Beyond Incomplete Spanning: Convenience Yields and Exchange Rate Disconnect*

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Abstract

We introduce convenience yields on dollar bonds into an incomplete-market equilibrium model of exchange rates and interest rates. The convenience yield enters as a stochastic wedge in the Euler equation for exchange rate determination. The model identifies a novel safe-asset convenience yield channel by which quantitative easing impacts the dollar exchange rate. Our model addresses three exchange rate puzzles. (1) The model can rationalize the low pass-through of SDF shocks to exchange rates and hence low exchange rate volatility. (2) It helps address but does not fully resolve the exchange rate disconnect puzzle. (3) The model generates an unconditional log currency expected return on the dollar that is in line with the data.

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1 Introduction

In the workhorse neoclassical model of international finance, exchange rates act as the only shock absorbers for innovations to the marginal utility growth rate of investors in different countries. The complete-market model falls short when confronted with the data. In this class of models, real exchange rates do not co-vary with macroeconomic quantities in the right way—the exchange rate disconnect puzzle. The model-implied real exchange rate appreciates when domestic investors experience high marginal utility growth. The model-implied real exchange rates are also too volatile—the exchange rate volatility puzzle.

Adopting a preference-free approach, Lustig and Verdelhan (2019) demonstrate that incomplete-market models cannot simultaneously address the U.I.P. (uncovered interest rate parity) violations, the exchange rate disconnect/the countercyclical variation puzzle, and the exchange rate volatility puzzle.[1] In the models developed by Gabaix and Maggiori (2015); Itskhoki and Mukhin (2019), the exchange rate is determined by the Euler equation of a specialized FX arbitrageur. These intermediaries give rise to a wedge in the Euler equation of standard investors who do not operate in foreign exchange markets and/or foreign bond markets.

In this paper, we take a different approach to modeling an Euler equation wedge. We develop an equilibrium model in which foreign investors earn convenience yields on their holdings of USD bonds. In our incomplete-market model, markets are not segmented. We allow home and foreign investors to trade the risk-free bonds of both currencies. The USD convenience yields introduce a stochastic wedge into the foreign investors' Euler equation and thereby affect exchange rate determination.

Since the Great Financial Crisis, sizable deviations from Covered Interest Parity have opened up in LIBOR markets (Du, Tepper, and Verdelhan, 2018b), but even before the GFC, there were large deviations from CIP in government bond markets (see Du, Im, and Schreger, 2018a; Jiang, Krishnamurthy, and Lustig, 2020a; Du and Schreger, 2021). Jiang et al. (2020a) infer the convenience yields earned on U.S. Treasurys by foreign investors from these deviations. Foreign investors earn convenience yields of around 200 basis points, significantly larger than the CIP deviations. Using a demand-system-based approach, Koijen and Yogo (2019) report similar estimates of the convenience yields.[2]

According to Jiang et al. (2020a), convenience yields account for a significant portion of variation in the dollar exchange rate.[3] In this paper, we develop a dynamic equilibrium model with stochastic convenience yields. We calibrate the model, and we use the model as a laboratory to

[1] A few recent papers show that market segmentation may offer a way forward. Alvarez, Atkeson, and Kehoe (2002); Gabaix and Maggiori (2015); Dou and Verdelhan (2015); Itskhoki and Mukhin (2019); Chien, Lustig, and Naknoi (2020); Sandulescu, Trojani, and Vedolin (2020) develop models with segmented asset markets.

[2] In related literature, Augustin, Chernov, Schmid, and Song (2020) study CIP deviations and convenience yields in a no-arbitrage framework. van Binsbergen, Diamond, and Grotteria (2019) infer the true risk-free rates and the implied convenience yields from option prices. Jiang, Richmond, and Zhang (2020b) study the implications of the flight-to-U.S. safety for portfolio imbalances and international capital flows.

[3] Engel and Wu (2018) find evidence that CIP deviations also have explanatory power for the exchange rate variation of other currencies.

address the exchange rate puzzles discussed above.

In our model, the USD convenience yields can act as shock absorbers, at least partly subsuming the role of exchange rates in standard neoclassical models. We report four key findings. First, the wedges introduced by convenience yields mitigate the pass-through of shocks from the stochastic discount factors (SDF) to exchange rates. As a result, the model-implied exchange rates are not as volatile as in the complete-market model. Second, the covariance between shocks to the USD convenience yield and the SDFs substantially reduces the counter-cyclicality of exchange rates. Third, the model generates an unconditional log currency expected return that is in line with the data. Fourth, we demonstrate a connection between quantitative easing and exchange rates via convenience yields, as opposed to the arbitrageur portfolio channels studied in Gourinchas, Ray, and Vayanos (2019); Greenwood, Hanson, Stein, and Sunderam (2019).

We adopt a preference-free approach to FX markets similar to Backus, Foresi, and Telmer (2001); Lustig, Roussanov, and Verdelhan (2011); Lustig and Verdelhan (2019), in the tradition of Hansen and Jagannathan (1991). We posit a pair of home (dollar) and foreign log SDFs. In addition, we assume that foreign investors receive a stochastic convenience yield on their holding of home (dollar) bonds. We then use the home and foreign Euler equation for each bond (four equations in total) to derive a closed-form expression for the exchange rate as a function of the histories of home and foreign SDF shocks and USD convenience yield shocks. Our development allows for a clean characterization of the sources of variation of the exchange rate. The long-run expected exchange rate level is well defined, which allows a Froot and Ramadorai (2005)-type representation. We also derive the risk premium implied by the model.

First, we make progress on the exchange rate volatility puzzle. Our model's equilibrium exchange rates in logs are less volatile than the difference of home and foreign log SDFs. This result, which is a convenience-yield variant of the result derived in Lustig and Verdelhan (2019), helps to resolve the volatility of exchange rates vis-à-vis stock prices (Brandt, Cochrane, and Santa-Clara, 2006). In our closed-form characterization, the covariance between the SDF shocks and the exchange rate is tightly connected to the covariance between the exchange rate and the convenience yield. In the model without convenience yields, exchange rates have to fully close the gap between the pricing kernels, absorbing all of the residual shocks. In the model with convenience yields, convenience yields can partially act as shock absorbers too. To calibrate the model, we match the comovement of convenience yields and exchange rates reported by Jiang, Krishnamurthy, and Lustig (2020a). This calibrated model then matches the volatility of exchange rates in the data. The convenience yields allow us to disentangle the volatility of the exchange rate from that of the SDFs.

Second, we make progress on the exchange rate disconnect puzzle. Convenience yield shocks impact exchange rates in our model. The equilibrium exchange rate reflects expected future interest rate spreads, currency risk premia and USD convenience yields. The dollar appreciates when dollar bonds carry a higher convenience yield. Depending on the covariance between convenience yield shocks and the SDFs, these shocks can carry a risk premium. In the case of foreign flight to the safety of USD Treasurys, the convenience yield shock has a higher covariance with the foreign SDF than the home SDF. This being the case, the convenience yield risk premium channel counteracts the standard complete markets channel. As a result, the dollar does not depreciate as much against the foreign currency when foreign investors experience higher than average marginal utility growth. We explore this countervailing force in a calibrated version of our model. The model generates an a-cyclical exchange rate, but cannot deliver a pro-cyclical exchange rate in line with the data (Backus and Smith, 1993; Kollmann, 1995). We make progress on the Backus and Smith (1993) puzzle but we do not fully resolve it.

Third, our model generates sizable deviations from U.I.P. The dollar has a negative unconditional expected excess return because it has a positive convenience yield and because it appreciates when the foreign SDF is high, thereby earning a negative risk premium. Our model delivers a realistic unconditional log currency risk premium while matching the volatility of exchange rates. In stark contrast, Lustig and Verdelhan (2019) show that these moments cannot be matched jointly in an incomplete-market model without convenience yields, as making progress on the risk premium puzzle immediately makes other puzzles worse. On the other hand, the baseline version of model does not feature time-varying prices or quantities of risk. We leave this out to keep the model tractable. As a result, our model does not generate time-variation in the conditional risk premium on foreign currencies, needed to replicate the failure of U.I.P. in the time series, first documented by Hansen and Hodrick (1980); Fama (1984). However, our model does generate time-variation in expected excess returns on long positions in foreign bonds, simply because of the variation in convenience yields.

There is a deep connection between bond markets and currency markets. In the data, when currency risk premia increase for a particular currency, local currency term premia or bond risk premia tend to decline (Lustig, Stathopoulos, and Verdelhan, 2019). In fact, over long horizons, the term premium and the currency risk premium contributions have to offset each other to restore long-run U.I.P. when the real exchange rate is stationary. As a result, currency carry trades are less profitable at longer maturities. Gourinchas et al. (2019); Greenwood et al. (2019) develop preferred habitat models of the term structure that replicate this stylized fact. In these models, changes to the supply of bonds alter the portfolio balance between home and foreign bonds for a hypothetical arbitrageur, necessitating a change in the exchange rate to accommodate the new portfolio. Quantitative easing works by changing the bond risk premium demanded by the arbitrageur. When the Fed buys Treasurys or Mortgage bonds, the bond risk premium on Treasurys shrinks, which is offset by an increase in the currency risk premia on dollars, to enforce long-run U.I.P. In the long-run, foreign investors expect to earn the same returns on dollar bonds as on home bonds. As a result, the dollar depreciates instantaneously.

Our work is the first to highlight a distinct convenience yield channel in FX markets, separate from the bond risk premium channel. There is both empirical and theoretical support for the proposition that shifts in the supply of safe assets induced by QE changes the convenience yield on safe bonds (Krishnamurthy and Vissing-Jorgensen, 2011). In our model, changes in the convenience yield also independently impact the exchange rate, even when bond and currency risk premia are constant. We explain and quantify this new convenience yield channel. When the Fed buys Treasurys, and reserves are more desirable as safe assets, then the convenience yield on dollar-denominated safe assets declines, and the dollar depreciates. If reserves are poor substitutes, then the convenience yields increase and the dollar appreciates. We simulate a convenience yield shock in our model and show that it lines up with the evidence from QE and exchange rates.

In closely related work, Dou and Verdelhan (2015); Itskhoki and Mukhin (2019); Chien, Lustig, and Naknoi (2020) develop international macro models with segmented markets to attack the exchange rate disconnect puzzle. Their models severs the equilibrium exchange rate from its macro-fundamentals by introducing market segmentation and deliver a pro-cyclical exchange rate based on the model's assumed patterns in the arbitrageur's portfolio. For example, Chien et al. (2020) consider a model in which only small pool of investors arbitrage between domestic and foreign securities. As a result, the real exchange rate is disconnected from the differences in aggregate consumption growth between home and foreign. Our model does not rely on market segmentation.[4]

Any no-arbitrage model with stationary exchange rate implies long-run U.I.P. (see Lustig et al., 2019). Backus, Boyarchenko, and Chernov (2018) show that the long-horizon returns on claims to cash flows that are not subject to permanent innovations converge to the returns on a long bond. When the exchange rate is stationary, investments in foreign bonds produce 'stationary' cash flows that are not subject to permanent innovations. Chinn and Meredith (2004) provide evidence that supports long-run U.I.P. Our model produces stationary exchange rates. A version of long-run U.I.P. holds in the model.

The rest of the paper proceeds as follows. Section 2 presents our model of exchange rate determination with convenience yields. Section 3 calibrates the model and then turns to examining the implications of the model for a collection of exchange rate phenomena. Section 4 analyzes how quantitative easing impacts currency markets in our model. Proofs of all propositions are in the Appendix.

Next, we develop an incomplete-markets, no-arbitrage model of the convenience yield channel in continuous time. To keep the analysis tractable, we do not allow for time variation in the price or quantity of risk. In our model, all variables are real, the long-run real exchange rate is stationary, and hence, long-run U.I.P. holds. The USD is special in that foreign investors only earn a convenience yield on USD bonds. The model creates a unique role for the Federal Reserve Bank to affect exchange rates through large-scale asset purchases.

[4]In earlier work, Colacito and Croce (2011) show that correlated long-run risks to consumption help to account for the exchange rate disconnect.

2 Model

We consider a continuous-time infinite horizon economy. We fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and assume that all stochastic processes are adapted to this space and satisfy the usual conditions. There are two countries, home (the U.S.) and foreign. Let s_t denote the log real exchange rate. A higher s_t means a stronger home currency (USD).

2.1 Asset Markets and SDF

In each country, agents can invest in home and foreign bonds. That is, we assume that the asset market is incomplete. We posit the following pair of log real SDFs for each of the agents in home and foreign, respectively:

$$
\begin{aligned}
dm_t &= -\mu dt - \sigma dZ_t, &(1)\\
dm_t^* &= \phi s_t dt - \sigma dZ_t^*, &(2)
\end{aligned}
$$

Here, $\{Z_t, Z_t^*\}$ are standard Brownian motion processes. The Brownian increments dZ_t and dZ_t^* represent shocks to the marginal utilities of each agent, which can captures business cycle shocks as well as changes in the agents' attitudes towards risk. The dynamics for the foreign SDF describe risk-free rate dynamics in the foreign country engineered to keep the real exchange rate stationary. The SDF dynamics describe an implicit monetary policy rule required for stationarity as in Engel and West (2005).

Assumption 1. *We assume that the mean-reversion parameter $\phi > 0$ is strictly positive.*

Assumption 1 implies that the foreign real interest rate is decreasing in the level of the home real exchange rate. In particular, if markets are complete, the log of the real exchange rate s_t^{cm} equals the difference in the log of the SDFs:

$$
\begin{aligned}
s_t^{cm} &= m_t - m_t^*, &(3)\\
ds_t^{cm} &= (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t), &(4)
\end{aligned}
$$

which is a simple stationary process.

2.2 Asset Pricing Conditions

Let P_t denote the cumulative return on the U.S. bond, and P_t^* denote the cumulative return on the foreign bond. These returns follow deterministic dynamics:

$$
\begin{aligned}
dP_t &= r_t P_t dt &(5)\\
dP_t^* &= r_t^* P_t^* dt. &(6)
\end{aligned}
$$

We assume that investors can trade both home and foreign risk-free assets. Let $S_t = \exp(s_t)$, $M_t = \exp(m_t)$ and $M_t^* = \exp(m_t^*)$. The home investors' pricing conditions give[5]

$$
\begin{aligned}
0 &= \mathbb{E}_t[d(M_t P_t)] && (7)\\
0 &= \mathbb{E}_t[d(M_t S_t^{-1} P_t^*)] && (8)
\end{aligned}
$$

and the foreign investors' pricing conditions give

$$
\begin{aligned}
0 &= \mathbb{E}_t[d(M_t^* P_t^*)] && (9)\\
0 &= \mathbb{E}_t[M_t^* S_t P_t \tilde{\lambda}_t dt + d(M_t^* S_t P_t)]. && (10)
\end{aligned}
$$

There is a flow convenience yield $S_t P_t \tilde{\lambda}_t dt$ in the foreigner investors' pricing condition for the home (dollar) risk-free asset. The discrete-time counterpart to this last equation is

$$
\exp(-\tilde{\lambda}_t) = \mathbb{E}_t \left[\frac{M_{t+1}^*}{M_t^*} \frac{S_{t+1} P_{t+1}}{S_t P_t} \right],
$$

which, as in Jiang et al. (2020a), states that the foreign investor's expected return from holding the USD bonds under the risk-neutral measure is lower than 1 because of the convenience yield $\tilde{\lambda}_t$. On the other hand, the U.S. investor receives no convenience yield on the USD bonds.[6]

The convenience yields can be inferred from the CIP deviations in government bond markets, denoted x_t: $(1-\beta)\lambda_t = -x_t$, where β denotes the dollarness of a synthetic Treasury constructed from a currency foreign bond. It measures the fraction of the convenience yield earned on this synthetic position.

We parameterize the convenience yield as follows:

$$
\tilde{\lambda}_t = \ell \frac{\exp(\lambda_t)}{\exp(\lambda_t) + 1}, \tag{11}
$$

which is bounded between 0 and ℓ. The variable λ_t satisfies

$$
d\lambda_t = -\theta \lambda_t dt + \nu dX_t, \tag{12}
$$

where dX_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$.

Finally, with slight abuse of notation, let $[dX_t, dY_t]$ denote the instantaneous conditional covariance between two diffusion processes X_t and Y_t. Formally, it is defined as $[dX_t, dY_t] = d[X_t, Y_t]/dt$ where $[X_t, Y_t]$ is the standard quadratic covariation between processes X_t and Y_t. We assume that

[5]With some abuse of notation we use the notation $\mathbb{E}_t[dX_t]$ to represent the infinitesimal generator of a stochastic process X_t. The formal notation, which is adopted in the proof, is $\mathcal{A}[X_t]$.

[6]At present, we have not studied the case where both U.S. and foreign investors receive convenience yields on USD bonds.

the convenience yield shock and the SDF shocks can be correlated:

$$[dZ_t, dX_t] = \rho, \qquad [dZ_t^*, dX_t] = \rho^*,$$

whereas the home and foreign SDF shocks are not correlated:

$$[dZ_t, dZ_t^*] = 0.$$

We assume that the correlation of SDF (which loads negatively on dZ or dZ^* shocks) and convenience yield innovations is positive ($\rho, \rho^* \leq 0$), so that the convenience yield tends to increase when the marginal utilities as represented by the pricing kernels rise. That is, in bad marginal utility states, there is an increased desire by foreign investors to own dollar bonds as in a "flight-to-Treasuries". Jiang et al. (2020a) present empirical evidence on this point.

2.3 Equilibrium Exchange Rate Dynamics

Without loss of generality, we write the real exchange as satisfying the following stochastic differential equation,

$$ds_t \;=\; \alpha_t dt + \beta_t \sigma (dZ_t^* - dZ_t) + \gamma_t \nu dX_t, \tag{13}$$

where α_t, β_t, and γ_t are $\mathcal{F}_t$-adapted stochastic processes. β_t governs the distance from complete markets. When $\beta_t \equiv 1$, and $\gamma_t \equiv 0$, we are back in the benchmark complete markets case.

Our objective is to present a solution to (13) that satisfies the four pricing conditions (7), (8), (9) and (10). In our incomplete market setting, there are many candidate solutions. We restrict attention to a class of these solutions that we are able to characterize and (as we explain) has economically sensible properties. We assume that the loading of the exchange rate on the aggregate shocks is time-invariant:

Assumption 2. *$\beta_t \equiv \beta$ is constant.*

Proposition 1. *Under Assumption 2, there is a class of solutions indexed by constant k so that,*

$$\beta \;=\; \frac{1}{2} \pm \frac{\sqrt{\sigma^2 - 2k}}{2\sigma}, \tag{14}$$

$$\gamma_t \;=\; \frac{(\rho^* - \rho)\sigma(1-2\beta) \pm \sqrt{(\rho^* - \rho)^2 \sigma^2 (1-2\beta)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}. \tag{15}$$

The log of the real exchange rate satisfies:

$$ds_t \;=\; \left(-\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*)\right) dt + \gamma_t \nu dX_t + \beta\sigma(dZ_t^* - dZ_t), \tag{16}$$

which loads on both the SDF shocks dZ and dZ^* and the convenience yield shock dX.

We explain this result in the next section, presenting the details of the proof in the appendix.

For each k, there are two solutions for β. One root is between $1/2$ and 1, and the other is between 0 and $1/2$. We will calibrate β based on regression results. As for γ_t, note that $(\rho^* - \rho)\sigma(1-2\beta)$ can be either positive or negative. We pick the root of γ_t with the positive sign:

$$\gamma_t = \frac{(\rho^* - \rho)\sigma(1-2\beta) + \sqrt{(\rho^* - \rho)^2\sigma^2(1-2\beta)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu},$$

so that for $k > \tilde{\lambda}_t$, we can guarantee $\gamma_t > 0$. We focus on solutions with $\gamma_t > 0$ to arrive at the natural result that the exchange rate appreciates when the foreign convenience yield for dollar bonds rises. Finally, note that unlike β_t, γ_t is not constant and varies with the convenience yield, $\tilde{\lambda}_t$.

The SDFs are highly volatile. When $\beta = 1$ and $\gamma_t = 0$, we are back in the workhorse complete-market model. As soon as markets are incomplete $0 < \beta < 1, \gamma > 0$ and the exchange rate responds to the convenience yield shocks; exchange rates no longer absorb all of the shocks to the SDFs. The convenience yields do some of the shock absorption.

Furthermore, we can solve the stochastic differential equation (16) to find a closed-form expression for the log of the real exchange rate.

Proposition 2. *The real exchange rate s_t can be expressed as*

$$s_t = f(\lambda_t) + H_t + \beta s_t^{cm}. \tag{17}$$

The first term $f(\lambda_t)$ is a function of the current convenience yield λ_t. Let $b = (\rho^* - \rho)\sigma(1-2\beta)$, then

$$\begin{aligned} f(\lambda) &= \frac{1}{2\nu}\{-\sqrt{b^2+4k}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) \\ &+ \sqrt{b^2+4k-4\ell}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k-4\ell}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-3\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) \\ &+ \lambda\left(\sqrt{b^2+4k}+b\right)\}. \end{aligned}$$

The second term H_t captures the history of past convenience yields:

$$\begin{aligned} H_t &= \exp(-\phi t)H_0 + \int_0^t \exp(-\phi(t-u))h(\lambda_u)du, \\ h(\lambda_t) &= -\frac{1}{2}\tilde{\lambda}_t - \phi f - (1-z)\mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*) + f'\theta\lambda_t - \frac{1}{2}f''\nu^2. \end{aligned}$$

The third term is the real exchange rate s_t^{cm} under complete markets scaled by β, where

$$ds_t^{cm} = (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t). \tag{18}$$

The proof is in the appendix.

This proposition shows that the real exchange rate level is determined by not only the relative pricing kernels, as summarized by the real exchange rate s_t^{cm} under complete markets, but also the current convenience yield and the history of the convenience yields λ_t.

We also note that since the exchange rate's long-run expectation $\lim_{T\to\infty}\mathbb{E}_t[s_T]$ is well-defined:

Lemma 1. *When the markets are incomplete, the exchange rate's long-run expectation* $\lim_{T\to\infty}\mathbb{E}_t[s_T]$ *is*

$$\bar{s} \equiv \lim_{T\to\infty}\mathbb{E}_t[s_T] = \frac{1}{\phi}\left(-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_t] - \mu + \frac{1}{2}\sigma\lim_{T\to\infty}\mathbb{E}_0[\gamma_t]\nu(\rho+\rho^*)\right).$$

In comparison, the complete-market counterpart is

$$\bar{s}^{cm} = \lim_{T\to\infty}\mathbb{E}_t[s_T^{cm}] = -\frac{\mu}{\phi},$$

which does not have the "convenience yield" term $-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_t]$ and the "risk premium" term $\frac{1}{2}\sigma\lim_{T\to\infty}\mathbb{E}_0[\gamma_t]\nu(\rho+\rho^*)$.

With this long-run expectation, s_t has a forward-looking representation:

$$s_t = \bar{s} - \mathbb{E}_t\int_t^\infty ds_u, \tag{19}$$

where the long-run expectation of the exchange rate $\bar{s}$ is derived in Appendix A.3. Following Froot and Ramadorai (2005); Jiang et al. (2020a), we can further decompose the exchange rate level as

$$s_t = \bar{s} + \mathbb{E}_t\int_t^\infty (r_u - r_u^*)du + \mathbb{E}_t\int_t^\infty \tilde{\lambda}_u du - \mathbb{E}_t\int_t^\infty rp_u du \tag{20}$$

where $\mathbb{E}_t\int_t^\infty (r_u - r_u^*)du$ captures expected future short rate differences,

$$r_u - r_u^* = \mu + \phi s_u,$$

while $\mathbb{E}_t\int_t^\infty \tilde{\lambda}_u du$ captures expected future convenience yields, and $-\mathbb{E}_t\int_t^\infty rp_u du$ captures expected future currency risk premia from the foreign perspective plus a Jensen's term,

$$rp_u = (\frac{1}{2}\tilde{\lambda}_u + \frac{1}{2}\sigma\gamma_u\nu(\rho+\rho^*)) = -([dm_t^*, ds_t] + \frac{1}{2}[ds_t, ds_t]).$$

This decomposition in equation (20) is the equivalent of a Campbell-Shiller decomposition for exchange rates. The exchange rate level today reflects future interest rate differences (cash flows), future convenience yields, minus future risk premia (discount rates). This expression is forward-looking, which complements the backward-looking expression for the exchange rate level in equation (17).

A version of this decomposition without convenience yields was derived by Campbell and Clarida (1987); Froot and Ramadorai (2005). The dollar exchange rate in logs today reflects future cash flows, given by the short rate differences, and future discount rates, given by the foreign currency risk premia earned by foreign investors going long in USD . The dollar appreciates when future U.S. short rates increase and dollar currency risk premia decline.

Jiang et al. (2020a) derive a version of this decomposition that allows for convenience yields. When foreign investors expect to earn larger convenience yields on USD bonds, the dollar appreciates in spot markets.

We use r_t^T to denote yield on a T-period zero coupon bond. rx_t^T to denote the conditional risk premium on T-period zero coupon bond. The USD long bond yields can be restated as the sum of the local currency bond risk premia and future risk-free rates: $(T-t)r_t^{T-t} = \int_t^T rx_u^{T-t} du + \mathbb{E}_t \int_t^T (r_u - r_u^*) du$. As a result, we can rewrite the exchange rate decomposition in equation (20) as follows:

$$\begin{aligned} s_t - \bar{s} &= \lim_{T\to\infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T\to\infty} \mathbb{E}_t \int_t^T (rx_u^{*,T-t-u} - rx_u^{T-t-u}) du \\ &- \lim_{T\to\infty} \mathbb{E}_t \int_t^T rp_u du + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du. \end{aligned} \tag{21}$$

When the exchange rate is stationary, the long-term USD bond and foreign bond have to carry the same risk premium in the limit (Backus et al., 2018; Lustig et al., 2019). There is no difference in riskiness between holding a U.S. and a foreign bond over long holding periods. In this case the sum of currency risk premia are exactly offset by the sum of local currency bond risk premium differentials between the two countries. We obtain that long-run U.I.P. holds in the absence of convenience yields:

$$s_t - \bar{s} = \lim_{T\to\infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du. \tag{22}$$

Here is a simple example. If the 20-year yield r_t^{20} declines by 5 bps, then we expect 5 bps p.a. appreciation of the USD over the next 20 years. The USD depreciates by 100 bps now. In the long-run, foreign investors do not accept lower local currency returns on their holdings of long USD bonds than on foreign bonds. There is no long-run exchange rate risk.

In the data, there is empirical evidence to support the notion that high foreign currency risk premia are offset by negative local currency bond risk premia. In the cross-section, there are no currency carry premium at long maturities. In the time series, current interest rates/term spread do not predict long foreign currency bond excess returns converted into USD (Lustig et al., 2019). This is consistent with evidence in favor of long-run U.I.P. (Chinn and Meredith, 2004; Boudoukh, Richardson, and Whitelaw, 2016).[7]

Lastly, we note that our set-up has both the "incomplete-market wedge" as in Backus et al. (2001); Lustig and Verdelhan (2019) and an additional convenience yield. These two exchange rate wedges are fundamentally different. Backus et al. (2001) consider the set-up without convenience

[7] Strictly speaking, this equation implies that exchange rates are spanned by long yields. Chernov and Creal (2018) find evidence against this spanning implication in the data: bond yields only explain a small fraction of the variation in exchange rates.

yield, in which the standard Euler equations are satisfied:

$$0 = \mathbb{E}_t[d(M_t S_t^{-1} P_t^*)], \qquad 0 = \mathbb{E}_t[d(M_t^* S_t P_t)].$$

In the presence of incomplete markets, an incomplete-market wedge $d\eta_t$ arises between the exchange rate movement and the SDF differential:

$$d\eta_t + (dm_t - dm_t^*) \quad = \quad ds_t.$$

In comparison, equations (8) and (10) in our set-up, reproduced below,

$$0 = \mathbb{E}_t[d(M_t S_t^{-1} P_t^*)], \qquad 0 = \mathbb{E}_t[M_t^* S_t P_t \tilde{\lambda}_t dt + d(M_t^* S_t P_t)]$$

implies a violation of the standard Euler equations. As a result, our equilibrium exchange rate dynamics (13) can be thought of as incorporating both the standard incomplete-market wedge and an additional convenience yield.

2.4 Family of Solutions

This section explains why there are multiple solutions to the model. Consider the pair of Euler equations for the home investor in the foreign bond (equation (8)) and the foreign investor in the home bond (equation (10)). We rewrite these equations to derive expressions for the currency risk premia on long positions in USD and foreign currency, respectively:

$$\begin{aligned} r_t - r_t^* + \mathbb{E}_t[ds_t] &= -\left(\frac{1}{2}[ds_t, ds_t] - [\sigma dZ_t^*, ds_t]\right) - \tilde{\lambda}_t, \\ r_t^* - r_t - \mathbb{E}_t[ds_t] &= -\left(\frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t, -ds_t]\right). \end{aligned}$$

These equations can be interpreted as follows. The expected log excess return on long positions in home bonds harvested by the foreign investor, given by the interest rate difference plus the expected rate of appreciation of the home currency, equals the log currency risk premium minus the convenience yield. The expression for the expected log excess return for the home investor is similar, but without the convenience yield. The sum of these two Euler equations produces the following condition:

$$-\tilde{\lambda}_t \quad = \quad \left(\frac{1}{2}[ds_t, ds_t] - \sigma[dZ_t^*, ds_t]\right) + \left(\frac{1}{2}[ds_t, ds_t] + \sigma[dZ_t, ds_t]\right). \tag{23}$$

The two terms in parentheses are respectively the log currency risk premium for the home investor going long foreign bonds and the foreign investor going long the home bond.

First, we consider the case where there are no convenience yields; $\tilde{\lambda}_t$ is always zero. In this symmetric case, these risk premia have to sum to zero. If one investor is earning a risk premium, the other investor must be paying the risk premium. In the case without convenience yields, consider the exchange rate process:

$$ds_t = \alpha_t dt + \beta\sigma(dZ_t^* - dZ_t^*).$$

That is, the only uncertainty is driven by the two Brownian motions driving the SDFs. Substituting into (23), we have that,

$$0 = \beta^2\sigma^2 - \beta\sigma^2.$$

This equation has two solutions: $\beta = 0$ and $\beta = 1$. The $\beta = 1$ case corresponds to the complete-market model. The exchange rate is volatile, and the volatility carries a risk premium that compensates for the volatility. The $\beta = 0$ case is also solution to all of the asset pricing equations. The exchange rate is non-stochastic and there is no risk premium in the model. All Euler equations are satisfied with a purely deterministic exchange rate (note that α_t will not equal zero). We can think of this case as corresponding to an autarchic allocation: each agent holds their own home bonds and the exchange adjusts deterministically to enforce uncovered interest parity.

Next, we consider a version of our model with stochastic convenience yields. With convenience yields, the foreign investor′ demand for dollar bonds necessarily reduces the foreign investor's pecuniary return to going long dollar bonds relative to foreign bonds (i.e., the non-pecuniary convenience yield partially offsets this reduced pecuniary return). But this means that the U.S. investor can earn an excess return by going long foreign bonds relative to dollar bonds. If the exchange rate is non-stochastic, this cannot be an equilibrium since the excess return to the U.S. investor offers an infinite Sharpe ratio. Thus the exchange rate must be stochastic, but as we show next, there is still a family of solutions that arises.

We substitute in the exchange rate process from (13) into (23) to give,

$$-\tilde{\lambda}_t = \gamma_t^2\nu^2 + 2\beta_t^2\sigma^2 + 2\gamma_t\nu\beta_t(\rho^* - \rho)\sigma - 2\beta_t\sigma^2 - (\rho^* - \rho)\sigma\gamma_t\nu. \tag{24}$$

Under our Assumption 2, we take β_t as constant and look for solutions for β that satisfy:

$$0 = 2\beta^2\sigma^2 - 2\beta\sigma^2 + k, \tag{25}$$

for constant k. Likewise, we look for solutions for γ_t that satisfy:

$$k - \tilde{\lambda}_t = \gamma_t^2\nu^2 + 2\gamma_t\nu\beta(\rho^* - \rho)\sigma - (\rho^* - \rho)\sigma\gamma_t\nu. \tag{26}$$

Then k indexes a family of solutions with varying pass-through from the convenience yield and SDF shocks to the exchange rate. Mathematically, the zero volatility case is no longer a solution because if $k = 0$, the solution for γ_t is imaginary. This latter point can be seen by inspecting (15). We note that k indexes the solution for both β and γ. A key property of these solutions is that higher β goes together with higher γ. The next section builds on this observation.

3 Quantitative Implications of Convenience Yields for Exchange Rates

This section discusses (1) the comovement between dollar exchange rate and flight-to-safety as in Jiang et al. (2020a), (2) the partial SDF-FX pass-through and the Brandt et al. (2006) puzzle, (3) the Backus-Smith puzzle, (4) currency risk premium in log and in level, and (5) the Froot-Ramadorai decomposition of exchange rate level. Our model provides a quantitative account of these patterns FX dynamics driven by our convenience model of exchange rates. We begin by explaining our calibration choices.

3.1 Calibration Choices

We calibrate the model at the annual frequency with the following parameter values: $\mu = 0$, $\sigma = 0.5$, $\phi = 0.135$, $\ell = 5\%$, $\theta = 3$, $\nu = 7.5$, $\rho = 0$, $\rho^* = -0.50$. This set of parameter values implies that the convenience yield $\tilde{\lambda}_t$ process has an unconditional mean of 1.9% and an unconditional standard deviation of 2.1% per annum. In Jiang et al. (2020a), we directly measure the U.S. Treasury basis, which we show under our theory will be proportional to the convenience yield, $\tilde{\lambda}_t$. We estimate the constant of proportionality to be $\frac{1}{1-0.9}$ so that the standard deviation of the Treasury basis of 0.23 implies a standard deviation of the convenience yield of $0.23/(1-0.9) = 2.3\%$ and the mean Treasury basis of 0.22 gives a mean convenience yield of $0.22/(1-0.9) = 2.2\%$. Moreover, the mean-reversion parameter $\theta = 3$ implies that the convenience yield shocks have a half-life of $\log(2)/\theta = 0.23$ years. In the data, we estimate an AR(1) model of the Treasury basis and find the estimated model to have a half-life of 0.24 years.

The pricing kernel volatility σ is calibrated to 50% per annum, which implies that the maximal annual Sharpe ratio permitted by either country's pricing kernel is roughly 0.5 as well. We further assume that the correlation between the home SDF shock and the convenience yield shock is $\rho = 0$, and the correlation between the foreign SDF shock and the convenience yield shock is $\rho^* = -0.50$. This assumption implies that the foreign agents' marginal utility goes up when the convenience yield increases.

The adjustment in interest rate in response to the exchange rate level is governed by the parameter ϕ, which we set to 0.135. This parameter value implies that the half life of the variation in a shock to the real exchange rate is $\log(2)/\phi = 5.13$ years. In the data, we estimate an AR(1) model of the log dollar index and find the estimated model to have a half-life of 5.18 years.

Note k can take values between $[\frac{\ell-(\rho^*-\rho)^2\sigma^2/4}{1-(\rho^*-\rho)^2/2}, \sigma^2/2]$. Equivalently, the equilibria in this system can be indexed by the value of β, which is bounded by $[0.09, 0.91]$. If β is below 0.09 or above 0.91, γ_t will have imaginary roots when λ_t is large.

3.2 Choosing k

We write the the innovation in the exchange rate in terms of the underlying economic shocks:

$$ds_t - \mathbb{E}_t[ds_t] = \beta(dm_t^* - dm_t - \mathbb{E}_t[dm_t^* - dm_t]) + \gamma_t \frac{\ell}{\tilde{\lambda}_t(\ell - \tilde{\lambda}_t)}(d\tilde{\lambda}_t - \mathbb{E}_t[d\tilde{\lambda}_t]).$$

The first term on the right-hand side is the exchange rate's exposure to the pricing kernel differential's shock. The second term is the exchange rate's exposure to the convenience yield shock.

We note that k indexes a family of solutions for the pricing kernel exposure β and the convenience yield exposure γ_t. Let us start with the calibration choices above, but with $\rho = \rho^* = 0$. In the left panel of Figure 1, we plot β against γ_t evaluated at $\lambda_t = 0$ for different values of β. This plot is generated by varying k over its domain. We see that the convenience yield loading γ_t is positively associated with the SDF loading β when $\beta < 0.5$, and is negatively associated with the SDF loading β when $\beta > 0.5$. When $\rho = \rho^* = 0$, our equations can be simplified to

$$\beta = \frac{1}{2} \pm \frac{\sqrt{\sigma^2 - 2k}}{2\sigma}, \text{ and } \gamma_t = \frac{\sqrt{4(k - \tilde{\lambda}_t)}}{2\nu};$$

when β takes the smaller root (i.e. less than 0.5), both β and γ_t are increasing in k. For $\beta > 0.5$, the β is decreasing in k, while γ_t is increasing in k.

In the panel on the right, we report the case with $\rho^* = -0.5$ which corresponds to our principal calibration. Over most of the range of β, the convenience yield loading γ_t is positively associated with the SDF loading β. Algebraically, from equation (26), when $\rho - \rho^* = 0.5$, the term on the right-hand side contributes to the relation, thus strengthening the relation between γ and β. In our calibration exercise, for each value of k, we compute γ_t and select the smaller root of β, and then simulate the model. The greater root of β will generate much greater exchange rate volatility

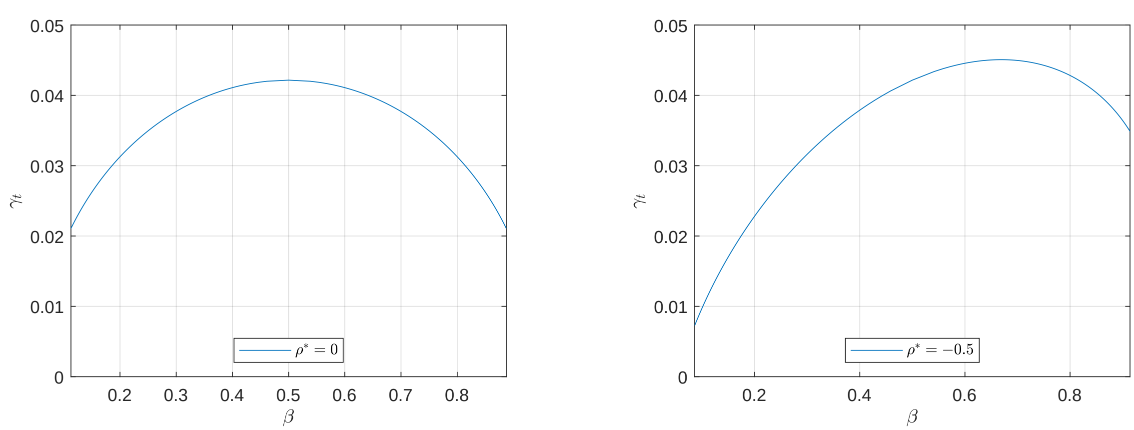

Figure 1: FX Loadings on the SDF and the Convenience Yield Shocks

that is counterfactual.

3.3 Exchange Rate and Flight-to-Safety

Jiang et al. (2020a) show that the dollar's real exchange rate is increasing in the convenience yield that foreign investors assign to the dollar risk-free bond. Specifically, when the U.S. Treasury's convenience yield increases by one standard deviation (0.23% as measured by Treasury basis), the dollar appreciates by 2.35%. In the post-2008 sample, the one-standard deviation shock leads to a dollar appreciation of 3.28%. We target this regression coefficient to pin down γ_t and then via the logic of the model, also pin down β.

We discretize the model by a time increment of $\Delta t = 0.001$ period and simulate 5000 periods. Table 1 presents regression results from the simulated sample. The top panel reports results for the case with flight to quality by foreign investors. The bottom panel reports results for the case

Table 1: Simulation Results

Columns (1) and (2) report the parameter values. (3) reports the slope coefficient in regression of Δs_t on $\Delta \tilde{\lambda}_t$. (4) reports annual FX volatility. (5) reports the slope coefficient in regression of Δs on $\Delta m - \Delta m^*$. (6) reports the annual expected log excess return on long position in the U.S. dollar. The regressions are run at quarterly frequency. Our simulation is based on a long sample of $T = 5000 \times 1000$ subperiods.

Panel A: $\rho^* = -0.5$					
(1)	(2)	(3)	(4)	(5)	(6)
k	β	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp.Log Return (%)
0.04	0.09	0.40	9.12	0.06	-1.73
0.05	**0.11**	**1.04**	**10.46**	**0.07**	**-2.44**
0.06	0.13	1.35	11.91	0.08	-2.77
0.07	0.18	2.04	16.21	0.10	-3.55
0.09	0.24	2.62	20.91	0.14	-4.27
0.11	0.31	3.14	26.38	0.19	-5.00
0.13	0.50	3.60	37.74	0.34	-6.11
Panel B: $\rho^* = 0$					
(1)	(2)	(3)	(4)	(5)	(6)
k	β	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp. Log Return (%)
0.05	0.11	2.79	17.40	0.11	-1.67
0.07	0.15	3.75	22.38	0.15	-1.73
0.08	0.20	4.45	26.96	0.19	-1.81
0.10	0.26	5.03	31.53	0.25	-1.91
0.11	0.33	5.55	36.65	0.32	-2.04
0.13	0.50	6.01	46.78	0.49	-2.32
Panel C: Comparisons					
(1)	(2)	(3)	(4)	(5)	(6)
k	β	FX-Conv Yield Coef	FX Vol (%)	SDF-FX Pass-Thru	Exp. Log Return (%)
Data	-	1.02—1.49	10.00	< 0	-1.89
1	0	-4.90	69.99	1.00	0.04

without flight to quality. We pick 7 different values of k, ranging from the minimum to the maximum possible values. Then, we run the regression of the exchange rate movement Δs_t on the change in the convenience yield $\Delta\tilde{\lambda}_t$, and report the regression coefficient in Column (3). In our preferred case in the second row of the top panel, with a low value of $\beta = 0.11$, this coefficient is 1.04. In comparison, the aforementioned empirical result in Jiang et al. (2020a) suggests that the slope coefficient should be between 1.02 and 1.49.

The lower panel of the Table reports the results for the case of $\rho^* = 0$. In the version of the model without flight-to-quality, the model generates too high a regression coefficient on the convenience yield innovation. As we discuss in the next sections, this parameterization also generates too high an exchange rate volatility, as shown in column (4) of the table.

3.4 Partial SDF-FX Pass-through and FX Volatility

Under complete markets, the real exchange rate follows

$$ds_t^{cm} = \alpha_t^{cm}dt + \beta_t^{cm}\sigma(dZ_t^* - dZ_t) + \gamma_t^{cm}\nu dX_t = (-\mu - \phi s_t^{cm})dt + \sigma(dZ_t^* - dZ_t), \tag{27}$$

which does not load on the convenience yield shock dX, i.e. $\gamma_t^{cm} = 0$, and moves one-to-one with the shocks to the pricing kernels, i.e. $\beta_t^{cm} = 1$.

In contrast, under incomplete markets with a convenience yield, the real exchange rate follows

$$ds_t = \left(-\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*)\right)dt + \gamma_t\nu dX_t + \beta_t\sigma(dZ_t^* - dZ_t), \tag{28}$$

which loads on the convenience yield shock dX while having only a partial pass-through governed by $0 < \beta < 1$ from the SDF shocks to the real exchange rate movement ds_t.

Lustig and Verdelhan (2019) provide a related result. They show that incomplete markets introduce a wedge in the exchange rate movement and this wedge is always negatively correlated with the SDF differential, which as a result partially offset the exchange rate movements induced by the SDF differential and lead to a less volatile exchange rate movement. In our model, we interpret this wedge as a convenience yield, and furthermore, we calibrate the relation between convenience yields and exchange rates based on the empirical analysis in Jiang et al. (2020a). This approach allows us to go further than Lustig and Verdelhan (2019) and nail down the extent of incomplete pass-through.

From Table 1 we see that in our preferred calibration, β equals 0.11. The SDF volatility is 50%, but the exchange rate volatility is only 10%. Higher values of k lead to higher values of β and higher exchange rate volatility. This partial SDF-FX pass-through result helps resolve the volatility puzzle of Brandt et al. (2006); the complete markets $dm - dm^*$ is more volatile than ds. In particular, the conditional variance of the exchange rate movement is

$$[ds_t, ds_t] = \gamma_t^2\nu^2 + 2\beta^2\sigma^2 + 2\gamma_t\nu\beta\sigma(\rho^* - \rho), \tag{29}$$

whereas under complete markets, it is

$$[ds_t^{cm}, ds_t^{cm}] \quad = \quad 2\sigma^2 \tag{30}$$

The reduced pass-through in our model is due to both a β that is much smaller than one, and $\rho^* - \rho = -0.5$, which reduces the volatility in equation (29).

3.5 Backus-Smith Puzzle

The exchange rate movement ds is exposed to both the SDF shock and the convenience yield shock. In relation to the Backus-Smith puzzle, we calculate the slope coefficient in a projection of the exchange rate changes the relative log SDF differential:

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} \quad = \quad \beta + \frac{\gamma_t \nu(\rho^* - \rho)}{2\sigma}. \tag{31}$$

From Table 1 we see that this coefficient is 0.1 in our model. For comparison, under complete markets, $\beta = 1$ and $\gamma_t = 0$, and therefore

$$\frac{[ds_t^{cm}, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} \quad = \quad 1. \tag{32}$$

Under incomplete markets, as β is below one, the first term in (31) shrinks the covariance $[ds_t, dm_t - dm_t^*]$ towards 0. This is the channel due to market incompleteness that was highlighted by Lustig and Verdelhan (2019).

The second term results from the correlation between the SDF shock and the convenience yield shock. If $\rho^* < \rho$, i.e. the foreign country's pricing kernel is more exposed to the convenience yield shock than the home country, this term is negative, which further reduces the slope coefficient in equation (31).

While our parameterization generates a coefficient near zero, it does not generate a negative coefficient. The exchange rate is still counter-cyclical: the model generates an appreciation of the foreign currency when the foreign investors experience higher marginal utility growth than the U.S. investors.

Can our model generate a negative slope coefficient in the projection of the rate of appreciation on the differences in the log pricing kernels? The negative second term in (31) suggests that it may. The economics here is that if the home exchange rate appreciates when the convenience yield increases, and convenience yield increases are correlated with worse economic conditions in foreign relative to home, then it may be possible to generate a procyclical exchange rate.

To see if it is possible to generate a negative regression coefficient, we plug γ_t into equation

(31):

$$\frac{[ds_t, dm_t - dm_t^*]}{[dm_t - dm_t^*, dm_t - dm_t^*]} = \beta + \frac{(\rho^* - \rho)}{2\sigma} \frac{(\rho^* - \rho)\sigma(1-2\beta) + \sqrt{(\rho^* - \rho)^2\sigma^2(1-2\beta)^2 + 4(k - \tilde{\lambda}_t)}}{2}.$$

When $\rho^* - \rho \geq 0$, this coefficient is guaranteed to be positive. So we want $\rho^* - \rho$ to be negative. If so, the coefficient is increasing in $\tilde{\lambda}_t$; we therefore pick the lowest possible $\tilde{\lambda}_t = 0$. Once β takes the smaller or the greater root, k and β are 1-to-1 increasing, but this coefficient is not monotone in either of them. So a numeric search is needed.

In Figure 2, we report the value of this coefficient while varying $\rho^* - \rho$ across $(-0.9, 0.9)$[8]. For each value of $\rho^* - \rho$, we vary k across its entire range $(\frac{\ell - (\rho^* - \rho)^2\sigma^2/4}{1 - (\rho^* - \rho)^2/2}, \sigma^2/2)$ and for each k, we allow β to take either root. As a result, we obtain a monotone sequence of β for each value of $\rho^* - \rho$. For example, when $\rho^* - \rho = -0.9$, the sequence of β is between 0 and 1. When $\rho^* - \rho = -0.5$, the sequence of β is between 0.10 and 0.90. This figure shows that under our specification, the Backus-Smith coefficient can be close to zero but is always positive. This is a quantitative result, not a theoretical one. Theory points to an economic force via the correlation between convenience yields and the SDF that can make the coefficient negative. Thus, it may be possible to consider alternative processes for λ_t or the SDF such that the coefficient is negative.

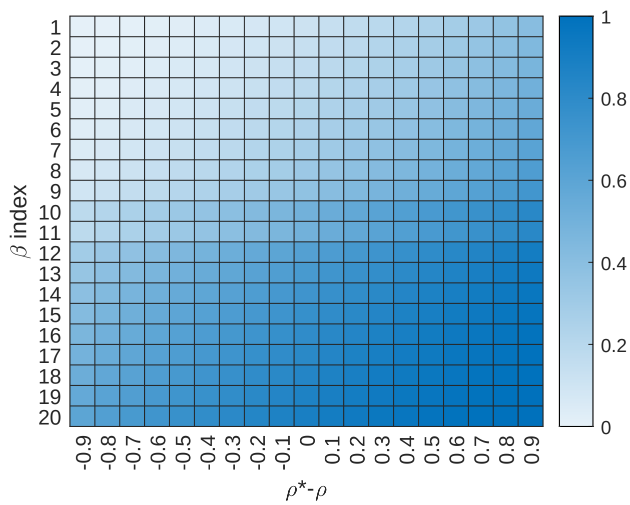

Figure 2: Backus-Smith Coefficient. We report the regression coefficient in equation (31) across different values of β and $\rho^* - \rho$.

[8] A positive semi-definite correlation matrix requires $1 - \sqrt{\rho^2 + \rho^{*2}} > 0$, i.e. $\rho^2 + \rho^{*2} < 1$. So this range for $\rho^* - \rho$ is allowed.

3.6 Currency Risk Premium

The expected log excess return on going long U.S. government bonds relative to foreign government bonds is given by:

$$\pi_t = \mathbb{E}_t[d\log(P_t S_t / P_t^*)] = \mathbb{E}_t[ds_t] + r_t - r_t^* = -\frac{1}{2}\tilde{\lambda}_t + \frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*) \tag{33}$$

The first term captures the dollar's convenience yield earned by foreign investors. The foreign investors derive non-pecuniary benefits from holding the dollar bond, and therefore require a lower expected return to hold dollar government bonds. The second term captures the dollar's currency risk premium. As the convenience yield shock is correlated with the SDF shocks, the magnitude of the risk premium depends on the correlations ρ and ρ^*.

This term is $-\frac{1}{2}\tilde{\lambda}_t$ instead of $-\tilde{\lambda}_t$ because the other half of the convenience yield is in the Jensen's term $\frac{1}{2}[ds_t, ds_t]$. In levels, the expected return on a long position in USD, in excess of the foreign risk-free rate, from the perspective of the foreign investor, loads on the convenience yield $\tilde{\lambda}_t$ with a coefficient of one:

$$\begin{aligned}
\Pi_t &= \mathbb{E}_t\left[d(P_t S_t / P_t^*)\right] = \mathbb{E}_t\left[dS_t\right] + r_t - r_t^* = \pi_t + \frac{1}{2}[ds_t, ds_t] \\
&= -\tilde{\lambda}_t + \beta\sigma^2 + \sigma\gamma_t\nu\rho^*.
\end{aligned}$$

The expected return in levels declines one-for-one with the dollar convenience yield. The expected return also declines as ρ^* declines, since higher foreign marginal utility growth coincides on average with higher convenience yields and smaller depreciation of the USD. On the other hand, the level of the expected foreign currency return, from the perspective of the U.S. investor, only reflects the covariance between the U.S. investor's SDF and the exchange rate movement:

$$\tilde{\Pi}_t = -\pi_t + \frac{1}{2}[ds_t, ds_t] = \beta\sigma^2 - \sigma\gamma_t\nu\rho.$$

This risk premium of the dollar is solely driven by the combination of market incompleteness and the cyclicality of the convenience yield. For comparison, if markets are complete, since the home and the foreign SDFs have the same volatilities, the log currency risk premium on USD is zero and the risk premium in levels equals the variance of the SDF (Bansal, 1997; Backus et al., 2001).

$$\pi_t^{cm} = 0 \tag{34}$$

$$\Pi_t^{cm} = \tilde{\Pi}_t^{cm} = \sigma^2. \tag{35}$$

In this case, the log currency risk premium is too small relative to the data whereas the level of currency risk premium is too large. The complete markets part of the model could be extended to generate non-zero complete markets currency risk premia by introducing asymmetries and time variation in the quantity and price of risk, as in the work of Verdelhan (2010); Colacito and Croce (2011); Farhi and Gabaix (2016).

In Table 1 we report that the expected log return in the model is -2.44%. For comparison, in Jiang et al. (2020a) we compute the returns for a foreign investor to owning the entire U.S. Treasury bond index relative to their home government bond index, over a sample from 1980 to 2019. We report that the dollar Treasury return is 1.89% lower than the foreign bond return, which is close to the model-implied estimate of 2.44%. According to equation (33), given an average convenience yield of $\mathbb{E}[\tilde{\lambda}_t] = 1.9\%$, our model indicates that about $\frac{1}{2}\mathbb{E}[\tilde{\lambda}_t] = 0.95\%$ in the expected log return is attributable to the convenience yield, and the remaining $2.44\% - 0.95\% = 1.49\%$ is attributable to the dollar's log risk premium.

Figures 3 and 4 further plot the currency risk premium as a function of β and for different values of $\tilde{\lambda}_t$. We plot both the level and the log expected return. As expected from equation (33), fixing β, the dollar's expected return declines with the current convenience yield $\tilde{\lambda}_t$.

Conditional Currency Risk Premium The SDFs have constant volatility in this model. The standard approach to introducing time variation in the conditional currency risk premium is to introduce time-varying volatility in the SDFs, which in turn can result from either changes in the quantities of risk or changes in the prices of risk. We have left out these features in order to derive a closed-form solution for the exchange rate dynamics. A more general model will be able to generate realistic variation in both the convenience yields and in the conditional currency risk premia.

That said, since the convenience yield and the SDF are correlated and the convenience yield has a time-varying volatility, our model does generate variation in the conditional currency risk premium. Under our calibration, γ_t is decreasing in $\tilde{\lambda}_t$, so the dollar exchange rate's loading on the convenience yield shock is lower when the convenience yield is higher. Since $\rho + \rho^* < 0$, the risk premium component in the dollar's expected log excess return, $\frac{1}{2}\sigma\gamma_t\nu(\rho + \rho^*)$, is increasing in $\tilde{\lambda}_t$. However, this effect is dwarfed by the convenience yield component, so the dollar's expected log excess return is still decreasing in $\tilde{\lambda}_t$ in Figure 3.

4 Quantitative Easing

Quantitative easing (QE) policies—that is, large scale purchases of long-term bonds matched by increases in bank reserves—have been shown to affect exchange rates (Neely, 2015). In this section, we show how our model can shed light on this connection.

4.1 Segmented Bond Markets and the Bond Risk Premium Channel

Gourinchas et al. (2019); Greenwood et al. (2019) bring an equilibrium model of the term structure with market segmentation along the lines of Vayanos and Vila (2021) to bear on FX markets. These authors explore the impact of downward sloping demand curves for Treasurys. An increase in net

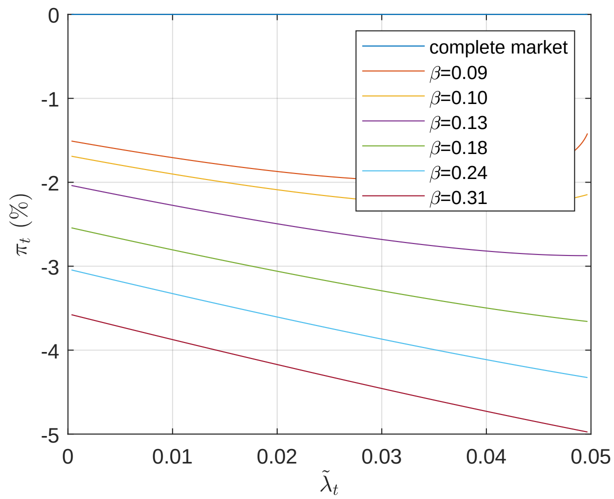

Figure 3: The Dollar's Expected Log Excess Return π_t

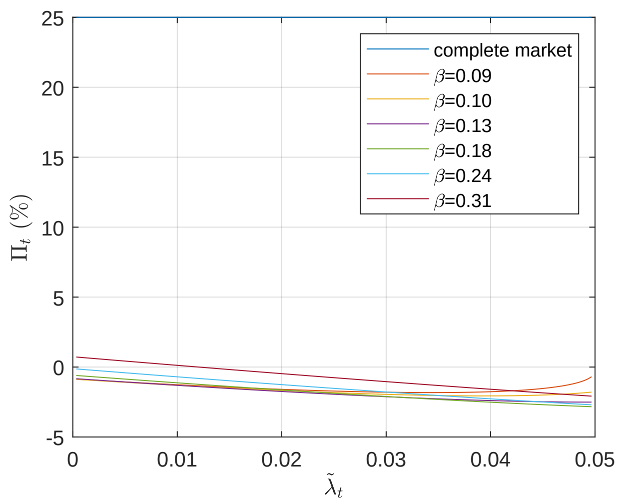

Figure 4: The Dollar's Expected Excess Return Level Π_t

U.S. supply of long bonds causes U.S. arbitrageurs demand larger bond risk premium on long USD bonds. As a result, policy makers can control long rates. By manipulating bond risk premia, policy makers will also change the equilibrium dollar exchange rate.

Consider what happens in the case of large-scale asset purchases in the U.S inside the Gourinchas et al. (2019); Greenwood et al. (2019) model. The central banks shrink the net supply of long bonds, U.S. arbitrageurs earn a smaller bond risk premium as a result. The decrease in local currency bond risk premia in the U.S. lowers USD long yields. In their model, the exchange rate is stationary. The USD depreciates right away to offset the effect of the lower USD yield: When the exchange rate is stationary, the exchange rate reflects differences in long yields:

$$s_t - \bar{s} = \lim_{T\to\infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}). \tag{36}$$

Inside this model, Fed can certainly lower long yields and cause the USD to depreciate. In this class of models, QE involves a redistribution of rents from Treasury arbitrageurs to FX arbitrageurs. This FX channel seems less potent because ECB, BoJ, BoE and others can and do respond. In addition, this bond risk premium channel is symmetric, in that the ECB, BoJ, BoE and others have the same control over local currency bond risk premia and the exchange rate.

4.2 Convenience Yield Channel

Our work identifies a novel convenience yield channel through which large scale asset purchases affect exchange rates. The dollar appreciates when future U.S. Treasury convenience yields λ increase, holding constant the long yields:

$$s_t - \bar{s} = \lim_{T\to\infty} (T-t)(r_t^{T-t} - r_t^{*,T-t}) + \lim_{T\to\infty} \mathbb{E}_t \int_t^T \tilde{\lambda}_u du. \tag{37}$$

The convenience yield channel creates a distinct role for flows/quantities. When the Federal Reserve buys MBS and issues reserves, his will tend to decrease convenience yields on USD bonds, and cause the USD to depreciate. When the Federal Reserve buys long-dated Treasurys and issues bank reserves, the effect on convenience yields depends on the substitutability of reserves and Treasurys. The convenience yield channel assigns a special role to the U.S., to the extent that the U.S. is the world's safe asset supplier.

Quantitative easing changes the supply of safe assets and the convenience yield on these assets. This channel is outlined in Krishnamurthy and Vissing-Jorgensen (2011), and as explained, can either increase or decrease the supply of safe assets. A swap of mortgage-backed securities for reserves likely increases the supply of safe assets, since reserves are a more convenient asset than mortgage-backed securities. A swap of Treasuries for reserves may increase or decrease the supply of safe assets depending on whether banks pass on the reserve expansion by expanding deposits, and the relative convenience of these deposits and Treasuries. Thus, convenience yields can either rise or fall with QE.

Our theory of exchange rate connects the convenience yield with exchange rates. That is, QE that increases the convenience yield on dollar bonds should be expected to appreciate the dollar, while QE that decreases the convenience yield should be expected to depreciate the dollar.

Figure 5 presents evidence linking changes in convenience yields around QE-event dates and changes in the dollar exchange rate. The dollar exchange rate is measured as the equal-weighted G-10 cross. The basis is the 1-year U.S. Treasury against an equal-weighted currency-hedged 1-year G-10 government bond. The data is from Krishnamurthy and Lustig (2019). As we show

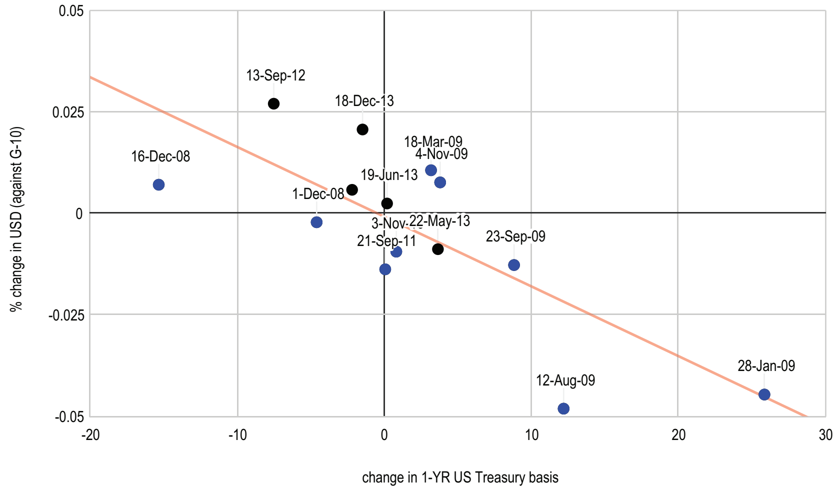

Figure 5: G-10 Dollar appreciation against change in basis around QE event dates. Sample of 14 QE event dates. 2-day window after QE-event dates. We include the event day and define the change in the basis (Δ Basis) and the change in the dollar from the close of trading on the day prior to the event day to the close of trading 2-days later.

theoretically in Jiang et al. (2020a), the basis is proportional to the convenience yield on U.S. Treasury bonds relative to foreign bonds.

We note two key patterns in this figure: the dollar appreciates in some of these events, while it depreciates in others; and both the sign and magnitude of the change in the dollar lines up with changes in the basis. Table 2 presents this evidence in a regression. We regress the 2-day (Panel A) and 3-day (Panel B) change in the exchange rate against the change in the basis, controlling for the change in the relative interest rates in home and foreign, which can control for shifts in the stance of monetary policy. At both horizons and measuring the basis using different maturity bonds, there is a strong relation between QE-induced changes in the basis and the dollar. Focusing on the 1-year basis in Panel A, we see that a 10 basis point change in the Treasury basis leads to a 1.66% appreciation in the dollar. From the results in Jiang et al. (2020a), a 10 basis point change in the basis is equal to 1% change in the convenience yield.

Next, we turn to our model to see how well it can capture these patterns. We do not explicitly model the relation between the convenience yield λ and the quantity of safe assets. Instead, we focus directly on inducing a shock to λ and tracing out the impact of this shock on the exchange rate. We discretize the model by a time increment of $\Delta t = 0.0025$ and start the model at $t = 0$. For initial values, we set $s_0 = s_0^{cm} = \bar{s}$ and $\lambda_0 = 0$, and set H_0 to satisfy

$$s_0 \quad = \quad f(\lambda_0) + H_0 + z s_0^{cm}. \tag{38}$$

We simulate dX_t, dZ_t and dZ_t^* under the normal distribution with mean zero and standard deviation $\sqrt{\Delta t}$. For the first quarter, i.e., periods $(0, 0.25]$, we introduce a positive impulse that

Table 2: QE, Basis, and Exchange Rate

Regression of changes in dollar (G-10) on QE-induced changes in U.S. Treasury basis and changes in yields. We include 14 QE event dates. We include the event day and define the change in the basis (Δ Basis) and the change in the dollar from the close of trading on the day prior to the event day to the close of trading x days later. Δ y-diff is the change in the 1-year interest rate differential between the U.S. and the G-10 average.

		3M	1Y	2Y	3Y	5Y	7Y	10Y
		Panel A: 2-day window						
Δ Basis	coeff	-0.247	-0.166	-0.240	-0.225	-0.170	-0.189	-0.152
	s.e.	0.057	0.028	0.035	0.037	0.034	0.047	0.050
Δ y-diff	coeff	20.012	31.381	17.501	16.338	12.568	12.857	11.231
	s.e.	9.066	8.031	3.092	2.951	2.610	2.624	3.195
	R^2	0.637	0.828	0.837	0.800	0.751	0.697	0.563
		Panel B: 3-day window						
Δ Basis	coeff	-0.219	-0.188	-0.175	-0.183	-0.135	-0.106	-0.083
	s.e.	0.051	0.027	0.036	0.037	0.036	0.037	0.043
Δ y-diff	coeff	15.319	22.568	15.494	13.861	12.186	12.068	11.944
	s.e.	7.054	6.307	3.227	2.541	2.064	2.253	2.685
	R^2	0.624	0.811	0.745	0.779	0.778	0.724	0.643

raises all realizations of the shocks dX_t by one standard deviation. This impulse simulates a positive convenience yield shock in the first quarter. Then, we average across 100,000 simulated paths of the shocks (dX_t, dZ_t, dZ_t^*). In this way, we estimate the average response following a positive convenience yield shock at date 0. We also simulate a benchmark case in which we draw from the normal distribution with mean 0 for the entire period $t \in (0, T]$. As expected, the average responses of exchange rate and convenience yield are close to zero in this benchmark case. We report the difference between the average responses in the case of a convenience yield shock and the benchmark case.

Figure 6 reports the result. In the top-left panel, we shock the convenience yield λ_t and then let the internal dynamics of mean reversion gradually bring the convenience yield to zero over the next 10 quarters. We can think of this shock as an announcement by the central bank to purchase assets at date 0, and then slowly unwind these purchases over the next 10 quarters.

The top-left panel of the figure graphs the instantaneous convenience yield over this path. The top-right panel plots the average convenience yield between time 0 and time t $(= \frac{1}{t}\int_{k=0}^{t} \tilde{\lambda}_k dt)$. This panel gives an expectations-hypothesis-type heuristic of how different maturity bases will react to this shock. We see that the largest response is in the short maturity bases with the effects dying out for longer maturity bases. At the one year point, the convenience yield rises by about 0.35% (given a 1:10 ratio between Treasury basis and convenience yield, this implies a widening in the Treasury basis of 3.5 basis points). The bottom-left panel plots the complete markets exchange

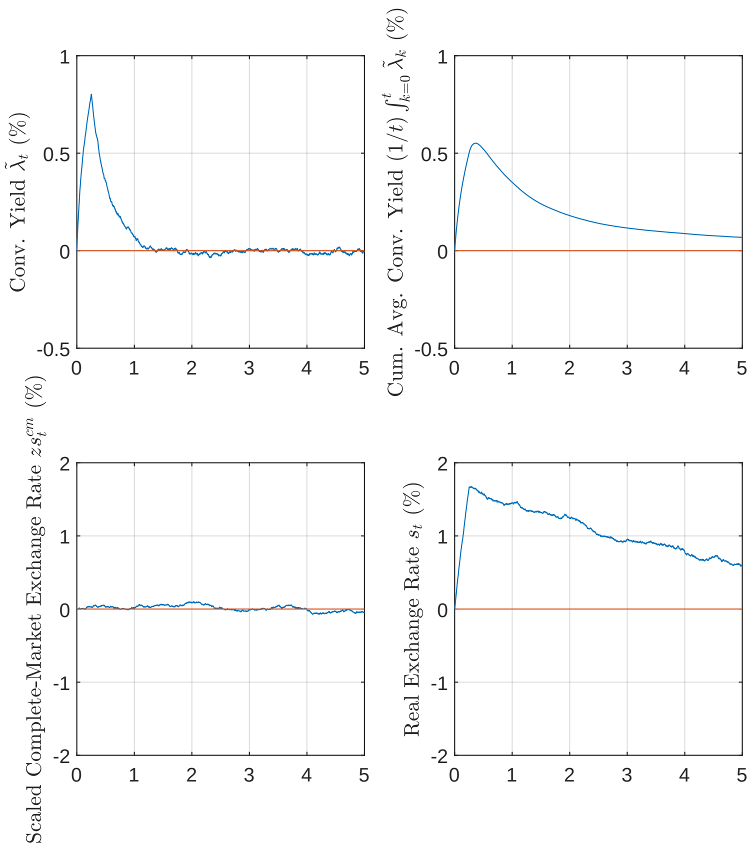

Figure 6: Impulse Response to a Convenience Yield Shock.

We report the average difference between simulations in which the convenience yield λ_t jumps up by 1 standard deviation ($\nu = 5$) in the period 0 and simulations in which all shocks have zero means. At the end of the 2nd quarter (flagged by the vertical line), the convenience yield $\tilde{\lambda}_t$ is around 2% and the real exchange rate is about 2% above the long-run average.

rate averaged across simulation paths. The last panel plots the exchange rate from the model. On impact, the exchange jumps by 1.7%, before gradually reverting to its long-run level. Thus quantitatively, our model generates a regression coefficient on the 1-year basis of -0.5, which is of the same magnitude but greater than the empirical estimates in Table 2.

The effect of this QE experiment unwinds gradually over the next several years. We note that the behavior in term H_t representing the cumulative convenience yields is also interesting. Since

$$H_t \quad = \quad \exp(-\phi t) H_0 + \int_0^t \exp(-\phi(t-u)) h(\lambda_u) du, \tag{39}$$

it aggregates influences of past convenience yields with an exponential decay. As a result, the half life of the response in real exchange rate is longer than the half life of the response in the spot convenience yield.

5 Conclusion

Our paper delivers a fully specified no-arbitrage model of exchange rates, interest rates and convenience yields. We show how convenience yields interact with incomplete markets and make progress on the exchange rate puzzles. Moreover, in our model, the U.S. central bank can directly affect the dollar exchange rate, not by changing bond currency risk premia, but by changing the convenience yields on dollar-denominated government bonds. We refer to this as the convenience yield channel. Our paper is the first to embed this convenience yield channel in a no-arbitrage model of exchange rates. This channel is complementary to the bond risk premium channel in models with bond market segmentation. The convenience yield channel imputes a unique role to the U.S. central bank in affecting the dollar exchange rate without changing U.S. interest rates, because it can change the convenience yields earned by foreign investors.

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A Appendix: Proof

A.1 Proof of Proposition 1 and 2

Recall that the real pricing kernels are

$$\begin{aligned} dM_t &= M_t(-\mu + \frac{1}{2}\sigma^2)dt - M_t \sigma dZ_t && \text{(A.1)} \\ dM_t^* &= M_t^*(\phi s_t + \frac{1}{2}\sigma^2)dt - M_t^* \sigma dZ_t^* && \text{(A.2)} \end{aligned}$$

Substitute the real pricing kernels into the FOCs. The first FOC becomes

$$\begin{aligned} 0 &= \mathcal{A}[(-\mu + \frac{1}{2}\sigma^2)dt - \sigma dZ_t + r_t dt] && \text{(A.3)} \\ r_t &= \mu - \frac{1}{2}\sigma^2 && \text{(A.4)} \end{aligned}$$

The second FOC becomes

$$\begin{aligned} 0 &= \mathcal{A}[M_t^* P_t^*] && \text{(A.5)} \\ r_t^* &= -\phi s_t - \frac{1}{2}\sigma^2 && \text{(A.6)} \end{aligned}$$

Notice

$$\begin{aligned} dS_t &= d\exp(s_t) = S_t ds_t + \frac{1}{2} S_t [ds_t, ds_t] dt && \text{(A.7)} \\ dS_t^{-1} &= d\exp(-s_t) = -S_t^{-1} ds_t + \frac{1}{2} S_t^{-1} [ds_t, ds_t] dt && \text{(A.8)} \end{aligned}$$

The third FOC becomes

$$\begin{aligned} 0 &= \mathcal{A}[\int M_t^* S_t P_t \tilde{\lambda}_t dt + M_t^* S_t P_t] && \text{(A.9)} \\ &= M_t^* S_t P_t \tilde{\lambda}_t + \mathcal{A}[\int S_t P_t dM_t^* + S_t M_t^* dP_t + M_t^* P_t dS_t + P_t [dM_t^*, dS_t] dt] && \text{(A.10)} \\ &= \tilde{\lambda}_t + \phi s_t + \frac{1}{2}\sigma^2 + r_t + \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] && \text{(A.11)} \end{aligned}$$

The fourth FOC becomes

$$\begin{aligned} 0 &= \mathcal{A}[M_t S_t^{-1} P_t^*] && \text{(A.12)} \\ &= \mathcal{A}[\int (S_t^{-1} dM_t + M_t dS_t^{-1} + [dM_t, dS_t^{-1}] dt) P_t^* + M_t S_t^{-1} P_t^* r_t^* dt] && \text{(A.13)} \\ &= -\mu + \frac{1}{2}\sigma^2 - \mathcal{A}[s_t] + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t, -ds_t] + r_t^* && \text{(A.14)} \end{aligned}$$

The sum of the third and the fourth FOC is

$$-\tilde{\lambda}_t = [ds_t, ds_t] - \sigma[dZ_t^* - dZ_t, ds_t] \tag{A.15}$$

Plug in the conjecture

$$ds_t = \alpha_t dt + \gamma_t \nu dX_t + \beta_t \sigma(dZ_t^* - dZ_t), \tag{A.16}$$

then

$$-\tilde{\lambda}_t = \gamma_t^2 \nu^2 + 2\beta_t^2 \sigma^2 + 2\gamma_t \nu \beta_t (\rho^* - \rho)\sigma - 2\beta_t \sigma^2 - (\rho^* - \rho)\sigma\gamma_t \nu \tag{A.17}$$

Suppose for a certain constant k,

$$-k = 2\beta_t^2 \sigma^2 - 2\beta_t \sigma^2 \tag{A.18}$$

$$k - \tilde{\lambda}_t = \gamma_t^2 \nu^2 + 2\gamma_t \nu \beta_t (\rho^* - \rho)\sigma - (\rho^* - \rho)\sigma\gamma_t \nu \tag{A.19}$$

The solution is

$$\beta_t = \frac{1}{2} \pm \frac{\sqrt{\sigma^2 - 2k}}{2\sigma}, \tag{A.20}$$

$$\gamma_t = \frac{(\rho^* - \rho)\sigma(1 - 2\beta_t) \pm \sqrt{(\rho^* - \rho)^2 \sigma^2 (1 - 2\beta_t)^2 + 4(k - \tilde{\lambda}_t)}}{2\nu}. \tag{A.21}$$

which has real roots for all possible values of λ_t if and only if

$$k < \sigma^2/2 \tag{A.22}$$

and

$$k > \frac{\ell - (\rho^* - \rho)^2 \sigma^2/4}{1 - (\rho^* - \rho)^2/2} \tag{A.23}$$

When the upper bound of k is obtained, $\beta_t = 1/2$. When the lower bound of k is obtained,

$$\beta_t = \frac{1}{2} \pm \frac{\sqrt{\frac{\sigma^2 - 2\ell}{1 - (\rho^* - \rho)^2/2}}}{2\sigma} \tag{A.24}$$

which bounds the range of possible value of β_t.

Lastly, we also solve α_t from

$$-\alpha_t = \tilde{\lambda}_t + \phi s_t + \mu + \frac{1}{2}[ds_t, ds_t] + [-\sigma dZ_t^*, ds_t] \tag{A.25}$$

$$= \frac{1}{2}\tilde{\lambda}_t + \phi s_t + \mu - \frac{1}{2}\sigma[dZ_t^*, ds_t] - \frac{1}{2}\sigma[dZ_t, ds_t] \tag{A.26}$$

$$
\begin{aligned}
\alpha_t &= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma(\gamma_t \nu \rho^* + \beta_t \sigma) + \frac{1}{2}\sigma(\gamma_t \nu \rho - \beta_t \sigma) && \text{(A.27)} \\
&= -\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma \gamma_t \nu (\rho + \rho^*) && \text{(A.28)}
\end{aligned}
$$

A.2 Proof of Proposition 2

Recall the definition of the real exchange rate under complete markets, we have

$$
d(s_t - \beta s_t^{cm}) = \left(-\frac{1}{2}\tilde{\lambda}_t - \phi(s_t - \beta s_t^{cm}) - (1-\beta)\mu + \frac{1}{2}\sigma \gamma_t \nu (\rho + \rho^*) \right) dt + \gamma_t \nu dX_t \quad \text{(A.29)}
$$

We conjecture

$$
\begin{aligned}
s_t - \beta s_t^{cm} &= f(\lambda_t) + H_t && \text{(A.30)} \\
H_t &= \exp(-\phi t) H_0 + \int_0^t \exp(-\phi(t-u)) h(\lambda_u) du && \text{(A.31)}
\end{aligned}
$$

which implies

$$
\begin{aligned}
dH_t &= \left(-\phi \exp(\phi(-t)) H_0 + h(\lambda_t) - \phi \int_0^t \exp(\phi(u-t)) h(\lambda_u) du \right) dt && \text{(A.32)} \\
&= \left(-\phi \exp(\phi(-t)) H_0 + h(\lambda_t) - \phi(H_t - \exp(\phi(-t)) H_0) \right) dt && \text{(A.33)} \\
&= \left(h(\lambda_t) - \phi H_t \right) dt && \text{(A.34)}
\end{aligned}
$$

We note

$$
\begin{aligned}
d(s_t - \beta s_t^{cm}) &= f' d\lambda_t + \frac{1}{2} f'' [d\lambda_t]^2 dt + dH_t && \text{(A.35)} \\
&= f'(-\theta \lambda_t dt + \nu dX_t) + \frac{1}{2} f'' \nu^2 dt + (h(\lambda_t) - \phi H_t)\, dt && \text{(A.36)}
\end{aligned}
$$

and this has to match equation (A.29).

Matching dX_t term,

$$
f' = \gamma_t = \frac{b + \sqrt{b^2 + 4(k - \tilde{\lambda}_t)}}{2\nu} \quad \text{(A.37)}
$$

where $b = (\rho^* - \rho)\sigma(1 - 2\beta_t)$. Then,

$$
\begin{aligned}
f(\lambda) &= \frac{1}{2\nu}\{-\sqrt{b^2+4k}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) && \text{(A.38)} \\
&+ \sqrt{b^2+4k-4\ell}\log\left(2e^{\lambda/2}\left(\cosh\left(\frac{\lambda}{2}\right)\left(\sqrt{b^2+4k-4\ell}\sqrt{b^2+4k-2\ell\tanh\left(\frac{\lambda}{2}\right)-2\ell}+b^2+4k-3\ell\right)-\ell\sinh\left(\frac{\lambda}{2}\right)\right)\right) \\
&+ \lambda\left(\sqrt{b^2+4k}+b\right)\}
\end{aligned}
$$

and

$$f''(\lambda) = \frac{\frac{\ell e^{2\lambda}}{\left(e^{\lambda}+1\right)^2} - \frac{\ell e^{\lambda}}{e^{\lambda}+1}}{\nu\sqrt{b^2 + 4\left(k - \frac{\ell e^{\lambda}}{e^{\lambda}+1}\right)}} \tag{A.39}$$

Matching dt term,

$$h(\lambda_t) \quad = \quad -\frac{1}{2}\tilde{\lambda}_t - \phi f - (1-\beta)\mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*) + f'\theta\lambda_t - \frac{1}{2}f''\nu^2 \tag{A.40}$$

Since γ_t is also a function of λ_t, we confirm the conjecture that $h(\lambda_t)$ is a function only of λ_t. So

$$s_t \quad = \quad f(\lambda_t) + H_t + \beta s_t^{cm} \tag{A.41}$$

A.3 Long-Term Expectation of Log Exchange Rate

Since,

$$ds_t \quad = \quad \left(-\frac{1}{2}\tilde{\lambda}_t - \phi s_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right)dt + \gamma_t\nu dX_t + \beta\sigma(dZ_t^* - dZ_t), \tag{A.42}$$

then,

$$d(e^{\phi t}s_t) = e^{\phi t}\left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right)dt + e^{\phi t}\gamma_t\nu dX_t + e^{\phi t}\beta\sigma(dZ_t^* - dZ_t) \tag{A.43}$$

The solution of the above Stochastic Differential Equation is:

$$s_T = e^{-\phi T}s_0 + \int_0^T e^{\phi(t-T)}\left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)\right)dt + \int_0^T e^{\phi(t-T)}\gamma_t\nu dX_t + \int_0^T e^{\phi(t-T)}\beta\sigma(dZ_t^* - dZ_t) \tag{A.44}$$

Recall that

$$\begin{aligned}
\gamma_t \quad &= \quad \frac{(\rho^*-\rho)\sigma(1-2\beta_t) \pm \sqrt{(\rho^*-\rho)^2\sigma^2(1-2\beta_t)^2 + 4(k-\tilde{\lambda}_t)}}{2\nu}, \\
|(1-2\beta_t)| \quad &\leq \quad \frac{\sqrt{\sigma^2-2k}}{\sigma},
\end{aligned}$$

then γ_t is bounded,

$$|\gamma_t| \quad \leq \quad \frac{(\rho^*-\rho)\sqrt{\sigma^2-2k} + \sqrt{(\rho^*-\rho)^2(\sigma^2-2k)^2 + 4k}}{2\nu}$$

Hence, for s_T, the integrands in the stochastic integrals are all $\mathcal{H}^2$, and the stochastic integrals are martingales with expectation 0. Then,

$$\begin{aligned}
\lim_{T\to\infty} \mathbb{E}_0[s_T] &= \lim_{T\to\infty} e^{-\phi T} s_0 + \lim_{T\to\infty} \mathbb{E}_0[\int_0^T e^{\phi(t-T)} \left(-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t \nu(\rho+\rho^*)\right) dt] \\
&= \frac{1}{\phi} \lim_{T\to\infty} \mathbb{E}_0[-\frac{1}{2}\tilde{\lambda}_t - \mu + \frac{1}{2}\sigma\gamma_t\nu(\rho+\rho^*)] \\
&= \frac{1}{\phi}\left(-\frac{1}{2}\lim_{T\to\infty}\mathbb{E}_0[\tilde{\lambda}_t] - \mu + \frac{1}{2}\sigma \lim_{T\to\infty}\mathbb{E}_0[\gamma_t]\nu(\rho+\rho^*)\right).
\end{aligned}$$