Price Setting with Strategic Complementarities as a Mean Field Game*

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Abstract

We analyze the impulse response of output after a monetary shock in a general equilibrium setup where firms set prices subject to adjustment costs and feature strategic complementarities with the decision of other firms. The firm's decision problem and the aggregate outcomes are cast as a Mean Field Game (MFG) and several analytical results are established. First, in a canonical menu cost setup featuring an sS rule the MFG framework allows us to study the effect of strategic complementarity/substitutability on the firm's optimal sS rules after the shock. We establish existence and uniqueness of the perturbed equilibrium (as long as the strategic complementarity is not too large) and analytically characterize the impulse response function (IRF) of output. We show that the presence of the strategic complementarity makes the IRF larger at each horizon. On the one hand, as the complementarity becomes large enough, the IRF diverges and at a critical point there is no equilibrium. On the other hand, as substitutability becomes arbitrarily large, the IRF converges to zero. Second, we extend the results to the Calvo$^+$ model, where we show that the cumulative impulse response is approximately proportional to the one without strategic complementarity.

JEL Classification Numbers: E3, E5

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1 Introduction

Following the seminal contributions of Bils and Klenow (2004) and Golosov and Lucas (2007) sticky price models have recorded an impressive progress during the last decade both in the theory, identifying what features are important for the propagation of monetary shocks, as well as in the empirics where new micro evidence has identified the desirable properties of theoretical models.[1] But in spite of the substantive progress, the need for tractability has led most theoretical models to abstract from the interdependence between the firms' decisions in price setting.

While such an abstraction can be justified in a class of general equilibrium models where demand has constant elasticity, exploring the robustness of the model results to the presence of strategic interactions is an important question. The question is relevant for macroeconomics because absent strategic complementarities the current quantitative macro models seem unable to produce the persistent non-neutrality of nominal shocks that is seen in the aggregate data. The idea that strategic complementarities may contribute to amplify the aggregate stickiness has a long tradition in macroeconomics, and was formalized by Cooper and John (1988) in a static setup, and also surveyed by Leahy (2011). Useful explorations can be found in the analysis of monetary shocks by Wang and Werning (2020) in a model of oligopoly, where however the optimal timing of the firm's decision is exogenous, and the where the strategic interactions are industy-wide. A pioneering exploration of a set up with economy wide strategic interactions, and where the occurrence of price changes is endogenous can be found in Klenow and Willis (2016), who numerically solve a state dependent pricing model with strategic complementarities calibrated on US data. Analytic results for dynamic price setting models where the timing of price changes is endogenous and that feature strategic complementarities, do not exist. The numerical analysis of models with strategic complementarities is very valuable, but obviously poses questions of existence and uniqueness of the

[1]See the contributions of Klenow and Malin (2010), Nakamura and Steinsson (2010), Caballero and Engel (1999, 2007), Midrigan (2011), Alvarez and Lippi (2014), Alvarez, Le Bihan, and Lippi (2016),Alvarez, Beraja, Gonzalez-Rozada, and Neumeyer (2019).

equilibrium that could (as we will show) undermine the results. Additionally there are no characterization of the equilibrium.

This paper presents a set of analytical result that allow us to solve for the optimal firm decisions in a dynamic environment in the presence of strategic complementarities (or substitutabilities) with the decision of other firms. The key breakthrough is obtained by formally casting the firm problem, as well as the aggregation of individual decisions, using the mathematical structure of Mean Field Games adapted to the case of problems with fixed costs where the optimal individual decisions follow an sS rule. Additionally it breaks new ground on the Mean Field Game theory by having a fully worked out equilibrium of a model with singular stochastic control, and also in models with strategic complementarity. A notable example of thorough study of a simplified Mean Field Game with impulse control is Bertucci (2017). In the Mean Field Game theory the famous monotonicity condition for Lasry-Lions, used in almost all papers in these area, corresponds to strategic substitutability, and rules out strategic complementarity, the case of substantive interest.

Several analytic results are established. First, we consider a canonical menu cost setup as in Golosov and Lucas (2007) extended to feature first-order strategic complementarities on the price setting. The MFG framework also allows us to study analytically the effect of strategic complementarity/substitutability on the firm's optimal sS rules after the shock. In particular, the firm objective function is to maximize expected discounted profits, net of menu costs. The firm's problem is to optimally decide when to pay a fixed cost and change prices, as well as to what values to set the prices at those times. By strategic complementarity (substitutability) we mean that the flow profit in each period depends on the firm's own markup and the markup of the average firm, with a positive (negative) cross derivative. We fist analyze the simple menu cost model. For the menu cost model, we establish existence and uniqueness of the perturbed equilibrium (as long as the strategic complementarity is not too large) and analytically characterize the impulse response function (IRF) of output. We show that the presence of the strategic complementarity makes the output IRF of a monetary shock larger at each horizon. Not only the effect is larger, but it is also convex on the degree of strategic complementarity/substitutability. Indeed, there is a critical value of the strength of the strategic complementarity at which the IRF becomes arbitrarily large, and then it the equilibrium cease to exists. On the other hand, as substitutability becomes arbitrarily large, the IRF converges to zero. Second, we consider a more general model of price setting, including a free adjustment probability of price adjustment, the so called Calvo plus model, proposed in Nakamura and Steinsson (2010) and solved analytically for the case without strategic complementarity in Alvarez, Le Bihan, and Lippi (2016) and Alvarez and Lippi (2019). This model spans the pure sS price setting model of Golosov and Lucas (2007) to the pure time dependent Calvo (1983) price setting model. We show that the same characterization of equilibrium extends to the Calvo plus model. We analyze the cumulative impulse response function for the Calvo plus model. If there is no strategic complementarity, as shown in Alvarez, Le Bihan, and Lippi (2016), the Cumulative Impulse Response (CIR) to a monetary shocks varies by a factor of 6 as we move form the pure sS Golosov and Lucas (2007) model to the pure time dependent model of Calvo (1983). When we add strategic complementarity/substitutability, we show that the CIR for a Calvo plus model is approximately proportional to CIR for the same Calvo plus model without strategic complementarity/substitutability. The constant of proportionality depends on the strength of the strategic complementarity/substitutability, but the it is (approximately) the same for any Calvo plus model.

2 Mean Field Game for a Price Setting Firm

We describe first the problem of a firm, whose value function u depends on state x and time t. The one dimensional state x represent a deviation from an ideal price, which when uncontrolled follows a Brownian motion with variance per unit of time σ^2 and no drift. The decision maker is a firm, who seeks to minimize the discounted value of the sum of flow cost

F and fixed cost of adjustment ψ. Additionally, at each instant, with a Poisson probability rate $\zeta > 0$, the firm firm can change its price without paying any cost. Let $\rho > 0$ be a discount rate and let $F : \mathbb{R}^2 \to \mathbb{R}$ be the flow return cost of the firm. The flow cost of $F(x, X)$ depends on the firm state x as well as given time path for a variable denote by X. The time dependence of the firm's problem, and in particular for the value function u arises from the path X.

The only actions the firms takes consists on deciding stopping times $\{\tau_j\}$ denoting when to pay a fixed cost $\psi > 0$ and change prices thereby changing the value of the state at $t = \tau_j$ from $x(\tau_j^-)$ to any desired value $x(\tau_j^+)$. Given the assumptions on F and the fixed cost, the optimal decision rule at each time t consists on dividing the state space on a region where control is not exercised, the inaction region, and a complementary region where control is exercised and the state is reset by an impulse. We can describe the optimal decision rules by three time paths $\bar{x}, \underline{x}$ and x^*. At a given time the optimal decision rule of the firm is represented by the interval $[\underline{x}(t), \bar{x}(t)]$ so that if $x(t)$ is in this interval the firm does *not* exercise control, i.e. inaction is optimal, but if $x(t) \notin (\underline{x}(t), \bar{x}(t))$, the firm exercises control, and changes its price as to immediately change its state from $x(t^-)$ to $x(t^+) = x^*(t)$. Additionally, the firm will change its price so that $x(t^+) = x^*(t)$, if t is a time where ther is a free adjustment opportunity. We refer to $\underline{x}^*(t)$ and $\bar{x}(t)$ as the boundaries of the range of inaction, to $x^*(t)$ as the optimal return point.

The essence of a Mean Field Game is that the path of X that is taken as given by each firm is given by the collective action the firms solving the same problem. We will let $m(x, t)$ be the density of x at time t generated by a collection of firms following the decision rules given by paths $\underline{x}, x^*, \bar{x}$, and a given distribution of x t time $t = 0$. In particular, given the path of the distribution m, we assume that $X(t)$ is the cross sectional average of x using density $m(\cdot, t)$ at each time t.

We will assume that after time $T > 0$, the firms value function becomes $\tilde{u}$, depends on x but does not depend on time. We will allow $T = \infty$. The thresholds $\underline{x}, x^*, \bar{x}$ and the coupling variable X map the time interval $[0,T]$ to the real line. The value function u and density m map $[0,T]\times\mathbb{R}$ into the real line and the positive real line respectively.

A classical solution for a Mean Field Game (MFG) for initial and final conditions m_0, u_T is given by functions $u, m, \underline{x}, \bar{x}, x^*, X$ satisfying for all $t\in[0,T]$:

$$\begin{aligned} 0 &= u_t(x,t) - \rho u(x,t) + \frac{\sigma^2}{2}u_{xx}(x,t) + F(x,X(t)) \\ &\quad + \zeta\left[u(x^*(t),t) - u(x,t)\right] \text{ for all } x\in[\underline{x}(t),\bar{x}(t)] \qquad (1) \\ 0 &= -m_t(x,t) + \frac{\sigma^2}{2}m_{xx}(x,t) - \zeta m(x,t) \\ &\quad \text{for all } x\in[\underline{x}(t),\bar{x}(t)] \text{ and } x\neq x^*(t) \qquad (2) \\ X(t) &= \int_{\underline{x}(t)}^{\bar{x}(t)} x\, m(x,t)dx \text{ for all } t\geq 0 \qquad (3) \end{aligned}$$

where $x^*(t) = \arg\min_x u(x,t)$. Additionally we have the boundary conditions for u for all $t\in[0,T]$:

$$u_x(\bar{x}(t),t) = u_x(\underline{x}(t),t) = u_x(x^*(t),t) = 0 \qquad (4)$$

$$u(\bar{x}(t),t) = u(\underline{x}(t),t) = u(x^*(t),t) + \psi \qquad (5)$$

with terminal condition at $t=T$ given by

$$u(x,T) = u_T(x) \text{ for all } x \qquad (6)$$

The boundary conditions for m for all $t\in[0,T]$ are

$$0 = m(\bar{x}(t),t) = m(\underline{x}(t),t) \text{ for all} \qquad (7)$$

$$1 = \int_{\underline{x}(t)}^{\bar{x}(t)} m(x,t)dx \qquad (8)$$

with initial condition at $t = 0$ given by

$$m(x, 0) = m_0(x) \text{ for all } x \tag{9}$$

We now comment of the interpretation and assumptions that we use for the MFG defined above.

Interpretation of the Flow cost. For the price setting model we interpret the flow cost F as the deviation from the static maximum profit for the firm, which depends on the deviation form the optimal price of the firm x as well as on the price charged by the rest of the firms X. Under this interpretation, during the times where the firm does not change its prices, the deviation of the firms ideal price evolves as a drift-less Brownian motion with variance per unit of time given by σ^2. Hence, $F(\cdot, X)$ has a minimum around the static maximizing x, given X. Our leading example for the function F is the case of a quadratic function of (x, X), which without loss of generality can be written as:

$$F(x, X) = B\,(x + \theta X)^2 \quad \text{with } B > 0 \tag{10}$$

In this case the static profits will be maximized by setting $x = -\theta X$. Thus, absent of adjustment cost the minimum cost is achieved by setting $x = -\theta X$, so if $\theta < 0$ the firms strategies exhibit strategic complementarity, and if $\theta > 0$ they exhibit strategic substitutability. In term of the notions used for MFGs, letting m_i be an arbitrary measure and $X_i \equiv \int x dm_i$, the definition of monotonicity applied to the period return $F(x, X) = B(x + \theta X)^2$ is that for any two $m_1 \neq m_2$ must satisfy the following inequality

$$0 < \int \left(B(x + \theta X_1)^2 - B(x + \theta X_1)^2 \right) (dm_1(x) - dm_2(x)) = 2B\theta(X_1 - X_2)^2$$

Hence, the monotonicity condition in MFGs corresponds to $\theta > 0$, or strategic substitutability.

Connected inaction region. We will assume throughout that the inaction region is connected, i.e. given by a single interval, namely $[\underline{x}(t), \bar{x}(t)]$. In principle, the inaction region could be a union of such intervals. For stationary case one can show that this is not the case, but in the MFG the argument is more involved. This is a moot point when we analyze the perturbation, since we explore variations of the problem nearby the stationary solution.

Boundary conditions for HBJ equation. The boundary conditions for the HBJ in equation (4) are typically referred to as "smooth pasting and "optimal return point", and the ones in equation (5) are referred as "value matching". They are a consequence of our assumption that for each t the value function $u(\cdot, t)$ is once differentiable for all x, and twice differentiable in the range of inaction. In particular, for any x outside the range of inaction, the value function must satisfy $u(x, t) = u(x^*(t), t) + \psi$. Thus, the boundary conditions imposed above assume that $u(\cdot, t)$ is once differentiable everywhere in x. This includes the so called "value matching" and "smooth pasting" conditions. Finally, the optimal return point is required since $x^*(t)$ achieves a minimum of $u(\cdot, t)$.

An alternative to the classical formulation of the HBJ in equation (1) and equation (4)-equation (5) is to write the following variational inequalities:

$$\rho u(x,t) = \tag{11}$$
$$\min\left\{u_t\left(x,t\right) + \frac{\sigma^2}{2}u_{xx}(x,t) + F(x, X(t)) + \zeta\left(\min_{x'} u(x',t) - u(x,t)\right),\ \rho\left(\psi + \min_{x'} u(x',t)\right)\right\}$$

which must hold for all $t \in [0, T]$ and for all x. We can define $x^*(t) = \arg\min_x u(x,t)$. Note that this formulation does not assume that $u(\cdot, t)$ is once differentiable, nor that range of inaction is given by a single interval.

Boundary conditions for KF equation. Under the assumption that the range of inaction is given by a single interval, then there should be zero mass everywhere else, so $m(x, t) = 0$ for all $x \notin [\underline{x}(t), \bar{x}(t)]$. Then, assuming continuity of $m(\cdot, t)$ for all x we obtain the boundary condition in equation (7). This is the condition to be expected at the boundaries of the range of inaction, since they are "exit" points, so that the density cannot accumulate there. Likewise, the Kolmogorov forward equation should not hold at $x = x^*(t)$ since this is an "entry" point, i.e. a point where the flux of density that exits from $x = \underline{x}(t)$ and $\bar{x}(t)$ is entering. The integral condition in equation (8) states that for every t, the function $m(\cdot, t)$ is a density and hence integrates to one, i.e mass is preserved. Finally, while we are not writing it, we require that $m(x,t) \geq 0$ for all x, t.

Differentiating the mass preservation condition equation (8) with respect to time, and using the p.d.e. equation (2) for m we obtain:

$$0 = -\zeta + \frac{\sigma^2}{2}\left[m_x\left(\bar{x}(t), t\right) - m_x^+\left(x^*(t), t\right) + m_x^-\left(x^*(t), t\right) - m_x\left(\underline{x}(t), t\right)\right] \tag{12}$$

where m_x^+ and m_x^- are the right and left derivatives of $m_x(\cdot, t)$.

No mass points. We have written the evolution of the cross sectional distribution under the assumption that it has no mass point for all times $t \geq 0$. This will follow if the initial distribution m_0 has no mass points, and if the equilibrium decision rules are such the distribution $m(\cdot, t)$ will not have mass points for all $t \geq 0$. This, in turns, requires that the support of m_0 is inside $[\underline{x}(0), \bar{x}(0)]$, or that $\int_{\underline{x}(0)}^{\bar{x}(0)} m_0(x)dx = 1$. Note that this last condition can only be verified after we solve for an equilibrium. We will return to this issue below.

Steady State: Initial and Final Conditions. We describe the stationary version of the MFG. Let $\bar{x}_{ss}$, $\underline{x}_{ss}$ and x^*_{ss} be three time-invariant thresholds, and let $\tilde{u}$ and $\tilde{m}$ be two time-invariant function with domain in $[\underline{x}_{ss}, \bar{x}_{ss}]$ solving:

$$0 = -\rho \tilde{u}(x) + \frac{\sigma^2}{2} \tilde{u}_{xx}(x) + F(x, X_{ss}) + \zeta \left(\tilde{u}(x^*_{ss}) - \tilde{u}(x) \right) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}] \tag{13}$$

$$0 = \frac{\sigma^2}{2} \tilde{m}_{xx}(x) - \zeta \tilde{m}(x) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}], x \neq x^*_{ss} \tag{14}$$

$$X_{ss} = \int_{\underline{x}_{ss}}^{\bar{x}_{ss}} x \, \tilde{m}(x) dx \tag{15}$$

with boundary conditions:

$$\tilde{u}_x \left(\bar{x}_{ss} \right) = \tilde{u}_x \left(\underline{x}_{ss} \right) = \tilde{u}_x \left(x^*_{ss} \right) = 0 \tag{16}$$

$$\tilde{u} \left(\bar{x}_{ss} \right) = \tilde{u} \left(\underline{x}_{ss} \right) = \tilde{u} \left(x^*_{ss} \right) + \psi \tag{17}$$

$$0 = \tilde{m}(\underline{x}_{ss}) = \tilde{m}(\bar{x}_{ss}) \tag{18}$$

If $\zeta = 0$, the stationary distribution $\tilde{m}$ given by a triangular tent-map:

$$\tilde{m}(x) = \begin{cases} \frac{2}{\bar{x}_{ss} - \underline{x}_{ss}} - (x - x^*_{ss}) \frac{2}{(\bar{x}_{ss} - \underline{x}_{ss})(\bar{x}_{ss} - x^*_{ss})} & \text{for } x \in [x^*_{ss}, \bar{x}_{ss}] \\ \frac{2}{\bar{x}_{ss} - \underline{x}_{ss}} + (x - \underline{x}_{ss}) \frac{2}{(\bar{x}_{ss} - \underline{x}_{ss})(x^*_{ss} - \underline{x}_{ss})} & \text{for } x \in [\underline{x}_{ss}, x^*_{ss}] \end{cases} \tag{19}$$

When $\zeta > 0$ we have the stationary distribution $\tilde{m}$ given by

$$\tilde{m}(x) = \begin{cases} L_1 e^{\ell x} + L_2 e^{-\ell x} & \text{for } x \in [0, \bar{x}_{ss}] \\ L_1 e^{-\ell x} + L_2 e^{\ell x} & \text{for } x \in [-\bar{x}_{ss}, 0] \end{cases}$$

and $\tilde{m}(x) = \tilde{m}(-x)$ for $x \in [-\bar{x}_{ss}, 0]$, and where the constant ℓ, L_2 and L_1 satisfy

$$\ell = \sqrt{\frac{2\zeta}{\sigma^2}} \, , \; L_1 e^{\ell \bar{x}_{ss}} + L_2 e^{-\ell \bar{x}_{ss}} = 0 \, , \text{ and } \frac{1}{2} = L_1 \frac{e^{\ell \bar{x}_{ss}} - 1}{\ell} + L_2 \frac{e^{-\ell \bar{x}_{ss}} - 1}{-\ell}$$

Our benchmark case of $F(x,X) = B(x+\theta X)^2$ we have that

$$X_{ss} = x^*_{ss} = 0 \text{ and } \bar{x}_{ss} = -\underline{x}_{ss}.$$

Note that the steady state is independent of the value of θ. In this case the solution for $\tilde{u}$ can be obtained, up to an implicit equation in $(\rho+\zeta)/\sigma^2$, a feature that we explore in Lemma 10.

For future reference we notice that in steady state there are two simple informative statistics of the distribution of price changes, When the firm decides to exercise control it changes x by changing prices. Thus, the distribution of price changes is the same as the distribution of adjustment of x. In the stationary case this distribution is extremely simple, it is given by a binomial distribution with two equal values, price increases equal to $\Delta p = x^*_{ss} - \underline{x}_{ss}$ and price increases equal to $\Delta p = x^*_{ss} - \bar{x}_{ss}$. We can summarize this distribution by its variance, i.e. $Var(\Delta p)$. The other interesting statistics is the average number of price changes per unit of time, which we denote as $N(\Delta p)$. We have $Var(\Delta p) N(\Delta p) = \sigma^2$.

$$Var(\Delta p) = \left(\bar{x}_{ss} - x^*_{ss}\right)^2 \text{ and } N(\Delta p) = \frac{\sigma^2}{\left(\bar{x}_{ss} - x^*_{ss}\right)^2} \tag{20}$$

Uniqueness of the stationary state. Here we argue that if $\theta \neq -1$ then the stationary solution displayed above is unique. On the other hand, if $\theta = -1$, then any number X_{ss} corresponds to a steady state.

To see this, first notice that in the static game, i.e. the case where $\rho \to \infty$ and $\psi = 0$, we have that the best response is $x^* = \arg\min_x B(x+\theta X)^2$ so $x^* = -\theta X$. Then requiring that $x^* = X$ we get that $X = -\theta X$, from which we obtain the desired result.

Now we argue that the result for the static game also holds for the stationary state. For this, define $w \equiv x + \theta X_{ss}$. Then the HBJ becomes:

$$(\rho+\zeta)\hat{u}(w) = Bw^2 + \hat{u}''(w)\tfrac{\sigma^2}{2} + \zeta u(w^*) \text{ for all } w \in [-\underline{w}, \bar{w}]$$

with boundary conditions:

$$\hat{u}(\bar{w}) = \hat{u}(\underline{w}) = \hat{u}(w^*) + \psi \text{ and } 0 = \hat{u}'(\bar{w}) = \hat{u}'(\underline{w}) = \hat{u}'(w^*)$$

and its unique solution satisfies $w^* = 0$ and $\underline{w} = -\underline{w} > 0$. The uniqueness of the solution follows from the fact that we can find a symmetric solution, and that this solution achieved the minimum of the control problem.

Given $\{\underline{w}, w^*, \bar{w}\}$ the density $\hat{m}(w)$ is the unique solution

$$0 = \hat{m}''(w)\tfrac{\sigma^2}{2} - \zeta \hat{m}(w) \text{ for all } w \in [\underline{w}, w^*) \cup (w^*, \bar{w}]$$

with boundary conditions:

$$0 = \hat{m}(\bar{w}) = \hat{m}(\underline{w}), \; \lim_{w \uparrow w^*} \hat{m}(w) = \lim_{w \downarrow w^*} \hat{m}(w), \; \text{ and } 1 = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) dw \,.$$

Note that since the solution for $\hat{m}$ is symmetric, centered at $w^* = 0$, and thus $\int_{\underline{w}}^{\bar{w}} w \, \hat{m}(w) dw = 0$. Thus, a stationary equilibrium solution of the original problem requires:

$$\begin{aligned}
& x^*_{ss} = w^* + \theta X_{ss}, \; \underline{x}_{ss} = \underline{w} + \theta X_{ss}, \; \bar{x}_{ss} = \bar{w} + \theta X_{ss}, \\
& X_{ss} = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) \left(w - \theta X_{ss} \right) dw = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) w dw - \theta X_{ss} \int_{\underline{w}}^{\bar{w}} \hat{m}(w) dw
\end{aligned}$$

and thus we can construct a stationary state if and only if:

$$X_{ss} = -\theta X_{ss}$$

Hence, just as in the static case with no adjustment cost, if $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state, and if $\theta = -1$ one can construct a stationary state for any X_{ss}.

Terminal and Initial conditions for MFG. We will use the stationary solution to define the initial density m_0 and the terminal value function u_T. For the initial condition we will consider an antisymmetric perturbation a of the stationary density $\tilde{m}$, where we let δ the parameter that indexes the size perturbation, so:

$$m_0(x) = \tilde{m}(x) + \kappa(x)\delta, \text{ with } \kappa(x) = -\kappa(-x) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}],\, x \neq x^*_{ss} \text{ and}$$
$$\kappa(\underline{x}_{ss}) = 0, \text{ and } |\kappa'(\underline{x}_{ss})| \leq \tilde{m}(\underline{x}_{ss}) \tag{21}$$

Note that under this conditions, $m_0(x)$ is a valid density for values of δ near zero. Below we will focus in a particular perturbation, that correspond to a small monetary shock.

For the terminal condition we set:

$$u_T(x) = \tilde{u}(x) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}] \text{ and}$$
$$u_T(x) = \tilde{u}(x^*_{ss}) + \psi \text{ for all } x \notin [\underline{x}_{ss}, \bar{x}_{ss}] \tag{22}$$

so that at time $t = T$. One interpretation of this terminal condition is that the firm problem is infinite horizon, but that the "coupling" with the rest of the firms, i.e. the strategic complementarity or substitutability, operates only until time $t = T$. In other words, after $t \geq T$ the value of θ becomes zero, and then each firms decision rules depends exclusively on its states.

Monetary Shock, Output IRF, and Initial Conditions. One can use the steady state to define interesting initial conditions for the MFG. A particularly interesting initial condition is the effect of an unanticipated aggregate nominal shock δ, which can be thought as $m_0(x) = \tilde{m}(x + \delta)$. The interpretation of this initial condition is that, after the monetary shock δ the nominal cost jumps immediately by this amount, and hence the value of state x of each firms goes to x to $x - \delta$, and hence the density before any decision is taken is $m_0(x) = \tilde{m}(x + \delta)$. We are particularly interested in the case where δ is small. Recall that the interpretation of x is the deviation of the optimal markup, i.e. it includes the difference between the price of the firms good minus its cost, and thus an increase in the common component of cost for al firms reduces the deviation of markup from its optimal value.

One of the most interesting features of the solution of the sS MFG interpreted as a price setting problem is the interpretation of the path of $X(t)$ after the small displacement $m_0(x) = \tilde{m}(x + \delta)$. In this case the resulting value of $X(t)$ is inversely proportional to the deviation from steady state output t periods after the a once and form all monetary shock δ, or the impulse response function for output –IRF for short.

No first order Strategic Complementarity/Substitutability, or no coupling. Let's consider the case where F does not depend on X, i.e. when $\theta = 0$. In this case, it is immediate to see that the solution of the value function u and the polices $\underline{x}, x^*, \bar{x}$ for all times is the stationary solution, i.e. $u(x,t) = \tilde{u}(x)$ at all t and x, and $\underline{x}(t) = \underline{x}_{ss}$, $x^*(t) = x^*_{ss}$ and $\bar{x}(t) = \bar{x}_{ss}$. Then we can use the stationary policies $\underline{x}_{ss}, x^*_{ss}, \bar{x}_{ss}$ and solve for $m(x,t)$, for a given m_0 at times $t \in [0,T]$, and obtain the implied behaviour $X(t)$ during that time interval. To be clear, this case assumes away the interesting coupling of a MFG between m and u. But perhaps this case can serve as the point (the paths) over which an expansion is taken. For the case in which $F(x)$ is symmetric, which implies symmetric policies, one can further simplify the solution to the path of $X(t)$ by solving a related problem for m, where there is no re-injection, so one can drop the condition equation (8), and assume that equation (2) holds also at $x(t) = x^*_{ss}$. This is due to the symmetry of the thresholds and the lack of drift. In Alvarez and Lippi (2021) we describe this path for a general class of models, which includes the simple sS model as a special case, and solving it by using the eigenfunctions of the related self-adjoint operator.

3 A small monetary shock in the sS MFG

In this section we set to analyze the effect of a monetary shock where the economy starts from its stationary state. We study the evolution of the MFG where the initial condition is given by small perturbation δ on the steady state distribution corresponding to a monetary shock of that size.

Consider an equilibrium with $\{\bar{x}(t,\delta), \underline{x}(t,\delta), x^*(t,\delta), X(t,\delta), u(x,t,\delta), m(x,t,\delta)\}$. We will linearize this equilibrium with respect to δ, and evaluate it at $\delta = 0$.

Normalization. To simplify the exposition we normalize the parameters of the problem so that at steady sate $\bar{x}_{ss} = 1$. In particular, given $\{\sigma^2, B, \rho, \zeta\}$ we set the fixed cost ψ so that $\bar{x}_{ss} = 1$. This amounts to measure the shock δ in units of standard deviation of steady state price changes, i.e. in units of $\sqrt{Var(\Delta p)}$. Moreover

Notation. We also let

$$k \equiv \frac{\sigma^2}{2}\ ,\ \eta \equiv \sqrt{\frac{\rho+\zeta}{k}}\ , \ell \equiv \sqrt{\frac{\zeta}{k}} \text{ and } C \equiv 2B\theta\,.$$

For future reference the average number of price changes in steady state is given by

$$N = \begin{cases} \zeta\left(\frac{\cosh(\ell)}{\cosh(\ell)-1}\right) & \text{for } \zeta > 0 \\ \sigma^2 = 2k & \text{for } \zeta = 0 \end{cases}$$

The initial conditions are

$$m_0(x) = \tilde{m}'(x) = \begin{cases} -1 & \text{if } x \in [-1,0) \\ 1 & \text{if } x \in (0,1] \end{cases} \tag{23}$$

for the case of $\zeta = 0$ and

$$m_0(x) = \tilde{m}'(x) = \begin{cases} \ell \left(L_1 e^{\ell x} - L_2 e^{-\ell x} \right) & \text{for } x \in [0,1] \\ -\ell \left(L_1 e^{-\ell x} + L_2 e^{\ell x} \right) & \text{for } x \in [-1,0] \end{cases}$$

for the case of $\zeta > 0$.

Second, we will let for all $t \in [0,T]$:

$$\begin{aligned} v(x,t) &\equiv \frac{\partial}{\partial \delta} u(x,t,\delta)|_{\delta=0} \text{ for all } x \in [-1,1] \\ n(x,t) &\equiv \frac{\partial}{\partial \delta} m(x,t,\delta)|_{\delta=0} \text{ for all } x \in [-1,1], x \neq 0 \\ \bar{z}(t) &\equiv \frac{\partial}{\partial \delta} \bar{x}(t,\delta)|_{\delta=0} \, , \; \underline{z}(t) \equiv \frac{\partial}{\partial \delta} \underline{x}(t,\delta)|_{\delta=0} \, , \; z^*(t) \equiv \frac{\partial}{\partial \delta} x^*(t,\delta)|_{\delta=0} \text{ and} \\ Z(t) &\equiv \frac{\partial}{\partial \delta} X(t,\delta)|_{\delta=0} \end{aligned}$$

Interpretation of the discounted case $\rho = \eta = 0$ case. We will consider the case where $\rho = \eta = 0$, which corresponds to the undiscounted case. The interpretation of such as is a the limit of the discount rate $\rho \to 0$, i.e. as the approximation for very small discount rate. The advantage of this limit is some calculations are simplified. We will show below that the expressions for different objects, in particular for $z^*, \underline{z}, \bar{z}$, as a function of ρ have a well defined limit as $\rho \to 0$. Technically, for some results –such as the compactness of the operator when $|\theta| > 1$, we will require that $T < \infty$, but T can be arbitrarily large.

Linearization of the HBJ and its boundary conditions. We differentiate the HJB equation for $u(x,t,\delta)$ with respect to δ at each (x,t) and use the boundary conditions to obtain

$$0 = -(\rho + \zeta) v(x,t) + v_t(x,t) + k v_{xx}(x,t) + C x Z(t) \text{ in } x \in [-1,1], t \in (0,T) \tag{24}$$

Furthermore, differentiating the two value matching boundary conditions for $u(\bar{x}(t,\delta),t,\delta) =$

$\psi + u(x^*(t,\delta), t, \delta)$ and $u(\underline{x}(t,\delta), t, \delta) = \psi + u(x^*(t,\delta), t, \delta)$ with respect to δ for each t and evaluating them at $\delta = 0$ we get for all $t \in (0, T)$:

$$v(-1,t) + \tilde{u}_x(-1)\underline{z}(t) = v(0,t) + \tilde{u}_x(0)z^*(t) \text{ and } v(1,t) + \tilde{u}_x(1)\bar{z}(t) = v(0,t) + \tilde{u}_x(0)z^*(t) \tag{25}$$

where we use the steady state value function $\tilde{u}(x)$. We prove the following:

Lemma 1. The function $v(x,t)$ is antisymmetric in x for each t, i.e. $v(x,t) = -v(-x,t)$ for all $x \in [-1, 1]$ and $t \in [0, T]$, and hence it satisfies the boundary condition:

$$0 = v(-1,t) = v(1,t) = v(0,t) \text{ all } t \in (0,T) \tag{26}$$

We also use the boundary condition at $t = T$, which imposing we go to steady state, or more generally to a function independent of δ, gives:

$$0 = v(x,T) \text{ all } x \in [-1,1] \tag{27}$$

3.1 Analysis of HBJ equation

We can solve the p.d.e. for v given by equation (24) for all t, x, which is the heat equation with source $CxZ(t)$, with a zero space boundary at $t = T$, and with the boundary conditions implied by value matching. We summarize this in the following lemma.

Lemma 2. Given the source $Z(t)$ for all $t \in [0, T]$, then the unique solution of the heat equation (24) with the two Dirichlet boundary conditions and the condition at $x = 0$ in equation (26) for all $t \in [0, T]$, and with the terminal space condition $v(x, T) = 0$ for all

$x \in [0,1]$ is:

$$v(x,t) = -4B\theta \int_t^T \sum_{j=1}^{\infty} e^{(\eta^2+(j\pi)^2)\,k(t-\tau)}\, Z(\tau) \frac{(-1)^j}{j\pi} \sin(j\pi x) d\tau \tag{28}$$

Given this lemma, the next proposition summarizes the nature of the optimal decision rules for a firm facing a path of future values for the cross sectional average price gap or markup:

PROPOSITION 1. Taking as given a path $Z(t)$ for $t \in [0,T]$ the solution to the firm's problem implies the following path for its optimal thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$:

$$\bar{z}(t) = \bar{T}(Z)(t) \equiv \theta \bar{A} \int_t^T \bar{H}(\tau - t) Z(\tau) d\tau \text{ for all } t \in [0,T) \tag{29}$$

$$z^*(t) = T^*(Z)(t) \equiv \theta A^* \int_t^T H^*(\tau - t) Z(\tau) d\tau \text{ for all } t \in [0,T) \tag{30}$$

where $\underline{z}(t) = \bar{z}(t)$ and where $\bar{H}$ and H^* are defined as:

$$\bar{H}(s) \equiv \sum_{j=1}^{\infty} e^{-(\eta^2+(j\pi)^2)\,k\,s} \geq 0 \;,\; H^*(s) \equiv \sum_{j=1}^{\infty} e^{-(\eta^2+(j\pi)^2)\,k\,s} (-1)^j \leq 0 \text{ for all } s > 0 \tag{31}$$

$$\bar{A} \equiv \frac{4B}{\tilde{u}_{xx}(1)} = k\,\tfrac{2\eta^2}{[1-\eta\coth(\eta)]} < 0\,,\text{ and } A^* \equiv \frac{4B}{\tilde{u}_{xx}(0)} = k\,\tfrac{2\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} > 0 \tag{32}$$

The ratio $A^*/|\bar{A}|$ is strictly increasing in η, with $\frac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} \to 6, |\frac{\eta^2}{[1-\eta\coth(\eta)]}| \to 3$ as $\eta \to 0$.

Note that, given the sign of the expression above, if $\theta < 0$, i.e. if there is strategic complementarity, a firm facing higher values of $Z(\tau)$ for $\tau \geq t$, sets a higher value of the optimal return $z^*(t)$, and a larger value of both the upper and lower values of the inaction band, $\bar{z}(t), \underline{z}(t)$. If $\theta > 0$ the result is the opposite. The strength of the result depends on θ as well as on $\eta = \sqrt{2(\rho+\zeta)/\sigma^2}$. Also, as expected, values of $Z(\tau)$ closer to t receive higher weight on the firm's decision for its optimal return point and width of the inaction band.

Furthermore, since $\bar{z}(t) = \underline{z}(t)$, the width of the inaction region is constant. The economics of this result is that the width of the inaction region reflects the option value of waiting, that is mainly affected by σ^2 and that is unaffected by the monetary shock. The parameter η also enters into the expressions for $\bar{A}$ and A^*, which reflect how the curvature of the value function changes as η changes. Equation (32) shows that the curvature of the steady state value function $\tilde{u}_{xx}$, characterized in Lemma 10, affects the speed of convergence.

Linearization of the KFE and its boundary conditions. We differentiate the KFE for $m(x, t, \delta)$ with respect to δ at each (x, t) to obtain:

$$0 = -n_t(x,t) + kn_{xx}(x,t) - \zeta n(x,t) \text{ in } x \in [-1,1], t \in (0,T), x \neq 0 \tag{33}$$

Differentiating with respect to δ the boundary conditions for each t for $m(\bar{x}(t,\delta), t, \delta) = 0$ and $m(\underline{x}(t,\delta), t, \delta) = 0$, and evaluating it at $\delta = 0$ we obtain:

$$0 = n(1,t) + \tilde{m}_x(1)\bar{z}(t) \quad \text{and} \quad 0 = n(-1,t) + \tilde{m}_x(-1)\underline{z}(t) \quad \text{all } t \in (0,T)$$

and using that $\tilde{m}_x(-1) = 1$ and $\tilde{m}_x(1) = -1$ from equation (23), we get for $\zeta = 0$:

$$n(1,t) = \bar{z}(t) \quad \text{and} \quad n(-1,t) = -\underline{z}(t) = -\bar{z}(t) \qquad \text{all } t \in (0,T) \tag{34}$$

and for $\zeta > 0$:

$$\begin{aligned} n(1,t) &= \frac{\ell^2}{2}\left(\frac{e^{\ell}}{(1-e^{\ell})^2} + \frac{e^{-\ell}}{(1-e^{-\ell})^2}\right)\bar{z}(t) \text{ all } t \in (0,T) \text{ and} \\ n(-1,t) &= -\bar{z}(t) \text{ all } t \in (0,T) \end{aligned} \tag{35}$$

where we used that $\bar{z}(t) = \underline{z}(t)$ from Proposition 1. Differentiating the mass preservation equation with respect to δ at each t we obtain:

$$0 = \int_{-1}^{1} n(x,t)dx \text{ all } t \in (0,T) \tag{36}$$

Differentiating this equation with respect to time and using the KFE we have:

$$0 = n_x(1,t) - n_x(0^+,t) + n_x(0^-,t) - n_x(-1,t) \text{ all } t \in (0,T) \tag{37}$$

The initial condition for n comes from differentiating $m(x,0) = \tilde{m}(x+\delta)$ with respect to δ, using the triangular shape of $\tilde{m}$, and evaluating it at $\delta = 0$

$$n(x,0) = \begin{cases} +1 & \text{if } x \in [-1,0) \\ -1 & \text{if } x \in (0,1] \end{cases} \tag{38}$$

using that $\tilde{m}_x(x) = \pm 1$ for the case of $\zeta = 0$ or

$$n(x,0) = \begin{cases} \ell L_1 e^{\ell x} - \ell L_2 e^{-\ell x} & \text{for } x \in [0,1] \\ -\ell L_1 e^{-\ell x} + \ell L_2 e^{\ell x} & \text{for } x \in [-1,0] \end{cases} \tag{39}$$

3.2 Analysis of the KF Equation

We develop an argument to determine the boundary of $n(x,t)$ at $x = 0$ to solve the Kolmogorov forward equation. We then present the solution as a function of given paths for the thresholds $z^*(t), \bar{z}(t)$.

Next we present two lemmas to establish that the function $n(x,t)$ is antisymmetric. Consider two functions $a(t)$ and $b(t)$ satisfying:

$$n(0^+,t) = b(t) \text{ all } t \geq 0 \tag{40}$$

$$n(0^-,t) = a(t) \text{ all } t \geq 0\,. \tag{41}$$

To determine a and b we study the following problem. Define n^+, n^- and $\hat{n}$ with domain in $[0,1] \times \mathbb{R}_+ \to \mathbb{R}$ as

$$n^+(x,t) = n(x,t) \qquad \text{for } x \geq 0 \text{ all } t$$

$$n^-(x,t) = -n(-x,t) \qquad \text{for } x \leq 0 \text{ all } t$$

$$\hat{n}(x,t) = n(x,t) + n(-x,t) \text{ for } x \leq 0 \text{ all } t$$

Note that $n^+(0,t) = b(t)$ and $n^-(0,t) = -a(t)$. The function $\hat{n}$ satisfies:

$$\hat{n}_t(x,t) = n_t(x,t) + n_t(-x,t) = k\left(n_{xx}(x,t) + n_{xx}(-x,t)\right) - \zeta\left(n(x,t) + n(-x,t)\right)$$

$$= k\hat{n}_{xx}(x,t) - \zeta\hat{n}(x,t) \text{ all } x,t \geq 0 \tag{42}$$

$$\hat{n}(1,t) = n(1,t) + n(-1,t) = -\frac{1}{m_x(1)}\left(\bar{z}(t) - \underline{z}(t)\right) = 0 \text{ all } t \tag{43}$$

$$\hat{n}(0,t) = n(0^+,t) + n(0^-,t) = b(t) + a(t) \text{ for all } t \tag{44}$$

$$\hat{n}(x,0) = n(x,0) + n(-x,0) = 0 \text{ for all } x \tag{45}$$

Note that the mass preservation in equation (36) is equivalent to:

$$0 = \int_0^1 n(x,t)dx + \int_0^1 n(-x,t)dx = \int_0^1 \hat{n}(x,t)dx = 0 \tag{46}$$

Lemma 3. (currently for $\zeta = 0$, need to rewrite for $\zeta \geq 0$). Consider the KFE for $\hat{n}$ given by:

$$\hat{n}_t(x,t) = k\hat{n}_{xx}(x,t) \text{ all } x \in [0,1] \text{ and } t > 0$$
$$\hat{n}(x,0) = 0 \text{ all } x \in [0,1]$$
$$\hat{n}(0,t) = (a+b)(t) \text{ all } t > 0 \text{ and}$$
$$\hat{n}(1,t) = 0 \text{ all } t > 0$$

Its solution is given by

$$\hat{n}(x,t) = r(x,t) + \sum_{j=1}^{\infty} c_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where}$$
$$r(x,t) = (a+b)(t)[1-x] \text{ all } x \in [0,1],\ t > 0$$

and where for all $j = 1, 2, \ldots$ we have:

$$\varphi_j(x) = \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h\rangle \equiv \int_0^1 h(x)\varphi_j(x)dx$$
$$c_j(t) = c_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau)e^{\lambda_j(\tau - t)}d\tau \text{ all } t > 0\,, \quad \text{where} \quad c_j(0) = \frac{\langle \varphi_j\,, -r(\cdot,0)\rangle}{\langle \varphi_j, \varphi_j\rangle} \text{ and } \lambda_j = (j\pi)^2 k$$
$$q_j(t) = \frac{\langle \varphi_j\,, -r_t(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = -\frac{2(a+b)'(t)}{j\pi} \text{ all } t > 0$$

We have the following

Lemma 4. Consider the p.d.e. in Lemma 3. Additionally, impose that $\hat{n}$ satisfies mass preservation, as given by equation (46). Then $b(t) = -a(t)$ for all $t \geq 0$.

Lemma 3 and Lemma 4 imply that $n(x,t) = -n(-x,t)$, i.e. that the function n is antisymmetric. We also have:

Lemma 5. Assume that $m(x^*(t,\delta),t,\delta)$ is continuous, and right and left differentiable at

$\delta = 0$. Then $z^*(t) = \frac{1}{-2\,\tilde{m}'(0)}(b(t) - a(t))$.

Lemma 4 and Lemma 5 have the important implication that:

$$n(0^+, t) = -\tilde{m}'(0^+)\, z^*(t) = -n(0^-, t) \text{ for all } t \geq 0$$

So we have

$$n(1, t) = -\tilde{m}_x(1)\bar{z}(t) = -n(-1, t) \text{ all } t \in (0, T)$$

We will now use these boundary conditions to obtain an explicit solution for $n(x, t)$.

Lemma 6. The solution for the KFE for n:

$$\begin{aligned}
n_t(x,t) &= k n_{xx}(x,t) - \zeta n(x,t) \text{ all } x \in [0,1] \text{ and } t > 0 \\
n(x,0) &= \tilde{m}'(x) = -\frac{\ell^2}{2}\left(\frac{e^{\ell x}}{(1-e^{\ell})^2} + \frac{e^{-\ell x}}{(1-e^{-\ell})^2}\right) \text{ all } x \in [0,1] \\
n(0,t) &= w^*(t) \text{ all } t > 0 \text{ and} \\
n(1,t) &= \bar{w}(t) \text{ all } t > 0
\end{aligned}$$

where

$$\begin{aligned}
w^*(t) &= -\tilde{m}'(0^+)\, z^*(t) = \frac{\ell^2}{2}\left(\frac{1}{(1-e^{\ell})^2} + \frac{1}{(1-e^{-\ell})^2}\right) z^*(t) \text{ and} \\
\bar{w}(t) &= -\tilde{m}'(1)\bar{z}(t) = \frac{\ell^2}{2}\left(\frac{e^{\ell}}{(1-e^{\ell})^2} + \frac{e^{-\ell}}{(1-e^{-\ell})^2}\right) \bar{z}(t)
\end{aligned}$$

is given by

$$\begin{aligned}
n(x,t) &= r(x,t) + \sum_{j=1}^{\infty} c_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where} \\
r(x,t) &= w^*(t) + x\left[\bar{w}(t) - w^*(t)\right] \text{ all } x \in [0,1],\ t > 0
\end{aligned}$$

and where for all $j = 1, 2, \dots$ we have:

$$\varphi_j(x) = \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h\rangle \equiv \int_0^1 h(x)\varphi_j(x)dx$$
$$c_j(t) = c_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau)e^{\lambda_j(\tau - t)}d\tau \text{ all } t > 0\,,\ \text{where } \lambda_j = (\ell^2 + (j\pi)^2)k \ ,$$
$$q_j(t) = \frac{\langle \varphi_j\,, -r_t(\cdot,t) - \zeta r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = 2\left[\frac{\cos(j\pi)-1}{j\pi}\right] w^{\star\prime}(t) + 2\frac{(-1)^j}{j\pi}\left[\bar{w}'(t) - w^{\star\prime}(t)\right]$$
$$+ 2\zeta\left[\frac{\cos(j\pi)-1}{j\pi}\right] w^{\star}(t) + 2\zeta\frac{(-1)^j}{j\pi}\left[\bar{w}(t) - w^{\star}(t)\right] \text{ all } t > 0$$
$$c_j(0) = \frac{\langle \varphi_j\,, \tilde{m}' - r(\cdot,0)\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j\,, \tilde{m}'\rangle}{\langle \varphi_j, \varphi_j\rangle} + 2\left[\frac{\cos(j\pi)-1}{j\pi}\right] w^{\star}(0) + 2\frac{(-1)^j}{j\pi}[w(0) - w^{\star}(0)]$$

Note that if $\zeta > 0$:

$$\frac{\langle \varphi_j\,, \tilde{m}'\rangle}{\langle \varphi_j, \varphi_j\rangle} = \begin{cases} -\frac{\ell^2 j\pi}{\ell^2 + (j\pi)^2}\left(\frac{1+e^{\ell}(-1)^{j+1}}{\left(1-e^{\ell}\right)^2} + \frac{1+e^{-\ell}(-1)^{j+1}}{\left(1+e^{-\ell}\right)^2}\right) \text{ if } \zeta > 0 \\ -\frac{2+(-1)^{j+1}}{(j\pi)^2} \text{ if } \zeta = 0 \end{cases} \tag{47}$$

We use Lemma 6 to solve for the quantity of interest, the impulse response of the mean $Z(t)$ for given path of the thresholds $\{\bar{z}(t), z^*(t)\}$, we have:

$$Z(t) = \int_{-1}^{1} xn(x,t)dx = 2\int_0^1 xn(x,t)dx \text{ all } t \in (0,T)$$

The next proposition summarizes our analysis of the KFE:

Proposition 2. Taking as given the paths of $\{z^*(t), \bar{z}(t)\}$, the solution of the Kolmogorov Forward equation implies the following path for the average value $\{Z(t)\}$:

$$Z(t) = T_Z(z^*, \bar{z})(t) \equiv Z_0(t) + 4k\int_0^t G^*(t-\tau)z^{\star}(\tau)d\tau + 4k\int_0^t \bar{G}(t-\tau)\bar{z}(\tau)d\tau \tag{48}$$

for all $t \in [0,T]$ and where Z_0, $\bar{G}$ and G^* are defined as

$$\bar{G}(s) \equiv -\tilde{m}'(1) \sum_{j=1}^{\infty} e^{-(\ell^2+(j\pi)^2)\,k\,s} \text{ and } G^*(s) \equiv -\tilde{m}'(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{-(\ell^2+(j\pi)^2)k\,s}$$

for all $s \geq 0$, where

$$-\tilde{m}'(1;0) = -\tilde{m}'(0^+;0) = 1 \text{ if } \zeta = 0 \text{ and otherwise}$$
$$-\tilde{m}'(1;\ell) = \frac{\ell^2}{2}\left(\frac{e^{\ell}}{(1-e^{\ell})^2} + \frac{e^{-\ell}}{(1-e^{-\ell})^2}\right) \text{ and } -\tilde{m}'(0^+;\ell) = \frac{\ell^2}{2}\left(\frac{1}{(1-e^{\ell})^2} + \frac{1}{(1-e^{-\ell})^2}\right)$$

and where

$$Z_0(t) = \begin{cases} 2\sum_{j=1}^{\infty} \frac{\ell^2}{\ell^2+(j\pi)^2}\left(\frac{(-1)^j - e^{\ell}}{\left(1-e^{\ell}\right)^2} + \frac{(-1)^j - e^{-\ell}}{\left(1+e^{-\ell}\right)^2}\right) e^{-(\ell^2+(j\pi)^2)kt} \text{ if } \zeta > 0 \\ -4\sum_{j=1}^{\infty} \frac{[1-\cos(j\pi)]}{(j\pi)^2} e^{-(j\pi)^2\,k\,t} \text{ if } \zeta = 0 \end{cases} \tag{49}$$

This proposition gives the evolution of the average price gap or markup, $Z(t)$, as a function of the path of decisions up to time t, summarized by the boundaries of the inaction and the optimal return point, i.e. $\{z^*(\tau), \bar{z}(\tau)\}$ for $0 \leq \tau \leq t$. As expected the mapping is monotone, in that larger values of past thresholds, lead to larger values of the current cross sectional average markup $Z(t)$. Moreover, its slope is bounded by one. Also as expected, the values of the pairs $(z^*(\tau), \bar{z}(\tau))$ for τ close to t have a higher weight than those further away in time. Given our normalization, the mapping T_Z depends only on $k \equiv \sigma^2/2$. As in the case of Proposition 1, note that the monotonicity and the bound of the slopes hold for any t.

For the case of $\zeta = 0$, the value of Z_0, which can be written as

$$Z_0(t) = -8\sum_{j=1}^{\infty} \frac{e^{-((2j-1)\pi)^2 kt}}{((2j-1)\,\pi)^2} \sim \hat{Z}_0(t) \equiv -\frac{8}{\pi^2} e^{-\pi^2 kt}$$

is proportional to (minus) the impulse response of output keeping the decisions rules fixed, as studied in Alvarez and Lippi (2021). Indeed the impulse response can be accurately summarized by the first term in the summation, i.e. $\hat{Z}_0(t) \equiv -8e^{-\pi^2 kt}/\pi^2$. This is a good approximation in the sense that $Z_0'(t)$ has the same sign as $\hat{Z}_0'(t)$ for all $t \geq 0$, they both decay at the same rate for large t, and $\left(\int_0^\infty \hat{Z}_0(t)dt\right) / \left(\int_0^\infty Z_0(t)dt\right) = 96/\pi^4 \approx 0.9855$. In Alvarez, Le Bihan, and Lippi (2016) it is shown that for $\zeta > 0$ and $T = \infty$, the cumulative area $\int_0^\infty Z_0(t)dt = -Kurt(\ell)/(6N)$ where $Kurt(\ell)$ is the kurtosis of the stationary distribution of price changes, which depends only on ℓ, divided by (six times) the frequency of price changes. In Alvarez and Lippi (2019) we analyzed the properties of $Z_0(t)$, not just its area, as function of ℓ.

A useful corollary of Lemma 6 is to extend the result of Proposition 2, from the analysis of the benchmark case with initial condition given by $n(x,0) = -sign(x)$ to the more general perturbation $n(x,0) = -a(x)$. This follows immediately from recomputing the projection for $c_j(0)$ in Lemma 6 and using it in the expression for $\int_0^1 n(x,t)dx$.

Corollary 1. Assume that $n(x,0) = \kappa(x)$ where $\kappa(x) = -\kappa(x)$ all $x \in [-1,0) \cup (0,1]$, $\kappa(-1) = 1$ and $\kappa'(-1) \leq 1$. Taking as given the paths of $\{z^*(t), \bar{z}(t)\}$, the solution of the Kolmogorov Forward equation implies the following path for the average value $\{Z(t)\}$:

$$Z(t) = T_Z(z^*, \bar{z})(t) \equiv Z_0^\kappa(t) + 4k \int_0^t G^*(t-\tau)z^\star(\tau)d\tau + 4k \int_0^t \bar{G}(t-\tau)\bar{z}(\tau)d\tau \qquad (50)$$

for all $t \in [0,T]$ and where $\bar{G}$ and G^* are defined as in Proposition 2 and where the only difference is that

$$Z_0^\kappa(t) \equiv 4 \sum_{j=1}^{\infty} \frac{e^{-(\ell+(j\pi)^2)\,k\,t}}{j\pi} \int_0^1 \sin(jx\pi)\kappa(x)dx\,.$$

3.3 Equilibrium Characterization for small shock

We characterize the linearized equilibrium as a path Z which is a fixed point of an operator $\mathcal{T}$ that maps the set of bounded paths on $[0,T]$ onto itself. The mapping $\mathcal{T}$ takes a path Z and produces a path $\mathcal{T}(Z)$ after applying the composition of the mappings described in Proposition 1 and in Proposition 2. In particular, we start with a path $Z = \{Z(t)\}_{t\in[0,T]}$, use equation (29) and equation (30) to produce two paths for the thresholds $\{\bar{z}(t), z^*(t)\}_{t\in[0,T]}$, and then use these two paths into equation (48) to produce a path which we label $\mathcal{T}(Z) = \{T(Z)(t)\}_{t\in[0,T]}$. The mapping $\mathcal{T}$ is simply the composition of T_Z with $\bar{T}$ and T^*, i.e.

$$\mathcal{T}(Z)(t) = T_z\left(T^*(Z), \bar{T}(Z)\right)(t) \text{ for all } t \in [0,T] \tag{51}$$

Given the linearity of the system we can explicitly write the kernel of $\mathcal{T}$

$$Z(t) = \mathcal{T}(Z)(t) \equiv Z_0(t) + \theta \int_0^T K(t,s) Z(s) ds \text{ all } t \in [0,T] \tag{52}$$

where Z_0 is the value in the case of no coupling, i.e. when $\theta = 0$, and where the kernel K is given by:

$$K(t,s) = 4k \int_0^{\min\{t,s\}} \left[\bar{A}\,\bar{H}(s-\tau)\bar{G}(t-\tau) + A^*\, H^*(s-\tau) G^*(t-\tau)\right] d\tau \leq 0$$

for all $(t,s) \in [0,T]^2$. Indeed note that $\bar{G}, \bar{H}$ and G^*, H^* satisfy:

$$\bar{H}(s) = -\tilde{m}'(1) e^{-\rho\, s}\, \bar{G}(s) \geq 0 \text{ and } H^*(s) = \tilde{m}'(0^+) e^{-\rho\, s}\, G^*(s) \leq 0 \text{ for all } s > 0 \tag{53}$$

Moreover, using the expressions for each of these functions we can write:

$$K(t,s) = 4\,k \int_0^{\min\{t,s\}} e^{-\rho\,(s-\tau)} \left[\bar{A}\,\bar{G}(s-\tau)\bar{G}(t-\tau) - A^*\, G^*(s-\tau) G^*(t-\tau)\right] d\tau \leq 0 \tag{54}$$

The kernel can be explicitly written as $K(t,s)$ as follows:

$$K(t,s) = 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A}_\ell - A^*_\ell\,(-1)^{j+i}\right]\frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2+\ell^2]k(t\wedge s)}-1\right]e^{-(j\pi)^2kt-\ell^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2+\ell^2}$$

$$\text{where } \bar{A}_\ell \equiv -\tilde{m}'(1;\ell)\,\bar{A} \text{ and } A^*_\ell \equiv -\tilde{m}'(0^+;\ell)\,A^*\,. \tag{55}$$

In these models output is negatively proportional to price gaps, so that letting $Y_\theta(t)$ the impulse response of output to a small monetary shock we have $Y_\theta(t) = -\varrho Z(t)$ where $\varrho > 0$ is a positive parameter and where we index the impulse response by the parameter θ. Note that $Y_0(t) \equiv -\varrho Z_0(t)$. The impulse response function solves $Y_\theta = \mathcal{T}\,Y_\theta$ as follows:

$$Y_\theta(t) = (\mathcal{T}\,Y_\theta)\,(t) \equiv Y_0(t) + \theta\int_0^T K(t,s)Y_\theta(s)ds \text{ all } t\in[0,T] \text{ where} \tag{56}$$

$$Y_0(t) = -\varrho Z_0(t)$$

3.4 Equilibrium Characterization

In this section we characterize the equilibrium of the model for a small shock. We concentrate most of the results into the more challenging case of $\zeta = 0$, which corresponds to the **?** model. All the results extend to the case of $\zeta > 0$, indeed under weaker conditions.

We gather several properties of the kernel K in the next lemma. When $\zeta = 0$, the kernel K is given by:

$$K(t,s) = 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A} - A^*\,(-1)^{j+i}\right]\frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)}-1\right]e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2} \tag{57}$$

We have the following characterization:

Lemma 7. Assume $\zeta = 0$. Define

$$\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T |K(t,s)|ds \text{ and } \text{Lip}_{K^\top} \equiv \sup_{s\in[0,T]} \int_0^T |K(t,s)|dt \tag{58}$$

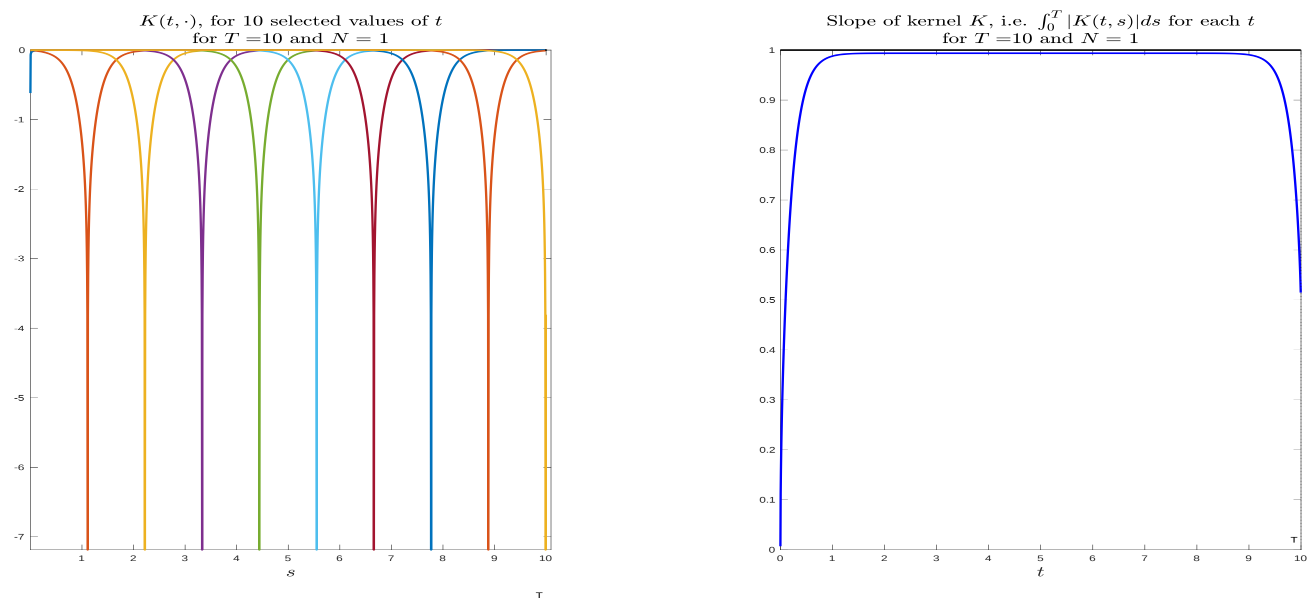
Then

1. K is negative, i.e. $K(t,s) < 0$ for all $(t,s) \in (0,T)^2$.
2. K is symmetric for the undiscounted case, i.e if $\eta = 0$, then $K(t,s) = K(s,t)$ for all $(t,s) \in [0,T]^2$,
3. $K(t,s)$ vanishes at $t = 0$, i.e. $K(0,s) = 0$ for all $s \in [0,T)$,
4. K is singular, diverging at the diagonal, $\lim_{s\to t} K(t,s) = -\infty$ for all $t \in (0,T)$,
5. $K(t,\cdot)$ has a uniform L_1 bound: $\text{Lip}_K < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\,\text{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$,
6. L_1 bound $K(t,\cdot)$ for large discount: $\lim_{\eta^2\to\infty} \text{Lip}_K = 0$,
7. L_1 bound $K(t,\cdot)$ for small discount: $\text{Lip}_K < 1 - \frac{7}{180}\eta^2 + o(\eta^2)$,
8. $K(\cdot,s)$ has a uniform L_1 bound: $\text{Lip}_{K^\top} < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\,\text{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$,
9. L^2 bound: $\int_0^T \int_0^T K^2(t,s)\,ds\,dt < c_0\,T\left(\frac{\eta^2}{[1-\eta\,\text{csch}(\eta)]} - \frac{\eta^2}{[1-\eta\coth(\eta)]}\right)$ for a constant $c_0 > 0$ independent of any other parameters.

Figure 1 illustrates several of the the properties of the Kernel established in Lemma 7. This figures displays $K(t,\cdot)$ for $s \in [0,T]$, for several values of t. This figures uses a finite value of T, and some chosen values of k and ρ.

Characterization of the IRF Y_θ. Now we characterize the impulse response function Y_θ as the solution to the integral equation (56). We study the existence and uniqueness of Y_θ, for different cases, depending on the value of θ. Depending of the case we have more or less complete characterization. Our main result consists, roughly speaking, on showing

Figure 1: Kernel $K(t, s)$



that equilibrium exists, it is unique and well posed provided that the strength of strategic complementarity is smaller than some critical value. On the other hand, equilibrium is unique and exists for any level of strategic substitutability. Finally, the size of the response to a monetary shock is higher, the larger the strength of strategic complementarity.

Our first, very simple result, shows that all IRF start at the same point.

Proposition 3. *Let Y_θ be the solution of equation (60). Then its value at $t = 0$ is the same as $Y_\theta(0) = Y_0(0) = \varrho$.*

The proof of Proposition 3 is immediate, since by Lemma 7, $K(0, s) = 0$ for all $s \in [0, T]$ hence $Y_\theta(0) = Y_0(0) + \theta \int_0^T K(0, s)Y_\theta(s)ds = Y_0(0)$.

We define $\mathcal{K}$ as

$$(\mathcal{K})\,(V)(t) \equiv \int_0^T K(t, s)V(s)ds \text{ for all } t \in [0, T] \tag{59}$$

for any function $V : [0, T] \to \mathbb{R}$, and $\mathcal{K}^r$ are defined recursively:

$$(\mathcal{K})^{r+1}(V)(t) \equiv \int_0^T K(t,s)\,(\mathcal{K})^r(V)(s)ds$$

For future reference $\mathcal{K}$ we define the series

$$S_\theta(t) = \sum_{r=0}^{\infty} \theta^r (\mathcal{K})^r (Y_0)(t) \text{ for all } t \in [0, T] \tag{60}$$

Our next result is an existence and uniqueness result for the undiscounted case. It provides a closed form expression of the IRF Y_θ in terms of the projections to an orthonormal base of $L^2([0, T])$.

Proposition 4. Assume that $T < \infty$ and $\rho = 0$. In this case the operator $\mathcal{K}$ is self-adjoint and compact, and thus it has eigenvalues and eigenfunctions which we denote by $\{\mu_j, \phi_j\}_{j=1}^{\infty}$. Then, if $1/\theta \neq \mu_j$ for all j, then unique solution of equation (56) given by

$$Y_\theta(t) = \sum_{j=1}^{\infty} \frac{\langle Y_0, \phi_j \rangle}{1 - \theta\mu_j}\,\phi_j(t) = \sum_{j:\mu_j<0} \frac{\langle Y_0, \phi_j \rangle}{1 - \theta\mu_j}\,\phi_j(t) \text{ for almost all } t \in (0, T]$$

since $\langle Y_0, \phi_j \rangle = 0$ if $\mu_j > 0$, where the equalities are in the $L^2([0, T])$ sense, and where $\langle \cdot, \cdot \rangle$ is the corresponding L^2 inner product.

The next proposition uses the characterization in Lemma 7 to verify the conditions for the Banach contraction fixed point theorem to establish existence and uniqueness of the solution of equation (56) for a range of θ including positive and negative values, roughly speaking values $|\theta| \leq 1$.

Proposition 5. Assume that $B > 0$, $k > 0$, $\zeta = 0$, and $T < \infty$ if $\rho = 0$, but otherwise these parameters take arbitrary values. A sufficient condition for the existence and uniqueness of the equilibrium IRF, i.e. of the uniqueness and existence of a solution to equation (56) is that $|\theta| \, \text{Lip}_K < 1$ or

$$|\theta| \frac{\eta^2}{18} \left(\frac{1}{1 - \eta \, \text{csch}(\eta)} - \frac{4}{1 - \eta \coth(\eta)} \right) < 1 \tag{61}$$

For small η^2 we have a simpler sufficient condition:

$$|\theta| \left(1 - \frac{7}{180} \eta^2 \right) < 1 \tag{62}$$

The proof of this proposition is immediate from the computations on bounds for Lip_K items 5 to 7 in Lemma 7. Note that since $\eta = \sqrt{2\rho/\sigma^2}$, since at steady state we have $\sigma^2 = N \, Var(\Delta p)$, and since for the purpose of computing the impulse response of output to a small monetary shock we have, without loss of generality, normalized $\bar{x}_{ss}$ so that $Var(\Delta p) = 1$, then we can set $\eta = \sqrt{2\rho/N}$. Thus η can be taken to be the square root of twice the discount factor 2ρ divided by the number of price changes per unit of time N. Note that the series expansions of the upper bound of $|\theta|$ in equation (62) gives

$$\frac{1}{|\theta|} > 1 - \frac{7}{90} \frac{\rho}{N}$$

Thus for practical purposes we can take the sufficient conditions for contraction can be taken to be $|\theta| \leq 1$. The previous proposition shows that under the stated conditions the fixed point satisfies $Y_\theta = S_\theta$.

The next proposition shows how the equilibrium behaves in the case of strategic complementarity, i.e negative θ. It shows that the higher the degree of strategic complementarity is, i.e. the higher values of $-\theta$ is, the higher is the impulse response of output $Y_\theta(t)$ to the same shock, at every horizon $t \in (0, T]$. The effect on $Y_\theta(t)$ is convex in θ, so for large enough value of $-\theta$ the value of Y_θ is arbitrarily large.

Proposition 6. Let $\theta \in (\underline{\theta}, 0]$, where $\underline{\theta}$ be the lower value for radius of convergence of the the series in equation (60). The unique solution of equation (56) has the following properties:

1. For each $t \in (0, T)$ the fixed point is positive, i.e. $Y_\theta(t) > 0$,

2. For each $t \in (0, T)$, the fixed point $Y_\theta(t)$ is (strictly) monotone decreasing in θ,

3. For each $t \in (0, T)$, the fixed point $Y_\theta(t)$ is (strictly) convex in θ.

While Proposition 6 is shown only for an interval of strictly negative values of θ. Nevertheless, the same properties holds in a neighborhood of at $\theta = 0$. In particular, we have:

$$\frac{\partial}{\partial \theta} Y_\theta(t)|_{\theta=0} = (\mathcal{K})\,(Y_0)(t) < 0 \text{ and } \frac{\partial^2}{\partial \theta^2} Y_\theta(t)|_{\theta=0} = 2\,(\mathcal{K})^2\,(Y_0)(t) > 0$$

and thus the monotonicity and convexity holds also at least in an interval of positive values. Indeed, numerically, we find all the properties in Proposition 6 hold for positive values of θ.

The next proposition shows that when strategic complementarities are large enough, i.e. large enough value of $-\theta$, then Y_θ must become negative for some t. Moreover, for the undiscounted case, i.e. when $\rho = 0$, we have a more precise characterization. As $\theta \downarrow \underline{\theta}$ the solution Y_θ goes from diverging to $+\infty$ to nonexistence, and to diverging to $-\infty$. These results, together with the previous proposition, is a sign that the problem is not well posed as $-\theta$ is large enough, since Y_θ goes from being positive and arbitrarily large, to being negative.

Proposition 7. Assume that $T < \infty$. In the undiscounted case, i.e. when $\rho = 0$, then

1. $\lim_{\theta \downarrow \underline{\theta}} Y_\theta(t) = +\infty$ for $t > 0$,

2. There is no solution in $Y_\theta \in L^2([0, T])$ of equation (56) for $\theta = \underline{\theta}$,

3. $\lim_{\theta \uparrow \underline{\theta}} Y_\theta(t) = -\infty$ for $t > 0$,

where $-\infty < \underline{\theta} = 1/\mu_1 < 0$ is the reciprocal dominant eigenvalue of the $\mathcal{K}$. In general, i.e. for any η if θ is negative and large enough in absolute value such that:

$$\theta \int_0^T K(t, s) ds > 1 \text{ for all } t \in (0, T)$$

and if Y_θ is a fixed point in equation (56) for such θ, then $\inf_{t\in(0,T)} Y_\theta(t) < 0$. Moreover, $\inf_{t\in(0,T)} Y_\theta(t) \to -\infty$ as $\theta \to \infty$.

We turn next to the uniqueness and existence for the case where $\theta > 0$, i.e. the case of strategic substitutability. We first establish a bound on the norm for any solution, by using a Lasry-Lions type of argument for the perturbed system. Then we use this bound and the Leray-Schauder fixed point theorem to establish existence. Uniqueness follows also from an application the Lasry-Lions argument for the perturbed system. For this purposes we establish two results. The first one is the compactness of the linear operator defined by K, which follows from 9 in Lemma 7. The second is a bound on the norm of the solutions. For the second result we define the following L_q norm:

$$||Y||_{L^q(\rho,T)} \equiv \left(\frac{\rho}{1 - e^{-\rho T}} \int_0^T e^{-\rho t} |Y(t)|^q dt \right)^{1/q} . \tag{63}$$

which we use in the following lemma. We show that the norm of Y is bounded in an equilibrium. For future reference we show this for a slightly more general initial condition.

Lemma 8. Assume that $\zeta = 0$. Let Y be a solution of $Y = \mathcal{T}Y$ as in equation (52), with initial condition $n(x, 0) = -\bar{\kappa}\, sign(x)$ for $\bar{\kappa} \in (0, 1]$. Assume that $\theta > 0$. Then $||Y||_{L^2(\rho,T)} < \bar{\kappa}|\varrho|$.

Now we use the previous lemma for the following result.

Proposition 8. Assume that $\theta > 0$. Then there exists a unique solution of equation (56).

We define the cumulative impulse response function as

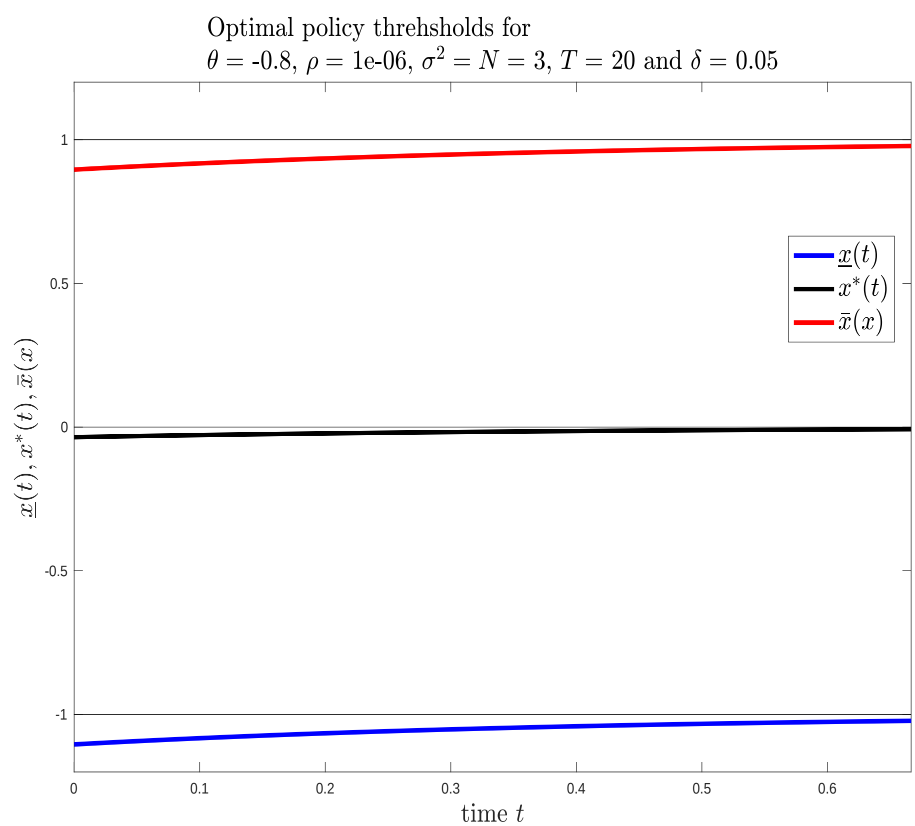
$$CIR_\theta \equiv \int_0^T Y_\theta(t) dt \tag{64}$$

The cumulative IRF is of economic interest, allowing to summarize the effect of the shock.

We use in Figure 4 to illustrate how θ affect $Y_\theta(t)$ in a simple way.

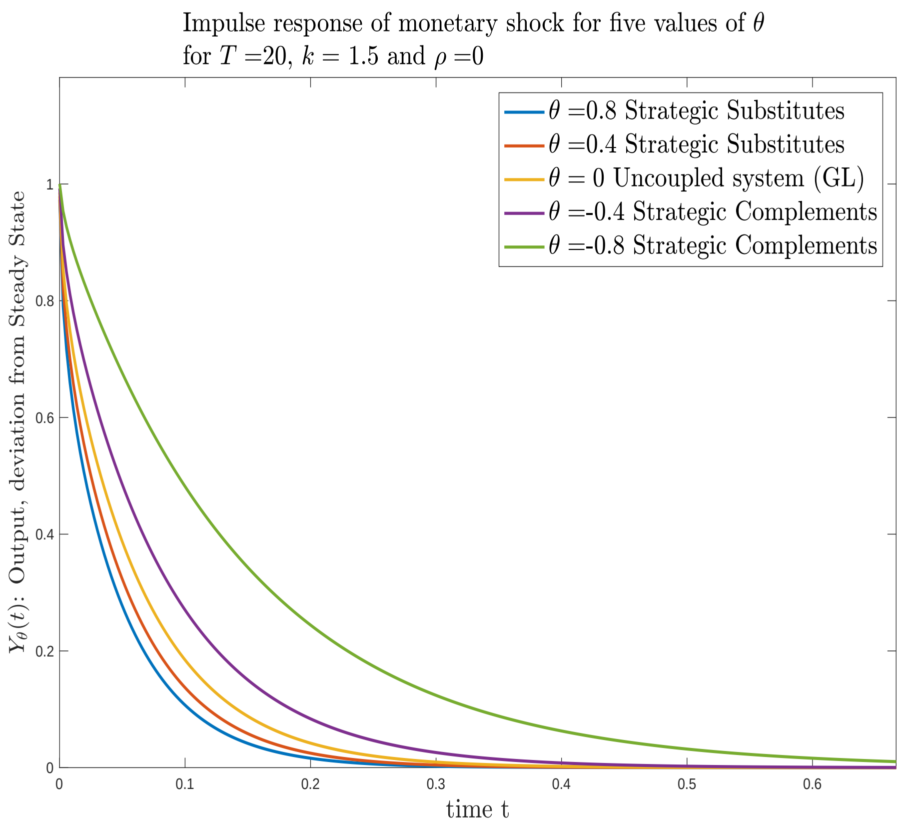
In Figure 2 we display the approximation to the equilibrium thresholds $\bar{x}(t), x^*(t)$ and $\underline{x}(t)$. The figure consider the case of $\delta = 0.05$ and $\theta = -.8$. The black thin lines are the steady sate values of the thresholds, and the color solid lines are the linear approximation to the equilibrium thresholds. The thresholds start just hedge of the initial distribution, and then they evolve according to the equilibrium. As shown above, the path for both boundaries of the range of inaction $\bar{x}(t)$ and $\underline{(}t)$, as well as the optimal return path $x^*(t)$ depart from the steady state with the same sign as θ. The fact with strategic complementarity the thresholds decrease is what it makes the impulse response larger, since fewer firms increase prices, and also when they do so, they do so, they return to a lower value of the price gap.

Figure 2: Equilibrium path of thresholds



In Figure 3 we display the IRF Y_θ for five different values of θ. It can be seen, as shown above, that $Y_\theta(t)$ is decreasing in θ for each t.

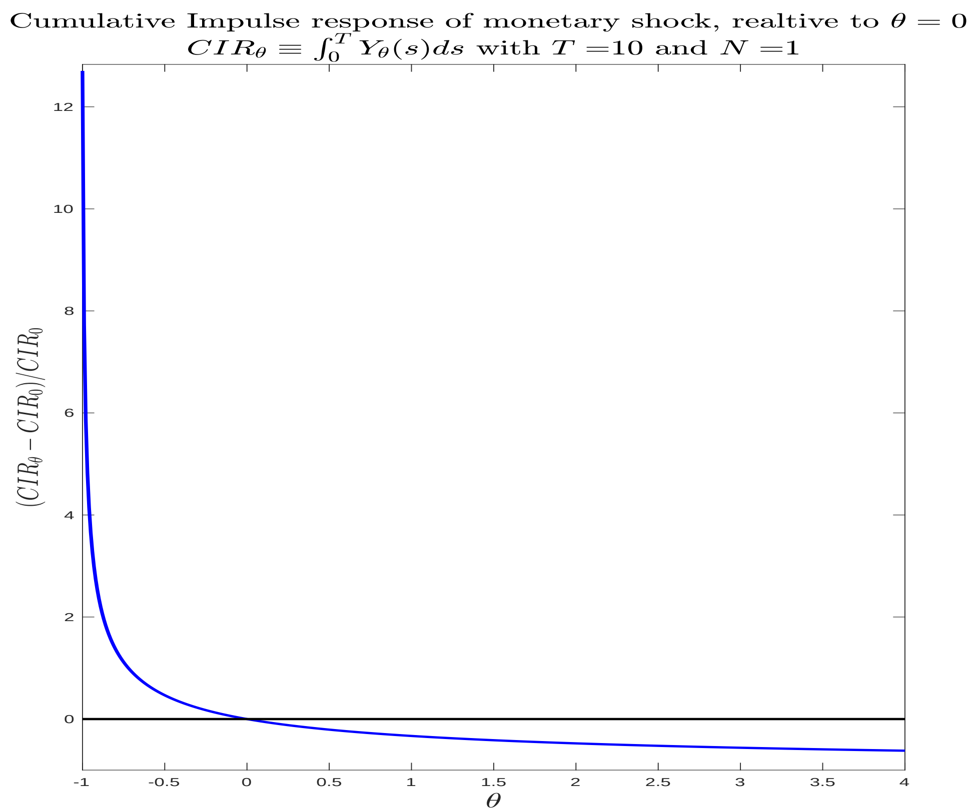
Figure 3: Impulse response of Monetary Shock



We have an explicit solution of for the undiscounted case where $\rho = 0$ and $T < \infty$ written in terms of the eigenvalues and eigenfunctions of $\mathcal{K}$.

Proposition 9. Assume that $\rho = 0$, $T < \infty$ and that $\theta < -1$. Let $\{\mu_j\}$ be the countably many real eigenvalues of the compact operator $\mathcal{K}$, and let $\{\phi_j\}$ be the associated set of eigenfunctions, which form an orthonormal base of $L_2([0, T])$. Then the solution Y_θ satisfies

Figure 4: Cumulative Impulse response of Monetary Shock



1. The square cumulative IRF, given by $CIR_\theta^2 \equiv \int_0^T Y_\theta(t)^2 dt$ decreases with θ, i.e.

$$CIR_\theta^2 = \sum_{j:\mu_j<0} \frac{\langle \phi_j, Y_0 \rangle^2}{(1-\theta\mu_j)^2} \text{ and } \frac{\partial}{\partial\theta} CIR_\theta^2 < 0$$

2. As $\theta \to \infty$ then $Y_\theta(t) \to 0$ for almost all t.

The next proposition shows the effect on the cumulative response function CIR_θ of small value of the coupling parameter θ. The approximation is based by differentiating $Y_\theta(t) = Y_0(t) + \theta \int_0^T K(t,s) Y_\theta(s) ds$ with respect to θ and evaluating at $\theta = 0$ obtaining $\frac{\partial}{\partial\theta} Y_\theta(t)|_{\theta=0} = \int_0^T K(t,s) Y_0(s) ds$.

Proposition 10. Consider the limit as $T \to \infty$ and $\rho \to 0$ of the CIR_θ. Then

$$\lim_{\rho\downarrow 0} \lim_{T\to\infty} \frac{1}{CIR_\theta} \frac{dCIR_\theta}{d\theta}|_{\theta=0} = 192 \sum_{m=1,3,5,\ldots} \left(\frac{1}{m\pi}\right)^5 [\text{csch}(m\pi) - \coth(m\pi)] \approx -0.578 \quad (65)$$

Figure 4 display the CIR_θ relative to CIR_0 for a range of θ. In particular it plots $CIR_\theta - CIR_0)/CIR_0$ for a range of θ that includes both strategic substitutes ($\theta > 0$) and complements ($\theta < 0$). It can be seen that the relative slope around θ is indeed approximately 0.578. Also we can see that as θ becomes more negative, and gets closer to the reciprocal of the dominant eigenvalue, then CIR_θ diverges towards $+\infty$.

3.5 Calvo$^+$ model

In this section we return to the analysis of the Calvo$^+$ model where we let $\zeta > 0$.

Pure Calvo Model. In this section we compare the results of the sS model obtained above with the results for a very simple time dependent pricing model.

In this simple time dependent model a can *only* change prices at exogenously randomly distributed times, independently of their state. In particular in each period a firm can change its price with probability $\zeta > 0$ per unit of time. The simple case of a time dependent model with a constant hazard rate is the most common case analyzed in the literature, due to its tractability, introduced by Calvo. The analysis we use here can be found in **?** obtain a very simple closed form expression in the presence of strategic complementarity/substitutability. We can recast the problem as a Mean Field Game, where the firm's problem becomes

$$\rho u(x,t) = B(x+\theta X(t))^2 + u_t(x,t) + \frac{\sigma^2}{2}u_{xx}(x,t) + \zeta\left(u(x^*(t),t) - u(x,t)\right) \text{ for all } x, \text{ and } t \in [0,T]$$
$$x^*(t) = \min_x u(x,t) \text{ for all } t \in [0,T]$$

and final boundary condition $u(x,T) = \tilde{u}(x)$, where $\tilde{u}$ is the stationary solution which corresponds to the problem with $\theta = 0$. The parameter $\zeta > 0$ is the exogenously given rate at which the firm can change its price.

The corresponding KFE for the measure $m(x,t)$ is:

$$0 = \frac{\sigma^2}{2}m_{xx}(x,t) - \lambda m(x,t) - m_t(x,t) \text{ for all } x \neq x^*(t), \text{ and } t \in [0,T]$$
$$1 = \int_{-\infty}^{\infty} m(x,t)dx \text{ for all } t \in [0,T]$$

with initial condition $m(x,0) = \tilde{m}(x+\delta)$, where $\tilde{m}$ is the stationary density of the problem with $\theta = 0$.

Since firms can only change prices at times independent to their state x, writing the control problem of the firm we obtain that the solution for $x^*(t)$ is:

$$\begin{aligned} x^*(t) &= \arg\min_x \int_t^{\infty} e^{-(\rho+\zeta)s}\mathbb{E}\left[\left(x + \sigma W(s) + \theta X(t+s)1_{\{t+s\leq T\}}\right)^2 \mid W(t) = 0\right] ds \\ &= -\theta(\zeta+\rho)\int_0^{T-t} e^{-(\zeta+\rho)\tau} X(t+\tau)d\tau \\ &= -\theta(\zeta+\rho)\int_t^{T} e^{-(\zeta+\rho)(s-t)} X(s)ds \text{ for all } t \geq 0 \end{aligned}$$

and thus we get the o.d.e.:

$$\frac{d}{dt}x^*(t) \equiv \dot{x}^*(t) = \theta(\zeta+\rho)X(t) + (\zeta+\rho)x^*(t) \text{ for all } t \geq 0$$

In this simple case we can solve for the dynamics of the cross-sectional average evolves $X(t)$ directly, without solving for the entire density. At time t a fraction $\zeta e^{-\zeta\tau}d\tau$ of firms have prices that have change at time $t-\tau$. At this times, they set the price to be $x^*(t-\tau)$. We also use that before the initial period, i.e. $t \leq 0$, the optimal reset price $x^*(t) = -0$, so boundary condition right after the shock is $X(0) = -1$, using the normalization $\delta = 1$. We thus have

$$X(t) = \zeta \int_0^t e^{-\zeta\tau} x^*(t-\tau)d\tau - e^{-\zeta t} \text{ for all } t \geq 0$$

which implies

$$\frac{d}{dt}X(t) \equiv \dot{X}(t) = \zeta\left(x^*(t) - X(t)\right) \text{ for all } t \geq 0$$

We can write a simple constant coefficient o.d.e. for the vector $(X(t), x^*(t))$ as

$$\begin{pmatrix} \dot{x}^*(t) \\ \dot{X}(t) \end{pmatrix} = \begin{pmatrix} \rho+\zeta & \theta(\rho+\zeta) \\ \zeta & -\zeta \end{pmatrix} \begin{pmatrix} x^*(t) \\ X(t) \end{pmatrix}$$

Letting μ the eigenvalues of the matrix, we have $(\mu-\rho-\zeta)(\zeta+\mu) - \theta(\rho+\zeta)\zeta = 0$. For instance if $\rho = 0$ we get $(\mu+\zeta)(\mu-\zeta) = \theta\zeta^2$, with solution $\mu = \pm\zeta a$, so that $(a+1)(a-1) = \theta$ or $a^2 - 1 = \theta$, so $\mu = \pm\sqrt{1+\theta}$. In the case of $\rho = 0$ and $T = \infty$ we get

$$\lim_{\rho\downarrow 0}\lim_{T\to\infty} Y_\theta^{Calvo}(t) = -X(t) = e^{-\zeta\sqrt{1+\theta}\,t} \text{ for all } t \geq 0 \text{ and thus} \tag{66}$$

$$\lim_{\rho\downarrow 0}\lim_{T\to\infty} CIR_\theta^{Calvo} = \frac{1}{\zeta\sqrt{1+\theta}}, \text{ and } \lim_{\rho\downarrow 0}\lim_{T\to\infty} \frac{1}{CIR_\theta^{Calvo}}\frac{dCIR_\theta^{Calvo}}{d\theta}|_{\theta=0} = -\frac{1}{2}$$

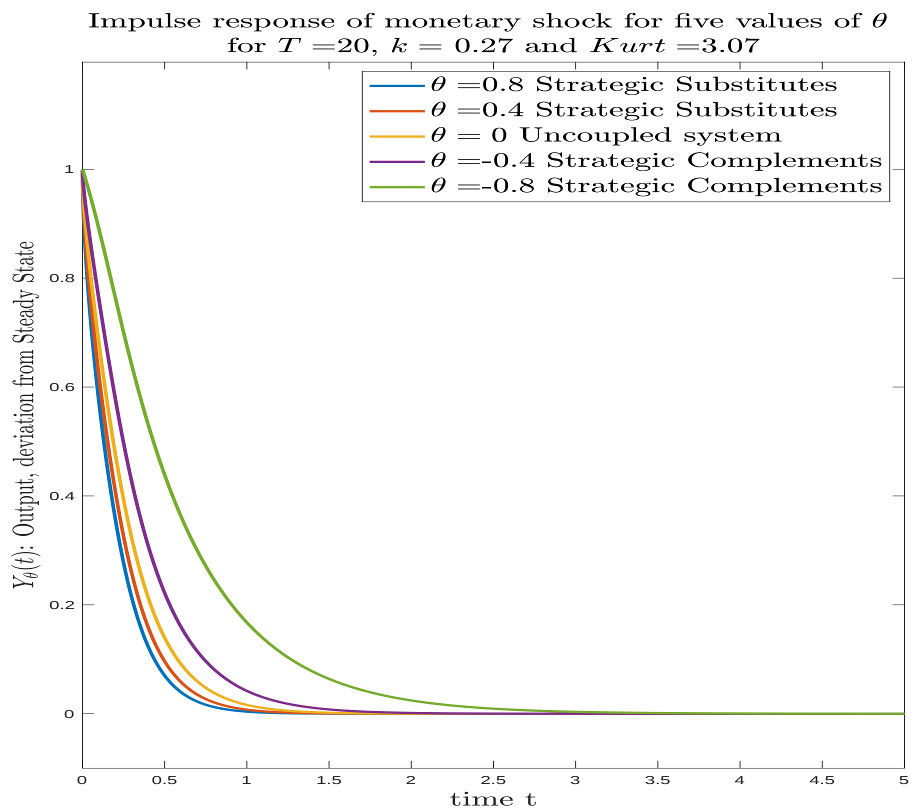
Note that in the Calvo model the proportional effect of θ in the cumulative impulse response CIR_θ for small values of θ is smaller, but very similar to the values we obtain in our baseline sS model. In particular, in the Calvo model this elasticity is -0.5, as shown in equation (66), where in the baseline sS model the elasticity is about -0.578 –see equation (65) in Proposition 10. It is intuitive that the elasticity will be higher in the baseline sS model, since the firm can also change when prices are changed. Note that while the elasticities are similar, the level of the CIR_0 are very different between the baseline sS model and the Calvo model, i,.e $CIR_0^{Calvo} = 6 \times CIR_0^{sS}$, provided that both models have the same steady state frequency of price changes –as can be seen in Alvarez, Le Bihan, and Lippi (2016). Interestingly, in Figure 6 we compare the cumulative impulse function, CIR for two models, our baseline sS model with the Calvo model. For both models we compute the CIR for a range of values of θ, and in each type of model we express the CIR as a ratio of the value for that model evaluated at $\theta = 0$. From our analysis of the baseline model and the expression for the Calvo model, in both cases the CIR_θ is decreasing and convex in θ, diverges towards $+\infty$ at a critical (negative) value of θ, and converges to zero as $\theta \to \infty$. What is remarkable is that the effect of θ in both models is very similar, as both curves are very close in the range of values of θ.

Calvo Plus model. In this section we display the results for the Calvo$^+$ model, where $\zeta > 0$ and $\bar{x}_{ss} < \infty$. As background, we first comment on known properties for the case of no strategic complementarity of substitutability. In the case of $\theta = 0$, Alvarez, Le Bihan, and Lippi (2016) showed that the scaled cumulative response function $CIR_0/N \equiv \int_0^\infty Y(t)dt/N$ depends only on $\ell^2 = \frac{\zeta \bar{x}_{ss}^2}{\sigma^2/2}$. Indeed, in that paper it is shown that $CIR_0 = Kurt(\ell)/(6N)$, where $Kurt(\ell)$ is the kurtosis of the price changes using the stationary distribution $\tilde{m}$, and statistic that depends only on ℓ. We keep the normalization that $\bar{x}_{ss} = 1$. We display the impulse response for different values of ℓ where for each ζ we adjust σ^2 so that we keep constant the steady state number of price changes N.

In this section we display impulse responses for several values of ℓ, going from the pure sS

($\ell = 0$), i.e. Golosov-Lucas model, to the pure time dependent, i.e. Calvo model ($\ell \to \infty$). As expected given the results from $\theta = 0$, the impulse response $Y_\theta(t)$ for a given θ, is increasing in ℓ at each t. Also, as indicated by the results below, for a given ℓ, impulse responses are decreasing in θ. This can be seen in Figure 5

Figure 5: IRF to a Monetary Shock, Calvo$^+$ model ($N = 2$)



In Figure 6 and Figure 7 we compute the cumulative impulse response CIR_θ for a range of values of θ for different models, each of them characterized by a different value of ℓ. For each one of the five values of ℓ, we compare the cumulative IRF for each value of θ with its value at $\theta = 0$. The lines for each ℓ are very similar, even though, as explained above, the values for CIR_0 varies by six times as ℓ goes from 0 to $\ell \to \infty$, or equivalently, as $Kurt(\ell)$ goes from 1 to 6. What it can be seen from this figures is that the effect of strategic complementarity/substitutability, while in principle very large is approximately multiplicative. This extend the results shown in Proposition 10 for $\ell = 0$ and for the case of $\ell \to \infty$ in equation (66), to the intermediate cases for ℓ.

Figure 6: Cumulative Impulse response of Monetary Shock

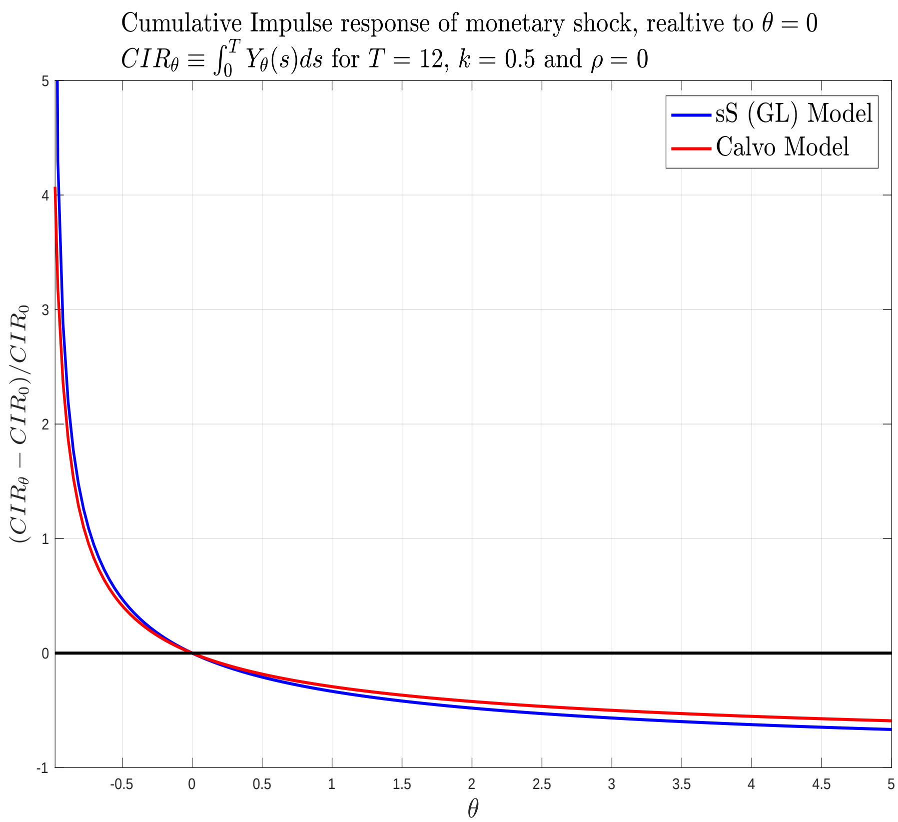
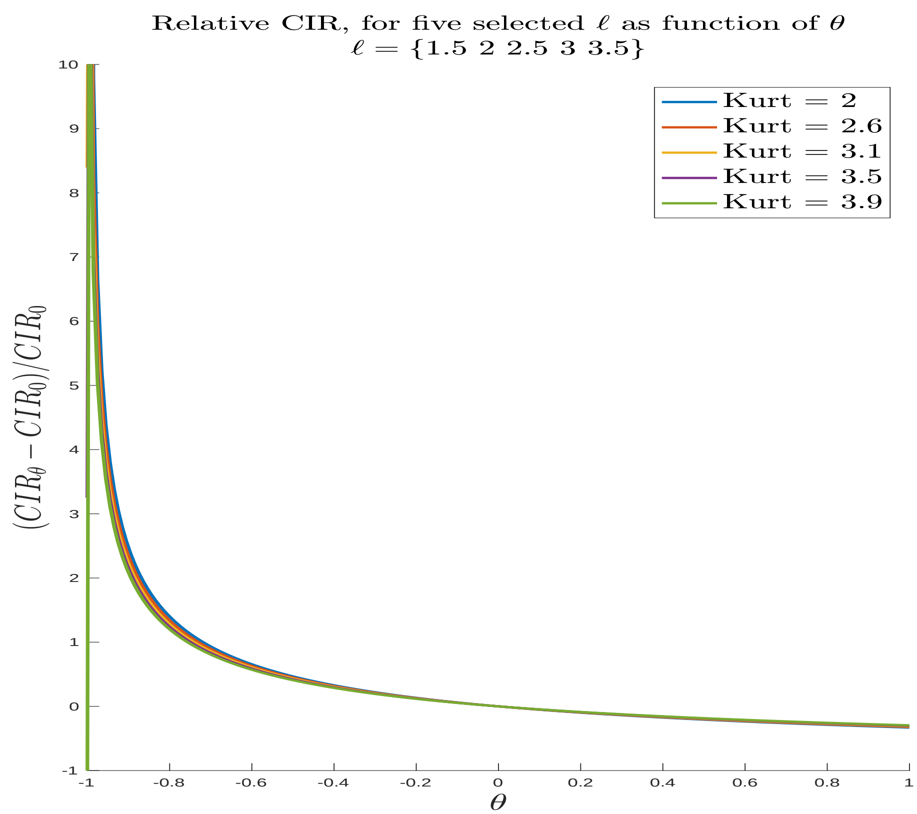


Figure 7: Cumulative IRF as function of θ for selected ℓ



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A Proofs

Proof. (of Lemma 1). First we show that v is antisymmetric. For that we use that the source $CZ(t)x$ is antisymmetric as a function of x. To see this, define $w : [0,1] \times [0,T]$ as $w(x,t) = v(x,t) + v(-x,t)$, which is identically zero and solves $0 = w_t(x,t) + kw_{xx}(x,t) - \rho w(x,t)$ with boundary conditions $w(1,t) = v(1,t) + v(-1,t) = 2v(0,t)$ and $w(0,t) = 2v(0,t)$ all t and $w(x,T) = 0$ all x.

We can use the maximum principle that shows that the maximum and minimum of w has to occur at the given boundaries, i.e. at either $x \in \{0,1\}$ and any $t \in [0,T)$ or at any $x \in [0,1]$ and $t = T$. To see this, notice that since $w(x,T) = 0$ for all $x \in [0,1]$, then if a minimum will be interior, i.e. if it will occur at $0 < \tilde{x} < 1$ and $0 \leq \tilde{t} < T$, then $w(\tilde{x},\tilde{t}) < 0$. Hence, $w_t(\tilde{x},\tilde{t}) = -kw_{xx}(\tilde{x},\tilde{t}) + \rho w(\tilde{x},\tilde{t}) < 0$ since $w_{xx}(\tilde{x},\tilde{t}) \leq 0$ because $(\tilde{x},\tilde{t})$ is an interior minimum and $k > 0$, and since $w(\tilde{x},\tilde{t}) < 0$. Hence $w(\tilde{x},t') < w(\tilde{x},\tilde{t})$ for t' close to $\tilde{t}$, a contradiction with $(\tilde{x},\tilde{t})$ being an interior minimum. A similar argument shows that there can't be an interior maximum.

Now we show that the maximum and minimum has to occur at $t = T$. For this we use that $w(x,t) = v(x,t) + v(-x,t)$ implies $w_x(0,t) = v_x(0,t) - v_x(0,t) = 0$ for all $t < T$. Thus, suppose that the minimum occurs at $(x,t) = (0,t_1)$ where $t_1 < T$. Then $w(0,t_1) = 2v(0,t_1)$ and $w_t(0,t_1) = 2v_t(0,t_1)$, so $2\rho v(0,t_1) = kw_{xx}(0,t_1) + 2v_t(0,t_1)$. Since $(0,t_1)$ is a minimum, we have $v_t(0,t_1) \geq 0$ and since the minimum occurs at $t_1 < T$, then $v(0,t_1) < 0$, so $w_{xx}(0,t_1) < 0$. But since $w_x(0,t_1) = 0$, then we obtain a contradiction with $(0,t_1)$ being a minimum. A similar argument shows that the maximum cannot occur at $(x,t) = (0,t_2)$ where $t_2 < T$. Thus the minimum and maximum occur at $t = T$, where $w(x,T) = 0$.

So we have shown that $w(x,t) = 0$ for all (x,t), and hence $v(x,t) = -v(-x,t)$ all (x,t). Since v is antisymmetric we have $v(0,t) = -v(-0,t)$ and hence $v(0,t) = 0$.

Second, using smooth pasting at the boundaries ($\tilde{u}_x(-1) = \tilde{u}_x(1) = 0$) and optimality at $x^* = 0$ ($\tilde{u}_x(0) = 0$) in equation (25), we can write the boundary conditions as

$$v(-1,t) = v(0,t) = v(1,t) = 0 \quad \text{all } t \in (0,T)$$

which gives the desired result. $\square$

Lemma 9. Let f be the solution of the heat equation

$$0 = f_t(x,t) + kf_{xx}(x,t) - \rho f(x,t) + s(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in [0,T) \tag{67}$$

and boundaries

$$f(1,t) = \bar{\phi}(t) \text{ and } f(-1,t) = \underline{\phi}(t) \text{ for all } t \in (0,T) \tag{68}$$

and

$$f(x,T) = \Phi(x) \text{ for all } x \in [-1,1] \tag{69}$$

for functions $\bar{\phi}, \underline{\phi}, \Phi$ and s. Assume that $\rho \geq 0$ and $k > 0$. The solution is unique.

Proof. Assume that there are two solutions f^1 and f^2. Let $F(x,t) \equiv f^2(x,t) - f^1(x,t)$. Note that the p.d.e. in equation (67) is linear, so that F must satisfy

$$0 = F_t(x,t) + kF_{xx}(x,t) - \rho F(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in (0,T) \tag{70}$$

with boundaries:

$$F(1,t) = 0 \text{ and } F(-1,t) = 0 \text{ for all } t \in (0,T) \text{ and} \tag{71}$$
$$F(x,T) = 0 \text{ for all } x \in [-1,1] \tag{72}$$

Define $I(t) \equiv \int_{-1}^{1}(F(x,t))^2 dx \geq 0$ for $t \in [0,T]$. Then use the boundary condition $I(T) = 0$ to write $0 = I(T) = I(0) + \int_0^T I'(t)dt$. Next compute:

$$\begin{aligned} I'(t) &= \int_{-1}^{1} \tfrac{d}{dt}(F(x,t))^2 dx = 2\int_{-1}^{1} F(x,t)F_t(x,t)dx = 2\int_{-1}^{1} F(x,t)[\rho F(x,t) - kF_{xx}(x,t)]dx \\ &= 2\rho \int_{-1}^{1} F(x,t)^2 dx + 2k \left(\int_{-1}^{1} F_x(x,t)^2 dx - F(x,t)F_x(x,t)\Big|_{-1}^{1} \right) \end{aligned}$$

where we have substituted the p.d.e. and integrated by parts. Using the boundary conditions in equation (71) we have:

$$I'(t) = 2\rho \int_{-1}^{1} F(x,t)^2 dx + 2k \int_{-1}^{1} F_x(x,t)^2 dx \geq 0$$

Thus $I(T) = 0$ only if I is zero for almost all t, and hence $F(x,t) = 0$ for almost all x, which in turns implies that $f^1 = f^2$. $\square$

Proof. (of Lemma 2) Uniqueness follows from the argument given in Lemma 9.

That equation (28) satisfies the zero boundary condition at $t = T$ follows immediately since at $t = T$ equation (28) becomes an integral with zero length. That the Dirichlet boundary condition holds at $x = 1$ and $x = -1$ follows since $\sin(xj\pi) = 0$ for all integers j. Note also that the $v(0,t) = 0$ since $\sin(0) = 0$. It only remains to show that equation (28) satisfies the heat equation with source $CxZ(t)$. Direct computation gives

$$\begin{aligned} v_t(x,t) &= CZ(t)\, 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} \sin(j\pi x) \\ &\quad - 2C \int_t^T \sum_{j=1}^{\infty} e^{(\rho + k(j\pi)^2)(t-\tau)} (\rho + k(j\pi)^2) Z(\tau) \frac{(-1)^j}{j\pi} \sin(j\pi x) d\tau \\ v_{xx}(x,t) &= 2C \int_t^T \sum_{j=1}^{\infty} e^{(\rho + k(j\pi)^2)(t-\tau)} Z(\tau) \frac{(-1)^j}{j\pi} (j\pi)^2 \sin(j\pi x) d\tau \end{aligned}$$

and notice that the Fourier series for x in the interval $[0,1]$ is $x = -2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} \sin(j\pi x)$, since $\int_0^1 x \sin(j\pi x) dx / \int_0^1 \sin^2(j\pi x) dx = -2\frac{(-1)^j}{j\pi}$. Replacing these expressions in the equation for $v_t(x,t)$ we can verify that $0 = v_t(x,t) + kv_{xx}(x,t) - \rho v(x,t) + CxZ(t)$ for all $x \in (-1,1)$ and $t \in [0,T)$. $\square$

For use in Proposition 1 we compute the expressions for the second derivative of $\tilde{u}$ when we use the normalization $\bar{x}_{ss} = 1$, i.e. the choice of ψ so that is attained.

Lemma 10. Fix the parameters σ, B and ρ and let ψ be such that $\bar{x}_{ss} = 1$. For such case the second derivatives of $\tilde{u}$ evaluated at the thresholds are given by:

$$0 < \tilde{u}_{xx}(0) = \frac{2B}{\rho}\left[1 - \eta \operatorname{csch}(\eta)\right] , \text{ and } 0 > \tilde{u}_{xx}(1) = \frac{2B}{\rho}\left[1 - \eta \coth(\eta)\right] \tag{73}$$

where $\eta \equiv \sqrt{\rho/k}$. Moreover $|\tilde{u}_{xx}(0)| < |\tilde{u}_{xx}(1)|$.

Proof. (of Lemma 10). The solution for $\tilde{u}$ is of the form of a sum of the particular solution $a_0 + a_2 x^2$ and the two homogenous solutions, which given the symmetry can be written as $A\cosh(\eta x)$, so that $\tilde{u}(x) = a_0 + a_2 x^2 + A\cosh(\eta x)$. From the o.d.e. of $\tilde{u}$ we obtain that $\eta = \sqrt{\rho/k}$. To determine the coefficients a_0, a_2 note the particular solution must satisfy:

$$\rho(a_0 + a_2 x^2) = Bx^2 + k2a_2$$

and hence $a_2 = B/\rho$ and $a_0 = 2kB/\rho^2$. It remains to find the value of A. For this we use smooth pasting at $\bar{x} = 1$. We have:

$$\tilde{u}_x(\bar{x}) = 0 = \frac{2B}{\rho}\bar{x} + A\eta \sinh(\eta \bar{x})$$

and using $\bar{x} = 1$ we get

$$A = -\frac{2B}{\rho\eta\sinh(\eta)} = \frac{2Bk^{1/2}}{\rho^{3/2}\sinh\left(\sqrt{\rho/k}\right)}$$

Since $\tilde{u}_{xx}(x) = \frac{2B}{\rho} + A\eta^2\cosh(\eta x)$ then the second derivatives are:

$$\tilde{u}_{xx}(0) = \frac{2B}{\rho} + A\eta^2 = \frac{2B}{\rho} - \frac{2B\eta^2}{\rho\eta\sinh(\eta)} = \frac{2B}{\rho}\left[1 - \eta\operatorname{csch}(\eta)\right]$$
$$\tilde{u}_{xx}(1) = \frac{2B}{\rho} + A\eta^2\cosh(\eta) = \frac{2B}{\rho} - \frac{2B\eta^2\cosh(\eta)}{\rho\eta\sinh(\eta)} = \frac{2B}{\rho}\left[1 - \eta\coth(\eta)\right]$$

The inequality is equivalent to:

$$1 - \frac{\eta}{\sinh(\eta)} < -1 + \frac{\eta\cosh(\eta)}{\sinh(\eta)} \text{ or } 2 < \eta\frac{1+\cosh(\eta)}{\sinh(\eta)} \text{ or } 2\sinh(\eta) < \eta(1+\cosh(\eta))$$

□

Proof. (of Proposition 1). Consider the smooth pasting and optimal return conditions from the original problem, i.e.

$$0 = u_x(\underline{x}(t,\delta), t, \delta)\,, \quad 0 = u_x(\bar{x}(t,\delta), t, \delta), \quad \text{and} \quad 0 = u_x(x^*(t,\delta), t, \delta)$$

Differentiate them w.r.t. δ to find $\bar{z}, \underline{z}$ and z^*:

$$\bar{z}(t) = -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)} \text{ for all } t \in [0,T)$$
$$\underline{z}(t) = -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)} = \bar{z}(t) \text{ for all } t \in [0,T)$$
$$z^*(t) = -\frac{v_x(0,t)}{\tilde{u}_{xx}(0)} \text{ for all } t \in [0,T)\,.$$

Differentiating equation (28) obtained in Lemma 2 we obtain:

$$v_x(1,t) = -2C \int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau) d\tau$$
$$v_x(0,t) = -2C \int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau)(-1)^j d\tau$$

The equality of $\bar{z} = \underline{z}$ follows from the antisymmetry of v established in Lemma 1 and from $\bar{z}(t) = -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)}$ and $\underline{z}(t) = -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)}$ since $\tilde{u}$ is symmetric, and hence $\tilde{u}_{xx}(-1) = \tilde{u}_{xx}(1)$.

The expressions for $\bar{A}$ and A^* in equation (32) follow from Lemma 10.

That $\bar{H}(s) > 0$ is immediate using that k and s are positive. That $H^*(s) < 0$ follows from grouping each pair of consecutive terms as in

$$H^*(s) = -\sum_{j=1,3,5,\ldots} e^{-(\eta^2+(j\pi)^2)\,k\,s}\left[1 - e^{-(\eta^2+((j+1)^2-j^2)\pi^2)\,k\,s}\right] < 0$$

where the inequality follows because k and s are strictly positive.

□

Proof. (of Lemma 3) The proof can be done by verifying that, given the expressions for r and φ, the conditions at the boundaries hold for all $t > 0$. To see that the p.d.e. holds in the interior, use that

$$c_j'(t) = -\lambda_j c_j(t) + q_j(t) \text{ for all } t > 0 \text{ and } j = 1, 2, \ldots$$

so replacing this into the expression for proposed solution for $\hat{n}_t$ and $\hat{n}_{xx}$, and using expression for λ_j and the second derivative of φ_j we end up with

$$(a+b)'(t)[1-x] + \sum_j q_j(t)\varphi_j(x) = 0$$

must hold for all x and t. But since $\{\varphi_j\}$ is an orthogonal base, $q_j(t)$ are the projections of $-(a+b)'(t)[1-x]$, this equation hold for all x and t.

□

Proof. (of Lemma 4) We first show that $a(t) + b(t) = 0$ for all t. Start by noticing using that $a(0) + b(0) = 0$, implies $c_0(j) = 0$. Then

$$\begin{aligned} c_j(t) &= -\frac{2e^{-\lambda_j t}}{j\pi} \int_0^t (a+b)'(\tau) e^{\lambda_j \tau} d\tau \\ &= \frac{2}{j\pi} \left[\lambda_j \int_0^t (a+b)(\tau) e^{\lambda_j(\tau - t)} d\tau - (a+b)(t) \right] \end{aligned}$$

We can then write:

$$\hat{n}(x,t) = (a+b)(t)[1-x] + \sum_{j=1}^{\infty} c_j(t) \sin(\pi j x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where}$$
$$c_j(t) = \frac{2}{j\pi} \left[\lambda_j \int_0^t (a+b)(\tau) e^{\lambda_j(\tau - t)} d\tau - (a+b)(t) \right]$$

Now we integrate $\hat{h}$ using mass preservation, obtaining:

$$0 = \int_0^1 \hat{n}(x,t) dt = \frac{1}{2}(a+b)(t) + \sum_{j=1}^{\infty} c_j(t) \frac{1 - \cos(\pi j)}{\pi j}$$

which can be written as:

$$\begin{aligned} (a+b)(t) &= -2 \sum_{j=1}^{\infty} c_j(t) \frac{1 - \cos(\pi j)}{\pi j} \\ &= -\sum_{j=1}^{\infty} \frac{4}{j\pi} \left[\lambda_j \int_0^t (a+b)(\tau) e^{\lambda_j(\tau - t)} d\tau - (a+b)(t) \right] \frac{1 - \cos(\pi j)}{\pi j} \end{aligned}$$

since the sum $\sum_{j=1}^{\infty} 4(1 - \cos(j\pi))/(j\pi)^2 = 1$ we can write:

$$\begin{aligned} 0 &= \sum_{j=1}^{\infty} \left[\lambda_j \int_0^t (a+b)(\tau) e^{\lambda_j(\tau - t)} d\tau \right] \frac{4[1 - \cos(\pi j)]}{(\pi j)^2} \\ &= e^{-\lambda_j t} \int_0^t \sum_{j=1}^{\infty} (a+b)(\tau) e^{\lambda_j \tau} \frac{\lambda_j 4[1 - \cos(\pi j)]}{(\pi j)^2} d\tau \text{ for all } t \geq 0 \end{aligned}$$

Hence

$$0 = \int_0^t \sum_{j=1}^{\infty} (a+b)(\tau) e^{\lambda_j \tau} \frac{\lambda_j 4[1 - \cos(\pi j)]}{(\pi j)^2} d\tau \text{ for all } t \geq 0$$

and differentiating with respect to time:

$$0 = (a+b)(t) \sum_{j=1}^{\infty} e^{\lambda_j t} \frac{\lambda_j 4[1 - \cos(\pi j)]}{(\pi j)^2} d\tau \text{ for all } t \geq 0$$

which requires that $(a+b)(t) = 0$ for all $t \geq 0$.
□

Proof. (of Lemma 5) In this lemma we use the requirement that $m(x,t,\delta)$ is continuous around $x = x^*(t,\delta)$ for all t and δ. Under the assumption that $m(x,t,\delta)$ is right and left differentiable at $x = x^*(t,\delta)$, we have

$$m(x,t,\delta) = \begin{cases} m(0,t,0) + m_x(0^-,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^-,t,0)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ m(0,t,0) + m_x(0^+,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^+,t,0)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

We can write these expressions in the notation developed above:

$$m(x,t,\delta) = \begin{cases} \tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

Using the continuity of m, we equate both expansions to obtain:

$$\tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) = \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta)$$

using that $\tilde{m}_x(0^-) = 1$ and $\tilde{m}_x(0^+) = -1$, and the notation $n(0^+,t) = b(t)$ and $a(t) = n(0^-,t)$ we have: $z^*(t) + a(t) + o(\delta)/\delta = -z^*(t) + b(t)$ or taking $\delta \to 0$:

$$z^*(t) = (b(t) - a(t))/2$$

□

Proof. (of Lemma 6) It is analogous to the proof of Lemma 3 with only minor modifications due to the difference in the boundary conditions. □

Proof. (of Proposition 2) We replace the expression from Lemma 6 for n into the integral for Z obtaining:

$$\begin{aligned} Z(t) &= 2\int_0^1 xn(x,t)dx = z^*(t)\frac{2}{2} + [\bar{z}(t) - z^*(t)]\frac{2}{3} + 2\sum_{j=1}^{\infty} c_j(t)\int_0^1 x\sin(j\pi x)dx \\ &= z^*(t) + [\bar{z}(t) - z^*(t)]\frac{2}{3} - 2\sum_{j=1}^{\infty} c_j(t)\frac{(-1)^j}{j\pi} \end{aligned}$$

Note that using the expression in Lemma 6 we can write

$$\begin{aligned} c_j(t) = 2&\left[\frac{\cos(j\pi) - 1}{j\pi}\right](1 + z^\star(0))\,e^{-k(j\pi)^2 t} + 2\frac{(-1)^j}{j\pi}[\bar{z}(0) - z^\star(0)]e^{-k(j\pi)^2 t} \\ &+ 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right]\int_0^t z^{\star\prime}(\tau)e^{k(j\pi)^2(\tau - t)}d\tau + 2\frac{(-1)^j}{j\pi}\int_0^t [\bar{z}'(\tau) - z^{\star\prime}(\tau)]e^{k(j\pi)^2(\tau - t)}d\tau \end{aligned}$$

Replacing the expression for $c_j(t)$:

$$\begin{aligned} Z(t) &= z^*(t) + [\bar{z}(t) - z^*(t)]\frac{2}{3} \\ &- 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right](1 + z^\star(0))\, e^{-\lambda_j t} \\ &- 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} 2\frac{(-1)^j}{j\pi}[\bar{z}(0) - z^\star(0)]e^{-\lambda_j t} \\ &- 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right]\int_0^t z^{\star\prime}(\tau)e^{\lambda_j(\tau - t)}d\tau \\ &- 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} 2\frac{(-1)^j}{j\pi}\int_0^t [\bar{z}'(\tau) - z^{\star\prime}(\tau)]e^{\lambda_j(\tau - t)}d\tau \end{aligned}$$

or

$$\begin{aligned} Z(t) &= z^*(t) + [\bar{z}(t) - z^*(t)]\frac{2}{3} \\ &- 4\,(1 + z^\star(0))\sum_{j=1}^{\infty}(-1)^j\frac{[\cos(j\pi) - 1]}{(j\pi)^2}e^{-\lambda_j t} - 4[\bar{z}(0) - z^\star(0)]\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}e^{-\lambda_j t} \\ &- 4\sum_{j=1}^{\infty}(-1)^j\frac{[\cos(j\pi) - 1]}{(j\pi)^2}\int_0^t z^{\star\prime}(\tau)e^{\lambda_j(\tau - t)}d\tau - 4\sum_{j=1}^{\infty}\int_0^t [\bar{z}'(\tau) - z^{\star\prime}(\tau)]\frac{1}{(j\pi)^2}e^{\lambda_j(\tau - t)}d\tau \end{aligned}$$

or

$$\begin{aligned} Z(t) &= z^*(t) + [\bar{z}(t) - z^*(t)]\frac{2}{3} \\ &- 4\,(1 + z^\star(0))\sum_{j=1}^{\infty}(-1)^j\frac{[\cos(j\pi) - 1]}{(j\pi)^2}e^{-k(j\pi)^2 t} - 4[\bar{z}(0) - z^\star(0)]\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}e^{-k(j\pi)^2 t} \\ &- 4\sum_{j=1}^{\infty}(-1)^j\frac{[\cos(j\pi) - 1]}{(j\pi)^2}\int_0^t z^{\star\prime}(\tau)e^{k(j\pi)^2(\tau - t)}d\tau - 4\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}\int_0^t [\bar{z}'(\tau) - z^{\star\prime}(\tau)]e^{k(j\pi)^2(\tau - t)}d\tau \end{aligned}$$

Integrating by parts we have:

$$\begin{aligned} &\int_0^t z^{\star\prime}(\tau)e^{k(j\pi)^2(\tau - t)}d\tau = z^\star(t) - z^\star(0)e^{-k(j\pi)^2 t} - k(j\pi)^2\int_0^t z^\star(\tau)e^{k(j\pi)^2(\tau - t)}d\tau \\ &\int_0^t [\bar{z}'(\tau) - z^{\star\prime}(\tau)]e^{k(j\pi)^2(\tau - t)}d\tau = [\bar{z}(t) - z^\star(t)] - [\bar{z}(0) - z^\star(0)]e^{-k(j\pi)^2 t} \\ &- k(j\pi)^2\int_0^t [\bar{z}(\tau) - z^\star(\tau)]e^{k(j\pi)^2(\tau - t)}d\tau \end{aligned}$$

and using that

$$\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} = \frac{1}{4} \text{ and } \sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} = \frac{1}{6}$$

then

$$\begin{aligned}
Z(t) &= z^*(t) + [\bar{z}(t) - z^*(t)]\frac{2}{3} \\
&\quad - 4\left(1 + z^\star(0)\right)\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} e^{-k(j\pi)^2 t} - 4[\bar{z}(0) - z^\star(0)]\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} e^{-k(j\pi)^2 t} \\
&\quad - z^*(t) - \frac{2}{3}[\bar{z}(t) - z^\star(t)] \\
&\quad + 4\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} k(j\pi)^2 \int_0^t z^\star(\tau) e^{k(j\pi)^2(\tau - t)} d\tau \\
&\quad + 4\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} k(j\pi)^2 \int_0^t [\bar{z}(\tau) - z^\star(\tau)] e^{k(j\pi)^2(\tau - t)} d\tau \\
&\quad + 4\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} z^\star(0) e^{-k(j\pi)^2 t} + 4\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} \int_0^t [\bar{z}(0) - z^\star(0)] e^{-k(j\pi)^2 t}
\end{aligned}$$

or

$$\begin{aligned}
Z(t) &= -4\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} e^{-k(j\pi)^2 t} \\
&\quad - 4z^\star(0)\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} e^{-k(j\pi)^2 t} \\
&\quad - 4[\bar{z}(0) - z^\star(0)]\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} e^{-k(j\pi)^2 t} \\
&\quad + 4k\sum_{j=1}^{\infty}(-1)^j [\cos(j\pi)-1] \int_0^t z^\star(\tau) e^{k(j\pi)^2(\tau - t)} d\tau \\
&\quad + 4k\sum_{j=1}^{\infty} \int_0^t [\bar{z}(\tau) - z^\star(\tau)] e^{k(j\pi)^2(\tau - t)} d\tau \\
&\quad + 4\sum_{j=1}^{\infty}(-1)^j \frac{[\cos(j\pi)-1]}{(j\pi)^2} z^\star(0) e^{-k(j\pi)^2 t} \\
&\quad + 4\sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} \int_0^t [\bar{z}(0) - z^\star(0)] e^{-k(j\pi)^2 t}
\end{aligned}$$

or

$$Z(t) = -4\sum_{j=1}^{\infty}\frac{[1-\cos(j\pi)]}{(j\pi)^2}e^{-k(j\pi)^2 t}$$
$$+ 4k\sum_{j=1}^{\infty}[1-\cos(j\pi)]\int_0^t z^\star(\tau)e^{k(j\pi)^2(\tau-t)}d\tau$$
$$+ 4k\sum_{j=1}^{\infty}\int_0^t [\bar{z}(\tau)-z^\star(\tau)]e^{k(j\pi)^2(\tau-t)}d\tau$$

which gives the expression for T_Z given the definitions of $\bar{G}, G^*$ and Z_0.

That $\bar{G}(s) > 0$ is immediate. That $G^*(s) \geq 0$ follows by noticing that we can write:

$$G^*(s) = \sum_{j=1,3,5,\dots} e^{-(j\pi)^2 k\, s}\left[1 - e^{-((j+1)^2-j^2)\pi^2 k\, s}\right]$$

and each term $\left[1 - e^{-((j+1)^2-j^2)\pi^2 k\, s}\right] > 0$ since k and s are positive.

□

Proof. (of Lemma 7.)

That $K \leq 0$ as in 1 uses the expression equation (54) and that $G^* \geq 0$, $A^* > 0$, $\bar{G} \geq 0$, and $\bar{A} < 0$.

The symmetry of K when $\eta = 0$ in 2 follows directly from its definition in equation (54).

That $K(0, s) = 0$ for all s as in 3 follows directly from its definition as an integral in equation (54).

The limit in 4 follows from evaluating equation (57) at $0 < t = s < \infty$, which for each j, i pair which gives

$$|K(t,t)| = \left|4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A} - A^*(-1)^{j+i}\right]\frac{1-e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}\right|$$
$$\geq 4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1-e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}$$
$$= -4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} + 4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}$$

The firs term of the last equality converges for $t > 0$, and j integer since

$$\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} < \frac{e^{-\pi^2ktj}}{(i\pi)^2}$$

and so

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} < \sum_{j=1}^{\infty}e^{-\pi^2ktj}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2} = \frac{1}{1-e^{-\pi^2kt}}\frac{1}{6}$$

The second term of the last equality diverges since the $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}$ diverges to $+\infty$.

The last part of the proof is to establish the bounds for the integral $\int_0^T|K(t,s)|ds$.

As a preliminary step we write $\int_0^T|K(t,s)|ds \leq \int_0^{\infty}|K(t,s)|ds$ as:

$$\int_0^{\infty}|K(t,s)|ds = 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A}\;-A^*\left(-1\right)^{j+i}\right]\kappa_{i,k} \text{ where}$$

$$\kappa_{i,j}(t) \equiv \int_0^{\infty}\frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)}-1\right]e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds$$

Direct computation gives

$$\begin{aligned}
\kappa_{i,j}(t) &= \int_0^t\frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds+\int_t^{\infty}\frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2ks]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds\\
&-\int_0^{\infty}\frac{e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds\\
&=\int_0^t\frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]ks}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds+\int_t^{\infty}\frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]kt}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds\\
&-e^{-(j\pi)^2kt}\int_0^{\infty}\frac{e^{-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds\\
&=e^{-(j\pi)^2kt}\int_0^t\frac{e^{(j\pi)^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds+e^{(i\pi)^2kt+\eta^2kt}\int_t^{\infty}\frac{e^{-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds\\
&-\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(j\pi)^2kt}}{k(i\pi)^2+k\eta^2}\\
&=\frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(j\pi)^2k}+\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(i\pi)^2k+\eta^2k}-\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(j\pi)^2kt}}{k(i\pi)^2+\eta^2k}\\
&=\frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\left(\frac{1}{(j\pi)^2k}+\frac{1}{(i\pi)^2k+\eta^2k}\right)\\
&=\frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{k}\left(\frac{(j\pi)^2+(i\pi)^2+\eta^2}{((j\pi)^2)\left((i\pi)^2+\eta^2\right)}\right)\\
&=\frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\,\left((i\pi)^2+\eta^2\right)}
\end{aligned}$$

Thus we get:

$$\kappa_{i,j}(t)=\frac{1-e^{-(j\pi)^2kt}}{k\,(j\pi)^2\,\left((i\pi)^2+\eta^2\right)}$$

We expand κ_{ij} around $\eta = 0$ to obtain:

$$\kappa_{i,j}(t) = \frac{1 - e^{-(j\pi)^2 kt}}{k\left((j\pi)^2\right)\left((i\pi)^2 + \eta^2\right)} = \frac{1 - e^{-(j\pi)^2 kt}}{k\left((j\pi)^2\right)\left((i\pi)^2\right)} \frac{(i\pi)^2}{\left((i\pi)^2 + \eta^2\right)}$$
$$= \frac{1 - e^{-(j\pi)^2 kt}}{k\,(j\pi)^2\,(i\pi)^2}\left(1 - \frac{\eta^2}{(i\pi)^2} + o(\eta^2)\right)$$

Thus

$$\int_0^T |K(t,s)|ds \leq -\int_0^\infty K(t,s)ds$$
$$= 4\sum_{j=1}^\infty \sum_{i=1}^\infty \left[-\bar{A} + A^*\,(-1)^{j+i}\right]\left(\frac{1 - e^{-(j\pi)^2 kt}}{k}\right)\frac{1}{(j\pi)^2 (i\pi)^2}\left(1 - \frac{\eta^2}{(i\pi)^2}\right) + o(\eta^2)$$
$$\leq 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^\infty \sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{\left(1 - e^{-(j\pi)^2 kt}\right)}{(j\pi)^2}\right]\left(1 - \frac{\eta^2}{(i\pi)^2}\right)$$
$$+ 4\frac{A^*}{k}\left[\sum_{j=1}^\infty \sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{\left(1 - e^{-(j\pi)^2 kt}\right)}{(j\pi)^2}\right]\left(1 - \frac{\eta^2}{(i\pi)^2}\right) + o(\eta^2)$$
$$< 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^\infty \sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1 - \frac{\eta^2}{(i\pi)^2}\right)$$
$$+ 4\frac{A^*}{k}\left[\sum_{j=1}^\infty \sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1 - \frac{\eta^2}{(i\pi)^2}\right) + o(\eta^2)$$

were we use that $1 - e^{-(j\pi^2)kt} < 1$ and that :

$$\frac{-\bar{A}}{k} = -\frac{2\eta^2}{1 - \eta\coth(\eta)} = 6 + \frac{2}{5}\eta^2 + o(\eta^2)$$
$$\frac{A^*}{k} = \frac{2\eta^2}{1 - \eta\,\mathrm{csch}(\eta)} = 12 + \frac{7}{5}\eta^2 + o(\eta^2)$$

to write:

$$
\begin{aligned}
\int_0^T |K(t,s)|ds &\leq -\int_0^\infty K(t,s)ds \\
&< 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\frac{A^*}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right)+o(\eta^2) \\
&= 4\left(6+\frac{2}{5}\eta^2\right)\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\left(12+\frac{7}{5}\eta^2\right)\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right)+o(\eta^2) \\
&= 4\times 6\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]+4\times 12\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right] \\
&+ 4\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(\frac{2}{5}-\frac{6}{(i\pi)^2}\right)\eta^2 \\
&+ 4\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(\frac{7}{5}-\frac{12}{(i\pi)^2}\right)\eta^2+o(\eta^2)
\end{aligned}
$$

Using the values for the following series into the the previous expression

$$
\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}=\frac{1}{6}\ ,\ \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(j\pi)^2}=\frac{1}{12}\ ,\ \sum_{j=1}^{\infty}\frac{1}{(j\pi)^4}=\frac{1}{90}=\frac{1}{6}\frac{1}{15}\ \text{and}\ \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(j\pi)^4}=\frac{7}{720}=\frac{1}{12}\frac{7}{60}
$$

we obtain:

$$
\begin{aligned}
\int_0^T |K(t,s)|ds &\leq -\int_0^\infty K(t,s)ds \\
&< 4\times 6\frac{1}{6^2}+4\times 12\frac{1}{6^2}\frac{1}{4}+4\left(\frac{1}{6^2}\frac{2}{5}-\frac{1}{6}\frac{6}{90}\right)\eta^2+4\left(\frac{7}{5}\frac{1}{6^2}\frac{1}{4}-\frac{1}{6}\frac{12}{12}\frac{7}{60}\right)\eta^2+o(\eta^2) \\
&= 1-\frac{7}{180}\eta^2+o(\eta^2)
\end{aligned}
$$

which is the expression for 7.

To obtain the bound in 5 for any η and $t \geq 0$ we note we note that

$$
\kappa_{i,j}(t)=\frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\left((i\pi)^2+\eta^2\right)}<\hat{\kappa}_{i,t}\equiv\frac{1}{k\,(j\pi)^2\,(i\pi)^2}
$$

hence

$$\int_0^{\infty} |K(t,s)|dt = 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A} - A^*(-1)^{j+i}\right]\kappa_{i,k}(t) \leq 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A} - A^*(-1)^{j+i}\right]\hat{\kappa}_{i,k}$$

Again, following the same steps as above we get:

$$\int_0^{T} |K(t,s)|ds \leq -\int_0^{\infty} K(t,s)ds \leq 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]$$
$$+ 4\frac{A^*}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]$$

Using the series obtained above we have:

$$\int_0^{T} |K(t,s)|ds < \frac{4}{6^2}\left(\frac{-\bar{A}}{k} + \frac{A^*}{k}\frac{1}{4}\right)$$

Using the expressions for $-\bar{A}/k$ and A^*/k we have:

$$\int_0^{T} |K(t,s)|ds < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\,\mathrm{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$$

To establish the bound in 8 we start with the direct of

$$\begin{aligned}
\kappa^{\top}_{i,j}(s) &= \int_0^s \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}dt + \int_s^\infty \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2ks]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}dt \\
&- \int_0^\infty \frac{e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}dt \\
&= \int_0^s \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]kt}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}dt + \int_s^\infty \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]ks}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}dt \\
&- e^{-(i\pi)^2ks-\eta^2ks}\int_0^\infty \frac{e^{-(j\pi)^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}dt \\
&= e^{-(i\pi)^2ks-\eta^2ks}\int_0^s \frac{e^{(i\pi)^2kt+\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}dt + e^{(j\pi)^2ks}\int_s^\infty \frac{e^{-(j\pi)^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}dt \\
&- \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(i\pi)^2ks-\eta^2ks}}{k(j\pi)^2} \\
&= \frac{\left(1-e^{-(i\pi)^2ks-\eta^2ks}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(i\pi)^2k+\eta^2k} + \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(j\pi)^2k} \\
&- \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(i\pi)^2ks-\eta^2ks}}{k(j\pi)^2} \\
&= \frac{\left(1-e^{-(i\pi)^2ks-\eta^2ks}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(i\pi)^2k+\eta^2k} + \frac{1-e^{-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{k(j\pi)^2} \\
&= \frac{\left(1-e^{-(i\pi)^2ks-\eta^2ks}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\left(\frac{1}{(i\pi)^2k+\eta^2k}+\frac{1}{k(j\pi)^2}\right)
\end{aligned}$$

Thus we get:

$$\kappa^{\top}_{i,j}(s) = \frac{1-e^{-(i\pi)^2ks-\eta^2ks}}{k\,(j\pi)^2\,\left((i\pi)^2+\eta^2\right)}$$

For future reference, note that for all $s \geq 0$ and $\eta \geq 0$

$$\kappa^{\top}_{i,j}(s) = \frac{1-e^{-(i\pi)^2ks-\eta^2ks}}{k\,(j\pi)^2\,\left((i\pi)^2+\eta^2\right)} \leq \hat{\kappa}_{i,j} \equiv \frac{1}{k\,(j\pi)^2\,(i\pi)^2}$$

Hence

$$\int_0^\infty |K(t,s)|dt = 4\sum_{j=1}^\infty\sum_{i=1}^\infty \left[\bar{A}\,-A^*\,(-1)^{j+i}\right]\kappa^{\top}_{i,k}(s) \leq 4\sum_{j=1}^\infty\sum_{i=1}^\infty \left[\bar{A}\,-A^*\,(-1)^{j+i}\right]\hat{\kappa}_{i,k}$$

The last expression is the same as the bound for $\int_0^\infty |K(t,s)|ds$, and hence the bound is the same, i.e.

$$\int_0^T |K(t,s)|dt < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\operatorname{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$$

Our proof that K^2 as a finite integral as in 9 consists on a long computation of the double integral. Moreover, for each t we can decompose K into the sum of a continuous function on s and one that is discontinuous at $s = t$.

Note that

$$|K(t,s)| \leq 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\left[\bar{A} - A^*(-1)^{j+i}\right]| \left|\frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)} - 1\right] e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}\right|$$

Thus using a change on variables we have:

$$\int_0^T\int_0^T K^2(t,s)dtds \leq \left[|\bar{A}| + |A^*|\right]\frac{4}{k^2\pi^6}\int_0^Q\int_0^Q \tilde{K}^2(t,s)dtds$$

where

$$\tilde{K}(t,s) \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{\left[e^{[j^2+i^2+d](t\wedge s)} - 1\right] e^{-j^2t-i^2s-ds}}{j^2+i^2+d} \text{ with } d \equiv \frac{\eta^2}{\pi^2} \text{ and } Q \equiv Tk\pi^2$$

We define

$$f(\tau) \equiv \left(e^{(j^2+i^2+d)\tau} - 1\right)\left(e^{(l^2+m^2+d)\tau} - 1\right)$$

and then write:

$$\tilde{K}^2(t,s) = \sum_j\sum_i\sum_l\sum_m \frac{f(t\wedge s)e^{-(j^2+l^2)t-(i^2+d+m^2+d)s}}{(j^2+i^2+d)(m^2+l^2+d)}$$

Fix j, i, m, l, and consider the double integral in s and t:

$$\int_0^Q\int_0^Q f(t\wedge s)e^{-(j^2+l^2)t-(i^2+d+m^2+d)s}dsdt = \mathcal{A} + \mathcal{B}$$
$$\equiv \int_0^Q\int_0^t f(t\wedge s)e^{-(j^2+l^2)t-(i^2+d+m^2+d)s}dsdt + \int_0^Q\int_t^Q f(t\wedge s)e^{-(j^2+l^2)t-(i^2+d+m^2+d)s}dsdt$$

where $\mathcal{A}$ and $\mathcal{B}$ were implicitly defined. Solving the integral for A by parts we have:

$$\begin{aligned}
\mathcal{A} &= \int_0^Q \left(\int_0^t f(s) e^{-(i^2+d+m^2+d)s} ds \right) e^{-(j^2+l^2)t} dt \\
&= \left(\int_0^{t'} f(s) e^{-(i^2+d+m^2+d)s} ds \right) \left(\frac{e^{-(j^2+l^2)t'}}{-(l^2+j^2)} \right) |_0^Q - \int_0^Q f(t) e^{-(i^2+d+m^2+d)t} \frac{e^{-(j^2+l^2)t}}{-(l^2+j^2)} dt \\
&= -\frac{e^{-(j^2+l^2)Q}}{(l^2+j^2)} \int_0^Q f(s) e^{-(i^2+d+m^2+d)s} ds + \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt
\end{aligned}$$

We also have:

$$\begin{aligned}
\mathcal{B} &= \int_0^Q f(t) e^{-(j^2+l^2)t} \left(\int_t^Q e^{-(i^2+d+m^2+d)s} ds \right) dt \\
&= \int_0^Q f(t) e^{-(j^2+l^2)t} \left(\frac{e^{-(i^2+d+m^2+d)Q} - e^{-(i^2+d+m^2+d)t}}{-(i^2+d+m^2+d)} \right) dt \\
&= \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&- \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+l^2)t} e^{-(i^2+d+m^2+d)Q} dt
\end{aligned}$$

Since $f(s) \geq 0$ we can write:

$$\begin{aligned}
\mathcal{A} &= -\frac{e^{-(j^2+l^2)Q}}{(l^2+j^2)} \int_0^Q f(s) e^{-(i^2+d+m^2+d)s} ds + \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&\leq \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&- \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+l^2)t} e^{-(i^2+d+m^2+d)Q} dt \\
&\leq \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt
\end{aligned}$$

Thus

$$\mathcal{A} + \mathcal{B} \leq \mathcal{C}(j,i,l,m) \equiv \left(\frac{1}{(l^2+j^2)} + \frac{1}{(i^2+d+m^2+d)} \right) \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt$$

Thus we want to compute the upper bound:

$$\int_0^Q \int_0^Q \tilde{K}^2(t,s)dsdt \leq \sum_j \sum_i \sum_l \sum_m \frac{\mathcal{C}(j,i,l,m)}{(j^2+i^2+d)(l^2+m^2+d)}$$

The next step is to compute the integral $\int_0^Q f(t)e^{-(j^2+i^2+d+l^2+m^2+d)t}dt$. We have

$$\begin{aligned}
&f(t)e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&\equiv \left(e^{(j^2+i^2+d)t}-1\right)\left(e^{(l^2+m^2+d)t}-1\right)e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&= \left[e^{(j^2+i^2+d+l^2+m^2+d)t}+1-e^{(j^2+i^2+d)t}-e^{(l^2+m^2+d)t}\right]e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&= 1+e^{-(j^2+i^2+d+l^2+m^2+d)t}-e^{-(l^2+m^2+d)t}-e^{-(j^2+i^2+d)t}
\end{aligned}$$

Now we compute the time integral:

$$\begin{aligned}
&\int_0^Q \left(1+e^{-(j^2+i^2+d+l^2+m^2+d)t}-e^{-(l^2+m^2+d)t}-e^{-(j^2+i^2+d)t}\right)dt \\
&= Q+\frac{1-e^{-(j^2+i^2+d+l^2+m^2+d)Q}}{(j^2+i^2+d+l^2+m^2+d)}-\frac{1-e^{-(l^2+m^2+d)Q}}{(l^2+m^2+d)}-\frac{1-e^{-(j^2+i^2+d)Q}}{(j^2+i^2+d)} \\
&\leq Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{C}(j,i,l,m) \leq &\left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right) \\
&\times\left(Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}\right)
\end{aligned}$$

and thus we have:

$$\begin{aligned}
&\int_0^Q \int_0^Q \tilde{K}^2(t,s)dsdt \\
&\leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right) \\
&\times\left(Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}\right)
\end{aligned}$$

We have

$$\int_0^Q \int_0^Q \tilde{K}^2(t,s)dsdt \leq 4\,Q\,\mathcal{D}$$

$$\mathcal{D} \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)} + \frac{1}{(i^2+d+m^2+d)}\right)$$

In turn, it suffices to show that

$$\mathcal{E} \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\frac{1}{(l^2+j^2)} < \infty$$

To find a bound for this series we use the following integral:

$$\mathcal{F} \equiv \int_1^\infty \int_1^\infty \int_1^\infty \int_1^\infty \frac{1}{(x_1^2+x_2^2+d)}\frac{1}{(y_1^2+y_2^2+d)}\frac{1}{(x_1^2+y_1^2)}dx_1dx_2dy_1dy_2$$

Thus using $\int_1^\infty 1/(z^2+a^2)dz = \tan^{-1}(a)/a$ we have:

$$\begin{aligned}
\mathcal{F} &= \int_1^\infty dx_1 \int_1^\infty dy_1 \frac{1}{(x_1^2+y_1^2)} \int_1^\infty \frac{1}{(x_1^2+x_2^2+d)}dx_2 \int_1^\infty \frac{1}{y_1^2+y_2^2}dy_2 \\
&= \int_1^\infty dx_1 \int_1^\infty dy_1 \frac{1}{(x_1^2+y_1^2)} \int_1^\infty \frac{1}{(x_1^2+x_2^2+d)}dx_2 \frac{\tan^{-1}(y_1)}{y_1^2} \\
&\leq \int_1^\infty dx_1 \int_1^\infty dy_1 \frac{1}{(x_1^2+y_1^2)} \int_1^\infty \frac{1}{(x_1^2+x_2^2)}dx_2 \frac{\tan^{-1}(y_1)}{y_1^2} \\
&= \int_1^\infty dx_1 \int_1^\infty dy_1 \frac{1}{(x_1^2+y_1^2)} \frac{\tan^{-1}(x_1)}{x_1}\frac{\tan^{-1}(y_1)}{y_1}
\end{aligned}$$

Using that $\tan^{-1}(z) \leq \pi/2$ for $z \geq 1$ we have

$$\mathcal{F} \leq \frac{\pi^2}{4}\int_1^\infty \int_1^\infty \frac{1}{(x_1^2+y_1^2)}\frac{1}{x_1}\frac{1}{y_1}dx_1dy_1$$

Using that $\int_1^\infty \frac{1}{(z^2+a^2)}\frac{1}{z}dz = \log(a^2+2)/(2a)$ we have

$$\begin{aligned}
\mathcal{F} &\leq \frac{\pi^2}{4}\int_1^\infty \frac{\log(y_1^2+2)}{2y_1}\frac{1}{y_1}dy_1 = \frac{\pi^2}{8}\int_1^\infty \frac{\log(y_1^2+2)}{y_1^2}dy_1 \\
&\leq \frac{\pi^2}{8}\int_1^\infty \frac{\log(y_1^2)}{y_1^2}dy_1 = \frac{\pi^2}{4}\int_1^\infty \frac{\log(y_1)}{y_1^2}dy_1 = \frac{\pi^2}{4} < \infty
\end{aligned}$$

since $\int_1^\infty \frac{\log(z)}{z^2}dz = 1$.

Combining all the inequalities obtained above we have:

$$\int_0^T \int_0^T K^2(t,s)dsdt \leq 4\,Q\frac{\pi^2}{4}\frac{4}{k^2\pi^6}\left(A^* - \bar{A}\right). = Tk\pi^2\frac{4}{k^2\pi^4}\left(A^* - \bar{A}\right)$$
$$= T\frac{4}{k\pi^2}\left(A^* - \bar{A}\right)$$

Since

$$\bar{A} = k\tfrac{2\eta^2}{[1-\eta\coth(\eta)]} < 0 \text{ and } A^* = k\,\tfrac{2\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} > 0$$

We have:

$$\int_0^T \int_0^T K^2(t,s)dsdt \leq \frac{8}{\pi^2}T\left(\tfrac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} - \tfrac{\eta^2}{[1-\eta\coth(\eta)]}\right)$$

□

Proof. (of Proposition 4). We order the eigenvalues a $|\mu_j| \geq |\mu_j + 1|$ for all j. Since $K < 0$, from Perron-Frobenious we know that $\mu_1 < 0$ and that $|\mu_1| > |\mu_j|$ for $j > 1$. Moreover, we have that $|\mu_1| = \mathrm{Lip}_K$, since:

$$||\mathcal{K}(Y)||_2^2 \equiv \int_0^T \left(\int_0^T K(t,s)\,Y(s)ds\right)^2 dt \leq \sqrt{\mathrm{Lip}_K\,\mathrm{Lip}_{K^\top}}\,||Y||_2^2$$
$$\leq \mathrm{Lip}_K\,||Y||_2^2$$

and $\mathrm{Lip}_K \geq 1$ by Lemma 7.

By a combination of Proposition 5 when $\theta \in [-1, 0)$ and of Proposition 8 when $\theta \geq 0$, we know that there exists a solution for all θ satisfying $\theta \geq -1$.

Assume that for some $j > 1$, we have $\mu_j > 0$. We know that $\mu_j < |\mu_1|$. Since $|\mu_1| \geq 1$ and we set $\theta = 1/\mu_j$ for which we have a solution Y_θ of the integral equation. By the Fredholm alternative, it must the the case that $\langle\phi_j, Y_0\rangle = 0$ for any of the eigenfunctions associated with any positive eigenvalue. Hence, either the eigenvalues are negative, or the corresponding eigenfunctions are orthogonal to Y_0. The solution is thus given by:

$$Y_\theta(t) = \sum_{j\in\mathbb{J}} \frac{\langle\phi_j, Y_0\rangle}{1-\theta\mu_j}\phi_j(t)$$

where $\mathbb{J} \equiv \{j : \mu_j < 0 \text{ or } \langle\phi, Y_0\rangle \neq 0 \text{ for any of the eigenfunctions } \phi \text{ of } \mu_j\}$. obtaining the desired result.

□

Proof. (of Proposition 7) To show that when $\theta = \underline{\theta}$ with $\underline{\mu}_1 = 1$ then, assume that that there is a solution of equation (56) in this case. Then we have

$$Y(t) = Y_0(t) + \theta\mathcal{K}(Y)(t)$$

where $Y \in L^2([0,T])$. Then form the inner product of Y with ϕ_1 to get:

$$\begin{aligned}\langle Y,\phi_1\rangle &= \langle Y_0,\phi_1\rangle + \theta\langle \mathcal{K}Y,\phi_1\rangle = \langle Y_0,\phi_1\rangle + \theta\langle Y,\mathcal{K}\phi_1\rangle \\ &= \langle Y_0,\phi_1\rangle + \theta\langle Y,\mu_1\phi_1\rangle = \langle Y_0,\phi_1\rangle + \theta\mu_1\langle Y,\phi_1\rangle \\ &= \langle Y_0,\phi_1\rangle + \langle Y,\phi_1\rangle\end{aligned}$$

where we have used that $\mathcal{K}$ is self-adjoint, that ϕ_1 is an eigenfunction of $\mathcal{K}$, and that $\theta\mu_1 = 1$. The last equality implies that $\langle Y_0,\phi_1\rangle = 0$. But since $Y_0 > 0$, and since $-\mathcal{K}$ is a positive operator, then by the Perron-Frobenious theorem the eigenfunction ϕ_1 associated with the dominant eigenvalue μ_1 does not change sign, and thus $\langle Y_0,\phi_1\rangle \neq 0$, arriving to a contradiction. Hence, there is no solution when $\theta = \underline{\theta} = 1/\mu_1$.

□

Proof. (of Lemma 8) Recall that $Y(t) = -\varrho Z(t)$, so we write the problem in terms of $Z's$. We take the system of coupled p.d.e's and boundary conditions:

$$\begin{aligned}
&0 = -\rho v(x,t) + v_t(x,t) + k v_{xx}(x,t) + 2B\theta x Z(t) \text{ for all } x\in[-1,1] \text{ and } t\\
&0 = v(-1,t) = v(1,t) = 0\,,\ v(x,t) = -v(-x,t) \text{ all } t, \text{ and } v(x,T) = 0 \text{ all } x\\
&0 = v_x(1,t) + \tilde{u}_{xx}(1)\bar{z}(t) = v_x(0,t) + \tilde{u}_{xx}(0)z^*(t) \text{ all } t\\
&0 = -n_t(x,t) + k n_{xx}(x,t) \text{ for all } x\in[-1,1], x\neq 0, \text{ and } t\\
&0 = n(1,t) = \bar{z}(t)\,,\ n(x,t) = -n(-x,t) \text{ all } t, \text{ and } n(x,0) = -\bar{\kappa}\, sign(x) \text{ all } x\\
&0 = n(0^+,t) - n(0^-,t) = 2z^*(t) \text{ all } t
\end{aligned}$$

To obtain the bound, we compute the Lasry-Lions energy type of integral, by multiplying the p.d.e for v times n and integrating in $[-1,1]$, and adding it to the product of the the p.d.e for n times v integrated in $[-1,1]$:

$$\begin{aligned}0 = \int_{-1}^{1} \big[&- \rho n(x,t)v(x,t) + n(x,t)v_t(x,t) + k n(x,t)v_{xx}(x,t) \\ &+ 2B\theta Z(t) x n(x,t) + n_t(x,t)v(x,t) - k n_{xx}(x,t)v(x,t)\big]dx \\ &= D_1(t) + D_2(t) + D_3(t)\end{aligned}$$

We analyze several terms separately. First,

$$D_1(t) \equiv \int_{-1}^{1} 2B\theta Z(t) x n(x,t)dx = 2B\theta (Z(t))^2 > 0$$

since we are assuming $\theta > 0$. Second

$$\begin{aligned}D_2(t) &\equiv \int_{-1}^{1} \big[- \rho n(x,t)v(x,t) + n(x,t)v_t(x,t) + n_t(x,t)v(x,t)\big]dx \\ &= e^{\rho t}\frac{d}{dt}\int_{-1}^{1} e^{-\rho t} n(x,t)v(x,t)dx\end{aligned}$$

Third, we write the last group of terms as

$$\begin{aligned} D_3(t) &\equiv \int_{-1}^{1} (n(x,t)v_{xx}(x,t) - n_{xx}(x,t)v(x,t))\,dx \\ &= \int_{-1}^{0} (n(x,t)v_{xx}(x,t) - n_{xx}(x,t)v(x,t))\,dx + \int_{-1}^{0} (n(x,t)v_{xx}(x,t) - n_{xx}(x,t)v(x,t))\,dx \end{aligned}$$

We use integration by parts to obtain:

$$D_3(t) = n(x,t)v_x(x,t)|_{-1}^{0^-} - n_x(x,t)v(x,t)|_{-1}^{0^-} + n(x,t)v_x(x,t)|_{0^+}^{1} - n_x(x,t)v(x,t)|_{0^+}^{1}$$

To evaluate D_3 we use the boundary conditions. In particular, since $v(1,t) = v(-1,t) = v(0,t) = 0$ thus:

$$\begin{aligned} D_3(t) &= n(x,t)v_x(x,t)|_{-1}^{0^-} + n(x,t)v_x(x,t)|_{0^+}^{1} \\ &= n(0^-,t)v_x(0^-,t) - n(-1,t)v_x(-1,t) + n(1,t)v_x(1,t) - n(0^+,t)v_x(0^+,t) \\ &= \big(n(0^-,t) - n(0^+,t)\big)\, v_x(0,t) + (n(1,t) - n(-1,t))\, v_x(1,t) \end{aligned}$$

where we use that v_x is continuous and that $v_x(1,t) = v_x(-1,t)$. Using the expression for these functions on the boundaries:

$$D_3(t) = -2z^*t(t)v_x(0,t) + 2\bar{z}(t)v_x(1,t) = 2\left(z^*(t)\right)^2 \tilde{u}_{xx}(0) - 2\left(\bar{z}(t)\right)^2 \tilde{u}_{xx}(1)$$

Recall that 0 is a minimum of $\tilde{u}$ and that ± 1 is (are) maximum of $\tilde{u} \in [-1,1]$, thus $\tilde{u}_{xx}(0) > 0$ and $\tilde{u}_{xx} < 0$.

$$D_3(t) = 2\left(z^*(t)\right)^2 \tilde{u}_{xx}(0) - 2\left(\bar{z}(t)\right)^2 \tilde{u}_{xx}(1) > 0.$$

Hence we can write

$$\begin{aligned} 0 &= D_3(t) + D_2(t) + D_1(t) \\ &= 2B\theta(Z(t))^2 + e^{\rho t}\frac{d}{dt}e^{-\rho t}\int_{-1}^{1} n(x,t)v(x,t)dx + 2\left(z^*(t)\right)^2 \tilde{u}_{xx}(0) - 2\left(\bar{z}(t)\right)^2 \tilde{u}_{xx}(1) \end{aligned}$$

or

$$\frac{d}{dt}e^{-\rho t}\int_{-1}^{1} n(x,t)v(x,t)dx = -e^{-\rho t}\left(2B\theta(Z(t))^2 + \left[2\left(z^*(t)\right)^2 \tilde{u}_{xx}(0) - 2\left(\bar{z}(t)\right)^2 \tilde{u}_{xx}(1)\right]\right)$$

Integrating with respect to t this expression in $[0,T]$ we get:

$$\begin{aligned} \int_0^T \frac{d}{dt}e^{-\rho t}\int_{-1}^{1} n(x,t)v(x,t)dxdt &= e^{-\rho T}\int_{-1}^{1} n(x,T)v(x,T)dx - \int_{-1}^{1} n(x,0)v(x,0)dx \\ = -\int_0^T e^{-\rho t}2B\theta(Z(t))^2dt &- \int_0^T e^{-\rho t}\left[2\left(z^*(t)\right)^2 \tilde{u}_{xx}(0) - 2\left(\bar{z}(t)\right)^2 \tilde{u}_{xx}(1)\right] dt \end{aligned}$$

Using the boundary condition $n(x,T)=0$ for all x we have:

$$\begin{aligned}&\int_{-1}^{1} n(x,0)v(x,0)dx\\&=\int_0^T e^{-\rho t}2B\theta(Z(t))^2dt+\int_0^T e^{-\rho t}\left[2\left(z^*(t)\right)^2\tilde{u}_{xx}(0)-2\left(\bar{z}(t)\right)^2\tilde{u}_{xx}(1)\right]dt\end{aligned}$$

We use equation (28) from Lemma 2 to evaluate $v(x,t)$ at $t=0$ obtaining:

$$v(x,0)=-4B\theta\int_0^T\sum_{j=1}^{\infty}e^{-(\eta^2+(j\pi)^2)\,k\tau}\,Z(\tau)\frac{(-1)^n}{j\pi}\sin(j\pi x)d\tau$$

that $n(\cdot,t)$ and $v(\cdot,t)$ are antisymmetric, and that $n(0,x)=-\bar{\kappa}$ for $x\in(0,1]$ so that

$$\begin{aligned}&\int_{-1}^{1} n(x,0)v(x,0)dx=2\int_0^1 n(x,0)v(x,0)dx=-2\bar{\kappa}\int_0^1 v(x,0)dx\\&=8B\theta\bar{\kappa}\int_0^T\sum_{j=1}^{\infty}e^{-(\eta^2+(j\pi)^2)\,k\tau}\,Z(\tau)\frac{(-1)^n}{j\pi}\int_0^1\sin(j\pi x)dxd\tau\end{aligned}$$

Using that

$$\int_0^1\sin(j\pi x)dx=\frac{1-\cos(\pi j)}{n\pi}=\begin{cases}\frac{2}{\pi j} & \text{if } j=1,3,\ldots\\ 0 & \text{if } n=2,4,\ldots\end{cases}$$

Thus we can write:

$$\int_{-1}^{1} n(x,0)v(x,0)dx=8B\theta\bar{\kappa}\sum_{j=1,3,5}^{\infty}\int_0^T e^{-(j\pi)^2\,k\tau}\,e^{-\rho\tau}Z(\tau)\frac{-2}{(j\pi)^2}d\tau$$

Replacing this expression we have:

$$\begin{aligned}&8\bar{\kappa}\sum_{j=1,3,5}^{\infty}\int_0^T e^{-(j\pi)^2\,kt}\,e^{-\rho t}Z(t)\frac{-1}{(j\pi)^2}dt\\&=\int_0^T e^{-\rho t}Z(t)^2dt+\int_0^T e^{-\rho t}\left[z^*(t)^2\frac{2\tilde{u}_{xx}(0)}{B\theta}-\bar{z}(t)^2\frac{2\tilde{u}_{xx}(1)}{\theta B}\right]dt\end{aligned}$$

Under the assumption that $\theta > 0$ and $z^* \neq 0$ or $\bar{z} \neq 0$ then

$$\begin{aligned}\int_0^T e^{-\rho t} Z(t)^2 dt &< 8\bar{\kappa} \sum_{j=1,3,5}^{\infty} \int_0^T e^{-(j\pi)^2\, kt}\, e^{-\rho t} Z(t) \frac{-1}{(j\pi)^2} dt \\ &\leq 8\bar{\kappa} \sum_{j=1,3,5}^{\infty} \int_0^T e^{-(j\pi)^2\, kt}\, e^{-\rho t} |Z(t)| \frac{1}{(j\pi)^2} dt \\ &< 8\bar{\kappa} \sum_{j=1,3,5}^{\infty} \int_0^T e^{-\rho t} |Z(t)| \frac{1}{(j\pi)^2} dt = 8\bar{\kappa} \sum_{j=1,3,5}^{\infty} \frac{1}{(j\pi)^2} \int_0^T e^{-\rho t} |Z(t)| dt\end{aligned}$$

where we used that $k > 0$. Using that $\sum_{j=1,3,5}^{\infty} \frac{1}{(j\pi)^2} = \frac{1}{8}$ so that

$$\int_0^T e^{-\rho t} Z(t)^2 dt < \bar{\kappa} \int_0^T e^{-\rho t} |Z(t)| dt$$

Multiplying both sides by $\frac{\rho}{1-e^{-\rho T}}$

$$\left(||Z||_{L^2(\rho,T)}\right)^2 \equiv \frac{\rho}{1-e^{-\rho T}} \int_0^T e^{-\rho t} Z(t)^2 dt < \bar{\kappa} \frac{\rho}{1-e^{-\rho T}} \int_0^T e^{-\rho t} \,|Z(t)| dt \equiv \bar{\kappa} ||Z||_{L^1(\rho,T)}$$

Using the relationship between L^1 and L^2 norms, i..e. that $||Z||_{L^1(\rho,T)} \leq ||Z||_{L^2(\rho,T)}$, then

$$\left(||Z||_{L^2(\rho,T)}\right)^2 < \bar{\kappa} ||Z||_{L^1(\rho,T)} \leq \bar{\kappa} ||Z||_{L^2(\rho,T)}$$

or dividing on both sides by $||Z||_{L^2(\rho,T)}$, we get $||Z||_{L^2(\rho,T)} < \bar{\kappa}$ or $||Y||_{L^2(\rho,T)} < \bar{\kappa} |\varrho|$.

Proof. (of Proposition 8)

Uniqueness follow from more general case in Proposition **??**. Existence follows from using Leray-Schauder fixed point. To apply this fixed point we define $T(\bar{\kappa}, Z)$ as:

$$T(\bar{\kappa}, Z) = \bar{\kappa} Z_0(t) + \bar{\kappa} \theta' \int_0^T K(t,s) Z(s) ds \text{ for all } t \in [0,T]$$

for any Z with $||Z||_{L^2(\rho,T)} < \infty$ and $\bar{\kappa} \in [0,1]$. The operator T is compact, as required in the Leray-Schauder theorem. The compactness follows from bound on $\int \int K^2 ds dt$ established in part 9 in Lemma 7. Next, using Corollary 1 the integral equation above corresponds to the case of $Z_0^{\kappa} = \bar{\kappa} Z_0$ and of $\theta = \theta' \bar{\kappa}$.

Finally, to apply the Leray-Schauder fixed point theorem we need to show that there is bound $C < \infty$ so that any fixed point $Z = T(\bar{\kappa}, Z)$ satisfies $||Z||_{L^2(\rho,T)} < C$. This is done in Lemma 8 letting $\theta = \theta' \bar{\kappa} > 0$.

□

Proof. (of Proposition 6)

That the series in equation (60), whenever it converges, is the solution of equation (56) follows from replacing the series into the integral equation.

That $Y_\theta(0) = 1$ follows from the fact that $Y_0(0) = 1$ and that $K(0,s) = 0$ for all $s \in (0,T)$.

To establish that $Y_\theta(t) > 0$ and $\theta < 0$, so we have $\theta K(t,s) > 0$ for all $(t,s) \in (0,T)^2$

and hence $(\theta\mathcal{K})^r(Y_0) > 0$ for $t \in (0,T)$. Note that, for each t, the sequence $S_n(\theta,t) \equiv \sum_{r=0}^{n} \theta^r(\mathcal{K})^r(Y_0)(t)$ is monotone increasing in n, and, by assumption converges. Hence, $Y_\theta(t) > 0$. Moreover if $\theta' < \theta < 0$ we have $S_n(\theta',t) > S_n(\theta,t)$. Thus, the limit preserves this inequality.

To establish that $Y_\theta(t)$ is convex, we differentiate twice the series with respect to θ, obtaining:

$$\frac{\partial^2}{\partial\theta^2} Y_\theta(t) = \sum_{r=2}^{\infty} r(r-1)\theta^{r-2} \left(\mathcal{K}\right)^r (Y_0)(t)$$

for $t \in (0,T)$. If r is even we have $\theta^{r-2} > 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) > 0$. If r is odd we have $\theta^{r-2} < 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) < 0$, hence all the terms in the sum are strictly positive, and thus $\frac{\partial^2}{\partial\theta^2} Y_\theta(t) > 0$.

□

Proof. (of Proposition 10) We set $T = \infty$. For this value we want to compute

$$\tfrac{d}{d\theta} CIR_\theta|_{\theta=0} = \int_0^\infty \tfrac{d}{d\theta} Y_\theta(t)|_{\theta=0} dt = \int_0^\infty \int_0^\infty K(t,s) Y_0(t)\, ds\, dt$$

which can be written as

$$Q \equiv \int_0^\infty \int_0^\infty K(t,s) Y_0(s) ds\, dt = \sum_{m=1}^{\infty} Q_m \text{ where } Q_m = 4 \int_0^\infty \int_0^\infty K(t,s) \frac{1-\cos(m\pi)}{(m\pi)^2} ds\, dt$$

where we have replaced the expression for Y_0

Replacing the expression for K we get that for each m

$$Q_m = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} 16 \left(1 - \cos(m\pi)\right) \left(\bar{A} - A^*(-1)^{i+j}\right) \tilde{\omega}_{i,j,m}$$

where $\tilde{\omega}_{i,j,m}$ is defined as

$$\begin{aligned}
\tilde{\omega}_{i,j,m} &= \frac{1}{k^2\pi^8} \frac{1}{(i^2+j^2+r^2)m^2}\, \omega_{i,j,m} \text{ and} \\
\omega_{i,j,m} &= \int_0^\infty \int_0^\infty \left(e^{(j^2+i^2+r^2)s\wedge t} - 1\right) e^{-j^2 t - i^2 s - r^2 s - m^2 s} ds\, dt
\end{aligned}$$

were we have used a change on variables for t, and where we use $r \equiv \eta^2/\pi^2$.

Now we compute $\omega_{i,j,m}$ letting $\rho \downarrow 0$, or equivalently $r \to 0$. For this note that we can write the inner integral in $\omega_{i,j,m}$ as follows:

$$\int_0^t e^{-j^2t}e^{-(m^2-j^2)s}ds + \int_t^\infty e^{i^2t}e^{-(i^2+m^2)s}ds - \int_0^\infty e^{-j^2t}e^{-(i^2+m^2)s}ds$$
$$= e^{j^2t}\frac{\left[1-e^{-(m^2-j^2)t}\right]}{(m^2-j^2)} + \frac{e^{i^2t}e^{-(i^2+m^2)t}}{(i^2+m^2)} - \frac{e^{-j^2t}}{(i^2+m^2)}$$
$$= \frac{e^{-j^2t}-e^{m^2t}}{(m^2-j^2))} + \frac{e^{-m^2t}-e^{-j^2t}}{(i^2+m^2)}$$

Then, integrating the resulting expression with respect to t between 0 and ∞ we get:

$$\omega_{i,j,m} = \frac{1}{(m^2-j^2)}\left[\frac{1}{j^2}-\frac{1}{m^2}\right] + \frac{1}{(i^2+m^2)}\left[\frac{1}{m^2}-\frac{1}{j^2}\right] = \frac{1}{m^2j^2} + \frac{1}{(i^2+m^2)}\frac{(j^2-m^2)}{m^2j^2}$$
$$= \frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)$$

Now we replace this expression into $\tilde{\omega}_{i,j,m}$

$$\omega_{i,j,m} = \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\omega_{i,j,m} = \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)$$
$$= \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{m^2j^2}\left(\frac{1}{i^2+m^2}\right) = \frac{1}{k^2}\frac{1}{(m\pi)^4}\frac{1}{(j\pi)^2}\frac{1}{(i^2\pi^2+m^2\pi^2)}$$

Finally we want to compute the infinite sums of the expression for $\omega_{i,j,m}$ over i, j, m. For this we will use that when m is odd:

$$\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2+m^2\pi^2} = \frac{m\pi\coth(m\pi)-1}{2m^2\pi^2}$$
$$\sum_{i=1}^{\infty}\frac{(-1)^i}{i^2\pi^2+m^2\pi^2} = \frac{m\pi\,\mathrm{csch}(m\pi)-1}{2m^2\pi^2}$$
$$\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i^2\pi^2+m^2\pi^2} = \frac{1-m\pi\,\mathrm{csch}(m\pi)}{2m^2\pi^2}$$

and we will also use that

$$\sum_{j=1}^{\infty}\tfrac{1}{(j\pi)^2} = \frac{1}{6} \text{ and } \sum_{j=0}^{\infty}\frac{1}{\pi^2(j+1)^2} = \frac{1}{8}\,.$$

We write $Q = \mathcal{Q}_I - \mathcal{Q}_{II}$:

$$\mathcal{Q}_I = \sum_{m=1,3,5,\dots} 2 \times 16\,\bar{A} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tilde{\omega}_{i,j,m} = \sum_{m=1,3,5,\dots} 32 \frac{\bar{A}}{k} \frac{1}{k} \frac{1}{(m\pi)^4} \sum_{j=1}^{\infty} \frac{1}{(j\pi)^2} \sum_{i=1}^{\infty} \frac{1}{(i^2\pi^2 + m^2\pi^2)}$$

$$= \sum_{m=1,3,5,\dots} \frac{32}{6} \frac{\bar{A}}{k} \frac{1}{k} \frac{1}{(m\pi)^4} \sum_{i=1}^{\infty} \frac{1}{(i^2\pi^2 + m^2\pi^2)}$$

$$= \sum_{m=1,3,5,\dots} \frac{32}{12} \frac{\bar{A}}{k} \frac{1}{k} \frac{1}{(m\pi)^6} \left(m\pi \coth(m\pi) - 1\right)$$

Now we write the second term of Q:

$$\mathcal{Q}_{II} = \frac{32}{k} \frac{A^*}{k} \sum_{1,3,5,\dots} \frac{1}{(m\pi)^4} \sum_{j=1}^{\infty} \frac{1}{j^2\pi^2} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{\pi^2 i^2 + \pi^2 m^2} = \frac{32}{k} \frac{A^*}{k} \sum_{m=1,3,5,\dots} \frac{1}{(m\pi)^4} \left(\mathcal{O} + \mathcal{E}\right) \text{ where}$$

$$\mathcal{O} = \sum_{j=1,3,5,\dots} \frac{1}{(\pi j)^2} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{(i^2\pi^2 + m^2\pi^2))} = \sum_{j=0}^{\infty} \frac{1}{\pi^2 (j+1)^2} \frac{(1 - m\pi \operatorname{csch}(m\pi))}{2m^2\pi^2}$$

$$= \frac{1}{8} \frac{(1 - m\pi \operatorname{csch}(m\pi))}{2m^2\pi^2} \text{ and}$$

$$\mathcal{E} = \sum_{j=2,4,6,\dots} \frac{1}{(\pi j)^2} \sum_{i=1}^{\infty} \frac{(-1)^i}{(i^2\pi^2 + m^2\pi^2))} = \left[\frac{1}{6} - \frac{1}{8}\right] \sum_{i=1}^{\infty} \frac{(-1)^i}{(i^2\pi^2 + m^2\pi^2))}$$

$$= \frac{1}{8} \frac{1}{3} \frac{(m\pi \operatorname{csch}(m\pi) - 1)}{2m^2\pi^2}$$

Thus

$$\mathcal{Q}_{II} = \frac{32}{k} \frac{A^*}{k} \sum_{m=1,3,5,\dots} \frac{1}{(m\pi)^4} \left(\mathcal{O} + \mathcal{E}\right) = \frac{32}{k} \frac{A^*}{k} \frac{1}{8} \left(\frac{1}{3} - 1\right) \sum_{m=1,3,5,\dots} \frac{1}{(m\pi)^4} \frac{(m\pi \operatorname{csch}(m\pi) - 1)}{2m^2\pi^2}$$

$$= \frac{32}{k} \frac{A^*}{k} \frac{1}{8} \frac{1}{3} \sum_{m=1,3,5,\dots} \frac{1 - m\pi \operatorname{csch}(m\pi)}{(m\pi)^6}$$

Recall that as $\rho \to 0$ then $\bar{A}/k \to -6$ and $A^*/k \to 12$, and thus

$$Q = \mathcal{Q}_I - \mathcal{Q}_{II} = \sum_{m=1,3,5,\dots} \frac{32}{12} \frac{\bar{A}}{k} \frac{1}{k} \frac{1}{(m\pi)^6} \left(m\pi \coth(m\pi) - 1\right) - \frac{32}{k} \frac{A^*}{k} \frac{1}{8} \frac{1}{3} \sum_{m=1,3,5,\dots} \frac{1 - m\pi \operatorname{csch}(m\pi)}{(m\pi)^6}$$

$$= \sum_{m=1,3,5,\dots} \frac{32}{12} 6 \frac{1}{k} \frac{1}{(m\pi)^6} \left(1 - m\pi \coth(m\pi)\right) - \frac{32}{k} 12 \frac{1}{8} \frac{1}{3} \sum_{m=1,3,5,\dots} \frac{1 - m\pi \operatorname{csch}(m\pi)}{(m\pi)^6}$$

$$= \frac{16}{k} \sum_{1,3,5,\dots} \left(\frac{1 - m\pi \coth(m\pi)}{(m\pi)^6} - \frac{1 - m\pi \operatorname{csch}(m\pi)}{(m\pi)^6} \right)$$

$$= \frac{16}{k} \sum_{m=1,3,5,\dots} \frac{\operatorname{csch}(m\pi) - \coth(m\pi)}{(m\pi)^5}$$

Finally we have:

$$
\begin{aligned}
CIR_0 &= \int_0^\infty Y_0(t)dt = \sum_{1,3,5,\ldots} 8 \int_0^\infty \frac{e^{-\pi^2 m^2 kt}}{(m\pi)^2} dt \\
&= \frac{8}{k} \sum_{1,3,5,\ldots} \frac{1}{(m\pi)^4} = \frac{8}{k}\frac{1}{96} = \frac{1}{12\,k}
\end{aligned}
$$

Thus

$$
\frac{1}{CIR_\theta}\frac{dCIR_\theta}{d\theta}|_{\theta=0} = \frac{Q}{CIR_0} = 16 \times 12 \sum_{m=1,3,5,\ldots} \frac{\operatorname{csch}(m\pi) - \coth(m\pi)}{(m\pi)^5}
$$

and using $16 \times 12 = 192$ we get our final result.

□

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Online Appendix:

Price Setting with Strategic Complementarities as a Mean Field Game

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It turns out that, after some derivations, both propositions boils down to solve the heat equation in the domain $(x,t) \in [0,1] \times \mathbb{R}_+$, with a source s, and with time boundaries given by the time varying functions A, B. In particular to solve for $w : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ given parameter $k > 0$, $\nu \geq 0$, source $s : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$, space boundary at time zero $f : [0,1] \times \mathbb{R}$, and value at the boundaries given by $a, b : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

$$
\begin{aligned}
0 &= -w_t(x,t) - \nu w(x,t) + k w_{xx}(x,t) + s(x,t) \text{ all } x \in [0,1] \text{ and } t > 0 \\
w(x,0) &= f(x) \text{ all } x \in [0,1] \\
w(0,t) &= A(t) \text{ all } t > 0 \text{ and} \\
w(1,t) &= B(t) \text{ all } t > 0
\end{aligned}
$$

LEMMA 11. The solution for the KFE equation for w is given by:

$$
\begin{aligned}
w(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t) \varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where} \\
r(x,t) &= A(t) + x[B(t) - A(t)] \text{ all } x \in [0,1],\ t > 0
\end{aligned}
$$

and where for all $j = 1, 2, \ldots$ we have:

$$
\begin{aligned}
\varphi_j(x) &= \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h \rangle \equiv \int_0^1 h(x) \varphi_j(x) dx \\
a_j(t) &= a_j(0) e^{-\lambda_j t} + \int_0^t q_j(\tau) e^{\lambda_j(\tau - t)} d\tau \text{ all } t > 0\,, \\
q_j(t) &= \frac{\langle \varphi_j\,,\, s(\cdot,t) - r_t(\cdot,t) - \nu r(\cdot,t) \rangle}{\langle \varphi_j, \varphi_j \rangle} \text{ all } t > 0 \\
\lambda_j &= \nu + (j\pi)^2 k \text{ and } a_j(0) = \frac{\langle \varphi_j\,,\, f - r(\cdot,0) \rangle}{\langle \varphi_j, \varphi_j \rangle}\,.
\end{aligned}
$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e. holds in the interior since

$$a'_j(t) = -\lambda_j a_j(t) + q_j(t) \text{ for all } t > 0 \text{ and } j = 1, 2, \ldots$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions on $\{h : [0,1] \to \mathbb{R}\}$, and finally that the boundary holds at $t = 0$ for all x.

Consider now the KBE equation, which only changes the sign of the time derivative, the range of time, and the time at which the space boundary condition holds, so $w : [0,1] \times [0,T] \to \mathbb{R}$, where:

$$\begin{aligned} 0 &= w_t(x,t) - \nu w(x,t) + k w_{xx}(x,t) + s(x,t) \text{ all } x \in [0,1] \text{ and } t > 0 \\ w(x,T) &= f(x) \text{ all } x \in [0,1], \\ w(0,t) &= A(t) \text{ all } t \in [0,T], \text{ and} \\ w(1,t) &= B(t) \text{ all } t \in [0,T] \end{aligned}$$

Lemma 12. The solution for the KBE for w is given by:

$$\begin{aligned} w(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t \in [0,T] \text{ where} \\ r(x,t) &= A(t) + x[B(t) - A(t)] \text{ all } x \in [0,1],\ t \in [0,T] \end{aligned}$$

and where for all $j = 1, 2, \ldots$ we have:

$$\begin{aligned} \varphi_j(x) &= \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h \rangle \equiv \int_0^1 h(x)\varphi_j(x)dx \\ a_j(t) &= a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau \text{ all } t \in [0,T)\,, \\ q_j(t) &= \frac{\langle \varphi_j\,,\, s(\cdot,t) + r_t(\cdot,t) - \nu r(\cdot,t) \rangle}{\langle \varphi_j, \varphi_j \rangle} \text{ all } t \in [0,T) \\ \lambda_j &= \nu + (j\pi)^2 k \text{ and } a_j(T) = \frac{\langle \varphi_j\,,\, f - r(\cdot,T) \rangle}{\langle \varphi_j, \varphi_j \rangle}\,. \end{aligned}$$

As in the previous case the proof can be done by verifying that the equation hold at the boundaries, that for $t \in [0,T]$ the p.d.e. holds in the interior since

$$-a'_j(t) = -\lambda_j a_j(t) + q_j(t) \text{ for all } t \in [0,T) \text{ and } j = 1, 2, \ldots$$

Note that $q_j(t)$ and $a_j(t)$ are also defined differently than for the KFE.