Managing Public Portfolios*

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Abstract

We develop a unified framework for optimal management of public portfolios for a general class of macro-finance models imposing very few restrictions on the market structure for financial assets and households' risk and liquidity preferences. We derive an expression for the optimal portfolio in terms of "sufficient statistics" that isolate main motives that the portfolio strives to achieve: hedging interest rate, primary deficit, and liquidity risks; exploiting liquidity benefits that government debts of different maturities provide; and internalizing equilibrium effects of public policies on financial asset prices. We use U.S. data to estimate these statistics and find that hedging interest rate risk plays a dominant role in shaping the optimal maturity structure of government debt.

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1 Introduction

This paper isolates and quantifies motives that shape optimal government portfolios of financial assets. We do this for a class of representative household general equilibrium models. This class of models is quite general and includes popular specifications of households' risk and liquidity preferences, sets of tradable securities, as well as restrictions that can limit access to some markets. We show that despite its generality, the main forces that shape an optimal portfolio can be summarized in a small number of objects – "sufficient statistics"– that can be measured in the data directly. We then apply our approach to study the optimal maturity structure of U.S. government debt.

Our framework consists of domestic households, foreign investors, and a benevolent government. Households are identical and derive utility from consumption and leisure; in addition, they may also derive indirect utility from holdings financial assets. This indirect utility summarizes shadow benefits and costs from holding assets that provide liquidity services, or affect borrowing constraints or trading frictions. A benevolent government planner uses distortionary taxes to finance exogenous public expenditures. Households, government and foreign investors trade an arbitrary set of financial assets. Our specification of household preferences and demand of foreign investors is flexible enough to incorporate a variety of models of asset price determination studied in the literature, such as recursive utility, discount factor shocks, ambiguity aversion, preferred habitat models, closed and open economies, etc.

We isolate key forces that determine optimal portfolio by considering implications of perturbing government portfolio at any history in the competitive equilibrium and then simplifying these expressions by applying a class of small noise expansions. This allows us to express the optimal portfolio as a function of sufficient statistics. These statistics can be measured directly in the data, and they do not require us taking a stance on the more primitive forces that drive asset pricing behavior. This is important since there is considerable disagreement in the asset pricing literature about the sources of asset price fluctuations in the data.

The key notion that emerges in our analysis is that of a *target portfolio*. The target portfolio is a portfolio that the government should choose in the absence of any costs of rebalancing of its holdings of financial assets. This portfolio captures a trade-off between hedging risks faced by the government in the future and providing liquidity services in the present. There are three future risks – interest rates, primary deficits, and liquidity – and they are summarized by covariances of returns on assets in government portfolios with various financial and macroeconomic variables. The value of the liquidity services in the present is summarized by a certain measure of liquidity premium on various assets.

If rebalancing of government portfolio has no effect on asset prices, as would be in the case of a small open economy, then it is optimal for the government to set its portfolio to the target portfolio. More generally, the formula for the optimal portfolio includes costs of rebalancing. We show that those costs are proportional to the distance between the target portfolio and the portfolio the government enters a period with, and the price elasticity of various assets. Our theory is general and can be applied to any set of securities, and all terms in the formulas we derive have direct empirical counterparts. The target portfolio is a useful measure even in suboptimal competitive equilibria as it provides a direction in which government portfolio can be rebalanced to improve welfare.

We then apply our framework to one particular market structure, in which the only securities that government holds are public debts of different maturities. We use data on the returns of U.S. government and corporate bonds, taxes, and primary deficits to estimate all components of the target portfolio. We find that only one component – interest rate risk – accounts for most of the shape of the target portfolio. Because of that, the target portfolio takes a very simple form, in which portfolio shares of debts decline roughly geometrically in their maturity, with the rate of decline given by households' discount factor. Moreover, maintaining this portfolio requires minimal rebalancing, which implies that the optimal portfolio is roughly equal to the target portfolio essentially for any price elasticity of assets.

This finding is driven by several observations. U.S. government debts are a poor hedge against the primary deficit and liquidity risks. Their returns in the data are much more volatile and not very correlated, with either future primary surpluses or various measures of future liquidity premium on government bonds. Moreover, primary surpluses are pro-cyclical while liquidity premium is counter-cyclical, which implies that these two risks have offsetting effects in the target portfolio. The liquidity premium also appears to be similar across different maturities of government bonds, which leaves interest rate risk as the only quantitatively meaningful term in the target portfolio.

Unlike primary surplus and liquidity risks, there exists a simple portfolio that can hedge interest rate risk quite well. The interest rate risk affects the government only when it needs to roll over its existing debt. By choosing a maturity structure that matches the duration of debts to the expected primary surpluses, the government can eliminate all expected debts rollovers, eliminating a large part of the interest rate risk. A simple back-of-the-envelope calculation shows that 99% of the composition of the target portfolio is comprised of hedging interest rate risk. A portfolio structured to minimize interest rate risk also minimizes the amount of rebalancing needed to maintain it. This, in turn, implies that costs of portfolio adjustments have little quantitative impact on the optimal portfolio. We illustrate this result both theoretically and quantitatively using estimates of price elasticities obtained by Greenwood and Vayanos (2014) and apply them to our formula.

Our paper is related to an extensive Ramsey literature on the optimal composition of government debt, such as Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004), Farhi (2010); Faraglia, Marcet, Oikonomou, and Scott (2018); Lustig, Sleet, and Yeltekin (2008), Bhandari, Evans, Golosov, and Sargent (2017b). All those authors used versions of standard neoclassical growth models to characterize optimal public portfolios. Nevertheless, it is well-known that such models fail spectacularly to match empirical relationships between asset prices, asset supply and macroeconomic variables, which are the main objects that determine how well different securities can hedge risks. We overcome this problem by considering a much more general specification of preferences and asset demands that includes *multiple* mechanisms that can account for the observed asset pricing behavior.

Realistic asset pricing dynamics dramatically change many insights about optimal public portfolios that emerge from that earlier literature. For example, in their quantitative model calibrated to the U.S. economy, Buera and Nicolini (2004) find that the government should issue long-term debt valued at tens or even hundreds times GDP while simultaneously taking an offsetting short (i.e., negative) positions in short-term debt of similar magnitudes. They also find that government holdings of debts of similar maturities may differ by hundreds percent of GDP; that the composition of the optimal portfolio is very sensitive to the menu of traded maturities; and that relatively small aggregate shocks caused very significant portfolio rebalancing. In contrast, our portfolio is very stable over time and has simple declining maturity weights qualitatively similar to that observed in the data. The differences in findings are driven by counterfactual asset pricing implications of the standard neoclassical growth model.

Our paper builds on a large literature in finance that focuses on understanding asset price determination, such as the work of Ai and Bansal (2018), Bansal and Yaron (2004), Albuquerque, Eichenbaum, Luo, and Rebelo (2016), Krishnamurthy and Vissing-Jorgensen (2012), Greenwood and Vayanos (2014). Those authors proposed a number of different mechanisms to explain the observed behavior of asset prices, and there is no consensus in the literature on which of those mechanisms is most relevant empirically. By setting up a framework that incorporates all of these mechanisms and obtaining expressions for the optimal portfolios using sufficient statistics, we sidestep the need to take a stance on their relative importance.

Work by Bohn (1990) is probably the closest in spirit to our approach. Similar to our work, he studied a representative agent model with distortionary taxes and obtained the optimal public portfolio expression in terms of different covariances that he then estimated in the U.S. data. However, in Bohn's model, consumers are risk-neutral, tax distortions are ad-hoc, financial securities provide no liquidity services, the set of those assets is restrictive, and all asset prices are exogenous. Our work is also related to a recent paper by Bigio, Nuno, and Passadore (2019) that studies the optimal composition of government portfolios of bonds of different maturities. They abstract from the interest rate risk, primary deficit, and liquidity channels that we emphasize and focus on understanding how price impacts to debt issuance affect portfolio composition. Furthermore, they impose an exogenous cap on the maturities that the government can issue, implying that the government needs to rebalance its portfolio even in the absence of all risks.

The optimal portfolio formulas that we obtain are related to the formulas that appear in the classical portfolio theory as Samuelson (1970), Merton (1969), Merton (1971), Campbell and Viceira (1999), Campbell and Viceira (2001), Viceira (2001). While both individual investors in the classical portfolio theory and the government in our model choose portfolios to hedge their risks, there are substantial differences in the forces that determine portfolio composition. Neither liquidity services nor price impacts feature in the classical portfolio theory where all investors are measure zero. The trade-off between risks and returns of various assets, captured by Sharpe ratios and the risk-aversion, that plays the central role in the classical portfolio theory, is entirely absent in the government problem. This is because the government is benevolent and shares the same preferences as agents. This implies that it cannot improve welfare by simply replicating any trade that households can do themselves. Instead, the government portfolio depends on a measure that captures additional costs (such as trading frictions) or benefits (such as liquidity services) that assets provide to agents beyond the pure transfer of resources across periods. We refer to this measure as a liquidity premium and provide a way to measure it in the data.

In a series of recent papers, Jiang, Lustig, Nieuwerburgh, and Xiaolan (2019, 2020) document a number of puzzling facts concerning the market value of total debt and primary surpluses in the U.S. These observations are puzzling when debt valuation is viewed from a lens of an arbitrage-free and frictionless asset pricing framework. Our setting departs from such a framework by incorporating market segmentation as well as a broad notion of liquidity services that U.S debts provide, and thus accounting for some of the puzzling observations. However, our focus in this paper is on how the market value of debt is optimally allocated across various securities and not much on the level itself.

Methodologically, we are related to two strands of literature. The ideas for the "sufficient statistics" approach we borrow from public finance literature, such as Saez (2001) and Chetty (2009). That literature generally focuses on settings where a government faces no risk. When applied to our problem directly, this approach yields no clear and transparent insights. We make progress by augmenting it with a certain class of small-noise approximations. The small noise approximations have been used frequently both in finance (e.g., Samuelson (1970), Devereux and Sutherland (2011)) and computational economics (e.g., Guu and Judd (2001), Schmitt-Grohe and Uribe (2004), Bhandari, Evans, Golosov, and Sargent (2021)). The particular class of expansions that we use does not require us to assume stationarity or abstract from heteroskedasticity, which makes it particularly suitable to study portfolio problems in dynamic stochastic economies.

The rest of the paper is organized as follows. In Section 2 we describe our economy. In Section 3 we describe the perturbations and approximations we use to study portfolio problems. In section 4 we focus on a special case of our model, dubbed "small open economy", in which we assume that equilibrium asset prices are unaffected by government policy. This model allows us to derive target portfolio in the most transparent settings and estimate it using U.S. data. In Section 5 we consider several models of how government policies affect asset prices, such as those implied by preferred habitat models in the spirit of Greenwood and Vayanos (2014) or closed economy models in which all assets are priced by the representative consumer. We show that the target portfolio continues to play central role in such models and that many quantitative insights derived in Section 4 continue to hold for realistic models of asset price behavior.

2 Baseline environment

Timing and shocks. Time is discrete and infinite. Exogenous disturbances in period t are summarized by state $s_t \subset \mathbb{R}^S$, where S is the number of shocks. S is countable but can be finite or infinite. Each shock takes values in a compact set. The initial state s_0 is predetermined. History of shocks is $s^t = (s_0, ..., s_t)$. We use $\Pr\left(s^t\right)$ and $\Pr\left(s^t | s^T\right)$ for $t > T$ to denote probabilities of s_t conditional on information in period 0 and s^T respectively. We say $s^{t+k} \succ s^t$ if we can write s^{t+k} as $\left(s^t, s_{t+1}, .., s_{t+k}\right)$.

Any variable x_t described below is a function of s^t. Most of the time, we omit explicit reference to the history and simple write x_t rather than $x_t\left(s^t\right)$. We use $\mathbf{x}$ to denote $\left\{x_t\left(s^t\right)\right\}_{t,s^t}$. We use interchangeably notations $\mathbb{E}_{s^t} x_{t+k}$ and $\mathbb{E}_t x_{t+s}$ to denote expectation of x_{t+k} conditional on history s^t.

Securities. There are three groups of agents in our economy: households, a government, and foreign investors. There is a fixed, finite or countably-infinite, set of securities that these agents can trade. Security i is characterized by some exogenous stream of dividends $\mathbf{d}^i$. The net supply of security i is exogenous and denoted by $\mathcal{B}^i \geq 0$.

Securities can be subject to additional constraints that restrict at which histories they can be traded and the set of agents who can trade them. *Government debt* is a type of security that is available in the zero net supply, and that can be held in negative quantity (i.e. "issued") only by the government. We use $\mathcal{G}$ to denote the set of securities i corresponding to government debt. We assume that the set of securities includes at least one type of government debt, called *short government bond.* The short government bond issued by the government in period t pays 1 unit of consumption good in period $t+1$. We use superscript rf for this security.

Price of security i in period t is denoted by q_t^i. We use convention $q_t^i = 0$ if security i cannot be traded in t by any agent. The *return* on security i that can be traded in period $t-1$ is $R_t^i \equiv \left(d_t^i + q_t^i\right)/q_{t-1}^i$. The return on the short government bond is called the short interest rate, R_t^{rf}, and satisfies $R_t^{rf} = 1/q_{t-1}^{rf}$. The *excess return* of security i is $r_t^i \equiv R_t^i - R_t^{rf}$.

Households. There is a unit measure of identical households. Each household has earnings $\mathbf{y}$, pays taxes $\boldsymbol{\tau}$, trades securities $\left\{\mathbf{b}^i\right\}_i$, and consumes consumption good $\mathbf{c}$. Household's budget constraint is

$$c_t + \sum_i q_t^i b_t^i = (1-\tau_t)\, y_t + \sum_i \left(q_t^i + d_t^i\right) b_{t-1}^i, \tag{1}$$

with some initial portfolio of assets $\left\{b_{-1}^i\right\}_i$. Household preferences in period t, V_t, are defined recursively via

$$V_t = U_t\left(c_t, y_t, \left\{q_t^i b_t^i\right\}_i\right) + \beta \mathbb{W}_t\left(V_{t+1}\right), \tag{2}$$

where U_t is the utility function and $\mathbb{W}_t$ is a functional that maps $t+1$ measurable random variables to real numbers. Households choose $\left(\mathbf{c}, \mathbf{y}, \left\{\mathbf{b}^i\right\}_i\right)$ to solve

$$\max_{\mathbf{c},\mathbf{y},\left\{\mathbf{b}^i\right\}_i} V_0 \tag{3}$$

subject to (1).

U_t is twice continuously differentiable in all arguments, strictly increasing in c_t and decreasing in y_t. $\mathbb{W}_t$ is twice continuously differentiable, strictly increasing, and increasing in the first- and second-order stochastic dominance.[1] Moreover, $\mathbb{W}_t$ has a property that $\mathbb{W}_t\left(x_{t+1}\right) = \mathbb{E}_t x_{t+1}$

[1]In other words, $\mathbb{W}_t\left(x_{t+1}^1\right) \geq \mathbb{W}_t\left(x_{t+1}^2\right)$ whenever random variable x_{t+1}^1 first- or second-order stochastically dominates x_{t+1}^2.

for any time-t measurable random variable x_{t+1}. U_t and $\mathbb{W}_t$ may be subject to shocks, i.e. depend on s_t.

Although problem (3) is written as if households can freely trade all securities, the dependence of U_t on $\left\{q_t^i b_t^i\right\}_i$ subsumes a variety of asset trading frictions. For example, the definition of government debt implies that agents must face constraints of the form $q_t^i b_t^i \geq 0$ for all $i \in \mathcal{G}$. Such constraints are associated with some Lagrange multipliers. One can bring these constraints with the corresponding Lagrange multipliers under the max operator and redefine function U_t to represent the household problem with constraints in the form (3). In the similar fashion, one can incorporate restrictions on household's ability to trade any security (e.g., $q_t^j b_t^j = 0$ for some j) or borrowing constraints (e.g., $\sum_i q_t^i b_t^i \geq -\underline{a}$ for some $\underline{a} \geq 0$) into the representation (3) of the household problem.

Similarly, the "bonds in the utility function" specification allows us to capture additional services that some securities may render to households beyond their role in transferring resources across periods. For example, Krishnamurthy and Vissing-Jorgensen (2012) argue that government bonds provide additional safety and liquidly services that cannot be provided by other securities such as private debts, and they model these services as a direct utility benefit. Their model is a special case of ours if we use a utility function of the form $U_t\left(c_t - \varphi_t\left(\left\{q_t^i b_t^i\right\}_{i\in\mathcal{G}}\right), y_t\right)$, where φ_t is some convex function. Similarly, other models of services of government debts, such as "government bonds in advance constraints", in the spirit of Andolfatto and Williamson (2015) or Bassetto and Cui (2018), also map into our framework by bringing these constraints, with appropriate Lagrange multipliers, under the max operator and redefining U_t.

We say that securities i and j are *perfect substitutes* if their contribution to household utility depends on their total market value, i.e., if can write $U_t\left(..., q_t^i b_t^i, q_t^j b_t^j\right)$ as $U_t\left(..., q_t^i b_t^i + q_t^j b_t^j\right)$. Otherwise, securities i and j are imperfect substitutes.

Our analysis will be substantially simplified if we abstract from income effects on labor supply. To this end, we assume that utility function can be represented as

$$U_t = U_t\left(c_t - \frac{\left(y_t/\theta_t\right)^{1+1/\gamma}}{1+1/\gamma}, \left\{q_t^i b_t^i\right\}_i\right),$$

where γ is the elasticity of labor supply and θ_t is a random variable.

Government. The government collects tax revenues $\mathcal{R}_t = \tau_t Y_t$, where Y_t is the aggregate output, to finance exogenous stochastic government expenditures G_t. The period government budget constraint is

$$\tau_t Y_t - G_t + \sum_i q_t^i B_t^i = \sum_i \left(q_t^i + d_t^i\right) B_{t-1}^i. \tag{4}$$

At this stage, we do not make any assumptions on which securities the government can and cannot trade, and equation (4) is merely an accounting identity. We use $X_t \equiv G_t - \mathcal{R}_t$ to denote primary deficit. All output is produced by households, so feasibility implies that $y_t = Y_t$ for all t. We use $B_t \equiv \sum_i q_t^i B_t^i$ to denote the market value of the outstanding government portfolio, and $\omega_t^i \equiv q_t^i B_t^i / B_t$ to denote the share of security i in that portfolio. Budget constraint (4) uses convention that if the government holds a long position in security i in period t then $B_t^i < 0$. We use this convention since the natural benchmark for us to examine is the case when government issues various debts. This way, all securities corresponding to government debts are take positive values. We refer to the ratio B_t/Y_t as the debt to GDP ratio.

Foreign investors. Foreign investors are a set of time-t measurable, twice continuously differentiable demand functions $\left\{D_t^i\left(\left\{\mathbf{q}^i\right\}_i\right)\right\}_t$. D_t^i may be subject to shocks and depend on s_t.

Competitive equilibrium.

Definition 1. *For given initial conditions* $\left\{b_{-1}^i, B_{-1}^i\right\}_i$, *and government policy* $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$ *a competitive equilibrium is a collection* $\left(\mathbf{c}, \mathbf{y}, \mathbf{Y}, \left\{\mathbf{b}^i, \mathbf{q}^i\right\}_i\right)$ *such that (i)* $\left(\mathbf{c}, \mathbf{y}, \left\{\mathbf{b}^i\right\}_i\right)$ *solves (3), (ii)* $\left(\boldsymbol{\tau}, \mathbf{Y}, \left\{\mathbf{q}^i, \mathbf{B}^i\right\}_i\right)$ *satisfies (4), (iii)* $\mathbf{y} = \mathbf{Y}$ *and* $\mathbf{b}^i + \mathbf{D}^i = \boldsymbol{\mathcal{B}}^i + \mathbf{B}^i$ *for all i.*

Discussion. Our model incorporates a variety of mechanisms for determining asset prices that are studied in the literature. The functional $\mathbb{W}_t$ is taken from the work of Ai and Bansal (2018), who show that it incorporates as special cases a large number of models: the recursive utility of Kreps and Porteus (1978) and Epstein and Zin (1989); the variational preferences of Maccheroni, Marinacci, and Rustichini (2006a), Maccheroni, Marinacci, and Rustichini (2006b); the multiplier preferences of Hansen and Sargent (2008) and Strzalecki (2011); the second-order expected utility of Ergin and Gul (2009); the smooth ambiguity preferences of Klibanoff, Marinacci, and Mukerji (2005), Klibanoff, Marinacci, and Mukerji (2009); the disappointment aversion preference of Gul (1991); the recursive smooth ambiguity preference of Hayashi and Miao (2011). Moreover, by relaxing the differentiability assumption on $\mathbb{W}_t$, one can extend them to the maxmin expected utility of Gilboa and Schmeidler (1989), Epstein and Schneider (2003). Similarly, since function U_t can depend on s_t, our specification incorporates discount factor shock model of Albuquerque, Eichenbaum, Luo, and Rebelo (2016).

Our specification of foreign investors allows us to cover a variety of ways in which changes in supply of assets (such as issuance of government debts) can affect asset prices. The two extreme cases are the closed economy ($D_t^i = 0$ for all i, t) and the small open economy (D_t^i is perfectly elastic with respect to $\mathbf{q}^i$). Moreover, this formulation also nests various segmented market models, such as noise traders as in Kyle (1985), or the preferred habitat investors as in Greenwood and Vayanos (2014) and Vayanos and Vila (2021).

Throughout our analysis we abstract from any default risk and assume that the government can commit to its policy $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$.

3 Techniques and key notions

3.1 Perturbations

In this section, we give a broad overview of our approach. Take any competitive equilibrium and the associated government policy $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$ that supports it. Consider a slight perturbation of $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$ and its impact on welfare. In the following sections, we study specific perturbations that we describe in details but for now it is useful to keep perturbations abstract and unspecified. To make the exposition simple, we consider perturbations parameterized by a scalar ϵ and study the effect of the perturbation as $\epsilon \to 0$. We use notation $\partial_\epsilon x_t$ to denote the derivative $\partial_\epsilon x_t \equiv \lim_{\epsilon \to 0} \left(x_{t,\epsilon} - x_t\right)/\epsilon$ for any variable x_t and assume that all derivatives exist.

Any perturbation must be feasible for the government, i.e., satisfy its budget constraint. This can be written in our notation as

$$\partial_\epsilon \left(\tau_t Y_t\right) + \sum_i q_t^i \partial_\epsilon B_t^i = \sum_i \partial_\epsilon q_t^i \left(B_t^i - B_{t-1}^i\right) + \sum_i \left(q_t^i + d_t^i\right) \partial_\epsilon B_{t-1}^i. \tag{5}$$

We are interested in the welfare impact of this perturbation, $\partial_\epsilon V_0$. Let $\beta^t \Pr\left(s^t\right) M_t\left(s^t\right)$ be the Lagrange multiplier on the household budget constraint (1) at history s^t, and let $a_t^i\left(s^t\right) \equiv \frac{\partial V_0 / \partial\left(q_t^i b_t^i\right)\left(s^t\right)}{\beta^t \Pr\left(s^t\right) M_t\left(s^t\right)}$ be the marginal utility of security i normalized by the marginal utility of consumption. The envelope theorem implies that $\partial_\epsilon V_0$ is given by

$$\partial_\epsilon V_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t M_t \left[-Y_t \partial_\epsilon \tau_t + \sum_i \left\{a_t^i b_t^i - \left(b_t^i - b_{t-1}^i\right)\right\} \partial_\epsilon q_t^i\right]. \tag{6}$$

Equation (6) shows that households' welfare does not depend on the perturbation in the government portfolio $\left\{\partial_\epsilon \mathbf{B}^i\right\}_i$ directly, but rather on the impact of this perturbation on taxes $\partial_\epsilon \boldsymbol{\tau}$ and asset prices $\left\{\partial_\epsilon \mathbf{q}^i\right\}_i$. The impact of taxes $\partial_\epsilon \tau_t$ is proportional to household earnings, and hence aggregate output, Y_t, adjusted by the shadow value of resources for the household, M_t.

The impact of asset prices $\partial_\epsilon q_t^i$ consists of two terms. One is the direct utility benefit that households may derive from holding security i. This benefit is proportional to the households' holding of security, b_t^i, adjusted by its marginal utility, $M_t a_t^i$. The second effect is related to what the household does with security i in equilibrium. If the household buys it in period t, $\left(b_t^i - b_{t-1}^i\right) > 0$, then higher asset prices reduce available resources by $\left(b_t^i - b_{t-1}^i\right) \partial_\epsilon q_t^i$, and the marginal impact of this reduction in resources on household utility is again captured by M_t.

A feasible perturbation increases welfare if $\partial_\epsilon V_0 > 0$ and lowers it if $\partial_\epsilon V_0 < 0$. If the government policy is optimal, then there is no welfare improving perturbation, and we must have

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t M_t \left[-Y_t \partial_\epsilon \tau_t + \sum_i \left\{ a_t^i b_t^i - \left(b_t^i - b_{t-1}^i \right) \right\} \partial_\epsilon q_t^i \right] = 0. \tag{7}$$

In what follows, we use equations (5) and (7), as well as its more general form (6), to characterize the optimal public portfolios $\left\{\mathbf{B}^i\right\}_i$ and the types of perturbations that can improve the sub optimal portfolios.

3.2 Approximations

The relationship between equations (5) and (7) is, in general, complex and non-linear. To simplify our analysis, we rely on a certain class of approximations that build on the ideas of Samuelson (1970), Devereux and Sutherland (2011), Schmitt-Grohe and Uribe (2004), Bhandari, Evans, Golosov, and Sargent (2021). Fix any history s^T. Without loss of generality, we can write vector s_t for $t \geq T$ as

$$s_t = \mathbb{E}_T s_t + \varepsilon_t \equiv \bar{s}_t + \varepsilon_t,$$

where $\mathbb{E}_T \varepsilon_t = 0$. Consider a sequence of stochastic processes, parameterized by scalar $\sigma \geq 0$, defined as

$$s_t\left(\sigma\right) = \bar{s}_t + \sigma \varepsilon_t.$$

Here $\sigma = 0$ corresponds to a deterministic economy in which all uncertainty is "shut down" after state s^T. Let $x_t\left(\sigma\right)$ be any equilibrium variable in the σ-economy. We use second order Taylor expansions of the equilibrium conditions with respect to σ around $\sigma = 0$.

Let $\bar{x}_t, \partial_\sigma x_t, \partial_{\sigma\sigma} x_t$ be the zeroth-, first- and second-order terms in expansion. We use signs "$\simeq$" and "$\approx$" to denote relationships that hold up to order $O\left(\sigma^3\right)$ and $O\left(\sigma\right)$ respectively. In this notation,

$$x_t\left(\sigma\right) \simeq \bar{x}_t + \sigma \partial_\sigma x_t + \frac{\sigma^2}{2} \partial_{\sigma\sigma} x_t, \quad x_t\left(\sigma\right) \approx \bar{x}_t.$$

Throughout our analysis, we assume that the equilibrium we study is sufficiently well-behaved. In particular, we assume that the equilibrium manifold is smooth in the neighborhood of policy $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$ that we consider, so that all derivatives exists for both positive and negative values of ϵ. Similarly, we assume that the equilibrium manifold is smooth in σ, in the neighborhood of $\sigma = 0$. Furthermore, we assume that the present value of the government budget constraint and multipliers M_t are finite at each history s^t. We call such economies *regular*. While it would be interesting to explore sufficient conditions for the existence of regular equilibria, that would require imposing additional structure on model primitives that would distract from the main focus of the paper.

3.3 Key notions

Household's optimality conditions are as follows. The optimal choice of labor satisfies

$$y_t = \theta_t^{1+\gamma} \left(1 - \tau_t\right)^{\gamma}, \tag{8}$$

and the optimal choice of savings satisfies

$$1 - a_t^i = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R_{t+1}^i. \tag{9}$$

Equation (8) helps to calculate the *tax revenue elasticity*, $\xi_t \equiv \partial \ln \mathcal{R}_t / \partial \ln \tau_t$. The tax revenue elasticity will play an important role in our analysis. Using the definition of tax revenues $\mathcal{R}_t$ and equation (8), it is easy to show that

$$\xi_t = 1 - \gamma \frac{\tau_t}{1 - \tau_t}. \tag{10}$$

The tax revenue elasticity is a useful measure for several reasons. First, it is easy to verify that if the government needs to return 1 unit of resources to the households, it would need to decrease marginal taxes by $1/\left(\xi_t Y_t\right)$ units. Second, this measure is a convenient way to summarize tax distortions, with $\xi_t = 1$ corresponding to no deadweight losses from transferring resources between households and the government.

We refer to a_t^i, that appears in equation (9), as the *liquidity premium* or *wedge* for security i. If security i can be freely traded by households and does not carry direct utility benefits then its liquidity premium is zero. Returns on government-issued debts are often lower than returns on comparable privately-issued debts. Through the lens of our definition, this implies that public debts have higher liquidity premia than private debts. We use A_t^k to denote the liquidity premium on short government bonds form periods t to $t+k$, i.e.

$$1 - A_t^k \equiv \left(1 - a_t^{rf}\right) \times ... \times \left(1 - a_{t+k}^{rf}\right),$$

with convention that $A_t^0 = 0$.

A useful benchmark to consider is the case when two securities are perfect substitutes. Applying the definition of perfect substitutes and equation (9) we immediately get that their liquidity premia must be the same:

Lemma 1. *Suppose securities i and j are perfect substitutes. Then $a_t^i = a_t^j$ for all t.*

4 Optimal public portfolios in a small open economy

We showed in the previous section that the welfare impact of any perturbation of government policy can be summarized by tax and price responses $\partial_\epsilon \boldsymbol{\tau}$ and $\left\{\partial_\epsilon \mathbf{q}^i\right\}_i$. To study their implications for the optimal public portfolio, it will be useful at first to abstract from price responses and consider the case when $\left\{\mathbf{q}^i\right\}_i$ are invariant to changes in government policy. We refer to this case as the *small open economy*. Focusing on the small open economy allows us to isolate most of the main forces determining the optimal public portfolio. Moreover, many of the insights that we obtain in this section continue to hold qualitatively and quantitatively once we extend our analysis to price effects in section5.

We first consider the following perturbation. Suppose that in some history s^T the government sells ϵ / q_T^j units of security j (where $\epsilon > 0$) and then buys it back in period $T+1$, keeping all other holdings of securities unchanged. For concreteness, we assume that security j is a government bond, so this transaction is equivalent to issuing some government bond j in history s^T and then buying it back in the next period. To make this perturbation feasible, taxes must be adjusted to satisfy the government budget constraint. It is easy to show that this tax adjustment satisfies, in the limit as $\epsilon \to 0$, $\partial_\epsilon \tau_T = -1/\left(Y_T \xi_T\right)$ in history s^T, $\partial_\epsilon \tau_{T+1} = R_{T+1}^j / \left(Y_T \xi_T\right)$ in $s^{T+1} \succ s^T$, and $\partial_\epsilon \tau_t = 0$ in all other histories. Substitute these responses in equation (6) to show that

$$\begin{aligned} \frac{\partial_\epsilon V_0}{\beta^T \Pr\left(s^T\right) M_T\left(s^T\right)} &= \frac{1}{\xi_T} - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R_{T+1}^j \frac{1}{\xi_{T+1}} \\ &= \left\{\left(\frac{1}{\xi_T} - 1\right) - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R_{T+1}^j \left(\frac{1}{\xi_{T+1}} - 1\right)\right\} + a_T^j. \end{aligned} \tag{11}$$

This equation shows that the welfare impact of this perturbation can be decomposed into two components: the effect on the intertermporal allocation of the deadweight losses from taxation (the expression in the curly brackets) and the liquidity premium a_T^j. To understand the intuition for this expression and its implications, suppose for a moment that taxes are not distortionary. In this case, $\xi_t = 1$ for all t, and issuing bond j is always welfare improving as long as its liquidity premium is strictly positive. In the absence of tax distortions, there are no costs in expanding the supply of public debts and it is optimal to issue enough debts to fully satiate liquidity demands.

When taxes are distortionary, the government needs to trade off liquidity benefits of debts with deadweight costs of taxation. Taxation has costs both because of the intertemporal allocation of distortions and the uncertainty about their allocation. To isolate the first effect, consider the zeroth order approximation of equation (11) when we shut down all the uncertainty after history s^T. Using notation introduced in section 3.2, we can write this approximation as

$$\frac{\overline{\partial_\epsilon V_0}}{\beta^T \Pr\left(s^T\right) \bar{M}_T\left(s^T\right)} = \left(\frac{1}{\bar{\xi}_T} - \frac{1}{\bar{\xi}_{T+1}}\right) + \frac{\bar{a}_T^j}{\bar{\xi}_{T+1}}. \tag{12}$$

This equation implies that in a deterministic economy optimal taxes must be increasing whenever any asset that government can trade has a strictly positive liquidity premium. The intuition for this result is closely related to the back-loading of incentive results derived by Acemoglu, Golosov, and Tsyvinski (2008) or Albanesi and Armenter (2012) in somewhat different settings. If we start with perfectly smoothed taxes, a slight tilt of the tax profile into the future has negligible impact on deadweight costs of taxation. However, this tilt allows to increase the level of government debt that has a non-negligible welfare effect if its liquidity premium is strictly positive.

Equation (12) holds for all securities that the government can trade. In particular, it implies that in the optimum, when the right hand side is set to zero, the liquidity premia for all government bonds must be equalized to the zeroth order, $\bar{a}_T^i = \bar{a}_T^{rf}$ for all $i \in \mathcal{G}$. Thus, equation (12) pins down the optimal transition path of government debt and its long-run level, but not its composition. Since the focus of our paper on the optimal composition of government debt, it will be convenient to abstract from transition dynamics. In particular, we focus on competitive equilibria that satisfy the following property

Definition 2. *Competitive equilibrium is stationary after period T if it satisfies*

$$\mathbb{E}_T \tau_{T+t} \approx \tau_T,\ \mathbb{E}_T \frac{X_{T+t}}{Y_{T+t}} \approx \frac{X_T}{Y_T},\quad \mathbb{E}_T \frac{Y_{T+t+1}}{Y_{T+t}} \approx \Gamma,\ \mathbb{E}_T q_{T+t}^{rf} \approx q, \tag{13}$$

for some Γ, q.

Stationarity assumes that tax rates, and primary deficits as fractions of GDP are approximately random walks, while the growth rates of output and short interest rates are expected to be approximately equal to the same values Γ and q in all future dates. We focus on stationary competitive equilibria in the body of the paper purely for expositional convenience. All our results in the appendix are derived without invoking this assumption, and the insights we discuss here do not depend on whether the equilibrium is stationary or not.

We now turn to the analysis of the optimal composition of government portfolio. The easiest starting point is to consider a perturbation in which in period T the government sell ϵ/q_T^j units of any security j that it can trade (where now ϵ can be positive or negative) and simultaneously buys ϵ/q_T^{rf} units of the one period bond, and then in period $T+1$ unwinds this transaction. The same arguments used to derive equation (11) now imply that welfare impact is given by

$$\frac{\partial_\epsilon V_0}{\beta^T \Pr\left(s^T\right) M_T\left(s^T\right)} = -sign\left(\epsilon\right) \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} r_{T+1}^j \frac{1}{\xi_{T+1}}. \tag{14}$$

This equation can be used to study how a public portfolio can be improved in any equilibrium as well as to characterize the optimum portfolio. For now, we focus on studying an optimum equilibrium, that is, the equilibrium in which government policies are set to maximize consumer welfare and discuss how to use equation (14) to improve an arbitrary portfolio in section 4.3. Since we restrict attention to regular equilibria, the perturbations we consider are feasible for both positive and negative small values of ϵ. No feasible perturbation can improve welfare in the optimum equilibrium, which implies that $\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} r_{T+1}^j \frac{1}{\xi_{T+1}} = 0$ for any security j that the government can trade. This condition can be equivalently written as

$$a_T^j - a_T^{rf} = cov_T\left(\frac{\beta M_{T+1}}{M_T} r_{T+1}^i, \frac{\left(\xi_{T+1}\right)^{-1}}{\mathbb{E}_T\left(\xi_{T+1}\right)^{-1}}\right).$$

To the second order of approximation, it is equivalent to (see appendix)

$$cov_T\left(\ln \xi_{T+1}, r_{T+1}^j\right) \simeq -R_{T+1}^{rf}\left(a_T^j - a_T^{rf}\right). \tag{15}$$

This equation shows the costs and benefits of issuing any bond j (or holding any security) relative to issuing a one period government bond. The benefits are proportional to the excess liquidity premium $a_T^j - a_T^{rf}$. The costs are proportional to the return risk r_{T+1}^j that subsequently requires tax adjustments. If the government adjusts taxes in period $T+1$ then these costs are summarized by $cov_T\left(\ln \xi_{T+1}, r_{T+1}^j\right)$ and is captured by the left hand side of (15).

It will be useful to generalize this pertubation to consider tax adjustments in arbitrary period $T+t$. Suppose that instead of returning the excess returns from $r_{T+1}^j \epsilon$ the government invests it into a one period bond that it rolls over for t more periods. Using the same arguments as before, one can show a generalization of equation (15)

$$cov_T\left(\ln \xi_{T+t}, r_{T+1}^j\right) \simeq -R_{T+1}^{rf}\left(a_T^j - a_T^{rf}\right) - cov_T\left(A_{T+1}^{t-1}, r_{T+1}^j\right) \text{ for all } t \geq 1. \tag{16}$$

The additional term on the right hand side of equation (15) captures liquidity benefits or losses from the need to adjust debt to roll over excess returns realized in period $T+1$.

We are now ready to use these optimality conditions to derive implications for the optimal public portfolio. Let

$$\begin{aligned} Q_{T+1}^{T+t} &\equiv 1 \times \frac{1}{R_{T+2}^{rf}} \times ... \times \frac{1}{R_{T+t}^{rf}}, \\ \mathcal{Q}_{T+1}^{T+t} &\equiv 1 \times \frac{1}{\sum_{i \geq 1} r_{T+2}^{i} \omega_{T+1}^{i} + R_{T+2}^{rf}} \times ... \times \frac{1}{\sum_{i \geq 1} r_{T+t}^{i} \omega_{T+t-1}^{i} + R_{T+t}^{rf}} \end{aligned}$$

be two stochatic discount rates between periods $T+1$ and $T+t$, define for all $t \geq 1$. The difference between the two notions is that Q_{T+1}^{T+t} uses short interest rates while $\mathcal{Q}_{T+1}^{T+t}$ uses returns on public portfolios. The government budget constraint in period $T+1$ can be written in a present value form as

$$\mathbb{E}_{T+1} \sum_{t=1}^{\infty} \mathcal{Q}_{T+1}^{T+t} X_{T+t} = -B_T \left[R_{T+1}^{rf} + \sum_{i \geq 1} \omega_T^i r_{T+1}^i \right], \tag{17}$$

where $\sum_{i \geq 1}$ denotes a sum over all assets $i \neq rf$. Multiply equation (17) by a return r_{T+1}^{j} of any security j that the government can trade, take expectation at time T, and use the fact that equation (11) implies that $\mathbb{E}_T r_{T+1}^{j} = O\left(\sigma^2\right)$ to write the second-order approximation of equation (17) as

$$\sum_{t=2}^{\infty} \mathbb{E}_T X_{T+t} cov_T \left(Q_{T+1}^{T+t}, r_{T+1}^{j} \right) + \sum_{t=1}^{\infty} \mathbb{E}_T Q_{T+1}^{T+t} cov_T \left(X_{T+t}, r_{T+1}^{j} \right) \simeq -B_T \sum_{i \geq 1} \omega_T^i cov_T \left(r_{T+1}^{i}, r_{T+1}^{j} \right). \tag{18}$$

This equation shows that any changes in the returns on public portfolio in period $T+1$ can be decomposed in changes in discount rates rates $\{\mathcal{Q}_{T+1}^{T+t}\}_{\ t \geq 1}$ and changes in primary deficits $\{X_{T+t}\}_{T \geq t}$. We can multiply this identity by returns on any security j and write it in form of covariances. The covariance of $\{\mathcal{Q}_{T+1}^{T+t}\}_{\ t \geq 1}$ and r_{T+1}^{j} consists of the covariances of risk-free rates, $cov_T\left(Q_{T+1}^{T+t}, r_{T+1}^{j}\right)$, and covariances of the returns on other securities adjusted by future portfolio holdings, $cov_T\left(r_{T+1+t}^{i} \omega_{T+t}^{i}, r_{T+1}^{j}\right)$. However, it can be shown that the latter covariances are of the order $O\left(\sigma^3\right)$ for $t \geq 1$ and thus they drop out from the approximated budget constraint (18).

At this stage, equation (18) is essentially an identify, written in an approximated form, and it holds irrespective of whether government's portfolio is optimal or not. In order to characterize the optimal portfolio, we need to combine this equation with the optimality condition (16). As the first step, observe that $cov_T\left(X_{T+t}, r^j_{T+1}\right)$ can be decomposed into covariance of returns r^j_{T+1} with tax rates τ_{T+t} (and, therefore, with tax revenue elasticities ξ_{T+t}) and covariance of returns with exogenous shocks to primary surpluses. The following lemma provides such a decomposition.

Lemma 2. *Define $X^{\perp}_{T+t}$ and ζ_{T+t} as*

$$\begin{aligned} X^{\perp}_{T+t} &\equiv X_{T+t} + \tau_{T+t}\mathbb{E}_T\left[\xi_{T+t}\right]\mathbb{E}_T\left[Y_{T+t}\right], \\ \zeta_{T+t} &\equiv \gamma^{-1}\left(1-(1+\gamma)\,\tau_{T+t}\right)^2. \end{aligned}$$

In the optimum equilibrium for all t, j,

$$cov_T\left(X_{T+t}, r^j_{T+1}\right) \simeq \mathbb{E}_T\zeta_{T+t}\mathbb{E}_T Y_{T+t} cov_T\left(\ln\xi_{T+t}, r^j_{T+1}\right) + cov_T\left(X^{\perp}_{T+t}, r^j_{T+1}\right).$$

Moreover, $cov_T\left(X^{\perp}_{T+t}, r^j_{T+1}\right)$, to the second order approximation, is a function of $cov_T\left(\theta_{T+t}, r^j_{T+1}\right)$ and $cov_T\left(G_{T+t}, r^j_{T+1}\right)$ but not $cov_T\left(\tau_{T+t}, r^j_{T+1}\right)$.

We are now ready to state the main result of this section. To do so, it will be convenient to write the present values expressions, such as the one that appears in equation (18), in a compact matrix form. Let $\mathbf{a}_T$ and $\boldsymbol{\omega}_T$ be vectors with elements $\mathbf{a}_T[i] = a^i_T - a^{rf}_T$, and $\boldsymbol{\omega}_T[i] = \omega^i_T$, all defined for all securities $i \geq 1$ that the government can trade.[2] Let $\mathbf{w}$ be a vector with elements $\mathbf{w}[t] = (q\Gamma)^t$ for $t \geq 1$, and let Σ^Q_T, Σ^X_T, Σ^a_T, Σ_T be matrices with elements defined in the following table

$\Sigma^Q_T[j,t] = cov_T\left(\ln Q^{T+1+t}_{T+1}, r^j_{T+1}\right)$	$\Sigma^X_T[j,t] = -cov_T\left(\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}, r^j_{T+1}\right)$
$\Sigma^A_T[j,t] = cov_T\left(A^t_{T+1}, r^j_{T+1}\right)$	$\Sigma_T[j,i] = cov_T\left(r^i_{T+1}, r^j_{T+1}\right)$

Combining lemma 2, optimality conditions (15) and (16), and the budget constraint (18), we obtain the following result.

Theorem 3. *In the stationary economy, the optimal portfolio satisfies*

$$\Sigma_T\boldsymbol{\omega}_T \simeq \left[\pi^Q\Sigma^Q_T + \pi^X_T\Sigma^X_T + \pi^A_T\Sigma^A_T\right]\mathbf{w} + \pi^a_T\mathbf{a}_T, \tag{19}$$

where $\pi^Q = (1-q\Gamma)/q$, $\pi^X_T = Y_T/(qB_T)$, $\pi^A_T = \zeta_T Y_T/B_T$, $\pi^a_T = \left(\frac{\Gamma}{1-q\Gamma}\right) \times \zeta_T Y_T/(qB_T)$.

The right hand side of equation (19) shows the main of objectives that the choice of the optimum portfolio strives to achieve. The government chooses its portfolio to hedge three

[2]In other words, we do not require that the government can trade all the securities that exist in the economy.

risks (the interest rate risk Σ_T^Q, primary surplus risk Σ_T^X, and liquidity risk Σ_T^A) as well as to exploit the "excess liquidity curve" $\mathbf{a}_T$ in choosing most efficient ways to provide liquidity. Costs of hedging and liquidity provision is captured by the measure of riskiness of different securities Σ_T. Vector $\mathbf{w}$ discounts future risk into present portfolio choice, and coefficients π^Q, π_T^X, π_T^A, π_T^a provides quasi-weights with which different objectives are weighted in the government portfolio.

Examining equation (19) reveals several insights. All things being equal, bonds with higher excess liquidity premium have higher share in government portfolio.[3] Weights π_T^X, π_T^A and π_T^a are all proportional to the GDP-to-debt ratio Y_T/B_T, but weight π^Q is not. Thus, the larger level of debt, *ceterus paribus*, implies bigger importance of hedging interest rate risk. The interest rate risk matters for the government because of the uncertainty it introduces about costs of future roll over of government debts. The more the debt that needs to be rolled over, the higher the costs are from any uncertainty about future interest rates. Weights π_T^A and π_T^a are also proportional to ζ_T. Simple algebra shows that $\zeta_T = \frac{(1+\gamma)^2}{\gamma}\left(\frac{1}{1+\gamma} - \tau_T\right)^2$. Coefficient ζ_T is decreasing in τ_T and reaches 0 at $\tau_T = \frac{1}{1+\gamma}$, which corresponds to the peak of the Laffer curve. To understand why current tax levels affect the importance of hedging liquidity risk, it is useful to consider the following thought experiment. Suppose that government can borrow at a cheaper rate than the household. Then the government can help households by borrowing on their behalf. The benefit from this transaction comes from lower interest rates that the government faces. The cost comes from distortions that arise from higher taxes that such borrowing must entail. The closer the taxes are to the peak of the Laffer curve, the larger is the cost of tax distortions relative to the benefits of liquidity provision. Hence, higher tax levels implies smaller weight on liquidity provision.

It is instructive to compare formula (19) to the optimal portfolios that emerge in the classical investment problems analyzed by such authors as Samuelson (1970), Campbell and Viceira (1999), Viceira (2001). Formula (19) misses the classical risk-return trade-off that is central to those problems. The relationship between the expected return and risk of any asset per se bears no implication on government portfolio: as long as households can trade the same asset, there is no benefit from government's trading it on their behalf. Instead, this trade-off is replaced with a term $\Sigma_T^{-1}\mathbf{a}_T$ that captures excess liqudity benefits relative to the returns uncertainty. Similarly, the weight π_T^a is unrelated to the coefficient of risk aversion,

[3] One natural case to consider is the one in which the one period government bond has highest liquidity premium. In this case $\mathbf{a}_T \leq \mathbf{0}$ and equation (19) shows the the vector of weights $\boldsymbol{\omega}_T$ must be reduced. Since vector $\boldsymbol{\omega}_T$ excludes the portfolio weight of the one period bond, and portfolio weights must sum to 1, this implies that portfolio share of the one period bond must be increased.

as it would be in the classical portfolio theory, and is determined instead by the level of tax distortions. Risks Σ_T^Q and Σ_T^X have analogues in the classical portfolio theory, with primary deficit risk corresponding to the "nontradable labor income" in Viceira (2001), but are weighted differently. Finally, the liquidity risk Σ_T^A does not appear in the classical problem.

4.1 Optimal portfolio of public debts

Theorem 3 does not take a stance on which securities the government can trade, and characterizes the optimal portfolio for an arbitrary set of such securities. The most common securities traded by the government are government debts of various maturities. In this section we explore the implications of theorem 3 for the optimal debt maturity.

We assume that government debts come in the form of pure discount bonds (that is, a bond that has no coupon payments and pays 1 unit of resources at some specified maturity date) and that the government can issue debt in any maturity. For the purposes of applying theorem 3, security i will correspond to a bond that matures in period $T+1+i$. When we refer bonds, we use $q_T^{(t)}, r_{T+1}^{(t)}$ to denote the period T price and period $T+1$ excess return of a pure discount bond that matures in period $T+1+t$. In this notation, security (0) and security rf in period T coincide, so that $q_T^{(0)}$ and q_T^{rf} can be used interchangeably. Also note that this structure implies that all bonds maturing in the same period are perfect substitutes, that is there is no difference, from period T point of view, between a 2 period bond issued in period $T-1$ and a 1 period bond issued in period T. In the appendix we show the following theorem.

Theorem 4. *Returns on public debts satisfy $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$. Therefore, the optimal portfolio of bonds in the stationary economy satisfies $\boldsymbol{\omega}_T \approx \boldsymbol{\omega}_T^*$, where*

$$\boldsymbol{\omega}_T^* = (1-q\Gamma)\,\mathbf{w} + \left[\pi_T^X \Sigma_T^{-1}\Sigma_T^X + \pi_T^A \Sigma_T^{-1}\Sigma_T^A\right]\mathbf{w} + \pi_T^a \Sigma_T^{-1}\mathbf{a}_T. \tag{20}$$

Portfolio $\boldsymbol{\omega}_T^*$ defined in equation (20) will play an important role in our subsequent analysis, when we study the effect of asset price responses to optimal portfolio. We refer to $\boldsymbol{\omega}_T^*$ as a *target portfolio.* Theorem 4 show that if in the small optimal economy, the optimal portfolio of government debt should be equal to the target portfolio.

The first part of this theorem shows that covariance of bond returns with themselves, Σ_T, is closely related to the covariance of discount rates $\left\{Q_{T+1}^{T+t}\right\}_t$ and bond returns, in the sense that $q_T^{rf}\Sigma_T$ is equal, to the second order of approximation, to Σ_T^Q. In the economies in which the expectations hypothesis holds, it follows naturally from the fact that $1/q_{T+1}^{(t)} = \mathbb{E}_{T+1}Q_{T+1}^{T+t}$, that is the interest rate between periods T and $T+1+t$ implied by the long bonds is equal to the expected product of short interest rates between these periods. Thus, changes in returns on bond with maturity in period $T+1+t$ is the same as changes in the expected product of short interest rates between period $T+1$ and $T+1+t$. In our economy the expectations hypothesis does not need to hold, but theorem 3 shows that deviations from it must be of the higher order of approximation than other effects.

One implication the result that $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$ is that the government can hedge the interest rate risk fully, at least to the order of approximation we consider. Recall that the interest rate risk emerges because the government needs to roll over its maturing debt. This risk can be greatly diminished if the government matches the maturity of its debts with expected surpluses. This way, there is no *expected* debt roll-over, and the interest rate risk has only third-order effect on welfare. In the stationary economy, expected surplus grow at an approximate rate Γ, while future periods are discounted at the rate q, and so full hedging of interest rate risk is achieved if portfolio weights on future maturities decline at a rate $q\Gamma$. This is captured by the term $(1-q\Gamma)\,\mathbf{w}$. Portfolio $(1-q\Gamma)\,\mathbf{w}$ is equivalent to a growth-adjusted consol, i.e. a consol which coupon payments grows with rate Γ.

How much the government should depart from full hedging of interest rate risk depends on how well government bonds can hedge primary deficit and liquidity risks (the expression in the square brackets of equation (20)), and on the shape of the "excess liquidity premium curve" $\mathbf{a}_T$. Recall from lemma 1 that $\mathbf{a}_T$ is also a measure of how substitutable different bonds are. Before proceeding with our analysis, it is instructive to evaluate these statistics in the U.S. data.

4.2 The target portfolio in the U.S. data

In this section, we use U.S. data to provide a rough estimate of the target portfolio $\boldsymbol{\omega}_T^*$ and its components. These estimations must necessarily be somewhat tentative. Most government bonds in the U.S. are nominal and their prices and returns are available for select maturities, while formulas in (20) are derived for real debts of all maturities. Thus, one needs to make some adjustments and interpolations in order to apply formula (20). That being said, this exercise allows us to highlight some salient features of the data that have important implications on the portfolio.

For almost all data, we use quarterly series from 1952 to 2017. The only exception is yields on high quality corporate bonds, that are available from 1984. Data on GDP, that we use as our measure of Y_t, and inflation is from national income and product accounts (BEA). The series for tax rates τ_t are from Barro and Redlick (2011). To measure returns on government bonds we use Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, out of which ten portfolios correspond to maturities of 6 to 60 months in 6 months intervals, and a final portfolio for maturities between 60 and 120 months. We add a twelveth portfolio, returns on a 3-Month Treasury Bill, published by the Federal Reserve Board of Governors. Returns on high quality corporate bonds are obtained from High Quality Market (HQM) Corporate Bond Yield Curve computed by the U.S. Treasury. Table 1 gives summary statistics of the nominal returns on U.S. treasuries and HQM bonds for different maturities.

Table 1: Summary Statistics for Returns

maturity	US Treasuries		HQM bonds	
(months)	mean	std	mean	std
3	1.08	0.77	1.02	0.83
6	1.16	0.85	1.08	0.80
12	1.22	1.03	1.20	0.78
18	1.28	1.30	1.32	0.83
24	1.31	1.51	1.44	0.92
30	1.34	1.79	1.56	1.03
36	1.38	2.00	1.67	1.15
42	1.41	2.17	1.78	1.28
48	1.41	2.37	1.88	1.40
54	1.44	2.51	1.98	1.51
60	1.37	2.81	2.07	1.61
120	1.52	3.21	2.40	2.04

Notes: This table records the number of observations, mean, standard deviation for holding period returns on US Treasuries and High Quality Corporate bonds. The sample for US Treasuries is 1952-2017 and the sample for HQM bonds is 1984-2017. The units of the returns are quarterly and in percents and the unit of maturity is month.

It is relatively straightforward to see how one can construct empirical counterparts of $cov_T\left(r^{(i)}_{T+1}, r^{(j)}_{T+1}\right)$ and $cov_T\left(\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}, r^{(j)}_{T+1}\right)$. It is less immediate how one can obtain a measure of liquidity premium a^i_T that is defined by equation (9). The shadow cost of resources M_t is not directly observable, and thus needs to be inferred somehow. Our approach to measuring $a^{(i)}_T$ is based on the following idea. Suppose we had a set of securities that households can trade, and that do not carry additional utility benefits such as liquidity services. Then returns on such securities would satisfy

$$1 = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R^i_{t+1}.$$

We can use this relationship to back out M_t, which we then can use together with observed returns on government bonds $R_{t+1}^{(i)}$ to estimate $a_t^{(i)}$ using (25).

In order to implement this procedure we use returns on high quality U.S. corporate bonds of different maturities. Such bonds seem to fit our requirements. They are issues by private sector, their default risk is extremely small, and they are hard to use as a collateral. They also allow a very simple and transparent way to estimate liquidity premium on government bonds. Let $R_{T+1}^{(j),AAA}$ be a period $T+1$ return on high quality corporate bond that matures in period $T+1+j$. Then the liquidity premium on the one period government bond is simply

$$a_T^{(0)} = 1 - \frac{q_T^{(0),AAA}}{q_T^{(0)}} = \frac{R_{T+1}^{(0),AAA} - R_{T+1}^{(0)}}{R_{T+1}^{(0),AAA}}. \tag{21}$$

The liquidity premia on other maturities satisfies

$$a_T^{(i)} = \frac{\mathbb{E}_T\left[R_{T+1}^{(i),AAA} - R_{T+1}^{(i)}\right]}{R_{T+1}^{(0),AAA}} + cov_T\left(\frac{\beta M_{T+1}}{M_T}, R_{T+1}^{(i),AAA} - R_{T+1}^{(i)}\right). \tag{22}$$

In Table 2 we present summary statistics of contemporary covariances and autocorrelations of the our data series that show up in our expressions. To convert nominal returns to real, we subtract expected inflation (see online appendix for all details about empirical implementations used in this section). Table 2 shows a number of salient features of the U.S. data. The returns on government bonds are much more volatile than primary surpluses, tax rates, or liquidity premia. Moreover, primary surpluses and liquidity premia have negative comovement. This negative comovement captures the fact to primary surpluses are pro-cyclical while liquidity premia are couter-cyclical. Expected returns on HQM are higher than on Treasuries, but they do not vary very systematically across different maturities.

Table 2: Covariance Matrix for Real Holding Period Excess Returns and Deficits

	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	X/GDP	100τ	$100a^{rf}$
6m	0.17	0.38	0.53	0.61	0.69	0.74	0.77	0.79	0.82	0.85	0.90	-0.03	0.01	0.35
12m		0.91	1.28	1.50	1.71	1.85	1.94	2.01	2.08	2.19	2.34	0.00	-0.02	0.88
18m			1.85	2.19	2.52	2.75	2.91	3.03	3.15	3.35	3.59	0.04	-0.02	1.37
24m				2.66	3.07	3.38	3.61	3.78	3.96	4.24	4.58	0.11	-0.04	1.64
30m					3.67	4.02	4.32	4.56	4.77	5.18	5.64	0.15	-0.05	1.98
36m						4.53	4.85	5.14	5.39	5.83	6.42	0.24	-0.03	2.18
42m							5.31	5.65	5.93	6.42	7.13	0.30	-0.07	2.39
48m								6.15	6.39	6.95	7.79	0.35	-0.08	2.46
54m									6.90	7.40	8.29	0.40	-0.04	2.70
60m										8.45	9.18	0.50	-0.07	2.73
120m											10.69	0.60	-0.07	3.07
X/GDP												3.99	-0.25	0.43
100τ													0.58	0.28
$100a^{rf}$														16.60
Autocorr	-0.16	-0.19	-0.2	-0.2	-0.19	-0.17	-0.17	-0.14	-0.15	-0.17	-0.13	0.96	0.76	
Difference in returns between HMQ and Treasury bonds, for different maturities														
Mean	0.042	0.044	0.046	0.048	0.05	0.052	0.054	0.056	0.058	0.059	0.064			
Std	0.036	0.032	0.031	0.031	0.032	0.032	0.031	0.031	0.03	0.03	0.027			

Notes: This table records the covariances of holding period excess returns, liquidity premium on the risk-free debt, and deficits normalized by GDP. The sample for US Treasuries is 1952-2017 and the sample for liquidity premium on the short bond is 1984-2017. All returns are quarterly and in percents. Deficits relative to GDP and tax rates are measured in percents.

We use this data to evaluate the target portfolio ω_T^* the expression for which is given in formula (20). This formula requires parameters γ, τ_T, Y_T/B_T, q and Γ. We set $\gamma = \frac{1}{2}$, which is consistent with common calibrations of the elasticity of labor supply. We consider quarterly frequencies for our analysis and set $Y_T/B_T = \frac{1}{4}$ and $\tau_T = \frac{1}{3}$ to be roughly in line with current use debt levels (relative to quarterly GDP) and tax rates. Any reasonable value for q and Γ must be close to 1. What matters for our quantitative analysis is not the values of q and Γ per se, but their product $q\Gamma$. Recall from the definition of q in equation (13) that it is the price of a one period bond in the zeroth order approximation of our economy, e.g. in the absence of any risk. It can be shown that it satisfies

$$q = \frac{\beta U_c\left(\bar{c}_{t+1} - \frac{\left(\bar{Y}_{t+1}/\bar{\theta}_{t+1}\right)^{1+1/\gamma}}{1+1/\gamma}, \ldots\right)}{U_c\left(\bar{c}_t - \frac{\left(\bar{Y}_t/\bar{\theta}_t\right)^{1+1/\gamma}}{1+1/\gamma}, \ldots\right)}.$$

In the stationary economy, consumption aggregator $\bar{c}_t - \left(\bar{Y}_t/\bar{\theta}_t\right)^{1+1/\gamma} \Big/ (1+1/\gamma)$ grows at a rate Γ, so this expression can equivalently be written as

$$q = \beta \Gamma^{-IES}, \tag{23}$$

where IES is the elasticity of the intertemporal substitution.[4] If we set $IES = 1$, to be in line with common parameterization of asset pricing models, this formula implies that $q\Gamma = \beta$. Common parameterizations of annual discount factor are in the range of 0.94 to 0.97. We pick the value of 0.96, so that at quarterly frequencies it implies $\beta = 0.99$. As long as we target $q\Gamma = 0.99$, the exact values of q and Γ play no substantial role, but for concreteness we set $\Gamma = 1.005$ (so that the annual GDP growth is 2 percent) and $q = 0.985$.

With these conventions, formula (20) can be written approximately as

$$\omega_T^* = \left[0.01 + \frac{1}{4}\Sigma_T^{-1}\Sigma_T^X + \frac{1}{8}\Sigma_T^{-1}\Sigma_T^A\right]\mathbf{w} + \frac{100}{8}\Sigma_T^{-1}\mathbf{a}_T. \tag{24}$$

Thus, the optimal composition of the target portfolio depends on the values of $\Sigma_T^{-1}\Sigma_T^X$, $\Sigma_T^{-1}\Sigma_T^A$, $\Sigma_T^{-1}\mathbf{a}_T$. We provide a more careful estimation of these objects below, but as a first pass it will be instructive to construct crude measures of these three objects using only the raw data we report in Table 2. The advantage of this back of the envelope approach is that it is very transparent and one can immediately see how the features of the data presented in Table 2 lead to the conclusion about optimal portfolio that we obtain below with more sophisticated techniques.

Each raw of vector $\mathbf{a}_T$ consists of the difference $a_T^{(i)} - a_T^{(0)}$ and can be written as

$$\begin{aligned} \mathbf{a}_T[i] \;\; = \;\; & q^{(0),AAA}\left\{\mathbb{E}_T\left[R_T^{(i),AAA} - R_T^{(i)}\right] - \mathbb{E}_T\left[R_T^{(0),AAA} - R_T^{(0)}\right]\right\} \\ & + cov_T\left(\frac{\beta M_{T+1}}{M_T}, R_T^{(i),AAA} - R_T^{(i)}\right) - cov_T\left(\frac{\beta M_{T+1}}{M_T}, R_T^{(0),AAA} - R_T^{(0)}\right). \end{aligned} \tag{25}$$

For our back of the envelope exercise we ignore covariances in the second line of this expression. In this case, $\mathbf{a}_T[i]$ is proportional to the difference of returns on $i+1$ and 1 period corporate bonds, adjusted by returns on corresponding government maturities. From the last rows of Table 2 we can see that the time average of $R_T^{(i),AAA} - R_T^{(i)}$ is not significantly different across

[4] This is just a slight generalization of a textbook equation familiar to many readers. For example, in many asset pricing models, the expression for the risk free interest rate is written as (see, e.g. equation (A26) in Bansal and Yaron (2004), equation (8) in Campbell and Cochrane (1999), or Chapter 14 in Ljungqvist and Sargent (2012))

$$\ln R^{rf} = -\ln\beta + \frac{1}{IES}\ln\Gamma + \text{terms that depend on risk.}$$

Take exponents of both sides of this equation, drop the risk terms, and use the fact that $q = 1/R^{rf}$ to obtain equation (23).

maturities, which suggests that $\mathbf{a}_T$ should be close to $\mathbf{0}$, and that government bonds are fairly good substitutes.

Expressions $\Sigma_T^{-1}\Sigma_T^X$ and $\Sigma_T^{-1}\Sigma_T^A$ are essentially a sophisticated way of taking ratios of different matrices. We crudely evaluate the magnitude of these ratios by computing in our sample

$$\frac{\text{average}\left\{-cov\left(\frac{X_{T+k}}{Y_{T+k}}, r_{T+1}^{(j)}\right)\right\}_{k,(j)}}{\text{average}\left\{cov\left(r_{T+1}^{(k)}, r_{T+1}^{(j)}\right)\right\}_{(k),(j)}} = -0.008, \quad \frac{\text{average}\left\{cov\left(A_{T+k}, r_{T+1}^{(j)}\right)\right\}_{k,(j)}}{\text{average}\left\{cov\left(r_{T+1}^{(k)}, r_{T+1}^{(j)}\right)\right\}_{(k),(j)}} = 0.013, \tag{26}$$

where averages are taken over different k and j. In these computations, $\frac{X_{T+k}}{Y_{T+k}}$ is simply a primary deficit to GDP ratio, and $A_{T+k} = \frac{1-\rho_a^k}{1-\rho_a} a_T^{(0)}$, where $a_T^{(0)}$ is computed from (21) and ρ_a is autocorrelation of $a_T^{(0)}$ taking from Table 2. Plugging these numbers, we obtain a back of the envelope expression for the target portfolio

$$\boldsymbol{\omega}_T^* = [0.01 - 0.002 + 0.001]\,\mathbf{w} = [0.01 - 0.001]\,\mathbf{w}.$$

These calculations suggest that 99% of the composition of the target portfolio is determined by hedging interest rate risk and only 1% on hedging other risks. Moreover, since $\mathbf{w}$ itself takes a very simply form, this implies that the target portfolio $\boldsymbol{\omega}_T^*$ in a stationary economy should put weight β^i on bonds of maturity i.

The advantage of the back of the envelope exercise is that it makes it very transparent the quantitative effect of some key salient features of the data. Table 2 shows that returns are an order of magnitude more volatile than primary deficits or liquidity premia. Thus, out of the three main risks for the government – interest rate, primary deficits, and liquidity – the interest rate risk is the most important one. Moreover, liquidity premium and primary surplus covary negatively in the data, which implies that target portfolio hedges almost exclusively interest rate risk. While returns on government debts are lower than returns on equivalent corporate bonds, these "liquidy premia" does not vary significantly across maturities, so that gains from exploiting the "excess liquidity curve" are small.

Our crude calculations abstracted from a number of important details emphasized by formula (20): we computed unconditional rather than conditional moments and ignored cross-correlations in equation (26); we did not take out fluctuations in tax rates necessary for proper computation of Σ_T^X; and we ignored covariances that appear in the second line of equation (25). While these details will play some role in calculations of the target portfolio, the stark implications that emerge from our back of the envelope exercise will change only if properly adjusted series look significantly different from the raw ones that we used. As we show next, we find no evidence for that.

4.2.1 Estimations of risks and excess liquidity premia

It is easiest to estimate $\Sigma_T^{-1}\Sigma_T^X$, $\Sigma_T^{-1}\Sigma_T^A$, $\mathbf{a}_T$ assuming that all variables are homoscedastic, and we focus on this case first. We then extend our analysis to estimating time-varying volatility.

Excess liquidity curve. The excess liquidity curve is obtained from equation (22). The first term on the right hand side of (22) can be measured directly from prices of the treasury bonds and high quality corporate bonds. To measure the second term, we follow Fama and MacBeth (1973) and posit a linear factor model for the inter temporal marginal rate of substitution as

$$\frac{\beta M_{t+1}}{M_t} = c_0 + \boldsymbol{c}_1' F_{t+1},$$

where F_{t+1} is m dimensional vector of factors. Denote the vector of excess holding period returns on the high quality corporate bonds as $\boldsymbol{r}_{t+1}^{AAA} \equiv \boldsymbol{R}_{T+1}^{AAA} - \mathbf{1}.R_{T+1}^{(0),AAA}$. The coefficients c_0 and c_1 are obtained by minimizing the pricing errors. In particular,

$$\min_{c_0,\boldsymbol{c}_1'} \|\mathbb{E}\left[c_0 + \boldsymbol{c}_1' F_{t+1}\right] \boldsymbol{r}_{t+1}^{AAA}\|^2.$$

It is easy to show that minimization leads to the following closed form solutions for the coefficients of interest with

$$c_0 = \left(\mathbb{E}R_{T+1}^{(0),AAA}\right)^{-1},$$

$$\boldsymbol{c}_1' = -c_0 \mathbb{E}\left(\boldsymbol{r}_{t+1}^{AAA}\right)^{\intercal} \left(\mathbb{E}\left[\boldsymbol{r}_{t+1}^{AAA}\right] F_{t+1}^{\intercal}\right) \left(\left[\mathbb{E}F_{t+1}\left(\left[\boldsymbol{r}_{t+1}^{AAA}\right]\right)^{\intercal}\right] \left[\mathbb{E}\left[\boldsymbol{r}_{t+1}^{AAA}\right] F_{t+1}^{\intercal}\right]\right)^{-1}.$$

To implement the above formulas, we need to take a stand on the factors F_{t+1}. There is a large literature on factor selection (see Jagannathan, Skoulakis, and Wang (2010) and Ludvigson and Ng (2009)). Motivated by this literature, we set the factors to the principal components extracted from $\boldsymbol{r}_{t+1}^{AAA}$. For our purposes, the first two principal components suffice as they capture almost all of the variation in returns and produce small pricing errors. We obtain the value of the intercept $c_0 = 1.01$ and two factor loadings c_1 are $[0.18, -0.01]$. We find that the values a^j vary little by maturity and equal to 0.3% per quarter.

Primary deficit and liquidity risks. We now turn to estimating $\Sigma_T^{-1}\Sigma_T^X$ and $\Sigma_T^{-1}\Sigma_T^A$. Matrix Σ_T^X contains covariances of $Z_{T+t} \equiv \frac{X_{T+t}^{\perp}}{\mathbb{E}_T Y_{T+t}}$ with returns $r_{T+1}^{(j)}$. In the stationary economy,

$$Z_{T+t} = \frac{(1+\Gamma)^{-t} X_{T+t}}{Y_T} + \tau_{T+t}\left(1 - \gamma\frac{\tau_T}{1-\tau_T}\right).$$

Thus, Z_t is a variable which fluctuations depends on detrended primary deficits $(1+\Gamma)^{-t} X_{T+t}$ and tax rates τ_{T+t}.

One approach to calculating $\Sigma_T^{-1}\Sigma_T^X$ and $\Sigma_T^{-1}\Sigma_T^A$ is to estimate the covariance matrices Σ_T, Σ_T^X and Σ_T^A separately and then to take the inverse of Σ_T. However, it is well-known that this approach is problematic. Returns are highly correlated in the data and taking inverses of their estimated covariance matrices is prone to large errors (see, e.g., Jagannathan and Ma (2003), DeMiguel, Garlappi, and Uppal (2007), Senneret, Malevergne, Abry, Perrin, and Jaffres (2016) for discussion). To overcome this problem, we follow the approach developed by Jagannathan and Ma (2003) and estimate a factor model of returns. For our baseline specification, we use a simple one factor specification

$$\mathbf{X}_t = \boldsymbol{\alpha} + \boldsymbol{\rho}\mathbf{X}_{t-1} + \boldsymbol{\delta} f_t + \boldsymbol{\varepsilon}_t, \tag{27}$$

where X_t is a stacked vector that consists of returns $r_t^{(j)}$ for different maturities j as well as $a_t^{(0)}$ and Z_t, f_t is the factor, $\boldsymbol{\varepsilon}_t$ is a vector of errors. Vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$ and a diagonal matrix $\boldsymbol{\rho}$ are the coefficients that we want to estimate. We use $\delta_{(i)}$, δ_Z, δ_a to correspond to the elements in $\boldsymbol{\delta}$ corresponding to the i-th maturity, Z and a. Analogous conventions apply to $\boldsymbol{\rho}$ and $\boldsymbol{\delta}_{(r)}$ denote a part of vector $\boldsymbol{\delta}$ that corresponds to only returns data.

The factor model (27) provides a simple way to express covariances that appear in formula (20). Since our baseline specification is homoscedastic, matrices Σ_T, Σ_T^X, and Σ_T^A are independent of T and can be written as

$$\begin{aligned}
\Sigma[i,j] &= cov\left(r_{T+1}^{(i)}, r_{T+1}^{(j)}\right) = \delta_{(i)}\delta_{(j)} var\left(f_t\right) + \mathbb{I}_{\{i=j\}} var\left(r_t^{(i)}\right), \\
\Sigma^X[j,t] &= cov\left(Z_{T+t}, r_{T+1}^{(j)}\right) = \rho_Z^k \delta_Z \delta_{(j)} var\left(f_t\right), \\
\Sigma^A[j,t] &= cov\left(A_{T+1}^t, r_{T+1}^j\right) = \delta_a \delta_{(j)} \frac{1-\rho_a^k}{1-\rho_a} var\left(f_t\right).
\end{aligned}$$

If Δ is a diagonal matrix with elements $\left\{var\left(r_t^{(i)}\right)\right\}_t$ then the inverse Σ^{-1} can be written as

$$\Sigma^{-1} = \Delta^{-1} - \frac{\left(\Delta^{-1}\boldsymbol{\delta}_{(r)}\right)\cdot\left(\Delta^{-1}\boldsymbol{\delta}_{(r)}\right)^{\intercal}}{\left[var\left(f_t\right)\right]^{-1} + \boldsymbol{\delta}_{(r)}^{\intercal}\Delta^{-1}\boldsymbol{\delta}_{(r)}}. \tag{28}$$

We set f_t to be the first principal component extracted out of all the observed excess returns, and estimate (27) using ordinary least squares using the set of bond maturities that we have data on. We then extrapolate our estimates for other maturities, use that to construct Σ^{-1} using equation (28), and compute $\Sigma^{-1}\Sigma^X$ and $\Sigma^{-1}\Sigma^A$.

Target portfolio. We use constructed $\mathbf{a}$, $\Sigma^{-1}\Sigma^X$, $\Sigma^{-1}\Sigma^A$ and $\Sigma^{-1}\mathbf{a}$ to calculate the target portfolio using formula (24). We also plot for comparison the portfolio that hedges only interest rate risk, $(1-q\Gamma)\,\mathbf{w} = \left[\beta\ \beta^2\ ...\right]^\intercal$. As one can see, the target portfolio $\boldsymbol{\omega}_T^*$ is very similar to this simple portfolio, consistent with our back of the envelope calculations.

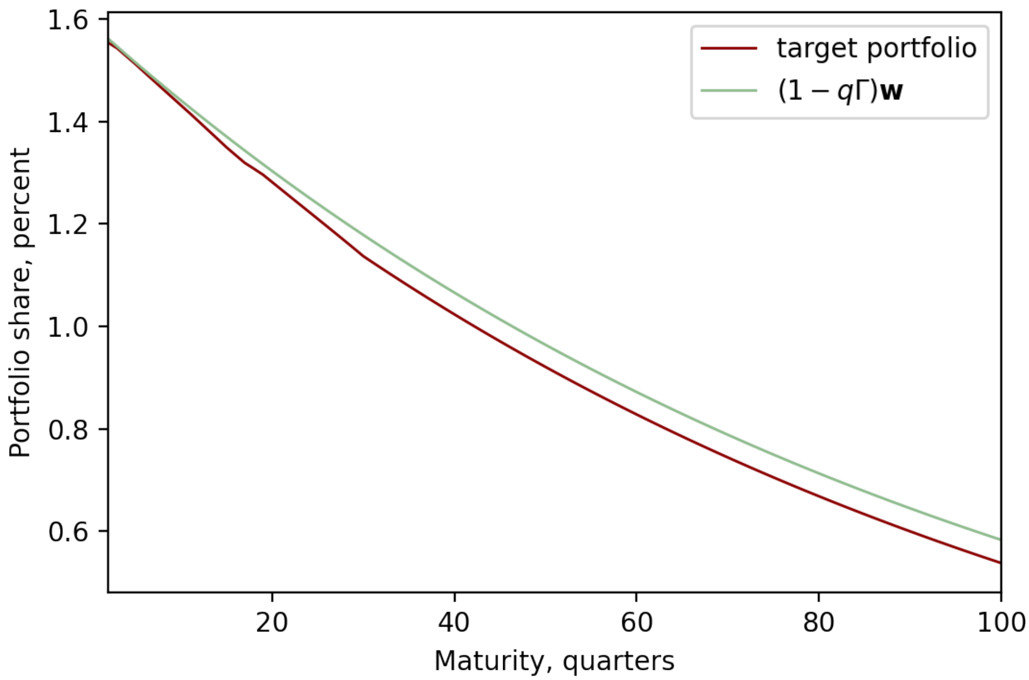

Figure 1: Portfolio shares of securities with maturities from 2 quarters to 121 quarters. The red line plots the target portfolio and the green line plots the portfolio that hedges interest rate risk.

Our conclusions remain unchanged if we extend model (27) to allow for time variation in the first moment $\boldsymbol{\alpha}_t$ or second moments, that is, the variance of the innovation $\boldsymbol{\varepsilon}_t$.

4.3 Comparison with the U.S. portfolio and normative implications

It is instructive to compare the target portfolio we computed above to the current U.S. portfolio of government bonds. We use CRSP to get the amount outstanding and Macaulay duration for all federally issued (marketable) debt. For a few bonds where the duration is absent, we set duration equal to maturity date minus current quotation date. Then for each date, we split the outstanding debt in bins indexed by maturities (at quarterly intervals). In Figure 2, we overlay the time-averaged US debt profile with the target portfolio profile. The CRSP database does not have outstanding amounts for bills. To address this, we supplement the

CRSP data with data from Monthly Statements of Public Debt issued by the US Treasury and fill in the amount outstanding in bills.

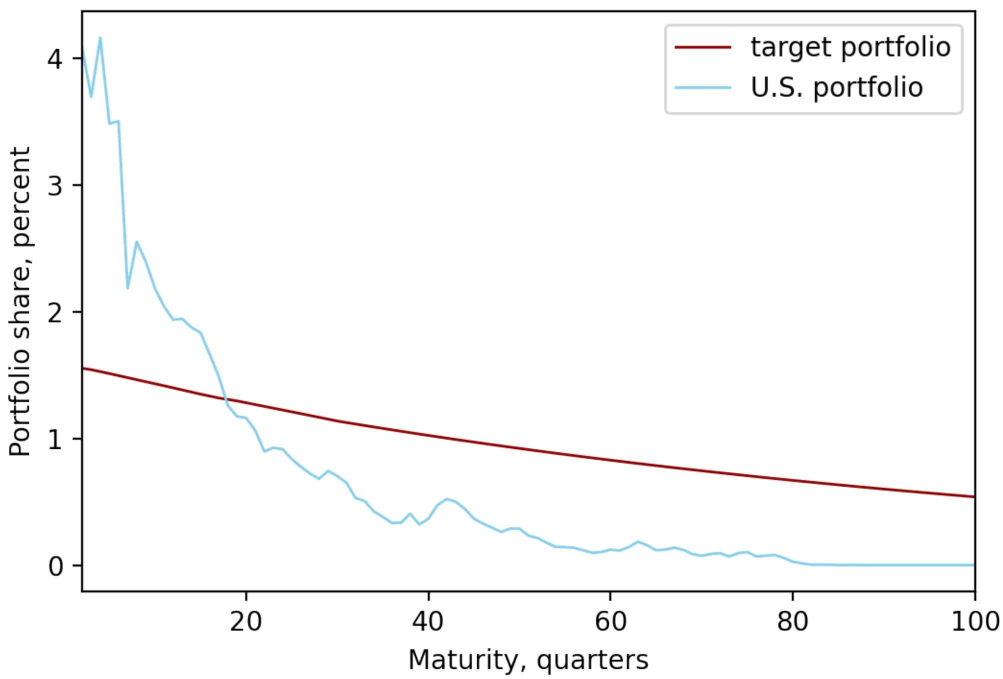

Figure 2: Portfolio shares of securities with maturities from 2 quarters to 121 quarters. The red line is the optimal portfolio and the blue line is the time-averaged US debt profile.

The U.S. debt profile starts above the optimal and curves cross each other at around 25 quarters. We find that the US overweights short maturities relative to the optimal. In terms of Macaulay duration, the optimal portfolio has a duration of about 13 years which is much larger than the range 5 years that we found for the U.S. debt profiles.

We now turn to the normative implications of this finding. Recall that we introduced the target portfolio in Theorem 4, where we showed that the optimal portfolio $\boldsymbol{\omega}_T$ should be equal to the target one, $\boldsymbol{\omega}_T^*$. Importantly, the moments $\Sigma^{-1}\Sigma^X$, $\Sigma^{-1}\Sigma^A$ and $\Sigma^{-1}\mathbf{a}$ that appear in that formula are evaluated at the optimum equilibrium. Our empirical approach estimated $\Sigma_T^{-1}\Sigma_T^X$, $\Sigma_T^{-1}\Sigma_T^A$ and $\mathbf{a}_T$ in the U.S. data, and therefore the target portfolio computed above, which we refer to as $\boldsymbol{\omega}_T^{data}$, is evaluated at the equilibrium corresponding to current policies. Since U.S. portfolio differs from $\boldsymbol{\omega}_T^{data}$, this suggests, at the very least, the the current U.S. portfolio is suboptimal.

More can be said about how portfolio $\boldsymbol{\omega}_T^{data}$ can be adjusted to improve welfare. Firstly, if

$\Sigma_T^{-1}\Sigma_T^X$, $\Sigma_T^{-1}\Sigma_T^A$ and $\mathbf{a}_T$ are approximately policy invariant then choosing portfolio $\boldsymbol{\omega}_T^{data}$ rather than current portfolio $\boldsymbol{\omega}_T$ will improve welfare. Although the condition that these covariances are policy invariance may appear stringent a-priori, it would holds in many asset pricing models where a combination of specifically chosen exogenous shocks and preferences serve as a device to match asset pricing moments. In such models, changes in taxes that are induced by switching from portfolio $\boldsymbol{\omega}_T$ to portfolio $\boldsymbol{\omega}_T^{data}$ do not materially affect stochastic properties of asset returns and risk premia. We verified this point in our paper Bhandari, Evans, Golosov, and Sargent (2017a), where we calibrated the bond pricing model of Albuquerque, Eichenbaum, Luo, and Rebelo (2016) under a stylized U.S. tax and portfolio policy, solved numerically the optimal portfolio, and showed that $\boldsymbol{\omega}_T^{data}$ in the calibrated equilibrium and the target portfolio $\boldsymbol{\omega}_T^{*}$ in the optimum are essentially the same.

Secondly, even without taking any stance on how $\Sigma_T^{-1}\Sigma_T^X$, $\Sigma_T^{-1}\Sigma_T^A$ and $\mathbf{a}_T$ are related to government policies, the target portfolio $\boldsymbol{\omega}_T^{data}$ computed above provides are useful information about a direction in which public portfolio should be reformed.

Theorem 5. *Consider a stationary competitive equilibrium in which government portfolio consists of pure discount bonds of various maturities. Let $\boldsymbol{\omega}_T$ be the government portfolio and $\boldsymbol{\omega}_T^{data}$ be the value of target portfolio evaluated at that competitive equilibrium values. Then portfolio $\boldsymbol{\omega}_T + \varepsilon\left(\boldsymbol{\omega}_T^{data} - \boldsymbol{\omega}_T\right)$ is welfare improving for sufficiently small $\varepsilon > 0$.*

This theorem shows that a small perturbation of the current portfolio $\boldsymbol{\omega}_T$ in the direction of the portfolio $\boldsymbol{\omega}_T^{data}$ (hence, the reason for us referring to it as a "target portfolio") is always welfare improving. The proof is essentially identical to the proofs of Theorems 3 and 4, except we do not impose that $\partial_\epsilon V_0 = 0$ but rather back out the implied value of $\partial_\epsilon V_0$ from this perturbation.

5 Optimal portfolios with price impacts

In the previous section, we focused on the formation of the optimal public portfolio in a small open economy, when asset prices do not respond to changes in the composition of that portfolio. In this section, we drop this assumption. Our approach will follow the same steps as in the previous section: we consider a portfolio swap of a one period government bond for another security in some history s^T, and subsequent smoothing of realized excess returns over future periods. In most applications we consider, the other security is a government bond of longer maturity, and so the portfolio swap closely resembles Quantitative Easing (QE) operations practices by several central banks around the world.

Without further restrictions, equilibrium prices $\left\{q_T^i\right\}_i$ at any history s^T are essentially an arbitrary function of government policies $\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)$ and its exact form depends on specification of the demands by foreign investors $\left\{\mathbf{D}^i\right\}_i$. In principle, such functions $\left\{\mathbf{q}^i\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)\right\}_i$ can be estimated empirically, in which case one can compute price responses $\left\{\partial_\epsilon \mathbf{q}_T^i\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)\right\}_i$ directly for any perturbation ϵ, and use this responses in the optimality condition (6) to find the target portfolio. In practice, data limitations make this approach impractical and additional structure must be imposed on the form of $\left\{\mathbf{q}_T^i\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)\right\}_i$.

In this section, we explore two models of asset price determination. First, we assume that asset prices in period T are only a (potentially stochastic) function of asset supply in period T. Since supply of assets available to households and foreign investors are proportional to government's asset holdings $\left\{B_T^i\right\}_i$, the asset prices can then be represented as $\left\{q_T^i\left(\left\{B_T^i\right\}_i\right)\right\}_i$. This is a popular specification used both in theoretical work and empirical work. See for instance the literature on non-Walrasian markets for securities as in Kyle (1985), Duffie, Garleanu, and Pedersen (2005, 2007), Lagos and Rocheteau (2009), and Bigio, Nuno, and Passadore (2019); segmented markets such as Greenwood and Vayanos (2014), Vayanos and Vila (2021), Gopinath and Stein (Forthcoming) with empirical applications as in Hamilton and Wu (2012), Allen, Kastl, and Wittwer (2020), and Gagnon, Raskin, Remache, and Sack (2011); and special cases of factor demand models for securities such as Koijen and Yogo (2019). We broadly refer to such models as "preferred habitat" models of asset prices.

Secondly, we assume that economy is closed, so that demand by foreign investors is always 0, and study the function $\left\{\mathbf{q}^i\left(\boldsymbol{\tau}, \left\{\mathbf{B}^i\right\}_i\right)\right\}_i$ implied by the competitive equilibrium of such economy. This is the approach that has been used in much of the Ramsey models of optimal public portfolios, such as Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004), Faraglia, Marcet, Oikonomou, and Scott (2018). We show that in both cases similar forces shape the optimal public portfolios. The closed economy model, however, implies price responses to QE type perturbations that seem at odds with empirical data.

5.1 Preferred habitat models

The simplest and most transparent model of prices determination is the one where prices of asset i depends only on supply of that asset, $q_T^i\left(B_T^i\right)$. Since our analysis is local, there is no loss from further assuming that $q_T^i\left(B_T^i\right)$ has a convenient log-linear relationship

$$\ln q_t^i = \lambda_{0,t}^i - \lambda_t^i B_t^i, \tag{29}$$

where $\lambda_{0,t}^i, \lambda_t^i$ are some exogenous and potentially stochastic parameters. One limitation of this specification as that it does not allow cross-price effects, e.g. the possibility that prices of

10 year bonds change if the government issues additional quantity of 11 year bonds. A more general version of this equation can be written as

$$\ln \mathbf{q}_t = \boldsymbol{\lambda}_{0,t} - \Lambda_t \mathbf{B}_t, \tag{30}$$

where $\boldsymbol{\lambda}_{0,t}$ and Λ_t are a vector and a matrix of coefficients, and $\mathbf{q}_t$ is a vector of asset prices and $\mathbf{B}_t$ is a vector of government holdings of securities. Let $\lambda_t^{ij} = \Lambda_t [i, j]$ be elements of Λ_t. For concreteness, in the discussion we will treat matrix Λ_t as being non-negative, although none of our results require that.

In this subsection, we assume that the relationship between prices and quantities of bonds is given by equation (29) or its generalization (30).[5] For now, we impose two more restrictions on function Λ_t : we assume that the row of matrix Λ_t corresponding to a one period government bond B_t^{rf}, is zero, and that the other rows are of the order $O\left(\sigma^2\right)$. While these assumptions are not necessary for our analysis, they substantially streamline exposition. In the end of this subsection, in Section 5.1.3, we explain why this is a natural benchmark to consider and how the optimal portfolio looks like without these restrictions.

To build the intuition for our results, we first consider a simple perturbation when in some state s^T the government issues a small additional quantity of bond j that it them buys it back in the next period. Let $\phi_t^i \equiv b_t^i / B_t^i$ be the fraction of government debt i held by households prior in the original, unperturbed equilibrium. Using equation (30) and the envelope condition (5), one can show that the welfare effect from this perturbation is

$$\begin{aligned}\frac{\partial_\epsilon V_0}{\beta^T \Pr\left(s^T\right) M_T\left(s^T\right)} &= \left\{ \left(\frac{1}{\xi_T} - 1\right) - \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R_{T+1}^j \left(\frac{1}{\xi_{T+1}} - 1\right) \right\} + a_T^j \\ - \sum_{i \geq 1} a_T^i \lambda_T^{ji} \phi_T^i B_T^i &+ \sum_{i \geq 1} \lambda_T^{ji} q_T^i \left[\left(\frac{1}{\xi_T} - \phi_T^i\right) B_T^i - \left(\frac{1}{\xi_T} - \phi_{T-1}^i\right) B_{T-1}^i \right].\end{aligned} \tag{31}$$

The first line of this expression is identical equation (11). The second line captures two additional effects that arise due to price adjustments. The first term on the second line capture direct household utility costs. When the government issues more bonds, their prices fall (provided that Λ_t is non-negative), which decreases the market value of any security i held by household by $\lambda_T^{ji} b_T^i = \lambda_T^{ji} \phi_T^i B_T^i$. The marginal fall in utility from this decrease is proportional

[5] The simplest way to rationalize the downward sloping relationship (29) is to assume that for each security i there is a short-lived foreign investor with (potentially stochastic) demand for that security, and that the demand of domestic households for securities is either sufficiently small or sufficiently inelastic relative to the foreign demand. Greenwood and Vayanos (2014) developed a generalized version of such a model, in which such short lived investors solve a portfolio problem ("arbitrageurs" in their terminology) that in equilibrium gives raise to equation (30).

to the liquidity premium a_T^i, and so the total effect is given by $\sum_{i\geq 1} a_T^i \lambda_T^{ji} \phi_T^i B_T^i$. The term in the second line of (31) captures the effect on government revenues from changes in asset prices. Note that one can write

$$\left(\frac{1}{\xi_T} - \phi_T^i\right) B_T^i = \left[\phi_T^i \left(\frac{1}{\xi_T} - 1\right) + \left(1 - \phi_T^i\right) \frac{1}{\xi_T}\right] B_T^i.$$

Fraction ϕ_T^i of government debt is held by domestic households. For such bonds, a dollar gain from the price impact for the government is a dollar loss for the households. The net welfare effect is not zero, however, since this extra dollar of revenues allows the government to decrease tax rates and lower distortions. Therefore, the welfare effect from the price impact on domestically held bond is proportional to the deadweight loss from taxes, $(1/\xi_T - 1)$. Fraction $\left(1 - \phi_T^i\right)$ of government debt is held by foreign investors. Since the government does not value income in the hand of the foreign investors, the welfare effect from the price impact on bonds held by foreigners is proportional to $1/\xi_T$. Analogous arguments apply to the term $\left(1/\xi_T - \phi_{T-1}^i\right) B_{T-1}^i$.

To study the optimal portfolio, we consider the same perturbation as in Section 4, when the government sells ϵ/q_T^j units of any security j and simultaneously purchases ϵ/q_T^{rf} units of a one period bond, and then in period $T+1$ unwinds this transaction and rolls over the realized excess returns for additional $k \geq 0$ periods. As in Section 4, we focus on the stationary economy in the main text, and present the extension to non-stationary model in the appendix. It would be useful to define a vector $\boldsymbol{\omega}_{T-1}^+$ to have elements $\boldsymbol{\omega}_{T-1}^+[i] = q_T^i B_{T-1}^i / B_T$. Thus, $\boldsymbol{\omega}_{T-1}^+[i]$ is a portfolio share of period $T-1$ holdings of security i at period T prices. We have the following result

Lemma 6. *In the optimal stationary equilibrium, the following condition holds*

$$cov_T\left(\ln \xi_{T+t}, \mathbf{r}_{T+1}\right) + cov_T\left(A_{T+1}^{t-1}, \mathbf{r}_{T+1}\right) \simeq -\frac{1}{q}\mathbf{a}_T - \xi_T B_T \Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^+\right), \qquad (32)$$

where Φ_T is a diagonal matrix with elements $\Phi_T[i,i] = 1/\xi_T - \phi_T^i$ and $\tilde{\Lambda}_T$ is a matrix with elements $\tilde{\Lambda}_T = Y_T\left(\Lambda_T[i,j]\, q_t^i/q_t^j - \Lambda_T[i,rf]\, q_t^i/q_t^{rf}\right)$

Equation (32) is the analogue of optimality condition (16) written in the matrix form. This equation shows that price responses add one extra term that is proportional to $\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^+$. To understanding this term, observe that $\boldsymbol{\omega}_T[i] - \boldsymbol{\omega}_{T-1}^+[i] = \left(B_T^i - B_{T-1}^i\right) q_T^i / B_T$, that it captures the effect of rebalancing of public portfolio in period T. If government does not change holdings of its security i in period T, change in the price of that security does not redistribute resources between government and other agent. The redistribution happens only if the government buys or sells additional securities, which is the effect proportional to $\left(B_T^i - B_{T-1}^i\right)$. The rebalancing term is adjusted by a matrix of price responses $\tilde{\Lambda}_T$ and Φ_T.

Using this result, we can characterize the optimal public portfolio in this economy

Theorem 7. *In the stationary economy with the price effects considered in this section, the optimal portfolio satisfies*

$$\Sigma_T \boldsymbol{\omega}_T \simeq \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A\right] \mathbf{w} + \pi_T^a \mathbf{a}_T + \pi_T^q \Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^+\right), \qquad (33)$$

where $\pi_T^q = q^{-2}\zeta_T \xi_T q\Gamma / (1 - q\Gamma)$.

The formula for the optimal portfolio derived in this theorem is the same as the analogous formula for the small open economy, equation (19), except it has an additional term that is proportional to the amount of rebalancing, $\left(\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^+\right)$. Stronger price effects, captured by matrix $\tilde{\Lambda}_T$, implies higher costs of rebalancing. We now turn to the portfolio of debts to further characterize the implications.

5.1.1 Optimal portfolios of public debts

We now apply this approach to reevaluate the optimal portfolio of public debts that we studied in Section 4.1. It will be more convenient to change the convention about how securities are arranged in vector ω_T and ω_{T-1}. In the previous section, we indexed all securities by i in period T, and maintained the same indices both in vector ω_T and ω_{T-1}. This approach introduces many redundancies when we apply it to study bonds of different maturities, since it treats a 3 period bond issued in $T-2$ and a 4 period bond issued in $T-3$ as different securities. It is natural to assume that those securities are perfect substitutes for both domestic households and foreign investors and thus treat them as one. In this case, the only relevant variable in the total number of securities that mature in any period $T+t$, but not the period in which they were issued.

For the purposes of this section, let an element $\omega_t[i]$ be the period t portfolio share of securities that mature in period $t+1+i$. In period $t+1$, the index of these securities declines by 1. Define vector $\boldsymbol{\omega}_{T-1}^{\#}$ by $\boldsymbol{\omega}_{T-1}^{\#}[i] = q_T^{(i)} B_{T-1}^{(i+1)} / B_T$, which is the analogue of $\boldsymbol{\omega}_{T-1}^+$. We have the following result

Corollary 8. *The optimal portfolio of public debts in a stationary economy satisfies*

$$\boldsymbol{\omega}_T - \boldsymbol{\omega}_T^* \approx \pi_T^q \Sigma_T^{-1} \Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^{\#}\right). \qquad (34)$$

In particular, in stationary economy if $\boldsymbol{\omega}_{T-1}^* \approx (1 - q\Gamma)\mathbf{w}$ *and* $R_T^{rf} = 1/q$ *then* $\boldsymbol{\omega}_{T-1}^{\#} \approx (1 - q\Gamma)\mathbf{w}$ *and, therefore,*

$$\boldsymbol{\omega}_T \approx \boldsymbol{\omega}_T^*.$$

The first part of this corollary simply combines insights of Theorems 4 and 7 and adjusts the result for the differences in indexing conventions. Equation (34) gives the law of motion for the optimal portfolio. In each period the government would like to set its portfolio $\boldsymbol{\omega}_T$ to the target portfolio $\boldsymbol{\omega}_T^*$ that we characterized in the previous section. However, portfolio rebalancing $\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^{\#}$ is costly due to price changes captured by $\tilde{\Lambda}_T$. Therefore, the optimal portfolio $\boldsymbol{\omega}_T$ is a weighted average between the target portfolio $\boldsymbol{\omega}_T^*$ and the value of the portfolio inherited from the previous period, $\boldsymbol{\omega}_{T-1}^{\#}$.

We showed in Section 4.2 that in the U.S. data, target portfolio $\boldsymbol{\omega}_T^*$ is approximately equal to $(1 - q\Gamma)\,\mathbf{w}$. The second part of Corollary 8 shows that in this case the distinction between target and optimal portfolio disappears and the two portfolios coincide. The intuition for this result is simple. The portfolio $(1 - q\Gamma)\,\mathbf{w}$ is such that it minimizes the need for rebalancing, and thus it remains optimal for any adjustment costs.

5.1.2 Greenwood-Vayanos price effects

In this section we consider the implications of price effects implied by the work of Greenwood and Vayanos (2014). Those authors developed a framework to account for observed price responses to changes in the composition of government debt and estimated their key parameters using U.S. data. In their model, government debt is prices by "arbitrageurs" that are equivalent to our foreign investors. Those arbitrageurs are the marginal investors who purchase debts of different maturities to maximize their mean-variance utility. "Other investors" can hold some of government debt but trading frictions prevent them from pricing debt on the margin. Those are equivalent to our households with trading frictions causing non-zero liquidity wedge.

Greenwood and Vayanos show that in their model price of each debt is a function of the overall duration of government portfolio, and in the empirical work they consider price functions of the form

$$\ln q_T^{(t+1)} = \lambda_{0,T}^{GV} + \lambda_T^{GV(t+1)} \left(\sum_{k=0}^{\infty} k B_T^{(k)} \right) \tag{35}$$

where the expression in the brackets is the duration of outstanding portfolio of government bonds.

It is easy to verify that in their model $\lambda_T^{GV,(t)}$ satisfy the restrictions we impose in section 5.1, that is, $\lambda_T^{GV,(t)}$ are of the order $O\left(\sigma^2\right)$ and the shortest maturity is price insensitive, corresponding to $\lambda_T^{GV,(0)} = 0$ in our discrete time model. Our analysis from section 5.1 extends with minimal changes to these settings. Consider the portfolio swap we discussed above, whereas the government buys ϵ units of bond of maturity $t+1$ and decreases holding of a one period bond by $-q_T^{(1+t)}/q_T^{(0)}\epsilon$. Verify that this increases duration, by

$$\partial_\epsilon D_T = \left[\frac{1+t}{q_T^{(t)}} - \frac{1}{q_T^{(0)}}\right] q_T^{(t)}.$$

and hence prices of each security change by

$$\tilde{\Lambda}_T\left[i,j\right] = q_T^{(i)}\left(\frac{1+j}{q_T^{(j)}} - \frac{1}{q_T^{(0)}}\right)\lambda_T^{(j)} Y_T.$$

We use estimates of a set of regressions of yields on duration of public debt that are reported in Greenwood and Vayanos (2014) to measure $\tilde{\Lambda}_T\left[i,j\right]$. More specifically, they regress

$$\ln \text{yield}_t^n = a^n + b^n D_t - c^n \ln \text{yield}_t^1 + \text{noise}$$

where D_t is the duration of public debt normalized by GDP. Using the equation (35), we get that $Y_T\lambda_T^{(n)} = -n \times b^n$. While point estimates of b^n vary across maturities, they are not that different statistically. We set all b^ns equal to 0.003, which is the mean across all maturities that they report.

Expression (33) can be expressed as a law of motion for $\boldsymbol{\omega}_T$

$$\boldsymbol{\omega}_T \simeq \boldsymbol{\omega}_T^{no,pe} + \pi_T^q \Sigma_T^{-1} \Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - L^+ \omega_{T-1}\right) \quad (36)$$

where $\boldsymbol{\omega}_T^{no,pe}$ is the portfolio when there are no price effects and the matrix L^+ is given by $\begin{bmatrix} 0 & \frac{1}{\Gamma q} & 0 & \dots \\ 0 & 0 & \frac{1}{\Gamma q} & \dots \\ .. & \dots & \dots & \dots \end{bmatrix}$. This gives us a mapping from an arbitrary portfolio at $T-1$ to the portfolio at T.

To describe how the price effects affect the optimal portfolio, we compute a "steady-state" of equation (36). The optimal steady state portfolio satisfies the previous equation when $\boldsymbol{\omega}_T^{no,pe}$, $\left\{\pi_T^q, \Phi_T, \tilde{\Lambda}_T\right\}$and Σ_T are independent of T. We set $\boldsymbol{\omega}_{T+k}^{no,pe} = \boldsymbol{\omega}_T^{no,pe}$ that we computed in section (4.2). Under the factor model $\Sigma_{T+k} = \Sigma$ when shocks are homoskedastic. We then assume that tax rates are stationary and $\zeta_{T+k} = \zeta_T$ and $\xi_{T+k} = \xi_T$. We also need to take a stand on how the debt is split between domestic and foreign holders to estimate the matrix Φ_T. We assume that the ratio domestically held debt is 70% of the total debt and this fraction is constant for all maturities and dates. Under these assumptions,

$$\omega^{pe} = \left(I - \pi^q \Sigma^{-1} \Phi \tilde{\Lambda}\right)^{-1} \left[\omega^{no,pe}\right]$$

In figure 3, we plot ω^{pe} and compare it to the no price effects formula $\omega^{no,pe}$. We find that the two are very similar to each other. This follows from proposition 8 and our earlier finding that the portfolio without price effects mirrors the portfolio that hedges the interest rate risk.

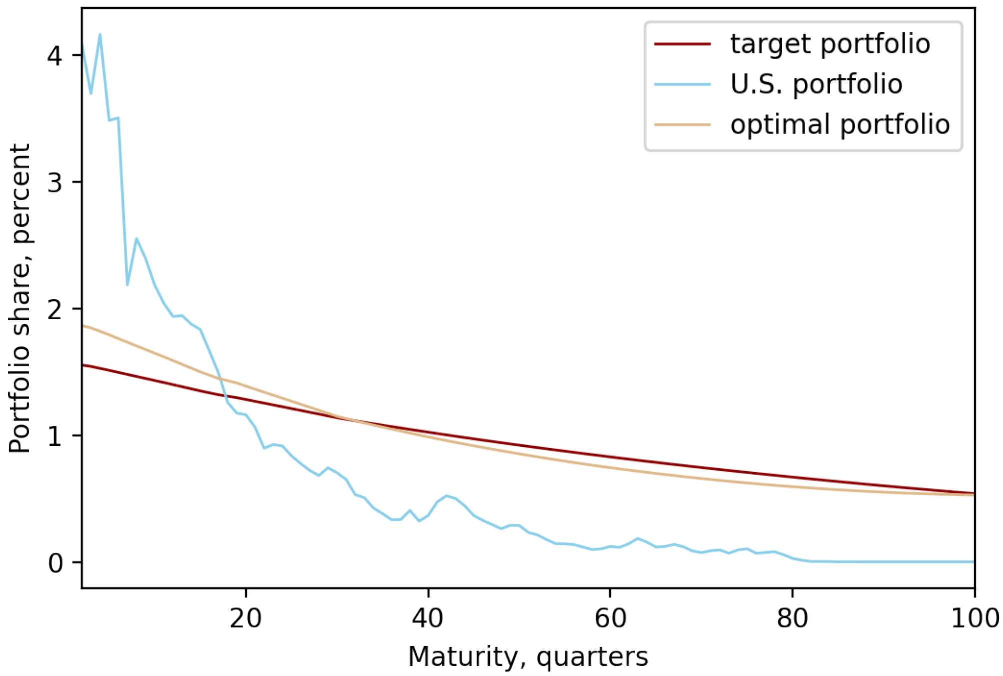

Figure 3: Portfolio shares of securities with maturities from 2 quarters to 121 quarters. The red line is the optimal portfolio without price effects, the orangel line is the optimal portfolio with price effects, and the blue line is the time-averaged US debt profile.

5.1.3 Further discussions

In Section 5.1 we made two simplifying assumptions about matrix Λ_t. The first assumption was that prices of the one period government bond do not depend on the perturbation we considered. This assumption was motivated in part by the empirical findings by Krishnamurthy and Vissing-Jorgensen (2012) and Greenwood and Vayanos (2014) that prices of short bonds appeared to be largely unaffected by QE programs in the U.S. None of our derivations required this assumption. Consider, for example, how our results would change in Section 5.1.1 if we drop this assumption. Let $\lambda_T^{(0),(i)} = \partial \ln q_T^{(0)} / \partial B_T^{(i)}$ be the price semi-elasticity of a one period bond with respect to changes in supply of $i+1$ period bonds, let $\varphi_T \equiv \left(1/\xi_T - \phi_T^{(0)}\right)$ and let $\tilde{\boldsymbol{\lambda}}_T$ be a vector with elements $\tilde{\boldsymbol{\lambda}}_T[j] = \lambda_T^{(0),(j)} q_T^{(0)} / q_T^{(j)} - \lambda_T^{(0),(0)}$. It is easy to verify that equation (34) becomes

$$\boldsymbol{\omega}_T - \boldsymbol{\omega}_T^* \approx \pi_T^q \left\{ \Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - \boldsymbol{\omega}_{T-1}^{\#} \right) + \varphi_T \tilde{\boldsymbol{\lambda}}_T \left(\omega_T^{(0)} - \omega_{T-1}^{(1),\#} \right) \right\},$$

where $\omega_T^{(0)} = 1 - \mathbf{1}'\boldsymbol{\omega}_T$. While this expression is slightly more complicated than equation (34), it convey the same message that rebalacing, this time also proportional to the rebalancing of a one period bond, $\omega_T^{(0)} - \omega_{T-1}^{(1),\#}$, determines how closely the portfolio should be adjusted to the target portfolio. Moreover, the second part of Corollary 8 continues to hold.

The second assumption we used is that all coefficients of matrix Λ_t are of the order $O\left(\sigma^2\right)$. We did it for two reasons. If price impact is of an order lower than σ^2 then the composition of the optimal public portfolio is determined, to the first order, exclusively by price responses.[6] More importantly, many commonly used models of asset price determination imply the price impact from QE-style asset swap should be zero to the zeroth and first order approximations. For example, this is true both in Greenwood and Vayanos (2014) model and in closed economy that we analyze in Section 5.2. The intution for this is as follows. To the zeroth order, there is no risk, and to the first order all risk is price by risk-neutral agents, so in both cases the transaction we consider involves swapping to assets with identical expected returns, and all economic agents are indifferent about such swaps.

5.2 Closed economy

In this section, we study price effects in a closed economy version of our model by imposing the resource constraint that every period, the sum of household consumption and public consumption equals total output. Unlike the simple price effects, a swap of securities at a particular history can affect asset prices at all other histories—past and future—due to general equilibrium effects on the stochastic discount factor, which now directly depends on the tax rates. To gain tractability in such a setting, we impose a few assumptions.

We analyze preferences such that the period utility is given by

$$U_t\left(c_{t+1} - \frac{\left(Y_{t+1}/\theta_{t+1}\right)^{1+1/\gamma}}{1+1/\gamma}, \dots\right) = \exp\left(\psi\left[c_t - \frac{\left(Y_{t+1}/\theta_{t+1}\right)^{1+1/\gamma}}{1+1/\gamma}\right]; s^t\right).$$

Thus we abstract from liquidity services and use a CARA aggregator for the period utility. We specialize the market structure such that a "consol" – a security that pays one unit of consumption in all future histories – can be replicated. This is clearly satisfied if the government has access to zero-coupon bonds of all maturities. We use the existence of a consol to make the discussion transparent. In the appendix, we show that all the results continue to hold if the government can only invest in a one period risk-free bond. We use the index $i = \infty$ to

[6] It is easy to study this case by simply omitting all the second order terms – covariances and excess liquidity premium – from equation (33) or (34). This analysis would be closely related to that in Bigio, Nuno, and Passadore (2019), except that those authors further impose an exogenous cap on the maximum bond maturity that the government can issue.

denote the consol. Finally, we assume that tax rates in the optimal allocation are stationary from date $t = 0$.

We study a modified version perturbation that ensures that the government can smooth out the excess returns over time using the consol. At date T, history s^T, the government reduces the market value of securities held in the risk free bond by ϵ and increases the market value of assets held in security j by ϵ. The direct effect of this perturbation generates stochastic return $\left[\frac{\epsilon}{q_T^j}\left(q_{T+1}^j\left(s^{T+1}\right)+d_{T+1}^j\left(s^{T+1}\right)\right)-\frac{\epsilon}{q_T^{rf}}\right]=r_{T+1}^j\epsilon$ in period $T+1$ which has to be returned to the households. Given a consol-replicating portfolio is available, it buys $\frac{r_{T+1}^j}{1+q_{T+1}^\infty}\epsilon$ units of consol in period $T+1$ at price q_{T+1}^∞. This portfolio insures that the direct effect of perturbation on tax revenues is completely smoothed for all $k \geq 1$, $\boldsymbol{s}^{T+k} \succ \boldsymbol{s}^T$.

Asset prices are forward looking and depend on the future path of consumption. Thus our perturbation in the closed economy can change asset prices at all histories even if the direct effect of the perturbation affects consumption only in $\boldsymbol{s}^{T+k} \succ \boldsymbol{s}^T$. The government in turn has to change tax rates at all histories to satisfy the budget constraint. To take into account all the welfare effects arising from price changes, we analyze the perturbation by applying our section 3.2 approximations from the perspective of history s^0. In the appendix, we provide a full description of the the changes in taxes and debts following the perturbation.

To state our main result and connect it to the discussion in section 5.1.1 we introduce a new object: "anticipation" effects, $\mathcal{A}_0$, that keeps track of the new forces that arise in the closed economy. Define $\Gamma_{\epsilon,t}^j\left(s^t\right)$ as the price effects on government portfolio at history s^t as follows

$$\Gamma_{\epsilon,t}^j \equiv B_t \sum_{i\geq 1} \mathbb{E}_0\omega_{t-1}^{\#}[i]\left(\frac{1}{\mathbb{E}_0\xi_t}-\phi_t^i\right)\left(\underbrace{\left[\partial_{\sigma\sigma}\partial_\epsilon^j \ln q_t^i - \partial_{\sigma\sigma}\partial_\epsilon^j \ln q_{t-1}^i - \left(\frac{d_t^i}{q_t^i}\right)\ln \partial_\epsilon^j q_t^i\right]}_{\partial_{\sigma\sigma}\partial_\epsilon^j R_t^i}\right)$$
$$- q_{t-1}^{rf}\mathbb{E}_0 B_{t-1}^{rf}\left(\frac{1}{\mathbb{E}_0\xi_t}-\phi^{rf}\right)\left[\frac{1}{q_{t-1}^{rf}}\right]\partial_{\sigma\sigma}\partial_\epsilon^j \ln q_{t-1}^{rf}.$$

The first line captures the price effects coming from the holdings of risky securities: the product of the portfolio shares $\mathbb{E}_0\omega_{t-1}^{\#}[i]$, adjusted by the deadweight losses $\left(\frac{1}{\mathbb{E}_0\xi_t}-\phi_t^i\right)$ and how the perturbation affects returns at s^t. The second line captures the price effects coming from how the perturbation affects the risk-free rate. The anticipation effects for a swap at history s^T with security j are price effects $\Gamma_{\epsilon,t}^j\left(s^t\right)$ for all histories other than those occurring between s^T

and $s^{T+1} \succ s^{T}$. Let $S \equiv \left\{s^{T}\right\} \cup \left\{s^{T+1} \succ s^{T}\right\}$. We define $\mathcal{A}_0\left(j, s^{T}\right)$ as

$$\mathcal{A}_0\left(j, s^{T}\right) \equiv \frac{\mathbb{E}_0 \sum_t M_t \Gamma^j_{\epsilon,t} I_{\{s^t \notin S\}}}{\Pr_0\left(\boldsymbol{s}^{T}\right) M_T\left(\boldsymbol{s}^{T}\right)}.$$

We use $\mathcal{A}_{0,T}$ without the argument j to denote the column vector of anticipation effects for swaps of all risky securities. The next proposition states the optimal formula in the closed economy.

Theorem 9. *The optimal portfolio in closed economy is given by*

$$\boldsymbol{\omega}_T - \boldsymbol{\omega}^*_T \approx \pi^q_T \Sigma^{-1}_T \left\{\Phi_T \tilde{\Lambda}_T \left(\boldsymbol{\omega}_T - \boldsymbol{\omega}^{\#}_{T-1}\right) + \varphi_T \tilde{\boldsymbol{\lambda}}_T \left(\omega^{(0)}_T - \omega^{(1),\#}_{T-1}\right) - \mathcal{A}_{0,T}\right\}$$

The overall structure of the formula and the insights are similar to section 5.1.1 with two exceptions. First, there is a new term $\mathcal{A}_{0,T}$ that shows up due to the general equilibrium effects. With CARA preferences, a large part of what drives the anticipation effect is the windfall in $t = 0$ from holding long-maturity debt. These forces arise because the government can commit to future tax policies and are well-studied in tax smoothing literature. Since that is not a focus of our paper, we can abstract from those forces if we additionally assume that initial debt is held in risk-free securities.

Theorem 10. *If $B^i_{-1} = b^i_{-1} = 0$ for all $i > 0$, then $\mathcal{A}_{0,T} = 0$.*

The second difference from section 5.1.1 is that closed economy model imposes tight restrictions on the matrix $\tilde{\Lambda}_T$ which are inconsistent with the empirical QE literature. In the closed economy, we can show that

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial^j_\epsilon r^i_{T+1} = \underbrace{2\psi \left(\frac{\bar{\xi} - 1}{\bar{\xi}}\right) \frac{1}{1 + \bar{q}^{\infty}_{T+1}}}_{>0} cov_T \left(r^i_{T+1}, r^j_{T+1}\right) > 0.$$

Thus a QE-type swap results in *higher* excess returns. In the closed economy, aggregate consumption moves roughly proportional to excess returns as the government returns the excess returns from the QE-swap via taxes. This makes the all risky assets worth lower and household demand a higher risk-premia. This is inconsistent with segmented market literature that finds the opposite, that is, excess returns on long maturity debt are *lower* after QE To see this use equation (30). As we discussed in section 5.1.1, the empirical literature on QE finds $\partial^j_\epsilon q^{rf}_T \approx 0 \quad \partial^j_\epsilon q^i_t > 0$, and estimate $\lambda^{i,j}_t > 0$ for $j > 0$ and $\lambda^{i,rf}_t \approx 0$. This gives

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial^j_\epsilon r^i_{T+1} = -\overline{R}^{rf}_{T+1} \lambda^{i,j}_T \frac{q^i_T}{q^j_T} < 0.$$

6 Conclusion

In this paper, we developed a methodology that combines sufficient statistics approach with small noise approximations to study the portfolio problem of a Ramsey planner. We found that the optimal portfolio is remarkably simple and stable over time, roughly replicating a growth-adjusted consol.

This conclusion stands in stark contrast with previous Ramsey models of government portfolios. For example, Angeletos (2002) showed theoretically that in standard neoclassical models the government can replicate complete market allocations by trading pure discount bonds of various maturities. Buera and Nicolini (2004) numerically characterized such portfolios and found that they take an extreme form: debts holdings of similar maturities may differ by thousands percent of GDP and even by sign, and small shock may lead to very large rebalancing.

It is useful to trace the sources of the differences in the conclusions. A small number of shocks drives all fluctuations in macroeconomic variables and financial returns in the standard neoclassical growth model. This implies that returns of various assets and shocks that affect primary deficits are highly correlated in such environments, and one can construct portfolios of pure discount bonds that replicate returns on Arrow-Debreu securities. The volatility of returns in such models is low, so in order to generate a sufficient income flow that hedges shocks to its primary surplus, the government needs to have very large quantities of securities. The net liability of the government can be low because those large debt positions have different signs and offset each other. The government in those models essentially runs a massive leveraged hedge fund, but the small number of shocks in the model implies that government portfolios can be fine-tuned to avoid any unintended risk.

In the data, the relationship between returns on government bonds and macroeconomic variables is quite different. Returns are very volatile and not very correlated with macroeconomic variables. This implies that they are a poor hedge against shocks to government revenues and expenditures. Large, unbalanced portfolios, such as the ones prescribed by the models of Angeletos (2002) and Buera and Nicolini (2004), thereby carry large risks.

Our approach also has implications about appropriate numerical methods needed to solve portfolio problems in Ramsey settings. The sufficient statistics we derive emphasize that second moments – such as conditional covariances – play a central role in determining such portfolios. One popular method of solving macroeconomic models is to discretize shocks by applying Tauchen algorithm. While such algorithm is designed to approximate well first-order moments of stochastic processes, it typically would not capture important second moments well. Alternative numerical methods, such as those based on small noise expansions, would be preferable.

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Online Appendix

Preliminary and Incomplete

7 Appendix: theoretical analysis

7.1 Proofs for section 4

We start with the following useful result

Lemma 11. *In the optimum equilibrium, $\mathbb{E}_T r^j_{T+1} = O\left(\sigma^2\right)$ for all j, T.*

Proof. Consider a perturbation that simultaneously buys security j and sells equal amount of security rf in period T, which is is then unwound in period $T+1$. The effect of these transactions on welfare can be obtained by subtracting (11) for security rf from the equation (11) for security j. In the optimum equilibrium the welfare effect of this transaction should be zero, i.e.

$$\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r^j_{T+1}}{\xi_{T+1}} = 0. \tag{37}$$

To the zeroth order, this equation reads

$$\frac{\beta \bar{M}_{T+1}}{\bar{M}_T} \frac{\bar{r}^j_{T+1}}{\bar{\xi}_{T+1}} = 0. \tag{38}$$

Since function $\frac{\tau}{1-\tau}$ is strictly increasing on $(-\infty, 1)$ interval and has range $(-1, \infty)$, we must have $\bar{\xi}_{T+1} = 1 - \gamma \frac{\bar{\tau}_{T+1}}{1-\bar{\tau}_{T+1}} \in (0, 1+\gamma)$. In the regular equilibrium $\bar{M}_T$ is positive and finite. Therefore, equation (38) implies that $\bar{r}^j_{T+1} = 0$. Therefore, the first order approximation of (37) is

$$\frac{\beta \bar{M}_{T+1}}{\bar{M}_T} \frac{1}{\bar{\xi}_{T+1}} \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0,$$

which implies that $\mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$. This establishes the result of the lemma. □

This lemma directly implies a number of corollaries that we will use extensively throughout our analysis. The proofs of the first two are obvious and they are omitted.

Corollary 12. *In the optimum equilibrium, $a^{rf}_T = a^j_T + O\left(\sigma^2\right)$ for all j, T. Moreover, in the stationary optimum equilibrium, $a^i_T = O\left(\sigma\right)$ for all i, T.*

Corollary 13. *In the optimum equilibrium, for all T, t*

$$\bar{\mathcal{Q}}_{T+1}^{T+t} = \frac{1}{\bar{R}_{T+2}^{rf}} \times ... \times \frac{1}{\bar{R}_{T+t}^{rf}} = \bar{Q}_{T+1}^{T+t}, \quad \mathbb{E}_{T+1}\partial_\sigma \mathcal{Q}_{T+1}^{T+t} = \mathbb{E}_{T+1}\partial_\sigma Q_{T+1}^{T+t} \text{ for } t > 1.$$

Corollary 14. *For any equilibrium variables x_t, z_t', z_t'', in the optimum equilibrium for any T, j we have*

$$\mathbb{E}_T\left[z_{T+1}' z_{T+1}''\right] cov_T\left(x_{T+1}, r_{T+1}^j\right) \simeq \mathbb{E}_T\left[z_{T+1}'\right]\mathbb{E}_T\left[z_{T+1}''\right] cov_T\left(x_{T+1}, r_{T+1}^j\right) \simeq \bar{z}_{T+1}'\bar{z}_{T+1}''\mathbb{E}_T\partial_\sigma x_{T+1}\partial_\sigma r_{T+1}^j.$$

Proof. We have

$$\mathbb{E}_T x_{T+1} r_{T+1}^j \simeq \mathbb{E}_T\left[\partial_\sigma x_{T+1}\partial_\sigma r_{T+1}^j + \frac{1}{2}\bar{x}_{T+1}\partial_{\sigma\sigma} r_{T+1}^j\right], \quad \mathbb{E}_T x_{T+1}\mathbb{E}_T r_{T+1}^j \simeq \mathbb{E}_T\left[\frac{1}{2}\bar{x}_{T+1}\partial_{\sigma\sigma} r_{T+1}^j\right],$$

and, therefore,

$$cov_T\left(x_{T+1}, r_{T+1}^j\right) = \left[\mathbb{E}_T x_{T+1} r_{T+1}^j - \mathbb{E}_T x_{T+1}\mathbb{E}_T r_{T+1}^j\right] \simeq \mathbb{E}_T\partial_\sigma x_{T+1}\partial_\sigma r_{T+1}^j.$$

Since $\overline{cov_T\left(x_{T+1}, r_{T+1}^j\right)} = 0$, we obtain

$$\begin{aligned}
\mathbb{E}_T\left[z_{T+1}' z_{T+1}''\right] cov_T\left(x_{T+1}, r_{T+1}^j\right) &\simeq \bar{z}_{T+1}'\bar{z}_{T+1}''\mathbb{E}_T\partial_\sigma x_{T+1}\partial_\sigma r_{T+1}^j, \\
\mathbb{E}_T\left[z_{T+1}'\right]\mathbb{E}_T\left[z_{T+1}''\right] cov_T\left(x_{T+1}, r_{T+1}^j\right) &\simeq \bar{z}_{T+1}'\bar{z}_{T+1}''\mathbb{E}_T\partial_\sigma x_{T+1}\partial_\sigma r_{T+1}^j.
\end{aligned}$$

□

We are now ready to characterize the optimality conditions.

Lemma 15. *In the optimum equilibrium, for all T, j, k*

$$\mathbb{E}_T\partial_\sigma \ln \xi_{T+1}\partial_\sigma r_{T+1}^j = \frac{\bar{R}_{T+1}^{rf}}{1-\bar{a}_T^{rf}}\frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2},$$

$$\mathbb{E}_T\partial_\sigma \ln \xi_{T+1+k}\partial_\sigma r_{T+1}^j = \frac{\bar{R}_{T+1}^{rf}}{1-\bar{a}_T^{rf}}\frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} + \frac{\bar{R}_{T+1}^{rf}}{1-\bar{a}_T^{rf}}\mathbb{E}_T\partial_\sigma \mathrm{A}_{T+1}^k\partial_\sigma r_{T+1}^j.$$

Proof. The second order expansion of (37) is

$$2\mathbb{E}_T\partial_\sigma\left(\frac{\beta M_{T+1}}{M_T}\right)\partial_\sigma r_{T+1}^j + 2\left(\overline{\frac{\beta M_{T+1}}{M_T}}\right)\frac{1}{\left(\overline{\xi_{T+1}^{-1}}\right)}\mathbb{E}_T\partial_\sigma \xi_{T+1}^{-1}\partial_\sigma r_{T+1}^j + \left(\overline{\frac{\beta M_{T+1}}{M_T}}\right)\mathbb{E}_T\partial_{\sigma\sigma} r_{T+1}^j = 0. \tag{39}$$

Applying these results to expansions of equation (9), we have

$$\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j = 2\mathbb{E}_T\partial_\sigma\left(\frac{\beta M_{T+1}}{M_T}\right)\partial_\sigma r_{T+1}^j + \left(\overline{\frac{\beta M_{T+1}}{M_T}}\right)\mathbb{E}_T\partial_{\sigma\sigma} r_{T+1}^j.$$

Combine with (39) and observe that $\frac{\partial_\sigma \xi_{T+1}^{-1}}{\left(\xi_{T+1}^{-1}\right)} = -\partial_\sigma \ln \xi_{T+1}$ to get

$$\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j = 2\left(\overline{\frac{\beta M_{T+1}}{M_T}}\right)\mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r_{T+1}^j = 2\left(\overline{\frac{1-a_T^{rf}}{R_{T+1}^{rf}}}\right)\mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r_{T+1}^j.$$

This yields the first equation.

When the government rolls over excess returns for additional k periods, the optimality condition reads

$$\mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r_{T+1}^j}{\xi_{T+1}} \left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R_{T+2}^{rf}\right) \times ... \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R_{T+1+k}^{rf}\right)\right] = 0.$$

Relative to equation (39), its second order approximation has an additional term can be written as

$$2\sum_{t=1}^{k} \mathbb{E}_T \mathbb{E}_{T+t} \partial_\sigma \ln\left(\frac{\beta M_{T+1+t}}{M_{T+t}} R_{T+1+t}^{rf}\right) \partial_\sigma r_{T+1}^j = 2\sum_{t=1}^{k} \mathbb{E}_T \partial_\sigma \ln\left(1 - a_{T+t}^{rf}\right) \partial_\sigma r_{T+1}^j = 2\mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^k \partial_\sigma r_{T+1}^j.$$

Therefore

$$\left(\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j\right) + 2\mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^k \partial_\sigma r_{T+1}^j = \left(\overline{\frac{1-a_T^{rf}}{R_{T+1}^{rf}}}\right)\mathbb{E}_T \partial_\sigma \ln \xi_{T+1} \partial_\sigma r_{T+1}^j.$$

Re-arrange to get the second equation. □

Corollary 16. *In the stationary optimum equilibrium, equations (15) and (16) hold.*

Proof. In the stationary optimum equilibrium, $\bar{a}_T^{rf} = 0$ by corollary 12. Combine this with lemma 15 and corollary 14 to show the result. □

Lemma 17. *Equation (18) holds.*

Proof. Lemma 11 and corollary 13 imply that the zeroth order approximation of (17) is

$$\sum_{t=1}^{\infty} \bar{Q}_{T+1}^{T+t} \bar{X}_{T+t} = -\bar{B}_T \bar{R}_{T+1}^{rf}. \tag{40}$$

Multiply equation (17) by r_{T+1}^j and take expectations at time T. The Law of the Iterated Expectations then implies that

$$\mathbb{E}_T \sum_{t=1}^{\infty} \mathcal{Q}_{T+1}^{T+t} X_{T+t} r_{T+1}^j = -\mathbb{E}_T B_T \left[R_{T+1}^{rf} + \sum_{i\geq 1} \omega_T^i r_{T+1}^i\right] r_{T+1}^j.$$

Take the second order expansion of this expressions, note that the terms multiplying $\partial_{\sigma\sigma} r^j_{T+1}$ cancel out due to (40) and that $\mathbb{E}_T \partial_\sigma \mathcal{Q}^{T+t}_{T+1} \partial_\sigma r^j_{T+1} = \mathbb{E}_T \left(\mathbb{E}_{T+1} \partial_\sigma \mathcal{Q}^{T+t}_{T+1} \right) \partial_\sigma r^j_{T+1} = \mathbb{E}_T \partial_\sigma Q^{T+t}_{T+1} \partial_\sigma r^j_{T+1}$ by corollary 13 to obtain

$$\mathbb{E}_T \sum_{t=1}^{\infty} \bar{X}_{T+t} \partial_\sigma Q^{T+t}_{T+1} \partial_\sigma r^j_{T+1} + \mathbb{E}_T \sum_{t=1}^{\infty} \bar{Q}^{T+t}_{T+1} \partial_\sigma X_{T+t} \partial_\sigma r^j_{T+1} = -\bar{B}_T \mathbb{E}_T \left[\sum_{i \geq 1} \bar{\omega}^i_T \partial_\sigma r^i_{T+1} \partial_\sigma r^j_{T+1} \right]. \tag{41}$$

Finally, note that $Q^{T+1}_{T+1} = 1$ and therefore $\partial_\sigma Q^{T+1}_{T+1} = 0$. This, together with corollary 14, establishes equation (18). □

Lemma 18. *Lemma 2 holds.*

Proof. Direct calculations show that $\partial_\sigma \ln \xi_t = -\frac{\bar{\gamma}}{\bar{\xi}_t (1-\bar{\tau}_t)^2} \partial_\sigma \tau_t$ and therefore we have

$$\mathbb{E}_T \partial_\sigma \ln \xi_{T+t} \partial_\sigma r^j_{T+1} = -\frac{\gamma}{\bar{\xi}_{T+t} \left(1 - \bar{\tau}_{T+t}\right)^2} \mathbb{E}_T \partial_\sigma \tau_{T+t} \partial_\sigma r^j_{T+1}. \tag{42}$$

Define function $\mathcal{X}$ as $\mathcal{X}(\tau, \theta, G) \equiv G - \tau (1-\tau)^\gamma \theta^{1+\gamma}$. It is easy to verify that $X_t = \mathcal{X}(\tau_t, \theta_t, G_t)$. Let $\mathcal{X}_{\tau,t}$, $\mathcal{X}_{\theta,t}$ and $\mathcal{X}_{G,t}$ be the first derivatives of $\mathcal{X}$ evaluated at a stochastic point (τ_t, θ_t, G_t). It is easy to verify that $\mathcal{X}_{\tau,t} = -\xi_t Y_t$. Therefore, using expressions for X and $X^\perp$, we have

$$\begin{aligned}
\mathbb{E}_T \partial_\sigma X_{T+t} \partial_\sigma r^j_{T+1} &= \bar{\mathcal{X}}_{\theta,T+t} \mathbb{E}_T \partial_\sigma \theta_{T+t} \partial_\sigma r^j_{T+1} + \bar{\mathcal{X}}_{G,T+t} \mathbb{E}_T \partial_\sigma G_{T+t} \partial_\sigma r^j_{T+1} - \bar{\xi}_{T+t} \bar{Y}_{T+t} \mathbb{E}_T \partial_\sigma \tau_{T+1} \partial_\sigma r^j_{T+1}, \\
\mathbb{E}_T \partial_\sigma X^\perp_{T+t} \partial_\sigma r^j_{T+1} &= \bar{\mathcal{X}}_{\theta,T+t} \mathbb{E}_T \partial_\sigma \theta_{T+t} \partial_\sigma r^j_{T+1} + \bar{\mathcal{X}}_{G,T+t} \mathbb{E}_T \partial_\sigma G_{T+t} \partial_\sigma r^j_{T+1}.
\end{aligned}$$

Combine these expressions with (42) to show that

$$\begin{aligned}
\mathbb{E}_T \partial_\sigma X_{T+t} \partial_\sigma r^j_{T+1} &= \mathbb{E}_T \partial_\sigma X^\perp_{T+t} \partial_\sigma r^j_{T+1} - \frac{\left(1 - \bar{\tau}_{T+t}\right)^2 \bar{\xi}^2_{T+t}}{\gamma} \bar{Y}_{T+t} \mathbb{E}_T \partial_\sigma \ln \xi_{T+t} \partial_\sigma r^j_{T+1} \quad (43) \\
&= \mathbb{E}_T \partial_\sigma X^\perp_{T+t} \partial_\sigma r^j_{T+1} + \zeta_{T+t} \bar{Y}_{T+t} \partial_\sigma \ln \xi_{T+t}, \partial_\sigma r^j_{T+1}.
\end{aligned}$$

Apply corollary 14 to prove lemma 2. □

We now prove a generalized version of Theorem 3, which does not assume stationarity. Define diagonal matrices Π^Q_T, Π^X_T, Π^A_T, Π^a_T with coefficients

$\Pi^Q_T[t,t] = \mathbb{E}_T \frac{q^{rf}_{T+t} X_{T+1+t}}{Y_{T+t}}$	$\Pi^X_T[t,t] = Y_T / \left(B_T q^{rf}_T \right)$
$\Pi^A_T[t,t] = \frac{\mathbb{E}_T q^{rf}_{T+t} \mathbb{E}_T \frac{Y_{T+1+t}}{Y_{T+t}} \mathbb{E}_T \zeta_{T+1+t}}{\left(1-a^{rf}_T\right) q^{rf}_T}$	$\Pi^a_T[t,t] = \frac{\mathbb{E}_T \zeta_{T+t}}{\left(1-a^{rf}_T\right) q^{rf}_T}$

Furthermore, let vector $\mathbf{w}$ be defined as $\mathbf{w}_T[t] = \mathbb{E}_T q^{rf}_T Q^{T+t}_{T+1} \frac{Y_{T+t}}{Y_T}$. We have

Theorem 19. *The optimal portfolio $\boldsymbol{\omega}_T^*$ satisfies*

$$\Sigma_T \boldsymbol{\omega}_T^* \simeq \left[\Pi_T^Q \Sigma_T^Q + \Pi_T^X \Sigma_T^X + \Pi_T^A \Sigma_T^A\right] \mathbf{w}_T + \Pi_T^a \mathbf{a}_T. \tag{44}$$

Proof. Combine the optimality conditions derived in lemma 15 with equation (43) to show that in the optimum

$$\mathbb{E}_{T\sigma} X_{T+t} \partial_\sigma r_{T+1}^j = \mathbb{E}_T \partial_\sigma X_{T+t}^\perp \partial_\sigma r_{T+1}^j + \bar{\zeta}_{T+t} \bar{Y}_{T+t} \left[\frac{\bar{R}_{T+1}^{rf}}{1-\bar{a}_T^{rf}} \frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} + \frac{\bar{R}_{T+1}^{rf}}{1-\bar{a}_T^{rf}} \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^k \partial_\sigma r_{T+1}^j\right].$$

Substitute that into the budget constraint (41) and re-arrange terms to obtain

$$\begin{aligned}
&\sum_{t=1}^\infty \bar{w}_{T+t} \left[\frac{\bar{q}_{T+t}^{rf} \overline{X}_{T+1+t}}{\bar{Y}_{T+t}}\right] \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln Q_{T+1}^{T+1+t} \\
&+ \sum_{t=1}^\infty \bar{w}_{T+t} \mathbb{E}_T \partial_\sigma r_{T+1}^j \frac{\partial_\sigma X_{T+t}^\perp}{\bar{Y}_{T+t}} + \frac{1}{1-\bar{a}_T^{rf}} \left(\frac{1}{\bar{q}_T^{rf}} \sum_{t=1}^\infty \bar{w}_{T+t} \left[\bar{\zeta}_{T+t}\right]\right) \frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} \\
&+ \frac{1}{1-\bar{a}_T^{rf}} \sum_{t=1}^\infty \bar{w}_{T+t} \left[\frac{\bar{q}_{T+t}^{rf}}{\bar{q}_T^{rf}} \frac{\bar{Y}_{T+1+t}}{\bar{Y}_{T+t}} \bar{\zeta}_{T+1+t}\right] \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^t \partial_\sigma r_{T+1}^j \\
= \; & -\left(\sum_{i\geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i\right) \bar{q}_T^{rf} \frac{\overline{B}_T}{\bar{Y}_T}.
\end{aligned} \tag{45}$$

Divide it by $\bar{q}_T^{rf} \frac{\overline{B}_T}{\bar{Y}_T}$, apply corollary 14 and write it in the matrix form to prove (44). ☐

Corollary 20. *In stationary economy, equation (19) holds.*

Proof. The definition of stationarity implies that

$$\bar{\xi}_{T+t} = \bar{\xi}_T, \quad \bar{\zeta}_{T+t} = \bar{\zeta}_T, \quad \frac{\bar{Y}_{T+t}}{\bar{Y}_T} = \Gamma^t, \quad \bar{q}_T^{rf} = q, \quad \overline{Q}_{T+1}^{T+t} = q^{t-1}, \quad \bar{w}_t = (q\Gamma)^t, \quad \frac{\bar{X}_{T+t}}{\bar{Y}_{T+t}} = \frac{\bar{X}_T}{\bar{Y}_T}.$$

Furthermore, corollary 12 implies that $\bar{a}_T^i = 0$ for all i, T. Therefore, equation (45) becomes

$$\begin{aligned}
&q\Gamma \frac{\bar{X}_T}{\bar{Y}_T} \sum_{t=1}^\infty (q\Gamma)^t \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j + \sum_{t=1}^\infty (q\Gamma)^t \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^\perp}{\bar{Y}_{T+t}} \partial_\sigma r_{T+1}^j \\
&+ \frac{\bar{\zeta}_T}{1-\bar{a}_T^{rf}} \left(\frac{1}{q} \sum_{t=1}^\infty (q\Gamma)^t\right) \frac{\partial_{\sigma\sigma} a_T^{rf} - \partial_{\sigma\sigma} a_T^j}{2} + \bar{\zeta}_T \Gamma \sum_{t=1}^\infty (q\Gamma)^t \mathbb{E}_T \partial_\sigma \mathrm{A}_{T+1}^t \partial_\sigma r_{T+1}^j \\
= \; & -\left(\sum_{i\geq 1} \mathbb{E}_T \partial_\sigma r_{T+1}^i \partial_\sigma r_{T+1}^j \overline{\omega}_T^i\right) q \frac{\overline{B}_T}{\bar{Y}_T}.
\end{aligned}$$

The zeroth order government budget constraint can be written as $\sum_{t=1}^\infty q^t \frac{\bar{Y}_{T+t}}{\bar{Y}_T} \frac{\bar{X}_{T+t}}{\bar{Y}_{T+t}} = -\frac{\bar{B}_T}{\bar{Y}_T}$. Applying stationarity conditions, we obtain $q\Gamma \frac{\bar{X}_T}{\bar{Y}_T} = -(1-q\Gamma) \frac{\bar{B}_T}{\bar{Y}_T}$. Substitute this into the previous equation, apply corollary 14 and write it in the matrix form to show (19). ☐

Corollary **??** follows from the following lemma.

Lemma 21. *Let $q_T^{(t)}, r_T^{(t)}$ be the period-T price and excess return of a pure discount bond that expires in period t. Then $q_T^{(T+1)} cov_T(r_{T+1}^{(T+1+t)}, r_{T+1}^j) \simeq cov_T\left(Q_{T+1}^{T+1+t}, r_{T+1}^j\right)$ for any security j that the government can trade. In particular, if the government can only trade pure discount bonds of all maturities and matrix Σ_T is arranged so that its i^{th} column corresponds to bonds expiring in period $T+i$, then $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$.*

Proof. We show that

$$\bar{q}_T^{(T+1)}\mathbb{E}_T\partial_\sigma r_{T+1}^{(T+1+t)}\partial_\sigma r_{T+1}^j = \mathbb{E}_T\partial_\sigma \ln Q_{T+1}^{T+1+t}\partial_\sigma r_{T+1}^j, \tag{46}$$

which is equivalent to $\bar{q}_T^{rf}\mathbb{E}_T\partial_\sigma r_{T+1}^i\partial_\sigma r_{T+1}^j = \mathbb{E}_T\partial_\sigma \ln Q_{T+1}^{T+1+t}\partial_\sigma r_{T+1}^j$ in the notation used in body of the paper. The latter equation implies that $q_T^{rf}\Sigma_T \simeq \Sigma_T^Q$ due to corollary 14.

Step 1. $\bar{q}_T^{(T+1)}\mathbb{E}_T\partial_\sigma r_{T+1}^{(T+1+t)}\partial_\sigma r_{T+1}^j = \mathbb{E}_T\left[\partial_\sigma \ln\frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^t \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right]\partial_\sigma r_{T+1}^j$.

The definition of returns and liquidity premium imply that

$$\begin{aligned} q_T^{(T+1+t)} &= \mathbb{E}_T\frac{\beta M_{T+1}}{M_T}q_{T+1}^{(T+1+t)}\frac{1}{1-a_T^{(T+1+t)}} \\ &= \mathbb{E}_T\left[\frac{\beta^{1+t}M_{T+1+t}}{M_T}\frac{1}{1-a_T^{(T+1+t)}}\times ... \times\frac{1}{1-a_{T+t}^{(T+1+t)}}\right]. \end{aligned}$$

Therefore, the excess return is

$$\begin{aligned} r_{T+1}^{(T+1+t)} &= \frac{q_{T+1}^{(T+1+t)}}{q_T^{(T+1+t)}} - \frac{1}{q_T^{(T+1)}} \\ &= \frac{1}{\frac{1}{1-a_T^{(T+1+t)}}}\frac{\mathbb{E}_{T+1}\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}}\frac{1}{1-a_{T+1}^{(T+1+t)}}\times ... \times\frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}{\mathbb{E}_T\left[\frac{\beta M_{T+1}}{M_T}\frac{\beta^t M_{T+1+t}}{M_{T+1}}\frac{1}{1-a_{T+1}^{(T+1+t)}}\times ... \times\frac{1}{1-a_{T+t}^{(T+1+t)}}\right]} - \frac{1}{\mathbb{E}_T\left[\frac{\beta M_{T+1}}{M_T}\frac{1}{1-a_T^{(T+1)}}\right]}. \end{aligned}$$

Its first order approximation terms can be written as

$$\begin{aligned} \partial_\sigma r_{T+1}^{(T+1+t)} &= \left(1-\bar{a}_T^{(T+1+t)}\right)\frac{\bar{M}_T}{\beta\bar{M}_{T+1}}\frac{\mathbb{E}_{T+1}\partial_\sigma\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}}\frac{1}{1-a_{T+1}^{(T+1+t)}}\times ... \times\frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}{\overline{\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}}\frac{1}{1-a_{T+1}^{(T+1+t)}}\times ... \times\frac{1}{1-a_{T+t}^{(T+1+t)}}\right]}} + \text{t.m.}T \\ &= \frac{\bar{M}_T}{\beta\bar{M}_{T+1}}\frac{1}{1-\bar{a}_T^{(T+1+t)}}\mathbb{E}_{T+1}\left[\partial_\sigma \ln\frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^t \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right] + \text{t.m.}T, \end{aligned}$$

where "t.m.T" denotes "terms measurable with respect to time T". Since $\bar{a}_T^{(T+1+t)} = \bar{a}_T^{(T+1)}$ and $\mathbb{E}_T\partial_\sigma r_{T+1}^j = 0$ for any j by lemma 11, this equation imply that

$$\mathbb{E}_T\partial_\sigma r_{T+1}^{(T+1+t)}\partial_\sigma r_{T+1}^j = \frac{1}{\bar{q}_T^{(T+1)}}\mathbb{E}_{T+1}\left[\partial_\sigma \ln\frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^t \partial_\sigma \ln\left(1-a_{T+k}^{(T+1+t)}\right)\right]\partial_\sigma r_{T+1}^j.$$

This proves Step 1.

Step 2. $\mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j = \mathbb{E}_T \left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^t \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+k)}\right)\right] \partial_\sigma r_{T+1}^j$.

By definition of Q_{T+1}^{T+1+t} we have

$$\begin{aligned} Q_{T+1}^{T+1+t} &= \mathbb{E}_{T+1} \frac{\beta M_{T+2}}{M_{T+1}} \frac{1}{1 - a_{T+1}^{(T+2)}} \times \mathbb{E}_{T+2} \frac{\beta M_{T+3}}{M_{T+2}} \frac{1}{1 - a_{T+2}^{(T+3)}} \times ... \times \mathbb{E}_{T+t} \frac{\beta M_{T+1+t}}{M_{T+t}} \frac{1}{1 - a_{T+t}^{(T+1+t)}} \\ &= \mathbb{E}_{T+1} \frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1 - a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1 - a_{T+t}^{(T+1+t)}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^{T+1+t} \partial_\sigma r_{T+1}^j &= \frac{\mathbb{E}_T \partial_\sigma \left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}} \right] \partial r_{T+1}^j}{\left[\frac{\beta^t M_{T+1+t}}{M_{T+1}} \frac{1}{1-a_{T+1}^{(T+2)}} \times ... \times \frac{1}{1-a_{T+t}^{(T+1+t)}} \right]} \\ &= \mathbb{E}_T \left[\partial_\sigma \ln \frac{\beta^t M_{T+1+t}}{M_{T+1}} - \sum_{k=1}^t \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+k)}\right) \right] \partial_\sigma r_{T+1}^j. \end{aligned}$$

Step 3. Equation (46) holds.

Note that

$$\begin{aligned} &\mathbb{E}_T \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+t)}\right) \partial_\sigma r_{T+1}^j = \mathbb{E}_T \left\{ \mathbb{E}_{T+k} \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+t)}\right) \right\} \partial_\sigma r_{T+1}^j \\ &= \mathbb{E}_T \left\{ \mathbb{E}_{T+k} \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+k)}\right) \right\} \partial_\sigma r_{T+1}^j = \mathbb{E}_T \partial_\sigma \ln \left(1 - a_{T+k}^{(T+1+k)}\right) \partial_\sigma r_{T+1}^j, \end{aligned}$$

where we applied the law of iterated expectations in the first and third equations, and lemma 11 in the second equations. This implies that the right hand sides of equations obtained in Step 1 and 2 are the same, proving (46). □