xing and Devereux, 2019); dividend taxes (Ohrn, 2018; Boissel and Matray, 2019); and corporate income tax (Yagan, 2015). Thanks to this work, we have a fairly good understanding of the short-run microelasticity of investment to taxes for concrete taxes, countries, and periods. We contribute by developing a structural investment model that is rich enough to capture complex taxation schedules and realistic firm heterogeneity; and parsimonious enough to characterize the long-run elasticities of macroeconomic outcomes to persistent tax reforms.

Understanding the short-run consequences of corporate tax reforms is undoubtedly important. But most corporate tax reforms are highly persistent, as shown in Figure I, and thus demand a long-run analysis. An early literature focused on the long-run consequences of corporate taxation, highlighting the effects on firm valuation and aggregate investment (Summers, 1981; Abel, 1982; Poterba and Summers, 1983; King and Fullerton, 1984; Auerbach, 1986; Auerbach and Hines, 1986; Hassett and Hubbard, 2002). Their analysis, framed in a neoclassical framework with a representative firm, precluded the incorporation of empirical relevant investment frictions and firm heterogeneity. Subsequent work made progress on these fronts by incorporating certain stances of firm heterogeneity as well as convex and non-convex adjustment frictions (Gourio and Miao, 2010, 2011; Altug, Demers and Demers, 2009; Winberry, 2021; Chen, Jiang, Liu, Suárez Serrato and Xu, 2019). We contribute by characterizing macroeconomic outcomes as a function of a few moments of the investment rate distribution, facilitating the analysis and measurement of the economic mechanisms that shape the effects of corporate tax reforms.

Lastly, we contribute to the literature studying the role of micro-level adjustment frictions for economic fluctuations (see Caplin and Spulber, 1987; Caballero and Engel, 1991, 1993, for early work). Recent theoretical work identifies a small set of observable micro-moments that capture the role of adjustment frictions for the propagation of aggregate shocks. For instance, Álvarez, Le Bihan and Lippi (2016) showed that in a large class of price-setting models, the cumulative output response following a monetary shock is proportional to the the expected price duration times the kurtosis of price changes; and Baley and Blanco (2021) showed that in models of lumpy investment the cumulative capital response following a productivity shock is proportional to the cross-sectional dispersion of capital-productivity ratios and the covariance of capital-productivity ratios with the time elapsed since their last adjustment. We contribute by extending this line of work to consider the role of corporate taxes in shaping the observable micro-moments. Additionally, we show how to handle the history dependence arising from partial irreversibility. In this way, we expand the breadth of problems that can be analyzed with this type of methodology to include history dependence, also labeled as problems with reinjection by Álvarez and Lippi (2021).

2 Investment With Fixed Costs and a Price Wedge

In this section, we develop a parsimonious investment model with the following features: idiosyncratic productivity shocks, fixed capital adjustment costs, a wedge between the purchase price and the sale price of capital that makes investment partially irreversible, and a constant interest rate.[1]

2.1 The problem of an individual firm

Time is continuous, extends forever, and it is denoted by s. The future is discounted at rate $\rho > 0$. We first present the problem of an individual firm and then consider a continuum of ex ante identical firms to characterize the aggregate behavior of the economy.

Technology and shocks. The firm produces output y_s using capital k_s according to a production function with decreasing returns to scale

$$(1) \qquad y_s \; = \; u_s^{1-\alpha} k_s^{\alpha}, \quad \alpha < 1.$$

Flow profits equal $\pi_s \equiv A y_s$, where A is a profitability parameter. Idiosyncratic productivity u_s follows a geometric Brownian motion with drift μ and volatility σ,

$$(2) \qquad \log u_s \; = \; \log u_0 + \mu s + \sigma W_s, \qquad W_s \sim Wiener.$$

The capital stock, if uncontrolled, depreciates at rate $\xi^k > 0$.

Investment frictions. The firm can control its capital stock through buying and selling investment goods at prices p^{buy} and p^{sell}. We assume that $p^{\text{buy}} > p^{\text{sell}}$ to reflect adverse selection, specificity of capital goods, or other frictions in the market for used capital that make investment partially irreversible (Abel and Eberly, 1996; Bertola and Caballero, 1994; Lanteri, 2018).[2] For every investment $i_s \equiv \Delta k_s = k_s - k_{s^-}$, the firm must pay an adjustment cost proportional to its productivity $\theta_s = \theta u_s$ where $\theta > 0$ is constant and it is measured in consumption units (Caballero and Engel, 1999).[3] To simplify notation, we define the price function

$$(3) \qquad p\left(i_s\right) \; = \; p^{\text{buy}} \mathbb{1}_{\{i_s>0\}} + p^{\text{sell}} \mathbb{1}_{\{i_s<0\}}.$$

Investment problem. Let $V(k, u)$ denote the value of a firm with capital stock k and productivity u. Given initial conditions (k_0, u_0), the firm chooses a sequence of adjustment dates $\{T_h\}_{h=1}^{\infty}$

[1] Our assumption of a constant interest rate is largely consistent with Koby and Wolf (2020), who show that micro data strongly reject the price elasticities required for a strong general equilibrium feedback.

[2] Chen, Jiang, Liu, Suárez Serrato and Xu (2019) show that non-deductible VAT tax generated a price wedge in China before its 2005 reform. Additionally, investment tax credits also contribute to the price wedge.

[3] For any stochastic process q_s, we use the notation $q_{s^-} \equiv \lim_{r\uparrow s} q_r$ to denote the limit from the left.

and investments $\{i_{T_h}\}_{h=1}^{\infty}$, where h counts the number of adjustments, to maximize its expected discounted stream of profits. The sequential problem is

$$V(k_0, u_0) \;=\; \max_{\{T_h, i_{T_h}\}_{h=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} e^{-\rho s}\pi_s \,\mathrm{d}s - \sum_{h=1}^{\infty} e^{-\rho T_h}\left(\theta_{T_h} + p\left(i_{T_h}\right) i_{T_h}\right)\right], \tag{4}$$

subject to the production technology (1), the idiosyncratic productivity shocks (2), the investment price function (3), and the law of motion for the capital stock

$$\log k_s \;=\; \log k_0 - \xi^k s + \sum_{h:T_h \le s} \left(1 + i_{T_h}/k_{T_h^-}\right), \tag{5}$$

which describes a period's capital as a function of its initial value k_0, the physical depreciation rate ξ^k, and the sum of all adjustments made at prior adjustment dates.

2.2 Capital-productivity ratios $\hat{k}$

To characterize the investment decision, it is convenient to reduce the state-space and recast the firm problem using a new state variable, the log capital-productivity ratio:

$$\hat{k}_s \;\equiv\; \log\left(k_s/u_s\right). \tag{6}$$

The problem admits this reformulation because the production function is homothetic and the adjustment costs are proportional to productivity. Note that in the absence of investment frictions, $\hat{k}_s$ is a constant. With investment frictions, between any two consecutive adjustment dates $[T_{h-1}, T_h]$, the capital-productivity ratio $\hat{k}$ follows a Brownian motion

$$\mathrm{d}\hat{k}_s \;=\; -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s, \tag{7}$$

where the drift $\nu \equiv \xi^k + \mu$ reflects the depreciation rate and productivity growth rate. At any adjustment date T_h, the log capital-productivity ratio changes by the amount

$$\Delta\hat{k}_{T_h} \;=\; \log\left(1 + i_{T_h}/k_{T_h^-}\right). \tag{8}$$

Using the Principle of Optimality, Lemma 1 rewrites the sequential problem in (4) as a recursive stopping-time problem. It also shows that the value of the firm equals a function of the log capital-productivity ratio $\hat{k}$ that scales with productivity, that is, $V(k, u) = uv(\hat{k})$. Since $\Delta\hat{k}_s$ and i_s have the same sign, we write the investment price as $p(\Delta\hat{k})$. All proofs appear in the Appendix.

Lemma 1. *Let $r \equiv \rho - \mu - \sigma^2/2$ be the adjusted discount factor and let $v(\hat{k}) : \mathbb{R} \to \mathbb{R}$ be a function of the log capital-productivity ratio equal to*

$$(9)\quad v(\hat{k}) = \max_{\tau, \Delta\hat{k}} \mathbb{E}\left[\int_0^\tau Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + e^{-r\tau}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}_\tau+\Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\Big|\hat{k}_0 = \hat{k}\right].$$

Then the firm value equals $V(k, u) = uv(\hat{k})$.

2.3 Optimal investment policy

The optimal investment policy is characterized by four numbers, $\mathcal{K} \equiv \{\hat{k}^- \leq \hat{k}^{*-} \leq \hat{k}^{*+} \leq k^+\}$, which correspond to the lower and upper borders of the inaction region

$$(10)\qquad \mathcal{R} = \left\{\hat{k} : \hat{k}^- < \hat{k} < \hat{k}^+\right\},$$

and two reset points $\hat{k}^{*-} < \hat{k}^{*+}$. A firm adjusts if and only if its log capital-productivity ratio falls outside the inaction region, that is, $\hat{k}_s \notin \mathcal{R}$. Conditional on adjusting, the firm purchases capital to bring its state up to $\hat{k}^{*-}$ when it hits the lower border $\hat{k}^-$, and sells capital to bring its state down to $\hat{k}^{*+}$ if it hits the upper border $\hat{k}^+$. Given $\mathcal{R}$, the optimal adjustment dates are

$$(11)\qquad T_h = \inf\left\{s \geq T_{h-1} : \hat{k}_s \notin \mathcal{R}\right\} \quad \text{with} \quad T_0 = 0.$$

The duration of a complete inaction spell τ_h and the time elapsed since the last adjustment a_s (or the age of the capital-productivity ratio), are given by:

$$(12)\qquad \tau_h = T_h - T_{h-1},$$

$$(13)\qquad a_s = s - \max\{T_h : T_h \leq s\}.$$

To save on notation, we write the reset points and the stopped capitals (an instant before adjustment) as functions of the adjustment sign:

$$(14)\qquad \hat{k}^*(\Delta\hat{k}) = \begin{cases} \hat{k}^{*-} & \text{if} \quad \Delta\hat{k} > 0, \\ \hat{k}^{*+} & \text{if} \quad \Delta\hat{k} < 0. \end{cases}$$

$$(15)\qquad \hat{k}_\tau(\Delta\hat{k}) = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k}.$$

Lemma 2 characterizes the value function and the optimal investment policy through the standard sufficient optimality conditions. The firm value and the policy must satisfy: (i) the Hamilton-Jacobi-Bellman equation in (16), which describes the evolution of the firm's value during periods of inaction, (ii) the value-matching conditions in (17) and (18), which set the value of adjusting equal to the value of not adjusting at the borders of the inaction region, and (iii) the smooth-pasting and optimality conditions in (19) and (20), which ensure differentiability at the borders of inaction and the two reset points.

Lemma 2. *The value function $v(\hat{k})$ and the optimal policy $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy:*

(i) In the inaction region $\mathcal{R}$, $v(\hat{k})$ solves the HJB equation:

$$rv(\hat{k}) \;=\; Ae^{\alpha\hat{k}} \;-\; \nu v'(\hat{k}) \;+\; \frac{\sigma^2}{2}v''(\hat{k}). \tag{16}$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v(\hat{k}^-) \;=\; v(\hat{k}^{*-}) \;-\; \theta \;-\; p^{buy}(e^{\hat{k}^{*-}} - e^{\hat{k}^-}), \tag{17}$$

$$v(\hat{k}^+) \;=\; v(\hat{k}^{*+}) \;-\; \theta \;+\; p^{sell}(e^{\hat{k}^+} - e^{\hat{k}^{*+}}). \tag{18}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$v'(\hat{k}) \;=\; p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{19}$$

$$v'(\hat{k}) \;=\; p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{20}$$

The optimal policy in terms of capital is recovered as $\{k^-, k^{*-}, k^{*+}, k^+\} = u \times \exp\{\hat{k}^-, \hat{k}^{*-}, k^{*+}, k^+\}$.

2.4 Individual Tobin's q

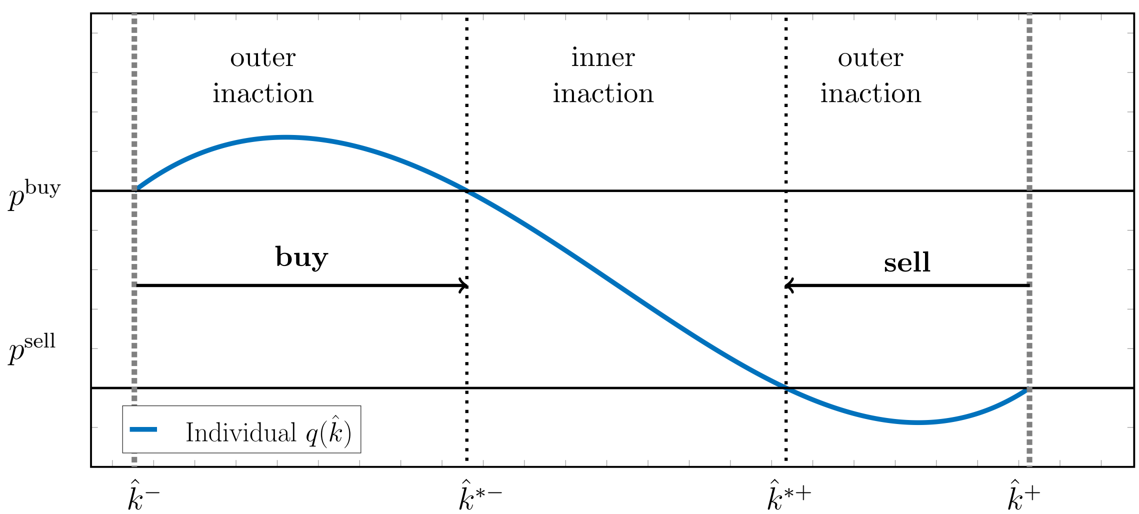
Next, we express the optimal investment decision using Tobin's q, namely, the shadow price of installed capital. Following Abel and Eberly (1994), we identify a firm's Tobin's q as the marginal valuation of an extra unit of installed capital, which is equal to

$$q(\hat{k}) \;\equiv\; \frac{\partial V(k,u)}{\partial k} \;=\; v'(\hat{k})e^{-\hat{k}}. \tag{21}$$

In contrast to its standard definition, Tobin's q in (21) is not divided by the replacement cost of capital (the investment price). The reason is that with partial irreversibility the price depends on the direction of the next adjustment and thus it varies depending on the position of $\hat{k}$.

Figure II describes the optimal investment policy using $q(\hat{k})$. It is useful to examine the role of each investment friction at a time. Without the fixed adjustment cost ($\theta = 0$), a firm purchases capital if $q(\hat{k}) \geq p^{\text{buy}}$ (or $\hat{k} \leq \hat{k}^{*-}$) and sells capital if $q(\hat{k}) \leq p^{\text{sell}}$ (or $\hat{k} \geq \hat{k}^{*+}$) without any delay. When $q(\hat{k})$ lies between the two prices (or the state between the two reset points), it is optimal to remain inactive: At that productivity level, it is too expensive to purchase capital and too cheap to sell it. This gives rise to an "inner" inaction region $[\hat{k}^{*-}, \hat{k}^{*+}]$ due exclusively to partial irreversibility. Without a price wedge ($p^{\text{buy}} = p^{\text{sell}}$), the "inner" inaction region collapses to a unique reset point k^*; however, the fixed adjustment cost generates an "outer" inaction region $[\hat{k}^-, \hat{k}^+]$ that prevents firms from adjusting, even if $q(\hat{k})$ lies above or below the investment price. When both frictions are active, the policy features both the "outer" and the "inner" inaction regions and two reset points.

Figure II – Optimal Investment Policy



Notes: The figure plots the individual $q(\hat{k}) = v'(\hat{k})/e^{\hat{k}}$ and the investment policy $\mathcal{K} = \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$.

The interaction of the investment frictions generates two interesting features in the optimal investment behavior. First, as argued by Caballero and Leahy (1996), individual $q(\hat{k})$ is *not monotonic* in $\hat{k}$. Without fixed costs, $q(\hat{k})$ monotonically decreases with $\hat{k}$ due to decreasing returns to scale $\alpha < 1$. With fixed costs, however, firms anticipate large adjustments when approaching the inaction thresholds. As $\hat{k}$ approaches the lower threshold $\hat{k}^-$, firms anticipate that a future tiny change $\mathrm{d}\hat{k} < 0$ will trigger a large positive investment $\Delta\hat{k} > 0$. The future positive investment lowers future $q(\hat{k})$ and feeds back into lower current $q(\hat{k})$, bending down the function. A reverse argument explains why $q(\hat{k})$ bends up as $\hat{k}$ approaches the upper threshold $\hat{k}^+$. As a result, individual $q(\hat{k})$ is *not a sufficient statistic for individual investment*, in contrast to the postulate in Tobin (1969).

Second, optimal investment features an *endogenous positive serial correlation in the sign of adjustments*. That is, it is more likely to buy capital if it was bought recently than if it was sold; and it more likely to sell capital if it was sold recently than if it was bought. The reason behind this correlation is that the inner inaction region generated by the price wedge widens the distance between the two borders of inaction but shortens the distance between each border of inaction and its corresponding reset point. Therefore, it is more likely to reach $\hat{k}^-$ from the nearby $\hat{k}^{*-}$ than from the further $\hat{k}^{*+}$. The serial correlation of the investment sign generates history dependence, which is technically challenging. However, we show below how to handle history dependence using distributions conditional on the last reset point.

2.5 Economy with a continuum of firms

Consider an economy populated by a continuum of ex ante identical firms that face the investment problem from the previous section. Idiosyncratic shocks W_t are independent across firms. The economy features stationary cross-sectional distributions of capital-productivity ratios and investment. Here we define and discuss at length these distributions, setting the stage for the results in later sections. The analytical characterization of the macroeconomic outcomes and the mappings to the microdata derived in Section 3 hinge completely on how we deal with partial irreversibility when characterizing the cross-sectional behavior of the economy. Specifically, our strategy consists on *conditioning on the last reset point* and using relative frequencies of upward and downward adjustment to back out the unconditional behavior.

Distribution of firms. Let $G(\hat{k})$ be the distribution of firms over their log capital-productivity ratio and let $g(\hat{k})$ be its continuous marginal density. Also, let $\mathcal{N}^-$, $\mathcal{N}^+$, and $\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+$ be the frequencies of positive, negative, and non-zero adjustments in the *total population*, which are equal to the mass of firms that adjust to $\hat{k}^{*-}$, to $\hat{k}^{*+}$, or to either point.[4] The density and frequencies solve the following system, which includes: a Kolmogorov forward equation that describes the evolution of capital-productivity ratios inside the inaction region (excluding the two reset points)

$$\nu g'(\hat{k}) \; + \; \frac{\sigma^2}{2} g''(\hat{k}) \; = \; 0, \quad \text{for all} \quad \hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \{\hat{k}^{*-}, \hat{k}^{*+}\}; \tag{22}$$

three border conditions

$$g(\hat{k}) \; = \; 0, \quad \text{for} \;\; \hat{k} \in \{\hat{k}^{*-}, \hat{k}^{*+}\}, \tag{23}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k}) \, \mathrm{d}\hat{k} \; = \; 1; \tag{24}$$

[4]To avoid any confusion with our notation, we emphasize that the sign in the exponent of an object refers to the last reset point, not to the sign of the adjustment.

two resetting conditions

$$\underbrace{\frac{\sigma^2}{2}\lim_{\hat{k}\downarrow\hat{k}^-} g'(\hat{k})}_{\mathcal{N}^-} = \frac{\sigma^2}{2}\left[\lim_{\hat{k}\uparrow\hat{k}^{*-}} g'(\hat{k}) - \lim_{\hat{k}\downarrow\hat{k}^{*-}} g'(\hat{k})\right], \tag{25}$$

$$\underbrace{-\frac{\sigma^2}{2}\lim_{\hat{k}\uparrow\hat{k}^+} g'(\hat{k})}_{\mathcal{N}^+} = \frac{\sigma^2}{2}\left[\lim_{\hat{k}\uparrow\hat{k}^{*+}} g'(\hat{k}) - \lim_{\hat{k}\uparrow\hat{k}^{*+}} g'(\hat{k})\right], \tag{26}$$

and two continuity conditions at the reset points (not reported). Condition (23) sets the mass of firms at the inaction thresholds equal to zero. Condition (24) ensures that g is a density. Conditions (25) and (26) relate the masses of upward and downward adjustments to the discontinuities in the derivative of g at the reset points. In a small period of time ds, the mass $\mathcal{N}^-$ that "exits" the inaction region by hitting the lower threshold—equal to $\frac{\sigma^2}{2}\lim_{\hat{k}\downarrow\hat{k}^-} g'(\hat{k})$—must coincide with the mass of firms that "enters" at the reset point $\hat{k}^{*-}$—equal to the jump in g'. This argument is analogous for $\mathcal{N}^+$; in fact, it is straightforward to verify that conditions (22) to (25) jointly imply condition (26), and thus it is redundant.

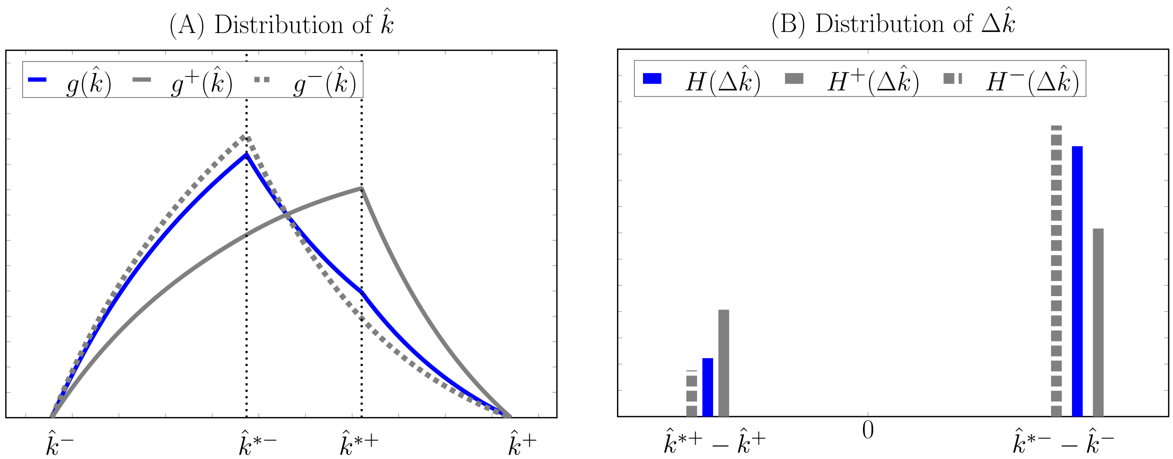
As we anticipated, to handle the history dependence that manifests in the autocorrelation of the investment sign, we define densities *conditional on the last reset point*. We let $g^-(\hat{k})$ and $g^+(\hat{k})$ denote the stationary density of $\hat{k}$ conditional on the last reset point being $\hat{k}^{*-}$ or $\hat{k}^{*+}$, respectively. In particular, g^- satisfies the same KFE in (22) for all $\hat{k} \in (\hat{k}^-, \hat{k}^+) \setminus \hat{k}^{*-}$ (there is no entry at $\hat{k}^{*+}$), the border conditions (23) and (24), and continuity at $\hat{k}^{*-}$.[5] An analogous characterization applies to g^+. Panel A in Figure III plots the three densities g, g^- and g^+ (these are proper densities and integrate to 1). We denote expectations computed with these distributions as $\mathbb{E}$, $\mathbb{E}^-$, and $\mathbb{E}^+$.

Distributions of actions. Next, we consider the distribution over actions—adjustment size and the duration of inaction—denoted by $H(\Delta\hat{k}, \tau)$, and the distributions of actions *conditional on the last reset point*: $H^-(\Delta\hat{k}, \tau)$ and $H^+(\Delta\hat{k}, \tau)$. Panel B of Figure III plots the marginal distributions of adjustment size, $H(\Delta\hat{k})$, $H^-(\Delta\hat{k})$, $H^+(\Delta\hat{k})$, where we have integrated out the duration τ; these distributions correspond to probability masses at two points $\Delta\hat{k} = \hat{k}^{*+} - \hat{k}^+ < 0$ and $\Delta\hat{k} = \hat{k}^{*-} - \hat{k}^- > 0$. We denote with bars the expectations computed with the distributions of adjusters: $\overline{\mathbb{E}}$, $\overline{\mathbb{E}}^-$ and $\overline{\mathbb{E}}^+$.

Two observations are in place. First, the mass of upward adjustments $H(\hat{k}^{*-} - \hat{k}^-)$ is larger than the mass of downward adjustments $H(\hat{k}^{*+} - \hat{k}^+)$; this is because the drift shrinks capital-productivity ratios over time prompting upward adjustments, and because partial irreversibility penalizes downward adjustments. This asymmetry is also observed in the firms' distribution, as g is closer to g^-. Second, the conditional masses reflect the autocorrelation in the investment sign;

[5] Besides the border conditions, there is one reseting condition relating the mass of adjusters to the unique discontinuity in the derivative of g^-, but it is implied by the border conditions.

Figure III – Unconditional and Conditional Distributions of $\hat{k}$ and $\Delta\hat{k}$



Notes: Panel A plots the unconditional density $g(\hat{k})$ and the densities conditional on the last reset $g^{\pm}(\hat{k})$. Panel B plots the unconditional distribution $H(\Delta\hat{k})$ and the distributions conditional on the last reset $H^{\pm}(\Delta\hat{k})$.

for instance, $H^{-} > H^{+}$ at $\Delta\hat{k} > 0$ means that the probability of reseting to $\hat{k}^{*-}$ is larger whenever the last reset point was also $\hat{k}^{*-}$. In other words, positive investments beget positive investments.

From conditional to unconditional distributions. Define the shares of upward $\mathcal{N}^{-}/\mathcal{N}$ and downward $\mathcal{N}^{+}/\mathcal{N}$ adjustments within the *population of adjusters*. By Bayes' law, the unconditional and conditional distribution of adjusters satisfy

$$H(\Delta\hat{k},\tau) \;=\; \frac{\mathcal{N}^{-}}{\mathcal{N}} H^{-}(\Delta\hat{k},\tau) \;+\; \frac{\mathcal{N}^{+}}{\mathcal{N}} H^{+}(\Delta\hat{k},\tau). \tag{27}$$

This relationship is useful to compute moments of adjusters. For example, the average duration of inaction equals the weighted sum of the average conditional durations:

$$\overline{\mathbb{E}}[\tau] \;=\; \overline{\mathbb{E}}\left[\mathbb{E}[\tau|\Delta k]\right] \;=\; \frac{\mathcal{N}^{-}}{\mathcal{N}}\overline{\mathbb{E}}^{-}[\tau] \;+\; \frac{\mathcal{N}^{+}}{\mathcal{N}}\overline{\mathbb{E}}^{+}[\tau]. \tag{28}$$

This approach, however, is incorrect when recovering the unconditional distribution of firms. In that case, the shares must be rescaled by the relative durations of inaction:

$$g(\hat{k}) \;=\; \frac{\mathcal{N}^{-}}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{-}[\tau]}{\overline{\mathbb{E}}[\tau]}\; g^{-}(\hat{k}) \;+\; \frac{\mathcal{N}^{+}}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{+}[\tau]}{\overline{\mathbb{E}}[\tau]}\; g^{+}(\hat{k}) \;=\; \mathcal{N}^{-}\;\overline{\mathbb{E}}^{-}[\tau]\; g^{-}(\hat{k}) \;+\; \mathcal{N}^{+}\;\overline{\mathbb{E}}^{+}[\tau]\; g^{+}(\hat{k}), \tag{29}$$

where we simplify the expression using $\overline{\mathbb{E}}[\tau] = \mathcal{N}^{-1}$, that is, the average duration of inaction equals the inverse of the total frequency of adjusters. This implies that the duration-adjusted frequencies also sum up to one, i.e., $\mathcal{N}^{-}\;\overline{\mathbb{E}}^{-}[\tau] + \mathcal{N}^{+}\;\overline{\mathbb{E}}^{+}[\tau] = 1$. Why do we need to rescale by duration? The answer is the *fundamental renewal property:* The average behavior in the economy is attributable to firms with longer periods of inaction (who are observed less frequently). Adjusting the shares with their relative duration corrects this observational bias. In the context of partial irreversibility, the slowly-adjusting firms are coincidentally those that make downward adjustments.[6]

Illustrative example. Consider an economy in which most firms do frequent upward adjustments. The durations of inaction are $\overline{\mathbb{E}}[\tau] = 2$, $\overline{\mathbb{E}}^-[\tau] = 1.5$, and $\overline{\mathbb{E}}^+[\tau] = 4$, and the frequencies are $\mathcal{N} = 0.5$, $\mathcal{N}^- = 0.4$, and $\mathcal{N}^+ = 0.1$ (note that $\overline{\mathbb{E}}[\tau] = 1/\mathcal{N}$ but $\overline{\mathbb{E}}^{\pm}[\tau] \neq 1/\mathcal{N}^{\pm}$). The shares of upward and downward adjustments are $\mathcal{N}^-/\mathcal{N} = 0.8$ and $\mathcal{N}^+/\mathcal{N} = 0.2$, and the relative durations are $\overline{\mathbb{E}}^-[\tau]/\overline{\mathbb{E}}[\tau] = 0.75$ and $\overline{\mathbb{E}}^+[\tau]/\overline{\mathbb{E}}[\tau] = 2$. While only 20% of adjustments are downward, they happen after inactions spells with twice the average duration, implying that the underlying states $\hat{k}$ generating those adjustments are occupied for longer periods of time. To account for this higher occupancy, the implied duration-modified frequencies, $\mathcal{N}^- \ \overline{\mathbb{E}}^-[\tau] = 0.6$ and $\mathcal{N}^+ \ \overline{\mathbb{E}}^+[\tau] = 0.4$, are the appropriate weights to recover the unconditional distribution of firms as $g = 0.6\ g^- + 0.4\ g^+$.

3 Three macroeconomic outcomes

How do investment frictions shape aggregate capital's allocation, valuation, and dynamics? This section defines these macroeconomic outcomes and characterizes them in terms of moments of the cross-sectional distribution $g(\hat{k})$ and observable statistics.[7]

3.1 Capital allocation

Following the development literature, we define *capital misallocation* as the cross-sectional variance of the log marginal revenue product of capital. In our model, all firms produce the same good and the output price is normalized to one; thus we measure instead the variance of marginal products. From the production function (1), the log of the marginal product of capital is collinear to a firm's capital-productivity ratio $\hat{k}$, that is, $\log mpk_s = \log \alpha - (1-\alpha)\hat{k}_s$. Therefore, misallocation is proportional to $\mathbb{V}ar[\hat{k}]$:

$$\mathbb{V}ar[\log mpk] \ = \ (1-\alpha)^2 \mathbb{V}ar[\hat{k}]. \tag{30}$$

In a frictionless environment, $\hat{k}_s$ is constant and $\mathbb{V}ar[\log mpk] = 0$. With frictions, however, dispersion in the marginal product of capital arises as in Asker, Collard-Wexler and De Loecker

[6]See Baley and Blanco (2021) for an application of the fundamental renewal property in the context of asymmetric fixed adjustment costs.

[7]We specialize investment frictions to a symmetric adjustment cost θ paid indistinctly for positive and negative investments, and a price wedge that gives rise to partial irreversibility. We abstract from other frictions to keep the presentation simple. However, we prove all the results for the generalized hazard model proposed by Caballero and Engel (1999, 2007) and examined in contemporaneous work by Álvarez, Lippi and Oskolkov (2020), which may accomodate other empirically-relevant frictions, such as random fixed costs and random opportunities for free adjustment, as in Baley and Blanco (2021).

(2014). Given the collinear relationship established in (30), we will use the term misallocation when referring to $\mathbb{V}ar[\hat{k}]$.

Measuring misallocation with microdata. The challenge in measuring misallocation stems from the fact that the distribution $g(\hat{k})$ is not observed. As economists, however, we have available detailed panel data $\Omega = \{\Delta\hat{k}, \tau\}$ with information on the actions of adjusters: the size of discrete adjustments $\Delta\hat{k}$ in (8) and the duration of completed inaction spells τ in (12). We present mappings that use micro investment data Ω to recover misallocation. We proceed in two steps. Proposition 1 recovers the parameters of the stochastic process and the two reset points through a system of equations that involve several moments from the distribution of adjusters. Then given the reset points, Proposition 2 recovers the population mean $\mathbb{E}[\hat{k}]$ and variance $\mathbb{V}ar[\hat{k}]$ of capital-productivity ratios.

Proposition 1. *Let* $\Phi(\nu, \sigma^2) \equiv \log\left(\alpha A/(r + \alpha\nu - \alpha^2\sigma^2/2)\right)$. *The parameters of the stochastic process for productivity* (ν, σ^2) *and the reset points* $(\hat{k}^{*-}, \hat{k}^{*+})$ *are recovered from the microdata* $\Omega \equiv (\Delta\hat{k}, \tau)$ *through the following system:*

$$\nu = \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}, \tag{31}$$

$$\sigma^2 = \frac{\overline{\mathbb{E}}[(\hat{k}_\tau + \nu\tau)^2] - \overline{\mathbb{E}}[(\hat{k}^*)^2]}{\overline{\mathbb{E}}[\tau]}, \tag{32}$$

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left[\Phi(\nu,\sigma^2) - \log(p^{buy}) + \log\left(\frac{1 - \overline{\mathbb{E}}^{-}\left[e^{-\hat{r}\tau + \alpha\left(\hat{k}_\tau - \hat{k}^{*+}\right)}\right]}{1 - \overline{\mathbb{E}}^{-}\left[\frac{p(\Delta\hat{k})}{p^{buy}} e^{-\hat{r}\tau + \hat{k}_\tau - \hat{k}^{*+}}\right]}\right)\right], \tag{33}$$

$$\hat{k}^{*+} = \frac{1}{1-\alpha}\left[\Phi(\nu,\sigma^2) - \log(p^{sell}) + \log\left(\frac{1 - \overline{\mathbb{E}}^{+}\left[e^{-\hat{r}\tau + \alpha\left(\hat{k}_\tau - \hat{k}^{*-}\right)}\right]}{1 - \overline{\mathbb{E}}^{+}\left[\frac{p(\Delta\hat{k})}{p^{sell}} e^{-\hat{r}\tau + \hat{k}_\tau - \hat{k}^{*-}}\right]}\right)\right]. \tag{34}$$

Expression (31) recovers the drift from the average adjustment size times the frequency of adjustment (the inverse of the expected duration of inaction $\mathcal{N} = \overline{\mathbb{E}}[\tau]^{-1}$); while expression (32) recovers the volatility from the variance in adjustment size.[8] Expressions (33) and (34) recover the reset points. The first term $\Phi(\nu, \sigma^2)$ reflects the ratio of marginal product to the user cost of capital. Through this ratio, both reset states increase with profitability A and idiosyncratic risk σ^2 and decrease with the discount r and the drift ν. The second term shows that reset points decrease with the corresponding investment price: firms invest more the lower is the purchasing price p^{buy} and disinvest less the lower is the selling price p^{sell}. Lastly, the third term shows how investment frictions affect the reset points through their effect on the marginal profits accrued during periods of inaction (in the numerator) and on the resale value (in the denominator).

[8]We obtained similar mappings from the data to the parameters in Baley and Blanco (2021) for the case without irreversibility. Irreversibility does not change the mapping to the drift, but it changes the mapping to the volatility.

With the reset points and parameters at hand, we proceed to recover the unconditional mean $\mathbb{E}[\hat{k}]$ and variance $\mathbb{V}ar[\hat{k}]$ of capital-productivity ratios $\hat{k}$.

Proposition 2. *Let $\Omega \equiv (\Delta\hat{k}, \tau)$ be a panel of observations. For each inaction spell find the departing point $\hat{k}^*$ and the ending point $\hat{k}_\tau$ using (14) and (15). Then the unconditional mean and variance of $\hat{k}$ are recovered from the microdata as follows:*

$$\mathbb{E}[\hat{k}] = \overline{\mathbb{E}}\left[\mathbb{E}\left[\left(\frac{\hat{k}^* + \hat{k}_\tau}{2}\right)\left(\frac{\hat{k}^* - \hat{k}_\tau}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\middle|\Delta\hat{k}\right]\right] + \frac{\sigma^2}{2\nu}, \tag{35}$$

$$\mathbb{V}ar[\hat{k}] = \overline{\mathbb{E}}\left[\mathbb{E}\left[\left((\hat{k}^* - \mathbb{E}[\hat{k}])(\hat{k}_\tau - \mathbb{E}[\hat{k}]) + \frac{(\hat{k}^* - \hat{k}_\tau)^2}{3}\right)\left(\frac{\hat{k}^* - \hat{k}_\tau}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\middle|\Delta\hat{k}\right]\right]. \tag{36}$$

The mapping in (35) recovers the population mean $\mathbb{E}[\hat{k}]$ from the average midpoint between the departing and the ending points of an inaction spell $(\hat{k}^* + \hat{k}_\tau)/2$, where the average is computed under a change of measure induced by the renewal weights $(\hat{k}^* - \hat{k}_\tau)/\overline{\mathbb{E}}[\Delta\hat{k}]$. To recover the population mean, the renewal measure overweighs the midpoints of adjusters with longer periods of inaction, which are more representative in the population.[9] The term $\sigma^2/2\nu$ corrects for the accumulated drift between adjustments. Similarly, the mapping in (36) recovers the population variance $\mathbb{V}ar[\hat{k}]$ from the average distance between the departing point and the mean $(\hat{k}^* - \mathbb{E}[\hat{k}])$, the ending point and the mean $(\hat{k}_\tau - \mathbb{E}[\hat{k}])$, and the between departing and ending points $(\hat{k}^* - \hat{k}_\tau)^2$, again computed using the renewal distribution. In these expressions, the inner expectation is computed with H^- or H^+ depending on the sign of the last adjustment, and the outer expectation is computed with shares of upward $\mathcal{N}^-/\mathcal{N}$ and downward $\mathcal{N}^+/\mathcal{N}$ adjustment in the population.

Economic forces shaping capital misallocation. Using the law of total variance, we decompose misallocation $\mathbb{V}ar[\hat{k}]$ into two terms that condition on the last adjustment:

$$\underbrace{\mathbb{V}ar[\hat{k}]}_{\text{total}} = \underbrace{\mathbb{E}\left[\mathbb{V}ar[\hat{k}|\Delta\hat{k}]\right]}_{\text{within}} + \underbrace{\mathbb{V}ar\left[\mathbb{E}[\hat{k}|\Delta\hat{k}]\right]}_{\text{between}}. \tag{37}$$

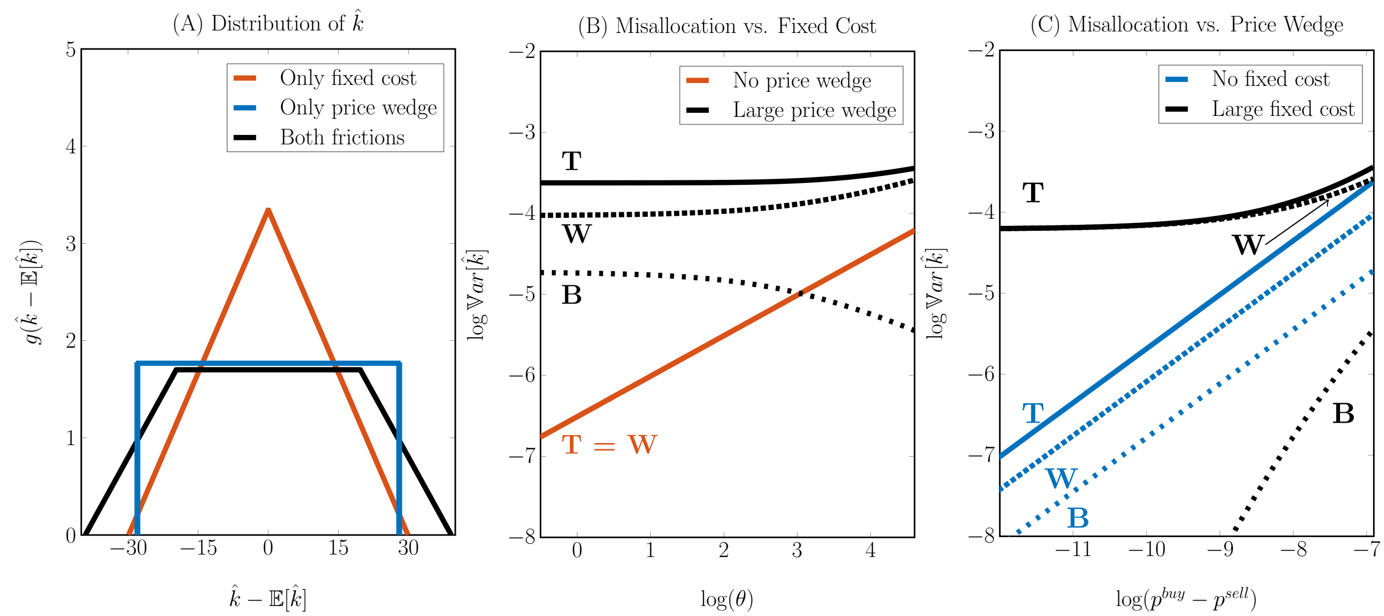
The decomposition in (37) is useful to assess the relative importance of each investment friction in generating capital misallocation. The first term is the average of the variance *within* each conditional distribution g^+ and g^-, that is, the average of $\mathbb{V}ar^-[\hat{k}]$ and $\mathbb{V}ar^+[\hat{k}]$ (computed from 36 conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure as in 28). Both investment frictions add to this dispersion. The second term reflects the distance *between* the conditional means $\mathbb{E}^-[\hat{k}]$ and $\mathbb{E}^+[\hat{k}]$ (computed from 35 conditioning on the sign of $\Delta\hat{k}$ and using the conditional renewal measure). This term arises exclusively from the price wedge that generates two different means. The larger the price wedge, the further apart are the conditional means and

[9] Without the price wedge, the renewal weights are equal to the relative size of adjustment $\Delta k/\overline{\mathbb{E}}[\Delta\hat{k}]$.

the larger the between variance. Note that this term is zero when only fixed costs are present as there is a unique reset point.

Figure IV illustrates the effects of each adjustment friction on the total, within, and between variances of capital-productivity ratios. To sharpen the exposition, we assume zero drift ($\nu = 0$) and a symmetric price wedge ($p^{buy} - p = p - p^{sell}$). Panel A plots the stationary density $g(\hat{k})$. Panels B and C are log-log plots of misallocation against one friction, setting the other friction either at zero or at a large value. We mark within (W), between (B), and total (T) variances.

Figure IV – Misallocation and Investment Frictions



Notes: Panel A plots the steady-state distribution of capital-productivity ratios, normalized by their mean. Triangle = only fixed costs; Rectangle = only price wedge; Parallelogram = both frictions. Panel B plots misallocation against the fixed cost for a zero (orange) and a large (black) price wedge. Panel C plots misallocation against the price wedge for a zero (blue) and a large (black) fixed cost. Variances: total (T), within (W) and between (B).

Consider only the fixed cost. The density is a triangle that concentrates at the unique reset point and decreases linearly toward the boundaries of the inaction region. A higher fixed cost widens the inaction region and increases misallocation in a log-log linear way (Panel B). Now consider only the price wedge. The density is a rectangle with uniform mass between the two reset/inaction points. In this case, a higher price wedge increases all components of misallocation (within and between) in a log-log linear way (Panel C).

With both frictions active, the density is a parallelogram. The relationship between misallocation and frictions is now flattened in the following sense. Consider the case with fixed costs and a large price wedge. Misallocation is at a higher level but the relationship between misallocation and the fixed cost flattens out. A higher fixed cost still widens the distance between inaction thresholds, increasing the within variance, but simultaneously reduces the distance between the reset points, decreasing the between variance. These opposing forces compensate each other cancelling the effects on misallocation (see the dotted and dashed black lines in Panel B which move in opposite directions). Next consider the case with partial irreversibility and a large fixed cost. Again, misallocation is at a higher level and the relationship between misallocation and the price wedge flattens out. The between variance disappears (its log becomes very negative).

The previous analysis teaches us that the relative size of frictions affects the response of misallocation to an increase of these frictions. Later in the paper, we come back to these effects to examine the effects of corporate taxation and its interaction with investment frictions.

3.2 Capital valuation

We define capital valuation as the weighted average of individual $q(\hat{k})$ in (21) with weights $\omega(\hat{k}) \equiv e^{\hat{k}}/\hat{K}$, divided by the average investment price $p \equiv \overline{\mathbb{E}}[p(\Delta\hat{k})]$:

$$q \;\equiv\; \frac{1}{p}\int_{\hat{k}^-}^{\hat{k}^+} q(\hat{k})\omega(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} \;=\; \frac{\mathbb{E}[v'(\hat{k})]}{p\hat{K}}. \tag{38}$$

In contrast to the individual $q(\hat{k})$, the definition of the aggregate q divides by the investment price. Aggregate q is a measure of the average propensity to invest. Without frictions, there is a unique investment price, and optimality implies that $q = 1$ always. That is, all investment or disinvestment opportunities are immediately implemented, eliminating any possibility for q to deviate from 1. With frictions, q may differ from one. If $q > 1$, the average marginal valuation of capital is larger inside the firms than outside them, and the average propensity to invest is positive.

Characterization of aggregate q. We proceed to characterize the aggregate q in terms of moments of $\hat{k}$. But first, we define an auxiliary pricing function that extends the investment price $p(\Delta\hat{k})$ to the inner action region $[\hat{k}^{*+}, \hat{k}^{*-}]$. Effectively, a firm will never face such a price, but it facilitates the computation of expected capital gains.

Let $\mathcal{P}(\hat{k}) \in \mathbb{C}^2$ be a twice continuously differentiable function in the domain $[\hat{k}^+, \hat{k}^-]$ such that: (i) to the left of the inner inaction region, it equals $p^{buy}/p - 1$; (ii) to the right of the inner action region, it equals $p^{sell}/p - 1$; and (iii) inside the inner inaction region, $\mathcal{P}(\hat{k})$ sets a price that a firm would face under a suboptimal purchase or sale of capital, which is determined by the differentiability requirement:[10]

$$\mathcal{P}(\hat{k}) \;\equiv\; \begin{cases} p^{\text{buy}}/p - 1 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}], \\ p^{\text{sell}}/p - 1 & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+]. \end{cases} \tag{39}$$

Since $\mathcal{P}(\hat{k})$ equals the deviations from the average price, it averages zero: $\overline{\mathbb{E}}[\mathcal{P}(\hat{k}^*)] = 0$.

[10]The exact definition of $\mathcal{P}(\hat{k})$ in the inner inaction region does not matter as firms never adjust in that region.

With the definition of the auxiliary price-deviation function, Proposition 3 expresses the aggregate q in terms of cross-sectional moments and parameters. The proof combines the HJB equation for $v'(\hat{k})$ in (16), which specifies firms' optimal behavior, with the KFE for $g(\hat{k})$ in (22), which describes the evolution of firms through the cross-sectional distribution, into a single "master equation". Then we integrate the master equation to eliminate idiosyncratic noise and recover aggregate variables.

Proposition 3. *Aggregate q equals:*

$$q \;=\; \frac{1}{r}\left(\underbrace{\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \left(\frac{\sigma^2}{2}-\nu\right)}_{\textit{productivity}} \;+\; \underbrace{\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\omega(\hat{k}_s))\right]\right]}_{\textit{irreversibility}}\right), \tag{40}$$

where aggregate productivity $\hat{Y}/\hat{K}$ is equal, up to second order, to

$$\frac{\hat{Y}}{\hat{K}} \;=\; \frac{\mathbb{E}[e^{\alpha\hat{k}}]}{\mathbb{E}[e^{\hat{k}}]} \;=\; \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}]+\frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \;+\; o(\hat{k}^3), \tag{41}$$

and the irreversibility term is negative, and up to first order, it is equal to

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{P}(\hat{k}_s)\omega(\hat{k}_s))\right]\right] \;\approx\; -\frac{\overline{\mathbb{C}ov}\left[\Delta\hat{k},\mathcal{P}(\Delta\hat{k})\right]}{\overline{\mathbb{E}}[\tau]} < 0. \tag{42}$$

Economic forces shaping capital valuation. Aggregate q in (40) equals the perpetuity value $(1/r)$ of three terms. The first term is aggregate productivity $\hat{Y}/\hat{K}$ equal to the average output-productivity ratio divided by the average capital-productivity ratio[11]. Observe that q increases with aggregate productivity; in turn, because of decreasing returns to scale $\alpha < 1$, aggregate productivity decreases with the average $\mathbb{E}[\hat{k}]$ and the dispersion $\mathbb{V}ar[\hat{k}]$ of capital-productivity ratios (see equation 41). As a consequence, aggregate q also decreases with the level and the dispersion of $\hat{k}$. Both the fixed cost and the price wedge affect q indirectly through this channel.

The second term reflects the expected change in the average capital-productivity ratio, which takes into account the deterministic trend ν and the risk σ^2. Since firms can upsize to exploit good outcomes and can downsize to insure against bad outcomes, they are effectively risk loving (Oi, 1961; Hartman, 1972; Abel, 1983). Thus higher idiosyncratic risk σ^2 directly pushes q up. At the same time, higher idiosyncratic risk has an indirect effect by increasing misallocation and thus lowering the aggregate output-capital ratio in (41). The overall effect of risk on q depends on the relative strength of these two opposing forces.

Besides the indirect effect that irreversibility plays through misallocation, it also has a direct effect on q measured by the last term. The irreversibility term equals the expected price deviations

[11] Aggregate productivity differs from the average output-capital ratio $\mathbb{E}[y/k] = \mathbb{E}[e^{(\alpha-1)\hat{k}}]$ due to heterogeneity.

from the average price weighted by the capital stock. Expression (42) maps it into minus the covariance of investment $\Delta\hat{k}$ and price deviations $\mathcal{P}(\Delta\hat{k})$. This covariance is positive: firms purchase capital at a price above the average, and sell capital at a price below the average. Since the covariance is positive, irreversibility reduces q. Intuitively, firms anticipate histories in which, after upsizing, negative productivity shocks force them to downsize and face the penalty of selling cheap, together with histories in which, after downsizing, positive productivity shocks force them to upsize and face the penalty of purchasing back capital at a higher price. To minimize the likelihood of these "switching" situations, firms under-invest and under-disinvest, effectively reducing capital valuation.

Individual vs. aggregate q. In Section 2.4 we showed that individual $q(\hat{k})$ is a non-monotonic function of $\hat{k}$. This observation has led some economists to argue that the individual non-monotonicity translates into aggregate non-monotonicity, discarding q as a sufficient statistic for aggregate investment; see Philippon (2009) for a survey. Expression (40) shows that this argument is flawed: fixed adjustment costs and partial irreversibility do not break the decreasing relation between aggregate q and aggregate $\hat{K}$. While this result appears to be counterintuitive, it is a natural consequence of aggregating individual units' behavior. The anticipatory effects that bend individual $q(\hat{k})$ in the vicinity of the borders of the inaction region disappear when aggregating the cross-section, as positive and negative stances of expected changes in $q(\hat{k})$ cancel each other out in the aggregate. As a result, aggregate q *is a sufficient statistic for aggregate investment.*[12]

3.3 Capital dynamics

Following Baley and Blanco (2021), our notion of capital dynamics is *the transitional dynamics of aggregate capital following an aggregate productivity shock.* Starting from the steady state, we introduce a small, permanent, and unanticipated decrease in the (log) level of productivity of size $\delta > 0$ to all firms. We normalize the arrival date of the aggregate shock to $s = 0$, so all firms' productivity and capital-productivity ratios change to

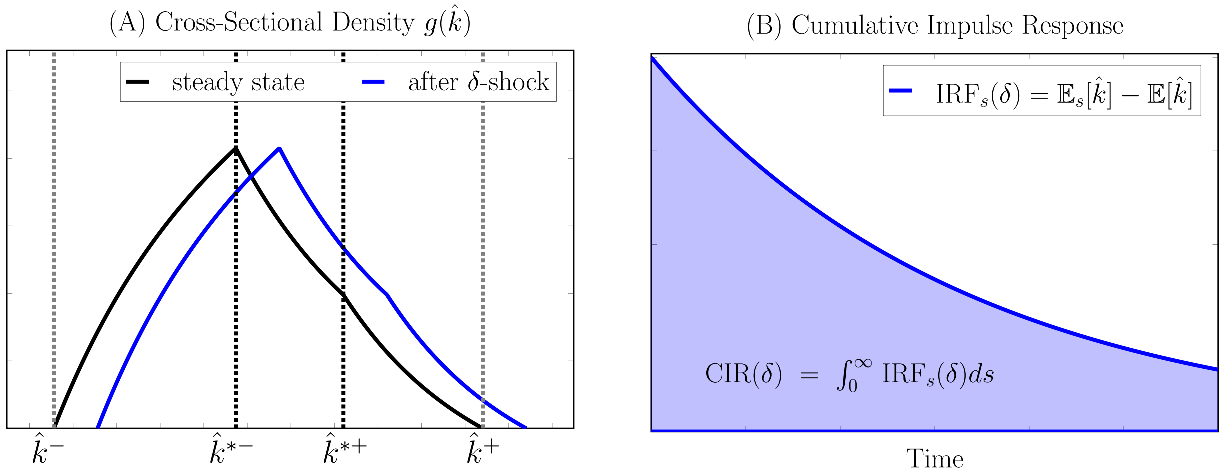
$$\log(u_0) = \log(u_{0^-}) - \delta; \qquad \log(\hat{k}_0) = \log(\hat{k}_{0^-}) + \delta. \tag{43}$$

Panel A of Figure V plots the initial density following the δ productivity shock (black line) next to the steady-state density $g(\hat{k})$ (blue line). The new distribution displaces horizontally to the right relative to the steady-state distribution. Our exercise consists in tracking the mean $\mathbb{E}_s[\hat{k}]$ as it makes its way back to its steady-state value $\mathbb{E}[\hat{k}]$. By assuming a constant interest rate, investment policies do not respond to changes in the distribution and they are fixed along the

[12] A similar aggregation result appears in Miao and Wang (2014) for the case with constant returns to scale and *iid* productivity shocks. Here we show that aggregation also applies with decreasing returns to scale and random walk productivity shocks.

transition path. Therefore, our analysis measures the strength of the partial equilibrium response to aggregate shocks.[13]

Figure V – Distribution Dynamics and Cumulative Impulse Response



Notes: Panel A shows the steady-state distribution $g(\hat{k})$ (blue line) and the initial distribution following a productivity shock (blue line). Panel B shows the $\mathrm{IRF}(\delta, s)$ (solid blue line) and the CIR (area).

We define the impulse-response function, denoted by $\mathrm{IRF}(\delta, s)$, measured s periods after an aggregate productivity shock of size δ as follows:

$$\mathrm{IRF}(\delta, s) \equiv \mathbb{E}_s[\hat{k}] - \mathbb{E}[\hat{k}], \tag{44}$$

where $\mathbb{E}_s[\cdot]$ denotes expectations with the time-s distribution. We define the cumulative impulse response $\mathrm{CIR}(\delta)$, as the area under the $\mathrm{IRF}_s(\delta)$ function across all dates $s \in (0, \infty)$

$$\mathrm{CIR}(\delta) \equiv \int_0^\infty \mathrm{IRF}_s(\delta) \, \mathrm{d}s. \tag{45}$$

Panel B in Figure V plots these two objects. The solid line is the impulse-response function $\mathrm{IRF}(\delta, s)$, and the area underneath it is cumulative impulse response function, $\mathrm{CIR}(\delta)$. The CIR is a useful metric: It summarizes both the impact and the persistence of the response in one scalar, eases the comparison across different models, and represents a "multiplier" of aggregate shocks. It is illustrative to compare the CIR with and without adjustment frictions. Without frictions, firms respond instantly to the aggregate shock and the CIR is zero. With frictions, the larger the CIR the longer it takes firms to respond to the aggregate shock and the slower the transitional dynamics.

[13]While assuming a constant interest rate (and investment policies) along the transition is an extreme assumption, Winberry (2021) shows that the interest rate response to aggregate productivity shocks is small and even countercyclical.

Characterization of the CIR. Next, we expresses the CIR as a function of cross-sectional moments of $\hat{k}$. The strategy is analogous to the one used for aggregate q, as we define an auxiliary function that allow us to characterize the role of irreversibility. As a first step, we define two values $\mathcal{M}^{buy} < 0 < \mathcal{M}^{sell}$ that measure the expected cumulative deviation of the capital-productivity ratio relative to the mean $\mathbb{E}[\hat{k}]$ conditional on the last adjustment:

$$\mathcal{M}^{buy} \equiv (\overline{\mathbb{E}}^{-}[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^{-}[\tau] \frac{\mathbb{E}[\mathbb{P}^{+}]}{\mathbb{P}^{-+}} < 0, \tag{46}$$

$$\mathcal{M}^{sell} \equiv (\overline{\mathbb{E}}^{+}[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^{+}[\tau] \frac{\mathbb{E}[\mathbb{P}^{-}]}{\mathbb{P}^{+-}} > 0. \tag{47}$$

Let us explain in detail $\mathcal{M}^{buy}$ in (46). Upsizing firms reset their capital-productivity ratio below the unconditional mean and, on average, remain below the mean for the duration of their inaction spell. The average deviation accumulated during one inaction spell is then $(\overline{\mathbb{E}}^{-}[\hat{k}] - \mathbb{E}[\hat{k}])\overline{\mathbb{E}}^{-}[\tau]$. Because the investment sign is serially correlated, upsizing firms remain in an upsizing phase contributing to negative deviations for several periods; they would only leave this phase after a series of negative shocks makes them downsize. In that case, they would enter a persistent downsizing phase with deviations above average $\mathcal{M}^{sell}$ in (47). The ratio $\mathbb{E}[\mathbb{P}^{+}]/\mathbb{P}^{-+}$ exactly reflects the average time spent in the transient upsizing phase, where $\mathbb{E}[\mathbb{P}^{+}] = \overline{\mathbb{E}}[\tau \mathbb{I}(\Delta\hat{k} < 0)]/\overline{\mathbb{E}}[\tau]$ is the unconditional probability of downsizing and $\mathbb{P}^{-+} = \Pr[\Delta\hat{k}' < 0 | \Delta\hat{k} > 0]$ is the probability of downsizing conditional on being currently in an upsizing phase.

As a second step, we define an auxiliary function $\mathcal{M}(\hat{k}) \in \mathbb{C}^2$ that extends the cumulative deviations to the inner inaction region. This function is identical to $\mathcal{P}(\hat{k})$ in (39), but replacing the price deviations with the quantities $\mathcal{M}^{buy}$ and $\mathcal{M}^{sell}$:

$$\mathcal{M}(\hat{k}) = \begin{cases} \mathcal{M}^{buy} & \text{if } \hat{k} \in [\hat{k}^{-}, \hat{k}^{*-}] \\ \mathcal{M}^{sell} & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^{+}]. \end{cases} \tag{48}$$

Note that $\overline{\mathbb{E}}[\mathcal{M}(\hat{k}^{*})] = 0$ because $\mathcal{M}(\hat{k})$ equals the cumulative capital deviations from the steady-state. Using the auxiliary capital-deviation function $\mathcal{M}(\hat{k})$, Proposition 4 characterizes the CIR.

Proposition 4. *The CIR of the average log capital-productivity ratio $\mathbb{E}[\hat{k}]$ following a marginal aggregate productivity shock of size $\delta > 0$ is equal, up to first order, to*

$$\frac{CIR(\delta)}{\delta} = \underbrace{\frac{\mathbb{V}ar[\hat{k}]}{\sigma^2}}_{variance} + \underbrace{\frac{\nu \mathbb{C}ov[\hat{k}, a]}{\sigma^2}}_{covariance} + \underbrace{\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)]\right]}_{irreversibility} + o(\delta), \tag{49}$$

where the variance is recovered from the microdata in (36), the covariance is recovered as

$$\mathbb{C}ov[\hat{k}, a] \;=\; \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] \;-\; \frac{\overline{\mathbb{E}}[(\hat{k}_\tau - \mathbb{E}[\hat{k}])^2\tau]}{\overline{\mathbb{E}}[\tau]} \;+\; \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}[\tau]}{2}(1+\overline{\mathbb{CV}}^2[\tau])\right), \tag{50}$$

and the irreversibility term is equal to

$$\mathbb{E}\left[\frac{1}{\mathrm{d}s}\mathbb{E}_s\left[\mathrm{d}(\mathcal{M}(\hat{k}_s)\hat{k}_s)\right]\right] \;=\; -\frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\Delta\hat{k})]}{\overline{\mathbb{E}}[\tau]} > 0. \tag{51}$$

Economic forces shaping capital dynamics. According to (49), the CIR equals a linear combination of two steady-state moments and an irreversibility term. The moments are the cross-sectional variance of capital-productivity ratios, $\mathbb{V}ar[\hat{k}]$, and the covariance of capital-productivity ratios $\hat{k}$ with the time elapsed since the last adjustment, $\mathbb{C}ov[\hat{k}, a]$. These steady-state moments are informative about transitional dynamics because aggregate shocks δ and idiosyncratic shocks u enter symmetrically into $\hat{k}$, and as a consequence, how firms respond to idiosyncratic shocks inform how they respond to aggregate shocks. Specifically, the variance $\mathbb{V}ar[\hat{k}]$ reflects insensitivity to idiosyncratic shock, while the covariance $\mathbb{C}ov[\hat{k}, a]$ reflects asymmetric costs of downsizing vs. upsizing. Our work in Baley and Blanco (2021) established the relationship between the CIR and these two steady-state moments in environments with drift, asymmetric fixed costs, and random opportunities of free adjustment, but without partial irreversibility.[14] There, the irreversibility term equals zero for marginal aggregate shocks $\delta \approx 0$.[15]

Let us now discuss the role of irreversibility for the CIR, which slows down the propagation of aggregate shocks according to positive sign of (51). The irreversibility term measures the change in cumulative deviations $\mathcal{M}^{sell}$ and $\mathcal{M}^{buy}$ due to the aggregate shock. Concretely, the aggregate shock δ increases the probability of downward adjustments as firms need to downsize to reflect the reduction in aggregate productivity. Due to the serial correlation of investment sign, a persistent downsizing phase activates. Recall that, after downsizing, firms remain above the steady-state average capital-productivity ratio. Thus after an aggregate shock, firms remain persistently above the average, slowing down the convergence of the average $\mathbb{E}_s[\hat{k}]$ to its long-run value $\mathbb{E}[\hat{k}]$. Besides this direct effect, irreversibility has an indirect effect by increasing the cross-sectional moments $\mathbb{V}ar[\hat{k}]$ and $\mathbb{C}ov[\hat{k}, a]$. In the data, this indirect effect dominates. Identifying and characterizing the irreversibility term in the CIR is one of the key contributions of our analysis, as it opens to door to study transitional dynamics in environments with history dependence (reinjection), that is, where the first stopping time does not fully absorb the effects of an aggregate shock.

[14]In Baley and Blanco (2021), we also provide expressions that CIR of higher-order moments $\mathbb{E}[\hat{k}^m]$.
[15]See Alexandrov (2021) for a characterization of the CIR for large shocks.

CIR of functions of $\hat{k}$. Our characterization of average capital dynamics in (49) can be generalized to consider the transitional dynamics of *any continuous function* $f(\hat{k})$ following an aggregate shock δ. In the generalized formula the values $\mathcal{M}^{buy}$ and $\mathcal{M}^{sell}$ behind the function $\mathcal{M}(\hat{k})$ are appropriately redefined to track the deviations of the time-t average value of f, $\mathbb{E}_t[f(\hat{k})]$, from its steady-state value, $\mathbb{E}[f(\hat{k})]$. This generalization could in principle be useful to study fluctuations in capital misallocation as in Ehouarne, Kuehn and Schreindorfer (2016) and Lanteri (2018) setting the function to $f(\hat{k}) = (\hat{k} - \mathbb{E}[\hat{k}])^2$; or dynamics of aggregate marginal valuation (the numerator of aggregate q) setting $f(\hat{k}) = v'(\hat{k})$.

4 The Macroeconomic Effects of Corporate Taxes

This section introduces a comprehensive tax schedule into the firm problem and characterizes analytically the role of corporate taxation in shaping macroeconomic outcomes. We do this in three steps. First, we show that taxes change four parameters: profitability A, the discount factor ρ, the fixed cots θ, and the investment prices $p(\Delta\hat{k})$. Once we redefine these parameters, the investment problem is identical as before. Second, we decompose the firm investment policy into a static component, which reflects the effects of taxation through the user cost of capital, and a dynamic component, which reflects the interaction of taxes and investment frictions. Third, we isolate the various mechanisms at play by considering two benchmark cases: a driftless case where irreversibility has an important role and a large-drift case where irreversibility is innocuous. Finally, using the insights derived from these special cases, we study the general environment.

4.1 A comprehensive tax schedule

Following Auerbach (1979), Summers (1981) and Abel (1982), we introduce a corporate tax system into the firm problem. It includes a corporate income tax t^c, a deduction allowance ξ^d, a personal income tax t^p, and a capital gains tax t^g.[16]

The firm pays the corporate income tax rate t^c on its cash flow Ay_s net of deductions $\xi^d k_s$, where ξ^d denotes the deduction rate. Since the physical and the legal depreciation rates differ, we distinguish deductions from the capital stock and denote these with d_s. The state space now includes deductions $V(k, u, d)$. The corporate income tax and deductions jointly determine the after–tax profit rate

$$\pi_s \; = \; Ay_s - t^c(Ay_s - \xi^d d_s) \; = \; (1 - t^c)Ay_s + t^c \xi^d d_s, \tag{52}$$

[16]This taxation schedule is also used by Poterba and Summers (1983); King and Fullerton (1984); Auerbach (1986); Auerbach and Hines (1986); Hassett and Hubbard (2002).

and the evolution of deductions[17]

$$(53)\qquad \log d_s \;=\; \log d_0 - \xi^d s + \sum_{h:T_h\le s}\left(1+\frac{p(i_{T_h})i_{T_h}+\theta_{T_h}}{d_{T_h^-}}\right).$$

The personal income tax t^p and the capital gain tax t^g alter the firms' discount factor. We assume that equity is purchased by a representative investor with access to a riskless bond with return ρ per unit of time. Let D_s be the dividend per share, P_s the equity price per share, and $E_s = 1$ the number of shares, which we normalize to unity without loss of generality. From the investor's perspective, dividends and bond returns are taxed at the rate t^p, while capital gains arising from changes in equity prices are taxed at the rate t^g. For any dividend process, no-arbitrage implies equal after-tax returns:

$$(54)\qquad (1-t^p)\rho\,\mathrm{d}s \;=\; (1-t^g)\frac{\mathbb{E}[\mathrm{d}P_s]}{P_s} \;+\; (1-t^p)\frac{D_s}{P_s}\,\mathrm{d}s.$$

Condition (54) pins down the time-0 value of the firm, which equals the equity price:

$$(55)\qquad V(k_0,u_0,d_0) \;=\; P_0 \;=\; \frac{1-t^p}{1-t^g}\,\mathbb{E}_0\left[\int_0^\infty e^{-\rho\frac{1-t^p}{1-t^g}s}D_s\,\mathrm{d}s\right].$$

This expression says that the firm maximizes the cum-dividends market value of equity P_0 using the investor's after-tax discount factor $\rho(1-t^p)/(1-t^g)$. We follow the "tax capitalization" view of the dividend decision and consider dividends as residuals, equal to the cash flow π_s net of investment and capital adjustment costs[18]

$$(56)\qquad D_s\,\mathrm{d}s \;=\; \pi_s\,\mathrm{d}s - (\theta_s + p(i_s)i_s)\mathcal{D}(s=T_h), \qquad \mathcal{D}(\cdot)\sim \mathit{Dirac}.$$

Given the tax schedule, Lemma 3 characterizes the problem with corporate taxation. The strategy consists in defining the discounted value of deductions per unit of investment t^d and using it to rewrite the 3-state problem as the 1-state problem with the capital-productivity ratio solved before in Section 2.3, under four parametric changes and an additive term that reflects deductions.

Lemma 3. *Define the discounted value of deductions as*

$$(57)\qquad t^d \;\equiv\; \frac{t^c\xi^d}{\rho\frac{1-t^p}{1-t^g}+\xi^d}.$$

[17] We assume that the fixed adjustment cost are capitalized and enter into the expression for deductions. We thank Jim Hines for helpful advice on this modelling assumption.

[18] In the previous sections without corporate taxes, the Modigliani-Miller theorem holds, that is, the firms' values and investment policies (and the implicit dividend policy) were independent of the capital structure. Introducing taxes, in principle, could break this independence (e.g., under the trade-off theory of the capital structure). Nevertheless, following the arguments in Miller (1977), we continue working under the Modigliani-Miller paradigm.

The firm value with taxes can be decomposed as:

$$V(k,u,d) \;=\; \frac{1-t^p}{1-t^g}\left[uv(\hat{k}) \;+\; t^d d\right], \tag{58}$$

where $v(\hat{k})$ solves the investment problem in Lemma 2 with the following parametric changes:

$$A \;\to\; (1-t^c)A, \tag{59}$$

$$\rho \;\to\; \frac{(1-t^p)}{(1-t^g)}\rho, \tag{60}$$

$$\theta \;\to\; (1-t^d)\theta, \tag{61}$$

$$p(\Delta\hat{k}) \;\to\; \left(1-t^d\right)p(\Delta\hat{k}). \tag{62}$$

The parametric changes established in Proposition 3 highlight the different channels through which taxes affect the firm value and optimal policy. The corporate income tax rate t^c directly affects after-tax profitability A in (59). The personal income tax rate t^p and the capital gains tax rate t^g directly affect the after-tax discount factor ρ in (60). The discounted value of deductions t^d affects the firm value through an income effect, as deductions increase additively the firm value in (58), and a substitution effect, as deductions promote investment by reducing the after-tax adjustment costs and after-tax prices in (61) and (62). Additionally, (t^c, t^p, t^g) operate indirectly through t^d. Next, we formalize the channels through which taxes affect investment through their interaction with investment frictions.

4.2 Static and dynamic effects of corporate taxation

Proposition 5 decomposes the optimal investment policy into a static frictionless component and a dynamic component that comprises the investment frictions. It shows that, from a firm's perspective, what matters for investment decisions is the fixed cost and the price wedge relative to *after-tax* profits. To simplify the notation, we define the *after-tax* discount $\tilde{r}$ and the *after-tax* user cost of capital $\tilde{\mathcal{U}}$ as:

$$\tilde{r} \;\equiv\; \frac{1-t^p}{1-t^g}\rho - \mu - \frac{\sigma^2}{2}, \tag{63}$$

$$\tilde{\mathcal{U}} \;\equiv\; \frac{1-t^p}{1-t^g}\rho + \xi^k - \sigma^2. \tag{64}$$

In particular, the after-tax user cost $\tilde{\mathcal{U}}$ is determined by the the personal income and capital gains taxes, the discount rate, the depreciation rate, and idiosyncratic volatility. For the problem to be well-defined, we assume $\tilde{r} > 0$ and $\tilde{\mathcal{U}} > 0$.

Proposition 5. *Let $\mathcal{K} \equiv \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ denote the firms' optimal investment policy. Consider the static log capital-productivity ratio $\hat{k}^{ss}$ that firms would set in the absence of the fixed costs and the price wedge:*

$$\hat{k}^{ss} = \frac{1}{1-\alpha} \log\left(\frac{1-t^c}{1-t^d}\, \frac{\alpha A}{p\tilde{\mathcal{U}}}\right). \tag{65}$$

With the static policy $\hat{k}^{ss}$, define the effective fixed cost $\tilde{\theta}$ (scaled by the after-tax static profits) and the effective prices $\tilde{p}^{sell}$ and $\tilde{p}^{buy}$ (scaled by the after-tax static profit-capital ratio):

$$\tilde{\theta} \equiv \frac{1-t^d}{1-t^c}\frac{\theta}{Ae^{\alpha\hat{k}^{ss}}}, \tag{66}$$

$$(\tilde{p}^{buy}, \tilde{p}^{sell}) \equiv \frac{1-t^d}{1-t^c}\frac{(p^{buy}-p, p^{sell}-p)}{Ae^{(\alpha-1)\hat{k}^{ss}}}. \tag{67}$$

Consider the normalized capital-productivity ratio $x \equiv \hat{k} - \hat{k}^{ss}$. Then the optimal investment policy can be decomposed as the sum of a static and a dynamic component

$$\mathcal{K} = \hat{k}^{ss} + \mathcal{X}(\tilde{\theta}, \tilde{p}^{buy}, \tilde{p}^{sell}), \tag{68}$$

where the dynamic component $\mathcal{X} \equiv \{x^-, x^{*-}, x^{*+}, x^+\}$ solves the following stopping problem:

$$\mathcal{V}(x) = \max_{\tau, \Delta x} \mathbb{E}\left[\int_0^\tau e^{-\tilde{r}\tau}\left(e^{\alpha x_s} - \alpha e^{x_s}\right)\mathrm{d}s + e^{\tilde{r}\tau}\left(-\tilde{\theta} + \tilde{p}(\Delta x)(e^{x_\tau+\Delta x} - e^{x_\tau}) + \mathcal{V}(x_\tau + \Delta x)\right)\Big| x_0 = x\right] \tag{69}$$

$$\mathrm{d}x_t = -\nu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{70}$$

$$\tilde{p}(\Delta x) = \tilde{p}^{buy}\mathbb{1}_{\{\Delta x>0\}} + \tilde{p}^{sell}\mathbb{1}_{\{\Delta x<0\}}. \tag{71}$$

Proposition 5 provides several insights regarding the effects of corporate taxation on investment. The static optimal policy $\hat{k}^{ss}$ in (65) sets firms' capital proportional to their current productivity, where the proportionality constant reflects after-tax profitability $(1-t^c)A$, the average after-tax user cost of capital $\tilde{\mathcal{U}}$ in (64), and the average after-tax investment price $(1-t^d)p$. Studying the effects of corporate taxes on a frictionless investment policy and its implications for aggregate capital accumulation have been widely studied (see Summers (1981) for early work and the vast literature that follows).

By definition, investment frictions do not affect the static choice $\hat{k}^{ss}$. In contrast, the dynamic policy $\mathcal{X}$ characterized by (69), (70), and (71) takes into account the fixed cost and the price wedge, but these frictions enter scaled by after-tax static profits or by after-tax output-capital ratio (see the definition of effective frictions in 66 and 67). Moreover, the flow payoff in the dynamic problem, that is $e^{\alpha x_s} - \alpha e^{x_s}$, only depends on the curvature of the profit function α. These observations imply that taxes have effects on the the dynamic component $\mathcal{X}$ of the optimal policy only through the effective investment frictions.

The fact that *after-tax* frictions are the key determinants for investment puts forward a novel channel for policy intervention: Corporate tax policy can effectively change the size fixed costs—a technological constraint typically considered outside the control of a policymaker—and the price wedge—a market determined outcome—and thus affect the dynamic component of investment. Proposition 6 formalizes the channels through which the corporate tax schedule shapes firm investment and signs the relationships with investment frictions.

Proposition 6. *Recall* $t^d \equiv t^c \xi^d / (\frac{1-t^p}{1-t^g}\rho + \xi^d)$ *and* $\tilde{\mathcal{U}} \equiv \frac{1-t^p}{1-t^g}\rho + \xi^k - \sigma^2$. *The effective fixed cost* $\tilde{\theta}$ *and effective price wedge* $\tilde{p}^{buy} - \tilde{p}^{sell}$ *relate to the fundamental frictions as follows:*

$$\tilde{\theta} = \left(\frac{1-t^d}{(1-t^c)A}\right)^{\frac{1}{1-\alpha}} \left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} \theta, \tag{72}$$

$$\tilde{p}^{buy} - \tilde{p}^{sell} = \frac{\alpha}{\tilde{\mathcal{U}}} \frac{p^{buy} - p^{sell}}{p}. \tag{73}$$

If $t^c > 0$ *and* $\tilde{\mathcal{U}} > 0$, *corporate taxes have the following effects on the effective investment frictions:*

1. *The effective fixed costs* $\tilde{\theta}$ *increases with* t^c *and* t^g; *it decreases with* ξ^d *and* t^p.
2. *The effective price wedge* $\tilde{p}^{buy} - \tilde{p}^{sell}$ *increases with* t^p *and decreases with* t^g. *It does not change with* t^c *or* ξ^d *as long as* $p = \overline{\mathbb{E}}[p(\Delta\hat{k})]$ *is fixed.*

We focus the explanation on the effects of the corporate income tax.[19] We derive three lessons from Proposition 6. First, a higher corporate income tax t^c reduces profits and therefore increases the effective fixed cost that are scaled by after-tax profits. This effect is mediated by depreciation allowance rate, being highest when $\xi^d = t^d = 0$. Second, the corporate income tax t^c does not affect the effective price wedge as long as the average price p remains fixed. This is because the profit-capital ratio $(1-t^c)Ae^{(\alpha-1)\hat{k}^{ss}}$ that divides the price wedge is invariant to t^c as it also enters the static policy $\hat{k}^{ss}$. If the average price p does change, which would only happen if the relative shares of upward and downward adjustments react to the tax, then the effective price wedge would change as well. However, this effect is second order and quantitatively small.

Lastly, the derivative of the effective fixed adjustment costs $\tilde{\theta}$ with respect to the corporate income tax t^c is increasing in the the fixed cost θ. This result suggests that firms, industries, or sectors with higher fixed costs will face relatively larger effective fixed costs when t^c increases. We will test this prediction using cross-sectoral data in Section 5.5.

[19]See Online Appendix for detail discussions on the effects of other taxes.

In summary, to study the effects of corporate taxation on the macroeconomy, one can separate the static and dynamic components. And to study the dynamic component, it suffices to assess how corporate taxes change the effective investment frictions.

4.3 Two benchmark cases

The dynamic policy $\mathcal{X}$ solves a stopping time problem that closely resembles the price-setting and investment problems with fixed costs, analyzed first by Barro (1972), Sheshinski and Weiss (1977), Dixit (1991), but with the addition of a price wedge. Below, we leverage on this previous work to characterize analytically the effect of taxes on individual policies and aggregate outcomes, extending previous results to the case with partial irreversibility. We consider two benchmark cases that isolate different mechanisms at play. Specifically, we show that the relative size of frictions matters and that the role of the price wedge crucially depends on the size of the drift.

Zero drift. We start characterizing the investment policy and the macroeconomic outcomes in driftless environments, that is, with zero drift and a symmetric price wedge. In this case, we demonstrate that capital misallocation $\mathbb{V}ar[x]$ is a sufficient statistic for the role of corporate taxation on capital valuation (q) and capital dynamics (CIR). Additionally, these driftless and symmetric environments clearly showcase the role of irreversibility: the price wedge constitutes an important friction as firms expect to purchase and sell capital with equal probability. As in Figure IV, we consider three cases: only fixed cost, only price wedge, and both frictions together. Proposition 7 characterizes these cases using a second-order approximation to the profit function.

Proposition 7. *Assume $\nu \to 0$ and symmetric effective price deviations $\tilde{p}^{buy} = -\tilde{p}^{sell} = \tilde{p}$. Without drift, the after-tax user cost of capital is $\mathcal{U} = \frac{1-t^p}{1-t^g}\rho - \sigma^2$ and the after-tax discount factor is $\tilde{r} = \frac{1-t^p}{1-t^g}\rho - \sigma^2/2$. In all symmetric cases we have: $\mathbb{E}[x] = 0$, $\mathbb{E}[\hat{k}] = \hat{k}^{ss}$ and $\mathbb{C}ov[x, a] = 0$.*

(i) ***Only fixed cost:*** *The inaction thresholds are $\bar{x} = \pm\left(\frac{6\tilde{\theta}\sigma^2}{\alpha(1-\alpha)}\right)^{1/4}$, the reset point is $x^* = 0$, and the macro outcomes are:*

$$\mathbb{V}ar[x] = \frac{\overline{x}^2}{6}; \qquad q = 1 - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]; \qquad \frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2}. \tag{74}$$

(ii) ***Only price wedge:*** *The inaction thresholds and reset points coincide $\overline{x}^* = \pm\left(\frac{3\tilde{p}\sigma^2}{2\alpha(1-\alpha)}\right)^{1/3}$, and the macro outcomes are:*

$$\mathbb{V}ar[x] = \frac{\overline{x}^{*2}}{3}; \quad q = 1 - \left(1 + \frac{2}{\alpha}\right)\frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]; \quad \frac{CIR(\delta)}{\delta} = \left(1 + \frac{1}{\sigma^2}\right)\mathbb{V}ar[x]. \tag{75}$$

(iii) ***Both frictions:*** *The thresholds of the inaction region* $\pm\overline{x}$ *and the reset points* $\pm x^*$ *solve:*

$$(76)\quad \overline{x}x^*(\overline{x}+x^*) = \frac{3\tilde{p}\sigma^2}{\alpha(1-\alpha)}; \qquad \overline{x}^4 - x^{*4} = \frac{3\tilde{p}\sigma^2}{\alpha(1-\alpha)}(\overline{x}-x^*)\left(1+\overline{x}+x^*\right) + \frac{6\tilde{\theta}\sigma^2}{\alpha(1-\alpha)},$$

and the macro outcomes are:

$$(77)\quad \mathbb{V}ar[x] = \frac{\overline{x}^2 + x^{*2}}{6}$$

$$(78)\quad q = 1 - \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\left(\mathbb{V}ar[x] + \frac{2}{\alpha}\frac{\overline{x}x^*}{3}\right)$$

$$(79)\quad \frac{CIR(\delta)}{\delta} = \frac{\mathbb{V}ar[x]}{\sigma^2} + \frac{x^*\overline{x}}{3}.$$

When only one friction is active, a marginal increase in the other friction has no effect on the macro outcomes:

$$(80)\quad \left.\frac{\mathrm{d}\mathbb{V}ar[x]}{\mathrm{d}\tilde{p}}\right|_{\tilde{\theta}>0,\,\tilde{p}=0} = 0; \quad \textit{and} \quad \left.\frac{\mathrm{d}Z}{\mathrm{d}\tilde{\theta}}\right|_{\tilde{\theta}=0,\,\tilde{p}>0} = 0, \quad \textit{for} \quad Z \in \{\mathbb{V}ar[x], q, CIR\}.$$

When only one friction is active, in cases (i) and (ii), there is a positive relationship between the corresponding effective investment friction ($\tilde{\theta}$ or $\tilde{p}$) and capital misallocation $\mathbb{V}ar[\hat{k}]$. This relationship is immediate from the expressions for the inaction region and the cross-sectional variance. In turn, higher misallocation reduces q by lowering aggregate productivity $\hat{Y}/\hat{K}$, and increases the CIR, slowing down the propagation of aggregate productivity shocks. If effective frictions were of the same size, that is $\tilde{\theta} = \tilde{p}$, expressions (74) and (75) reveal that a price wedge generates a higher $\mathbb{V}ar[x]$, a lower q, and a larger CIR compared to the case with only fixed costs.

Now let us discuss case (iii) in which both frictions are present. In this case, the sufficient statistics for q and CIR are misallocation $\mathbb{V}ar[x]$ and the product $\overline{x}x^*$. When $\overline{x} \approx x^*$ their product is proportional to $\mathbb{V}ar[x]$ (as in the case with only partial irreversibility). The first observation is that the inaction region $(-\overline{x}, \overline{x})$ and the reset points $\{-x^*, x^*\}$ are jointly determined by the size of both frictions. Frictions have opposing effects on the within and between components of misallocation, so the effect on the total misallocation is ambiguous. When the price wedge is positive, introducing a fixed cost shrinks the distance between the two reset points, reducing between variance. When the fixed costs is positive, introducing a price wedge generates two different reset points, increasing the between variance. In the limits where only one friction is active, the result in (80) teaches us that a marginal increase in the other friction has no effect on the macro outcomes (recall our earlier discussion around Figure IV).

Large drift. Next we characterize the case with a large drift relative to idiosyncratic shocks. In this case, we demonstrate that the price wedge is innocuous. The reason is that firms upsize by actively purchasing capital but downsize by letting the drift shrink its capital-productivity ratio. Thus the purchase price $\tilde{p}^{buy}$ is the only relevant price. Proposition 8 shows this result.

Proposition 8. *Let $\nu > 0$ and $\sigma^2 \to 0$ such that $\nu/\sigma^2 \to \infty$. Without idiosyncratic shocks, the after-tax user cost is $\tilde{\mathcal{U}} = \frac{1-t^p}{1-t^g}\rho + \xi^k$ and the after-tax discount is $\tilde{r} = \frac{1-t^p}{1-t^g}\rho - \mu$. The policy is a one-sided inaction region with lower threshold x^- and one reset point x^*. The cross-sectional distribution is Uniform over $[x^-, x^*]$ with moments:*

$$\mathbb{E}[x] = \frac{(x^* + \overline{x})}{12}; \qquad \mathbb{V}ar[x] = \frac{(x^* - \overline{x})^2}{12}. \tag{81}$$

The policy solves the non-linear system

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{\tilde{r}\tilde{\theta}}{\sqrt{12}\alpha(1-\alpha)}; \qquad \frac{\mathbb{E}[x]}{\mathbb{V}ar[x] + \mathbb{E}[x]^2} = -\left(\frac{\tilde{r}}{\nu} + \frac{\alpha+1}{2}\right), \tag{82}$$

and the macro outcomes are

$$q = 1 - \frac{\frac{1-t^p}{1-t^g}\rho + \xi^k}{\frac{1-t^p}{1-t^g}\rho - \mu}(1-\alpha)\left(\mathbb{E}[x] + \frac{\alpha}{2}\mathbb{V}ar[x]\right); \qquad \frac{CIR(\delta)}{\delta} = 0. \tag{83}$$

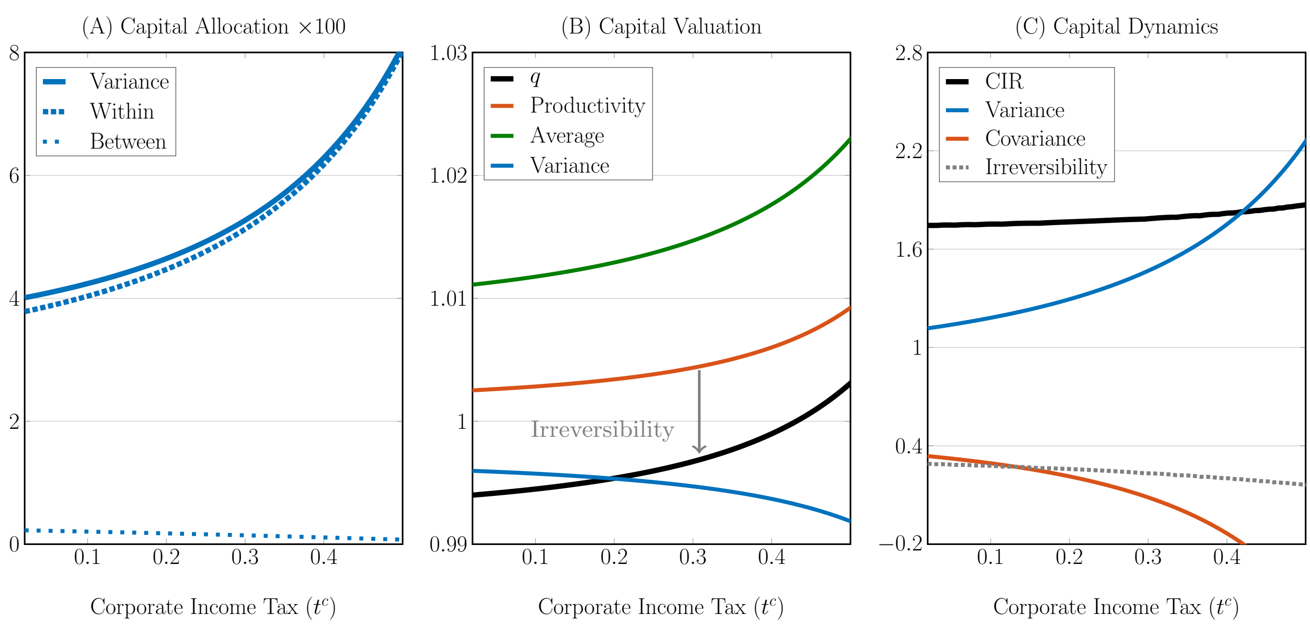
The case with a large drift reveals new mechanisms relative to the ones explored in the symmetric environments above. First, as already mentioned, the price wedge has no effect. Second, comparing the expression for aggregate q and the CIR with large drift against the three driftless cases in Proposition 7 we see that now the average $\mathbb{E}[x]$ matters (before it was zero). The non-linear system in (82) that pins down the investment policy shows that larger effective fixed costs $\tilde{\theta}$ increase both the average $\mathbb{E}[x]$ (in absolute value) and the variance $\mathbb{V}ar[x]$ of the normalized capital-productivity ratios x. In fact, the first equation is an indifference curve that mediates the trade-off between these two moments. A larger fixed cost lengthens the inaction period and firms accumulate more drift, reducing the average capital-productivity ratio relative to a frictionless economy. As firms anticipate a larger drift, they increase the reset point widening the distance between x^- and x^*, increasing the variance. The same system shows that the average $\mathbb{E}[x]$ is negative, thus $\mathbb{E}[\hat{k}] = \hat{k}^{ss} + \mathbb{E}[x] < \hat{k}^{ss}$. How do changes in the mean and variance—generated by higher effective fixed costs $\tilde{\theta}$—affect the macro outcomes? As the mean becomes more negative, q goes up; but as the variance increases, q goes down. The overall effect depends on the relative elasticities of these moments with respect to $\tilde{\theta}$. Lastly, as shown in Corollary 2 of Baley and Blanco (2021), the CIR equals zero: aggregate productivity shocks are immediately absorbed by firms and there are no deviations from steady-state.

The benchmark cases with zero and large drift teach us two key lessons. First, the importance of the effective price wedge (and the taxes that shape it) crucially depends on the drift. In a driftless environment, the price wedge is an important source of misallocation; in environments with large drift, it is not. Second, the effect of taxes on aggregate q depend on the relative size of the average $\mathbb{E}[x]$ and the variance $\mathbb{V}ar[x]$ of centralized capital-productivity ratios. In symmetric environments, the average is zero $\mathbb{E}[x] = 0$ and thus higher misallocation always decreases q. With a large drift, the mean is negative $\mathbb{E}[x] < 0$, reflecting capital scarcity. Scarcity increases q and could in principle dominate the misallocation effect that reduces q. Next, we consider an intermediate case with positive drift in the range of values observed in the data.

4.4 Corporate income tax and macro outcomes

We conclude the this section exploring the macroeconomic effects of corporate taxes in a general environment where both investment frictions are active and under parametric assumptions that are consistent with the data. Concretely, we use the parameters in Table I which are recovered from the Chilean microdata and described in detail in the next section. Here, the purpose is to illustrate the response of macro outcomes to a tax change and dissect the various channels discovered in the theoretical analysis. To simplify the exposition, we focus on the corporate income tax rate t^c and present a detailed analysis for the other taxes in the Appendix. Figure VI illustrates the macroeconomic consequences of an increase in the corporate tax rate from 0 to 0.5.

Figure VI – Macro Outcomes for Different Corporate Income Tax Rates



Notes: Panels A, B, and C plot capital allocation, capital valuation, and capital dynamics for various levels of the corporate income tax rate in the range $t^c \in [0, 0.5]$. The calibration follows Table I together with $\tilde{\theta} = 0.0175$.

Panel A shows capital allocation, measured by the variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ and decomposed into within and between variances following (37). In this parameterization with a relatively large drift, fixed costs are the primary investment friction and the within variance is almost the only source of misallocation. We observe that a higher corporate income tax distorts the allocation of capital. As predicted by equation (73) and Propositions 7 and 8, higher t^c raises the effective fixed cost $\tilde{\theta}$ and widens inaction regions, raising firms' tolerance for mismatch between their capital and their productivity and increasing the dispersion of capital-productivity ratios.

Panel B shows capital valuation, measured by the aggregate q and decomposed into a productivity and an irreversibility component according to (40). As predicted by the theory, the irreversibility term is negative and reduces q below the perpetuity value of productivity (and shifts q below 1 for most levels of t^c). We discover that q increases with the corporate income tax. To understand why this is the case, we follow expression (83) and plot the two margins affecting productivity $\hat{Y}/\hat{K}$ in (41): the average, $\exp(-(1-\alpha)\mathbb{E}[x])$, and the variance, $\exp(-(1-\alpha)\alpha\mathbb{V}ar[x]/2)$, of centralized of capital-productivity ratios. When t^c increases, firms invest less and the average capital-productivity ratio $\mathbb{E}[\hat{k}]$ falls. Scarce capital makes it more valuable and q goes up. At the same time, the allocation of capital worsens, $\mathbb{V}ar[\hat{k}]$ increases (see Panel A) and q goes down. In this parameterization, the first effect dominates and q is an increasing function of t^c.

Panel C shows capital dynamics, measured by the CIR and decomposed into three terms following (49): variance $\mathbb{V}ar[\hat{k}]/\sigma^2$, covariance $\nu\mathbb{C}ov[\hat{k}, a]/\sigma^2$, and irreversibility. The variance of capital-productivity ratios reflects misallocation (see Panel A). The covariance of capital-productivity ratios with the time elapsed since last adjustment reflects asymmetries in capital adjustment. A positive covariance (as observed in most of the tax domain considered) means that firms with old capital have a large desire to downsize, but they do not sell capital to avoid the penalty of the price wedge. And the irreversibility term is positive and increases the CIR beyond the sum of the other two terms. We find that the CIR increases with the corporate income tax, meaning that aggregate productivity shocks propagate more slowly when taxes are high.[20] The variance and covariance terms move in opposite directions with t^c: the variance increases because of higher effective fixed costs $\tilde{\theta}$, whereas the covariance decreases (and turns negative) because the price wedge plays a relatively smaller role vis-à-vis the fixed costs. Overall, for this parameterization, the first effect dominates and the CIR increases with t^c.

In summary, this calibrated version of our parsimonious model (informed by the Chilean data) suggests that a higher corporate income tax rate increases capital misallocation, increases capital valuation (due to lower capital accumulation), and slows down the propagation of aggregate shocks. In the next section, we provide suggestive evidence of these mechanisms using establishment-level data from Chile.

[20] The CIR goes from 1.73 at $t^c = 0$ to 1.96 at $t^c = 0.5$.

5 Suggestive empirical evidence

In this section, we put the theory to work. First, we describe the Chilean investment micro data and the country-specific tax schedule used in all empirical exercises. Second, we show how to exploit the micro data to recover the macro outcomes. To do this, we externally set the tax rates and technology parameters to replicate the average Chilean experience in the last 40 years. Using these parameters, we apply the mappings to recover the average macro outcomes in Chile between 1990 and 2018. Lastly, we provide suggestive evidence of the main implications of the theory through low-frequency time-series correlations and cross-sectorial data.

5.1 Data description

Using Chilean data has several advantages to evaluate our theory. Chile is a small open economy with an exogenous interest rate as our theory assumes. Chile does not feature a specific tax on capital gains t^g, which are taxed at the personal income tax rate t^p. As these rates are identical, the ratio $(1-t^p)/(1-t^g)$ equals one and does not affect the discount factor. Finally, the corporate income tax rate has varied in the last four decades, departing from 40% in the 1980, reaching 10% in 1984, and increasing consistently until 20% in 2020.

Data sources. We use yearly investment data on manufacturing plants in Chile from the Annual National Manufacturing Survey (*Encuesta Nacional Industrial Anual*) for the period 1980 to 2011. To construct the capital series, we use information on depreciation rates and price deflators from Chilean national accounts and Penn World Tables. The sample considers plants that appear in the sample for at least 10 years (more than 60% of the sample) and have more than 10 workers. Data on the corporate income tax comes from Vegh and Vuletin (2015), which we cross-checked using several sources. The Data Appendix provides details.

Capital stock and investment rates. We construct the capital stock series using the perpetual inventory method. We include structures, machinery, equipment, and vehicles. Given an initial level of capital k_0, a plant's capital stock in year t evolves as

$$k_t \;=\; (1-\xi^k)k_{t-1} \;+\; i_t/(p(i_t)D_t), \tag{84}$$

where ξ^k is the physical depreciation rate, D_t is the gross fixed capital formation deflator, and initial capital k_0 is a plant's self-reported nominal capital stock at current prices for the first year in which it is nonnegative. Following the theory, i_t/D_t is the real investment in output units, and $i_t/(p(i_t)D_t)$ is the real investment in capital units, which considers different prices for capital purchases and sales. We set a price wedge of 5%, which is an intermediate value in the literature.[21] We construct gross nominal investment i_t with information on purchases, reforms, improvements, and sales of fixed assets, and define the investment rate ι_t as the ratio of real gross investment to the capital stock:[22]

$$\iota_t \equiv \frac{i_t/(p(i_t)D_t)}{k_{t-1}}. \tag{85}$$

Variable construction. For each plant and each inaction spell h, we record the change in the capital-productivity ratio upon action $\Delta\hat{k}_h$ and the spell's duration τ_h. We construct $\Delta\hat{k}_h$ with investment rates from (85):

$$\Delta\hat{k}_h = \begin{cases} \log(1+\iota_h) & \text{if} \quad |\iota_h| > \underline{\iota}, \\ 0 & \text{if} \quad |\iota_h| < \underline{\iota}. \end{cases} \tag{86}$$

The threshold $\underline{\iota} > 0$ reflects the idea that small maintenance investments should be excluded. Following Cooper and Haltiwanger (2006), we set $\underline{\iota} = 0.01$, such that all investment rates below 1% in absolute value are considered to be part of an inaction spell. Then we define an adjustment date T_h from $\Delta\hat{k}_{T_h} \neq 0$ and compute a spell's duration as the difference between two adjacent adjustment dates: $\tau_h = T_h - T_{h-1}$. Finally, we truncate the investment distribution to lie in the range $[-0.3, 1]$ to eliminate outliers.[23]

Figure VII plots the resulting cross-sectional distribution of non-zero changes of the capital-productivity ratios $\Delta\hat{k}$ and completed inaction spells τ, conditional on a past positive or negative investment. The data shows investment patterns that are consistent with partial irreversibility. In particular, the distribution of investment conditional on a last negative investment $H^+(\Delta k)$ is skewed toward the left of the distribution conditional on a last positive investment H^-, meaning that the probability of a negative investment is larger after a negative investment and vice versa.

5.2 Calibration

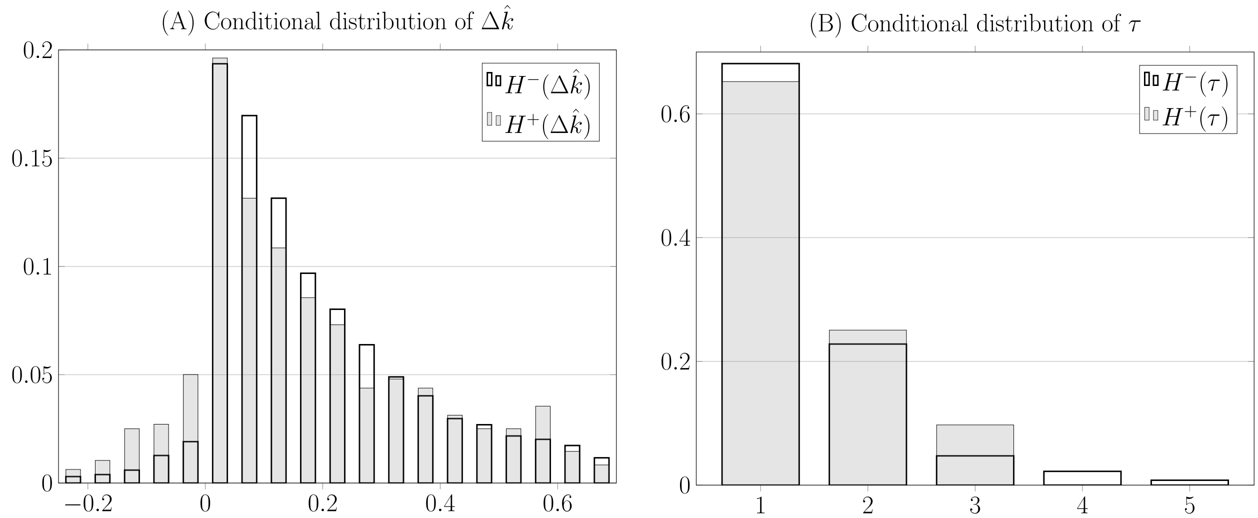
There are several parameters that we must calibrate externally before applying the theoretical mappings to recover the capital behavior from the data. These parameters include taxes, the

[21] Ramey and Shapiro (2001) study the reallocation of capital previously operated by closing aerospace plants. They find a discount of 0.28 cent per dollar. Lanteri, Medina and Tan (2019) set a discount of 0.07 cent per dollar and Khan and Thomas (2013) set a discount of 0.03 cent per dollar. We set a value in between these studies. The Appendix shows robustness exercises with respect to the price wedge. Note that only the price wedge matters for computing investment, not the price level.

[22] Note that the investment rate equals $\iota_{T_h} \equiv i_{T_h}/k_{T_h^-} = (k_{T_h} - k_{T_h^-})/k_{T_h^-}$, where $k_{T_h^-} = \lim_{t \uparrow T_h} k_t$. In contrast to the continuous-time model, in which investment is computed as the difference in the capital stock between two consecutive instants, in the data we compute it as the difference between two consecutive years.

[23] Table I in the Data Appendix presents descriptive statistics on investment rates. In particular, the inaction rate ($|\iota| < 0.01$) equals 40.1%.

Figure VII – Empirical Distribution of Observable Actions



Notes: Own calculations using establishment data from Chile. Panel A plots the distribution of *non-zero* changes in capital-productivity ratios and Panel B plots the duration of inaction spells. Solid bars = conditional on departing from $\hat{k}^{*+}$ (last negative investment); white bars = conditional on departing from $\hat{k}^{*-}$ (last positive investment). Sample: Firms with at least 10 years of data, truncation at $[-0.3, 1]$, and inaction threshold of $\underline{\iota} = 0.01$.

discount factor, technological constants, and investment prices. We set these parameters to match several average statistics from the Chilean economy in the last 40 years. The parameterization is summarized in Table I. These values are used to produce Figure VI above.

Taxes. We set the tax schedule to match the average values in Chile between 1980 to 2018. The personal income and capital gain tax rates are identical and equal to $t^p = t^g = 0.471$; the corporate income tax rate (the highest marginal rate) is on average $t^c = 0.26$. We set the depreciation deduction rate to $\xi^d = 0.07$ to match the PDV of depreciation allowances under the Chilean straight-line system.

Externally-calibrated parameters. One period equals one year. We set the real interest rate to 5.8% ($\rho = 0.058$) to match the average lending nominal rate minus realized inflation. The productivity growth rate is set to 3.3% ($\mu = 0.033$) to match the average GDP growth rate. The returns-to-scale parameter is set to $\alpha = 0.72$ to match an elasticity of output to capital of 0.3 taking into account labor and decreasing returns to scale.[24] We set investment prices to $p^{\text{buy}} = 2$ and $p^{\text{sell}} = 1.91$ to match an average aggregate output-capital ratio of $\hat{Y}/\hat{K} = 0.36$ and an irreversibility wedge of 5% as above.

[24] To set this parameter, we consider a generalized production function that includes labor l as a frictionless input, $y = u^{1-\eta\tilde{\alpha}}\left(k^{\tilde{\alpha}} l^{1-\tilde{\alpha}}\right)^{\eta}$. Static maximization over labor implies $y \propto k^{\frac{\eta\tilde{\alpha}}{1-(1-\tilde{\alpha})\eta}}$. Assuming standard parameters in the literature, $\eta = 0.90$ and $\tilde{\alpha} = 0.3$, the implied value for the output-capital elasticity is $\alpha = (\eta\tilde{\alpha})/(1-(1-\tilde{\alpha})\eta) = 0.72$.

Estimated parameters. Using the mappings from data to the parameters of the productivity process in (31) and (32), we recover a drift of $\nu = 0.086$ and a volatility of $\sigma^2 = 0.036$. Together with the productivity growth rate, the value for the drift ν implies a physical depreciation rate of $\xi^k = \nu - \mu = 0.053$. Given these values, the implied after-tax discount is $\tilde{r} = \rho - \mu - \sigma^2/2 = 0.007$ and the after-tax user cost is $\tilde{\mathcal{U}} = \rho + \xi^k - \sigma^2 = 0.075$.

Table I – Parameterization

Taxes				Technology					Productivity	
τ^p	τ^g	τ^c	ξ^d	μ	α	ρ	p^{buy}	p^{sell}	ν	σ^2
0.471	0.471	0.260	0.070	0.033	0.720	0.058	2.000	1.910	0.086	0.036

Notes: Baseline parameterization of the model. Average tax rates in Chile in the period 1980-2018. Other parameters are set externally or recovered from the microdata through the lens of the theory, see text for details.

5.3 Average capital behavior

The first empirical assesses the nature and magnitude of investment frictions and the role of the various components in shaping the average macro outcomes (misallocation, irreversibility, productivity, etc.). This exercise uses the average levels of taxes during the sample period. Table II shows the average outcomes recovered from the data through the lens of the theory.

We begin by examining the investment policy. We recover reset capitals from (33) and (34). The reset capital after positive investments is $\hat{k}^{*-} = 4.24$ and the reset capital after negative investments is $\hat{k}^{*+} = 4.60$. Their difference, $\hat{k}^{*+} - \hat{k}^{*-} = 0.36$, is explained by the exogenous price wedge $\log(p^{\text{buy}}/p^{\text{sell}}) = 0.16$ and the endogenous behavior of firms reflected in the PDV of the capital-productivity ratio, which is equal to 0.19 and computed as a residual. Next, we recover the average macro outcomes during the sample period.

Using (35) and (36), we estimate an average misallocation of $\mathbb{V}ar[\hat{k}] = 0.068$, where 76% comes from within-dispersion and 24% comes from between-dispersion (driven by the price wedge). As expected, the relative importance of the price wedge diminishes in environments with drift. We use (40) to recover an average $q = 1.009$, which is not far from its frictionless value. We back out its productivity component 1.028 from (41) and its irreversibility component -0.02 from (42). As predicted by the theory, irreversibility decreases q. Lastly, using (49), we recover an average of CIR $= 2.76$, meaning that a 1% decrease in aggregate productivity generates a total deviation of aggregate capital above its steady-state value of 2.7%. Thus the average multiplier of aggregate productivity shocks is 2.7. We decompose the CIR into its three components: variance 1.88 from (36), covariance 0.70 from (50), and irreversibility 0.18 from (51). As predicted by the theory, the CIR's irreversibility term is positive; however, it is quantitatively small and most of its effects operate indirectly by increasing the steady-state moments (specifically, the price wedge generates a positive covariance of capital-productivity ratios with age, $\mathbb{C}ov[\hat{k}, a]$, which would otherwise be negative without it).

Table II – Aggregate Capital Behaviour: 1980-2011

Partial Irreversibility		**Capital Allocation**	
Reset capital after negative investment ($\hat{k}^{*+}$)	4.592	Variance	0.068
Reset capital after positive investment ($\hat{k}^{*-}$)	4.236	Within	0.052
Difference in reset capitals ($\hat{k}^{*+} - \hat{k}^{*-}$)	0.356	Between	0.016
Exogenous price wedge	0.164	Mean	4.258
PDV of capital-productivity ratio	0.191		

Capital Valuation		**Capital Dynamics**	
Tobins q	1.009	CIR	2.762
Productivity	1.028	Variance	1.885
Irreversibility	-0.020	Covariance	0.702
		Irreversibility	0.176

Notes: Own calculations using establishment-level data from Chile. Uses externally-calibrated parameters described in Table I.

Let us compare the values for the macro outcomes reported in Table II—recovered directly from the microdata mappings assuming a corporate tax rate of $t^c = 0.26$—with the corresponding values in Figure VI in Section 4.4—obtained by simulating the model. All values in the data are consistently larger than those produced by the model. The reason for this discrepancy is that the parsimonious model (with a symmetric fixed cost) cannot reproduce the large variance of capital-productivity ratios $\mathbb{V}ar[\hat{k}]$ and the large covariance of capital-productivity ratios and their age $\mathbb{C}ov[\hat{k}, a]$ recovered from the data. In fact, we calibrated the effective fixed cost $\tilde{\theta} = 0.0175$ to strike a balance between the two moments in the data.

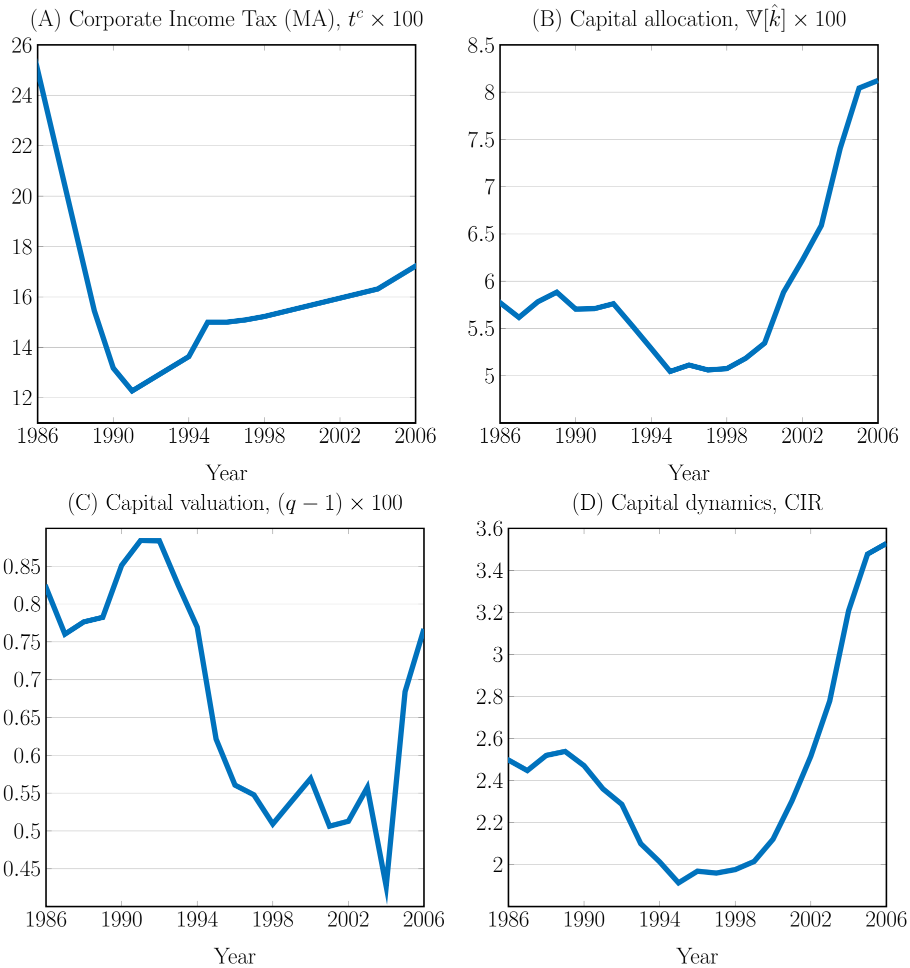
In previous work (Baley and Blanco, 2021), we demonstrated that the symmetric fixed cost model is unable to replicate the empirical values of these two moments and showed how it should be augmented in order to match them. Introducing a time-dependent component in adjustment, such as random opportunities for free adjustment, increases the variance $\mathbb{V}ar[\hat{k}]$. Introducing asymmetric fixed costs that depend on the adjustment sign increases the covariance $\mathbb{C}ov[\hat{k}, a]$ (the price wedge already pushes the variance up, but it is not quantitatively enough). Augmenting the model in these two directions is straightforward and necessary to conduct a fully-fledged quantitative analysis. Nevertheless, we have opted to keep the model as simple as possible to facilitate its exposition and to highlight the key mechanisms at work in the cleanest way.

5.4 Low-frequency time-series correlations

The second exercise aims to provide suggestive evidence of the correlations between capital misallocation, valuation, and dynamics obtained in Section 4.4. We see this exercise as proof of concept that the endogenous objects move in the direction implied by the theory. We compute 5-year moving averages of the time-series of the Chilean corporate income tax rate and the data-implied time-series of the three macro outcomes. Figure VIII plots the results.

We confirm that an increase in the corporate income tax (Panel A) is associated with an increase in misallocation (Panel B) and an increase in the CIR (Panel D), meaning that the propagation speed of aggregate productivity shocks has slowed down. The correlation with the aggregate q (Panel C) is more nuanced. Recall that q is driven by both scarcity and misallocation, which operate in opposing directions. Moreover, q is a forward-looking variable and thus it responds to expectations of future taxes, while here we assume constant taxes.

Figure VIII – Time series of corporate income tax and long-run outcomes



Notes: 5-year moving averages of (A) corporate income tax rate, (B) cross-sectional variance of capital-productivity ratios ($\times 100$), (C) aggregate q (minus 1×100), and (D) CIR.

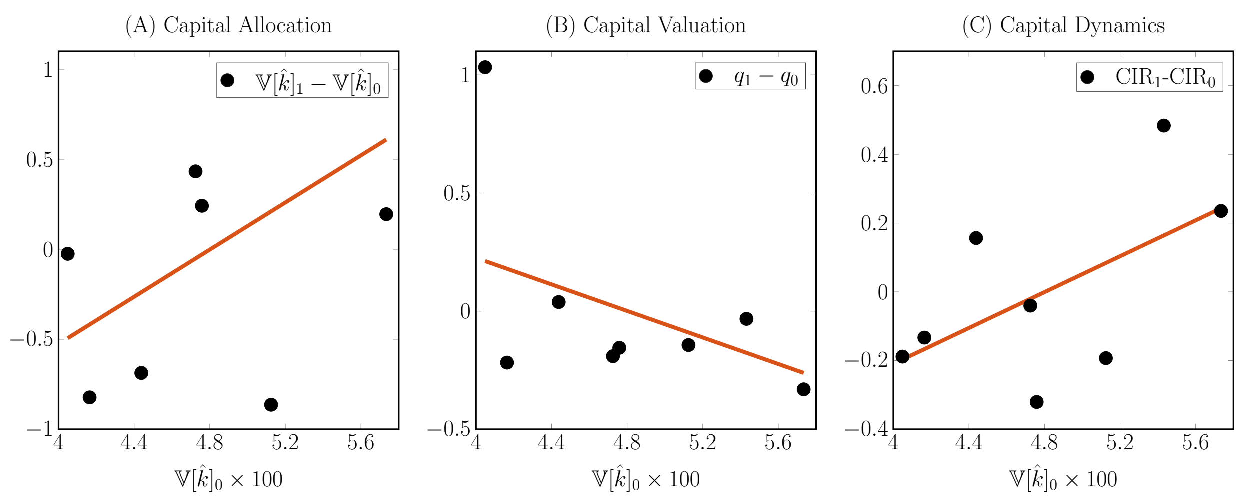
5.5 Cross-sectoral heterogeneity

When exploiting time variation, there is the concern that other trends could be driving the correlations. Some of these trends may include the decline in interest rates, the rise of market power, the use of intangible capital, or other trends specific to the Chilean context. To alleviate some of these concerns, the third exercise exploits heterogeneity across industries, in the spirit of the empirical analysis in House and Shapiro (2008) and Zwick and Mahon (2017).

According to Proposition 6, the derivative of the effective fixed cost $\tilde{\theta}$ with respect to the corporate income tax t^c is increasing in the underlying fixed cost θ. Thus the theory predicts that industries with larger fixed costs (and therefore larger misallocation) should be more affected by a change in taxation. To test this prediction, we consider 8 industries within the Chilean manufacturing sector. We split the sample period into two regimes: a low-tax regime (period 0) going from 1991 to 2001, and a high-tax regime (period 0) going from 2002 to 2011. For each industry, we compute the initial level of misallocation $\mathbb{V}ar[\hat{k}]_0$ and consider it a proxy for the pervasiveness of investment frictions in that sector. Then, we compute the changes in the macro outcomes between the two tax regimes. Figure IX plots the results.

We discover that industries with higher misallocation suffer a larger increase in misallocation and in the CIR, confirming the predictions from the theory. In contrast, industries with higher misallocation suffer a larger decrease in q. This suggests (as in the time-series exercise) that the scarcity effect dominates.

Figure IX – Sectorial variation in the macro response to a permanent increase in t^c



Notes: Panels A, B, and C plot the change in the macro outcomes between two periods against the initial level of misallocation $\mathbb{V}ar[\hat{k}]_0$. Each dot corresponds to a manufacturing industry: Non-metallic mineral, machinery, textile, basic metal, chemistry, paper, and food.

6 Conclusions

We develop a novel methodology for evaluating the effects of corporate tax reforms, highlighting the interaction with investment frictions and the interconnectedness between macroeconomic outcomes.

References

Abel, A. B. (1979). *Investment and the Value of Capital*. 65, Dissertations-G.

— (1982). Dynamic effects of permanent and temporary tax policies in aq model of investment. *Journal of Monetary Economics*, **9** (3), 353–373.

— (1983). Optimal investment under uncertainty. *The American Economic Review*, **73** (1), 228–233.

— and Eberly, J. C. (1994). A unified model of investment under uncertainty. *American Economic Review*, **84** (5), 1369.

— and — (1996). Optimal investment with costly reversibility. *The Review of Economic Studies*, **63** (4), 581–593.

Alexandrov, A. (2021). *The Effects of Trend Inflation on Aggregate Dynamics and Monetary Stabilization*. Tech. rep., University of Bonn and University of Mannheim, Germany.

Altug, S., Demers, F. S. and Demers, M. (2009). The investment tax credit and irreversible investment. *Journal of Macroeconomics*, **31** (4), 509–522.

Álvarez, F., Beraja, M., Gonzalez-Rozada, M. and Neumeyer, P. A. (2018). From hyperinflation to stable prices: Argentina's evidence on menu cost models. *The Quarterly Journal of Economics*, **134** (1), 451–505.

—, Le Bihan, H. and Lippi, F. (2016). The real effects of monetary shocks in sticky price models: a sufficient statistic approach. *American Economic Review*, **106** (10), 2817–51.

— and Lippi, F. (2021). The analytic theory of a monetary shock. *NBER Working Paper*.

—, Lippi, F. and Oskolkov, A. (2020). *The Macroeconomics of Sticky Prices with Generalized Hazard Functions*. Tech. rep., National Bureau of Economic Research.

Asker, J., Collard-Wexler, A. and De Loecker, J. (2014). Dynamic inputs and resource (mis) allocation. *Journal of Political Economy*, **122** (5), 1013–1063.

Auerbach, A. J. (1979). Wealth maximization and the cost of capital. *The Quarterly Journal of Economics*, **93** (3), 433–446.

— (1986). Tax reform and adjustment costs: The impact on investment and market value.

— and Hines, J. R. (1986). *Tax reform, investment, and the value of the firm*. Tech. rep., National Bureau of Economic Research.

BALEY, I. and BLANCO, A. (2019). Firm uncertainty cycles and the propagation of nominal shocks. *American Economic Journal: Macroeconomics*, **11** (1), 276–337.

— and — (2021). Aggregate dynamics in lumpy economies. *Econometrica*, **89** (3), 1235–1264.

BARRO, R. J. (1972). A theory of monopolistic price adjustment. *The Review of Economic Studies*, pp. 17–26.

BERTOLA, G. and CABALLERO, R. J. (1994). Irreversibility and aggregate investment. *The Review of Economic Studies*, **61** (2), 223–246.

BOISSEL, C. and MATRAY, A. (2019). Higher dividend taxes, no problem! evidence from taxing entrepreneurs in france. *Evidence from Taxing Entrepreneurs in France (July 10, 2019).*

CABALLERO, R. J. and ENGEL, E. (1991). Dynamic (s,s) economies. *Econometrica*, **59** (6), 1659–1686.

— and — (1993). Microeconomic adjustment hazards and aggregate dynamics. *The Quarterly Journal of Economics*, **108** (2), 359–383.

— and — (1999). Explaining investment dynamics in u.s. manufacturing: A generalized (s, s) approach. *Econometrica*, **67** (4), 783–826.

— and — (2007). Price stickiness in ss models: New interpretations of old results. *Journal of monetary economics*, **54**, 100–121.

— and LEAHY, J. V. (1996). Fixed costs: The demise of marginal q. *NBER working paper*, (w5508).

CAPLIN, A. S. and SPULBER, D. F. (1987). Menu costs and the neutrality of money. *The Quarterly Journal of Economics*, **113**, 287–303.

CHEN, Z., JIANG, X., LIU, Z., SUÁREZ SERRATO, J. C. and XU, D. (2019). *Tax Policy and Lumpy Investment Behavior: Evidence from China's VAT Reform*. Tech. rep., National Bureau of Economic Research.

COOPER, R. W. and HALTIWANGER, J. C. (2006). On the nature of capital adjustment costs. *The Review of Economic Studies*, **73** (3), 611–633.

DAVID, J. M. and VENKATESWARAN, V. (2019). The sources of capital misallocation. *American Economic Review*, **109** (7), 2531–67.

DIXIT, A. (1991). Analytical approximations in models of hysteresis. *The Review of Economic Studies*, **58** (1), 141–151.

EHOUARNE, C., KUEHN, L.-A. and SCHREINDORFER, D. (2016). *Misallocation cycles*. Tech. rep., Working Paper.

GOURIO, F. and MIAO, J. (2010). Firm heterogeneity and the long-run effects of dividend tax reform. *American Economic Journal: Macroeconomics*, **2** (1), 131–68.

— and — (2011). Transitional dynamics of dividend and capital gains tax cuts. *Review of Economic Dynamics*, **14** (2), 368–383.

HALL, R. E. and JORGENSON, D. W. (1967). Tax policy and investment behavior. *The American Economic Review*, **57** (3), 391–414.

HARTMAN, R. (1972). The effects of price and cost uncertainty on investment. *Journal of economic theory*, **5** (2), 258–266.

HASSETT, K. A. and HUBBARD, R. G. (2002). Tax policy and business investment. In *Handbook of public economics*, vol. 3, Elsevier, pp. 1293–1343.

HAYASHI, F. (1982). Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica: Journal of the Econometric Society*, pp. 213–224.

HOUSE, C. L. and SHAPIRO, M. D. (2008). Temporary investment tax incentives: Theory with evidence from bonus depreciation. *American Economic Review*, **98** (3), 737–68.

HSIEH, C.-T. and KLENOW, P. J. (2009). Misallocation and manufacturing tfp in china and india. *The Quarterly journal of economics*, **124** (4), 1403–1448.

KHAN, A. and THOMAS, J. K. (2013). *Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity*. Working paper.

KING, M. A. and FULLERTON, D. (1984). *The taxation of income from capital: A comparative study of the United States, the United Kingdom, Sweden and West Germany*. University of Chicago Press.

KOBY, Y. and WOLF, C. (2020). Aggregation in heterogeneous-firm models: Theory and measurement. *Manuscript, July*.

LANTERI, A. (2018). The market for used capital: Endogenous irreversibility and reallocation over the business cycle. *American Economic Review*, **108** (9), 2383–2419.

—, MEDINA, P. and TAN, E. (2019). Capital-reallocation frictions and trade shocks. *Economic Research Initiatives at Duke (ERID) Working Paper*, (293).

LERCHE, A. (2019). Investment tax credits and the response of firms.

MAFFINI, G., XING, J. and DEVEREUX, M. P. (2019). The impact of investment incentives: Evidence from uk corporation tax returns. *American Economic Journal: Economic Policy*, **11** (3), 361–89.

MIAO, J. and WANG, P. (2014). Lumpy investment and corporate tax policy. *Journal of Money, Credit and Banking*, **46** (6), 1171–1203.

MILGROM, P. and SEGAL, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, **70** (2), 583–601.

MILLER, M. H. (1977). Debt and taxes. *the Journal of Finance*, **32** (2), 261–275.

OHRN, E. (2018). The effect of corporate taxation on investment and financial policy: Evidence from the dpad. *American Economic Journal: Economic Policy*, **10** (2), 272–301.

OI, W. Y. (1961). The desirability of price instability under perfect competition. *Econometrica: journal of the Econometric Society*, pp. 58–64.

Oksendal, B. (2007). *Stochastic Differential Equations*. Springer, 6th edn.

Philippon, T. (2009). The bond market's q. *The Quarterly Journal of Economics*, **124** (3), 1011–1056.

Poterba, J. M. and Summers, L. H. (1983). Dividend taxes, corporate investment, and 'q'. *Journal of Public Economics*, **22** (2), 135–167.

Ramey, V. A. and Shapiro, M. D. (2001). Displaced capital: A study of aerospace plant closings. *Journal of political Economy*, **109** (5), 958–992.

Restuccia, D. and Rogerson, R. (2013). Misallocation and productivity. *Review of Economic Dynamics*, **1** (16), 1–10.

Sheshinski, E. and Weiss, Y. (1977). Inflation and costs of price adjustment. *The Review of Economic Studies*, **44** (2), 287–303.

Stokey, N. (2009). *The Economics of Inaction*. Princeton University Press.

Stokey, N. L. (1989). *Recursive methods in economic dynamics*. Harvard University Press.

Summers, L. (1981). Taxation and corporate investment: A q-theory approach. *Brookings Papers on Economic Activity*, **1981** (1), 67–140.

Tobin, J. (1969). A general equilibrium approach to monetary theory. *Journal of money, credit and banking*, **1** (1), 15–29.

Vegh, C. A. and Vuletin, G. (2015). How is tax policy conducted over the business cycle? *American Economic Journal: Economic Policy*, **7** (3), 327–70.

Veracierto, M. L. (2002). Plant-level irreversible investment and equilibrium business cycles. *American Economic Review*, **92** (1), 181–197.

Winberry, T. (2021). Lumpy investment, business cycles, and stimulus policy. *American Economic Review*, **111** (1), 364–96.

Yagan, D. (2015). Capital tax reform and the real economy: The effects of the 2003 dividend tax cut. *American Economic Review*, **105** (12), 3531–63.

Zwick, E. and Mahon, J. (2017). Tax policy and heterogeneous investment behavior. *American Economic Review*, **107** (1), 217–48.

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A Supplementary Appendix

A.1 Auxiliary Theorems

Auxiliary Theorem 1. *Let X be a (sub)martingale on the filtered space $(\Omega, \mathbb{P}, \mathcal{F})$ and let τ be a stopping time. If $(\{X_t\}_t, \tau)$ is a well-defined stopping process, then*

$$\mathbb{E}[X_\tau](\geq) = \mathbb{E}[X_0]. \tag{A.1}$$

This is the Optional Sampling Theorem (OST), see Theorem 4.4 in Stokey (2009) for the proof.

Auxiliary Theorem 2. *Let g be a real valued function of a Brownian motion x_t, F the ergodic distribution of x, and τ a stopping time. Assume that $\int g(x)\,\mathrm{d}F(x) = \lim_{T\to\infty} T^{-1}\int_0^T g(x_t)\,\mathrm{d}t$ for all initial x_0 and a constant reset state $x_\tau = x^*$. The following relationship holds:*

$$\underbrace{\mathbb{E}\left[\int_0^\tau g(x_t)\,\mathrm{d}t\middle| x_0 = x^*\right]}_{\textit{occupancy measure}} = \underbrace{\int g(x)\,\mathrm{d}F(x)}_{\textit{steady-state mass}} \quad \underbrace{\mathbb{E}\left[\tau\middle| x_0 = x^*\right]}_{\textit{proportionality constant}}. \tag{A.2}$$

Proof. This result establishes the equivalence between the steady-state distribution and the occupancy measure. The occupancy measure (LHS) is the average time an agent's state spends at a given value. It is proportional to the stationary mass of agents at that particular state (RHS), with a proportionality constant equal to the expected time between adjustments. For example, if $g(x) = x^m$, then $\mathbb{E}\left[\int_0^\tau x_t^m\,\mathrm{d}t | x_0 = x^*\right] = \mathbb{E}[x^m]\mathbb{E}[\tau|x_0 = x^*]$. For notation clarity, we use $\overline{\mathbb{E}}[\cdot] \equiv \mathbb{E}[\cdot|x_0 = x^*]$. See Appendix B.2 for the proof and Stokey (2009) for more details.

A.2 Proof of Lemmas 1, 2, and 3

The proof has 3 steps. First, we derive the sufficient conditions that characterize the optimal value function $V(k, u, d)$ and the policy $(k^-, k^{*-}, k^{*+}, k^+)$. Second, to reduce the dimensionality of the state space, we guess that V can be separated as $V(k, u, d) = uv(\hat{k}) + t^d d$, where $v(\hat{k})$ is a function of the log capital-to-productivity ratio $\hat{k} \equiv \log(k/u)$ and t^d is the PDV of depreciation deductions. The corresponding policy is $(\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+)$. Third, we verify the guess by showing that the sufficient conditions for V are equivalent to those satisfied by v. Lastly, we express the optimal policy as a system of equations in fundamental parameters.

To simplify the notation, we denote the state evaluated at the lower border of inaction as $S^- \equiv (k^-, u, d)$ and evaluated after a positive adjustment as $S^{*-} \equiv \left(k^{*-}, u, d + p^{buy}\Delta k\right)$, where $\Delta k = k^{*-} - k^-$. Analogously, we denote the state evaluated at the upper border of inaction as $S^- \equiv (k^-, u, d)$ and evaluated after a negative adjustment as $S^{*+} \equiv \left(k^{*+}, u, d + p^{sell}\Delta k\right)$, where $\Delta k = k^{*+} - k^+$.

Step 1: Characterize the value function $V(k, u, d)$. Recall the recursive problem in (4). Substituting output (1), after-tax profits (52), and adjustment costs $\theta_s = \theta u_s$ we obtain

$$\begin{aligned} V(k_0, u_0, d_0) = \max_{\{T_h,\ i_{T_h}\}_{h=1}^\infty} \mathbb{E}\Bigg[&\int_0^\infty e^{-\rho\frac{1-t^p}{1-t^g}s}[(1-t^c)Au_s^{1-\alpha}k_s^\alpha + t^c\xi^d d_s]\,\mathrm{d}s \\ &- \sum_{h=1}^\infty e^{-\rho\frac{1-t^p}{1-t^g}T_h}\left(\theta u_{T_h} + p\left(i_{T_h}\right)i_{T_h}\right)\Bigg] \end{aligned} \tag{A.3}$$

Using the principle of optimality, we write the problem recursively as a stopping-time problem with initial conditions $(k_0, u_0, d_0) = (k, u, d)$:

$$\begin{aligned} V(k, u, d) = \max_{\tau,\ \Delta k_\tau} \mathbb{E}\Bigg[&\int_0^\tau e^{-\rho\frac{1-t^p}{1-t^g}s}[(1-t^c)Au_s^{1-\alpha}k_s^\alpha + t^c\xi^d d_s]\,\mathrm{d}s \\ &+ e^{-\rho\frac{1-t^p}{1-t^g}\tau}\left(-\theta u_\tau - p(\Delta k_\tau)\Delta k_\tau + V\left(k_\tau + \Delta k_\tau, u_\tau, d_\tau + \tilde{p}(\Delta k_\tau)\Delta k_\tau\right)\right)\Bigg]. \end{aligned} \tag{A.4}$$

The optimal policy $(k^-, k^{*-}, k^{*+}, k^+)$ satisfies the system of sufficient conditions (A.5) to (A.15).

1. In the interior of the inaction region, i.e., $k \in (k^-, k^+)$, $V(S)$ solves the HJB equation:

$$\rho \frac{1-t^p}{1-t^g} V(S) = u(1-t^c)A\left(\frac{k}{u}\right)^{\alpha} + t^c \xi^d d - \xi^k k \frac{\partial V(S)}{\partial k} - \xi^d d \frac{\partial V(S)}{\partial d} + \left(\mu + \frac{\sigma^2}{2}\right) u \frac{\partial V(S)}{\partial u} + \frac{(\sigma u)^2}{2} \frac{\partial^2 V(S)}{\partial u^2}. \tag{A.5}$$

2. The value matching conditions for all (u, d), that equalize the value of adjusting to the value of inaction at the borders of the inaction region $\{k^-, k^+\}$:

$$V(S^{*-}) - p^{buy}\Delta k - \theta u = V(S^-), \tag{A.6}$$

$$V(S^{*+}) - p^{sell}\Delta k - \theta u = V(S^+). \tag{A.7}$$

3. The two optimality conditions for the reset capitals $\{k^{*-}, k^{*+}\}$:

$$\frac{\partial V(S^{*-})}{\partial k} = \left[1 - \frac{\partial V(S^{*-})}{\partial d}\right] p^{buy}, \tag{A.8}$$

$$\frac{\partial V(S^{*+})}{\partial k} = \left[1 - \frac{\partial V(S^{*+})}{\partial d}\right] p^{sell}. \tag{A.9}$$

4. The six smooth pasting conditions:

$$\frac{\partial V(S^{-})}{\partial k} = \left[1 - \frac{\partial V(S^{-})}{\partial d}\right] p^{buy}, \tag{A.10}$$

$$\frac{\partial V(S^{+})}{\partial k} = \left[1 - \frac{\partial V(S^{+})}{\partial d}\right] p^{sell}, \tag{A.11}$$

$$\frac{\partial V(S^{*-})}{\partial u} = \theta + \frac{\partial V(S^{-})}{\partial u}, \tag{A.12}$$

$$\frac{\partial V(S^{*-})}{\partial u} = \theta + \frac{\partial V(S^{-})}{\partial u}, \tag{A.13}$$

$$\frac{\partial V(S^{*+})}{\partial d} = \frac{\partial V(S^{+})}{\partial d}. \tag{A.14}$$

$$\frac{\partial V(S^{*+})}{\partial d} = \frac{\partial V(S^{+})}{\partial d}. \tag{A.15}$$

For additional details about the sufficiency of HJB equations, value matching, optimality and smooth pasting conditions, see Oksendal (2007) and Baley and Blanco (2019).

Step 2: Guess of separable value function. We guess a separable value function:

$$V(S) = uv(\hat{k}) + t^d d, \tag{A.16}$$

where the function $v(\hat{k})$ is defined as

$$v(\hat{k}) \equiv \max_{\tau, \Delta\hat{k}} \mathbb{E}\left[\int_0^{\tau} (1-t^c)Ae^{-\tilde{r}s + \alpha\hat{k}_s}\, \mathrm{d}s + e^{-\tilde{r}\tau}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}_\tau + \Delta\hat{k}} - e^{\hat{k}_\tau}) + v(\hat{k}_\tau + \Delta\hat{k})\right)\right]. \tag{A.17}$$

and the new parameters are defined as

$$t^d \equiv \frac{t^c \xi^d}{\rho\frac{1-t^p}{1-t^g} + \xi^d}; \qquad \tilde{r} \equiv \rho\frac{1-t^p}{1-t^g} - \mu - \sigma^2/2; \qquad \nu \equiv \mu + \xi^k. \tag{A.18}$$

The value function $v(\hat{k})$ and the optimal policy $\{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}$ satisfy the following three conditions:

(i) In the interior of the inaction region, $v(\hat{k})$ solves the HJB equation:

$$\tilde{r}v(\hat{k}) \;=\; Ae^{\alpha\hat{k}} \;-\; \nu v'(\hat{k}) \;+\; \frac{\sigma^2}{2}v''(\hat{k}). \tag{A.19}$$

(ii) At the borders of the inaction region, $v(\hat{k})$ satisfies the value-matching conditions:

$$v(\hat{k}^-) \;=\; v(\hat{k}^{*-}) \;-\; \theta \;+\; (1-t^d)p^{buy}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}), \tag{A.20}$$

$$v(\hat{k}^+) \;=\; v(\hat{k}^{*+}) \;-\; \theta \;+\; (1-t^d)p^{sell}(e^{\hat{k}^+} - e^{\hat{k}^{*+}}). \tag{A.21}$$

(iii) At the borders of the inaction region and the two reset states, $v(\hat{k})$ satisfies the smooth-pasting and the optimality conditions:

$$v'(\hat{k}) \;=\; (1-t^d)p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \tag{A.22}$$

$$v'(\hat{k}) \;=\; (1-t^d)p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^+\right\}. \tag{A.23}$$

Step 3: Verify the guess. We verify that the guess in (A.16) and (A.17) is correct by showing the equivalence between the sufficient conditions of V, (A.5) to (A.15), and the sufficient conditions of v, (A.19) to (A.23). Given our guess, the following relationships between the derivatives of V and the derivatives of v hold:

$$\frac{\partial V(S)}{\partial u} \;=\; v(\hat{k}) - v'(\hat{k}), \tag{A.24}$$

$$\frac{\partial^2 V(S)}{\partial u^2} \;=\; -\frac{v'(\hat{k})}{u} + \frac{v''(\hat{k})}{u}, \tag{A.25}$$

$$\frac{\partial V(S)}{\partial k} \;=\; \frac{u}{k}v'(\hat{k}), \tag{A.26}$$

$$\frac{\partial V(S)}{\partial d} \;=\; t^d. \tag{A.27}$$

Verify HJB. We start with the HJB by substituting the guess into (A.5)

$$\begin{aligned}\rho\frac{1-t^p}{1-t^g}\left(uv(\hat{k}) + t^d d\right) \;=\;& u(1-t^c)Ae^{\alpha\hat{k}} + t^c\xi^d d - \xi^k k\frac{u}{k}v'(\hat{k}) - t^d\xi^d d \\ &+ \left(\mu + \frac{\sigma^2}{2}\right)u(v(\hat{k}) - v'(\hat{k})) + \frac{\sigma^2 u^2}{2}\left(\frac{v''(\hat{k})}{u} - \frac{v'(\hat{k})}{u}\right).\end{aligned}$$

Using the definition of t^d to cancel the term $t^c\xi^d d$ on both sides, and rearranging we obtain:

$$\left(\rho\frac{1-t^p}{1-t^g} - \mu - \frac{\sigma^2}{2}\right)uv(\hat{k}) \;=\; u(1-t^c)Ae^{\alpha\hat{k}} - (\mu+\xi^k)uv'(\hat{k}) + \frac{\sigma^2}{2}uv''(\hat{k}).$$

Using the new parameters $\nu \equiv \mu + \xi^k$ and $\tilde{r} \equiv \rho\frac{1-t^p}{1-t^g} - \mu - \sigma^2/2$, and cancelling u, we obtain the HJB in (A.19):

$$\tilde{r}v(\hat{k}) \;=\; (1-t^c)Ae^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}).$$

Verify value matching. We verify the value matching conditions by substituting the guess into (A.6) and (A.7).

$$\begin{aligned}uv(\hat{k}^{*-}) \;+\; t^d(d+\Delta k) \;-\; p^{buy}\Delta k \;-\; \theta u \;&=\; uv(\hat{k}^-) \;+\; t^d d,\\ uv(\hat{k}^{*+}) \;+\; t^d(d+\Delta k) \;-\; p^{sell}\Delta k \;-\; \theta u \;&=\; uv(\hat{k}^-) \;+\; t^d d.\end{aligned}$$

Writing positive investment as $\Delta k = u(e^{\Delta\hat{k}-\hat{k}^-} - e^{\hat{k}^-})$, negative investment as $\Delta k = u(e^{\Delta\hat{k}-\hat{k}^+} - e^{\hat{k}^+})$, cancelling

$t^d d$, dividing both sides by u, we obtain (A.20) and (A.21):[25]

$$\begin{aligned} v(\hat{k}^{*-}) - (1-t^d)p^{buy}(e^{\Delta\hat{k}-\hat{k}^-} - e^{\hat{k}^-}) - \theta &= v(\hat{k}^-), \\ v(\hat{k}^{*+}) - (1-t^d)p^{sell}(e^{\Delta\hat{k}-\hat{k}^+} - e^{\hat{k}^+}) - \theta &= v(\hat{k}^-). \end{aligned}$$

Verify optimality and smooth pasting. We verify the optimality conditions and the smooth pasting conditions for capital by substituting the guess into (A.8), (A.9), (A.10) and (A.11)

$$\begin{aligned} \frac{u}{k}v'(\hat{k}^{*-}) &= (1-t^d)p^{buy} \iff v'(\hat{k}^{*-}) &= e^{\hat{k}^{*-}}(1-t^d)p^{buy}, \\ \frac{u}{k}v'(\hat{k}^{*-}) &= (1-t^d)p^{sell} \iff v'(\hat{k}^{*-}) &= e^{\hat{k}^{*+}}(1-t^d)p^{sell}, \\ \frac{u}{k}v'(\hat{k}^{-}) &= (1-t^d)p^{buy} \iff v'(\hat{k}^{-}) &= e^{\hat{k}^{-}}(1-t^d)p^{buy}, \\ \frac{u}{k}v'(\hat{k}^{+}) &= (1-t^d)p^{sell} \iff v'(\hat{k}^{-}) &= e^{\hat{k}^{+}}(1-t^d)p^{sell}. \end{aligned}$$

which are equal to the expressions in (A.22) and (A.23). To verify the smooth-pasting for idiosyncratic productivity, we substitute the guess into (A.12) and (A.13) and then substitute (A.22) and (A.23) to rewrite $v'(\cdot)$ in terms of prices, which yields the conditions in (A.20) and (A.21):

$$\begin{aligned} v(\hat{k}^{*-}) &= \theta + v(\hat{k}^-) + (1-t^d)p^{buy}(e^{\hat{k}^{*-}-\hat{k}^-}) \\ v(\hat{k}^{*+}) &= \theta + v(\hat{k}^+) + (1-t^d)p^{sell}(e^{\hat{k}^{*+}-\hat{k}^+}) \end{aligned}$$

Finally, the smooth pasting conditions for deductions (A.14) and (A.15) are trivially satisfied given the linearity of the guess. We conclude that the guess is correct.

A.3 Cross-sectional distributions

Let $g(\hat{k})$ be the cross-sectional distribution of $\hat{k}$. It satisfies a Kolmogorov forward equation with its boundary conditions.

Without partial irreversibility.

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k}), \quad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^*\} \tag{A.28}$$

$$g(\hat{k}^\pm) = 0 \tag{A.29}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{A.30}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^*\}), \mathbb{C}^2(\{\hat{k}^*\}) \tag{A.31}$$

With partial irreversibility.

$$\Lambda(\hat{k})g(\hat{k}) = \nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k}), \quad \forall \hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\} \tag{A.32}$$

$$g(\hat{k}^\pm) = 0 \tag{A.33}$$

$$\int_{\hat{k}^-}^{\hat{k}^+} g(\hat{k})\,\mathrm{d}\hat{k} = 1, \tag{A.34}$$

$$g(\hat{k}) \in \mathbb{C}, \mathbb{C}^1(\{\hat{k}^{*-}, \hat{k}^{*+}\}), \mathbb{C}^2(\{\hat{k}^{*-}, \hat{k}^{*+}\}) \tag{A.35}$$

[25] To express investment in terms of $\hat{k}$, we start from the expression $\Delta\hat{k} = \log(1 + \Delta k/k_\tau)$, which yields $\Delta k = e^{\Delta\hat{k}}k_\tau - k_\tau$; multiply and divide by u_τ and substitute the definition of $\hat{k}$ to get: $\Delta k = u_\tau e^{\Delta\hat{k}+\hat{k}_\tau} - e^{\hat{k}_\tau}$. Then we use $\hat{k}_\tau = \hat{k}^+$ or $\hat{k}_\tau = \hat{k}^-$ accordingly.

A.4 Proof of Proposition 1

This proposition expresses the parameters of the stochastic process (ν, σ^2) as functions of the data. The proof follows the strategy in Baley and Blanco (2021), but taking into account the two reset points arising from partial irreversibility. Let $\hat{k}_s$ follow a Brownian motion with drift ν and volatility σ and two reset states $\hat{k}^{*-} < \hat{k}^{*+}$, where $\hat{k}^{*-} = \hat{k}^*(\Delta\hat{k} > 0)$ and $\hat{k}^{*+} = \hat{k}^*(\Delta\hat{k} < 0)$.

A.4.1 Drift

Evaluating the law of motion $\hat{k}_s = \hat{k}^*(\Delta\hat{k}) - \nu t + \sigma W_t$ at a stopping time $s = \tau$,

$$\hat{k}_\tau - \hat{k}^*(\Delta\hat{k}) + \nu\tau \;=\; \sigma W_\tau. \tag{A.36}$$

Since the stopped capital depends on the adjustment sign, we take conditional using the distribution of adjusters $H(\Delta\hat{k}, \tau)$:

$$\mathbb{E}\left[\hat{k}_\tau \middle| \Delta\hat{k}\right] - \hat{k}^*(\Delta\hat{k}) + \nu\mathbb{E}\left[\tau\middle|\Delta\hat{k}\right] \;=\; 0. \tag{A.37}$$

Taking expectation again with H and using the law of iterating expectation, we obtain

$$\underbrace{\overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}_\tau\middle|\Delta\hat{k}\right]\right]}_{\overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right]\;-\;\overline{\mathbb{E}}[\Delta\hat{k}]} \;-\; \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] \;+\; \nu\,\underbrace{\overline{\mathbb{E}}\left[\mathbb{E}\left[\tau\middle|\Delta\hat{k}\right]\right]}_{\mathbb{E}[\tau]} \;=\; 0 \tag{A.38}$$

To compute the first term, we substitute the relationship between stopped capital, reset state and adjustment size, $\hat{k}_\tau = \hat{k}^*(\Delta\hat{k}) - \Delta\hat{k}$, and then use the Markovian property of adjustments:

$$\overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}_\tau\middle|\Delta\hat{k}\right]\right] \;=\; \overline{\mathbb{E}}\left[\mathbb{E}\left[\hat{k}^*(\Delta\hat{k}')\middle|\Delta\hat{k}\right]\right] - \overline{\mathbb{E}}\left[\mathbb{E}\left[\Delta\hat{k}'\middle|\Delta\hat{k}\right]\right] \;=\; \overline{\mathbb{E}}\left[\hat{k}^*(\Delta\hat{k})\right] - \overline{\mathbb{E}}[\Delta\hat{k}] \tag{A.39}$$

To compute the second term, we use the law of iterated expectations and obtain:

$$\overline{\mathbb{E}}\left[\mathbb{E}\left[\tau'\middle|\Delta\hat{k}\right]\right] = \overline{\mathbb{E}}\left[\tau\right] \tag{A.40}$$

Substituting (A.39) and (A.40) into (A.38) and rearranging, we obtain the result

$$\nu \;=\; \frac{\overline{\mathbb{E}}[\Delta\hat{k}]}{\overline{\mathbb{E}}[\tau]}. \tag{A.41}$$

A.4.2 Idiosyncratic volatility

Let $Y_s \equiv (\hat{k}_s + \nu s)^2$. Applying Itō's lemma to Y_s,

$$\mathrm{d}Y_s \;=\; 2(\hat{k}_s + \nu s)(\mathrm{d}\hat{k}_s + \nu\,\mathrm{d}s) \;+\; (\mathrm{d}\hat{k}_s)^2 \;=\; 2(\hat{k}_s + \nu s)\sigma\,\mathrm{d}W_s + \sigma^2\,\mathrm{d}s \tag{A.42}$$

We integrate both sides from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0 = \hat{k}^*(\Delta\hat{k})$, which is a function of Δk. Then, we use the OST to set the martingale to zero, $\overline{\mathbb{E}}[\int_0^\tau(\hat{k}_s + \nu s)\,\mathrm{d}W_s] = 0$, and obtain:

$$\overline{\mathbb{E}}\left[Y_\tau(\Delta k) - Y_0\middle|\Delta\hat{k}\right] \;=\; 2\sigma\overline{\mathbb{E}}\left[\int_0^\tau(\hat{k}_s + \nu s)\,\mathrm{d}W_s\middle|\Delta k\right] + \sigma^2\overline{\mathbb{E}}\left[\int_0^\tau 1\,\mathrm{d}s\middle|\Delta k\right] \;=\; \sigma^2\overline{\mathbb{E}}[\tau|\Delta k] \tag{A.43}$$

Substituting $Y_\tau \equiv (\hat{k}_\tau(\Delta k) + \nu\tau)^2$ and $Y_0 \equiv \hat{k}^*(\Delta\hat{k})^2$, and taking expectations again to average across positive and negative adjustments, we get:

$$\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[(\hat{k}_\tau(\Delta\hat{k}') + \nu\tau')^2|\Delta\hat{k}\right]\right] - \mathbb{E}\left[\hat{k}^*(\Delta\hat{k}')^2\middle|\Delta k\right] \;=\; \sigma^2\overline{\mathbb{E}}\left[\overline{\mathbb{E}}\left[\tau'\middle|\Delta k\right]\right]. \tag{A.44}$$

By the Markovian property of Δk and τ, the previous expression simplifies to:

$$\overline{\mathbb{E}}[(\hat{k}_\tau(\Delta\hat{k})+\nu\tau)^2] - \overline{\mathbb{E}}[\hat{k}^*(\Delta\hat{k})^2] \;=\; \sigma^2\overline{\mathbb{E}}[\tau] \tag{A.45}$$

Rearranging, we obtain the mapping to σ^2:

$$\sigma^2 \;=\; \frac{\overline{\mathbb{E}}[(\hat{k}_\tau(\Delta\hat{k})+\nu\tau)^2] - \overline{\mathbb{E}}[\hat{k}^*(\Delta\hat{k})^2]}{\overline{\mathbb{E}}[\tau]}. \tag{A.46}$$

A.4.3 Reset states

To characterize the reset states as a function of the data, we first apply the envelope theorem.

Envelope theorem. Start from the recursive definition of $v(\hat{k})$ in (9) for an arbitrary initial condition $\hat{k}_0$ and evaluate it at the optimal policy $(\tau^*, \hat{k}^*)$

$$v(\hat{k}_0) \;=\; \mathbb{E}\left[\int_0^{\tau^*} Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; e^{-r\tau^*}\left(-\theta - p(\Delta\hat{k})(e^{\hat{k}^*} - e^{\hat{k}_{\tau^*}}) + v(\hat{k}^*)\right)\right] \tag{A.47}$$

Substitute the evolution of log capital-to-productivity ratios $\hat{k}_s = \hat{k}_0 - \nu s + \sigma W_s$ and take the derivative with respect to the initial state $\hat{k}_0$ (note that the terms $-\theta - p(\Delta\hat{k})e^{\hat{k}^*} + v(\hat{k}^*)$ are independent of $\hat{k}_0$) to get

$$\begin{aligned} v'(\hat{k}_0) &= \mathbb{E}\left[\int_0^{\tau^*} \frac{\mathrm{d}Ae^{-rs+\alpha(\hat{k}_0-\nu s+\sigma W_s)}}{\mathrm{d}\hat{k}_0}\,\mathrm{d}s \quad + p(\Delta\hat{k})e^{-r\tau^*}\frac{\mathrm{d}e^{\hat{k}_0-\nu\tau^*+\sigma W_{\tau^*}}}{\mathrm{d}\hat{k}_0}\right] \\ &= \mathbb{E}\left[\int_0^{\tau^*} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})e^{-r\tau^*+\hat{k}_{\tau^*}}\right] \end{aligned} \tag{A.48}$$

In the second line apply the envelope condition for arbitrary choice sets (Milgrom and Segal, 2002) to ignore the derivative of the optimal policies with respect to the initial condition.

Because there are two reset points, at this step we must condition on the appropriate initial condition to evaluate (A.48). If the last reset point is $\hat{k}_0 = \hat{k}^{*-}$ (there was a capital purchase), then we use the optimality condition $v'(\hat{k}^{*-}) = p^{buy}e^{\hat{k}^*}$ in the LHS to get:

$$p^{buy}e^{\hat{k}^*} \;=\; \mathbb{E}^-\left[\int_0^{\tau^*} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})e^{-r\tau^*+\hat{k}_{\tau^*}}\right]. \tag{A.49}$$

Analogously, if the last reset point is $\hat{k}_0 = \hat{k}^{*+}$ (there was a capital sale), then we use the optimality condition $v'(\hat{k}^{*+}) = p^{sell}e^{\hat{k}^*}$ in the LHS to get:

$$p^{sell}e^{\hat{k}^{*+}} \;=\; \mathbb{E}^+\left[\int_0^{\tau^*} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s \;+\; p(\Delta\hat{k})\;e^{-r\tau^*+\hat{k}_{\tau^*}}\right]. \tag{A.50}$$

Reset state without partial irreversibility. We depart from the optimality condition

$$pe^{\hat{k}^*} = \mathbb{E}\left[\int_0^{\tau} \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + pe^{-r\tau+\hat{k}_\tau}\Big|\hat{k}_0 = \hat{k}^*\right] \tag{A.51}$$

Now we write the RHS as a function of observable data. Define $Y_s = e^{-rs+\alpha\hat{k}_s}$ and apply Ito's lemma:

$$\mathrm{d}Y_s \;=\; Y_s\left(-r\,\mathrm{d}s + \alpha\,\mathrm{d}\hat{k}_s + \frac{\alpha^2}{2}\,\mathrm{d}\hat{k}_s^2\right) \;=\; \left(\frac{\alpha^2\sigma^2}{2} - r - \alpha\nu\right)Y_s\,\mathrm{d}s + \alpha\sigma Y_s\,\mathrm{d}B_s. \tag{A.52}$$

Let $\phi \equiv r + \alpha\nu - \alpha^2\sigma^2/2$. Integrating both sides from 0 to τ, taking expectations conditional on adjustment, i.e., with respect to the initial condition $\hat{k}_0 = \hat{k}^*$, and using the OST to set the expectation of martingales to zero, we obtain:

$$\text{(A.53)}\qquad \overline{\mathbb{E}}\left[\int_0^\tau \mathrm{d}Y_s\,\mathrm{d}s\right] \;=\; -\phi\overline{\mathbb{E}}\left[\int_0^\tau Y_s\,\mathrm{d}s\right] + \alpha\sigma\,\underbrace{\overline{\mathbb{E}}\left[\int_0^\tau Y_s\,\mathrm{d}W_s\right]}_{=0}.$$

Since $Y_\tau = e^{-r\tau+\alpha\hat{k}_\tau} = e^{-r\tau+\alpha(\hat{k}^*-\Delta\hat{k})}$ and $Y_0 = e^{\alpha\hat{k}^*}$

$$\text{(A.54)}\qquad \overline{\mathbb{E}}\left[Y_\tau - Y_0\ \right] \;=\; -\phi\overline{\mathbb{E}}\left[\int_0^\tau e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s\right] \iff e^{\alpha\hat{k}^*}\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{\phi} = \overline{\mathbb{E}}\left[\int_0^\tau Y_s\,\mathrm{d}s\right].$$

Since $\hat{k}_\tau = \hat{k}^* - \Delta\hat{k}$

$$\text{(A.55)}\qquad \mathbb{E}\left[pe^{-r\tau+\hat{k}_\tau}|\hat{k}_0=\hat{k}^*\right] = pe^{\hat{k}^*}\,\mathbb{E}\left[e^{-r\tau-\Delta\hat{k}}|\hat{k}_0=\hat{k}^*\right].$$

From equations (A.51) to (A.55),

$$\begin{aligned} pe^{\hat{k}^*} &= \mathbb{E}\left[\int_0^\tau \alpha Ae^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s + pe^{-r\tau+\hat{k}_\tau}|\hat{k}_0=\hat{k}^*\right] \\ \text{(A.56)}\qquad &= \alpha Ae^{\alpha\hat{k}^*}\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{\phi} + pe^{\hat{k}^*}\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right] \end{aligned}$$

Operating from the previous equation

$$\text{(A.57)}\qquad \hat{k}^* = \frac{1}{1-\alpha}\left[\log\left(\frac{\alpha A}{r+\alpha\nu-\alpha^2\sigma^2/2}\right) - \log(p) + \log\left(\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{1-\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right]}\right)\right]$$

Finally, letting $\Phi \equiv \log\left(\frac{\alpha A}{\phi}\right) = \log\left(\frac{\alpha A}{(r+\alpha\nu-\alpha^2\sigma^2/2)}\right)$, we obtain the result

$$\text{(A.58)}\qquad \hat{k}^* = \frac{1}{1-\alpha}\left[\Phi - \log(p) + \log\left(\frac{1-\overline{\mathbb{E}}[e^{-r\tau-\alpha\Delta\hat{k}}]}{1-\overline{\mathbb{E}}\left[e^{-r\tau-\Delta\hat{k}}\right]}\right)\right].$$

Reset states with partial irreversibility. We follow similar steps but taking into account that adjustments happen at two different reset points. Consider the optimality condition of a firm that has sold capital at price p^{sell} and resetting to $\hat{k}^{*+}$

$$\begin{aligned} \text{(A.59)}\qquad p^{\text{sell}}e^{\hat{k}^{*+}} &= \alpha A\overline{\mathbb{E}}^+\left[\int_0^\tau e^{-rs+\alpha\hat{k}_s}\,\mathrm{d}s\right] \;+\; \overline{\mathbb{E}}^+\left[p(\Delta\hat{k})e^{-r\tau+\hat{k}_\tau}\right] \\ p^{\text{sell}}e^{\hat{k}^{*+}} &= e^{\Phi}e^{\alpha\hat{k}^{*+}}\left(1-\overline{\mathbb{E}}^+[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]\right) + p^{\text{sell}}e^{\hat{k}^{*+}}\overline{\mathbb{E}}^+\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right] \\ e^{(1-\alpha)\hat{k}^{*+}} &= \frac{e^{\Phi}}{p^{\text{sell}}}\frac{1-\overline{\mathbb{E}}^+[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]}{1-\mathbb{E}^+\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]} \end{aligned}$$

Taking logs, we obtain the reset point for negative investments:

$$\text{(A.60)}\qquad \hat{k}^{*+} \;=\; \frac{1}{1-\alpha}\left[\Phi - \log(p^{\text{sell}}) + \log\left(\frac{1-\overline{\mathbb{E}}^+[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*+})}]}{1-\mathbb{E}^+\left[\frac{p(\Delta\hat{k})}{p^{\text{sell}}}e^{-r\tau+\hat{k}_\tau-\hat{k}^{*+}}\right]}\right)\right]$$

With analogous steps we obtain the reset point for positive investments:

$$\hat{k}^{*-} = \frac{1}{1-\alpha}\left[\Phi - \log(p^{\text{buy}}) + \frac{1-\overline{\mathbb{E}}^{-}\left[e^{-r\tau+\alpha(\hat{k}_\tau-\hat{k}^{*-})}\right]}{1-\mathbb{E}^{-}\left[\frac{p(\Delta\hat{k})}{p^{\text{buy}}}e^{-r\tau+(\hat{k}_\tau-\hat{k}^{*-})}\right]}\right] \tag{A.61}$$

A.5 Proof of Proposition 2

This proposition expresses the cross-sectional moments of $\hat{k}$ and joint moments of $(\hat{k}, a)$ as functions of the data. The proof follows the strategy in Baley and Blanco (2021), but taking into account the two reset points arising from partial irreversibility.

A.5.1 Moments of $\hat{k}$.

We apply Itō's lemma to $\hat{k}_s^n$ for $n \geq 2$:

$$\text{(A.62)} \qquad \mathrm{d}\hat{k}_s^{n+1} \;=\; -\nu(n+1)\hat{k}_s^n \,\mathrm{d}s \;+\; \sigma(n+1)\hat{k}_s^n \,\mathrm{d}s \;+\; \frac{\sigma^2 n(n+1)}{2}\hat{k}_s^{n-1}\,\mathrm{d}s.$$

We integrate this expression from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0 = \hat{k}^*$, which is a function of Δk:

$$\text{(A.63)} \qquad \overline{\mathbb{E}}^{\pm}\left[\hat{k}_\tau^{n+1} - (\hat{k}^*)^{n+1}\right] = -\nu(n+1)\overline{\mathbb{E}}^{\pm}\left[\int_0^\tau \hat{k}_s^n \,\mathrm{d}s\right] + \frac{\sigma^2 n(n+1)}{2}\overline{\mathbb{E}}^{\pm}\left[\int_0^\tau \hat{k}_s^{n-1}\,\mathrm{d}s\right].$$

Next, take law of iterated expectation to average $\overline{\mathbb{E}}^-$ and $\overline{\mathbb{E}}^+$, divide both sides by $\overline{\mathbb{E}}[\tau]$, and then use Auxiliary Theorem 2 to recover steady-state moments using the occupancy measure:

$$\text{(A.64)} \qquad \frac{\overline{\mathbb{E}}\left[\hat{k}_\tau^{n+1} - (\hat{k}^*)^{n+1}\right]}{\overline{\mathbb{E}}[\tau]} \;=\; -\nu(n+1)\frac{\overline{\mathbb{E}}\left[\int_0^\tau \hat{k}_s^n \,\mathrm{d}s\right]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2 n(n+1)}{2}\frac{\overline{\mathbb{E}}\left[\int_0^\tau \hat{k}_s^{n-1}\,\mathrm{d}s\right]}{\overline{\mathbb{E}}[\tau]}$$

$$\text{(A.65)} \qquad = -\nu(n+1)\mathbb{E}[\hat{k}^n] + \frac{\sigma^2 n(n+1)}{2}\mathbb{E}[\hat{k}^{n-1}].$$

Solving for $\mathbb{E}[\hat{k}^n]$ and substituting $\nu = \overline{\mathbb{E}}[\Delta\hat{k}]/\overline{\mathbb{E}}[\tau]$:

$$\text{(A.66)} \qquad \mathbb{E}[\hat{k}^n] \;=\; \frac{1}{n+1}\frac{\overline{\mathbb{E}}\left[(\hat{k}^*)^{n+1} - \hat{k}_\tau^{n+1}\right]}{\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2 n}{2\nu}\mathbb{E}[\hat{k}^{n-1}]$$

Applying similar steps as before, we compute the centered moments:

$$\text{(A.67)} \qquad \mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^n] \;=\; \frac{1}{n+1}\frac{\overline{\mathbb{E}}[(\hat{k}^* - \mathbb{E}[\hat{k}])^{n+1}] - \overline{\mathbb{E}}\left[(\hat{k}_\tau - \mathbb{E}[\hat{k}])^{n+1}\right]}{\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2 n}{2\nu}\mathbb{E}[(\hat{k} - \mathbb{E}[\hat{k}])^{n-1}].$$

Mean of $\hat{k}$. Evaluate expression (A.66) at $n = 1$ and use the formula for the drift (A.41) to obtain:

$$\text{(A.68)} \qquad \mathbb{E}[\hat{k}] \;=\; \frac{\overline{\mathbb{E}}\left[(\hat{k}^*)^2 - \hat{k}_\tau^2\right]}{2\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2}{2\nu}.$$

Without a price wedge (only fixed costs), there is a unique reset point $\hat{k}^*$. Factorize the quadratic difference $(\hat{k}^*)^2 - \hat{k}_\tau^2$ and substitute $\Delta\hat{k} = \hat{k}^* - \hat{k}_\tau$ to obtain:

$$\text{(A.69)} \qquad \mathbb{E}[\hat{k}] = \frac{\overline{\mathbb{E}}\left[(\hat{k}^* + \hat{k}_\tau)(\hat{k}^* - \hat{k}_\tau)\right]}{2\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\sigma^2}{2\nu} \;=\; \overline{\mathbb{E}}\left[\frac{\hat{k}^* + \hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\right] + \frac{\sigma^2}{2\nu}.$$

Conditional means:

$$\mathbb{E}^-[\hat{k}] = \overline{\mathbb{E}}^-\left[\frac{\hat{k}^{*-}+\hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}^-[\Delta\hat{k}]}\right)\right] + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^-[\tau]}{\overline{\mathbb{E}}^-[\Delta\hat{k}]} \tag{A.70}$$

$$\mathbb{E}^+[\hat{k}] = \overline{\mathbb{E}}^+\left[\frac{\hat{k}^{*+}+\hat{k}_\tau}{2}\left(\frac{\Delta k}{\overline{\mathbb{E}}^+[\Delta\hat{k}]}\right)\right] + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}^+[\tau]}{\overline{\mathbb{E}}^+[\Delta\hat{k}]} \tag{A.71}$$

Variance of $\hat{k}$. To compute the variance of $\hat{k}$, we evaluate expression for centered moments (A.67) at $n=2$:

$$\mathbb{V}ar[\hat{k}] = \frac{\overline{\mathbb{E}}[(\hat{k}^*-\mathbb{E}[\hat{k}])^3] - \overline{\mathbb{E}}\left[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3\right]}{3\overline{\mathbb{E}}[\Delta\hat{k}]} \tag{A.72}$$

Without a price wedge (only fixed costs), there is a unique reset point $\hat{k}^*$ and we can simplify the previous expression. Inside the first term, add and subtract $\hat{k}_\tau$ and substitute $\Delta\hat{k} = \hat{k}^* - \hat{k}_\tau$. Expand the cubic polynomials, cancel terms, and rearrange:

$$\begin{aligned}
\mathbb{V}ar[\hat{k}] &= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}]+\Delta\hat{k})^3-(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]} \\
&= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3+3(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\Delta\hat{k}+3(\hat{k}_\tau-\mathbb{E}[\hat{k}])\Delta\hat{k}^2+\Delta\hat{k}^3-(\hat{k}_\tau-\mathbb{E}[\hat{k}])^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]} \\
&= \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\Delta\hat{k}+(\hat{k}_\tau-\mathbb{E}[\hat{k}])\Delta\hat{k}^2]}{\overline{\mathbb{E}}[\Delta\hat{k}]} + \frac{\overline{\mathbb{E}}[\Delta\hat{k}^3]}{3\overline{\mathbb{E}}[\Delta\hat{k}]} \\
&= \frac{\overline{\mathbb{E}}[\Delta\hat{k}(\hat{k}_\tau-\mathbb{E}[\hat{k}])(\hat{k}^*-\mathbb{E}[\hat{k}])+\Delta\hat{k}(\hat{k}^*-\hat{k}_\tau)^2/3]}{\overline{\mathbb{E}}[\Delta\hat{k}]} \\
&= \overline{\mathbb{E}}\left[\left((\hat{k}_\tau-\mathbb{E}[\hat{k}])(\hat{k}^*-\mathbb{E}[\hat{k}])+\frac{(\hat{k}^*-\hat{k}_\tau)^2}{3}\right)\left(\frac{\Delta\hat{k}}{\overline{\mathbb{E}}[\Delta\hat{k}]}\right)\right].
\end{aligned}$$

A.5.2 Joint moments of $(\hat{k}, a)$.

To compute the joint moments of $\hat{k}$ and its age a, we consider the function $Y_s = (\hat{k}_s - \mathbb{E}[\hat{k}])^{n+1}s$. Applying Itō's lemma, we obtain:

$$\begin{aligned}
\mathrm{d}Y_s &= (\hat{k}_s-\mathbb{E}[\hat{k}])^n\,\mathrm{d}s - \nu n(\hat{k}_s-\mathbb{E}[\hat{k}])^{n-1}s\,\mathrm{d}s + \sigma n(\hat{k}_s-\mathbb{E}[\hat{k}])^{n-1}s\,\mathrm{d}W_s \\
&\quad + \frac{\sigma^2}{2}n(n-1)(\hat{k}_s-\mathbb{E}[\hat{k}])^{n-2}s\,\mathrm{d}s.
\end{aligned} \tag{A.73}$$

We integrate this expression from 0 to τ and take conditional expectations with respect to the initial condition $\hat{k}_0 = \hat{k}^*$, which is a function of Δk.

$$\frac{\overline{\mathbb{E}}\left[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^{n+1}\tau\right]}{\overline{\mathbb{E}}[\tau]} = \mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^{n+1}] - \nu(n+1)\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^n a] + \frac{\sigma^2}{2}n(n+1)\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}a] \tag{A.74}$$

Rearranging:

$$\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^n a] = \frac{1}{\nu(n+1)}\left[\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^{n+1}] - \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^{n+1}\tau]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2}{2}n\mathbb{E}[(\hat{k}-\mathbb{E}[\hat{k}])^{n-1}a]\right] \tag{A.75}$$

Finally, to compute the covariance between $(\hat{k}, a)$, we evaluate expression (A.75) at $n=1$ to obtain

$$\mathbb{C}ov[\hat{k}, a] = \frac{1}{2\nu}\left(\mathbb{V}ar[\hat{k}] - \frac{\overline{\mathbb{E}}[(\hat{k}_\tau-\mathbb{E}[\hat{k}])^2\tau]}{\overline{\mathbb{E}}[\tau]} + \frac{\sigma^2}{2}\frac{\overline{\mathbb{E}}[\tau]}{2}(1+\mathbb{CV}^2[\tau])\right) \tag{A.76}$$

A.6 Proof of Proposition 3

Proposition 3 characterizes the aggregate Tobin's q as a function of the aggregate output-to-capital ratio $\hat{Y}/\hat{K}$ and the expected capital gains. Let us depart from the definition of aggregate Tobin's q in (38)

$$q \equiv \frac{1}{p}\int_{\hat{k}^-}^{\hat{k}^+} \omega(\hat{k})\frac{v'(\hat{k})}{e^{\hat{k}}} \, \mathrm{d}G(\hat{k}) = \frac{\mathbb{E}[v'(\hat{k})]}{p\hat{K}}. \tag{A.77}$$

We characterize the numerator,

$$\mathbb{E}[v'(\hat{k})] = \int_{\hat{k}^-}^{\hat{k}^+} v'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}, \tag{A.78}$$

by combining the HJB equation for $v'(\hat{k})$ and the KFE for $g(\hat{k})$ into a single "master equation".

A.6.1 Without partial irreversibility

The value $v'(k)$ satisfies the following conditions:

$$rv'(\hat{k}) = \alpha A e^{\alpha\hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2}v'''(\hat{k}) + \Lambda(\hat{k})\left[pe^{\hat{k}} - v'(\hat{k})\right], \forall \hat{k}\in(\hat{k}^-,\hat{k}^+) \tag{A.79}$$

$$v'(\hat{k}) = pe^{\hat{k}}, \quad \text{for} \quad \hat{k}\in\{\hat{k}^-,\hat{k}^*,\hat{k}^+\}, \tag{A.80}$$

$$v'(\hat{k}) \in \mathbb{C},\mathbb{C}^1 \tag{A.81}$$

and $g(\hat{k})$ satisfies a KFE and boundary conditions in (A.28) to (A.31). From equation (A.28), we solve for the adjustment hazard $\Lambda(\hat{k}) = (\nu g'(\hat{k}) + (\sigma^2/2)g''(\hat{k})/g(\hat{k})$; then substitute the hazard into (A.79), multiply both sides by $g(\hat{k})$, and join common terms, to obtain the Master equation that is valid for all $\hat{k}\in(\hat{k}^-,\hat{k}^+)/\{\hat{k}^*\}$:

$$\begin{aligned} rv'(\hat{k})g(\hat{k}) &= \alpha A e^{\alpha\hat{k}}g(\hat{k}) - \nu v''(\hat{k})g(\hat{k}) + \frac{\sigma^2}{2}v'''(\hat{k})g(\hat{k}) + \left[\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right]\left[pe^{\hat{k}} - v'(\hat{k})\right] \\ &= \alpha A e^{\alpha\hat{k}}g(\hat{k}) - \nu\left(v'(\hat{k})g(\hat{k})\right)' + \frac{\sigma^2}{2}\left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)' + pe^{\hat{k}}\left[\nu g'(\hat{k}) + \frac{\sigma^2}{2}g''(\hat{k})\right]. \end{aligned} \tag{A.82}$$

Next, we integrate both sides in the interval $[\hat{k}^-,\hat{k}^+]$.

$$\begin{aligned} \underbrace{r\int_{\hat{k}^-}^{\hat{k}^+} v'(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}}_{qp\hat{K}} &= \alpha A\underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\alpha\hat{k}}g(\hat{k})\,\mathrm{d}\hat{k}}_{\hat{Y}} - \nu\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}\left(v'(\hat{k})g(\hat{k})\right)'\mathrm{d}\hat{k}}_{T_1} \\ &+ \frac{\sigma^2}{2}\underbrace{\int_{\hat{k}^-}^{\hat{k}^+}\left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)'\mathrm{d}\hat{k}}_{T_2} + p\left[\nu\underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k}}_{T_3} + \frac{\sigma^2}{2}\underbrace{\int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}}g''(\hat{k})\,\mathrm{d}\hat{k}}_{T_4}\right]. \end{aligned} \tag{A.83}$$

In the LHS, we recognize aggregate q. In the first term of the RHS, we recognize the average output-to-productivity ratio in (41). The remaining terms, labelled T_1, T_2, T_3 and T_4, are computed following a common strategy: split the domain into two regions, $[\hat{k}^-,\hat{k}^+] = [\hat{k}^-,\hat{k}^*]\cup[\hat{k}^*,\hat{k}^{*+}]$; integrate by parts if needed; use the border conditions

$g(\hat{k}^{\pm}) = 0$ in (A.29); and use continuity conditions for g in (A.31) and for v' in (A.81).

$$T_1 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} \Big(v'(\hat{k})g(\hat{k})\Big)' \,\mathrm{d}\hat{k} \;=\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{v'(\hat{k}^*)g(\hat{k}^*)} \;+\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-v'(\hat{k}^*)g(\hat{k}^*)} \;=\; 0. \tag{A.84}$$

$$T_3 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g'(\hat{k}) \,\mathrm{d}\hat{k} \;=\; \underbrace{e^{\hat{k}} g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{e^{\hat{k}^*} g(\hat{k}^*)} \;+\; \underbrace{e^{\hat{k}} g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-e^{\hat{k}^*} g(\hat{k}^*)} - \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g(\hat{k}) \,\mathrm{d}\hat{k} = -\hat{K}. \tag{A.85}$$

$$T_4 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}} g''(\hat{k}) \,\mathrm{d}\hat{k} \;=\; e^{\hat{k}} g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} \;+\; e^{\hat{k}} g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+} \;-\; T_3. \tag{A.86}$$

$$\begin{aligned} T_2 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \Big(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\Big)' \,\mathrm{d}\hat{k} \;=\; \underbrace{v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*}}_{v''(\hat{k}^*)g(\hat{k}^*)} \;+\; \underbrace{v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}}_{-v''(\hat{k}^*)g(\hat{k}^*)} \\ &-\; \left[v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^*} \;+\; v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^*}^{\hat{k}^+}\right] \\ &=\; -p\left[e^{\hat{k}^*} g'(\hat{k}^*) \;-e^{\hat{k}^-} g'(\hat{k}^-) \;+\; e^{\hat{k}^+} g'(\hat{k}^+) \;-e^{\hat{k}^*} g'(\hat{k}^*)\right] \;=\; -p(T_3 \;+\; T_4). \end{aligned} \tag{A.87}$$

Substituting equations (A.84) to (A.86) into (A.83):

$$rqp\hat{K} \;=\; \alpha A\hat{Y} \;-\; \frac{\sigma^2}{2}p(T_3 + T_4) \;+\; p\left[\nu T_3 \;+\; \frac{\sigma^2}{2}T_4\right] \;=\; \alpha A\hat{Y} \;+\; p\hat{K}\left(\frac{\sigma^2}{2} - \nu\right).$$

Dividing both sides by $p\hat{K}$ we obtain the result:

$$q \;=\; \frac{1}{r}\left[\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \frac{\sigma^2}{2} - \nu\right]. \tag{A.88}$$

A.6.2 With partial irreversibility

The proof follows the same steps as without partial irreversibility, but using the auxiliary pricing function $\mathcal{P}(\hat{k})$.

Auxiliary pricing function $\mathcal{P}(\hat{k})$. We propose the following auxiliary price-deviation function $\mathcal{P}(\hat{k}) \in \mathbb{C}^2$ that satisfies all the requirements and is a member of the polynomial family:

$$\mathcal{P}(\hat{k}) \;\equiv\; \begin{cases} p^{\text{buy}}/p - 1 & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}] \\ \frac{(p^{\text{sell}} - p^{\text{buy}})}{p} \sum_{i=0}^{5} \mathcal{P}_i \left(\frac{\hat{k} - \hat{k}^{*-}}{\hat{k}^{*+} - \hat{k}^{*-}}\right)^i & \text{if } \hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+}) \\ p^{\text{sell}}/p - 1 & \text{if } \hat{k} \in [\hat{k}^{*+}, \hat{k}^+] \end{cases} \tag{A.89}$$

with coefficients $\mathcal{P}_0 = \frac{p^{\text{buy}} - p}{p^{\text{sell}} - p^{\text{buy}}}, \mathcal{P}_1 = \mathcal{P}_2 = 0, \mathcal{P}_3 = 10, \mathcal{P}_4 = -15$, and $\mathcal{P}_5 = 6$.

The function is twice continuously differentiable in the domain $(\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\}$. We need to check that it is twice continuously differentiable at the two reset points $\{\hat{k}^{*-}, \hat{k}^{*+}\}$. We need to find the solution of $\mathcal{P}_i$ that satisfies the following conditions:

- At the lower reset point, $\hat{k}^{*-}$, the function must satisfy:

$$\mathcal{P}(\hat{k}^{*-}) \;=\; \sum_{i=0}^{5} \mathcal{P}_i(\hat{k}^{*-} - \hat{k}^{*-})^i \;=\; p^{\text{buy}}/p - 1, \tag{A.90}$$

$$\mathcal{P}'(\hat{k}^{*-}) \;=\; \sum_{i=1}^{5} \mathcal{P}_i i(\hat{k}^{*-} - \hat{k}^{*-})^{i-1} \;=\; 0, \tag{A.91}$$

$$\mathcal{P}''(\hat{k}^{*-}) \;=\; \sum_{i=2}^{5} \mathcal{P}_i i(i-1)(\hat{k}^{*-} - \hat{k}^{*-})^{i-2} \;=\; 0 \tag{A.92}$$

From the first three equations, we confirm the values for $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}_2$ that make $\mathcal{P}(\hat{k})$ twice continuously differentiable at $\hat{k}^{*-}$:

$$\mathcal{P}_0 = p^{\text{buy}}, \quad 1 \times \mathcal{P}_1 = 0, \quad 2 \times 1 \times \mathcal{P}_2 = 0. \tag{A.93}$$

- At the upper reset point, $\hat{k}^{*+}$, the function must satisfy:

$$\mathcal{P}(\hat{k}^{*+}) \;=\; \sum_{i=0}^{5} \mathcal{P}_i(\hat{k}^{*+} - \hat{k}^{*-})^i \;=\; p^{\text{sell}}/p - 1, \tag{A.94}$$

$$\mathcal{P}'(\hat{k}^{*+}) \;=\; \sum_{i=1}^{5} \mathcal{P}_i i(\hat{k}^{*+} - \hat{k}^{*-})^{i-1} \;=\; 0, \tag{A.95}$$

$$\mathcal{P}''(\hat{k}^{*+}) \;=\; \sum_{i=2}^{5} \mathcal{P}_i i(i-1)(\hat{k}^{*+} - \hat{k}^{*-})^{i-2} \;=\; 0. \tag{A.96}$$

To simplify the notation, we define the irreversibility wedge ω as the length of the inner inaction region, i.e., $\omega \equiv \hat{k}^{*+} - \hat{k}^{*-} > 0$. With this definition, we can write the last three equations as

$$b_3\omega^3 + b_4\omega^4 + b_5\omega^5 = (p^{\text{sell}} - p^{\text{buy}})/p \tag{A.97}$$

$$b_3 3\omega^2 + b_4 4\omega^3 + b_5 5\omega^4 = 0 \tag{A.98}$$

$$b_3 6\omega^1 + b_4 12\omega^2 + b_5 20\omega^3 = 0 \tag{A.99}$$

and we can write it with matrices as

$$\begin{bmatrix} \omega^3 & \omega^4 & \omega^5 \\ 3\omega^2 & 4\omega^3 & 5\omega^4 \\ 6\omega^1 & 12\omega^2 & 20\omega^3 \end{bmatrix} \begin{bmatrix} b_3 \\ b_4 \\ b_5 \end{bmatrix} = \frac{p^{\text{sell}} - p^{\text{buy}}}{p} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{A.100}$$

If we take the first rows and multiply by $-31/\omega$ and sum to the second row, and we take the first rows and multiply by $-61/\omega^2$ and sum to the second row, we have

$$\begin{bmatrix} \omega^3 & \omega^4 & \omega^5 \\ 0 & \omega^3 & 2\omega^4 \\ 0 & 6\omega^2 & 14\omega^3 \end{bmatrix} \begin{bmatrix} b_3 \\ b_4 \\ b_5 \end{bmatrix} = \frac{p^{\text{sell}} - p^{\text{buy}}}{p} \begin{bmatrix} 1 \\ -3/\omega \\ -6/\omega^2 \end{bmatrix}. \tag{A.101}$$

The solution to this system is given by

$$b_4 \;=\; \frac{-3\frac{p^{\text{sell}}-p^{\text{buy}}}{\omega}14\omega^3 + 6\frac{p^{\text{sell}}-p^{\text{buy}}}{\omega^2}2\omega^4}{\omega^6[14-12]} = \left(p^{\text{sell}} - p^{\text{buy}}\right)\frac{-3\times 14 + 12}{\omega^4 2} = \frac{-15}{\omega^4}\left(p^{\text{sell}} - p^{\text{buy}}\right) \tag{A.102}$$

$$b_5 \;=\; \frac{-6\frac{p^{\text{sell}}-p^{\text{buy}}}{\omega^2}\omega^3 + 3\frac{p^{\text{sell}}-p^{\text{buy}}}{\omega}6\omega^2}{\omega^6[14-12]} = (p^{\text{sell}} - p^{\text{buy}})\frac{-6+18}{\omega^5 2} = \frac{6}{\omega^5}\left(p^{\text{sell}} - p^{\text{buy}}\right) \tag{A.103}$$

$$b_3 \;=\; \frac{p^{\text{sell}} - p^{\text{buy}}}{\omega^3} - b_4\omega - b_5\omega^2 = \frac{p^{\text{sell}} - p^{\text{buy}}}{\omega^3}[1 + 15 - 6] = \frac{10}{\omega^3}\left(p^{\text{sell}} - p^{\text{buy}}\right). \tag{A.104}$$

The auxiliary pricing function $\mathcal{P}(\hat{k})$ satisfies the following conditions:

1. $\mathcal{P}(\hat{k}) \in \mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2$.
2. $\mathcal{P}(\hat{k})$ is decreasing for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$.
3. $\mathcal{P}(\hat{k})$ is concave then convex for all $\hat{k} \in (\hat{k}^{*-}, \hat{k}^{*+})$.

To show the other two properties, we simplify notation and let $x \equiv \frac{\hat{k}-\hat{k}^{*-}}{\hat{k}^{*+}-\hat{k}^{*-}}$. We find the signs of the first and second derivatives:

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{P}(\hat{k})}{\mathrm{d}\hat{k}} &\propto (p^{\text{sell}} - p^{\text{buy}})(10 \times 3 \times x^2 - 15 \times 4 \times x^3 + 6 \times 5 \times x^4) \\
&\propto -30x^2 + 60x - 30 \\
&= -x^2 + 2x - 1 = (x-1)(-x+1) < 0 \ \ \forall x
\end{aligned} \tag{A.105}$$

$$\begin{aligned}
\frac{\mathrm{d}^2\mathcal{P}(\hat{k})}{\mathrm{d}\hat{k}^2} &\propto (p^{\text{sell}} - p^{\text{buy}})(10 \times 3 \times 2 \times x - 15 \times 4 \times 3 \times x^2 + 6 \times 5 \times 4 \times x^3) \\
&\propto -30 \times 4x^2 + 60 \times 3x - 30 \times 2 \\
&\propto -2x^2 + 3x - 1 \\
&= -2(x-1)(x-1/2) = \begin{cases} < 0 & \text{if } x < 1/2 \\ \geq 0 & \text{if } x \geq 1/2 \end{cases}
\end{aligned} \tag{A.106}$$

Characterization of q with irreversibility. Inside the inaction region, $\hat{k} \in (\hat{k}^-, \hat{k}^+)$, the value $v'(k)$ satisfies the following conditions:

$$rv'(\hat{k}) = \alpha A e^{\alpha \hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2} v'''(\hat{k}) + \begin{cases} \Lambda(\hat{k}) \left[p^{\text{buy}} e^{\hat{k}} - v'(\hat{k}) \right] & \text{if } \hat{k} \in [\hat{k}^-, \hat{k}^{*-}) \\ 0 & \text{if } \hat{k} \in [\hat{k}^{*-}, \hat{k}^{*+}] \\ \Lambda(\hat{k}) \left[p^{\text{sell}} e^{\hat{k}} - v'(\hat{k}) \right] & \text{if } \hat{k} \in (\hat{k}^{*+}, \hat{k}^+] \end{cases} \tag{A.107}$$

$$v'(\hat{k}) = p^{\text{buy}} e^{\hat{k}}, \quad \text{for } \hat{k} \in \{\hat{k}^-, \hat{k}^{*-}\}, \tag{A.108}$$

$$v'(\hat{k}) = p^{\text{sell}} e^{\hat{k}}, \quad \text{for } \hat{k} \in \{\hat{k}^{*+}, \hat{k}^+\}, \tag{A.109}$$

$$v'(\hat{k}) \in \mathbb{C}, \mathbb{C}^1. \tag{A.110}$$

Since $\Lambda(\hat{k}) = 0$ between the two reset points, $[\hat{k}^{*-}, \hat{k}^{*+}]$, the HBJ can be written compactly as

$$rv'(\hat{k}) = \alpha A e^{\alpha \hat{k}} - \nu v''(\hat{k}) + \frac{\sigma^2}{2} v'''(\hat{k}) + \Lambda(\hat{k}) \left[\mathcal{P}(\hat{k}) e^{\hat{k}} - v'(\hat{k}) \right], \ \forall \hat{k} \in (\hat{k}^-, \hat{k}^+) \tag{A.111}$$

$$v'(\hat{k}) = \mathcal{P}(\hat{k}) e^{\hat{k}}, \quad \text{for } \hat{k} \in \{\hat{k}^-, \hat{k}^{*-}, \hat{k}^{*+}, \hat{k}^+\}. \tag{A.112}$$

where $\mathcal{P}(\hat{k})$ is defined in (A.89). In turn, the $g(\hat{k})$ satisfies the KFE and boundary conditions in (A.32) to (A.35). From equation (A.32), we solve for the adjustment hazard $\Lambda(\hat{k})$, substitute it into (A.111), and obtain the Master Equation that is valid for all $\hat{k} \in (\hat{k}^-, \hat{k}^+)/\{\hat{k}^{*-}, \hat{k}^{*+}\}$. Integrating:

$$\begin{aligned}
rq\hat{K} = \alpha A \hat{Y} &- \nu \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \left(v'(\hat{k}) g(\hat{k}) \right)' \mathrm{d}\hat{k}}_{T_1} + \frac{\sigma^2}{2} \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \left(v''(\hat{k}) g(\hat{k}) - v'(\hat{k}) g'(\hat{k}) \right)' \mathrm{d}\hat{k}}_{T_2} \\
&+ \nu \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \mathcal{P}(\hat{k}) e^{\hat{k}} g'(\hat{k}) \, \mathrm{d}\hat{k}}_{T_3} + \frac{\sigma^2}{2} \underbrace{\int_{\hat{k}^-}^{\hat{k}^+} \mathcal{P}(\hat{k}) e^{\hat{k}} g''(\hat{k}) \, \mathrm{d}\hat{k}}_{T_4}
\end{aligned} \tag{A.113}$$

In this case, to compute the terms T_1, T_2, T_3, T_4, we must split the domain into three regions, $[\hat{k}^-, \hat{k}^+] = [\hat{k}^-, \hat{k}^{*-}) \cup [\hat{k}^{*-}, \hat{k}^{*+}] \cup (\hat{k}^*, \hat{k}^{*+}]$; integrate by parts when needed; use the border conditions $g(\hat{k}^{\pm}) = 0$ in (A.33); and use continuity conditions for g in (A.35) and for v' in (A.110).

(A.114)

$$T_1 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} \left(v'(\hat{k})g(\hat{k})\right)' \mathrm{d}\hat{k} \;=\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}}}_{v'(\hat{k}^{*-})g(\hat{k}^{*-})} \;+\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}}}_{v'(\hat{k}^{*+})g(\hat{k}^{*+})-v'(\hat{k}^{*-})g(\hat{k}^{*-})} \;+\; \underbrace{v'(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{-v'(\hat{k}^{*+})g(\hat{k}^{*+})} \;=\; 0.$$

(A.115)

$$\begin{aligned} T_2 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \left(v''(\hat{k})g(\hat{k}) - v'(\hat{k})g'(\hat{k})\right)' \mathrm{d}\hat{k} \\ &=\; \underbrace{\left[v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v''(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right]}_{=\,0} - \left[v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v'(\hat{k})g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] \\ &=\; -\left[\mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}\right] \end{aligned}$$

(A.116)

$$\begin{aligned} T_3 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} \;=\; \underbrace{\mathcal{P}(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} \;+\; \mathcal{P}(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} \;+\; \mathcal{P}(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=\,0} \\ &\quad - \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left[\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right] g(\hat{k})\,\mathrm{d}\hat{k} \;=\; -\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right)\right]. \end{aligned}$$

(A.117)

$$\begin{aligned} T_4 \;&=\; \int_{\hat{k}^-}^{\hat{k}^+} \mathcal{P}(\hat{k})e^{\hat{k}}g''(\hat{k})\,\mathrm{d}\hat{k} \;=\; \underbrace{\mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} \;+\; \mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} \;+\; \mathcal{P}(\hat{k})e^{\hat{k}}g'(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{T_2} \\ &\quad - \left[\int_{\hat{k}^{*+}}^{\hat{k}^+} \left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right) e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} \;+\; \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right) e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} \;+\; \int_{\hat{k}^-}^{\hat{k}^{*-}} e^{\hat{k}}\left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right) g'(\hat{k})\,\mathrm{d}\hat{k}\right] \\ &=\; -T_2 - T_3 - \int_{\hat{k}^-}^{\hat{k}^+} \mathcal{P}'(\hat{k})e^{\hat{k}}g'(\hat{k})\,\mathrm{d}\hat{k} \\ &=\; -T_2 \;-\; T_3 - \left[\underbrace{\mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \mathcal{P}'(\hat{k})e^{\hat{k}}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=\,0} - \int_{\hat{k}^-}^{\hat{k}^+} e^{\hat{k}}\left[\mathcal{P}'(\hat{k}) + \mathcal{P}''(\hat{k})g(\hat{k})\right]\mathrm{d}\hat{k}\right] \\ &=\; -T_2 \;-\; T_3 \;+\; \mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k}) + \mathcal{P}''(\hat{k})\right)\right] \end{aligned}$$

Substituting (A.114) to (A.117) into (A.113)

(A.118)

$$\begin{aligned} rq\hat{K} \;&=\; \alpha A\hat{Y} \;+\; \nu T_3 \;+\; \frac{\sigma^2}{2}(T_2 \;+\; T_4) \\ &=\; \alpha A\hat{Y} \;+\; \nu T_3 \;-\; \frac{\sigma^2}{2}\left(T_3 \;-\; \mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k}) + \mathcal{P}''(\hat{k})\right)\right]\right) \\ &=\; \alpha A\hat{Y} \;+\; \left(\frac{\sigma^2}{2} - \nu\right)\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right)\right] \;+\; \frac{\sigma^2}{2}\mathbb{E}\left[e^{\hat{k}}\left(\mathcal{P}'(\hat{k}) + \mathcal{P}''(\hat{k})\right)\right] \end{aligned}$$

Dividing both sides by $\hat{K} = \mathbb{E}[e^{\hat{k}}]$ and defining the weights $\omega(\hat{k}) = e^{\hat{k}}/\hat{K}$ we obtain the result:

$$q \;=\; \frac{1}{r}\left[\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \left(\frac{\sigma^2}{2} - \nu\right)\mathbb{E}\left[\omega(\hat{k})\left(\mathcal{P}(\hat{k}) + \mathcal{P}'(\hat{k})\right)\right] \;+\; \frac{\sigma^2}{2}\mathbb{E}\left[\omega(\hat{k})\left(\mathcal{P}'(\hat{k}) + \mathcal{P}''(\hat{k})\right)\right]\right] \tag{A.119}$$

A.6.3 Aggregate productivity

Finally, we approximate the aggregate productivity term. Using a second-order Taylor expansion, we obtain:

$$\frac{\hat{Y}}{\hat{K}} \;=\; \frac{\mathbb{E}[e^{\alpha\hat{k}}]}{\mathbb{E}[e^{\hat{k}}]} \;=\; \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}] + \frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \;+\; o(2). \tag{A.120}$$

A.7 Proof of Proposition 4

Proposition 4 characterizes the CIR as a function of cross-sectional steady-state moments. Let $g(\hat{k})$ be the capital-productivity steady-state distribution and $g_t(\hat{k})$ the distribution t-periods after an aggregate productivity shock of size $\delta > 0$, with $g_0(\hat{k}) = g(\hat{k} - \delta)$. Let $f(\hat{k})$ be a continuous function of $\hat{k}$. Define the cumulative impulse response of the function f as follows:

$$\text{CIR}(f, \delta) \equiv \int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \; \left[g_s(\hat{k}) - g(\hat{k})\right] \mathrm{d}\hat{k} \, \mathrm{d}s. \tag{A.121}$$

Note that in the main text we take $f(\hat{k}) = \hat{k}$.

Strategy. The proof has four steps. Step 1 we express the CIR as the integral of a value function $m(\hat{k})$ and $g'(\hat{k})$. Step 2 characterizes the terminal value of the value function. Step 3 constructs the master equation. Step 4 characterizes the CIR as a function of steady-state moments.

First order approximation. Start from the definition fo the CIR, (1) operates over the integral; (2) uses conditional expectation, where $g_s(\hat{k}|\hat{k}_0)d\hat{k}$ is the probability of the state $\hat{k}$ at time s with initial condition $\hat{k}_0$; (3) uses the definition of the initial condition; (4) and (5) apply Fubini's theorem and the definition of limit of an integral; (7) applies a Taylor approximation over δ.

$$\begin{aligned}
\text{CIR}(f, \delta) &= \int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} f(\hat{k}) \left(g_s(\hat{k}) - g(\hat{k})\right) \mathrm{d}\hat{k} \, \mathrm{d}s \\
&=^{(1)} \int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}) \, \mathrm{d}\hat{k} \, \mathrm{d}s \\
&=^{(2)} \int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g_0(\hat{k}_0) \, \mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k} \, \mathrm{d}s \\
&=^{(3)} \int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) g(\hat{k}_0 - \delta) \, \mathrm{d}\hat{k}_0\right] \mathrm{d}\hat{k} \, \mathrm{d}s \\
&=^{(4)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\int_0^{\infty} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) \, \mathrm{d}\hat{k} \, \mathrm{d}s\right] g(\hat{k}_0 - \delta) \, \mathrm{d}\hat{k}_0 \\
&=^{(5)} \int_{\hat{k}^-}^{\hat{k}^+} \left[\lim_{\mathcal{T} \to \infty} \underbrace{\int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) \, \mathrm{d}\hat{k} \, \mathrm{d}s}_{\equiv \, m_{\mathcal{T}}(\hat{k})}\right] g(\hat{k}_0 - \delta) \, \mathrm{d}\hat{k}_0 \\
&=^{(6)} \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T} \to \infty} m_{\mathcal{T}}(\hat{k}) g(\hat{k} - \delta) \, \mathrm{d}\hat{k} \\
&=^{(7)} -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T} \to \infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k}) \, \mathrm{d}\hat{k} + o(\delta^2)
\end{aligned}$$

where we define

$$m_{\mathcal{T}}(\hat{k}) \equiv \int_0^{\mathcal{T}} \int_{\hat{k}^-}^{\hat{k}^+} \left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right) g_s(\hat{k}|\hat{k}_0) \, \mathrm{d}\hat{k} \, \mathrm{d}s \tag{A.122}$$

A.7.1 Without partial irreversibility

Without partial irreversibility there is one investment price p and one reset state $\hat{k}^*$.

Step 1: Up to first order, the CIR equals:

$$\text{CIR}(f,\delta) \;=\; -\delta\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + o(\delta^2), \tag{A.123}$$

where the function $m(\hat{k}) \in \mathcal{C}, \mathcal{C}^1$ satisfies the following HJB and border conditions:

$$0 \;=\; f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2}m''(\hat{k}) + \Lambda(\hat{k})(m(\hat{k}^*) - m(\hat{k})), \tag{A.124}$$

$$0 \;=\; m(\hat{k}^*) - m(\hat{k}^\pm), \tag{A.125}$$

$$0 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}. \tag{A.126}$$

1. Show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$ for all $\hat{k}$.
   See Baley and Blanco (2021).

2. We show that $\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$.
   Substitute the definition of $m_{\mathcal{T}}(\hat{k})$ in the integral. In the following equalities, (1) and (2) use Fubini's theorem and Bayes' theorem; (3) uses the fact that $g(\hat{k})$ is the steady-state distribution; (4) solves the first and second integrals.

$$\begin{aligned}\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} &= \int_{\hat{k}^-}^{\hat{k}^+}\left[\int_0^{\mathcal{T}}\int_{\hat{k}^-}^{\hat{k}^+}\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)g_s(\hat{k}|\hat{k}_0)\,\mathrm{d}\hat{k}\,\mathrm{d}s\right]g(\hat{k})\,\mathrm{d}\hat{k}_0 \\ &=^{(1)} \int_0^{\mathcal{T}}\int_{\hat{k}^-}^{\hat{k}^+}\int_{\hat{k}^-}^{\hat{k}^+}\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)g_s(\hat{k}|\hat{k}_0)g(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}\hat{k}_0\,\mathrm{d}s \\ &=^{(2)} \int_0^{\mathcal{T}}\int_{\hat{k}^-}^{\hat{k}^+}\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)\left[\int_{\hat{k}^-}^{\hat{k}^+} g_s(\hat{k}|\hat{k}_0)g(\hat{k})\,\mathrm{d}\hat{k}_0\right]\mathrm{d}\hat{k}\,\mathrm{d}s \\ &=^{(3)} \int_0^{\mathcal{T}}\int_{\hat{k}^-}^{\hat{k}^+}\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)g(\hat{k})\,\mathrm{d}\hat{k}\,\mathrm{d}s =^{(4)} 0.\end{aligned} \tag{A.127}$$

3. Show that $\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$.
   Write $m(\hat{k}) = \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})$ inside the integral, pull the limit outside the integral, and use the previous result in (A.127) to get:

$$\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} \;=\; \int_{\hat{k}^-}^{\hat{k}^+}\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} \;=\; \lim_{\mathcal{T}\to\infty}\int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0 \tag{A.128}$$

4. HJB and border conditions for $m_{\mathcal{T}}(\hat{k})$.
   The function $m_{\mathcal{T}}(\hat{k})$ satisfies the following HBJ with border conditions:

$$\begin{aligned}0 &= f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}} - \nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}} + \frac{\sigma^2}{2}\frac{\mathrm{d}^2 m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2} + \Lambda(\hat{k})(m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k})) \\ 0 &= m_{\mathcal{T}}(\hat{k}^*) - m_{\mathcal{T}}(\hat{k}^\pm) \\ 0 &= \int_{\hat{k}^-}^{\hat{k}^+} m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}\end{aligned}$$

5. HJB and border conditions for $m(\hat{k})$.
   Taking the limit to $\mathcal{T}\to\infty$, and using point-wise convergence of $m_{\mathcal{T}}(\hat{k})$, we have the result.

Step 2: We characterize the terminal value $m(\hat{k}^*) = -\mathbb{C}ov[a, f(\hat{k})]$.

Proof of Step 2. Observe that $m(\hat{k})$ satisfies the following recursive representation

$$m(\hat{k}) \;=\; \mathbb{E}\left[\int_0^\tau (f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + m(\hat{k}^*)\middle|\,\hat{k}_0 = \hat{k}\right]. \tag{A.129}$$

Define the following auxiliary function

$$z(\hat{k}|\varphi) \;=\; \mathbb{E}\left[\int_0^\tau e^{\varphi s}(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})])\,\mathrm{d}s + e^{\varphi\tau} m(\hat{k}^*)\middle|\,\hat{k}_0 = \hat{k}\right]. \tag{A.130}$$

and note that $z(\hat{k}|0) = m(\hat{k})$. The auxiliary function $z(\hat{k}|\varphi)$ satisfies the following HBJ and border condition

$$-\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k})\left(z(\hat{k}|\varphi) - m(\hat{k})\right) = f(\hat{k})) - \mathbb{E}[f(\hat{k})] - \nu\frac{\partial z(\hat{k}|\varphi)}{\partial \hat{k}} + \frac{\sigma^2}{2}\frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}^2}, \tag{A.131}$$

$$z(\hat{k}^{\pm}, \varphi) \;=\; m(\hat{k}^*). \tag{A.132}$$

Since $z(\hat{k}|0) = m(\hat{k})$, (A.128) implies that $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0)g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (A.131), we have that

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial\varphi} - z(\hat{k}|\varphi) \;=\; -\nu\frac{\partial^2 z(\hat{k},\varphi)}{\partial\hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial\hat{k}^2\partial\varphi}, \tag{A.133}$$

$$\frac{\partial z(\hat{k}^{\pm}|\varphi)}{\partial\varphi} \;=\; m(\hat{k}^*). \tag{A.134}$$

Using the Schwarz's theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $z(\hat{k}|0) = m(\hat{k})$

$$\Lambda(\hat{k})\frac{\partial z(\hat{k}|0)}{\partial\varphi} - m(\hat{k}) = -\nu\frac{\partial}{\partial\hat{k}}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial\hat{k}^2}\left(\frac{\partial z(\hat{k}|0)}{\partial\varphi}\right), \tag{A.135}$$

$$\frac{\partial z(\hat{k}^{\pm}|0)}{\partial\varphi} \;=\; m(\hat{k}^*). \tag{A.136}$$

Evaluating (A.129) at $\hat{k}^*$, and using the occupancy measure, we write the previous equation as:

$$\frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau m(\hat{k}_s)\,\mathrm{d}s\middle| k_0 = \hat{k}^*\right] \;=\; \mathbb{E}[\tau]\mathbb{E}[m(\hat{k})] \;=\; 0 \tag{A.137}$$

At the same time, we have that

$$\frac{\partial z(\hat{k}^*|0)}{\partial\varphi} \;=\; \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \tau m(\hat{k}^*)\middle|\,\hat{k}_0 = \hat{k}^*\right], \tag{A.138}$$

Together (A.137) and (A.138) imply:

$$0 = \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\,\hat{k}_0 = \hat{k}\right] + \mathbb{E}\left[\tau\middle|\,\hat{k}_0 = \hat{k}\right] m(\hat{k}^*). \tag{A.139}$$

Solving for $m^{\hat{k}^*}$:

$$m(\hat{k}^*) = -\frac{\mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s\middle|\,\hat{k}_0 = \hat{k}\right]}{\mathbb{E}\left[\tau\middle|\,\hat{k}_0 = \hat{k}\right]}. \tag{A.140}$$

Thus we find the terminal value:

$$m(\hat{k}^*) = -\mathbb{E}[a(f(\hat{k}) - \mathbb{E}[f(\hat{k})])] \;=\; -\mathbb{C}ov[a, f(\hat{k})]. \tag{A.141}$$

A.7.2 With partial irreversibility

Step 1: Up to first order, the CIR equals:

$$\text{CIR}(f,\delta) \;=\; -\delta \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k}) \,\mathrm{d}\hat{k} + o(\delta^2) \tag{A.142}$$

where the function $m(\hat{k}) \in \mathcal{C}, \mathcal{C}^1$ satisfies the following HJB and border conditions:

$$0 \;=\; f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu m'(\hat{k}) + \frac{\sigma^2}{2} m''(\hat{k}) + \Lambda(\hat{k})(\mathcal{M}(\hat{k}^*) - m(\hat{k})), \tag{A.143}$$

$$0 \;=\; \mathcal{M}(\hat{k}^*) - m(\hat{k}^{\pm}), \tag{A.144}$$

$$0 \;=\; \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g(\hat{k}) \,\mathrm{d}\hat{k}. \tag{A.145}$$

Proof of Step 1. Following the same steps as in the case without irreversibility, we obtain:

$$\text{CIR}(f,\delta) \;=\; -\delta \int_{\hat{k}^-}^{\hat{k}^+} \lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) g'(\hat{k}) \,\mathrm{d}\hat{k} + o(\delta^2). \tag{A.146}$$

1. Show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) = m(\hat{k})$ for all $\hat{k}$.

   Let $\{T_i\}_{i=0}^{N(\mathcal{T})}$ be the adjustment dates between 0 and $\mathcal{T}$ for all $i = 1, 2, \ldots, N(\mathcal{T}) - 1$ and $T_0 = 0$ and $T_{N(\mathcal{T})} = \mathcal{T}$. Then, we can write $m_{\mathcal{T}}(\hat{k})$ as

   $$m_{\mathcal{T}}(\hat{k}) = \mathbb{E}\left[\sum_{i=1}^{N(\mathcal{T})} \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \middle| \hat{k}_0 = \hat{k} \right] \tag{A.147}$$

   Taking the limit $\mathcal{T} \to \infty$, we obtain the following equalities: (1) divides the sum; (2) uses the indicator function to write the finite sum; (3) uses the fact that

   $$\mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \mathbb{I}(N(\mathcal{T}) \geq i+1) \middle| \hat{k}_0 = \hat{k} \right] = 1, \forall i;$$

   (4) uses the fact that $\mathbb{E}\left[\sum_{i=1}^{\infty} \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right]$ is an independent of $\mathcal{T}$.

   (A.148)

   $$\begin{aligned}
   &lim_{\mathcal{T}\to\infty} m_{\mathcal{T}}(\hat{k}) \\
   &= \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{i=1}^{N(\mathcal{T})} \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right] \\
   &=^{(1)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{i=1}^{N(\mathcal{T})-1} \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s + \lim_{\mathcal{T}\to\infty} \int_{N(\mathcal{T})-1}^{\mathcal{T}} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right] \\
   &=^{(2)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \sum_{i=1}^{\infty} \mathbb{I}(N(\mathcal{T}) \geq i+1) \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s + \lim_{\mathcal{T}\to\infty} \int_{N(\mathcal{T})-1}^{\mathcal{T}} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right] \\
   &=^{(3)} \mathbb{E}\left[\sum_{i=1}^{\infty} \int_{T_{i-1}}^{T_i} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s + \lim_{\mathcal{T}\to\infty} \int_{N(\mathcal{T})-1}^{\mathcal{T}} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right] \\
   &=^{(4)} \mathbb{E}\left[\lim_{\mathcal{T}\to\infty} \int_{\mathcal{T}_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}} \left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})] \right) \mathrm{d}s \middle| \hat{k}_0 = \hat{k} \right] + \text{terms independent of } \mathcal{T}
   \end{aligned}$$

2. Show that $\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{N(\mathcal{T})-1}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\,\hat{k}_0=\hat{k}\right]$ is independent of $\hat{k}$ and $\mathcal{T}$. Let us define

$$\text{(A.149)}\qquad \mathbb{V}(\hat{k}^{*\pm},\mathcal{T})=\mathbb{E}\left[\int_{\tau_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\,\hat{k}_{\tau^-_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}=\hat{k}^{*\pm}\right].$$

Let $P_N^+(\hat{k})=\mathbb{E}\left[\mathbb{E}[\hat{k}_{\tau^-_N}\geq\hat{k}^{*+}\middle|\,\hat{k}_0=\hat{k}\right]$ and $P_N^-(\hat{k})=\mathbb{E}\left[\mathbb{E}[\hat{k}_{\tau^-_N}\leq\hat{k}^{*-}\middle|\,\hat{k}_0=\hat{k}\right]$. Then, we using conditional expectation, we can write

$$\begin{aligned}
&=\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\int_{\tau_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}}^{\mathcal{T}}\left(f(\hat{k}_s)-\mathbb{E}[f(\hat{k})]\right)\mathrm{d}s\middle|\,\hat{k}_0=\hat{k}\right],\\
&=^{(1)}\mathbb{E}\left[\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*+},\mathcal{T})\lim_{\mathcal{T}\to\infty}P^+_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)+\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*-},\mathcal{T})\lim_{\mathcal{T}\to\infty}P^-_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)\middle|\,\hat{k}_0=\hat{k}\right],\\
&=^{(2)}\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*+},\mathcal{T})P^{+,\infty}+\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*-},\mathcal{T})P^{-,\infty},\\
&=^{(3)}\mathbb{V}^\infty(\hat{k}^{*+})P^{+,\infty}+\mathbb{V}^\infty(\hat{k}^{*-})P^{-,\infty}.
\end{aligned}$$

Here, in step (1) we use law of iterated expectation. Step (2) comes from convergence of discrete Markov chains (see chapter 11 of Stokey (1989)). To show this claim, define $P_N(\hat{k})=\left[P_N^-(\hat{k});P_N^+(\hat{k})\right]\in\mathbb{R}^{2\times1}$, then

$$\text{(A.150)}\qquad P_N(\hat{k})=P^TP_{N-1}(\hat{k}),$$

where $P=[P_1^-(\hat{k}^{*-}),1-P_1^+(\hat{k}^{*-});1-P_1^-(\hat{k}^{*+});P_1^+(\hat{k}^{*+})]\in\mathbb{R}^{2\times2}$ is a 2×2 transition probability where the rows are the transition probability. Under the assumption that $P_1^-(\hat{k}^{*-}),P_1^+(\hat{k}^{*+})\in(0,1)$, we that that

$$\text{(A.151)}\qquad \lim_{N\to\infty}P_N(\hat{k})=\lim_{N\to\infty}P^{N-1}P_1(\hat{k})=[P^{-\infty};P^{+\infty}].$$

where the last equality comes from theorem 11.1 of Stokey (1989). So, $\lim_{\mathcal{T}\to\infty}P^+_{\max\{\mathcal{N}(\mathcal{T})-1,0\}}(\hat{k}_0)$ is independent of $\mathcal{T}$ and $\hat{k}_0$. See Baley and Blanco (2021) and **?** for the convergence of $\lim_{\mathcal{T}\to\infty}\mathbb{V}(\hat{k}^{*\pm},\mathcal{T})=\mathbb{V}^\infty(\hat{k}^{*\pm})$.

3. Show that $\int_{\hat{k}^-}^{\hat{k}^+}m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}=0$.
Using the property that $\Lambda(\hat{k})=0$ for all $\hat{k}\in(\hat{k}^{*-},\hat{k}^{*+})$, we can write in a simple form the HJB and border conditions satisfied by $m_{\mathcal{T}}(\hat{k})$:

$$\begin{aligned}
&\text{(A.152)}\qquad & 0 &= f(\hat{k})-\mathbb{E}[f(\hat{k})]-\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\mathcal{T}}-\nu\frac{\mathrm{d}m_{\mathcal{T}}(\hat{k})}{d\hat{k}}+\frac{\sigma^2}{2}\frac{\mathrm{d}^2m_{\mathcal{T}}(\hat{k})}{\mathrm{d}\hat{k}^2}+\Lambda(\hat{k})(\mathcal{M}_{\mathcal{T}}(\hat{k})-m_{\mathcal{T}}(\hat{k}))\\
&\text{(A.153)}\qquad & 0 &= \mathcal{M}_{\mathcal{T}}(\hat{k}^{*\pm})-m_{\mathcal{T}}(\hat{k}^{\pm}),\\
&\text{(A.154)}\qquad & 0 &= \int_{\hat{k}^-}^{\hat{k}^+}m_{\mathcal{T}}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k}.
\end{aligned}$$

Taking the limit to $\mathcal{T}\to\infty$ and using point-wise convergence of $m_{\mathcal{T}}(\hat{k})$, we have the result.

Up to first order, the CIR is equal to:

$$\text{(A.155)}\qquad \frac{\mathrm{CIR}(f,\delta)}{\delta}=\mathbb{C}ov\left[f(\hat{k}),\frac{\hat{k}}{\sigma^2}\right]-\frac{\nu}{\sigma^2}\mathbb{E}\left[\tilde{\mathcal{M}}(\hat{k})+\hat{k}\tilde{\mathcal{M}}'(\hat{k})\right]+\frac{1}{2}\mathbb{E}\left[2\tilde{\mathcal{M}}'(\hat{k})+\hat{k}\tilde{\mathcal{M}}''(\hat{k})\right]+o(\delta)$$

where the function $\tilde{\mathcal{M}}(\hat{k})$ is defined as

$$\text{(A.156)}\qquad \tilde{\mathcal{M}}(\hat{k})\;=\;\begin{cases} m(\hat{k}^{*-}) & \text{if } \hat{k}\in[\hat{k}^-,\hat{k}^{*-}]\\ (m(\hat{k}^{*+})-m(\hat{k}^{*-}))\sum_{i=0}^5\mathcal{M}_i\left(\frac{\hat{k}-\hat{k}^{*-}}{\hat{k}^{*+}-\hat{k}^{*-}}\right)^i & \text{if } \hat{k}\in(\hat{k}^{*-},\hat{k}^{*+})\\ m(\hat{k}^{*+}) & \text{if } \hat{k}\in[\hat{k}^{*+},\hat{k}^+]\end{cases}$$

with $\mathcal{M}_0 = \frac{m(\hat{k}^{*+})}{m(\hat{k}^{*-})-m(\hat{k}^{*+})}$, $\mathcal{M}_1 = \mathcal{M}_2 = 0, \mathcal{M}_3 = 10, \mathcal{M}_4 = -15$, and $\mathcal{M}_5 = 6$; and the two values $m(\hat{k}^{*-}), m(\hat{k}^{*+})$ that measure the expected cumulative deviation of the capital-productivity ratio relative to the steady-state average $\mathbb{E}[\hat{k}]$ *conditional* on the sign of the last adjustment:

$$m(\hat{k}^{*-}) \;=\; -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^-[\tau]}{1-P^{--}}. \tag{A.157}$$

$$m(\hat{k}^{*+}) \;=\; -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1-P^{++}}. \tag{A.158}$$

Step 2: We characterize the continuation value in HBJ of $m(\hat{k})$. Define the following objects

$$\mathbb{E}^{\pm}[f(\hat{k})] \equiv \frac{\mathbb{E}\left[\int_0^\tau f(\hat{k}_s)\,\mathrm{d}s \middle| \hat{k}_0 = \hat{k}^{*\pm}\right]}{\mathbb{E}\left[\tau \middle| \hat{k}_0 = \hat{k}^{*\pm}\right]}, \tag{A.159}$$

$$P^{\pm +} \equiv \mathbb{E}\left[\mathbb{1}_{\{\hat{k}_\tau \geq \hat{k}^{*+}\}} \middle| \hat{k}_0 = \hat{k}^{*\pm}\right]. \tag{A.160}$$

where $\mathbb{E}^{\pm}[f(\hat{k})]$ is the mean $f(k)$ conditional of a positive or negative last investment and $P^{\pm+}$ is the probability of a negative investment after a positive or negative investment. Then

$$m(\hat{k}^{*-}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^-[\tau]}{1-P^{--}},$$

$$m(\hat{k}^{*+}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1-P^{++}}. \tag{A.161}$$

Observe that $\mathcal{M}(f, \hat{k})$ satisfies the following recursive representation

$$\mathcal{M}(\hat{k}) = \mathbb{E}\left[\int_0^\tau \left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right)\mathrm{d}s + \mathcal{M}(\hat{k}) \middle| \hat{k}_0 = \hat{k}\right]. \tag{A.162}$$

Define an auxiliary function $z(\hat{k}|\varphi)$ as follows:

$$z(\hat{k}|\varphi) \equiv \mathbb{E}\left[\int_0^\tau e^{\varphi s}\left(f(\hat{k}_s) - \mathbb{E}[f(\hat{k})]\right)\mathrm{d}s + e^{\varphi\tau}\mathcal{M}(\hat{k}) \middle| \hat{k}_0 = \hat{k}\right]. \tag{A.163}$$

and note the relationship: $z(\hat{k}|0) = \mathcal{M}(\hat{k})$, $z(\cdot|\varphi) \in \mathcal{C}, \mathcal{C}^1$ for all φ, and

$$-\varphi z(\hat{k}|\varphi) + \Lambda(\hat{k})\left(z(\hat{k}|\varphi) - v^{f*}(\hat{k})\right) = f(\hat{k})) - \mathbb{E}[f(\hat{k})] - \nu z'(\hat{k}|\varphi) + \frac{\sigma^2}{2} z''(\hat{k}|\varphi), \tag{A.164}$$

$$z(\hat{k}^{\pm}|\varphi) = m(\hat{k}^*). \tag{A.165}$$

Since $z(\hat{k}|0) = \mathcal{M}(\hat{k})$, then we have $\int_{\hat{k}^-}^{\hat{k}^+} z(\hat{k}|0)g(\hat{k})\,\mathrm{d}\hat{k} = \int_{\hat{k}^-}^{\hat{k}^+} \mathcal{M}(\hat{k})g(\hat{k})\,\mathrm{d}\hat{k} = 0$. Taking the derivative with respect to φ in (A.164), we have that

$$(\Lambda(\hat{k}) - \varphi)\frac{\partial z(\hat{k}|\varphi)}{\partial \varphi} - z(\hat{k}|\varphi) = -\nu\frac{\partial^2 z(\hat{k}|\varphi)}{\partial \hat{k}\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3 z(\hat{k}|\varphi)}{\partial \hat{k}^2 \partial \varphi}, \tag{A.166}$$

$$\frac{\partial z(\hat{k}^{\pm}, \varphi)}{\partial \varphi} = m(\hat{k}^*). \tag{A.167}$$

Using the Schwarz's theorem to exchange partial derivatives and evaluating at $\varphi = 0$:

$$\Lambda(\hat{k}) \left.\frac{\partial z(\hat{k},\varphi)}{\partial \varphi}\right|_{\varphi=0} - m(\hat{k}) = -\nu \frac{\partial \frac{\partial \left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial \varphi}}{\partial \hat{k}} + \frac{\sigma^2}{2} \frac{\partial^2 \left.\frac{\partial z(\hat{k},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\partial \hat{k}^2}, \tag{A.168}$$

$$\left.\frac{\partial z(\hat{k},\varphi)}{\partial \varphi}\right|_{\varphi=0} = m(\hat{k}^*). \tag{A.169}$$

From the previous equation, using the occupancy measure and the renewal distribution, we have that

$$\begin{aligned}
\frac{\mathcal{R}^- \frac{\partial z(\hat{k}^{*-}|0)}{\partial\varphi} + \mathcal{R}^+ \left.\frac{\partial z(\hat{k}^{*+},\varphi)}{\partial \varphi}\right|_{\varphi=0}}{\mathbb{E}[\tau]} &= \frac{\mathcal{R}^- \mathbb{E}\left[\int_0^\tau m(\hat{k}_s) ds | k_0 = \hat{k}^{*-}\right] + \mathcal{R}^+ \mathbb{E}\left[\int_0^\tau m(\hat{k}_s) ds | k_0 = \hat{k}^{*+}\right]}{\mathbb{E}[\tau]}, \\
&= \mathbb{E}[m(\hat{k})], \\
&= 0. \qquad \text{(A.170)}
\end{aligned}$$

Therefore, $\mathcal{R}^- \left.\frac{\partial z(\hat{k}^{*-},\varphi)}{\partial\varphi}\right|_{\varphi=0} + \mathcal{R}^+ \left.\frac{\partial z(\hat{k}^{*+},\varphi)}{\partial\varphi}\right|_{\varphi=0} = 0$ and

$$\begin{aligned}
0 &= \frac{\mathcal{R}^- \left.\frac{\partial z(\hat{k}^{*-},\varphi)}{\partial\varphi}\right|_{\varphi=0} + \mathcal{R}^+ \left.\frac{\partial z(\hat{k}^{*+},\varphi)}{\partial\varphi}\right|_{\varphi=0}}{\mathbb{E}[\tau]}, \\
&= \frac{\mathcal{R}^- \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right) \mathrm{d}s + \tau m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k}^{*+}\right] + \mathcal{R}^+ \mathbb{E}\left[\int_0^\tau s\left(f(\hat{k}_s) - \mathbb{E}\left[f(\hat{k})\right]\right) \mathrm{d}s + \tau m(\hat{k}^*) \middle| \hat{k}_0 = \hat{k}^{*-}\right]}{\mathbb{E}[\tau]}, \\
&= \mathbb{E}\left[a\left(f(\hat{k}) - \mathbb{E}[f(\hat{k})]\right)\right] + \frac{\mathcal{R}^- \mathbb{E}^-[\tau v^{f*}(\hat{k}_\tau)] + \mathcal{R}^+ \mathbb{E}^+[\tau v^{f*}(\hat{k}_\tau)]}{\mathbb{E}[\tau]}, \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + \frac{\mathcal{R}^- \mathbb{E}^-[\tau v^{f*}(\hat{k}_\tau)] + \mathcal{R}^+ \mathbb{E}^+[\tau v^{f*}(\hat{k}_\tau)]}{\mathbb{E}[\tau]}, \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + \frac{\mathcal{R}^- \mathbb{E}^-[\tau\left(v^f(\hat{k}^{*+})\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+}) + v^f(\hat{k}^{*-})\left(1 - \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})\right)\right)]}{\mathbb{E}[\tau]} \dots \\
&+ \frac{\mathcal{R}^+ \mathbb{E}^+[\tau\left(v^f(\hat{k}^{*+})\mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+}) + v^f(\hat{k}^{*-})\left(1 - \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})\right)\right)]}{\mathbb{E}[\tau]} \\
&\text{(A.171)} \\
&= \mathbb{C}ov\left[a, f(\hat{k})\right] + v^f(\hat{k}^{*-}) + (v^f(\hat{k}^{*+}) - v^f(\hat{k}^{*-})) \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]}
\end{aligned}$$

To characterize $v^f(\hat{k}^{*+}) - v^f(\hat{k}^{*-})$, observe that

$$v^f(\hat{k}^{*-}) = \left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right) \mathbb{E}^-[\tau] + (1 - P^{--}) v^f(\hat{k}^{*+}) + P^{--} v^f(\hat{k}^{*-}) \tag{A.172}$$

where $\mathbb{E}^-[f(\hat{k})]$ is the expected $\hat{k}$ conditional of a positive investment. Thus,

$$-\left(v^f(\hat{k}^{*+}) - v^f(\hat{k}^{*-})\right) = \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right) \mathbb{E}^-[\tau]}{1 - P^{--}} \tag{A.173}$$

From (A.171) and (A.173), we have that

$$v^f(\hat{k}^{*-}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right) \mathbb{E}^-[\tau]}{1 - P^{--}}. \tag{A.174}$$

With similar steps as before, it is easy to show that

$$v^f(\hat{k}^{*+}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\mathcal{C}^+ \left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right) \mathbb{E}^+[\tau]}{1 - P^{++}}. \tag{A.175}$$

Step 3: In this step we show

$$-\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k}) g'(\hat{k}) \, \mathrm{d}\, \hat{k} = \delta \left[\mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{1}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right]\right] + o(\delta^2) \tag{A.176}$$

Using (A.32) to get $\Lambda(k) = \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})}$ and using equation (A.143)

$$\frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} \mathcal{M}(\hat{k}) = f(\hat{k}) - \mathbb{E}[f(\hat{k})] - \nu \mathcal{M}'(\hat{k}) + \frac{\sigma^2}{2}\mathcal{M}''(\hat{k}) + \frac{\nu g'(\hat{k}) + \frac{\sigma^2}{2} g''(\hat{k})}{g(\hat{k})} v^{f**}(\hat{k}). \tag{A.177}$$

Multiplying by $g(\hat{k})\hat{k}$ and taking the integral between $\hat{k}^-$ and $\hat{k}^+$

$$0 = \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2} T_2 + \nu T_3 + \frac{\sigma^2}{2} T_4 \tag{A.178}$$

$$T_1 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[\mathcal{M}'(\hat{k}) g(\hat{k}) + \mathcal{M}(\hat{k}) g'(\hat{k})\right] \mathrm{d}\hat{k} \tag{A.179}$$

$$T_2 = \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[\mathcal{M}''(\hat{k}) g(\hat{k}) - \mathcal{M}(\hat{k}) g''(\hat{k})\right] \mathrm{d}\hat{k} \tag{A.180}$$

$$T_3 = \int_{\hat{k}^-}^{\hat{k}^+} v^{f**}(\hat{k})\hat{k} g'(\hat{k}) \, \mathrm{d}\hat{k} \tag{A.181}$$

$$T_4 = \int_{\hat{k}^-}^{\hat{k}^+} v^{f**}(\hat{k})\hat{k} g''(\hat{k}) \, \mathrm{d}\hat{k}. \tag{A.182}$$

Next, we compute each of the terms T_j, for $j \in \{1, 2, 3, 4\}$.

$$\begin{aligned} T_1 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[\mathcal{M}'(\hat{k}) g(\hat{k}) + \mathcal{M}(\hat{k}) g'(\hat{k})\right] \mathrm{d}\hat{k} \\ &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left(\mathcal{M}(\hat{k}) g(\hat{k})\right)' \mathrm{d}\hat{k} \\ &= \underbrace{\hat{k}\mathcal{M}'(\hat{k}) g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}\mathcal{M}'(\hat{k}) g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\mathcal{M}'(\hat{k}) g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} \mathcal{M}'(\hat{k}) g(\hat{k}) \, \mathrm{d}\hat{k} = 0 \end{aligned} \tag{A.183}$$

$$
\begin{aligned}
T_2 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}\left[\mathcal{M}''(\hat{k})g(\hat{k}) - \mathcal{M}(\hat{k})g''(\hat{k})\right]\mathrm{d}\hat{k} \\
&= \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}\frac{\mathrm{d}\left[\mathcal{M}'(\hat{k})g(\hat{k}) - \mathcal{M}(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}\frac{d\left[\mathcal{M}'(\hat{k})g(\hat{k}) - \mathcal{M}(\hat{k})\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\right]}{\mathrm{d}\hat{k}}d\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}\frac{\mathrm{d}\left[\mathcal{M}'(\hat{k})g(\hat{k}) - \mathcal{M}(\hat{k})g'(\hat{k})\right]}{\mathrm{d}\hat{k}}\mathrm{d}\hat{k} \\
&= \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+} \\
&\quad \cdots - \left[\int_{\hat{k}^-}^{\hat{k}^{*-}}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\ \hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\ \hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+}\left[\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k}) - m(\hat{k})g'(\hat{k})\right]\mathrm{d}\ \hat{k}\right] \\
&= \underbrace{\hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^-}^{\hat{k}^+} + \hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + \hat{k}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \left[m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^-}\right] \\
&\quad \cdots - \left[\int_{\hat{k}^-}^{\hat{k}^+}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\,\mathrm{d}\ \hat{k} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})d\hat{k}\right] \\
&= -\left[v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^f(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}\right] - \int_{\hat{k}^-}^{\hat{k}^+}\frac{dm(\hat{k})}{d\hat{k}}g(\hat{k})\,\mathrm{d}\ \hat{k} + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})\frac{dg(\hat{k})}{d\hat{k}}\,\mathrm{d}\ \hat{k} \\
&= -\left[v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^f(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&\quad - \left[\underbrace{m(\hat{k})g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + m(\hat{k})g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k}\right]
\end{aligned}
$$

(A.184)

$$
= -\left[v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v^f(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^f(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^+}\right] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\ \hat{k}.
$$

The term T_3 is equal to

$$
\begin{aligned}
T_3 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k}v^{f*}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(1)} \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k}v^{f*}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k}v^{f*}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k}v^{f*}(\hat{k})g'(\hat{k})\,\mathrm{d}\hat{k} \\
&=^{(2)} \left[\underbrace{v^{f*}(\hat{k})\hat{k}g(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f*}(\hat{k})\hat{k}g(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f*}(\hat{k})\hat{k}g(\hat{k})\Big|_{\hat{k}^{*+}}^{\hat{k}^+}}_{=0} - \int_{\hat{k}^-}^{\hat{k}^+}\left[v^{f*}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]g(\hat{k})\,\mathrm{d}\hat{k}\right]
\end{aligned}
$$

(A.185) $$=^{(3)} -\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]$$

Step (1) divide the integration domain in the discontinuity points. Step (2) uses continuity of $v^{f*}(\hat{k})$ and $g(\hat{k})$, together with the boundaries conditions of $g(\hat{k}^{\pm})=0$. Finally, T_4 is equal to

$$
\begin{aligned}
T_4 &= \int_{\hat{k}^-}^{\hat{k}^+} \hat{k} v^{f*}(\hat{k}) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k} \\
&= \int_{\hat{k}^-}^{\hat{k}^{*-}} \hat{k} v^{f*}(\hat{k}) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*-}}^{\hat{k}^{*+}} \hat{k} v^{f*}(\hat{k}) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k} + \int_{\hat{k}^{*+}}^{\hat{k}^+} \hat{k} v^{f*}(\hat{k}) \frac{d^2 g(\hat{k})}{\mathrm{d}\hat{k}^2}\,\mathrm{d}\hat{k} \\
&= v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}} - \int_{\hat{k}^-}^{\hat{k}^+}\left[v^{f*}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\,\mathrm{d}\hat{k} \\
&= v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}} \\
&\quad \cdots \underbrace{- \left[v^{f*}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]\hat{k}g(\hat{k})\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} - \left[v^{f*}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]\hat{k}g(\hat{k})\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} - \left[v^{f*}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right]\hat{k}g(\hat{k})\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}}}_{=0} \\
&\quad \cdots + \int_{\hat{k}^-}^{\hat{k}^+}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right] g(\hat{k})\,\mathrm{d}\hat{k} \\
&= v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}} + \mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right]
\end{aligned}
\tag{A.186}
$$

From equations (A.182) to (A.186)

$$
\begin{aligned}
0 &= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu T_1 + \frac{\sigma^2}{2}T_2 + \nu T_3 + \frac{\sigma^2}{2}T_4 \\
&= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] - \nu 0 + \frac{\sigma^2}{2}\Bigg[-\Bigg[v^{f}(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f}(\hat{k})\hat{k}g'(\hat{k})\Big|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}}\Bigg] + 2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k}\Bigg] \\
&\quad \cdots - \nu\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{\sigma^2}{2}\Bigg[v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^-}^{\hat{k}^{*-}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*-}}^{\hat{k}^{*+}} + v^{f*}(\hat{k})\hat{k}\frac{\mathrm{d}g(\hat{k})}{\mathrm{d}\hat{k}}\Bigg|_{\hat{k}^{*+}}^{\hat{k}^{+}} + \mathbb{E}\Bigg[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}}{\mathrm{d}\hat{k}^2} \\
&= \mathbb{E}\left[f(\hat{k})\hat{k}\right] - \mathbb{E}\left[\hat{k}\right]\mathbb{E}\left[f(\hat{k})\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} - \nu\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right] \\
&= \mathbb{C}ov\left[f(\hat{k}), \hat{k}\right] + \sigma^2\int_{\hat{k}^-}^{\hat{k}^+} m(\hat{k})g'(\hat{k})\,\mathrm{d}\,\hat{k} - \nu\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{\mathrm{d}^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right].
\end{aligned}
\tag{A.187}
$$

Therefore, we obtain:

$$
\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{1}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{\mathrm{d}^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right] + o(\delta)
\tag{A.188}
$$

Step 4: Finally, we express the CIR in a convenient way. Until now we have that

$$
\frac{\mathrm{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[v^{f**}(\hat{k}) + \hat{k}\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}}\right] + \frac{1}{2}\mathbb{E}\left[2\frac{\mathrm{d}v^{f**}(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2 v^{f**}(\hat{k})}{\mathrm{d}\hat{k}^2}\right] + o(\delta)
\tag{A.189}
$$

where

$$v^{f*}(\hat{k}) = \begin{cases} v(\hat{k}^{*-}) & \text{if } \hat{k}^- \leq \hat{k} \leq \hat{k}^{*-} \\ (v(\hat{k}^{*+}) - v(\hat{k}^{*-})) \sum_{i=0}^{5} \mathcal{P}_i \left(\frac{\hat{k}-\hat{k}^{*-}}{\hat{k}^{*+}-\hat{k}^{*-}} \right)^i & \text{if } \hat{k}^{*-} < \hat{k} < \hat{k}^{*+} \\ v(\hat{k}^{*+}) & \text{if } \hat{k}^{*+} \leq \hat{k} \leq \hat{k}^+ \end{cases} \tag{A.190}$$

$$v^f(\hat{k}^{*-}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^-[\tau]}{1 - P^{--}}. \tag{A.191}$$

$$v^f(\hat{k}^{*+}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1 - P^{++}}. \tag{A.192}$$

with $\mathcal{P}_0 = \frac{v(\hat{k}^{*+})}{v(\hat{k}^{*-}) - v(\hat{k}^{*+})}$, $\mathcal{P}_1 = \mathcal{P}_2 = 0$, $\mathcal{P}_3 = 10$, $\mathcal{P}_4 = -15$, and $\mathcal{P}_5 = 6$. given this expressions, we can re-express

$$v^f(\hat{k}) = -\mathbb{C}ov\left[f(\hat{k}), a\right] + \tilde{v}^f(\hat{k}). \tag{A.193}$$

$$\tilde{v}^{f*}(\hat{k}) = \begin{cases} \tilde{v}^f(\hat{k}^{*-}) & \text{if } \hat{k}^- \leq \hat{k} \leq \hat{k}^{*-} \\ (\tilde{v}^f(\hat{k}^{*+}) - \tilde{v}^f(\hat{k}^{*-})) \sum_{i=0}^{5} \mathcal{P}_i \left(\frac{\hat{k}-\hat{k}^{*-}}{\hat{k}^{*+}-\hat{k}^{*-}} \right)^i & \text{if } \hat{k}^{*-} < \hat{k} < \hat{k}^{*+} \\ \tilde{v}^f(\hat{k}^{*+}) & \text{if } \hat{k}^{*+} \leq \hat{k} \leq \hat{k}^+ \end{cases} \tag{A.194}$$

$$\tilde{v}^f(\hat{k}^{*-}) = \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \geq \hat{k}^{*+})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^-[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^-[\tau]}{1 - P^{--}}. \tag{A.195}$$

$$\tilde{v}^f(\hat{k}^{*+}) = \frac{\mathbb{E}[\tau \mathbb{I}(\hat{k}_\tau \leq \hat{k}^{*-})]}{\mathbb{E}[\tau]} \frac{\left(\mathbb{E}^+[f(\hat{k})] - \mathbb{E}[f(\hat{k})]\right)\mathbb{E}^+[\tau]}{1 - P^{++}}. \tag{A.196}$$

With this transformation, we have that

$$\frac{\text{CIR}(f,\delta)}{\delta} \;=\; \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right] - \frac{\nu}{\sigma^2}\mathbb{E}\left[\tilde{v}^f(\hat{k}) + \hat{k}\frac{\mathrm{d}\tilde{v}^f(\hat{k})}{\mathrm{d}\hat{k}}\right] \;+\; \frac{1}{2}\mathbb{E}\left[2\frac{\mathrm{d}\tilde{v}^f(\hat{k})}{\mathrm{d}\hat{k}} + \hat{k}\frac{d^2\tilde{v}^f(\hat{k})}{\mathrm{d}\hat{k}^2}\right] \tag{A.197}$$

If we apply Ito's lemma to $\hat{k}\tilde{v}^f(\hat{k})$, we have that

$$\mathbb{E}_s[\mathrm{d}(\hat{k}_s\tilde{v}^f(\hat{k}_s))] = \left[-\nu\left[\tilde{v}^f(\hat{k}_s) + \hat{k}_t\frac{\mathrm{d}\tilde{v}^f(\hat{k}_s)}{\mathrm{d}\hat{k}}\right] + \frac{\sigma^2}{2}\mathbb{E}\left[2\frac{\mathrm{d}\tilde{v}^f(\hat{k}_s)}{\mathrm{d}\hat{k}} + \hat{k}_s\frac{d^2\tilde{v}^f(\hat{k}_s)}{\mathrm{d}\hat{k}^2}\right]\right]\mathrm{d}s \tag{A.198}$$

Therefore

$$\frac{\text{CIR}(f,\delta)}{\delta} = \mathbb{C}ov\left[f(\hat{k}), \frac{\hat{k}+\nu a}{\sigma^2}\right] + \mathbb{E}\left[\frac{\mathbb{E}_0^{\hat{k}}[\mathrm{d}(\hat{k}_0\tilde{v}^f(\hat{k}_0))]}{\mathrm{d}s}\right] \tag{A.199}$$

A.8 Proof of Proposition 5

This proposition splits the investment problem into static and dynamic components. Then substitutes the parametric changes introduced by taxation and redefines the investment frictions in terms of after-tax profits.

A.8.1 Flow profits with user cost of capital

Depart from the sufficient optimality conditions (HJB, value matching, optimality and smooth pasting) established in Lemma 2:

$$
\begin{aligned}
&\text{(A.200)} \qquad & rv(\hat{k}) &= Ae^{\alpha\hat{k}} - \nu v'(\hat{k}) + \frac{\sigma^2}{2}v''(\hat{k}), \quad \hat{k} \in (\hat{k}^-, \hat{k}^+), \\
&\text{(A.201)} \qquad & v(\hat{k}^-) &= v(\hat{k}^{*-}) - \theta + p^{buy}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}), \\
&\text{(A.202)} \qquad & v(\hat{k}^+) &= v(\hat{k}^{*+}) - \theta + p^{sell}(e^{\hat{k}^+} - e^{\hat{k}^{*+}}), \\
&\text{(A.203)} \qquad & v'(\hat{k}) &= p^{buy}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \\
&\text{(A.204)} \qquad & v'(\hat{k}) &= p^{sell}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^{+}\right\}.
\end{aligned}
$$

Define the auxiliary function:

$$
\text{(A.205)} \qquad \mathcal{W}(\hat{k}) \equiv \frac{v(\hat{k}) - pe^{\hat{k}}}{A} \implies v(\hat{k}) = A\mathcal{W}(\hat{k}) + pe^{\hat{k}}.
$$

Substitute the new expression for $v(\hat{k})$ in (A.205) into both sides of (A.200):

$$
\text{(A.206)} \qquad rA\mathcal{W}(\hat{k}) + rpe^{\hat{k}} = Ae^{\alpha\hat{k}} - \nu A\mathcal{W}'(\hat{k}) - \nu pe^{\hat{k}} + \frac{\sigma^2}{2}A\mathcal{W}''(\hat{k}) + \frac{\sigma^2}{2}pe^{\hat{k}}.
$$

Divide by A, join terms with $pe^{\hat{k}}$, and rearrange, to obtain:

$$
\text{(A.207)} \qquad r\mathcal{W}(\hat{k}) = \underbrace{e^{\alpha\hat{k}} - \left(r + \nu - \frac{\sigma^2}{2}\right)\frac{p}{A}e^{\hat{k}}}_{\Pi(\hat{k})} - \nu\mathcal{W}'(\hat{k}) + \frac{\sigma^2}{2}\mathcal{W}''(\hat{k}),
$$

where flow profits including the user cost are defined as:

$$
\text{(A.208)} \qquad \Pi(\hat{k}) \equiv e^{\alpha\hat{k}} - \left(r + \nu - \frac{\sigma^2}{2}\right)\frac{p}{A}e^{\hat{k}}.
$$

Substitute the auxiliary function into the value matching conditions (A.201) to (A.202) and simplify:

$$
\begin{aligned}
&\text{(A.209)} \qquad & \mathcal{W}(\hat{k}^-) &= \mathcal{W}(\hat{k}^{*-}) - \frac{\theta}{A} + \frac{(p^{buy} - p)}{A}(e^{\hat{k}^-} - e^{\hat{k}^{*-}}) \\
& & \mathcal{W}(\hat{k}^+) &= \mathcal{W}(\hat{k}^{*+}) - \frac{\theta}{A} + \frac{(p^{buy} - p)}{A}(e^{\hat{k}^+} - e^{\hat{k}^{*-}})
\end{aligned}
$$

Substitute the auxiliary function into the smooth pasting and optimality conditions (A.203) to (A.204) and simplify:

$$
\begin{aligned}
&\text{(A.210)} \qquad & \mathcal{W}'(\hat{k}) &= \frac{(p^{buy} - p)}{A}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^-, \hat{k}^{*-}\right\}, \\
& & \mathcal{W}'(\hat{k}) &= \frac{(p^{sell} - p)}{A}e^{\hat{k}}, \quad \hat{k} \in \left\{\hat{k}^{*+}, \hat{k}^{+}\right\}.
\end{aligned}
$$

Summarizing, the optimality conditions satisfied by $\mathcal{W}(\hat{k})$ are: (A.207), (A.208), (A.209), and (A.210).

A.8.2 Static component

Define the static policy $\hat{k}^{ss}$ as the capital-productivity ratio that maximizes $\Pi(\hat{k})$ in (A.208):

$$\hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{\alpha A}{p(r+\nu-\sigma^2/2)}\right) = \frac{1}{1-\alpha}\log\left(\frac{\alpha A}{p(\rho+\xi^k-\sigma^2)}\right), \tag{A.211}$$

where in the second equality we have substituted $r = \rho-\mu-\sigma^2/2$. Next, do an infinite-order Taylor approximation of flow profits around the static policy $\hat{k}^{ss}$, denoting the n-th derivative with $\Pi^n(\cdot)$:

$$\Pi(\hat{k}) =^{(1)} \sum_{n=0}^{\infty}\frac{\Pi^n(\hat{k}^{ss})}{n!}(\hat{k}-\hat{k}^{ss})^n =^{(2)} e^{\alpha\hat{k}^{ss}}\sum_{n=0}^{\infty}\frac{(\alpha^n-\alpha)}{n!}(\hat{k}-\hat{k}^{ss})^n =^{(3)} e^{\alpha\hat{k}^{ss}}\left(e^{\alpha(\hat{k}-\hat{k}^{ss})}-\alpha e^{(\hat{k}-\hat{k}^{ss})}\right). \tag{A.212}$$

where equality (2) substitutes the derivatives of flow profits evaluated at the static policy

$$\Pi^n(\hat{k}^{ss}) = \alpha^n e^{\alpha\hat{k}} - \left(r+\nu-\frac{\sigma^2}{2}\right)\frac{p}{A}e^{\hat{k}}\bigg|_{\hat{k}=\hat{k}^{ss}} = e^{\alpha\hat{k}^{ss}}(\alpha^n-\alpha); \tag{A.213}$$

and equality (3) applies the definition of the exponential function $f(z) = e^z = \sum_{n=0}^{\infty} z^n/n!$ evaluated at the points $z = \alpha(\hat{k}-\hat{k}^{ss})$ and $z = (\hat{k}-\hat{k}^{ss})$. Thus we obtain the following expression for flow profits:

$$\Pi(\hat{k}) = e^{\alpha\hat{k}^{ss}}\left(e^{\alpha(\hat{k}-\hat{k}^{ss})}-\alpha e^{(\hat{k}-\hat{k}^{ss})}\right). \tag{A.214}$$

A.8.3 Dynamic component

Consider the normalized capital-productivity ratios $x = \hat{k}-\hat{k}^{ss}$. Its dynamics are given by $\mathrm{d}x_s = \mathrm{d}\hat{k}_s = -\nu\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$. Using the normalized ratios x, the auxiliary function $\mathcal{W}(\hat{k})$ in (A.205), and the static policy $\hat{k}^{ss}$ in (A.211), define the function $\mathcal{V}(x)$, the flow profits $\Pi(x)$, and the dynamic policy $\mathcal{X}$ as follows:

$$\mathcal{V}(x) \equiv \frac{\mathcal{W}(x+\hat{k}_{ss})}{e^{\alpha\hat{k}^{ss}}}, \tag{A.215}$$

$$\pi(x) \equiv \frac{\Pi(x+\hat{k}_{ss})}{e^{\alpha\hat{k}^{ss}}} = e^{\alpha x}-\alpha e^x, \tag{A.216}$$

$$\mathcal{X} \equiv (x^-, x^{*-}, x^{*+}, x^+) = (\hat{k}^- - \hat{k}^{ss}, \hat{k}^{*-} - \hat{k}^{ss}, \hat{k}^{*+} - \hat{k}^{ss}, \hat{k}^+ - \hat{k}^{ss}). \tag{A.217}$$

Rewrite the optimality conditions (A.207), (A.208), (A.209), and (A.210) using $\mathcal{V}(x)$, $\pi(x)$, and $\mathcal{X}$ as follows:

$$\begin{aligned}
r\mathcal{V}(x) &= \pi(x) - \nu\mathcal{V}'(x) + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad x \in (x^-, x^+),\\
\mathcal{V}(x^-) &= \mathcal{V}(x^{*-}) - \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} + \frac{p^{buy}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}(e^{x^-}-e^{x^{*-}}),\\
\mathcal{V}(x^+) &= \mathcal{V}(x^{*+}) - \frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} + \frac{p^{sell}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}(e^{x^+}-e^{x^{*+}}),\\
\mathcal{V}'(x) &= \frac{p^{buy}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}e^x, \quad x \in \{x^-, x^{*-}\},\\
\mathcal{V}'(x) &= \frac{p^{sell}-p}{Ae^{(\alpha-1)\hat{k}^{ss}}}e^x, \quad x \in \{x^+, x^{*+}\}.
\end{aligned} \tag{A.218}$$

A.8.4 Adding corporate taxes

Substitute the parametric changes that incorporate corporate taxation in the problem:

$$A \to (1-t^c)A, \qquad \rho \to \left(\frac{1-t^p}{1-t^g}\right)\rho, \qquad p(\Delta\hat{k}) \to (1-t^d)p(\Delta\hat{k}), \qquad \theta \to (1-t^d)\theta, \tag{A.219}$$

into the static policy:

$$\hat{k}^{ss} = \frac{1}{1-\alpha} \log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right) \tag{A.220}$$

where $\tilde{\mathcal{U}} \equiv \frac{1-t^p}{1-t^g}\rho + \xi^k - \sigma^2$ is the after-tax user cost of capital. Then, define the effective fixed cost $\tilde{\theta}$, which is scaled by after-tax profits; the effective price wedge $(\tilde{p}^{buy}, \tilde{p}^{sell})$, which is scaled by after-tax output-capital ratio; and the effective discount factor $\tilde{r}$ as follows:

$$\tilde{\theta} \equiv \frac{1-t^d}{1-t^c}\frac{\theta}{Ae^{\alpha \hat{k}^{ss}}}, \tag{A.221}$$

$$(\tilde{p}^{buy}, \tilde{p}^{sell}) \equiv \frac{1-t^d}{1-t^c}\frac{(p^{buy}-p, p^{sell}-p)}{Ae^{(\alpha-1)\hat{k}^{ss}}}, \tag{A.222}$$

$$\tilde{r} \equiv \left(\frac{1-t^p}{1-t^g}\right)\rho - \mu - \frac{\sigma^2}{2}. \tag{A.223}$$

Using these definitions, convert the system in (A.218) simplifies to the final expressions:

$$\begin{aligned}
\tilde{r}\mathcal{V}(x) &= e^{\alpha x} - \alpha e^{x} - \nu \mathcal{V}'(x) + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad x \in (x^-, x^+), \\
\mathcal{V}(x^-) &= h(x^{*-}) - \tilde{\theta} + \tilde{p}^{buy}(e^{x^-} - e^{x^{*-}}), \\
\mathcal{V}(x^+) &= h(x^{*+}) - \tilde{\theta} + \tilde{p}^{sell}(e^{x^+} - e^{x^{*+}}), \\
\mathcal{V}'(x) &= \tilde{p}^{buy}e^{x}, \quad x \in \left\{x^-, x^{*-}\right\}, \\
\mathcal{V}'(x) &= \tilde{p}^{sell}e^{x}, \quad x \in \left\{x^+, x^{*+}\right\}.
\end{aligned} \tag{A.224}$$

The previous system solves the stopping problem:

$$\begin{aligned}
\mathcal{V}(x) = \max_{\tau, \Delta x} \mathbb{E}\Bigg[&\int_0^\tau e^{-\tilde{r}\tau}\left(e^{\alpha x_s} - \alpha e^{x_s}\right) \mathrm{d}s \\
&+ e^{\tilde{r}\tau}\left(-\tilde{\theta} + \tilde{p}(\Delta x)(e^{x_\tau + \Delta x} - e^{x_\tau}) + \mathcal{V}(x_\tau + \Delta x)\right) \Big| x_0 = x \Bigg]
\end{aligned} \tag{A.225}$$

$$\tilde{p}(\Delta x) = \tilde{p}^{\text{buy}}\mathbb{1}_{\{\Delta x > 0\}} + \tilde{p}^{\text{sell}}\mathbb{1}_{\{\Delta x < 0\}}. \tag{A.226}$$

A.9 Proof of Proposition 6

This proposition expresses the effective investment frictions in terms of fundamental parameters and taxes. Then it computes the derivatives of after-tax frictions with respect to taxes.

A.9.1 Effective frictions

Let $\tilde{\mathcal{U}} \equiv \frac{(1-t^p)}{(1-t^g)}\rho + \xi^k - \sigma^2$ be the after-tax user cost of capital. Substitute the static policy in (65), given by

$$\hat{k}^{ss} = \frac{1}{1-\alpha}\log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}},\right) \tag{A.227}$$

into the after-tax static profits and into the after-tax profit-capital ratio:

$$(1-t^c)Ae^{\alpha\hat{k}^{ss}} = (1-t^c)A\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{(1-t^c)A}{(1-t^d)^\alpha}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha}{1-\alpha}} \tag{A.228}$$

$$\frac{(1-t^c)}{(1-t^d)}Ae^{(\alpha-1)\hat{k}^{ss}} = \frac{(1-t^c)}{(1-t^d)}A\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)^{\frac{\alpha-1}{1-\alpha}} = \frac{(1-t^c)}{(1-t^d)}A\frac{(1-t^d)p\tilde{\mathcal{U}}}{(1-t^c)\alpha A} = \frac{p\tilde{\mathcal{U}}}{\alpha}. \tag{A.229}$$

Therefore, the scaled investment frictons are equal to:

$$\tilde{\theta} \equiv \frac{1-t^d}{1-t^c}\frac{\theta}{Ae^{\alpha\hat{k}^{ss}}} = \left(\frac{(1-t^d)^\alpha}{(1-t^c)A}\right)^{\frac{1}{1-\alpha}}\left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}(1-t^d)\theta = \left(\frac{(1-t^d)}{(1-t^c)A}\right)^{\frac{1}{1-\alpha}}\left(\frac{p\tilde{\mathcal{U}}}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}\theta, \tag{A.230}$$

$$\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}} \equiv \frac{1-t^d}{1-t^c}\frac{(p^{\text{buy}}-p, p^{\text{sell}}-p)}{Ae^{(\alpha-1)\hat{k}^{ss}}} = \frac{\alpha}{\tilde{\mathcal{U}}}\frac{p^{\text{buy}}-p^{\text{sell}}}{p}. \tag{A.231}$$

A.9.2 Derivatives of frictions wrt. taxes

Now we compute the derivatives with respect to taxes $\mathbb{T}$. Assume $t^c > 0$, $\tilde{\mathcal{U}} > 0$ and p fixed. To correctly isolate the tax effects, we define the following parameters:

$$t^\rho \equiv \frac{1-t^p}{1-t^g}, \qquad \tilde{\xi}^d \equiv \frac{\xi^d}{t^\rho\rho+\xi^d}, \qquad t^d = t^c\tilde{\xi}^d. \tag{A.232}$$

To ease the computations, we sign the tax effects on the log of $\tilde{\theta}$:

$$\log\tilde{\theta} = \frac{1}{1-\alpha}\log\left(1-\frac{t^c\xi^d}{t^\rho\rho+\xi^d}\right) - \frac{1}{1-\alpha}\log(1-t^c)A + \frac{\alpha}{1-\alpha}\log R + \frac{\alpha}{1-\alpha}\log\left(\frac{p}{\alpha}\right) + \log\theta. \tag{A.233}$$

- Effects of taxes on the effective fixed cost in (A.230).

$$\begin{aligned}
\frac{\partial\log\tilde{\theta}}{\partial t^c} &= \frac{1}{(1-\alpha)(1-t^c)}\left[1-\frac{\xi^d(1-t^c)}{t^\rho\rho+\xi^d(1-t^c)}\right] > 0,\\
\frac{\partial\log\tilde{\theta}}{\partial\xi^d} &= \frac{1}{1-\alpha}\left[\frac{1-t^c}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d}\right] < 0,\\
\frac{\partial\log\tilde{\theta}}{\partial t^p} &= -\frac{1}{1-\alpha}\frac{\rho}{1-t^g}\left[\frac{1}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d} + \frac{1}{R}\right] < 0,\\
\frac{\partial\log\tilde{\theta}}{\partial t^g} &= \frac{1}{1-\alpha}\frac{\rho(1-t^p)}{(1-t^g)^2}\left[\frac{1}{t^\rho\rho+\xi^d(1-t^c)} - \frac{1}{t^\rho\rho+\xi^d} + \frac{1}{R}\right] > 0.
\end{aligned}$$

- Effects of taxes on the effective price wedge in (A.231)

$$
\begin{aligned}
\frac{\partial(\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}})}{\partial t^c} &= 0, \\
\frac{\partial(\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}})}{\partial \xi^d} &= 0, \\
\frac{\partial(\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}})}{\partial t^p} &= \frac{\alpha}{pR^3}\frac{\rho}{1-t^g}(p^{\text{buy}} - p^{\text{sell}}) > 0, \\
\frac{\partial(\tilde{p}^{\text{buy}} - \tilde{p}^{\text{sell}})}{\partial t^g} &= -\frac{\alpha}{pR^3}\frac{\rho(1-t^p)}{(1-t^g)^2}(p^{\text{buy}} - p^{\text{sell}}) < 0.
\end{aligned}
$$

A.10 Proof of Proposition 7

This proposition computes the optimal investment policy and the macro outcomes for driftless symmetric models. Assume $\nu \to 0$ (symmetry stochastic process) and $\tilde{p}^{buy} = -\tilde{p}^{sell} = \tilde{p}$. Since optimal dynamic policies $\mathcal{X}$ are symmetric in all the following cases, then $\mathbb{E}[x] = 0$ and $\mathbb{C}ov[x, a] = 0$. We characterize the case with both frictions active. It nests the solution for only fixed costs (setting $x^* = 0$) and only price wedge (setting $x^* = \overline{x}$).

A.10.1 Sufficient optimality conditions

The solution is characterized by a symmetric inaction region with borders $\pm\overline{x}$ and two reset points $\pm x^*$ that satisfy the following system of equations:

$$\tilde{r}\mathcal{V}(x) = e^{\alpha x} - \alpha e^x + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad \forall x \in (-\overline{x}, \overline{x}), \tag{A.234}$$

$$\mathcal{V}(-\overline{x}) = \mathcal{V}(-x^*) - \tilde{p}(e^{-x^*} - e^{-\overline{x}}) - \tilde{\theta}, \tag{A.235}$$

$$\mathcal{V}(\overline{x}) = \mathcal{V}(x^*) - \tilde{p}(e^{\overline{x}} - e^{x^*}) - \tilde{\theta}, \tag{A.236}$$

$$\mathcal{V}'(x) = \tilde{p}e^x, \quad x \in \{-\overline{x}, -x^*\}, \tag{A.237}$$

$$\mathcal{V}'(x) = -\tilde{p}e^x, \quad x \in \{\overline{x}, x^*\}. \tag{A.238}$$

A.10.2 Optimal policy

Assume $\tilde{p}$ and $\tilde{\theta}$ are small. A second order Taylor approximation of the flow profits $\pi(x) = e^{\alpha x} - \alpha e^x$ in (A.216) around the (frictionless) point $x = 0$ yields:

$$\pi(x) = \pi(0) + \pi'(0)x + \frac{\pi''(0)}{2}x^2 = 1 - \alpha + \frac{\alpha^2 - \alpha}{2}x^2 = (1 - \alpha) - \frac{\alpha(1 - \alpha)}{2}x^2. \tag{A.239}$$

For the characterization, we ignore the constant term. Next, a first order approximation of the exponential function $e^x = 1 + x$ yields $e^{\overline{x}} - e^{x^*} = \overline{x} - x^*$ and $e^{-x^*} - e^{-\overline{x}} = \overline{x} - x^*$. Under these approximations, we write the system of equations as:

$$\tilde{r}\mathcal{V}(x) = -\frac{\alpha(1 - \alpha)}{2}x^2 + \frac{\sigma^2}{2}\mathcal{V}''(x), \quad \forall x \in (-\overline{x}, \overline{x}), \tag{A.240}$$

$$\mathcal{V}(-\overline{x}) = \mathcal{V}(-x^*) - \tilde{p}(\overline{x} - x^*) - \tilde{\theta}, \tag{A.241}$$

$$\mathcal{V}(\overline{x}) = \mathcal{V}(x^*) - \tilde{p}(\overline{x} - x^*) - \tilde{\theta}, \tag{A.242}$$

$$\mathcal{V}'(x) = \tilde{p}(1 + x), \quad x \in \{-\overline{x}, -x^*\}, \tag{A.243}$$

$$\mathcal{V}'(x) = -\tilde{p}(1 + x), \quad x \in \{\overline{x}, x^*\}. \tag{A.244}$$

Since the problem is symmetric, we only work with the positive domain and use equations (A.240), (A.242), and (A.244). Next, we derive two conditions that pin down the optimal policy by approximating the value function and its derivatives. We denote the n^{th} derivative with $\mathcal{V}^n(x)$ for $n > 2$.

To derive the first condition, we approximate the value function with a 4^{th} order Taylor expansion around $x = 0$, noting that odd derivatives evaluated at zero are equal to zero by symmetry:

$$\mathcal{V}(x) = \mathcal{V}(0) + \frac{\mathcal{V}''(0)}{2!}x^2 + \frac{\mathcal{V}^4(0)}{4!}x^4. \tag{A.245}$$

The second derivative of (A.240) evaluated at zero equals:

$$\tilde{r}\mathcal{V}''(0) = -\alpha(1 - \alpha) + \frac{\sigma^2}{2}\mathcal{V}^4(0). \tag{A.246}$$

Taking the limit $\tilde{r} \to 0$ in the previous expression, we find $\mathcal{V}^4(0)$:

$$\mathcal{V}^4(0) = \frac{2\alpha(1 - \alpha)}{\sigma^2}. \tag{A.247}$$

Substitute the approximation to $\mathcal{V}(x)$ into the value matching condition in (A.241) and simplify:

$$\begin{aligned}
\mathcal{V}(\overline{x}) &= \mathcal{V}(x^*) \;-\; \tilde{p}(\overline{x}-x^*)-\tilde{\theta} \\
\mathcal{V}(0)+\frac{\mathcal{V}''(0)}{2}\overline{x}^2+\frac{\mathcal{V}^4(0)}{24}\overline{x}^4 &= \mathcal{V}(0)+\frac{\mathcal{V}''(0)}{2}x^{*2}+\frac{\mathcal{V}^4(0)}{24}x^{*4} \;-\; \tilde{p}(\overline{x}-x^*)-\tilde{\theta} \\
\frac{\mathcal{V}''(0)}{2}(\overline{x}^2-x^{*2})+\frac{\alpha(1-\alpha)}{12\sigma^2}(\overline{x}^4-x^{*4}) &= \;-\;\tilde{p}(\overline{x}-x^*)-\tilde{\theta} \\
\frac{\mathcal{V}''(0)}{2}(\overline{x}^2-x^{*2})+\frac{\alpha(1-\alpha)}{12\sigma^2}(\overline{x}^2+x^{*2})(\overline{x}^2-x^{*2}) &= \;-\;\tilde{p}(\overline{x}-x^*)-\tilde{\theta} \\
\mathcal{V}''(0)+\frac{\alpha(1-\alpha)}{6\sigma^2}(\overline{x}^2+x^{*2}) &= \;-\;2\left(\frac{\tilde{p}(\overline{x}-x^*)+\tilde{\theta}}{(\overline{x}^2-x^{*2})}\right)
\end{aligned}$$

To derive a second condition, we approximate the first derivative of the value function with a 3^rd order Taylor approximation, noting that odd derivatives evaluated at zero are zero by symmetry:

(A.248) $$\mathcal{V}'(x)=\frac{\mathcal{V}''(0)}{1!}x+\frac{\mathcal{V}^4(0)}{3!}x^3$$

Evaluating at $\overline{x}$ and x^* and substituting the expression for the fourth derivative:

(A.249) $$-\tilde{p}(1+\overline{x})=\mathcal{V}''(0)\overline{x}+\frac{\alpha(1-\alpha)}{3\sigma^2}\overline{x}^3; \qquad -\tilde{p}(1+x^*)=\mathcal{V}''(0)x^*+\frac{\alpha(1-\alpha)}{3\sigma^2}x^{*3}.$$

Taking the difference between the two previous conditions and subtracting $\tilde{\theta}$

(A.250) $$-\tilde{p}(\overline{x}-x^*)=\mathcal{V}''(0)(\overline{x}-x^*)+\frac{\alpha(1-\alpha)}{3\sigma^2}(\overline{x}^3-x^{*3})$$

and solving for $\mathcal{V}''(0)$:

(A.251) $$-\frac{\mathcal{V}''(0)}{2}=\frac{\tilde{p}}{2}+\frac{\alpha(1-\alpha)}{6\sigma^2}(\overline{x}^2+x^{*2}+\overline{x}x^*)$$

Back into optimality:

(A.252) $$-\tilde{p}(1+x^*)=\left[-\tilde{p}-\frac{\alpha(1-\alpha)}{3\sigma^2}(\overline{x}^2+x^{*2}+\overline{x}x^*)\right]x^*+\frac{\alpha(1-\alpha)}{3\sigma^2}x^{*3}$$

(A.253) $$-\tilde{p}=\frac{\alpha(1-\alpha)}{3\sigma^2}\left[x^{*2}-(\overline{x}^2+x^{*2}+\overline{x}x^*)\right]x^*$$

(A.254) $$\frac{3\tilde{p}\sigma^2}{\alpha(1-\alpha)}=\overline{x}x^*(\overline{x}+x^*)$$

Check this

(A.255) $$\overline{x}x^*(\overline{x}+x^*)=\frac{3\sigma^2\tilde{p}}{\alpha(1-\alpha)}$$

(A.256) $$\overline{x}^4-x^{*4}=\overline{x}x^*(\overline{x}+x^*)(\overline{x}-x^*)\left(1+\overline{x}+x^*\right)+\frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)}$$

The optimal policy nests the two extreme cases:

1. With only a fixed cost, $\tilde{p}=0$ and $x^*=0$. The first condition is irrelevant and the second condition pins down the optimal policy:

   (A.257) $$\overline{x}^4=\frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)} \quad\Longrightarrow\quad \overline{x}=\left(\frac{6\sigma^2\tilde{\theta}}{\alpha(1-\alpha)}\right)^{1/4}.$$

2. With only a price wedge, $\tilde{\theta}=0$ and $\overline{x}^*=x^*=\overline{x}$. The second condition is irrelevant and the first condition

pins down the optimal policy:

$$2\overline{x}^{*3} = \frac{3\sigma^2\tilde{p}}{\alpha(1-\alpha)} \quad\Longrightarrow\quad \overline{x}^* = \left(\frac{3\sigma^2\tilde{p}}{2\alpha(1-\alpha)}\right)^{1/3}. \tag{A.258}$$

A.10.3 Cross-sectional and renewal distributions and adjustment probabilities

Cross-sectional distribution. The stationary density $g(x)$ solves the KFE with border, continuity, and reinjection (exit mass equals entry mass) conditions:

$$0 = \frac{\sigma^2}{2} g''(x), \tag{A.259}$$

$$g(\overline{x}) = g(-\overline{x}) = 0, \tag{A.260}$$

$$\int_{-\overline{x}}^{\overline{x}} g(x)\,\mathrm{d}x = 1, \tag{A.261}$$

$$\lim_{x\downarrow -x^*} g(x) = \lim_{x\uparrow -x^*} g(x), \qquad \lim_{x\downarrow x^*} g(x) = \lim_{x\uparrow x^*} g(x), \tag{A.262}$$

$$\lim_{x\downarrow -\overline{x}} g'(x) = \lim_{x\uparrow -x^*} g'(x) - \lim_{x\downarrow -x^*} g'(x), \qquad \lim_{x\uparrow \overline{x}} g'(x) = \lim_{x\downarrow x^*} g'(x) - \lim_{x\uparrow x^*} g'(x). \tag{A.263}$$

Solving for $g(x)$, we obtain a linear function:

$$g''(x) = 0, \quad g'(x) = A, \quad g(x) = Ax + B. \tag{A.264}$$

We split the state-space into three segments $[-\overline{x}, -x^*] \cup [-x^*, x^*] \cup [x^*, \overline{x}]$ and consider three different functions $g_k(x) = A_k x + B_k$ for $j = 1, 2, 3$, one for each segment. Evaluating at the border conditions, we obtain relationships for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{array}{r} -A_1\overline{x} + B_1 = 0 \\ A_3\overline{x} + B_3 = 0 \end{array}\right\} \quad\Longrightarrow\quad \overline{x} = B_1/A_1 = -B_3/A_3. \tag{A.265}$$

Evaluating at the reinjection conditions, we obtain A_2:

$$\left.\begin{array}{r} A_1 = A_1 - A_2 \\ A_3 = A_3 - A_2 \end{array}\right\} \quad\Longrightarrow\quad A_2 = 0. \tag{A.266}$$

Evaluating at the continuity conditions, using $A_2 = 0$ we obtain for (A_1, B_1) and (A_3, B_3):

$$\left.\begin{array}{r} B_2 = -A_1 x^* + B_1 \\ B_2 = A_3 x^* + B_3 \end{array}\right\} \quad\Longrightarrow\quad x^* = \frac{B_1 - B_2}{A_1} = \frac{B_2 - B_3}{A_3}. \tag{A.267}$$

Finally, we use the fact that the density integrates to one:

$$\begin{aligned} 1 &= \int_{-\overline{x}}^{-x^*} (A_1 x + B_1)\,\mathrm{d}x + \int_{-x^*}^{x^*} B_2\,\mathrm{d}x + \int_{x^*}^{\overline{x}} (A_3 x + B_3)\,\mathrm{d}x \\ &= \left(A_1\frac{x^2}{2} + B_1 x\right)\Big|_{-\overline{x}}^{-x^*} + B_2 x\Big|_{-x^*}^{x^*} + \left(A_3\frac{x^2}{2} + B_3 x\right)\Big|_{x^*}^{\overline{x}} \\ &= A_1\left(\frac{x^{*2} - \overline{x}^2}{2}\right) + B_1(\overline{x} - x^*) + 2B_2 x^* + A_3\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + B_3(\overline{x} - x^*) \\ &= (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2 x^* + (B_1 + B_3)(\overline{x} - x^*). \end{aligned} \tag{A.268}$$

Substituting $B_1 = \overline{x}A_1$ and $B_3 = -\overline{x}A_3$ from (A.265) into the previous expression:

$$1 = (A_3 - A_1)\left(\frac{\overline{x}^2 - x^{*2}}{2}\right) + 2B_2x^* - \overline{x}(A_3 - A_1)(\overline{x} - x^*) \tag{A.269}$$

$$= (A_3 - A_1)(\overline{x} - x^*)\left[\frac{\overline{x} + x^*}{2} - \overline{x}\right] + 2B_2x^*$$

$$= (A_3 - A_1)\frac{(\overline{x} - x^*)^2}{2} + 2B_2x^*.$$

Therefore, the cross-sectional density is equal to:

$$g(x) = \frac{1}{\overline{x}^2 - x^{*2}}\begin{cases} \overline{x} + x & \text{for} \quad x \in [-\overline{x}, -x^*] \\ \overline{x} - x^* & \text{for} \quad x \in [-x^*, x^*] \\ \overline{x} - x & \text{for} \quad x \in [x^*, \overline{x}]. \end{cases} \tag{A.270}$$

Renewal probabilities and relative shares. The renewal probabilities (the mass of adjusters from each reset point) are equal to:

$$\mathcal{N}^- = \frac{\sigma^2}{2}\lim_{x \downarrow -\overline{x}} g'(x) = \frac{\sigma^2}{2}A_1 = \frac{\sigma^2}{2}\frac{1}{(\overline{x}^2 - x^{*2})} \tag{A.271}$$

$$\mathcal{N}^+ = -\frac{\sigma^2}{2}\lim_{x \uparrow \overline{x}} g'(x) = -\frac{\sigma^2}{2}A_3 = \frac{\sigma^2}{2}\frac{1}{(\overline{x}^2 - x^{*2})}. \tag{A.272}$$

The shares of total, upward, and downward adjustment are:

$$\mathcal{N} = \mathcal{N}^- + \mathcal{N}^+ = \frac{\sigma^2}{(\overline{x}^2 - x^{*2})} \tag{A.273}$$

$$\frac{\mathcal{N}^-}{\mathcal{N}} = \frac{1}{2}; \qquad \frac{\mathcal{N}^+}{\mathcal{N}} = \frac{1}{2}. \tag{A.274}$$

Probability of negative adjustment. Let $\mathbb{P}^+(x) \equiv \Pr[\Delta x < 0|x]$ denote the probability of doing a negative adjustment (after hitting the upper bound) conditional on the state x. It solves the HJB with border conditions:

$$0 = \mathbb{P}^{+\prime\prime}(x); \quad \mathbb{P}^+(\overline{x}) = 1; \quad \mathbb{P}^+(-\overline{x}) = 0 \tag{A.275}$$

Solving for $\mathbb{P}^+(x) = Ax + B$ and evaluating at the border conditions:

$$\left.\begin{array}{r} A\overline{x} + B = 1 \\ -A\overline{x} + B = 0 \end{array}\right\} \implies \left.\begin{array}{c} A = 1/2\overline{x} \\ B = 1/2 \end{array}\right\} \implies \mathbb{P}^+(x) = \frac{\overline{x} + x}{2\overline{x}} = \frac{1}{2} + \frac{x}{2\overline{x}}. \tag{A.276}$$

The unconditional probability of a negative adjustment is:

$$\mathbb{E}[\mathbb{P}^+] = \frac{1}{2} + \frac{1}{2\overline{x}}\mathbb{E}[x] = \frac{1}{2}. \tag{A.277}$$

The probability of a negative adjustment conditional on the last adjusting being positive (a switch in adjustment sign) equals:

$$\mathbb{P}^+(-x^*) \equiv \Pr[\Delta x < 0| - x^*] = \frac{\overline{x} - x^*}{2\overline{x}}. \tag{A.278}$$

Probability of positive adjustment. Let $\mathbb{P}^-(x) \equiv \Pr[\Delta x > 0|x]$ denote the probability of doing a positive adjustment (after hitting the lower bound) conditional on the state x. It solves the HJB with border conditions:

$$0 = \mathbb{P}^{-\prime\prime}(x); \quad \mathbb{P}^-(-\overline{x}) = 1; \quad \mathbb{P}^-(\overline{x}) = 0. \tag{A.279}$$

Solving for $\mathbb{P}^-(x) = Ax + B$ and evaluating at the border conditions:

$$\left.\begin{aligned} -A\overline{x} + B = 1 \\ A\overline{x} + B = 0 \end{aligned}\right\} \quad \left.\begin{aligned} A = -1/2\overline{x} \\ B = 1/2 \end{aligned}\right\} \quad \mathbb{P}^-(x) = \frac{\overline{x} - x}{2\overline{x}} \;=\; \frac{1}{2} - \frac{x}{2\overline{x}}. \tag{A.280}$$

The unconditional probability of a positive adjustment is:

$$\mathbb{E}[\mathbb{P}^-] \;=\; \frac{1}{2} - \frac{1}{2\overline{x}}\mathbb{E}[x] \;=\; \frac{1}{2}. \tag{A.281}$$

The probability of a positive adjustment conditional on the last adjusting being negative (a switch in adjustment sign) equals:

$$\mathbb{P}^-(x^*) \equiv \Pr[\Delta x > 0 | x^*] \;=\; \frac{\overline{x} - x^*}{2\overline{x}}. \tag{A.282}$$

A.10.4 Expected duration of inaction

Let $T(x) \equiv \mathbb{E}[\tau | x]$. It solves the HJB with border conditions:

$$0 \;=\; 1 + \frac{\sigma^2}{2} T''(x), \qquad T(\overline{x}) \;=\; T(-\overline{x}) \;=\; 0. \tag{A.283}$$

Solving for $T(x)$:

$$T''(x) \;=\; -\frac{2}{\sigma^2}, \quad T'(x) \;=\; -\frac{2}{\sigma^2}x + A, \quad T(x) \;=\; -\frac{x^2}{\sigma^2} + Ax + B. \tag{A.284}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\left.\begin{aligned} -\frac{\overline{x}^2}{\sigma^2} + A\overline{x} + B = 0 \\ -\frac{\overline{x}^2}{\sigma^2} - A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} 2A\overline{x} = 0 \\ -\frac{2\overline{x}^2}{\sigma^2} + 2B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = 0 \\ B = \frac{\overline{x}^2}{\sigma^2} \end{aligned}\right\} \implies T(x) = \frac{\overline{x}^2 - x^2}{\sigma^2}. \tag{A.285}$$

The expected duration of inaction given the current state $\mathbb{E}[\tau|x]$, the expected duration of a complete inaction spell conditional on the last reset point $(\overline{\mathbb{E}}^+[\tau], \overline{\mathbb{E}}^-[\tau])$, and the unconditional expected duration of inaction $\overline{\mathbb{E}}[\tau]$ are given by:

$$\mathbb{E}[\tau|x] \;=\; \frac{\overline{x}^2 - x^2}{\sigma^2}, \tag{A.286}$$

$$\overline{\mathbb{E}}^+[\tau] \;=\; \overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{A.287}$$

$$\overline{\mathbb{E}}[\tau] \;=\; \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[\tau] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[\tau] = \frac{\overline{x}^2 - x^{*2}}{\sigma^2}, \tag{A.288}$$

where the shares of upward and downward adjustment are identical: $\mathcal{N}^+/\mathcal{N} = \mathcal{N}^-/\mathcal{N} = 1/2$.

A.10.5 Cross-sectional means

Let $m(x) \equiv \mathbb{E}\left[\int_0^\tau x_s \,\mathrm{d}s | x_0 = x\right]$. It solves the HJB with border conditions:

$$0 = x + \frac{\sigma^2}{2} m''(x), \quad m(\overline{x}) = m(-\overline{x}) = 0. \tag{A.289}$$

Solving for $m(x)$:

$$m''(x) = -\frac{2}{\sigma^2}x, \qquad m'(x) = -\frac{x^2}{\sigma^2} + A, \qquad m(x) = -\frac{x^3}{3\sigma^2} + Ax + B. \tag{A.290}$$

Evaluating at the border conditions, we obtain values for A and B:

$$\text{(A.291)}\qquad \left.\begin{aligned} -\frac{\overline{x}^3}{3\sigma^2} + A\overline{x} + B = 0 \\ \frac{\overline{x}^3}{3\sigma^2} - A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = \frac{\overline{x}^2}{3\sigma^2} \\ B = 0 \end{aligned}\right\} \implies m(x) = \frac{\overline{x}^2 x - x^3}{3\sigma^2} = \frac{x}{3}\frac{\overline{x}^2 - x^2}{\sigma^2} = \frac{x}{3}\mathbb{E}[\tau|x]$$

Unconditional means. Using the occupancy measure, we obtain the means conditional on the last rest point:

$$\text{(A.292)}\qquad \overline{\mathbb{E}}^-[x] = \frac{m(-x^*)}{\overline{\mathbb{E}}^-[\tau]} = -\frac{x^*}{3}; \qquad \overline{\mathbb{E}}^+[x] = \frac{m(x^*)}{\overline{\mathbb{E}}^+[\tau]} = \frac{x^*}{3},$$

where $\overline{\mathbb{E}}^-[\tau] = \mathbb{E}[\tau| - x^*]$ and $\overline{\mathbb{E}}^+[\tau] = \mathbb{E}[\tau|x^*]$.

Unconditional mean. By symmetry, $\mathbb{E}[x] = 0$. To show this formally, we use the conditional means and the renewal distribution:

$$\text{(A.293)}\qquad \mathbb{E}[x] = \frac{\mathcal{N}^+}{\mathcal{N}}\overline{\mathbb{E}}^+[x] + \frac{\mathcal{N}^-}{\mathcal{N}}\overline{\mathbb{E}}^-[x] = \frac{1}{2}\left(\frac{x^*}{3}\right) + \frac{1}{2}\left(\frac{-x^*}{3}\right) = 0.$$

A.10.6 Cross-sectional variances

Unconditional variance. Since $\mathbb{E}[x] = 0$, then $\mathbb{V}ar[x] = \mathbb{E}[x^2]$. Using the cross-sectional distribution, the second moment equals:

$$\text{(A.294)}\qquad \begin{aligned} \mathbb{V}ar[x] &= \int_{-\overline{x}}^{\overline{x}} x^2 g(x)\,\mathrm{d}x \\ &= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\int_{-\overline{x}}^{-x^*} x^2(\overline{x}+x)\,\mathrm{d}x + (\overline{x}-x^*)\int_{-x^*}^{x^*} x^2\,\mathrm{d}x + \int_{x^*}^{\overline{x}} x^2(\overline{x}-x)\,\mathrm{d}x\right] \\ &= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\left(\frac{x^3\overline{x}}{3} + \frac{x^4}{4}\right)\Big|_{-\overline{x}}^{-x^*} + (\overline{x}-x^*)\frac{x^3}{3}\Big|_{-x^*}^{x^*} + \left(\frac{x^3\overline{x}}{3} - \frac{x^4}{4}\right)\Big|_{x^*}^{\overline{x}}\right] \\ &= \frac{1}{(\overline{x}^2 - x^{*2})}\left[\frac{-x^{*3}\overline{x}}{3} + \frac{x^{*4}}{4} + \frac{\overline{x}^{*4}}{3} - \frac{\overline{x}^4}{4} + (\overline{x}-x^*)\frac{x^{*3}+x^{*3}}{3} + \frac{\overline{x}^4}{3} - \frac{\overline{x}^4}{4} - \frac{x^{*3}\overline{x}}{3} + \frac{x^{*4}}{4}\right] \\ &= \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{\overline{x}^4 - x^{*4}}{6}\right) = \frac{1}{(\overline{x}^2 - x^{*2})}\left(\frac{(\overline{x}^2 - x^{*2})(\overline{x}^2 + x^{*2})}{6}\right) \\ &= \frac{\overline{x}^2 + x^{*2}}{6} \end{aligned}$$

Conditional variance. Let $m_2(x) \equiv \mathbb{E}\left[\int_0^\tau x_s^2\,\mathrm{d}s|x_0 = x\right]$. It solves the HJB with borders:

$$\text{(A.295)}\qquad 0 = x^2 + \frac{\sigma^2}{2}m''(x), \quad m_2(\overline{x}) = m_2(-\overline{x}) = 0.$$

Solving for $m_2(x)$:

$$\text{(A.296)}\qquad m_2''(x) = -\frac{2x^2}{\sigma^2}, \qquad m_2'(x) = -\frac{2x^3}{3\sigma^2} + A, \qquad m(x) = -\frac{x^4}{6\sigma^2} + Ax + B.$$

Evaluating at the border conditions, we obtain values for A and B:

$$\text{(A.297)}\qquad \left.\begin{aligned} -\frac{\overline{x}^4}{6\sigma^2} + A\overline{x} + B = 0 \\ -\frac{\overline{x}^4}{6\sigma^2} - A\overline{x} + B = 0 \end{aligned}\right\} \implies \left.\begin{aligned} A = 0 \\ B = \frac{\overline{x}^4}{6\sigma^2} \end{aligned}\right\} \implies m_2(x) = \frac{\overline{x}^4 - x^4}{6\sigma^2} = \frac{\overline{x}^2 + x^2}{6}\mathbb{E}[\tau|x].$$

Using the occupancy measure, the second moments conditional on the last rest point are:

$$\overline{\mathbb{E}}^{+}[x^2] \;=\; \frac{m_2(x^*)}{\overline{\mathbb{E}}[\tau|x^*]} \;=\; \frac{\overline{x}^2+x^{*2}}{6}; \qquad \overline{\mathbb{E}}^{-}[x^2] \;=\; \frac{m_2(-x^*)}{\overline{\mathbb{E}}[\tau|-x^*]} \;=\; \frac{\overline{x}^2+x^{*2}}{6}. \tag{A.298}$$

Finally, by definition of the variance, we have that:

$$\overline{\mathbb{V}ar}^{+}[x] \;=\; \overline{\mathbb{E}}^{+}[x^2]-(\overline{\mathbb{E}}^{+}[x])^2 \;=\; \frac{\overline{x}^2+x^{*2}}{6}-\frac{x^{*2}}{9} \tag{A.299}$$

$$\overline{\mathbb{V}ar}^{-}[x] \;=\; \overline{\mathbb{E}}^{-}[x^2]-(\overline{\mathbb{E}}^{-}[x])^2 \;=\; \frac{\overline{x}^2+x^{*2}}{6}-\frac{x^{*2}}{9} \tag{A.300}$$

where the conditional means are computed in (A.292).

Variance decomposition. According to the law of total variance:

$$\underbrace{\mathbb{V}ar[x]}_{total} \;=\; \underbrace{\mathbb{E}[\mathbb{V}ar[x|\Delta x]]}_{within} \;+\; \underbrace{\mathbb{V}ar[\mathbb{E}[x|\Delta x]]}_{between}. \tag{A.301}$$

The *within variance* is computed as:

$$\begin{aligned}\mathbb{E}[\mathbb{V}ar[x|\Delta x]] &= \frac{\mathcal{N}^+}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{+}[\tau]}{\overline{\mathbb{E}}[\tau]}\mathbb{V}ar^{+}[x]+\frac{\mathcal{N}^-}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{-}[\tau]}{\overline{\mathbb{E}}[\tau]}\mathbb{V}ar^{-}[x] \\ &= 2\left(\frac{1}{2}\right)(1)\left(\frac{\overline{x}^2+x^{*2}}{6}-\frac{x^{*2}}{9}\right)=\frac{\overline{x}^2+x^{*2}}{6}-\frac{x^{*2}}{9}.\end{aligned} \tag{A.302}$$

The *between variance* is computed as:

$$\begin{aligned}\overline{\mathbb{V}ar}[\mathbb{E}[x|\Delta x]] &= \frac{\mathcal{N}^+}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{+}[\tau]}{\overline{\mathbb{E}}[\tau]}(\mathbb{E}^{+}[x]-\mathbb{E}[x])^2+\frac{\mathcal{N}^-}{\mathcal{N}}\frac{\overline{\mathbb{E}}^{-}[\tau]}{\overline{\mathbb{E}}[\tau]}(\mathbb{E}^{-}[x]-\mathbb{E}[x])^2 \\ &= \frac{1}{2}(1)\left(\frac{x^*}{3}\right)^2+\frac{1}{2}(1)\left(-\frac{x^*}{3}\right)^2 \;=\; \frac{x^{*2}}{9}.\end{aligned} \tag{A.303}$$

In the benchmark cases we decompose the total variance as:

$$\mathbb{V}ar[x] \;=\; \begin{cases} \underbrace{\frac{\overline{x}^2}{6}}_{total} = \underbrace{\frac{\overline{x}^2}{6}}_{within} + \underbrace{0}_{between} & \text{if} \quad \tilde{p}=0 \\ \underbrace{\frac{\overline{x}^2}{3}}_{total} = \underbrace{\frac{2\overline{x}^2}{9}}_{within} + \underbrace{\frac{\overline{x}^2}{9}}_{between} & \text{if} \quad \tilde{\theta}=0. \end{cases} \tag{A.304}$$

A.10.7 Aggregate q.

From (40) and (42) aggregate q with taxes and without drift equals:

$$q \;=\; \frac{1}{\tilde{r}}\left[\frac{1-t^c}{1-t^d}\frac{\alpha A\hat{Y}}{p\hat{K}} \;+\; \frac{\sigma^2}{2} \;-\; \frac{\overline{\mathbb{C}ov}\left[\Delta\hat{k},\mathcal{P}(\Delta\hat{k})\right]}{\overline{\mathbb{E}}[\tau]}\right]. \tag{A.305}$$

We first compute the aggregate productivity term and the irreversibility terms.

Aggregate productivity. The numbers refer to the steps below. In equality (1) we start from the approximation of $\hat{Y}/\hat{K}$; in equality (2) we substitute $\mathbb{E}[\hat{k}]=\hat{k}^{ss}+\mathbb{E}[x]=\frac{1}{1-\alpha}\log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)+\mathbb{E}[x]$ and $\mathbb{V}ar[\hat{k}]=\mathbb{V}ar[x]$; in equality (3) we simplify, in equality (4) we do a Taylor approximation to the exponential function, and in equality (5) we substitute $\mathbb{E}[x]=0$:

$$\begin{aligned}
\frac{\hat{Y}}{\hat{K}} &=^{(1)} \exp\left\{-(1-\alpha)\left(\mathbb{E}[\hat{k}]+\frac{\alpha}{2}\mathbb{V}ar[\hat{k}]\right)\right\} \\
&=^{(2)} \exp\left\{-(1-\alpha)\left(\frac{1}{1-\alpha}\log\left(\frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{\mathcal{U}}}\right)+\mathbb{E}[x]+\frac{\alpha}{2}\mathbb{V}ar[x]\right)\right\} \\
&=^{(3)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\exp\left\{-(1-\alpha)\mathbb{E}[x]-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right\} \\
&=^{(4)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\left(1-(1-\alpha)\mathbb{E}[x]-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) \\
&=^{(5)} \frac{p\tilde{\mathcal{U}}}{\alpha A}\frac{1-t^d}{1-t^c}\left(1-\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right). \qquad \text{(A.306)}
\end{aligned}$$

Irreversibility term. The numerator in the irreversibility term equals the covariance of adjustment size and the auxiliary pricing function $\mathcal{P}(\Delta x)$ defined in (A.89). We compute this term using conditioning on the last reset point and then averaging with the conditional renewal distribution:

$$\begin{aligned}
\overline{\mathbb{C}ov}\left[\Delta x,\mathcal{P}(\Delta x)\right] &= \overline{\mathbb{E}}\left[\Delta x\mathcal{P}(\Delta x)\right]-\overline{\mathbb{E}}\left[\Delta x\right]\overline{\mathbb{E}}\left[\mathcal{P}(\Delta x)\right] \\
&= \frac{1}{2}\left(\overline{\mathbb{E}}^{-}\left[\Delta x\mathcal{P}(\Delta x)\right]+\overline{\mathbb{E}}^{+}\left[\Delta x\mathcal{P}(\Delta x)\right]\right) \\
&= \frac{1}{2}\left((\overline{x}-x^*)\left(\frac{p^{buy}}{p}-1\right)+(x^*-\overline{x})\left(\frac{p^{sell}}{p}-1\right)\right) \\
&= \frac{(\overline{x}-x^*)}{2}\left[\frac{p^{buy}-p^{sell}}{p}\right] \\
&= (\overline{x}-x^*)\frac{\tilde{\mathcal{U}}}{\alpha}\tilde{p}.
\end{aligned}$$

In the last line we use the relationship between effective and fundamental price wedge in (A.231) and the assumption of a symmetric wedge:

$$\frac{p^{buy}-p^{sell}}{p}=\frac{\tilde{\mathcal{U}}}{\alpha}(\tilde{p}^{buy}-\tilde{p}^{sell})=\frac{2\tilde{\mathcal{U}}\tilde{p}}{\alpha}. \qquad \text{(A.307)}$$

The denominator of the irreversibility term is the expected duration of inaction in (A.288): $\overline{\mathbb{E}}[\tau]=(\overline{x}^2-x^{*2})/\sigma^2$. Therefore, the irreversibility term for aggregate q equals:

$$-\frac{\overline{\mathbb{C}ov}\left[\Delta x,\mathcal{P}(\Delta x)\right]}{\overline{\mathbb{E}}[\tau]} = -\frac{(\overline{x}-x^*)\tilde{\mathcal{U}}\tilde{p}}{\alpha}\frac{\sigma^2}{\overline{x}^2-x^{*2}} = -\frac{\tilde{\mathcal{U}}}{\alpha}\frac{\tilde{p}\sigma^2}{\overline{x}+x^*}=-\frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)<0. \qquad \text{(A.308)}$$

where the last equality uses the first condition for the optimal policy in (A.255). Finally, substituting the aggregate productivity term (A.306) and the irreversibility term (A.308) into the expression for q (without drift), simplifying, and using the driftless aftertax user cost $\tilde{\mathcal{U}} = \tilde{\rho} - \sigma^2$ and the driftless after-tax discount $\tilde{r} = \tilde{\rho} - \sigma^2/2$:

$$\begin{aligned} q &= \frac{1}{\tilde{r}}\left[\frac{1-t^c}{1-t^d}\frac{\alpha A}{p}\frac{\hat{Y}}{\hat{K}} + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\ &= \frac{1}{\tilde{r}}\left[\frac{1-t^c}{1-t^d}\frac{\alpha A}{p}\frac{1-t^d}{1-t^c}\frac{p\tilde{\mathcal{U}}}{\alpha A}\left(1 \;-\; \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\ &= \frac{1}{\tilde{r}}\left[\tilde{\mathcal{U}}\left(1 \;-\; \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) + \frac{\sigma^2}{2} - \frac{\overline{x}x^*}{3}\tilde{\mathcal{U}}(1-\alpha)\right] \\ &= \frac{\tilde{\mathcal{U}} + \sigma^2/2}{\tilde{r}} \;-\; \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x] - \frac{\overline{x}x^*}{3\tilde{r}}\tilde{\mathcal{U}}(1-\alpha) \\ &= 1 \;-\; \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\left[\mathbb{V}ar[x] + \frac{2}{\alpha}\frac{\overline{x}x^*}{3}\right]. \end{aligned}$$

In the benchmark cases:

$$\text{(A.309)} \qquad q \;=\; \begin{cases} 1 \;-\; \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x] & \text{if} \quad \tilde{p} = 0, \\ 1 \;-\; \frac{\tilde{\mathcal{U}}}{\tilde{r}}\frac{\alpha(1-\alpha)}{2}\left(1 + \frac{2}{\alpha}\right)\mathbb{V}ar[x] & \text{if} \quad \tilde{\theta} = 0, \end{cases}$$

where for the case $\tilde{\theta} = 0$ we note that $\mathbb{V}ar[x] = \overline{x}^{*2}/3$.

A.10.8 CIR

From (49) and (51), the CIR without drift equals:

$$\text{(A.310)} \qquad \frac{\text{CIR}(\delta)}{\delta} \;=\; \frac{\mathbb{V}ar[x]}{\sigma^2} \;-\; \frac{\overline{\mathbb{C}ov}[\Delta\hat{k}, \mathcal{M}(\Delta\hat{k})]}{\overline{\mathbb{E}}[\tau]} \;+\; o(\delta).$$

Cumulative deviations. Recall the values for the unconditional probabilities of a negative and a positive adjustment $\mathbb{E}[\mathbb{P}^+] = \mathbb{E}[\mathbb{P}^-] = 1/2$ in (A.277) and (A.281), and the conditional probabilities of switching adjustment sign $\mathbb{P}^+(-x^*) = \mathbb{P}^-(x^*) = (\overline{x} - x^*)/2\overline{x}$ in (A.278) and (A.282). Substituting these probabilities, the conditional means $\mathbb{E}^-[x] = -x^*/3$ and $\mathbb{E}^-[x] = x^*/3$ in (A.292), and the conditional durations $\mathbb{E}^-[\tau] = \mathbb{E}^+[\tau] = (\overline{x}^2 - x^{*2})/\sigma^2$ in (A.287) into the definition of cumulative deviations $\mathcal{M}^{buy}$ in (46) and $\mathcal{M}^{sell}$ in (47) yields:

$$\text{(A.311)} \quad \mathcal{M}^{buy} = \mathbb{E}[\mathbb{P}^-]\frac{1}{\mathbb{P}^+(-x^*)}(\mathbb{E}^-[x] - \mathbb{E}[x])\mathbb{E}^-[\tau] \;=\; \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x} - x^*}\right)\left(-\frac{x^*}{3} - 0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} \;=\; -\frac{x^*\overline{x}(\overline{x} + x^*)}{3\sigma^2},$$

$$\text{(A.312)} \quad \mathcal{M}^{sell} = \mathbb{E}[\mathbb{P}^+]\frac{1}{\mathbb{P}^-(x^*)}(\mathbb{E}^+[x] - \mathbb{E}[x])\mathbb{E}^+[\tau] \;=\; \frac{1}{2}\left(\frac{2\overline{x}}{\overline{x} - x^*}\right)\left(\frac{x^*}{3} - 0\right)\frac{\overline{x}^2 - x^{*2}}{\sigma^2} \;=\; \frac{x^*\overline{x}(\overline{x} + x^*)}{3\sigma^2}.$$

Irreversibility term. The irreversibility term for the CIR equals the covariance of the adjustment size and the auxiliary capital-deviation deviation function $\mathcal{M}(\Delta x)$ defined in (48). Recall $x^* = \overline{x} - \Delta x$ for $\Delta x < 0$ and $-x^* = -\overline{x} - \Delta x$ for $\Delta x > 0$ and by symmetry $\overline{\mathbb{E}}[\Delta x] = 0$. The numerator of the irreversibility term equals:

$$\begin{aligned} \overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)] &= \overline{\mathbb{E}}[\Delta x \mathcal{M}(\Delta x)] - \overline{\mathbb{E}}[\Delta x]\,\overline{\mathbb{E}}[\mathcal{M}(\Delta x)] \\ &= \frac{1}{2}\left[\overline{\mathbb{E}}^-[\Delta x\mathcal{M}(\Delta x)] + \overline{\mathbb{E}}^+[\Delta x\mathcal{M}(\Delta x)]\right] \\ &= \frac{1}{2}\left[(\overline{x} - x^*)\mathcal{M}^{buy} + (x^* - \overline{x})\mathcal{M}^{sell}\right] = (\overline{x} - x^*)\mathcal{M}^{buy}, \\ &= -(\overline{x} - x^*)\frac{x^*\overline{x}(\overline{x} + x^*)}{3\sigma^2} = -\frac{x^*\overline{x}}{3}\left(\frac{\overline{x}^2 - x^{*2}}{\sigma^2}\right) \\ &= -\frac{x^*\overline{x}}{3}\overline{\mathbb{E}}[\tau]. \end{aligned}$$

Therefore, the irreversibility term of the CIR equals:

$$-\frac{\overline{\mathbb{C}ov}[\Delta x, \mathcal{M}(\Delta x)]}{\overline{\mathbb{E}}[\tau]} \;=\; \frac{x^*\overline{x}}{3} > 0. \tag{A.313}$$

Finally, substituting the expression for the cross-sectional variance in (A.294) and the irreversibility term in (A.313) into the CIR yields:

$$\frac{\text{CIR}(\delta)}{\delta} \;=\; \frac{\overline{x}^2 + {x^*}^2}{6\sigma^2} \;+\; \frac{x^*\overline{x}}{3} \;+\; o(\delta). \tag{A.314}$$

In the benchmark cases:

$$\text{CIR} \;=\; \begin{cases} \frac{\overline{x}^2}{6\sigma^2} \;+\; o(\delta) & \text{if} \quad \tilde{p} = 0, \\[2ex] \frac{(1+\sigma^2){\overline{x}^*}^2}{3\sigma^2} \;+\; o(\delta) & \text{if} \quad \tilde{\theta} = 0, \end{cases} \tag{A.315}$$

A.11 Proof of Proposition 8

This proposition computes the optimal investment policy and the macro outcomes for models with very large drift relative to idiosyncratic costs. Assume $\nu \to \infty$. If $\nu \to \infty$ then economy converges to an economy with $\nu > 0$ and $\sigma \to 0$ since idiosyncratic shocks are small relative to the drift—see Álvarez, Beraja, Gonzalez-Rozada and Neumeyer (2018) for details in the context of price-setting. The firm investment problem is given by:

$$\mathcal{V}(x) \;=\; \max_{\tau,\Delta x} \mathbb{E}\left[\int_0^\tau e^{-\tilde{r}\tau}(x_s^\alpha - \alpha x_s)\,\mathrm{d}s + e^{-\tilde{r}\tau}\left(-\tilde{\theta} \;+\; \mathcal{V}(x_\tau)\right)\Big| x_0 = x\right], \tag{A.316}$$

$$\mathrm{d}x_s \;=\; -\nu x_s\,\mathrm{d}s. \tag{A.317}$$

A.11.1 Sufficient optimality conditions

This problem was studied by Sheshinski and Weiss (1977). They show that the optimal policy consists of a one-sided inaction region with lower threshold x^- and a reset point x^*. Since there are no idiosyncratic shock, there is no mass above x^* and the cross-sectional distribution is uniform in the range $[x^-, x^*]$.

Let $h(x) \equiv x^\alpha - \alpha x$. Following Sheshinski and Weiss (1977), the optimal policy satisfies the following conditions (in our notation):

$$h(x^*) \;-\; h(x^-) \;=\; \tilde{r}\tilde{\theta}, \tag{A.318}$$

$$\int_{e^{x^-}}^{e^{x^*}} h'(x)\; x^{\frac{\tilde{r}}{\nu}}\,\mathrm{d}x \;=\; 0. \tag{A.319}$$

The first optimality condition equals:

$$e^{\alpha x^*} - \alpha e^{x^*} = \tilde{r}\tilde{\theta} + e^{\alpha x^-} - \alpha e^{x^-}. \tag{A.320}$$

The second optimality condition equals:

$$\int_{e^{x^-}}^{e^{x^*}} \alpha(x^{\alpha-1} - 1)x^{\frac{\tilde{r}}{\nu}}\,\mathrm{d}x \;=\; \frac{x^{\tilde{r}/\nu} + \alpha +}{\tilde{r}/\nu + \alpha +} - \frac{x^{\tilde{r}/\nu} + 1}{r/\nu + 1}\Bigg|_{e^{x^-}}^{e^{x^*}} = 0, \tag{A.321}$$

or equivalently:

$$\frac{e^{(\tilde{r}/\nu+\alpha)x^*} - e^{(\tilde{r}/\nu+\alpha)x^-}}{r/\nu + \alpha} \;=\; \frac{e^{(\tilde{r}/\nu+1)x^*} - e^{(\tilde{r}/\nu+1)x^-}}{r/\nu + 1}. \tag{A.322}$$

A.11.2 Optimal investment policy

Next we approximate the decision problem. As in (A.239), we do a second-order Taylor approximation of the flow profits $\pi(x) = e^{\alpha x} - \alpha e^x$ around $x = 0$: $\pi(x) \;=\; (1-\alpha) - \frac{\alpha(1-\alpha)}{2}x^2$. Applying this approximation to the first optimality condition in (A.320) yields:

$${x^*}^2 - {x^-}^2 \;=\; -\frac{2\tilde{r}\tilde{\theta}}{\alpha(1-\alpha)}. \tag{A.323}$$

Since the cross-sectional distribution is uniform in the range $[x^-, x^*]$, it has the following moments:

$$\mathbb{V}ar[x] \;=\; \frac{(x^* - x^-)^2}{12}; \qquad \mathbb{E}[x] \;=\; \frac{x^* + x^-}{2}. \tag{A.324}$$

Thus we can write the first optimality condition in (A.323) as:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} \;=\; -\frac{\tilde{r}\tilde{\theta}}{2\sqrt{3}\alpha(1-\alpha)}. \tag{A.325}$$

Next, consider a third-order approximation to the following exponential function:

$$\frac{e^{\beta x}}{\beta} = 1 + x + \frac{\beta}{2}x^2 + \frac{\beta^2}{6}x^2. \tag{A.326}$$

Apply this approximation to both sides of the second optimality condition, using $\beta \in \{\tilde{r}/\nu + \alpha, \tilde{r}/\nu + 1\}$, and rearrange to obtain:

$$\left[\frac{(\tilde{r}/\nu + \alpha) - (\tilde{r}/\nu + 1)}{2}\right](x^{*2} - x^{-2}) + \left[\frac{(\tilde{r}/\nu + \alpha)^2 - (\tilde{r}/\nu + 1)^2}{6}\right](x^{*3} - x^{-3}) = 0 \tag{A.327}$$

$$\frac{(x^* + x^-)}{2} + \frac{(2\tilde{r}/\nu + \alpha + 1)}{6}((x^* + x^-)^2 - x^* x^-) = 0. \tag{A.328}$$

Note the following relationship:

$$-x^* x^- = \frac{(x^* - x^-)^2 - (x^* + x^-)^2}{4} \tag{A.329}$$

Substituting back this expression into (A.328) and also $\mathbb{E}[x] = (x^* + x^-)/2$ we get:

$$\mathbb{E}[x] + \frac{(2\tilde{r}/\nu + \alpha + 1)}{6}\left((x^* + x^-)^2 + \frac{(x^* - x^-)^2 - (x^* + x^-)^2}{4}\right) = 0 \tag{A.330}$$

$$6\mathbb{E}[x] + (2\tilde{r}/\nu + \alpha + 1)\left(\frac{3(x^* + x^-)^2 + (x^* - x^-)^2}{4}\right) = 0 \tag{A.331}$$

$$\mathbb{E}[x] = -\left(\frac{\tilde{r}}{\nu} + \frac{\alpha + 1}{2}\right)\mathbb{E}[x^2]. \tag{A.332}$$

Therefore, together with (A.324), the optimal policy (x^-, x^*) is characterized by the following 2×2 non-linear system:

$$\mathbb{E}[x]\sqrt{\mathbb{V}ar[x]} = -\frac{\tilde{r}\tilde{\theta}}{\sqrt{12}\alpha(1-\alpha)}; \qquad \frac{\mathbb{E}[x]}{\mathbb{V}ar[x] + \mathbb{E}[x]^2} = -\left(\frac{\tilde{r}}{\nu} + \frac{\alpha + 1}{2}\right). \tag{A.333}$$

A.11.3 Macro outcomes

(i) Misallocation. The cross-sectional variance is the variance of a uniform distribution in the range $[x^-, x^*]$:

$$\mathbb{V}ar[x] = \frac{(x^* - x^-)^2}{12}. \tag{A.334}$$

(ii) Aggregate q. Let $\tilde{\rho} \equiv \rho(1-t^p)/(1-t^g)$. Without idiosyncratic shocks, the after-tax user cost is $\tilde{\mathcal{U}} = \tilde{\rho} + \xi^k$ and the after-tax discount is $\tilde{r} = \tilde{\rho} - \mu$ and simplifying:

$$\begin{aligned} q &= \frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{r}}\frac{\hat{Y}}{\hat{K}} - \frac{\nu}{\tilde{r}} \\ &= \frac{1-t^c}{1-t^d}\frac{\alpha A}{p\tilde{r}}\frac{1-t^d}{1-t^c}\frac{p\tilde{\mathcal{U}}}{\alpha A}\left(1 - (1-\alpha)\mathbb{E}[x] - \frac{\alpha(1-\alpha)}{2}\mathbb{V}ar[x]\right) - \frac{\nu}{\tilde{r}} \\ &= 1 - \frac{\tilde{\rho} + \xi^k}{\tilde{\rho} - \mu}(1-\alpha)\left(\mathbb{E}[x] + \frac{\alpha}{2}\mathbb{V}ar[x]\right) \end{aligned}$$

(iii) CIR: From Corollary 2 in Baley and Blanco (2021):

$$\frac{\text{CIR}(\delta)}{\delta} = 0. \tag{A.335}$$