Solving Heterogeneous Agent Models with the Master Equation

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Introduction

- Modern macro $\equiv$ IRFs in models with rich cross-sectional heterogeneity
  - HANK, macro-search, spatial/trade
- Recent computational advances: Ahn et al. (2018), Auclert et al. (2019)
  - Reiter (2008) on steroids
  - Mostly numerical, restricted to first order
- Is there an underlying **conceptual framework**?
  1. Expand solvable models: GE feedback w/ many prices, entire distribution
  2. Improve accuracy & capture nonlinearities: 2nd-order
  3. Economic interpretation of components of IRFs
  4. Improve computation: accelerate, simplify

This paper: The idea

Analytic foundation for perturbation methods with heterogeneous agents

1. Include **entire distribution as state variable** into individual decision
   - Bellman eq. on infinite-dim. space of distribution: the **Master Equation**
   - Introduced in maths/mean field games literature (Cardaliaguet et al. 2019)
   - Fully recursive/Markovian representation of the economy
2. **Analytically perturb** the Master Equation in the **distrib. & ag. shocks**
   - Continuous time key for tractability
   - First/Second-order Approximation to the Master Equation (FAME, SAME)
   - Leverages generalized derivatives in infinite-dimensional spaces

This paper: The benefits

- The FAME
  - Single Bellman equation that embeds all equilibrium relationships
  - Depends on steady-state objects only, w/ **explicit** expressions
  - Dimension reduced from ∞ to 2 x idiosyncratic states
- Impulse Responses
  - Block-recursive structure: FAME $\rightarrow$ KF $\rightarrow$ IRF
  - A priori speed & conv. conditions w/ explicit steady-state objects
- Transparent implementation with standard Bellman equation methods
- The SAME is virtually the same

The plan

This talk

1. Derive the Master Equation in Krusell-Smith (1998) economy
2. Derive the FAME in Krusell-Smith (1998) economy
3. Derive the SAME in Krusell-Smith (1998) economy

In paper but not in talk

- Provide plug-and-play formulae for much more flexible setup
- 2 applications
  - Application 1: welfare gains from state-dependent UI
  - Application 2: dynamic spatial/migration model

▸ Literature

The Master Equation

in Krussell Smith (1998)

Setup

- Continuous time
- Individuals solve a standard income fluctuation problem
  - No borrowing constraint for now
  - Uninsurable income risk $\Rightarrow$ asset distribution matters for interest rate
- A representative firm rents capital and labor from households
- No aggregate shocks for now
  - Deterministic transition from out of steady-state

Individual decision problem

- Individual decision problem (HJB)

$$\rho V_t(a,y) - \frac{\partial V_t}{\partial t}(a,y) \quad = \quad \max_{c\geq 0} u(c) + (r_t a + w_t y - c)\frac{\partial V_t}{\partial a}(a,y) + L_0(y)[V_t]$$

  where functional operator $L_0(y)[\cdot]$ encodes productivity changes, e.g.

$$L_0(y)[V] = \mu(y)\frac{\partial V}{\partial y} + \frac{\sigma(y)^2}{2}\frac{\partial^2 V}{\partial y^2}$$

  and V has at most linear growth at infinity ($\equiv$ No-Ponzi condition)

- Collect individual states, prices and define operator

$$\begin{aligned} x &\equiv (a,y) \\ L_t(x,c)[V] &\equiv (r_t a + w_t y - c)\frac{\partial V}{\partial a}(x) + L_0(y)[V] \end{aligned}$$

- HJB writes more compactly

$$\rho V_t(x) - \frac{\partial V_t}{\partial t}(x) \quad = \quad \max_{c\geq 0} u(c) + L_t(x,c)[V_t]$$

Firms and evolution of distribution

- Firm decision problem

$$\max_{K,N} \bar{Z} K^{\alpha} N^{1-\alpha} - r_t K - w_t N$$

- Evolution of distribution (KF)

$$\begin{aligned} \frac{\partial g_t}{\partial t}(x) &= -\frac{\partial}{\partial a}\Big(s_t(x) g_t(x)\Big) + L_0^*(x)[g_t] \\ &\equiv L_t^*(x, \hat{c}_t(x))[g_t] \end{aligned}$$

where

  - $s_t(a, y) = r_t a + w_t y - \hat{c}_t(x)$: savings rate
  - $\hat{c}_t(x)$: optimal consumption decision
  - $L^*(x)[\cdot]$ denotes the adjoint of functional operator $L(x)[\cdot]$

A quick refresher on functional operators

- Analogy between **functions, operators** and **vectors, matrices**
- If instead we had a discrete state space or discretized on the computer
  - Functions $V(x), g(x)$ $\iff$ vectors V_i, g_i
  - Operator $L(x)[\cdot]$ $\iff$ matrix L_{ij}, where $x \Leftrightarrow i$
  - Action of operator on function $L(x)[V]$ $\iff$ matrix multiplication $L \cdot V$
  - Adjoint $L^*(x)[\cdot]$ $\iff$ matrix transpose L^T

Step 1/3: Find "prices" that affect individual decisions

- In this example, immediate: $\boldsymbol{r_t}, \boldsymbol{w_t}$

- In spatial models, one or more prices per location

- In labor market models, "prices" $\equiv$ entire wage distribution

Step 2/3: Express prices as functionals of distribution

- From firm's FOC

$$r_t = \boldsymbol{\mathcal{R}(g_t)} \qquad\qquad w_t = \boldsymbol{\mathcal{W}(g_t)}$$

- $\mathcal{R}, \mathcal{W}$ are simple functionals, e.g.

$$\boldsymbol{\mathcal{R}(g_t)} = \alpha \left(\frac{\iint y g_t(a,y) dyda}{\iint a g_t(a,y) dyda} \right)^{1-\alpha}$$

- Individual decision problem becomes

$$\begin{aligned} \rho V_t(x) - \frac{\partial V_t}{\partial t}(x) &= \max_{c \geq 0} u(c) + (\boldsymbol{\mathcal{R}(g_t)} a + \boldsymbol{\mathcal{W}(g_t)} y - c) \frac{\partial V_t}{\partial a}(x) \\ &\quad + L_0(y)[V_t] \\ &\equiv \max_{c \geq 0} u(c) + L(x, c, \boldsymbol{g_t})[V_t] \end{aligned}$$

- **No "t" subscript on L_0 anymore!**

Step 3/3: Change variables

- Re-write the **value function** as a **functional of the distribution**

$$V_t(x) \equiv V(x, g_t)$$

- Obtain the time derivative with the **chain rule**

$$\frac{\partial V_t}{\partial t}(x) = \int \frac{\partial V}{\partial g}(x, x', g_t) \frac{\partial g_t}{\partial t}(x') dx'$$

  - $\frac{\partial V}{\partial g}$ = Frechet derivative of V w.r.t. g: derivative w.r.t. functions
  - Recall analogy with discrete case $g \equiv (g_j)_j$, would have

$$\frac{\partial V_{it}}{\partial t} = \sum_j \frac{\partial V_i}{\partial g_j}(g_t) \frac{\partial g_{jt}}{\partial t}$$

- Recognize that $\frac{\partial g_t}{\partial t}$ given by the KF equation:

$$\frac{\partial V_t}{\partial t}(x) = \int \frac{\partial V}{\partial g}(x, x', g_t) L^*(x', \hat{c}(x', g_t), g_t)[g_t] dx'$$

Putting it all together: The Master Equation

- The individual decision problem becomes

$$\rho V(x,g) = \overbrace{\max_{c\geq 0} u(c) + L(x,c,g_t)[V]}^{\text{Standard flow utility and continuation value}} \\ \underbrace{+ \int \frac{\partial V}{\partial g}(x,x',g) L^*(x',\hat{c}(x',g),g)[g] dx'}_{\text{State-space representation of } \frac{\partial V_t}{\partial t}}$$

- This is the **Master Equation** (Cardaliaguet et al. 2019)
- Fully recursive/Markovian representation of the economy
- Integro-PDE in infinite dimension
- Not very practical

The FAME

The key simplification: Linearize in the distribution

- Suppose there exists a steady-state $\boldsymbol{V^{SS}(x), g^{SS}(x)}$
- Consider small perturbations in the distribution g around $\boldsymbol{g^{SS}}$:

$$g = \boldsymbol{g^{SS}} + \boldsymbol{h}\text{ , with } h \text{ small in some metric}$$

- To **first order**

$$V(x, \boldsymbol{g^{SS}} + \boldsymbol{h}) \approx \boldsymbol{V^{SS}(x)} + \int \boldsymbol{v(x, x')} \boldsymbol{h(x')} dx'$$

- $\boldsymbol{v}$ is the "Impulse Value"
  - Frechet derivative of the value function **at steady-state distribution**

$$\boldsymbol{v(x, x')} = \frac{\partial V}{\partial g}(x, x', \boldsymbol{g^{SS}})$$

  - Represents how value function **locally** reacts to a distributional impulse $\boldsymbol{h}$

Strategy

- Substitute first-order approximation

$$V(x, \boldsymbol{g^{SS}} + \boldsymbol{h}) \approx \boldsymbol{V^{SS}(x)} + \int \boldsymbol{v(x, x')} \boldsymbol{h(x')} dx'$$

  into the **Master Equation**

- Then "identify coefficients" on $\boldsymbol{h(x')}$

- "Coefficients" on $\boldsymbol{h(x')}$ are **functions**

The FAME

$$\rho \boldsymbol{v}(\boldsymbol{x},\boldsymbol{x'}) = \underbrace{u'(c^{SS}(x))D(x,x')}_{\text{Direct price impact}} + \underbrace{\mathcal{L}(\boldsymbol{x})[\boldsymbol{v}(\cdot,\boldsymbol{x'})]}_{\substack{\text{Continuation value from}\\ \text{idios. shocks to } x}} + \underbrace{\mathcal{L}(\boldsymbol{x'})[\boldsymbol{v}(\boldsymbol{x},\cdot)]}_{\substack{\text{Continuation value from}\\ \text{propagation of impulse at } x'}}$$

$$+ \underbrace{\int \boldsymbol{v}(\boldsymbol{x},\boldsymbol{x''}) \frac{\partial}{\partial a''}\Big(g^{SS}(x'') \underbrace{\Big(\underbrace{\mathcal{M}(x'',x',\boldsymbol{v})}_{\text{distributional MPC}} - D(x'',x')\Big)}_{\substack{\text{Change in savings rate of HH } x''\\ \text{in response to impulse at } x'}} dx''}_{\text{Weighted average of changes in savings rates of other HHs}}$$

where

$$D(x,x') = \big(\mathcal{R}_0 a' + \mathcal{R}_1 y'\big)a + \big(\mathcal{W}_0 a' + \mathcal{W}_1 y'\big)y$$

$$\mathcal{R}_0 = -(1-\alpha)\alpha\left(Y^{SS}/K^{SS}\right)^{1-\alpha}/K^{SS}$$

$$\mathcal{L}(\boldsymbol{x}) = L(x, c^{SS}(x), g^{SS}) = \big(r^{SS}a + w^{SS}y - c^{SS}(x)\big)\partial_a + L(y)$$

$$\mathcal{M}(x'',x',\boldsymbol{v}) = \frac{1}{u''(c^{SS}(x''))}\frac{\partial \boldsymbol{v}}{\partial a}(x'',x')$$

Properties of the FAME

- **Standard HJB**
- **Block-recursive**
  - Single Bellman equation that embeds the evolution of the distribution
  - No extra fixed point on prices: has been merged into HJB
- From infinite dimension to **finite dimension**
  - To first order, only need perturbations in distribution point by point x'
- **Explicit steady-state dependence**
  - Analytic local perturbation
- **Computation:** standard finite differences & only steady-state dimension
  - Leverages analytic structure

▸ Relation to sequence space

Discretizing the Impulse Value

- Discretize $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{x}')$ into a **matrix** $\mathbf{v}_{\boldsymbol{ij}} \equiv \boldsymbol{v}(\boldsymbol{x}_{\boldsymbol{i}}, \boldsymbol{x}_{\boldsymbol{j}})$
- Discretized FAME

$$\begin{aligned} \rho \mathbf{v} &= \mathrm{diag}(\mathrm{u}'^{SS}) \cdot \mathrm{D} + \mathbf{L} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{L}^T \\ &+ \mathbf{v} \cdot \mathrm{d}_a \cdot \Big[\mathrm{diag}(\mathrm{g}^{SS}) \cdot \Big(\mathrm{diag}(1/\mathrm{u}''^{SS}) \cdot \mathbf{v} - \mathrm{D} \Big) \Big] \end{aligned}$$

- Written compactly

$$\mathrm{M}\mathbf{v} + \mathbf{v}\mathrm{N} + \mathbf{v}\mathrm{P}\mathbf{v} = \mathrm{Q}$$

for known matrices $\mathrm{M}, \mathrm{N}, \mathrm{P}, \mathrm{Q}$ that depend only on steady-state objects

Computing the Impulse Value

- Need to solve for square matrix $\mathbf{v}$ in

$$\mathsf{M}\mathbf{v} + \mathbf{v}\mathsf{N} + \mathbf{v}\mathsf{P}\mathbf{v} = \mathsf{Q}$$

- Suppose that $\mathsf{P} = 0$
  - Obtain a **Sylvester matrix equation** $\mathsf{M}\mathbf{v} + \mathbf{v}\mathsf{N} = \mathsf{Q}$
  - Well-studied problem with established routines in most programming languages
  - Much more efficient than stacked system $\left(\mathsf{M} \otimes \mathsf{Id} + \mathsf{Id} \otimes \mathsf{N}\right)\mathsf{vec}(\mathbf{v}) = \mathsf{vec}(\mathsf{Q})$
- Since $P \neq 0$, need to iterate

A numerical scheme

- Guess an initial matrix $\mathbf{v}^{(0)}$
- Given a matrix $\mathbf{v}^{(n)}$, solve the **Sylvester matrix equation** in $\mathbf{v}^{(n+1)}$

$$\mathsf{M}\mathbf{v}^{(n+1)} + \mathbf{v}^{(n+1)}\left[\mathsf{N} + \mathsf{P}\mathbf{v}^{(n)}\right] = \mathsf{Q}$$

  - Important to treat the "sandwich" term this way
  - Similar to implicit scheme $\Longrightarrow$ stability
- Keep iterating until $\mathbf{v}^{(n)}$ and $\mathbf{v}^{(n+1)}$ close enough
- Examples
  - Krussel Smith (1998) model: ~ 0.1 seconds, 200 lines Matlab code
  - Krussel Smith (1998)+ frictional job ladder: ~ 5 sec., 300 lines Matlab code

The distribution and impulse response functions

- **After** solving for the Impulse Value $\mathbf{v}$, linearize KF equation
- Obtain

$$\underbrace{\frac{\partial \boldsymbol{h_t}}{\partial t}(x)}_{\substack{\text{Change in density} \\ \text{at } x}} = \underbrace{\mathcal{L}^*(\boldsymbol{x})[\boldsymbol{h_t}]}_{\substack{\text{Propagation of impulse} \\ \text{holding savings at SS}}} + \underbrace{\mathcal{K}(\boldsymbol{x})[\boldsymbol{h_t}]}_{\substack{\text{Response of savings} \\ \text{to impulse}}}$$

where

$$\mathcal{K}(\boldsymbol{x})[\boldsymbol{h}] \equiv \int \boldsymbol{K}(\boldsymbol{x}, \boldsymbol{x'}) \boldsymbol{h}(\boldsymbol{x'}) dx'$$

$$\boldsymbol{K}(\boldsymbol{x}, \boldsymbol{x'}) \equiv \frac{\partial}{\partial a}\Big(g^{SS}(x)\big(\mathcal{M}(x, x', \boldsymbol{v}) - D(x, x')\big)\Big)$$

- Similarly discretize and compute any deterministic IRF through

$$\mathbf{h}_{t+\Delta} = \mathbf{h}_t + \boldsymbol{\Delta}\left[\mathbf{L}^T + \mathbf{K}\right]\mathbf{h}$$

Aggregate shocks

Aggregate shocks

- Introduce aggregate productivity shocks $d\log Z_t = -\mu \log Z_t dt + \boldsymbol{\varepsilon} dW_t$
- Define **rescaled** aggregate productivity $z_t = \frac{1}{\varepsilon}\log\frac{Z_t}{\bar{Z}}$ so that

$$dz_t = -\mu z_t dt + dW_t$$

- **Master Equation** with aggregate shocks: $V(x, g, \varepsilon, z)$ solves

$$\begin{aligned}\rho V(x,\varepsilon,z,g) &= \max_c u(c) + L(x,c,\varepsilon z,g)[V] + \boldsymbol{\mathcal{A}(z)[V]} \\ &+ \int \frac{\partial V}{\partial g}(x,x',\varepsilon,z,g) L^*(x',\hat{c}(x',\varepsilon z,g),\varepsilon z,g)[V]dx'\end{aligned}$$

The FAME with Aggregate Shocks

- Take limit $\varepsilon \to 0$, $g \approx \boldsymbol{g^{SS}} + \varepsilon \boldsymbol{h}$:

$$V(x, \varepsilon, z, g) \approx \boldsymbol{V^{SS}(x)} + \varepsilon \left\{ \int \boldsymbol{v(x, x')} \boldsymbol{h(x')} dx' + \boldsymbol{\omega(x, z)} \right\}$$

where $\boldsymbol{\omega}$ is the "aggregate shock Impulse Value"

- Same strategy as in deterministic case
  - Substitute 1st-order approximation in Master Equation
  - Identify coefficients
  - Obtain one FAME for $\boldsymbol{v(x, x')}$, one FAME for $\boldsymbol{\omega(x, z)}$

- Distributional Impulse Value $\boldsymbol{v(x, x')}$ still satisfies the **deterministic FAME**
  - Block-recursive structure again
  - Start with deterministic FAME
  - Then only need to solve for $\boldsymbol{\omega(x, z)}$ w/ aggregate shock FAME

The FAME with Aggregate Shocks

- Aggregate shocks Impulse Value ω satisfies

$$\rho\omega(x,z) = \underbrace{z\Omega_0(x)u'(c^{SS}(x))}_{\text{Direct aggregate shock impact}} + \underbrace{\mathcal{L}(x)[\omega(\cdot,z)]}_{\substack{\text{Continuation value from}\\ \text{idios. shocks to } x}} + \underbrace{\mathcal{A}(z)[\omega(x,\cdot)]}_{\substack{\text{Continuation value from}\\ \text{aggregate shocks}}}$$

$$+ \underbrace{\int v(x,x')\frac{\partial}{\partial a'}\Big(g^{SS}(x')\underbrace{\big(\underbrace{\mathcal{M}(x',\omega(\cdot,z))}_{\text{Aggregate shock MPC}} - \Omega_0(x)z\big)}_{\text{Change in savings of HH } x'}\Big)dx'}_{\text{Weighted average of changes in savings rates of other HHs}}$$

where $\Omega_0(x) = \mathcal{R}_2 a + \mathcal{W}_2 y$

- **Standard HJB** that depends only on **known steady-state objects**

A numerical scheme

- Discretize $\omega(x, z)$ into a matrix **w**
- **w** solves a **standard Sylvester matrix equation**

$$\overline{\mathsf{M}}\mathbf{w} + \mathbf{w}\overline{\mathsf{N}} = \overline{\mathsf{Q}}$$

  for known matrices $\overline{\mathsf{M}}, \overline{\mathsf{N}}, \overline{\mathsf{Q}}$ that depend only on known steady-state objects
  - Block-recursive structure
  - The distributional Impulse Value **v** is already known
- Solve directly for **w**, no need to iterate
- **Examples**
  - KS98 model: $\sim$ 0.05 sec., 50 extra lines Matlab code
  - KS98 + frictional job ladder: $\sim$ 0.3 sec., 50 extra lines Matlab code

IRFs with aggregate shocks

- Linearized KF equation with aggregate shocks = SPDE

$$\underbrace{d\boldsymbol{h_t(x)}}_{\substack{\text{Change in density}\\ \text{at } x}} = \Big\{ \underbrace{\boldsymbol{\mathcal{L}^*(x)}[\boldsymbol{h_t}]}_{\substack{\text{Prop. of distr. impulse}\\ \text{holding savings at SS}}} + \underbrace{\boldsymbol{\mathcal{K}(x)}[\boldsymbol{h_t}]}_{\substack{\text{Response of savings}\\ \text{to distr. impulse}}} + \underbrace{\boldsymbol{S(x, z_t)}}_{\substack{\text{Response of savings}\\ \text{to ag. shock}}} \Big\} dt$$

where

$$\boldsymbol{S(x,z)} = \frac{\partial}{\partial a}\Big(g^{SS}(x)\big(\mathcal{M}(x, \boldsymbol{\omega(\cdot, z)}) - \Omega_0(x) z\big)\Big)$$

- Steady-state is stochastically stable if $\lambda^{dom}(\mathcal{K} + \mathcal{K}^*) < 0$

- Can similarly discretize and compute any IRF through

$$\mathbf{h}_{t+\Delta} = \mathbf{h}_t + \boldsymbol{\Delta}\big[\mathbf{L}^T + \mathbf{K} + \mathbf{S}_t\big]\mathbf{h}$$

for a given sequence of aggregate shocks $\mathbf{z}_t$

▸ Stochastic steady-state

The SAME

The SAME

- So far only considered **first-order** perturbations of the Master Equation
- Now **second-order** perturbations: **same logic**, just more components
- Again take limit $\varepsilon \to 0$, $g \approx \boldsymbol{g^{SS}} + \varepsilon \boldsymbol{h}$:

$$V(x,\varepsilon,z,g) \approx \underbrace{\boldsymbol{V^{SS}(x)}}_{\text{Steady-state}} + \underbrace{\varepsilon \left\{ \int \boldsymbol{v(x,x')} \boldsymbol{h(x')} dx' + \boldsymbol{\omega(x,z)} \right\}}_{\text{First order}}$$

$$+ \underbrace{\frac{\varepsilon^2}{2} \left\{ \iint \underbrace{\boldsymbol{\mathcal{V}(x,x',x'')}}_{\substack{\text{2nd-order effect} \\ \text{of distribution alone}}} \boldsymbol{h(x')} \boldsymbol{h(x'')} dx' dx'' + 2 \int \underbrace{\boldsymbol{\Gamma(x,x',z)}}_{\substack{\text{Cross effect} \\ \text{of ag. shock. \& distrib.}}} \boldsymbol{h(x')} dx' + \underbrace{\boldsymbol{\Omega(x,z)}}_{\substack{\text{2nd-order effect} \\ \text{of ag. shock alone}}} \right\}}_{\text{Second order}}$$

- 3 unknown functions $\boldsymbol{\mathcal{V}(x,x',x'')}, \boldsymbol{\Gamma(x,x',z)}, \boldsymbol{\Omega(x,z)}$

The SAME: Strategy

- Same strategy as in FAME
  - Substitute 2nd-order approximation in Master Equation
  - Identify coefficients

- **Block-recursive** structure again
  1. Enough to start with SAME for $\mathcal{V}(x, x', x'')$
  2. Then solve SAME for $\Gamma(x, x', z)$
  3. Finally solve SAME for $\Omega(x, z)$

The SAME: Bellman equation

$$
\begin{aligned}
\rho\mathcal{V}(x,x',x'') = & \underbrace{T(x,x',x'')}_{\text{Exogenous 2nd-order impact}} \\
& + \underbrace{\mathcal{L}(x)[\mathcal{V}(\cdot,x',x'')]}_{\text{Continuation value from changes to own state } x} + \underbrace{\mathcal{L}(x')[\mathcal{V}(x,\cdot,x'')] + \mathcal{L}(x'')[\mathcal{V}(x,x',\cdot)]}_{\text{Continuation value from propagation in pair of impulses } h(x') \text{ and } h(x'')} \\
& + \underbrace{\int \Big(\mathcal{V}(x,t,x'')\sigma(t,x') + \mathcal{V}(x,x',t)\sigma(t,x'')\Big)dt}_{\text{GE: 2}^{\text{nd}}\text{-order valuation of 1}^{\text{st}}\text{-order changes in other HHs' savings}} \\
& + \underbrace{\int \mathcal{V}(t,x',x'')\tau(x,t)dt}_{\text{GE: 1}^{\text{st}}\text{-order valuation of 2}^{\text{nd}}\text{-order changes in other HHs' savings}}
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma(y,t) &= \partial_y\left[g^{SS}(y)(b_g(y,t) - \mathcal{M}(y,t,v))\right] \\
\tau(x,y) &= \partial_y\Big(v_y(x,y)g^{SS}(y)k^{SS}(y)\Big) \\
T(x,z,t) &= \text{similar combination of steady-state objects and } v
\end{aligned}
$$

▸ Details

The SAME: Computation

- Discretize $\mathcal{V}(x, x', x'')$ into a **tensor $\mathbf{V}_{ijk}$**

- Obtain a **generalized Sylvester tensor equation**

$$\mathbf{V} \times_1 \hat{\mathsf{P}} + \mathbf{V} \times_2 \hat{\mathsf{Q}} + \mathbf{V} \times_3 \hat{\mathsf{R}} = \mathsf{T}$$

  where
  - $\hat{\mathsf{P}}, \hat{\mathsf{Q}}, \hat{\mathsf{R}}, \mathsf{T}$ are known matrices that depend on steady-state and $\mathbf{v}$
  - $\times_\ell$ denotes sum along index $\ell \in \{1, 2, 3\}$ of tensor and first index of matrix
  - $\times_\ell$ simply generalizes matrix product to tensors

- Well-established algorithms to solve the **Sylvester tensor equation**
  - Unpack tensor along any dimension
  - Recover sequence of standard Sylvester matrix equations

- Example: KS98: ~ 0.5 seconds

- Similar Bellman equations and discretization for $\Gamma(x, x', z)$ and $\Omega(x, z)$

Scope

Generalization

In paper, extend all results to general joint framework with

- Arbitrary controlled jump-diffusion process for state $x_t \in \mathbb{R}^{D_X}$
  - Wage ladder, different types, location/industry/occupation choice
- State constraints
  - Borrowing constraints
- Mass points in the distribution
  - Borrowing constraints, kinks in interest rate
- Value enters in flow payoff & generator
  - Epstein-Zin, bargaining models
- Intuition the same, just more notation
- Provide plug-and-play formulae

Conclusion

Conclusion

- FAME/SAME = recursive approach to dynamic economies w/ heterogeneity
- Crux of approach: work with full distribution & perturb analytically
- Outcomes
  - Ready-to-use formulae
  - Efficient, block-recursive & easy-to-code algorithm
  - 2nd-order perturbation
- Applicable to a wide range of settings
  - HANK + frictional labor markets
  - Dynamic discrete choice / spatial / trade
- Analytic PDE structure opens promising synergies for large-scale models
  - Sparse grids
  - Neural networks

Thank you!

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Appendix

Literature

- **Ahn et al. (2018)**
  - FAME $\equiv$ analytic foundation for Ahn et al.
  - Some dimension reduction for free
  - Bypasses automatic differentiation and Blanchard-Kahn steps
- **Auclert et al. (2019)**
  - Sequence-space FAME $\equiv$ analytic foundation for Auclert et al.
  - Bypasses automatic differentiation
- **Bandhari et al. (2018)**
  - FAME preserves full nonlinearity in idiosyncratic decisions
- **Alvarez et al. (2021)**
  - FAME applicable more broadly
- Handles **2nd-order perturbations**: SAME

Back to main presentation

Sequence-space representation: PE

- The distributional Impulse value v satisfies

$$v(a,y,a',y') = \sum_{p\in\{r,w\}} \int_0^\infty e^{-\rho t} \underbrace{v_t^p(a,y)}_{\substack{\text{Response of value}\\ \text{to price impulse at } t}} \underbrace{\bar{v}_t^p(a',y')}_{\substack{\text{Response of price at } t\\ \text{to distr. impulse}}} dt$$

- $v_t^p(a,y)$, $p \in \{r,w\}$ are the price Impulse Values

- To first order, for price sequences $\hat{r}_t, \hat{w}_t, t \geq 0$,

$$V_t(a,y) = V^{SS}(a,y) + \sum_{p\in\{r,w\}} \int_0^\infty e^{-\rho\tau} v_\tau^p(a,y)\hat{p}_{t+\tau} d\tau$$

- $v_t^p(a,y)$, $p \in \{r,w\}$ satisfy standard HJBs

$$\begin{aligned} -\frac{\partial v_t^p}{\partial t}(a,y) &= \mathcal{L}(a,y)[v_t^p] \\ v_0^r(a,y) &= au'(c^{SS}(a,y)) \ , \ v_0^w(a,y) = yu'(c^{SS}(a,y)) \end{aligned}$$

Back to main presentation

Sequence-space representation: GE

- Compute first-order consumption response from price Impulse Values
- Linearize KF equation analytically in prices
- Obtain equilibrium linear system in prices, e.g.

$$\hat{r}_t = \sum_{p \in \{w,r\}} \left(\underbrace{J_t^{0,r,p}}_{\text{Initial distrib.}} + \underbrace{\int_0^t J_{t-\tau}^{1,r,p} \hat{p}_\tau \, d\tau}_{\substack{\text{Cumul. effect of past prices} \\ \text{through past savings rates}}} + \underbrace{\int_t^\infty J_{t,\tau-t}^{2,r,p} \hat{p}_\tau \, d\tau}_{\substack{\text{Cumul. effect of future prices} \\ \text{through expectations}}} \right)$$

- Sequence-space Jacobians J have explicit expressions with
  - Price Impulse Values v^p
  - Steady-state distribution $g^{SS}(a, y)$ and transition probabilities $\mathcal{L}(a, y)$
  - Initial distribution $h_0(a, y)$

Back to main presentation

Stochastic steady-state

- Invariant distribution in stochastic steady-state is high-dimensional
  - Essentially $\mathbb{P}[h_t = h, z_t = z]$
  - Probability distribution over functions $h(x)$
  - Impractical

- Instead focus on **unconditional distribution** over indiv. and ag. states
  - Essentially $\boldsymbol{\bar{h}(x, z)} \equiv \mathbb{P}[x_t = x, z_t = z]$
  - Implicitly integrates over randomness in h_t conditional on $z_t = z$
  - Much more practical

- **Unconditional distribution** enough to first order
  - Enough to compute first-order moments e.g.

$$\mathbb{E}[a^n|z] = \int a^n \Big(g^{SS}(x) + \varepsilon \bar{h}(x, z)\Big) dx$$

  - Business cycle moments require second order anyway

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Stochastic steady-state

- The **unconditional** stochastic steady-state distribution $\bar{h}(x, z)$ solves

$$\mathcal{L}^*(x)[\bar{h}(\cdot, z)] + \mathcal{K}(x)[\bar{h}(\cdot, z)] + \mathcal{A}^*(z)[\bar{h}(x, \cdot)] + S(x, z) = 0$$

- Depends only on known steady-state objects

- Discretized: obtain a **Sylvester matrix equation**

$$\left[\mathbf{L}^T + \mathbf{K}\right] \cdot \mathbf{h} + \mathbf{h} \cdot \mathbf{A} = -\mathbf{S}$$

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The SAME: Details

$$
\begin{aligned}
\boldsymbol{T(x,z,t)} \quad = \quad & \underbrace{\mathcal{L}_{gg}(x,z,t)[V^{SS}]}_{\text{Direct price impact}} + \underbrace{\mathcal{L}_g(x,z)[\boldsymbol{v(\cdot,t)}] + \mathcal{L}_g(x,t)[\boldsymbol{v(\cdot,z)}]}_{\text{Cross price-continuation value}} \\
+ \quad & \underbrace{u''(c^{SS}(x))\mathcal{M}(x,z,\boldsymbol{v})\mathcal{M}(x,t,\boldsymbol{v})}_{\text{Cross consumption-continuation value}} \\
- \quad & \underbrace{\left[\boldsymbol{v_z(x,z)}\big(b_g(z,t) - \mathcal{M}(z,t,\boldsymbol{v})\big) + \boldsymbol{v_t(x,t)}\big(b_g(t,z) - \mathcal{M}(t,z,\boldsymbol{v})\big)\right]}_{\text{GE: change in propagation of impulse due to change in savings}} \\
- \quad & \underbrace{\int \boldsymbol{v_y(x,y)} g^{SS}(y)\Big[b_{gg}(y,z,t) - k_p^{SS}\boldsymbol{v_y(y,z)}\boldsymbol{v_y(y,t)}\Big]dy}_{\text{GE: 1}^{\text{st}}\text{-order valuation of 2}^{\text{nd}}\text{-order changes in others' savings}}
\end{aligned}
$$

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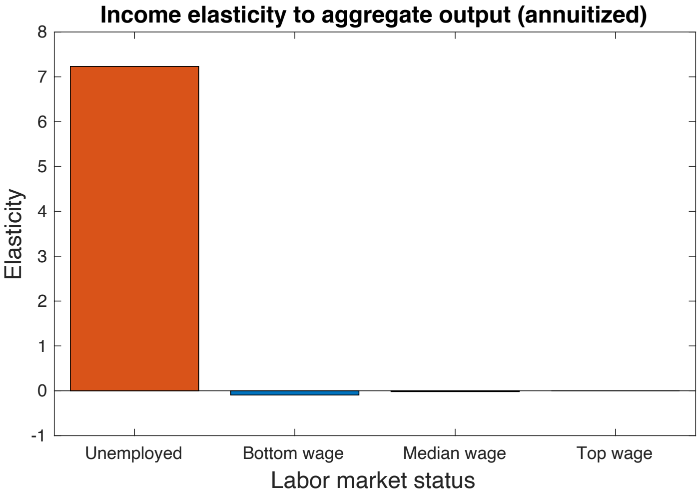
Applications

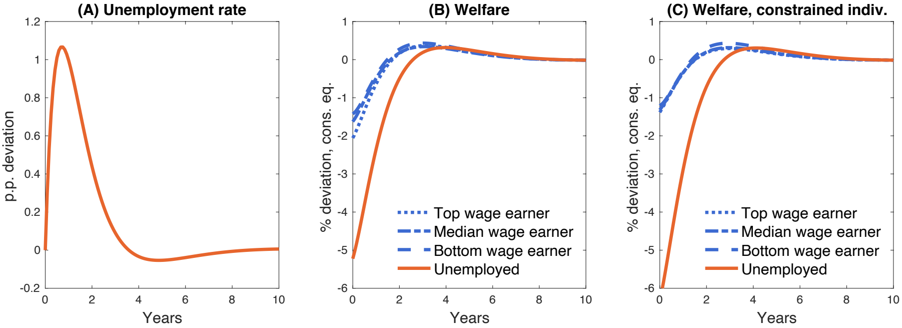
Frictional credit and labor markets

- Setup
  - Firms use capital & post wages à la Burdett-Mortensen
  - Frictional unemployment + JtJ search $\rightarrow$ uninsurable income risk
  - Borrowing constraint
  - State-dependent UI
  - Calibrated to MPC $= 0.2$, u-rate $= 0.1$
- Implementation
  - Distributional Impulse Value: 4s
  - Aggregate shock Impulse Value: 0.1s
  - Any IRF: $<$1s
  - Stochastic steady-state distribution: $<$1s
  - $\sim$ 200 lines of Matlab code w/ only matrix products and linear systems

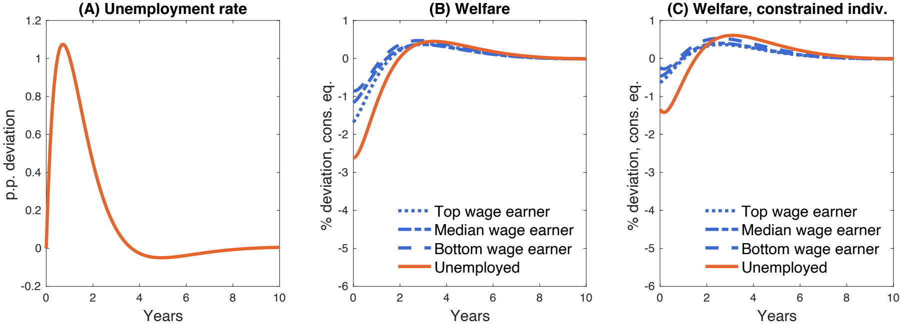
Unemployed bear the brunt of recessions



Impulse response to TFP shock with constant UI



Impulse response to TFP shock with countercyclical UI



- UI elasticity to u-rate calibrated to a 15% increase