Sovereign default and the decline in interest rates*

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Abstract

Yields on sovereign debt have declined dramatically across the developed world over the last half-century. Standard explanations of this decline include a change in discount rates due to an aging population or increased demand for assets from abroad. We show that these explanations encounter difficulties when confronted with the full range of evidence across asset classes. We propose that this decline was due to a decline in inflation expectations/default risk on sovereign debt. We argue that this explanation has a better chance of capturing an important feature of the decline in interest rates: namely that it has spanned centuries. We incorporate this explanation into an otherwise standard model of asset prices, augmented with inventory storage. An effective lower bound implies the existence of such a storage technology; otherwise, there are arbitrage opportunities within the model. Including storage in a production-based model allows us to match the reduction in investment and GDP growth observed over the last three decades.

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1 Introduction

Over the last three decades, the developed world has experienced a stark decline in interest rates. This decline, together with low output growth subsequent to the Great Recession of 2009, has evoked, for some, the possibility of "secular stagnation," a term coined by Hansen (1939) to describe a persistent period of low investment, employment, and growth. Summers (2015) and Gordon (2015) argue for the relevance of Hansen's concept from two angles: demand-side — an increase in demand for savings arising from changing demographics or growing inequality — and supply-side — arising from a decline in the ideas and dynamism that have fueled the economic growth of the last half-century. A complementary idea is that of a "global savings glut" (Bernanke, 2005): that there is too great a supply of savings, mainly from patient investors from outside the United States, compared to the demand, which would arise from the need to fund productive activities (ideas for which may be lacking). Caballero et al. (2008) propose that this excess supply of savings entering the US is due to a combined lack of pledgeability of cash flows and increasing wealth in the developing world.

However, is a greater desire for savings in fact what lies behind the decline in interest rates? On some level, the link appears too obvious to be worth questioning. Yet any explanation based on a greater desire for savings run into a significant problem when one also considers evidence from the US stock market. A greater desire for savings should have pushed up stock prices as well as bond prices, but it did not. From the point of view of the literature on increased desire for savings, low interest rates, and low growth, the behavior of the aggregate stock market is a puzzle. For a careful explanation of this puzzle, one can look to Farhi and Gourio (2018), who jointly consider growth, interest rates, and stock valuations in a neoclassical growth model. In addition, they allow for rare disasters, of the type considered by Barro (2009). Farhi and Gourio show that a significant increase in the risk of rare disasters is what is necessary to jointly reconcile the level of interest rates and the level of stock prices.

While Farhi and Gourio (2018) succeed in jointly explaining stock prices and interest rates, basing this explanation on increased fears of disaster runs into its own problems. The nature of rare disasters means that increased disaster fears are hard to falsify, but one would expect to find evidence in option prices. Yet options evidence suggests remarkable stability in fears of rare disasters. Second, implications of increased risk of disasters are fragile in that they depend directionally on whether the elasticity of intertemporal substitution (EIS) is above or below one. If the EIS is below, rather than above one, increased risk of rare disasters, together with low growth, require that agents must have become less patient, not more. Yet the arguments in Summers (2015) and Caballero et al. (2008) point unambiguously toward an increase in patience.

We therefore propose a different explanation, one based on a decline in the risk of inflation. There is substantial evidence for a steady decline in inflation expectations, spanning the 30 years over which interest rates have declined. More recently, evidence form options markets suggest that inflation expectations have become "anchored" — that is, investors do not fear either very high or very low inflation (Reis, 2020). When one takes this evidence into account, it is not difficult to jointly explain a decline in interest rates and the stability of stock valuation ratios. Because the true real rate has not declined, valuation ratios are unchanged, and there is no need to assume a large increase in the probability of a rare disaster to explain the evidence.

One may wonder: if it is simply inflation expectations that have declined, why is it that the *measured* real rate, namely nominal rates minus ex post realized inflation, have also declined? But this apparent disconnect disappears if one accounts for inflation *risk*. Indeed, if inflation were perfectly forecastable, then a change in inflation expectations should not impact ex post real rates. But historically, inflation has on occasion come as a surprise. A decline in inflation risk will lead investors to require less of a premium to hold nominal securities. Interest rates will decline if this premium declines, even if measured in real terms ex post. This effect is more pronounced if investors fear inflation that, in sample, does not occur. From the point of view of cash flows, and given that the sovereign has control over the money supply, inflation risk is essentially risk of default. A decline in inflation is thus a decline in the probability of default, and thus should also be expected to impact rates on securities that are said to be inflation-protected. The first contribution of our paper is to show that a model with rare disasters and a decline in inflation expectations can explain the decline in interest rates and the stability of valuation ratios. Because sovereign risk depends on institutions that have altered substantially over the centuries, this explanation could account for the striking fact that current rates are low, not just relative to the last 30 years, but to the last 300 years.

We also show that fears of rare disasters, together with low inflation risk, leads to nominal rates that are at or below zero even without the need to assume an increased desire for savings. In practice, the existence of cash creates an effective lower bound on interest rates. Such a lower bound is absent in traditional asset pricing models. Thus a second contribution of our paper is to augment a traditional asset pricing model with cash. We introduce cash in a way that does not require any change to preferences, or demand for liquidity. In our model, cash is a storage technology (inventory). When interest rates are sufficiently low, agents have an incentive to hold cash, which becomes a positive net supply asset.

When we consider storage of output, together with productive technologies in a general equilibrium model, we can jointly match growth, as well as valuation ratios and the riskfree rate. Including inventory in this framework allows us to obtain dynamics even within a framework with independent and identically distributed shocks. In addition, this set-up also allows us to match the decline in the investment-capital ratio: in low interest rate regimes, resources that would have been spent on capital are endogenously funneled into non-productive inventory.

This mechanism comes into play when expected returns on the risky asset do not rise fast enough as real rates fall; in the absence of attractive enough investment opportunities, investors look to hoard funds through unproductive avenues. In this way, we provide a new mechanism through which low growth can be compounded by low interest rates in a general equilibrium environment.

The remainder of this paper is organized as follows. In Section 2, we briefly summarize the evidence. In Section 3, we consider the ability of an endowment economy to match this evidence, either with changes in the probability of disaster, or changes in the probability of default. In Section 4, we solve the model with an inventory technology and show additional implications. Section 5 concludes.

2 Summary of the data

Figure 1 shows nominal government rates in a seven-century-long data set collected by Schmelzing (2020). Interest rates are highly volatile, as Jordà et al. (2019) emphasize.[1] Periods of extreme

[1] Jordà et al. (2019) note that prior observations of a real rate of zero are not unusual. However, these are observations after subtracting ex post realized inflation, not ex ante yields. While it is true that both returns are zero from an investor's perspective, one was a realization of zero because of high inflation, whereas the other is an

spikes, and also low rates, occurred around the Napoleonic Wars and World War II, reflecting a tension between an increase in risk of sovereign default and precautionary savings around disasters. High rates in the 1970s and 1980s clearly stand out. Nonetheless, the figure shows a steady decline. Perhaps a more dramatic demonstration comes from Figure 2, which shows the Bank of England lending rate, from the start of when the series was available. Only in the very most recent period did this rate reach a zero lower bound. Figure 3 zooms in on the last thirty years, the focus of much of the literature. As is well known, the federal funds rate declined sharply from 10% to 2% at present (right panel). On the other hand, the price-dividend ratio has gone from around 20 to 50, implying a dividend yield of approximately 5% going to 2% — a smaller decline. Figure 4 shows a longer time series of the price-dividend ratio, and also includes the price to cyclically adjusted earnings (CAPE) ratio and the price-dividend ratio from the United Kingdom.[2] It shows that the price-dividend ratio shifted upward in the late 1990s. This pattern does not appear in the CAPE ratio, nor in the UK, and therefore may reflect a use of repurchases rather than cash payments as a means of returning cash to shareholders, and not a decline in interest rates.

3 An endowment-economy model with rare disasters

To interpret these data, we first consider an endowment economy with a representative agent with Epstein and Zin (1989) preferences. For simplicity, we follow Farhi and Gourio (2018) (henceforth FG), and calibrate the model separately to two sample periods (1984–2000 and 2001–2016), assuming constant parameters. While this approach does mean that certain features of the data (such as volatility of prices and interest rates) will remain outside the scope of the analysis, it does allow us to consider the possibility of long-run unforeseen changes in parameters that may be realistic. Farhi and Gourio (2018) assume a neoclassical growth model. We will return to such a model in the next section, but for the purpose of this section, the extra degree of complication is not necessary. As far as prices and interest rates are concerned, and in this i.i.d. growth rate setting, the production model and the endowment model yield the same predictions.

ex ante value of zero. The economic interpretation is zero.

[2] US CAPE ratio data are from Shiller (2000) and UK price-dividend ratio data are from Jordà et al. (2019).

Let C_t denote the time-t endowment. Let $\Delta c_{t+1} \equiv \log(C_{t+1}/C_t)$, and assume

$$\Delta c_{t+1} = \mu + \eta_{t+1}, \tag{1}$$

where η_{t+1} is an independent and identically distributed "disaster" term of the form

$$\eta_{t+1} = \begin{cases} 0 & \text{with probability} \quad 1-p, \\ -Z & \text{with probability} \quad p, \end{cases} \tag{2}$$

and $Z > 0$. Under Epstein and Zin (1989) preferences, the stochastic discount factor is given by

$$M_{t+1} = \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta-1}, \tag{3}$$

where $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$. This means that the Euler equation with respect to the consumption claim is

$$1 = E_t \left[\beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta} \right], \tag{4}$$

for the return on the claim to consumption $R_{w,t+1}$.

As shown in Appendix Section A, this setup gives a constant price-consumption ratio κ equal to

$$\kappa = \frac{\beta e^{(1-\frac{1}{\psi})\mu} \left[1 + p(e^{-(1-\gamma)Z} - 1) \right]^{\frac{1}{\theta}}}{1 - \beta e^{(1-\frac{1}{\psi})\mu} \left[1 + p(e^{-(1-\gamma)Z} - 1) \right]^{\frac{1}{\theta}}}. \tag{5}$$

Given the return on the wealth portfolio, we can use the Euler equation to price the one-period riskless bond:

$$\log R_{f,t+1} \equiv r_f = -\log \beta + \frac{1}{\psi}\mu - \log(1 + p(e^{\gamma Z} - 1)) + \left(\frac{\theta - 1}{\theta} \right) \log(1 + p(e^{-(1-\gamma)Z} - 1)), \tag{6}$$

where r_f is the continuously-compounded riskfree rate.

Equations (5) and (6) constitute a system of two equations in two unknowns. We can in principle solve for p and β.[3] Further, combining these, we arrive at the risk premium on the wealth claim, given by

$$\log E_t[R_{w,t+1}] - r_f = \log(1 + p(e^{-Z} - 1)) + \log(1 + p(e^{\gamma Z} - 1)) - \log(1 + p(e^{-(1-\gamma)Z} - 1)) \quad (7)$$

We calibrate this model assuming measured growth rates of $\mu = 0.0350$ from 1984 to 2000 and $\mu = 0.0282$ from 2000 to 2016. In what follows, we refer to μ as expected growth, even though it is in fact expected growth in the absence of disasters. Table 1 shows the results. For greatest comparability, we consider $\gamma = 12$, $\psi = 2$, and a disaster size $Z = -\log 0.85$, the same parameters used by FG.[4] We find we can match the data using a discount factor of 0.967 in the early period and 0.979 in the later period, and a disaster probability of 3.4% in the early period, going to 6.6% in the later period. These results line up very well with those of FG.

These results depend on both the decrease in growth and the EIS ψ. Suppose first that growth had not declined. In this case, the disaster probability in the second period would need to be 1.3 percentage points higher. Under a calibration with $\psi > 1$, lower expected growth does help to explain stable valuation ratios, in that equalizing growth rates requires the disaster probability to do more work. We consider the sensitivity of this result to the EIS. When $\psi = 1/2$ rather than 2, the change in the disaster probability is roughly the same, but now investors need to be less patient, not more, in the second period. This result does not come about only because of the need to accomodate lower growth: when we equalize growth for $\psi = 1/2$, again, we find that investors must be less patient over the second period to explain the data. Moreover, Figure 6 shows what must happen to the disaster probability as a function of risk aversion. Lower values of risk aversion require a greater shift in the disaster probability (as well as an increase in levels).

Given the potentially anomalous behavior of the price-dividend ratio, in Table 2 we replace it with the CAPE ratio or the UK price-dividend ratio. Because these ratios are more stable, a

[3] There are some parameter combinations for which this is not possible. For example, when we try to set $\beta = .967$ for both 1984–2000 and 2001–2016 and then estimate μ and p, we are not able to obtain a solution for the 2001–2016 period.

[4] FG estimate the growth rate within a neoclassical growth model with rare disasters. Their growth rate is the composition of three different growth rates: namely, the growth rate of TFP, the growth rate of the population, and the growth in investment prices. We instead take this growth rate as being exogenously set at the same level. We also use identical values for the price-dividend ratio and the riskfree rate, which are reported in Panel A of Table 1.

much greater increase in the disaster probability is required to match the data.

Interpretation In this analysis, three parameters change across the two periods and affect the price-dividend ratio and the riskfree rate: the patience parameter β, the drift term in growth μ, and the probability of a disaster p. We now drill down to find out the role of each of the three. Table 3 reports the results. Note that the price-dividend ratio κ takes the form $\kappa = R/(1-R)$, for R a function of the primitive parameters. R must lie between 0 and 1, and the closer to 1, the greater is the price-dividend ratio. By taking the log of R, we can decompose the effect of the parameters in the price-dividend ratio into a term that depends only on the riskfree rate, a term that depends on the risk premium, and a term that depends on expected growth:

$$\begin{aligned}\log R = &-\underbrace{\left(-\log\beta + \frac{1}{\psi}\mu - \log(1+p(e^{\gamma Z}-1)) + \left(\frac{\theta-1}{\theta}\right)\log(1+p(e^{-(1-\gamma)Z}-1))\right)}_{\text{riskfree rate effect}} \\ &-\underbrace{\left(\log(1+p(e^{-Z}-1)) + \log(1+p(e^{\gamma Z}-1)) - \log(1+p(e^{-(1-\gamma)Z}-1))\right)}_{\text{risk premium effect}} \\ &+\underbrace{\left(\mu + \log(1+p(e^{-Z}-1))\right)}_{\text{cash flow effect}}.\end{aligned} \tag{8}$$

This is analogous to the decomposition in Campbell and Shiller (1988).

Using this decomposition, we can see that a change in β only affects the price-dividend ratio through its effect on the riskfree rate. Greater patience lowers the riskfree rate, and thus the rate at which investors discount all future cash flows. It thus raises the price-dividend ratio. Because the price-dividend ratio is a convex function of R, at values of R close to 1 (for which the price-dividend ratio is high), what appear to be small shifts in β cause massive changes in the ratio. One way to understand this result is duration: when the price-dividend ratio discounts cash flows in the distant future, their valuation will be very sensitive to small changes in rates. Thus, just to send the riskfree rate even half of the distance between the two samples would send the price-dividend ratio soaring to nearly 100. This is the fundamental problem with basing the decline in interest rates on an increased desire to save. Taking the declining growth rate into account helps to lower the price-dividend ratio, provided that the EIS is above 1.

Note that the growth rate μ enters in the price-dividend ratio in two ways, once multiplied by $1/\psi$, which represents its effect on the riskfree rate, and once multiplied by unity, representing its effect on future cash flows. A decrease in μ decreases the interest rate, following the usual consumption Euler equation intuition: the greater is expected growth, the greater the desire to save for the future, and hence the lower the riskfree rate must be. Or, in a production economy, the lower is growth, the lower the demand for borrowing, and the lower the riskfree rate. Either way, low interest rates and low growth clearly go together. However, when the EIS is above 1, the effect of growth on the interest rate is small. Unlike patience, the decrease in growth lowers the interest rate, but raises the price-dividend ratio, again, because the EIS is above 1, and the cash flow effect dominates the interest rate effect.

Taking both an increase in β and a decline in growth into account leads to a price-dividend ratio of about 70, not 50 as the data require. The remainder must be filled in by an increase in the risk premium (and a further decrease in expected future cash flows) through the disaster probability. Here, the model is helped by the fact that the increase in the disaster probability also causes a decline in the riskfree rate.

When the EIS is below 1, any decrease in the growth rate μ will lead to an increase in the price-dividend ratio, as the interest rate effect will dominate the cash flow effect. It will also, through the channel described above, lead to a decline in interest rates, and the decline should be larger than the EIS greater than 1 case. Besides the fact that an EIS below 1 delivers counterfactual predictions in asset pricing and production-based models, and for this reason is generally avoided, it does not appear to be possible to match the data in this case either, simply through a decrease in μ. The decline in μ is simply not sufficient to match the decline in the interest rate, even at a low interest rate. It is therefore necessary again to raise the disaster probability. But an increase in the disaster probability, for an EIS below one, also raises valuations. In order to match both facts, one must start the exercise with agents becoming less patient rather than more. The demand and supply reasons for low interest rates are at odds in this case.

A robust finding is that one requires higher disaster probability to explain the joint evidence on interest rates and stock returns. The higher disaster probability is so useful because it changes the risk premium and expected cash flows, both of which dampen the effects of the change in the riskfree rate.[5] Thus the change in the equity premium is important. Yet other studies have argued that the equity premium has fallen over the sample, not risen (Avdis and Wachter, 2017, van Binsbergen and Koijen, 2010, 2011, Fama and French, 2002, Lettau et al., 2008).

It is also difficult to reconcile an increased equity premium with evidence from options markets. Virtually any explanation for an ex ante equity premium will be reflected in greater risk. And while it is possible that such risk is not realized in sample, it should be visible in ex ante measures in risk, which one can obtain through options prices. Figure 7 shows that volatility, proxied by the Volatility Index (VIX) from the Chicago Board Options Exchange (CBOE), shows no increasing pattern. While the VIX itself is highly volatile, the average is remarkably stable between the two periods: equal to 21 in both. A higher disaster probability implies a significantly higher VIX, not only because the ex ante volatility is higher (due to the realization of disasters), but because the risk-neutral volatility is higher still. We show that it is possible to formally reject an increase in the disaster probability needed to jointly account for the price-dividend ratio and the riskfree rate.[6]

Inflationary default risk The typical empirical estimate of the equilibrium riskfree rate is the real return on short-term government debt; however, this return is not necessarily riskless, as the government can default either outright or through inflation. We now price this claim by including partial default that co-occurs with disasters. A decline in this partial default risk can explain the secular trends in asset returns since 1980.

Let Π_t denote the gross inflation rate. Suppose that, for the standard normal shock ε_t and the

[5] In contrast, a change in μ also dampens the change in the riskfree rate, but only through cash flows.

[6] We model dividends as levered consumption: $D_t = C_t^{\phi}$. The model-implied VIX, derived in Appendix B, is given by

$$VIX_t^2 = \phi^2 \left(\sigma^2 + \lambda e^{\gamma Z} Z^2\right) \tau, \tag{9}$$

where τ denotes the time interval over which volatility is priced and σ^2 is normal-times consumption volatility. Note that this is true for both power utility and recursive preferences. We choose parameters according to our calibration in Table 1: disaster size $Z = -\log 0.85$, relative risk aversion coefficient $\gamma = 12$, consumption volatility $\sigma^2 = 0.02$, first sample disaster intensity $\lambda_1 = 0.03$, and second sample disaster intensity $\lambda_2 = 0.07$. These are annualized parameters, so $\tau = 1/12$ matches the time interval used to calculate the VIX. We then choose ϕ^2 such that (9) with λ_1 is equal to the empirically observed value 0.2056^2 in the first sample. Given this calibration — which implies $\phi^2 = 19.8$ — we calculate that the implied VIX with $\lambda_2 = 0.07$ is 23.36 compared to the empirical average of 20.66. Using Newey-West standard errors with two lags on the monthly VIX, the t-statistic on this test is 2.66.

disaster shock given in (2),

$$\log \Pi_{t+1} \equiv \pi_{t+1} = \mu_{\pi,t} + \sigma_\pi \varepsilon_{t+1} - \zeta \eta_{t+1}. \tag{10}$$

The term $\mu_{\pi,t}$ represents some deterministic component — for example, an autoregressive process — and the coefficient ζ scales the percentage inflation that occurs in a disaster. Specifically, when there is a consumption disaster, there is also an added inflation equal to $-\zeta\eta_{t+1} = \zeta Z$. In Appendix C.1, we show that, in a sample with no disaster realizations, the difference between realized and expected inflation is given by $-p\zeta Z$. This difference, estimated using the one year inflation forecast of the Survey of Professional Forecasters, is plotted in Figure 8. The horizontal dashed lines show the average difference in each of our respective samples along with two-standard-error confidence intervals. These averages could be interpreted as estimates of $-p\zeta Z$. If we assume a disaster probability of $p = 0.03$ and a disaster size of throughout both periods, then $\zeta Z = 0.33$ from 1980 to 2000 and $\zeta Z = 0$ from 2000 to present. Quantitatively, this means an inflation disaster of 33% and 0% in the first and second samples, respectively.

In Appendix C.2, we show that there is a mapping between partial default through inflation and default through an outright failure to make promised payments. We take the latter approach to modelling short-term debt. Suppose that, instead of defaulting indirectly through inflation, the government defaults outright by paying only a fraction of its promised debt payments in a disaster. The payoff to the bond is now $e^{\eta^b_{t+1}}$, where

$$\eta^b_{t+1} = \begin{cases} 0 & \text{with probability} \quad 1-p, \\ -\zeta Z & \text{with probability} \quad p, \end{cases} \tag{11}$$

and is $|\eta^b_{t+1}|$ perfectly positively correlated with $|\eta_{t+1}|$. This means that $\eta^b_{t+1} = \zeta\eta_{t+1}$. Note that this collapses to the riskfree case when $\zeta = 0$. This means that the Euler equation for the one-period bond is now

$$Q_t = E_t\left[\beta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R^{\theta-1}_{w,t+1} e^{\eta^b_{t+1}}\right] \tag{12}$$

where Q_t is the time t price of the risky bond. This gives an expected return on the one-period bond as

$$\log E_t\left[R_{b,t+1}\right] = r_f + \underbrace{\log\left(\frac{(1+p(e^{-\zeta Z}-1))(1+p(e^{\gamma Z}-1))}{1+p(e^{-(\zeta-\gamma)Z}-1)}\right)}_{\text{risk premium}} \tag{13}$$

where r_f is the same as in (6). The expected cash flow loss term comes from the fact that the payoff on the bond is lower in the case of partial default. If there are no disasters in sample, then the ex post realized return will equal only the sum of the riskfree rate and the risk premium terms.

We can compare this directly to the real return on the nominally riskfree bond. Suppose that log inflation follows the process $\pi_{t+1} = \mu_\pi - \eta^b_{t+1}$. Then the ex ante real return on this bond is identical to that given in (13). In this case, the expected cash flow loss term is an effective loss from unexpected inflation disasters. The deterministic component μ_π is complete predictable, so it has no affect on the real return. Suppose there are no inflation disasters in sample, then the observed ex post real return (i.e., the nominal return less realized inflation) equals

$$\log R_{b,t+1} = r_f + \underbrace{\log\left(\frac{(1+p(e^{-\zeta Z}-1))(1+p(e^{\gamma Z}-1))}{1+p(e^{-(\zeta-\gamma)Z}-1)}\right)}_{\text{risk premium}} - \underbrace{\log(1+p(e^{-\zeta Z}-1))}_{\text{difference in-sample (no disaster)}} \quad . \tag{14}$$

Note that, if $\zeta > 0$, then this is strictly larger than the ex ante real return on the bond $\log E_t[R_{b,t+1}]$.

Calibration and results We now calibrate this model in a similar way to the disaster model above, but instead of varying p and β, we keep p constant and allow ζ to vary. Table 4 shows the results. We see that β need not rise as much as before, because it is now needed to explain only a slight increase in the price-dividend ratio. From 1984 to 2000, the risky bond has a positive risk premium coming from the fact that the bond partially defaults in the disaster state. However, from 2001 to 2016, the bond acts a hedge against the disaster, offering a slightly higher return when the disaster occurs.[7] One can interpret this through the lens of nominal debt and inflation. In a model with inflation, the value of nominal debt is eroded by unexpected increases in inflation, and made more valuable with unexpected deflation. In the early part of the sample, it is plausible that unexpected inflation correlated negatively with consumption, leading to positive risk premia. However, if states of low consumption correspond with deflation, the risk premium would be

[7] For example in Panel E of Table 4, the defaultable claim is required to have a return of approximately 1% in the event of a disaster.

negative. The calibration suggests that ζ has become negative, suggesting that we moved from a regime with inflationary disasters into one with slightly deflationary disasters.

Another advantage of this model is that it gives reasonable solutions in the special case of log utility. In this case, the return on the defaultable claim is given by

$$\log E_t[R_{b,t+1}] = -\log \beta + \mu - \log(1 + p(e^{(1-\zeta)Z} - 1)), \tag{15}$$

and the price-dividend ratio by $\kappa = \frac{\beta}{1-\beta}$. The results for this case are presented in Panel E of Table 4. Provided that enough deflation accompanies a rare disaster, the return on the defaultable claim would be high enough to account for the low observed returns.

To ensure an effective lower bound on the real riskfree rate, we augment the usual endowment economy with a storage technology. The absence of arbitrage in the model with inventory implies a zero lower bound.

4 A model with production

Assume that there exists a productive capital asset, and let K_t denote the quantity of capital. Let δ denote the depreciation rate, and A the productivity, so that output equals

$$Y_t = AK_t. \tag{16}$$

Following Gomes et al. (2019), we define *planned capital* to be the sum of the previous period's investment and depreciated capital stock:

$$\tilde{K}_t \equiv X_{t-1} + (1 - \delta)K_{t-1}, \tag{17}$$

where X_{t-1} denotes investment. As in Barro (2009), Gabaix (2011), and Gourio (2012) planned capital is subject to a capital quality shock:

$$K_t = \tilde{K}_t e^{\eta_t} \tag{18}$$

where η_t is iid, and, as in Section 3, has the distribution:

$$\eta_t = \begin{cases} 0 & \text{with probability} \quad 1-p, \\ -Z & \text{with probability} \quad p. \end{cases} \tag{19}$$

For simplicity, we consider a log utility investor. We rewrite the agent's consumption and investment problem as a standard infinite-horizon asset allocation problem (Samuelson, 1969), in which the agent chooses how much to invest in the capital asset and how much to invest in a riskfree asset. Again, let C_t denote consumption. Let W_t denote wealth. The agent solves

$$V(W_t) \equiv \max_{\{C_s,\theta_s\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^s \log C_s \tag{20}$$

subject to

$$W_{t+1} = (W_t - C_t)(1 + r_f) + (W_t - C_t)\theta_t(r_{K,t+1} - r_f), \tag{21}$$

where r_f is the riskfree asset return (this will turn out to be a constant, so we simplify notation and do not use a time subscript), $r_{K,t+1}$ is the return on the capital asset between t and $t+1$, and θ_t is the allocation (as a percentage of investable wealth $W_t - C_t$), to the capital asset.

Equation (21) is equivalent to (16–19) provided that

$$1 + r_{Kt} \equiv (1 - \delta + A)e^{\eta_t}. \tag{22}$$

To see this, let

$$W_t = C_t + I_t + \tilde{K}_{t+1}, \tag{23}$$

where I_t is the dollar investment in the riskless asset. Then

$$W_{t+1} = I_t(1 + r_f) + \tilde{K}_{t+1}(1 + r_{K,t+1}) \tag{24}$$

Substituting (23) into (24) implies (21), with

$$\theta_t = \frac{\tilde{K}_{t+1}}{W_t - C_t}$$

also note that substituting (22) into (24) implies

$$W_{t+1} = I_t(1 + r_f) + K_t(1 - \delta + A).$$

Finally, note that

$$C_t + I_t + \tilde{K}_{t+1} = I_{t-1}(1 + r_f) + \tilde{K}_t(1 + r_{K,t+1})$$

The left hand side is simply (23), whereas the right-hand side is (24), but at t rather than $t+1$. Defining investment X_t as

$$X_t = \tilde{K}_{t+1} - (1 - \delta)K_t$$

allows us to rewrite the agent's budget constraint as:

$$C_t + I_t + X_t = Y_t + I_{t-1}(1 + r_f).$$

To solve the model, rewrite (20) as a recursion:

$$V(W_t) = \max_{C_t,\theta_t} \log C_t + \beta E_t \left[V(W_{t+1})\right], \tag{25}$$

subject to (21). The solution implies a constant θ_t,

$$V(W_t) = (1 - \beta)^{-1} \log W_t, \tag{26}$$

and

$$C_t = (1 - \beta)W_t. \tag{27}$$

For any return $r_{i,t+1}$, it must be the case that

$$E_t \left[\beta \frac{W_t}{W_{t+1}}(1 + r_{i,t+1})\right] = 1 \tag{28}$$

and

$$E_t\left[\beta\frac{W_t}{W_{t+1}}(r_{i,t+1}-r_{f,t+1})\right]=0. \tag{29}$$

We now consider two sets of equilibrium restrictions. The first, and standard one, assumes the riskfree asset is in zero net supply and will give us an economy that is equivalent to that in Section 3.

4.1 Equilibrium with no inventory

If the riskfree asset is in zero supply, then, in equilibrium, $\theta = 1$. Equation (28) implies

$$1+r_f = E_t\left[(1+r_{K,t+1})^{-1}\right]^{-1} = (1-\delta+A)(1+p(e^Z-1))^{-1}. \tag{30}$$

Given what consumption growth must equal, this is the same as what one finds in Section 3. The risk premium is also the same as in Section 3:

$$\log E_t\left[\frac{1+r_{K,t+1}}{1+r_{f,t+1}}\right] = \log\left(1+p(e^{-Z}-1)\right)+\log\left(1+p(e^Z-1)\right). \tag{31}$$

4.2 Equilibrium with inventory

Now assume that the riskfree asset is in positive supply. Why would one have a positive supply riskfree asset? Conceptually, anything that is a store of value from one period to another could be a riskfree asset, provided that it is in fact riskfree and can be frictionlessly interchanged between consumption and investment. Many consumption goods would not fit this description because they cannot easily be changed into something other than what they are. Money does fit this description provided that there is no unexpected inflation (in which case it ceases to be riskfree). To keep things simple, we will think of inventory as money. Strictly speaking then, our analysis applies only to the second year period, in which we have estimated inflation risk to be negative. This turns out to make no difference – when the real interest rate (30) is greater than zero in the equilibrium without inventory, inventory can exist but there is no reason for agents to hold it.[8] Again, strictly speaking, if the asset is cash, given that we require deflation to fit the observed

[8]Liquidity services could be such a reason, but for simplicity we do not assume these.

government bond yield, we should have a positive return to inventory. However, the deflation we require to fit these yields is small, and thus allowing for a positive return in inventory would make little difference. Like all valuation equations, the existence of this riskfree storage is predicated on investors' (subjective) expectations about inflation. Evidence suggests (Reis, 2020) that in the later period it is reasonable to assume that investors believed inflation would be low and stable, and thus consistent with our assumptions on the existence of inventory. The interesting question of what determines inflation expectations is beyond the scope of this paper.

In the economy with inventory, (28) and (29) still hold. We look for an equilibrium with $r_f = 0$ and θ between zero and 1. Note that, if inventory exists, no-arbitrage implies that bonds cannot trade at a riskfree rate below zero. Thus $r_f = 0$ is a requirement of equilibrium. The budget constraint is now:

$$W_{t+1} = (W_t - C_t)(1 + \theta_t r_{K,t+1}), \tag{32}$$

with the return on capital the same as before. Define r_f^* to be (30). Provided that $r_f^* < 0$, there will exist such a solution. Imposing $r_f = 0$ implies that (28) and (29) reduce to:

$$E_t \left[\frac{1}{1 + \theta r_{K,t+1}} (1 + r_{i,t+1}) \right] = 1, \tag{33}$$

$$E_t \left[\frac{1}{1 + \theta r_{K,t+1}} r_{i,t+1} \right] = 0, \tag{34}$$

As a last step, we can explicitly define the solution for θ given the return on capital using (33) or (34), as both give the same answer. Taking the expectation of (33) and setting $r_{i,t+1} = r_f = 0$ gives

$$\left(\frac{1-p}{1 + \theta r_{K,0}} \right) + \left(\frac{p}{1 + \theta r_{K,Z}} \right) = 1. \tag{35}$$

where $r_{K,0} = (1 - \delta + A) - 1$ and $r_{K,Z} = (1 - \delta + A)e^{-Z} - 1$ are the net returns in the non-disaster and disaster states, respectively. Solving this shows that zero is one of the roots for θ, which can be seen immediately from (33) where $r_{i,t+1} = 0$. The other root gives the economically meaningful solution

$$\theta = -\frac{(1-p)r_{K,0} + p r_{K,Z}}{r_{K,0} r_{K,Z}}. \tag{36}$$

Finally, we price assets in an economy that allows for inventory. Note that we will consider two calibrations, one in which $r_f^* > 0$, and therefore that inventory holdings are zero in equilibrium (this corresponds to the first half of the sample), and the other in which $r_f^{<}0$ in which inventory holdings are positive. The primary reason for the change in r_f^* is the lower growth in the second half of the sample. An alternative interpretation, yielding the same results given the calibration, is that the inventory technology did not exist in the first half of the sample due to inflation risk.

While the true riskfree rate cannot go below zero, the observed real yield on defaultable claims may be positive or negative, depending on the risk premium. Further, the representative investor can smooth the effects of disaster risk by storing consumption in inventory. We calibrate this model to match the observed price-dividend ratio, government bond rate, investment, and GDP growth.

To do this, we need to first determine the growth rate of output and capital within the model. These will allow us to solve for the price-dividend ratio of the market, which we take to be the claim to aggregate output Y_t. Note that in this economy the growth rates of wealth and consumption are the same:

$$\frac{C_{t+1}}{C_t} = \frac{W_{t+1}}{W_t} = \beta(1 + \theta r_{K,t+1}) = \beta(\theta e^{\eta_{t+1}}(1 - \delta + A) + 1 - \theta). \tag{37}$$

Likewise, the growth rates of capital and output are equal:

$$\frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t} = \beta\Big(\theta e^{\eta_{t+1}}(1 - \delta + A) + e^{\eta_{t+1} - \eta_t}(1 - \theta)\Big), \tag{38}$$

as shown in Section E.1. Note that when $\theta = 1$, all of the growth rates in the economy are equal. Moreover, note from (38) that the rate of capital growth depends on whether a disaster has occurred at time t. This also means that the investment-capital ratio varies in the disaster state and is given by

$$\frac{X_t}{K_t} = \beta\theta\left((1 - \delta + A) + \frac{1 - \theta}{\theta e^{\eta_t}}\right) - (1 - \delta). \tag{39}$$

It may seem surprising that this ratio varies (despite the existence of a balance-growth path). In a state in which a disaster has occurred, the agent begins with a relatively high inventory-capital ratio. Therefore, the agent reallocates inventory to increase capital such that θ is optimal. Since the initial level of capital K_t is low, investment is comparatively high. This is shown in Figure 13.

Using these results, we can solve for the price-dividend ratio of the market, defined as the claim to output Y_t. We conjecture that the price-dividend ratio depends only on the current realization of the disaster η_t. Thus, we have that

$$1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)\left(\frac{\kappa^Y(\eta_{t+1})+1}{\kappa^Y(\eta_t)}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\right]. \tag{40}$$

Solving for the price-divided ratio (shown in Section E.2) gives that

$$\kappa^Y(0) = \frac{\beta}{1-\beta}\left(1+p\left(\left(1+\frac{(1-\theta)(e^Z-1)}{1+\theta r_{K,0}}\right)^{-1}-1\right)\right), \tag{41}$$

$$\kappa^Y(-Z) = \left(1+\frac{(1-\theta)(e^Z-1)}{1+\theta r_{K,0}}\right)\kappa^Y(0). \tag{42}$$

Note that the price-dividend ratio is higher following the occurrence of a disaster. This comes through the endogenous cashflow channel: due to the presence of inventory, growth in capital, and therefore in output, is high when the disaster occurs and capital is destroyed. This means that in the disaster state the output claim pays off more than in the non-disaster state. This abnormally high cashflow leads to higher prices in states where the disaster has occurred, even under log utility. This can be seen in Figure 14 where the price-dividend ratio is much higher in the state where the disaster has occurred.

The solution for the expected return to the defaultable claim is simple in the economy without inventory, and is given by

$$E_t[R_{b,t+1}] = (1-\delta+A)(1+p(e^{(1-\zeta)Z}-1))^{-1}. \tag{43}$$

In the economy with inventory, this is slightly different, and is given by the longer expression

$$E_t[R_{b,t+1}] = \left[(1-p)\left(\frac{1}{1+\theta r_{K,0}}\right)+p\left(\frac{e^{-\zeta Z}}{1+\theta r_{K,Z}}\right)\right]^{-1}. \tag{44}$$

4.3 Calibration and results

Using these results, we calibrate this model to match the real interest rate, price-dividend ratio, and GDP growth in the US, as in the sections above. To do this, we solve a system of three equations in three unknowns, where the unknowns are the calibrated parameters, β, ζ, and δ. The three equations are given by

$$E_t[R_{b,t+1}] = \left[(1-p)\left(\frac{1}{1+\theta r_{K,0}}\right) + p\left(\frac{e^{-\zeta Z}}{1+\theta r_{K,Z}}\right)\right]^{-1}, \tag{45}$$

$$\kappa^Y(0) = \frac{\beta}{1-\beta}\left(1 + p\left(\left(1 + \frac{(1-\theta)(e^Z-1)}{1+\theta r_{K,0}}\right)^{-1} - 1\right)\right), \tag{46}$$

$$\frac{Y_{t+1}}{Y_t}(0) = \beta\Big(\theta(1-\delta+A) + (1-\theta)\Big), \tag{47}$$

where $E_t[R_{b,t+1}]$ is the expected return on the defaultable claim, $\kappa^Y(0)$ is the price-dividend ratio given that a disaster has not occurred, and $\frac{Y_{t+1}}{Y_t}(0)$ is output growth given no disaster. Calibrating to these no-disaster moments is consistent with the fact that we do not observe any realized disasters in our sample. We then find the values of the parameters of interest that make it such that the data moments match their corresponding model moments.

The results are displayed in Table 7. The model with inventory is able to match these moments with a reasonable calibration of β, ζ, and δ, even under log utility. The disaster size used here is relatively large (it corresponds to a destruction of 65% of the capital stock), but this is nearly isomorphic to allowing for higher risk aversion and a smaller disaster.[9]

Further note that the AK production model with inventory does far better than the one without inventory. The presence of inventory allows the model to match the low growth from 2001 to 2016 with a smaller rise in depreciation than the model without it. This is because inventory leads to less growth in output, as the movement of funds to inventory crowds out investment. This can be seen by the lower investment-capital ratio in the model with inventory. As a caveat, both growth and the investment-capital ratio are calculated in the states where a disaster has not occurred. Expected growth in this model is much lower than realized growth much of the time, precisely because the disaster is rare. The opposite is the case for the investment-capital ratio; the

[9]In the case of Epstein and Zin (1989) with unit EIS this would likely be the case.

agent diverts investment from capital to inventory in the non-disaster states, anticipating that a disaster may occur. In the state where the disaster does occur, the agent invests heavily, as seen in Figure 13.

This reduction in the investment-capital ratio in normal states allows the model to match the observed reduction in the data *without targeting it*. This is not possible in the model without inventory, as the agent want to invest more to offset the effects of falling growth.

5 Concluding remarks

The puzzle of low interest rates is a puzzle not only from the point of view of the last quarter century, but over a much longer horizon. It is also a joint puzzle: why have low interest rates not been accompanied by higher valuation ratios?

The purpose of this article is to argue that the most natural explanation is not an increased demand for savings, which would lower interest rates and raise valuation ratios, nor a decrease in growth, which is hardly enough on its own to account for the observed change, nor an increase in the risk premium, as there is no evidence that risk has increased by nearly the required amount. These joint phenomena have a simple explanation, which is that *the true riskfree rate has hardly changed at all*. Short-term debt claims are defaultable, and investors have come to require a lower premium for this risk of default.

Because our explanation requires that the true riskfree rate has remained roughly constant, we require a framework that allows for (a) a riskfree rate that is sufficiently low to explain nominal debt yields at zero and (b) an explanation that survives the existence of a zero lower bound. We accomplish the former using a model with a risk of rare disasters. In a rare disaster model, investors' precautionary savings pushes the riskfree rate below zero. We accomplish the latter by introducing a costless storage technology into an endowment economy. When parameters are such that the true riskfree rate is below zero, agents choose to store the consumption good until markets clear at a riskfree rate of zero.

What we do not model is the cause for the decline in investor expectations of sovereign default. Evidence suggests that this decline both has a relatively short-term component based on the history of the last 30 years and a long-term component spanning centuries, based on a growing faith over time in the stability of sovereigns. The forces determining this shift in expectations are an interesting topic for further research.

APPENDIX

A Endowment economy with rare disasters

A.1 Price-consumption ratio

Section 3 estimates a simple model of rare disasters which is stylistically similar, in an asset pricing sense, to the model of Farhi and Gourio (2018). Here, we solve the model explicitly. Under (3) the Euler equation with respect to the consumption claim is

$$1 = E_t\left[\beta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^\theta\right], \tag{A.1}$$

meaning that the price-consumption ratio is given by

$$\Rightarrow \left(\frac{P_t}{C_t}\right)^\theta = \beta^\theta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\theta(1-\frac{1}{\psi})}\left(\frac{P_{t+1}}{C_{t+1}}+1\right)^\theta\right], \tag{A.2}$$

which we conjecture to be a constant κ. To verify this conjecture, we solve

$$\kappa^\theta = \beta^\theta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{\theta(1-\frac{1}{\psi})}(\kappa+1)^\theta\right]. \tag{A.3}$$

This implies that

$$\frac{\kappa}{\kappa+1} = \beta e^{(1-\frac{1}{\psi})\mu}\left[1+p(e^{-(1-\gamma)Z}-1)\right]^{\frac{1}{\theta}}, \tag{A.4}$$

which simplifies to

$$\kappa = \frac{\beta e^{(1-\frac{1}{\psi})\mu}\left[1+p(e^{-(1-\gamma)Z}-1)\right]^{\frac{1}{\theta}}}{1-\beta e^{(1-\frac{1}{\psi})\mu}\left[1+p(e^{-(1-\gamma)Z}-1)\right]^{\frac{1}{\theta}}}, \tag{A.5}$$

verifying the conjecture.

A.2 Riskfree rate

Now, we use the price-consumption ratio to solve for the riskfree rate. The riskfree rate is given by solving the following Euler equation

$$R_f^{-1} = E_t\left[\beta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta-1}\right]. \tag{A.6}$$

After some algebra, we arrive at

$$R_f = \beta^{-1} e^{\frac{1}{\psi}\mu}\left[1 + p(e^{\gamma Z} - 1)\right]^{-1}\left[1 + p(e^{-(1-\gamma)Z} - 1)\right]^{\frac{\theta-1}{\theta}} \tag{A.7}$$

which implies that the log riskfree rate is given by

$$r_f = -\log\beta + \frac{1}{\psi}\mu - \log(1 + p(e^{\gamma Z} - 1)) + \left(\frac{\theta - 1}{\theta}\right)\log(1 + p(e^{-(1-\gamma)Z} - 1)). \tag{A.8}$$

A.3 Expected return on defaultable claim

The price of the defaultable claim is obtained by solving the Euler equation

$$Q_t = E_t\left[\beta^\theta \left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{w,t+1}^{\theta-1} e^{\eta_{t+1}^b}\right]. \tag{A.9}$$

This gives the price of the one period bond as

$$Q_t = \beta e^{-\frac{1}{\psi}\mu}\left[1 - p + pe^{-(1-\gamma)Z}\right]^{\frac{1-\theta}{\theta}}\left[1 - p + pe^{-(\zeta-\gamma)Z}\right] \tag{A.10}$$

meaning that the log expected return on the one period bond would be

$$\begin{aligned}\log E_t[R_{b,t+1}] &= \log E_t[e^{\eta_{t+1}^b}] - \log\beta + \frac{1}{\psi}\mu + \left(\frac{\theta-1}{\theta}\right)\log(1 + p(e^{-(1-\gamma)Z} - 1)) \\ &\quad - \log(1 + p(e^{-(\zeta-\gamma)Z} - 1)) \\ &= r_f + \log(1 + p(e^{-\zeta Z} - 1)) + \log(1 + p(e^{\gamma Z} - 1)) - \log(1 + p(e^{-(\zeta-\gamma)Z} - 1))\end{aligned} \tag{A.11}$$

B Volatility Index in a disaster economy

For tractability, we adapt the simple disaster model to continuous time, following Seo and Wachter (2019). Suppose consumption follows the jump-diffusion process

$$\frac{dC_t}{C_{t^-}} = \mu dt + \sigma dB_t + (e^{-Z_t} - 1)dN_t, \tag{B.1}$$

where B_t is a standard Brownian motion, N_t is a Poisson process with constant intensity λ, and Z_t has time-invariant distribution υ. As in Abel (1999) and Campbell (2003), we model dividends as levered consumption: $D_t = C_t^{\phi}$. Under both power utility and recursive preferences, it follows that the price of the claim to the dividend stream follows the process

$$\frac{dS_t}{S_{t^-}} = \mu_S dt + \phi\sigma dB_t + (e^{-\phi Z_t} - 1)dN_t. \tag{B.2}$$

The quadratic variation is then given by

$$QV_{t,t+\tau} \equiv \int_t^{t+\tau} d[\log S, \log S]_s = \phi^2\sigma^2\tau + \int_t^{t+\tau} \phi^2 Z_s^2 dN_s. \tag{B.3}$$

For risk-neutral measure Q, the VIX is then given by

$$VIX_t^2 \equiv E_t^Q[QV_{t,t+\tau}] = \phi^2\left(\sigma^2 + \lambda E_\upsilon\left[e^{\gamma Z_t} Z_t^2\right]\right)\tau, \tag{B.4}$$

where the last term is derived from Girsanov's theorem:

$$E_{t^-}^Q\left[\phi^2 Z_s^2 dN_s\right] = E_{t^-}\left[\frac{\pi_t}{\pi_{t^-}}\phi^2 Z_s^2 dN_s\right] = \lambda\phi^2 E_\upsilon\left[e^{\gamma Z_t} Z_t^2\right]. \tag{B.5}$$

Note that this is true for both power utility and recursive preferences.

C Inflationary and outright default

C.1 Empirical evidence

Consider the inflation process (10). A rational agent would predict inflation to be

$$E_t[\pi_{t+1}] = \mu_{\pi,t} + p\zeta Z. \tag{C.1}$$

Now suppose there are no disasters in-sample, in which case the observed difference between realized and expected inflation

$$\pi_{t+1} - E_t[\pi_{t+1}] = \sigma_\pi\varepsilon_{t+1} - p\zeta Z, \tag{C.2}$$

and its unconditional average

$$\frac{1}{T}\sum_{t=0}^{T}(\pi_{t+1} - E_t[\pi_{t+1}]) \rightarrow -p\zeta Z. \tag{C.3}$$

The difference (C.2), estimated using the one year inflation forecast of the Survey of Professional Forecasters, is plotted in Figure 8. The horizontal dashed lines represent the estimates of $-p\zeta Z$ in our respective samples, along with two-standard-error confidence intervals. If we assume a disaster probability of $p = 0.03$ throughout both periods, then $\zeta Z = 0.33$ from 1980 to 2000 and $\zeta Z = 0$ from 2000 to present. Quantitatively, this means an inflation disaster of 33% and 0% in the first and second samples, respectively.

C.2 Pricing bonds: Inflationary default

Suppose inflation follows the process (10) and output follows the process

$$\log Y_{t+1} = \log Y_t + \mu + \eta_{t+1}. \tag{C.4}$$

Now consider a bond that promises a nominal payoff of 1 in all states. For simplicity, let $\sigma_\pi = 0$ and $\mu_{\pi,t} = \mu_\pi$. The real payoffs are thus

$$x_{t+1} = e^{-\pi_{t+1}} = \begin{cases} e^{-\mu_\pi} & \text{if} \quad \eta_{t+1} = 0, \\ e^{-\mu_\pi - \zeta Z} & \text{if} \quad \eta_{t+1} = -Z. \end{cases} \tag{C.5}$$

The nominal return on this claim is thus

$$R^{\$}_{f,t+1} = E_t \left[\beta \frac{u'(C_{t+1})}{u'(C_t)} e^{-\pi_{t+1}} \right]^{-1}. \tag{C.6}$$

If the agent has power utility, then this return is

$$R^{\$}_{f,t+1} = E_t \left[\beta e^{-\gamma \Delta y_{t+1} - \pi_{t+1}} \right]^{-1} = \beta e^{\gamma\mu + \mu_\pi} \left[1 + p(e^{(\gamma - \zeta) Z} - 1) \right]^{-1}. \tag{C.7}$$

The real excess return on this bond is thus the deflated nominal return less the real risk-free rate:

$$r_{e,t+1} = r^{\$}_{f,t+1} - r_{f,t+1} - \pi_{t+1} = -\log \left[1 + p(e^{(\gamma-\zeta)Z} - 1) \right] + \log \left[1 + p(e^{\gamma Z} - 1) \right] - \zeta \eta_{t+1}, \tag{C.8}$$

which implies that the log of the expected excess return is

$$\log E_t[R_{e,t+1}] = \underbrace{\log \left[\frac{1 + p(e^{\gamma Z} - 1)}{1 + p(e^{(\gamma-\zeta)Z} - 1)} \right]}_{\text{inflation risk premium}} + \underbrace{\log \left(1 + p(e^{-\zeta Z} - 1 \right)}_{\text{expected inflation}}. \tag{C.9}$$

For $\zeta < 0$, the risk premium on the defaultable claim is negative, indicating that the bond acts as a hedge against the risk of unexpected inflation in the disaster state. For $\zeta > 0$, the risk premium is positive and represents the risks investors face from inflation eroding their future payoff.

Note the components of this expression: the first term is compensation for the risk that high inflation co-occurs with low consumption growth; the second term is the expected loss of real cash flows from inflationary default.

D General theory in an economy with inventory

D.1 Agent's problem in two periods

There are two periods $t \in \{0, 1\}$. Let Ω represent the set of states ω at time $t = 1$. Markets are completed by a set of state-contingent claims with prices $\delta(\omega)$ and unit payoffs. Let P_i denote the price of a portfolio of these claims with payoff $x_i(\omega)$. The representative agent chooses consumption $\{C_t\}$, inventory I, and allocations $\{\theta(\omega)\}$ to maximize

$$(1-\beta)\log C_0 + \beta E_0\left[\log C_1(\omega)\right] \tag{D.1}$$

subject to[10]

$$C_0 + I + \int_{\omega\in\Omega} \delta(\omega)\theta(\omega)d\omega = Y_0, \tag{D.2}$$

$$C_1(\omega) = Y_1(\omega) + I + \theta(\omega), \tag{D.3}$$

$$I \geq 0. \tag{D.4}$$

The contingent claims are in zero net supply, so markets clear when

$$C_0 = Y_0 - I, \quad C_1(\omega) = Y_1(\omega) + I. \tag{D.5}$$

Let λ_0, $\lambda_1(\omega)$, and ξ represent the Lagrange multipliers on (D.2), (D.3), and (D.4), respectively. When $\xi = 0$, (D.4) does not bind. We then have the first-order conditions with respect to C_0, $C_1(\omega)$, I, and $\theta(\omega)$, respectively:

$$(1-\beta)C_0^{-1} = \lambda_0, \tag{D.6}$$

$$\beta C_1(\omega)^{-1} = \lambda_1(\omega), \tag{D.7}$$

$$\lambda_0 = E_0\left[\lambda_1(\omega)\right] + \xi, \tag{D.8}$$

$$\delta(\omega)\lambda_0 = \lambda_1(\omega). \tag{D.9}$$

[10] Because marginal utility approaches infinity as consumption approaches zero, the constraint $I \leq Y_0$ will never bind, so we omit it from the list of constraints.

Note that (D.8) holds only in expectation, since inventory is chosen before the realization of ω. Note also that combining (D.6), (D.7), and (D.9) gives us the traditional Euler equation

$$\delta(\omega) = \frac{\beta}{1-\beta}\left(\frac{C_0}{C_1(\omega)}\right). \tag{D.10}$$

Moreover, we can combine the first-order conditions, take expectations, and impose market clearing to get two conditions (suppressing the argument of $Y_1(\omega)$):

$$1 = E_0\left[\frac{\beta}{1-\beta}\left(\frac{Y_0 - I}{Y_1 + I}\right)\right] + \xi\frac{(Y_0 - I)}{1-\beta}, \tag{D.11}$$

$$1 = E_0\left[\frac{\beta}{1-\beta}\left(\frac{Y_0 - I}{Y_1 + I}\right)R_i\right], \tag{D.12}$$

where R_i is the gross return of any asset. Furthermore, suppose that output Y is distributed according to

$$\log Y_1 = \log Y_0 + \mu + \eta, \tag{D.13}$$

where η is defined as in equation (2) for disaster probability p and disaster magnitude Z.

D.1.1 Riskfree rate

Now suppose there is a riskfree asset. Its return is characterized by

$$1 = E_0\left[\frac{\lambda_1(\omega)}{\lambda_0}R_f\right], \tag{D.14}$$

which gives us

$$\frac{\xi}{\lambda_0} = \begin{cases} 1 - \frac{1}{R_f} & \text{if} \quad R_f > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{D.15}$$

In the first case, the constraint binds, so we know that $I = 0$. In the second case, the constraint does not bind, so $I > 0$. This implies the following important result: when the constraint binds, R_f is the same as in the usual endowment economy.

The inventory asset serves as zero-return endowment storage, and thus the absence of arbitrage ensures that the riskfree rate is bounded below. Intuitively, if the unconstrained (i.e., absent inventory) gross riskfree rate $R_f < 1$, then $\exists I > 0$ such that $R_f = 1$. This intuition is formalized as follows.

Proposition D.1. *Suppose the agent has strictly positive marginal utility $u'(C_t)$. When $I > 0$, the gross riskfree rate $R_f = 1$. When $I = 0$, $R_f > 1$ and is equal to the riskfree rate in a no-inventory endowment economy.*

Proof. If $I > 0$, then $\xi = 0$ and (D.11) and (D.12) combine to give us $R_f = 1$. If $I = 0$, then $\xi > 0$ and

$$R_f = 1 + \frac{\xi}{\beta E_0\left[u'(C_1)\right]}, \tag{D.16}$$

which is greater than 1 by the assumption that $u'(C_t) > 0$. Moreover, if $I = 0$, then the Euler equation (D.12) yields

$$R_f = E_0\left[\frac{\beta}{1-\beta}\left(\frac{u'(Y_1)}{u'(Y_0)}\right)\right]^{-1}, \tag{D.17}$$

which is the same as in the no-inventory endowment economy. ∎

Note that this result is independent of the stochastic process governing output Y. If we impose (D.13), then we can calibrate the model and do comparative statics for different disaster probabilities. Figure 9 plots the riskfree rate as a function of the disaster probability p. The ratio of inventory to initial output $i \equiv I/Y_0$ is given implicitly by the relation

$$1 = E_0\left[\frac{\beta}{1-\beta}\left(\frac{1-i}{e^{\mu+\eta}+i}\right)\right]. \tag{D.18}$$

when the constraint does not bind ($\xi = 0$), and $i = 0$ otherwise. Given the process (D.13), we have an explicit solution for i, which we plot along with the unconstrained riskfree rate in Figure 9. As suggested by the above proposition, inventory imposes a lower bound on the riskfree rate. For low values of p, the equilibrium riskfree rate is positive and there is no demand for inventory; for high values of p, the agent invests in inventory such that $R_f = 1$.

D.1.2 Risk premia

We define two risky assets and solve for their expected returns and implied risk premia. The first is the consumption claim, which pays a dividend of $Y_1 + I$; the second is the market (the output claim), which pays Y_1. Given our log utility setup, the price of the consumption claim is

$$P_C = \frac{\beta}{1-\beta}(Y_0 - I). \tag{D.19}$$

This is independent of Y_1, a result of log utility. This security's return is

$$R_C = \frac{1-\beta}{\beta}\left(\frac{Y_1+I}{Y_0-I}\right). \tag{D.20}$$

So the return on the consumption claim equals the inverse of the agent's intertemporal marginal rate of substitution. Similarly, the price of the market claim is

$$P_M = \frac{\beta}{1-\beta}(Y_0 - I)E_0\left[\left(\frac{Y_1}{Y_1+I}\right)\right] = P_C E_0\left[\left(\frac{Y_1}{Y_1+I}\right)\right], \tag{D.21}$$

and the market return is thus

$$R_M = \frac{1-\beta}{\beta} \left(\frac{Y_1}{Y_0 - I} \right) E_0 \left[\left(\frac{Y_1}{Y_1 + I} \right) \right]^{-1}. \tag{D.22}$$

Note that, in the absence of inventory, the consumption claim and the market are the same security. Thus, we can consider three securities: the consumption claim with inventory, the market claim with inventory, and the market claim without inventory. Figure 10 plots the expected returns and the implied risk premia for each of these securities under the same calibration as in the previous section. Notice that the risk premia of all securities are virtually the same, but due to the lower bound on the riskfree rate, the gross returns on these securities diverge for high disaster probabilities.

D.1.3 Price-dividend ratio

We can calculate the price-dividend ratio for each of the three securities in the previous section. For the consumption claim, we retrieve the usual log utility result:

$$\frac{P_C}{C_0} = \frac{\beta}{1-\beta}. \tag{D.23}$$

The price-dividend ratio for the market is

$$\frac{P_M}{Y_0} = \frac{\beta}{1-\beta} E_0 \left[\left(\frac{Y_1}{Y_1 + I} \right) \right]. \tag{D.24}$$

Evidently, these ratios are the same when there is no inventory asset or when $I = 0$. Figure 11 plots these ratios for each of the three securities. Note that the price-dividend ratio absent inventory equals the price-consumption ratio with inventory, and both are constant. Note also that, as inventory becomes more positive due to an increasing disaster probability, the price-dividend ratio on the market begins to decline. This can be thought of as a leverage effect of inventory.

D.2 Real inventory

Consider an infinite horizon economy. The agent has access to an inventory technology with which he or she can store consumption across periods. It follows from this definition that inventory is a real asset. Let $\Omega_{t+1|t}$ represent the set of time-$(t+1)$ states ω_{t+1} that can be reached from state ω_t. Markets are completed by a set of state-contingent claims. Let $\varphi(\omega_{t+1}|\omega_t)$ and $\nu(\omega_{t+1}|\omega_t)$ represent the price and quantity, respectively, of the state-ω_{t+1}-contingent claim in state ω_t. Let $\mathbb{P}(\omega_{t+1}|\omega_t)$ represent the conditional probability of state ω_{t+1}. The representative agent chooses consumption $\{C(\omega_t)\}$, inventory $\{I(\omega_t)\}$, investment $\{X(\omega_t)\}$, and asset allocations

$\{\nu(\omega_{t+1}|\omega_t)\}$ to maximize[11]

$$E_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t)\right] \tag{D.25}$$

subject to

$$C_t + I_t + X_t + \int_{\Omega_{t+1|t}} \varphi_t(\omega_{t+1})\nu_t(\omega_{t+1})d\omega_{t+1} = Y_t + I_{t-1} + \nu_{t-1}(\omega_t), \tag{D.26}$$

$$I_t \geq 0, \tag{D.27}$$

$$I_t \leq Y_t + I_{t-1} - X_t. \tag{D.28}$$

Note the generality of this setup: in a production economy, output Y_t is a function of last period's investment X_{t-1} and capital stock K_{t-1}; in an endowment economy, we let Y_t follow some given process and set $X_t = 0$, $\forall t$. Now define $\Delta I_t \equiv I_t - I_{t-1}$. The contingent claims are in zero net supply, so markets clear when

$$C_t = Y_t - \Delta I_t - X_t. \tag{D.29}$$

For Lagrange multipliers $\lambda(\omega_t)$ and $\xi(\omega_t)$ on (D.26) and (D.27), respectively, we get first-order conditions

$$\beta^t u'(C_t) = \lambda_t, \tag{D.30}$$

$$E_t[\lambda_{t+1}] + \xi_t = \lambda_t, \tag{D.31}$$

$$E_t\left[\lambda_{t+1}\frac{\partial Y_{t+1}}{\partial X_t}\right] = \lambda_t, \tag{D.32}$$

$$\delta_t(\omega_{t+1})\lambda_t = \mathbb{P}_t(\omega_{t+1})\lambda_{t+1}. \tag{D.33}$$

Note that (D.31) and (D.32) hold only in expectation, since inventory I_t and investment X_t are chosen before the realization of state ω_{t+1}. Note also that combining (D.30) and (D.33) gives us the traditional Euler equation

$$\delta(\omega_{t+1}|\omega_t) = \mathbb{P}(\omega_{t+1}|\omega_t)\beta\frac{u'(C(\omega_{t+1}))}{u'(C(\omega_t))}. \tag{D.34}$$

[11]For notational simplicity, we suppress the argument ω_t except where necessary for clarity. For example, we write C_t instead of $C(\omega_t)$ and $\varphi_t(\omega_{t+1})$ instead of $\varphi(\omega_{t+1}|\omega_t)$.

Moreover, we can combine the first-order conditions and take expectations to get two conditions:

$$E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}\right]+\frac{\xi_t}{\beta^t u'(C_t)}=1, \tag{D.35}$$

$$E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}R_{i,t+1}\right]=1, \tag{D.36}$$

where $R_{i,t+1}$ is the gross return of any asset, including the return on investment $R_{X,t+1}\equiv\frac{\partial Y_{t+1}}{\partial X_t}$. We can use these equilibrium results to prove the following proposition.

Proposition D.2. *Suppose the agent has strictly positive marginal utility $u'(C_t)$ and inventory is a real asset. When $I_t>0$, the gross real riskfree rate $R_{f,t+1}=1$. When $I_t=0$, $R_{f,t+1}>1$ and is equal to the real riskfree rate in a no-inventory economy.*

Proof. If $I_t>0$, then $\xi_t=0$ and (D.35) and (D.36) combine to give us $R_{f,t+1}=1$. If $I_t=0$, then $\xi_t>0$ and

$$R_{f,t+1}=1+\frac{\xi_t}{\beta^{t+1}E_t\left[u'(C_{t+1})\right]}, \tag{D.37}$$

which is greater than 1 by the assumption that $u'(C_t)>0$. Moreover, if $I_t=0$, then the Euler equation (D.36) yields

$$R_{f,t+1}=E_t\left[\beta\frac{u'(Y_{t+1}-X_{t+1})}{u'(Y_t-X_t)}\right]^{-1}, \tag{D.38}$$

which is the same as in the no-inventory economy. ■

D.3 Nominal inventory

Consider the same setup as above, but now we have a price level P_t and assume that inventory is exposed to inflation $\Pi(\omega_{t+1}|\omega_t)\equiv\frac{P(\omega_{t+1})}{P(\omega_t)}$. Under this definition, we can think of inventory as cash. The only difference is that the budget constraint (D.26) now becomes

$$C_t+I_t+X_t+\int_{\Omega_{t+1|t}}\delta_t(\omega_{t+1})\theta_t(\omega_{t+1})d\omega_{t+1}=Y_t+\Pi(\omega_t|\omega_{t-1})^{-1}I_{t-1}+\theta_{t-1}(\omega_t). \tag{D.39}$$

The defining equilibrium conditions then become

$$E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}\Pi_{t+1}^{-1}\right]+\frac{\xi_t}{\beta^t u'(C_t)}=1, \tag{D.40}$$

$$E_t\left[\beta\frac{u'(C_{t+1})}{u'(C_t)}R_{i,t+1}\right]=1, \tag{D.41}$$

where $R_{i,t+1}$ is a real return. As we state in the following proposition, we now get that the effective lower bound is on the nominal riskfree rate, not the real rate.

Proposition D.3. *Suppose the agent has strictly positive marginal utility $u'(C_t)$ and inventory is a nominal asset. When $I_t > 0$, the gross nominal riskfree rate $R^{\$}_{f,t+1} = 1$. When $I_t = 0$, $R^{\$}_{f,t+1} > 1$ and is equal to the riskfree rate in a no-inventory economy.*

Proof. The real return on a nominally riskfree asset is $\Pi^{-1}_{t+1} R^{\$}_{f,t+1}$. Given this fact, we can apply the proof method for Proposition D.2 to the equilibrium conditions (D.40) and (D.41) directly. ■

Something worth noting is that the market clearing condition in a nominal inventory economy now becomes

$$C_t = Y_t - I_t + \Pi_t^{-1} I_{t-1} - X_t. \tag{D.42}$$

Thus, for an agent with positive inventory supply, inflation has immediate real effects on consumption.

E Production model with inventory

E.1 Output growth and the investment-capital ratio

Under $Y_t = AK_t$, we have that

$$\frac{K_{t+1}}{K_t} = \frac{Y_{t+1}}{Y_t}. \tag{E.1}$$

To find an explicit expression for these two growth rates, note that the definition of θ implies that $\tilde{K}_{t+1} = \theta(W_t - C_t) = \beta\theta W_t = \beta\theta(\tilde{K}_t(1+r) + I_{t-1})$. Dividing by K_t yields

$$\frac{\tilde{K}_{t+1}}{K_t} = \beta\theta\left((1 - \delta + A) + \frac{I_t}{K_t}\right). \tag{E.2}$$

Next, note that

$$\frac{I_{t+1}}{K_{t+1}} = \frac{(1-\theta)(W_t - C_t)}{\theta(W_t - C_t)e^{\eta_{t+1}}} = \frac{(1-\theta)}{\theta e^{\eta_{t+1}}} \tag{E.3}$$

which implies that

$$\frac{\tilde{K}_{t+1}}{K_t} = \beta\theta\left((1 - \delta + A) + \frac{1-\theta}{\theta e^{\eta_t}}\right). \tag{E.4}$$

Finally, this can be used to understand the growth rate of capital in the economy

$$\frac{K_{t+1}}{K_t} = \beta\theta e^{\eta_{t+1}}\left((1 - \delta + A) + \frac{1-\theta}{\theta e^{\eta_t}}\right). \tag{E.5}$$

E.2 Price-dividend ratio

We conjecture that the price-dividend ratio depends only on the current state η_t (i.e. whether the disaster occurred or not). Thus,

$$1 = E_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)\left(\frac{\kappa^Y(\eta_{t+1})+1}{\kappa^Y(\eta_t)}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\right]. \tag{E.6}$$

This implies that

$$\kappa^Y(\eta_t) = \beta\left[(1-p)\left(1+\frac{(1-\theta)(e^{-\eta_t}-1)}{1+\theta r_{K,0}}\right)(\kappa^Y(0)+1) + p\left(1+\frac{(1-\theta)(e^{-Z-\eta_t}-1)}{1+\theta r_{K,Z}}\right)(\kappa^Y(-Z)+1)\right], \tag{E.7}$$

which gives a set of two equations in two unknowns:

$$\kappa^Y(0) = \beta\left[(1-p)(\kappa^Y(0)+1) + p\left(1+\frac{(1-\theta)(e^{-Z}-1)}{1+\theta r_{K,Z}}\right)(\kappa^Y(-Z)+1)\right], \tag{E.8}$$

$$\kappa^Y(-Z) = \beta\left[(1-p)\left(1+\frac{(1-\theta)(e^{Z}-1)}{1+\theta r_{K,0}}\right)(\kappa^Y(0)+1) + p(\kappa^Y(-Z)+1)\right]. \tag{E.9}$$

The solution to this system is

$$\kappa^Y(0) = \frac{\beta}{1-\beta}\left(1+p\left(\left(1+\frac{(1-\theta)(e^{Z}-1)}{1+\theta r_{K,0}}\right)^{-1}-1\right)\right), \tag{E.10}$$

$$\kappa^Y(-Z) = \left(1+\frac{(1-\theta)(e^{Z}-1)}{1+\theta r_{K,0}}\right)\kappa^Y(0). \tag{E.11}$$

F Data appendix

Figure 1 data on nominal sovereign borrowing rates come from Schmelzing (2020). The UK borrowing rates come from the Calendar of State Papers and the Bank of England. Data for the Netherlands come from from various sources from Leiden, Haarlem, Utrecht, Schiedam, and Amsterdam, Dormans (1991), Weeveringh (1852), and the ECB. German data come from various sources of various German principalities. US data come from Durand and Winn (1947), Homer and Sylla (2005), Sylla et al. (2005), the NBER Macrohistory database, and FRED. Data on the Bank of England lending rate comes from Global Financial Data.

Figure 2 data on Bank of England lending rates come from Global Financial Data. From 1694 to 1800 the rate on foreign bills is used, from 1800 to 1972 the bank rate is used, from 1972 to 1981 the minimum lending rate is used, from 1981 to 1997 the BoE base rate is used, and from 1997 to the present the BoE Operational interest rate is used.

Figure 3 interest rate data come from FRED and the cum- and ex-dividend returns from The Center for Research in Security Prices (CRSP). Figure 4 data come from Jordà et al. (2019) for the US and UK price-dividend ratio and Shiller (2000) for the cyclically adjusted price-earnings ratio.

Data for Figure 5 come from FRED and the Survey of Professional Forecasters. Figure 7 data are taken from

Global Financial Data and are from the Chicago Board Options Exchange (CBOE).

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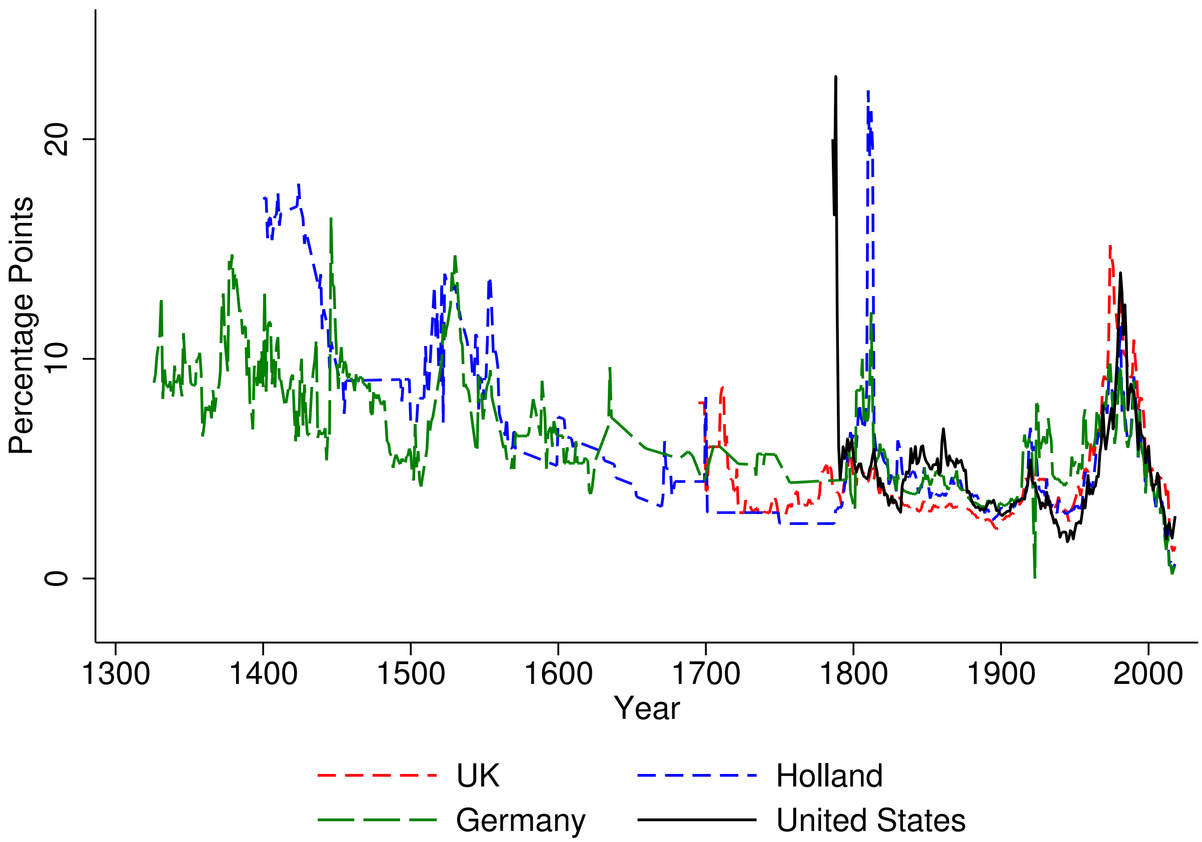
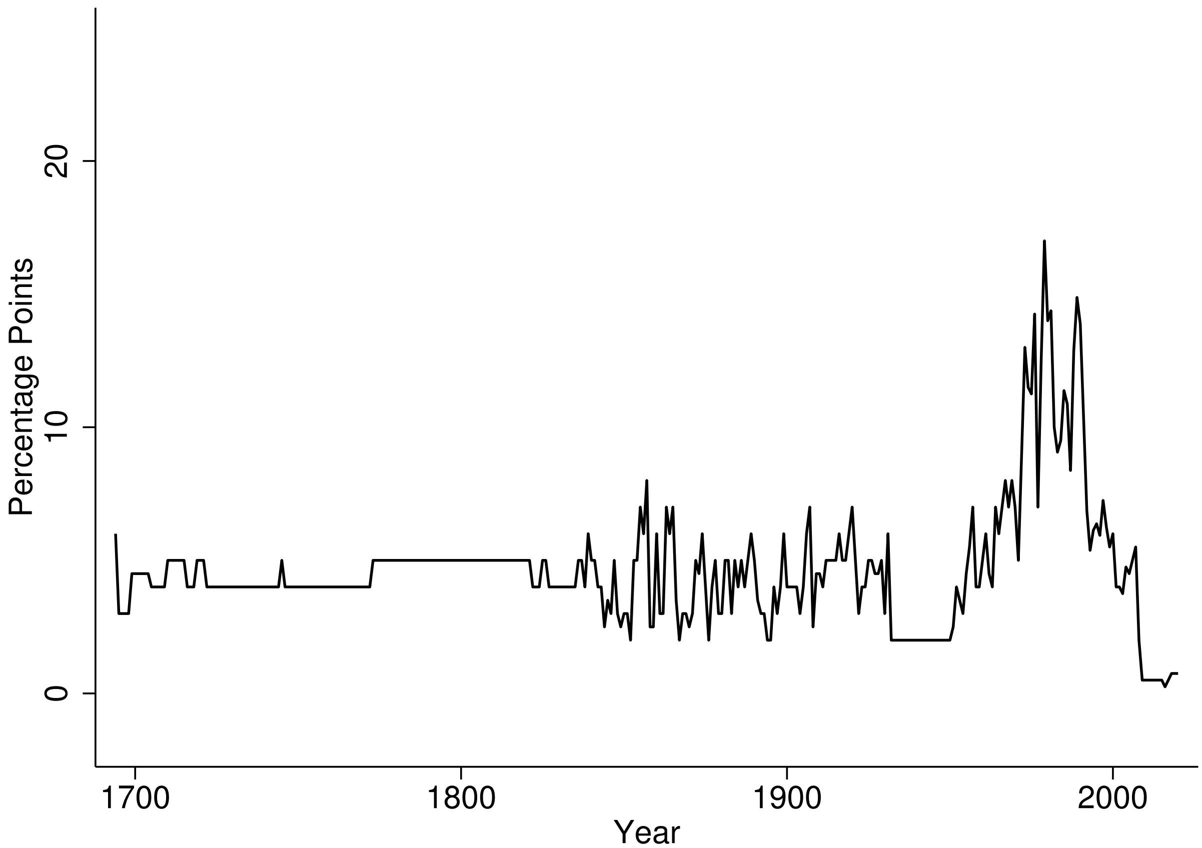
Tables and figures

Figure 1: Nominal government rates



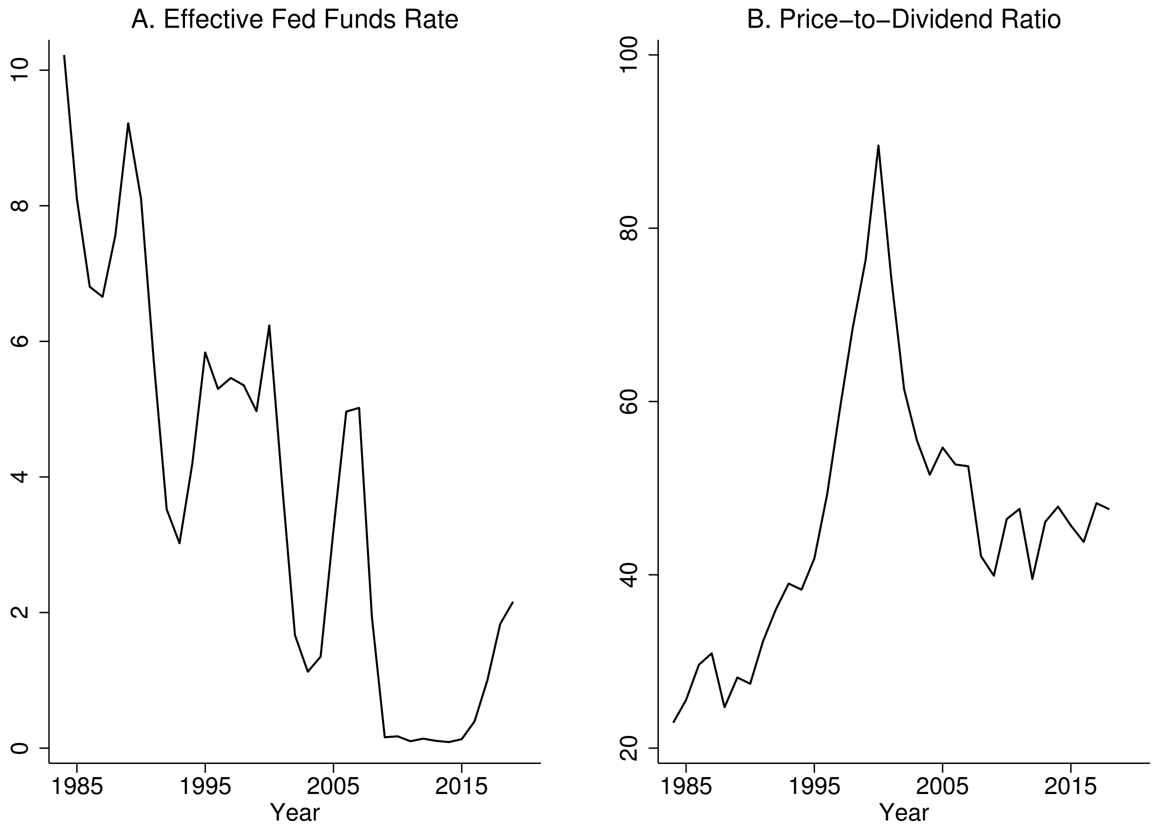
NOTES: The figure shows the nominal sovereign borrowing rate in the United Kingdom, Holland, Germany, and the United States from 1311–2018. Rates are from Schmelzing (2020) and are in annual terms. Data before 1694 for the UK (before the founding of the Bank of England) are not used as the data are incomplete. Rates come from a variety of archival, primary, and secondary sources. For information on the construction and contents of the data, visit the paper webpage.

Figure 2: Bank of England lending rate



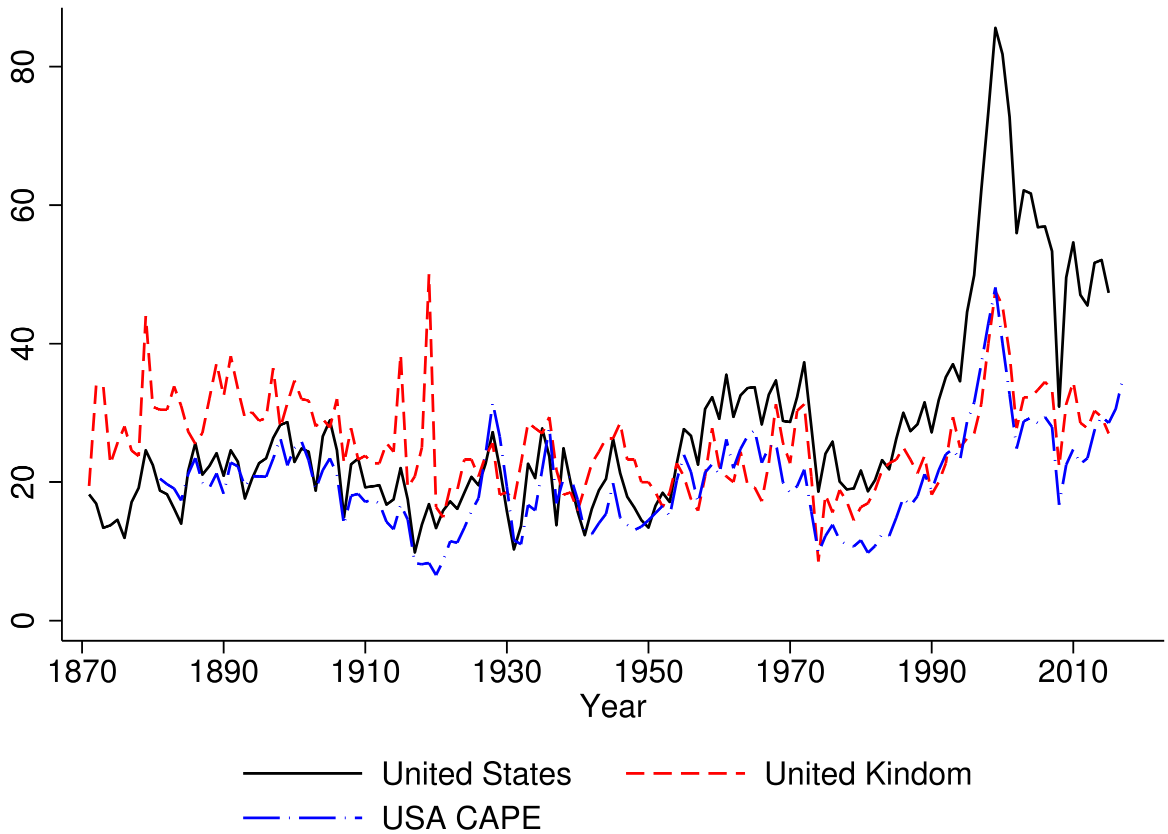
NOTES: The figure shows the nominal lending rate, expressed in annual terms of the Bank of England (BoE). From 1694 to 1800 the rate on foreign bills is used, from 1800 to 1972 the bank rate is used, from 1972 to 1981 the minimum lending rate is used, from 1981 to 1997 the BoE base rate is used, and from 1997 to the present the BoE Operational interest rate is used. Data comes from Global Financial Data.

Figure 3: Riskfree rate and price-dividend ratio in the United States from 1984–2018



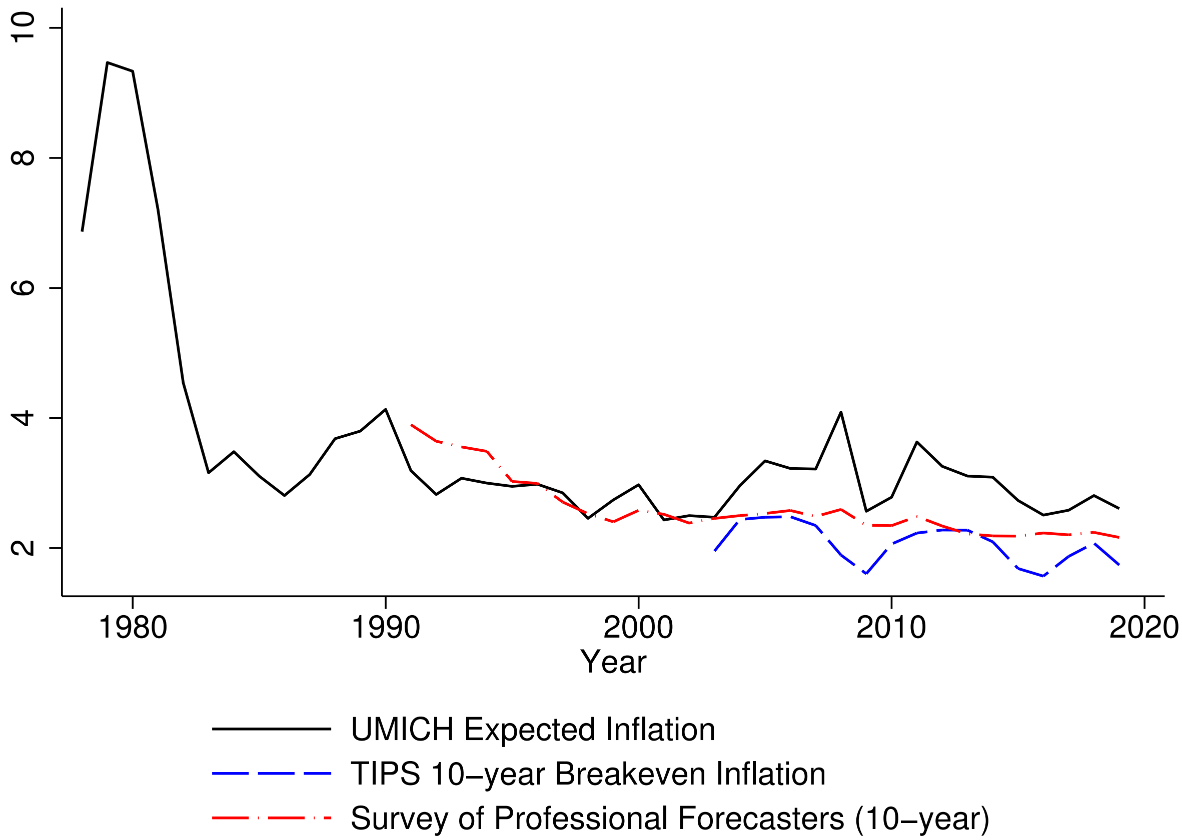
Notes: The figure shows the effective Fed funds rate (FRED series FEDFUNDS) and the price-dividend ratio for the United States on the value-weighted CRSP index. We compute the annual dividends based on the difference between ex- and cum-dividend returns from the CRSP database.

Figure 4: Price-dividend and CAPE ratios: United States and United Kingdom



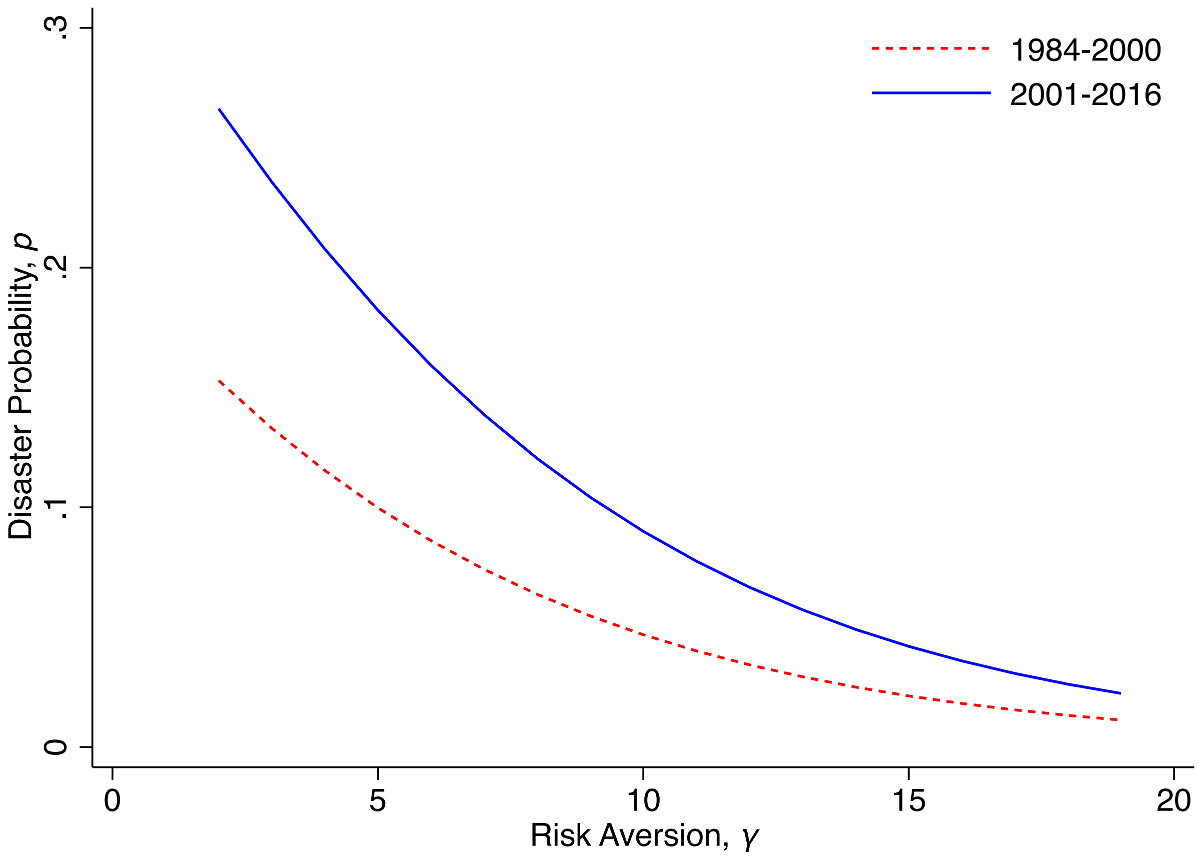
NOTES: The figure shows the price-dividend ratio for the United States and United Kingdom since 1870 from Jordà et al. (2019) and the cyclically adjusted price-earnings ratio from Shiller (2000). The black, solid line shows data for the United States price-dividend ratio and the red, solid line shows data for the price-dividend ratio of the United Kingdom. Price-dividend ratios are the end of year price divided by the aggregate dividends from the preceding year. The blue dashed-dotted line shows the CAPE ratio for the United States as taken from Shiller (2000). The methodology for these data can be found at `http://www.econ.yale.edu/~shiller/data.htm`.

Figure 5: Expected inflation in the United States



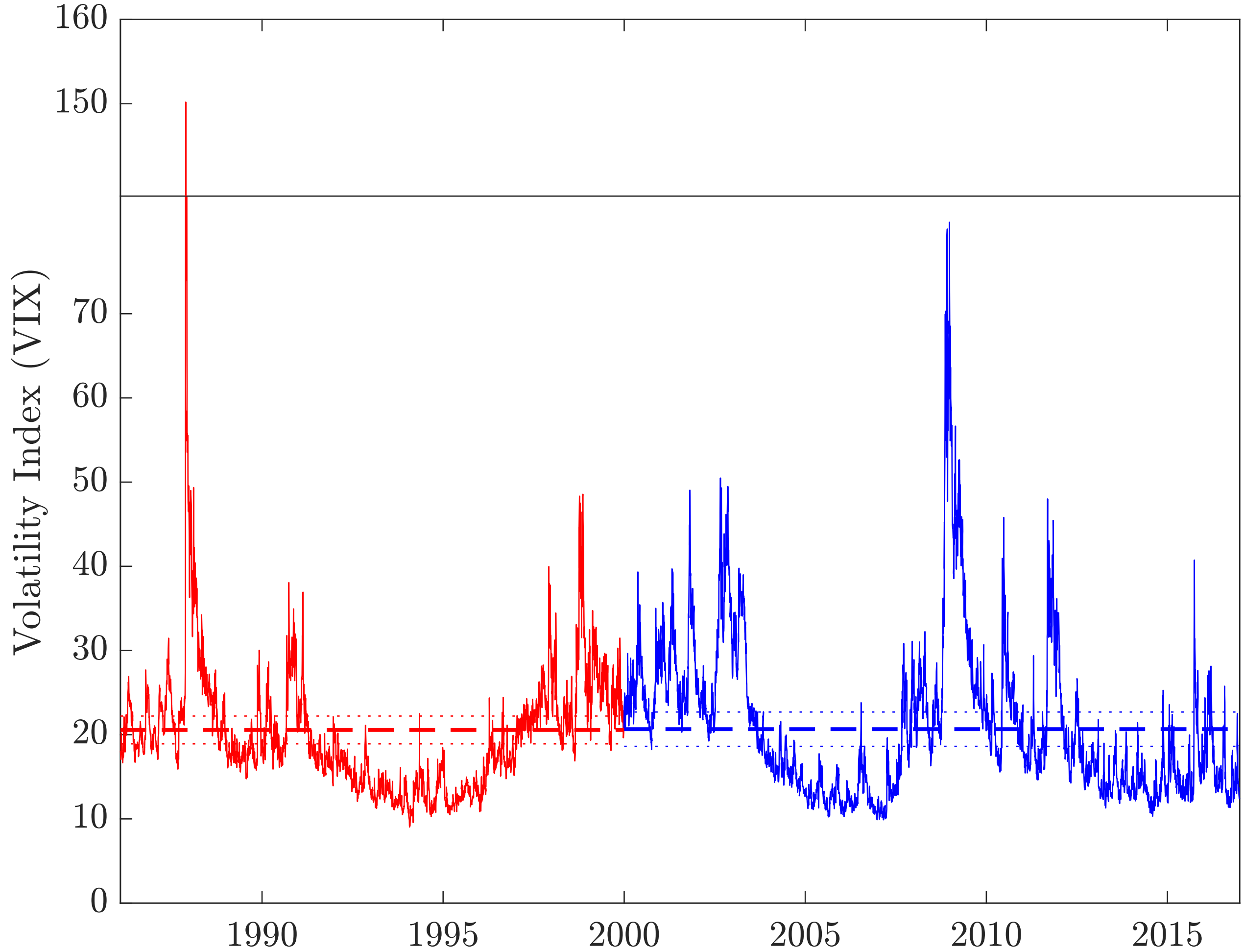
Notes: The solid black line shows expected inflation from the Surveys of Consumers of University of Michigan (FRED Series MICH). The dashed blue line shows the 10-year breakeven inflation rate computed from Treasury Inflation-Indexed Constant Maturity Securities (FRED Series T10YIE).

Figure 6: Disaster probability in Farhi and Gourio (2018) under different risk aversion



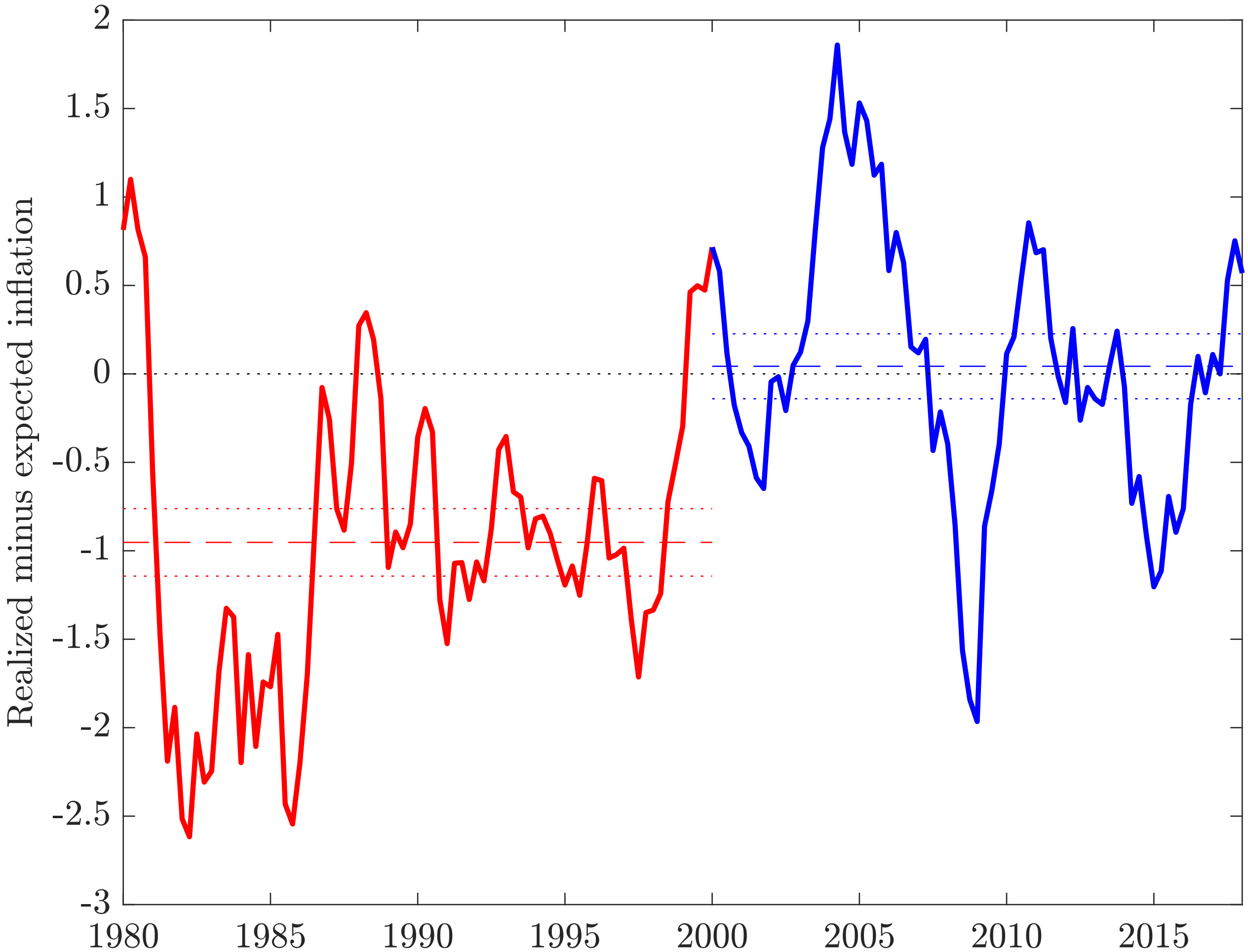
NOTES: In this figure, we match the price-dividend ratio and the riskfree rates computed over 1984–2000 (blue, solid line) and 2001–2016 (red, dashed line) using the model in Farhi and Gourio (2018). We fix the EIS $\psi = 2$ and the disaster size $Z = -\log(0.85)$. Following Farhi and Gourio (2018), we use the growth rates as estimated in the data (3.5% and 2.8%, respectively). We let $\beta = 0.967$ in the first period and let $\beta = 0.979$ in the second period. The figure shows the disaster probability necessary to match the data for a given coefficient of relative risk aversion. A risk aversion coefficient of 12 is used in Table 1 and by Farhi and Gourio (2018).

Figure 7: CBOE Volatility Index (VIX)



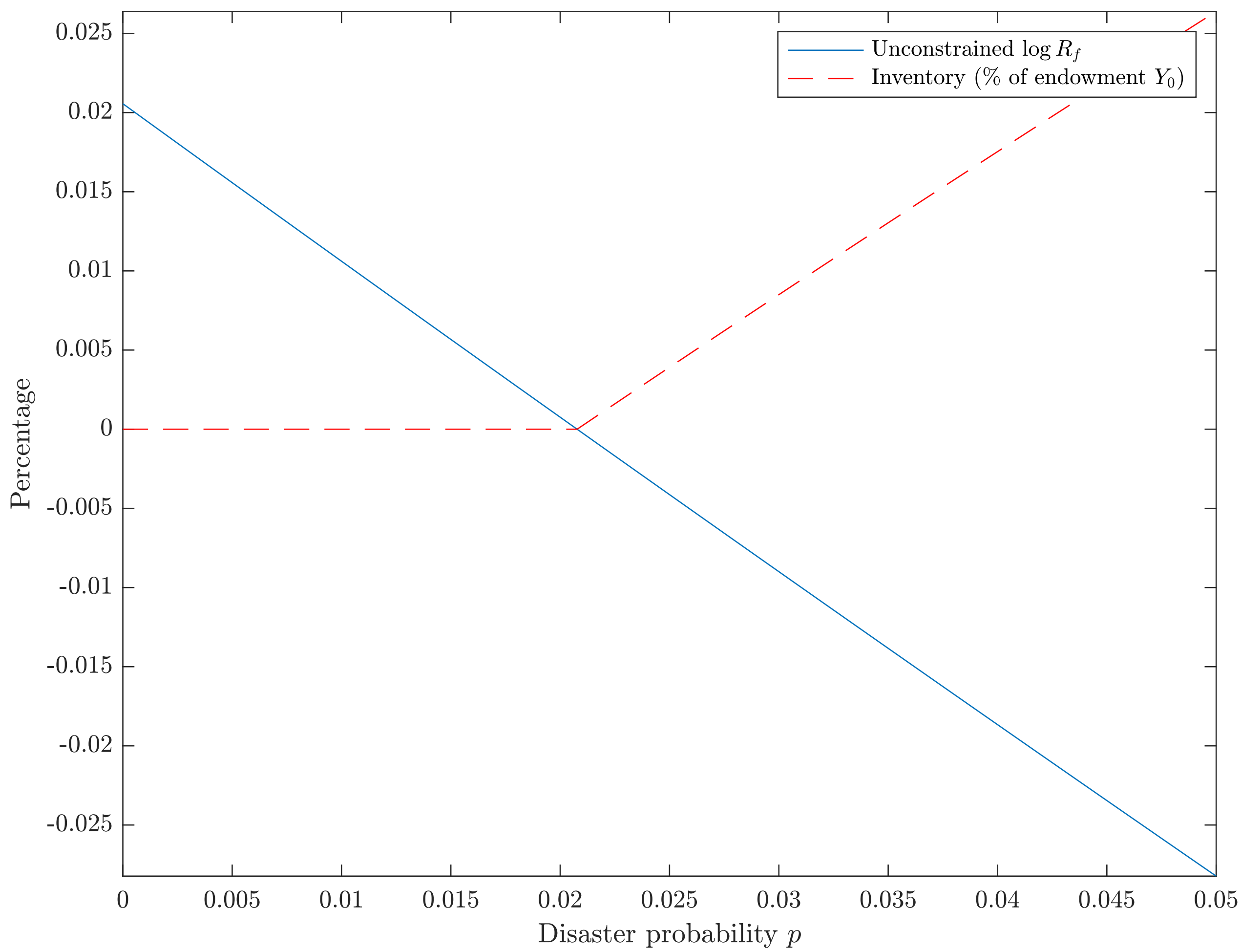
NOTES: The figure plots the VIX series from 1986 to 2020 from the Chicago Board Options Exchange (CBOE). The solid, black line is the VIX index, which represents the market's expectation of 30-day ahead volatility. The long dashed red line is the average VIX from the beginning of the series to the end of the year 2000. The long dashed blue line shows the average VIX from the beginning of 2001 onward. Estimated averages in both samples are plotted with a two-standard-error confidence interval using Newey-West standard errors with two lags on the monthly VIX.

Figure 8: Realized versus expected one-year inflation



NOTES: The figure plots the difference between realized and expected one-year inflation, where expectations are taken from the Survey of Professional Forecasters. The horizontal dashed lines show the average difference in each of our respective samples along with two-standard-error confidence intervals. These averages could be interpreted as estimates of $-p\zeta$ in our model.

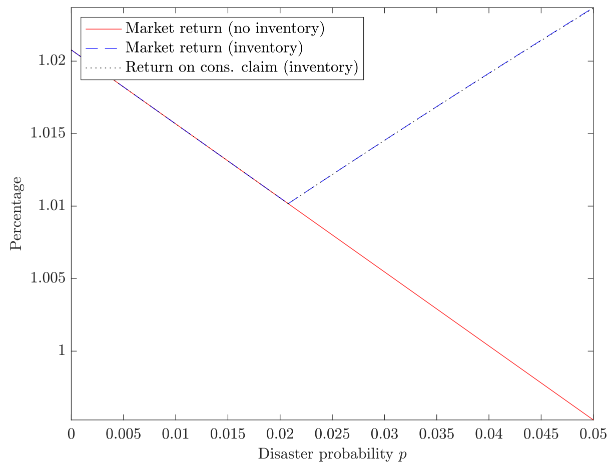
Figure 9: Riskfree rate with and without inventory



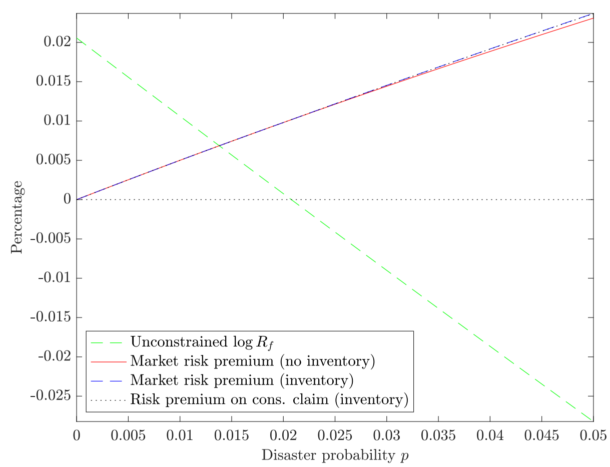
NOTES: The figure shows the unconstrained riskfree rate and inventory demand implied by our two-period model. The unconstrained riskfree rate is defined as the riskfree rate in an economy with no inventory asset. The inventory demand is represented by the ratio of inventory to the initial endowment I/Y_0 in an economy with the inventory asset. Both values are calculated under the same calibration with log utility.

Figure 10: Risky asset returns with and without inventory

(a) Expected returns

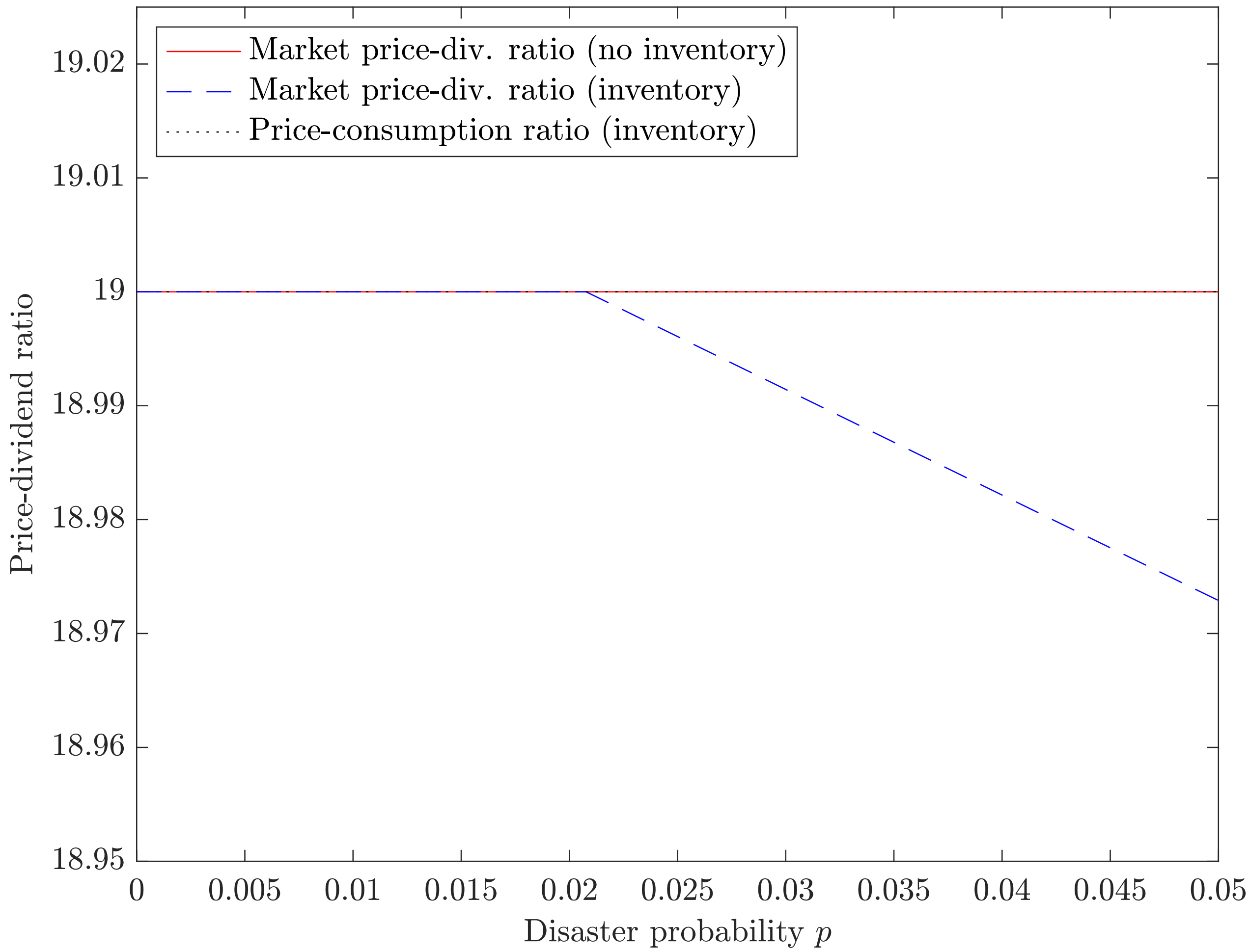


(b) Risk premia



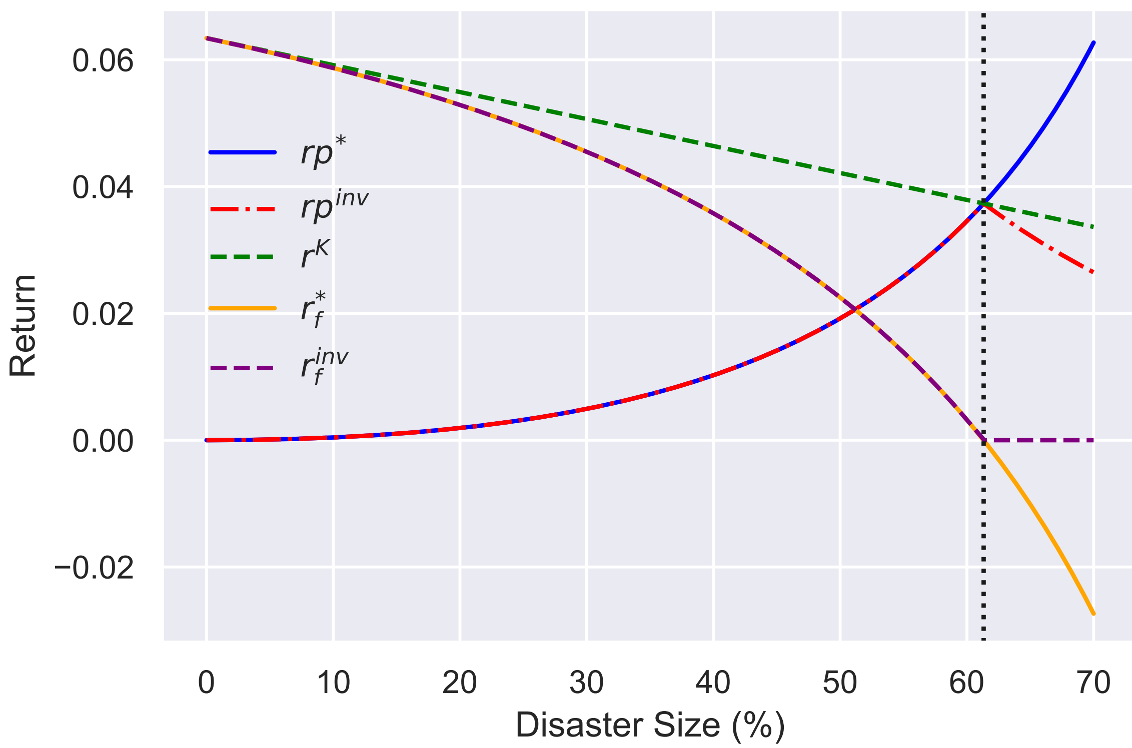
NOTES: The figure shows the expected returns (Panel **(a)**) and risk premia (Panel **(b)**) of three risky assets in our two-period model. Specifically, we consider two economies: one with and one without the inventory asset. For each economy, we consider the consumption claim, which pays a dividend of $Y_1 + I$, and the market (the output claim), which pays Y_1. The consumption and output claims are the same asset in the economy with no inventory. Panel **(b)** also shows the unconstrained riskfree rate, defined as the riskfree rate in an economy with no inventory asset. All values are calculated under the same calibration with log utility.

Figure 11: Price-dividend ratios



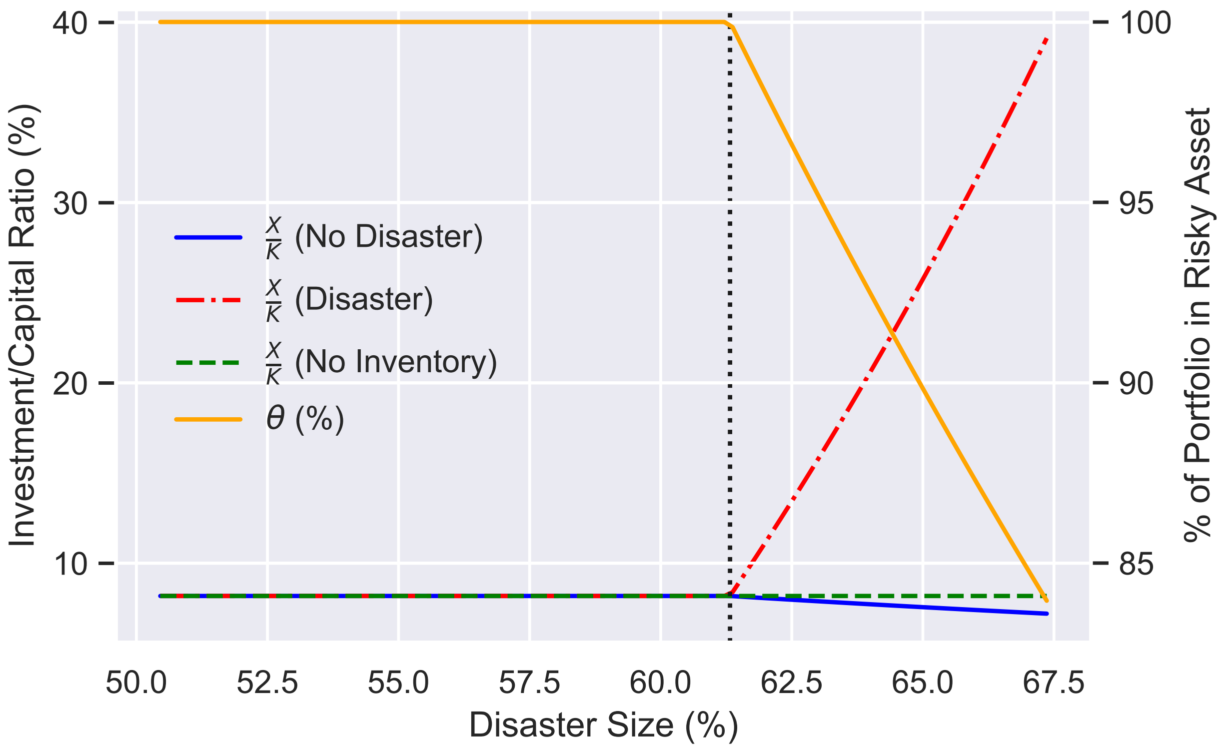
NOTES: The figure shows the price-dividend ratio of three risky assets in our two-period model. Specifically, we consider two economies: one with and one without the inventory asset. For each economy, we consider the consumption claim, which pays a dividend of $Y_1 + I$, and the market (the output claim), which pays Y_1. The consumption and output claims are the same asset in the economy with no inventory. The price-consumption ratio with inventory and the market price-dividend ratio with no inventory are both constant and equal to $\beta/(1-\beta)$. All values are calculated under the same calibration with log utility.

Figure 12: Investment-capital ratio in the model



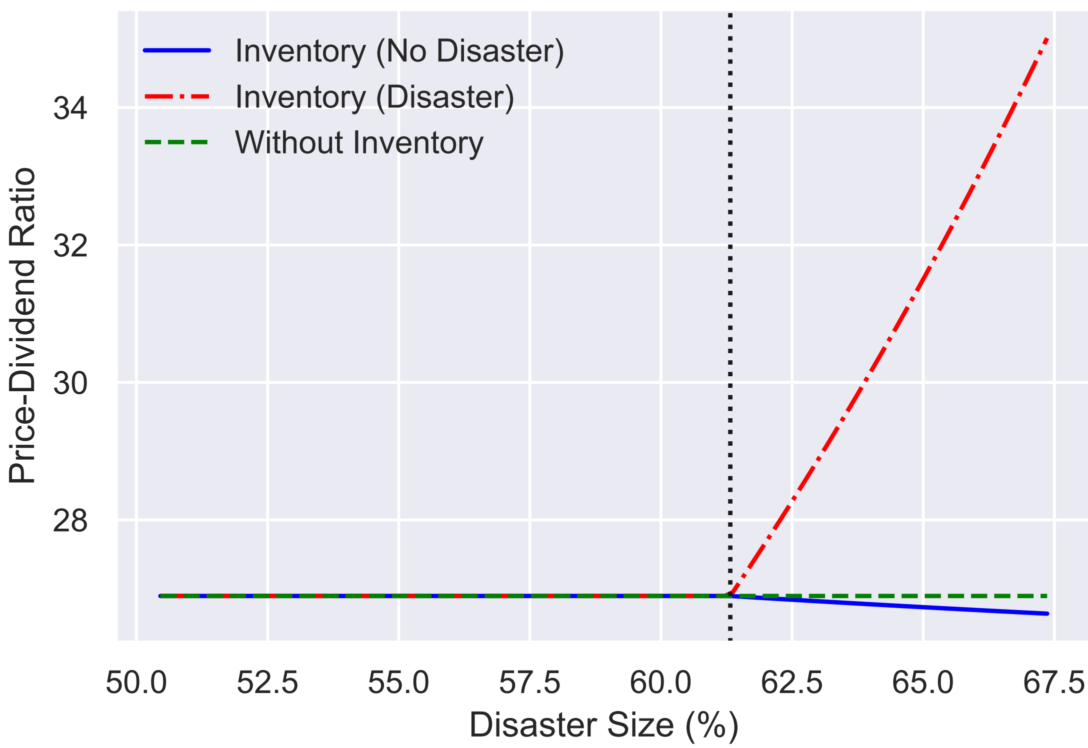
NOTES: This figure shows the riskfree rate and risk premium for the AK model with log utility for varying levels of the disaster shock. The solid blue line represents the risk premium when $\theta = 1$ is imposed, rp^*, (meaning that this is the standard AK model with log utility and capital quality shocks). The dashed and dotted red line represents the risk premium on the output claim in the model with inventory rp^{inv}. The dashed green line shows the return on capital, r^K, which, to the right of the dotted black line, is the risk premium on capital in the model with inventory. The solid orange line shows the riskfree rate in the model where the capital share of invested wealth $\theta = 1$ is imposed r_f^*, and the dashed purple line is the riskfree rate when the agent chooses to hold inventory, r_f^{inv}. For this diagram, we set patience parameter $\beta = .964$, depreciation $\delta = .057$, the probability of disaster $p = .04$, and the marginal product of capital $A = .12$. The dotted black line represents the point at which the riskfree rate is equal to 0 in the model without inventory.

Figure 13: Investment capital ratio in the model



Notes: This figure shows how capital investment varies in the AK model with log utility for different levels of the disaster shock. The solid blue line shows the investment-capital ratio when there is no disaster, while the dashed and dotted red line shows the price-dividend ratio in the state where there is a disaster. The dashed green line shows the investment-capital ratio in the model without inventory. Finally, the solid orange line shows θ, the fraction of capital invested wealth over invested wealth, in percentage terms. For this diagram, we set patience parameter $\beta = .964$, depreciation $\delta = .057$, the probability of disaster $p = .04$, and the marginal product of capital $A = .12$. The dotted black line represents the point at which the riskfree rate is equal to 0 in the model without inventory.

Figure 14: Price-dividend ratio in the model



NOTES: This figure shows how the price-dividend ratio varies in the AK model with log utility for different levels of the disaster shock. The solid blue line shows the price-dividend ratio when there is no disaster, while the dashed and dotted red line shows the price-dividend ratio in the state where there is a disaster. The dashed green line price dividend ratio in the model without inventory (here when $\theta = 1$). For this diagram, we set patience parameter $\beta = .964$, depreciation $\delta = .057$, the probability of disaster $p = .04$, and the marginal product of capital $A = .12$. The dotted black line represents the point at which the riskfree rate is equal to 0 in the model without inventory.

Table 1: Calibration for disaster probability

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
Price-dividend ratio κ	42.34	50.11
Riskfree rate r_f	0.0279	-0.0035
Panel B: Baseline Model Parameters		
μ	0.0350	0.0282
β	0.967	0.979
p	0.0343	0.0663
Panel C: No Growth Change		
μ	0.0350	0.0350
β	0.967	0.978
p	0.0343	0.0793
Panel D: $\psi = 0.5$		
μ	0.0350	0.0282
β	0.997	0.983
p	0.0343	0.0667
Panel E: No Growth Change, $\psi = 0.5$		
μ	0.0350	0.0350
β	0.997	0.985
p	0.0343	0.0793

Notes: Unless otherwise noted, risk aversion $\gamma = 12$, EIS $\psi = 2$, and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.85)$. In Panel B, consumption growth, μ, is taken from the data and the disaster probability, p, and the patience parameter, β, are calibrated to match the data moments. In Panel C, growth is held constant, and p and β are calibrated to match the data moments. In Panel D, β is held constant, and we attempt to match the data moments given μ and p. Panel E repeats the exercise of Panel B, except with $\psi = 0.5$. All parameters and returns are annual.

Table 2: Calibration for disaster probability under alternative moments

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
US CAPE ratio κ_{US}^{PE}	25.97	26.73
UK Price-dividend ratio κ_{UK}	27.78	30.86
US riskfree rate r_{US}^{f}	0.0279	-0.0035
UK riskfree rate r_{UK}^{f}	0.0503	0.0042
Panel B: US Moments, CAPE Ratio		
μ	0.0350	0.0282
β	0.957	0.968
p	0.0556	0.101
Panel C: UK Moments, PD Ratio		
μ	0.0278	0.0156
β	0.955	0.971
p	0.0134	0.0533

Notes: Unless otherwise noted, risk aversion $\gamma = 12$, EIS $\psi = 2$, and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.85)$. The patience parameter, β, and the probability of disaster, p, are calibrated to match the data moments. Panel B is calibrated to the riskfree rate and uses the same consumption growth, μ, as in Farhi and Gourio (2018) and the cyclically adjusted price-to-earnings ratio of Shiller (2000). Panel C is calibrated to the price-dividend ratio, average GDP growth, and real riskfree rate (average nominal rate less average inflation) from Jordà et al. (2019) for the United Kingdom. All parameters and returns are annual.

Table 3: Contribution of each parameter

Panel A: $\psi = 2$

	Parameter Values			Data Moments	
	β	μ	p	κ	r_f
Baseline Calibration (1984–2000)	0.967	0.0350	0.0343	42.34	0.0279
Higher β	0.979	0.0350	0.0343	94.74	0.0151
Higher β & lower μ	0.979	0.0282	0.0343	71.44	0.0117
Baseline Calibration (2001–2016)	0.979	0.0282	0.0663	50.11	-0.0035

Panel B: $\psi = 0.5$

	Parameter Values			Data Moments	
	β	μ	p	κ	r_f
Baseline Calibration (1984–2000)	0.997	0.0350	0.0343	42.34	0.0279
Lower β	0.983	0.0350	0.0343	25.63	0.0428
Lower β & lower μ	0.983	0.0282	0.0343	31.27	0.0292
Baseline Calibration (2001–2016)	0.983	0.0282	0.0667	50.11	-0.0035

Notes: Risk aversion $\gamma = 12$ and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.85)$. The first panel shows what happens what parameters are changed sequentially with $\psi = 2$ in Panel A and $\psi = 0.5$ in Panel B. The first row in each panel represents the Farhi and Gourio (2018) calibration from 1984–2000. The second row in each panel uses the same calibration, but move the patience parameter, β, to its 2001–2016 level. The third in each panel row has the same calibration as the second, aside from setting consumption growth, μ, to be the 2001–2016 level. Finally, the fourth row in each panel allows β, μ, and the disaster probability, p, to go to their 2001–2016 levels.

Table 4: Inflation default risk

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
Price-dividend ratio κ	42.34	50.11
Yield on defaultable bond r_b	0.0279	-0.0035
Panel B: Baseline Model Parameters		
μ	0.0350	0.0282
β	0.968	0.975
ζ	0.137	-0.533
Panel C: $\psi = 0.5$		
μ	0.0350	0.0282
β	0.995	0.991
ζ	0.137	-0.533
Panel D: $\gamma = 5$, $Z = -\log(.7)$		
μ	0.0350	0.0282
β	0.974	0.981
ζ	0.594	0.185
Panel E: $\psi = 1$, $\gamma = 5$, & $Z = -\log(.7)$		
μ	0.0350	0.0282
β	0.977	0.980
ζ	0.594	0.185

Notes: Unless otherwise noted, risk aversion $\gamma = 12$, EIS $\psi = 2$, the disaster probability $p = 0.04$, and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.85)$. In each Panel, growth is taken from the data and the relative bond payoff in disasters, ζ, and and the patience parameter, β, are calibrated to match the data moments. Consumption growth, μ, is taken from Farhi and Gourio (2018). All parameters and returns are annual.

Table 5: Calibration for inflation default risk under alternative moments

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
US CAPE ratio κ_{US}^{PE}	25.97	26.73
UK Price-dividend ratio κ_{UK}	27.78	30.86
US bond rate r_{US}^{b}	0.0279	-0.0035
UK bond rate r_{UK}^{b}	0.0503	0.0042
Panel B: US Moments, CAPE Ratio		
μ	0.0350	0.0282
β	0.954	0.958
ζ	-0.333	-1.043
Panel C: UK Moments, PD Ratio		
μ	0.0278	0.0156
β	0.960	0.969
ζ	0.738	-0.289

Notes: Unless otherwise noted, risk aversion $\gamma = 12$, EIS $\psi = 2$, the disaster probability $p = 0.04$, and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.85)$. The patience parameter, β, and the relative bond payoff in disasters, ζ, are calibrated to match the data moments. Panel B is calibrated to the riskfree rate used in Farhi and Gourio (2018) and the cyclically adjusted price-to-earnings ratio of Shiller (2000). Consumption growth, μ, is taken from Farhi and Gourio (2018). Panel C is calibrated to the price-dividend ratio and real riskfree rate (average nominal rate less average inflation) from Jordà et al. (2019) for the United Kingdom. Consumption growth is also taken from Jordà et al. (2019). All parameters and returns are annual.

Table 6: Inflationary default under different disaster probabilities

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
US CAPE ratio κ_{US}^{PE}	25.97	26.73
US bond rate r_{US}^{b}	0.0279	-0.0035
Panel B: $p = 0.03$		
β	0.963	0.964
ζ	0.0646	-0.430
Panel C: $p = 0.04$		
β	0.963	0.964
ζ	0.305	-0.118
Panel D: $p = 0.05$		
β	0.963	0.964
ζ	0.463	0.090

Notes: Unless otherwise noted, risk aversion $\gamma = 5$, EIS $\psi = 1$, and disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.70)$. The patience parameter, β, and the relative bond payoff in disasters, ζ, are calibrated to match the data moments. All panels are calibrated to the riskfree rate and consumption growth, μ, used in Farhi and Gourio (2018) and the cyclically adjusted price-to-earnings ratio of Shiller (2000). In each panel, the 1984–2000 period has $\mu = 0.0350$ and the 2001–2016 period has $\mu = 0.0282$.

Table 7: Inventory and inflationary default in a model with production

	Values	
	1984–2000	2001–2016
Panel A: Moments in the data		
US CAPE ratio κ_{US}^{PE}	25.97	26.73
US bond rate r_{US}^{b}	0.0279	-0.0035
Growth rate μ	0.0368	0.0189
Panel B: With Inventory		
β	0.963	0.964
ζ	0.247	-0.036
δ	0.043	0.057
θ	1.000	0.898
$\frac{X}{K}$	0.080	0.076
r_f^*	0.003	-0.010
Panel C: Without Inventory		
β	0.963	0.964
ζ	0.247	0.100
δ	0.043	0.061
θ	1.000	1.000
$\frac{X}{K}$	0.080	0.082
r_f^*	0.003	-0.014

Notes: The model is solved under log-utility so risk aversion $\gamma = 1$ and EIS $\psi = 1$. The disaster size (decline in log consumption in the event of a disaster) $Z = -\log(.35)$, the probability of disaster $p = .04$, and the marginal product of capital $A = .12$. The patience parameter, β, the relative bond payoff in disasters, ζ, and depreciation, δ, are calibrated to match the data moments. All panels are calibrated to the riskfree rate used in Farhi and Gourio (2018), the cyclically adjusted price-to-earnings ratio of Shiller (2000), and the real GDP growth rates from FRED series GDPC1. Each of these are averages over the two periods, 1984–2000 and 2001–2016. Panel B shows the model solved assuming the agent can hold inventory, and Panel C assumes this is not the case. In each panel, we report the capital share of invested wealth, θ, and the investment-capital ratio, $\frac{X}{K}$, and the unconstrained riskfree rate, r_f^*.