Search Complementarities, Aggregate Fluctuations, and Fiscal Policy

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Abstract

We develop a quantitative business cycle model with search complementarities in the inter-firm matching process that entails a multiplicity of equilibria. An active static equilibrium with strong joint venture formation, large output, and low unemployment can coexist with a passive static equilibrium with low joint venture formation, low output, and high unemployment. Changes in fundamentals move the system between the two static equilibria, generating non-linear, large, and persistent business cycle fluctuations and a bimodal distribution of output. The volatility of shocks is important for the selection and duration of each static equilibrium. Sufficiently adverse shocks, such as a financial crisis or a pandemic, in periods of low macroeconomic volatility trigger severe and protracted downturns. The magnitude of a fiscal stimulus is critical to foster economic recovery in the passive static equilibrium, while it plays a limited role in the active static equilibrium.

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1 Introduction

Search often involves two parties. Workers search for firms and firms search for workers. Customers search for shops and shops search for customers. Entrepreneurs search for venture capitalists and venture capitalists search for entrepreneurs.

Two-sided searches generate a strategic complementarity. If the probability of a match is supermodular on the search effort exerted by the parties, an increase in the search effort by one party will raise (under some conditions on the search costs function) the other party's search effort. Conversely, a decrease in the search effort by one party will lower the other party's search effort. Depending on fundamentals (i.e., payoff-relevant variables such as productivity or the discount factor), this strategic complementarity begets a unique static Nash equilibrium (i.e., an equilibrium for the current period) where both agents search with low effort, a unique static Nash equilibrium where both agents search with high effort, or multiple static Nash equilibria with different search efforts.

In this context, shocks to fundamentals have direct and indirect effects. For example, if matches are persistent, the direct effect of a higher discount factor is to increase the search effort by agents, independently of what other agents do. But since all the agents are searching more, the supermodularity of matching probabilities kicks in. We may get the indirect effect of a switch from the static Nash equilibrium with low search effort to the static Nash equilibrium where the search effort is higher than if we only had the direct effect. In other words, search complementarities amplify and propagate shocks to fundamentals. Loosely speaking, search complementarities provide a microfoundation for what would appear, at first sight, to be increasing returns to matching in the spirit of Diamond (1982).

To explore this amplification and propagation mechanism, we build a quantitative business cycle model and calibrate it to U.S. data. Firms post job vacancies and fill them with workers from households in an off-the-shelf Diamond-Mortensen-Pissarides (DMP) frictional labor market. The DMP block of the model gives us a natural framework to analyze unemployment and vacancies but, for simplicity, will not present search complementarities. Once vacancies are filled, firms must match among themselves in long-lasting relationships to produce output. This mechanism captures the inter-firm linkages embedded in contemporary value-added chains. For example, an airplane manufacturer must find a producer of carbon fiber reinforced thermoplastics

(CFRPs) to complete a plane and a CFRP producer must find an airplane manufacturer that will purchase its products. In our model, the search effort among firms is supermodular. When airplane manufacturers and CFRP manufacturers search for each other with high effort, output is high and unemployment low. Otherwise, output is low and unemployment high.

We interpret the search friction among firms as reflecting the process of firms engaging in complex relationships, which goes well beyond locating potential partners. We have in mind, among others, the costly effort by buyers in analyzing vendors (in our example, assessing the quality of CFRPs delivered by a new supplier and checking their suitability for proprietary and well-guarded production processes) and in completing all the required contractual arrangements, certifications, and regulatory compliance procedures. For the suppliers, we have in mind the costly effort related to advertising and branding, participating in trade fairs, tendering offers, adapting production processes to highly specific buyer requirements, and setting up supply procedures to process and track orders from a new buyer. The ample space dedicated to these topics in operations management textbooks (e.g., Heizer et al., 2016, or Stevenson, 2018) is compelling evidence of how seriously the industry takes this friction. The authors' insistence on the importance of investing enough resources in building a supply chain demonstrates the role of effort in succeeding or failing to create an inter-firm match.

In terms of exogenous movements in fundamentals, households are subject to discount factor shocks, and firms experience productivity shocks. Since households own the firms in the economy, the discount factor shocks also affect how firms discount the future.

Thus, in our model, the return from establishing a joint venture between firms depends on fundamentals *and* on the search effort of potential partners. The latter dependence generates a region of state variable values where there is a unique *passive* static equilibrium (where firms search for partners in the current period with zero effort), a region where there is a unique *active* static equilibrium (where firms search for partners in the current period with positive effort), and a region where both static equilibria exist. In this case, we will assume that the economy stays in the same static equilibrium as in the previous period: if yesterday firms did not search, today firms still do not search; if yesterday firms searched with positive effort, today firms still search. History dependence is both an intuitive and transparent equilibria selection device and a well-documented predictor of empirical behavior in coordination games similar to ours (see the classic findings in Van Huyck et al., 1990, 1991).

Since in the active static equilibrium, firms post more vacancies, output is higher, and unemployment lower than in the passive static equilibrium, shocks to the discount factor induce large aggregate fluctuations by switching the economy between the regions of uniqueness and the multiplicity of static equilibria. Furthermore, once the economy is at one static equilibrium, it remains there until a sufficiently large discount factor shock, such as those triggered by a financial crisis or a pandemic, terminates the equilibrium. In the meantime, even if the alternative static equilibrium reappears, the economy is stuck in the previous static equilibrium. Hence, search complementarities can transform transitory negative shocks into protracted slumps. This phenomenon might explain the aftermath of the Great Recession in the U.S., where output remained below trend and employment-to-population ratios were depressed for a decade. Through the lenses of our model, the economy moved in 2008 to a static equilibrium with less search, and it did not abandon it even after the original adverse shocks evaporated.

If the model starts from the active equilibrium deterministic steady state, a one-period adverse shock to the discount factor of 12% moves the system to the passive static equilibrium, increasing the unemployment rate by 3.2% and reducing output by roughly 15%. The drop in output is in the ballpark of the one observed for the U.S. in the Great Recession measured as a deviation with respect to trend (which we ignore in our model to ease notation).[1] Justiniano and Primiceri (2008) estimate a standard deviation of the discount factor equal to 5% in the U.S. post-war period. A reduction of 12% in the discount factor is around a two-and-a-half standard deviation fall in the discount factor, a low probability but not a rare event. Smaller shocks to the discount factor fail to move the system away from the original static equilibrium, and the properties of the system are similar to those of conventional business cycle models. By comparison, the observed U.S. standard deviation of productivity shocks is too small to generate productivity realizations that move the economy from one static equilibrium to the other.

The model matches U.S. business cycle statistics, in particular along two key dimensions. First, the economy generates a strong internal propagation of shocks. The autocorrelation of the variables is larger and closer to the observed data than in standard models without the need to assume highly persistent exogenous shocks. In our model, instead, persistence comes from history dependence. Second, our economy generates endogenous movements in labor productivity and

[1]Between 2007.Q4 and 2014.Q4, output per capita fell 12.4% in the U.S. with respect to its post-war trend. In comparison, unemployment increased, at its peak, from 4.4% to 10.0%, around 50% more than in our model.

more realistic volatility of unemployment than alternative business cycle models.

Even more interestingly, our model creates strong non-linearities and bimodal ergodic distributions of endogenous variables such as output. As Adrian et al. (2019) have documented, output is highly non-linear and the non-linearity is strongly linked to financial conditions (a natural interpretation of our discount factor shock). Second, again as in our model, we show that the pattern of output bimodularity is significant only in periods of high volatility. Most business cycle models fail at accounting for these features of the data.

All our results come without adding expectational shocks to the model as in Kaplan and Menzio (2016). While we could include those, we prefer not to do so to focus more sharply on the interaction between shocks to fundamentals and search complementarities. For the same reason, we will postpone for future research the study of non-Markov strategies by firms, alternative static equilibrium selection devices, and limit cycles.

The data support the two central mechanisms in our model: search complementarities and shocks to the discount factor. To document the existence of search complementarities, we use the Occupational Employment Statistics (OES) database constructed by the BLS. We show how increases in search effort by suppliers (measured as the number of workers involved in advertising, marketing, sales, demonstration, and promotion) correlate strongly with increases in the search effort of industry buyers (measured as the number of workers involved in ordering, buying, purchasing, and procurement). Shocks to the discount factor —proxied by a broad range of indexes— are tightly correlated with the volume of intermediate input, output, unemployment, and partnership creation. Indeed, observed fluctuations in intermediate inputs account for 71% of fluctuations in total industry gross output.

We also explore how the volatility of shocks shapes fluctuations under search complementarities. A fall in macroeconomic volatility, such as the Great Moderation, leads to increased persistence in labor market downturns (see Liu et al., 2019, for evidence that the Great Moderation continued until the current health crisis). Since large shocks are unlikely when volatility is low, once the economy is pushed into the passive static equilibrium due to a rare negative shock, it takes a long time before a new large rare positive shock allows the economy to abandon the passive static equilibrium. Under the Great Moderation, recessions are rarer, but their consequences more severe. Thus, our model suggests that the long-lasting weak recovery from the financial crisis is a direct consequence of the Great Moderation, albeit an unwelcome one.

Finally, we investigate the role of fiscal policy in our model. In our example above, a CFRP producer can supply an airplane manufacturer or provide materials for the construction of a new, seismic-resistant public school in California. If the government increases its expenses (modeled as a rise in government-owned firms such as a new public school), the search incentives for private firms increase, and the economy can switch from a passive static equilibrium to an active one. Thus, the fiscal multipliers can be as high as 3.5 when the fiscal stimulus is of just the right size to move the economy from the passive to the active static equilibrium. Thus, our model supports large fiscal packages after particularly large shocks. On the other hand, if search effort is already high (or the fiscal expansion too small in a passive static equilibrium), the fiscal multiplier will be as low as 0.15.

There is a long tradition in economics of linking strategic complementarities to aggregate fluctuations, going back to Diamond (1982), Weitzman (1982), Howitt (1985), and Diamond and Fudenberg (1989) and explored by Cooper and John (1988), Chatterjee et al. (1993), Huo and Ríos-Rull (2013), and Kaplan and Menzio (2016). Recent papers with strategic complementarities, but with mechanisms different from ours, include Taschereau-Dumouchel and Schaal (2015) (with complementarities in production capacity), Sterk (2016) (with complementarities created by the lost skills of unemployed workers), and Eeckhout and Lindenlaub (2018) (with complementarities between on-the-job search and vacancy posting by firms).

How does our paper add to this tradition? First, we analyze how strategic complementarities interact with shocks to fundamentals in an otherwise standard quantitative business cycle model. Our model, while preserving parsimony, improves upon the empirical performance of conventional business cycle models. Thus, our experiments regarding, for example, fiscal policy provide quantitative guidance for policymakers. Second, by highlighting complementarities in search effort and providing empirical evidence for it, we dispense with increasing returns to scale on production or trading. Third, we postulate an empirically plausible mechanism for static equilibrium switches through variations in the discount factor of the household. Fourth, we show the effects of changes to the exogenous volatility of shocks on our economy, with consequences for the length of static equilibria spells and their switches. Fifth, we go beyond recent models that have highlighted the role of search intensity or vacancy posting (Kehoe et al., 2019, and Leduc and Liu, 2020) in that we generate the sharp non-linear responses and bimodal distribution of output reported by Adrian et al. (2019).

2 A simple model with search complementarities

To build intuition, we present a simple model with search complementarities. This environment embodies the mechanisms at work in our fully-fledged model with greater transparency, but at the cost of quantitative implications that are not designed to account for the data.

2.1 Environment

We start with a deterministic version of the model. The economy is composed of a continuum of islands of unit measure where time is discrete and infinite. Two risk-neutral firms populate each island. Both firms are owned by a representative household, whose only task is to consume the aggregate net profits of all firms in the economy. At the start of the period, firms are in two separate locations within the island, and they must meet to engage in production. If they do not meet, each firm produces zero output. If they do meet, they jointly produce 2 units of output that they split into equal parts. At the end of the period, the match is dissolved, and each firm moves to a new, separate location to search in the next period *ex novo*. Since we will analyze symmetric equilibria where all firms exert the same search effort, we drop the island index. Although realizations of meetings will differ among islands, a law of large numbers will hold in the aggregate economy and individual matching probabilities will equal the aggregate share of islands where matches occur. Similarly, since there are no payoff-relevant state variables carrying information across periods and given our focus on static Nash equilibria for each period, we do not need to specify a discount factor. Thus, for the moment, we drop the time index of each variable.

The probability of meeting is given by a matching function that depends on the search effort of each firm within the island. Specifically, for a search effort $\sigma_1 \in [0, 1]$ of firm 1 and a search effort $\sigma_2 \in [0, 1]$ of firm 2, the matching probability function is:

$$\pi(\sigma_1, \sigma_2) = \frac{1 + \sigma_1 + \sigma_2 + \sigma_1\sigma_2}{4}. \tag{1}$$

This function yields a matching probability of 1/4 when $\sigma_1 = \sigma_2 = 0$, a probability of 1 when $\sigma_1 = \sigma_2 = 1$, and probabilities between 1/4 and 1 in the intermediate cases of search effort.

For an $\alpha \in [0, 1)$, the cost of search effort for firm $i \in \{1, 2\}$ is $c(\sigma_i) = \frac{1+\alpha}{4}\sigma_i + \frac{\sigma_i^3}{3}$.

2.2 Nash equilibria

To find the set of Nash equilibria in our model, we look at the problem of firm 1 when it takes the search effort of firm 2, $\overline{\sigma}_2$, as given. The expected profit function of firm 1 is:

$$J\left(\sigma_1, \overline{\sigma}_2\right) = \frac{1 + \sigma_1 + \overline{\sigma}_2 + \sigma_1 \overline{\sigma}_2}{4} - \frac{1+\alpha}{4}\sigma_1 - \frac{\sigma_1^3}{3}.$$

Maximizing $J\left(\sigma_1, \overline{\sigma}_2\right)$ with respect to σ_1 and noticing that the optimal solution is, for some values of $\overline{\sigma}_2$, at a corner of zero optimal search effort, we get the best response function $\Pi\left(\sigma_2\right)$ for firm 1:

$$\sigma_1^* = \begin{cases} 0 & \text{if}\, \sigma_2 \leq \alpha \\ \frac{1}{2}\sqrt{\sigma_2 - \alpha} & \text{if}\, \sigma_2 > \alpha. \end{cases} \tag{2}$$

Analogously, the best response function $\Pi\left(\sigma_1\right)$ for firm 2 is:

$$\sigma_2^* = \begin{cases} 0 & \text{if}\, \sigma_1 \leq \alpha \\ \frac{1}{2}\sqrt{\sigma_1 - \alpha} & \text{if}\, \sigma_1 > \alpha. \end{cases} \tag{3}$$

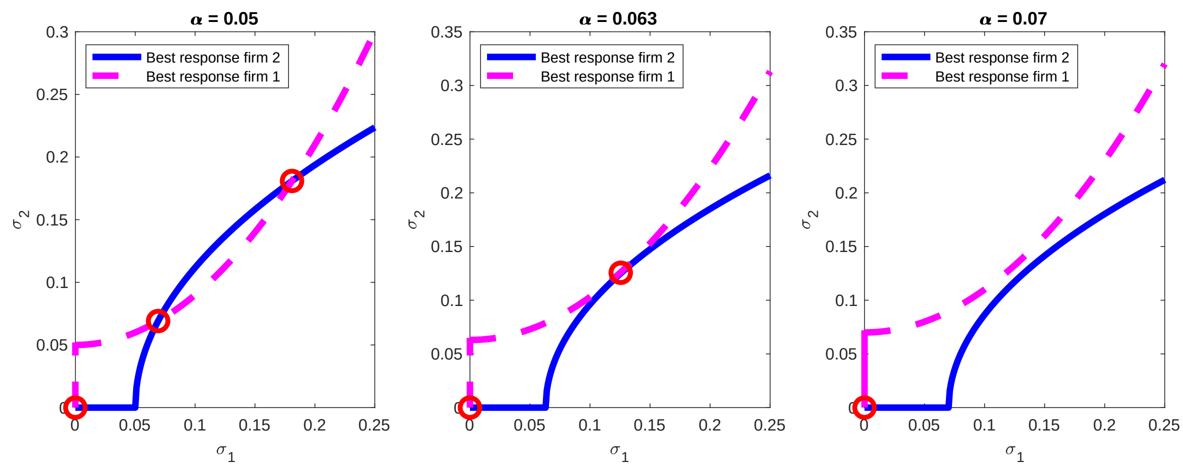
These best response functions explain why we assumed that $\alpha \in [0,1)$. Values of $\alpha < 0$ imply that there is a unique static Nash equilibrium and that such an equilibrium has positive search effort. Values of $\alpha \geq 1$ also yield a unique static Nash equilibrium, but with zero search effort. Only for $\alpha \in [0,1)$ can we have multiple static Nash equilibria.

A tuple $\{\sigma_1^*, \sigma_2^*\}$ is a static pure Nash equilibrium if it is a fixed point of the product of the best response functions (2) and (3) (Footnote 5 explains why we ignore mixed-strategies equilibria; also from here on, we will omit "static" when no ambiguity occurs). Clearly, for all $\alpha \in [0,1)$, $\{\sigma_1^*, \sigma_2^*\} = \{0,0\}$ is a Nash equilibrium. We call this case a *passive* equilibrium, where the matching probability is 1/4, aggregate output y is 1/2, and consumption c is 1/2.

Depending on the value of α, we might have one or two more equilibria in pure strategies with a positive search effort of $\sigma^* = \sigma_1^* = \sigma_2^* > 0$. The matching probability is now given by $\frac{1+2\sigma^*+(\sigma^*)^2}{4}$, gross aggregate output y by $\frac{1+2\sigma^*+(\sigma^*)^2}{2}$, and consumption c by $\frac{1+2\sigma^*+(\sigma^*)^2}{2} - \frac{1+\alpha}{2}\sigma^* - \frac{2}{3}\left(\sigma^*\right)^3$. To derive c, we subtracted the search costs of both firms from output. We call equilibria with positive search effort *active*.

Figure 1 draws three cases: $\alpha = 0.05$ (panel on the left), $\alpha = 0.063$ (central panel), and

Figure 1: Three cases of cost parameter α



$\alpha = 0.07$ (panel on the right). The dashed line plots the best response function of firm 1, the solid line the best response function of firm 2, and the red circles each Nash equilibrium. When $\alpha = 0.05$, there are three Nash equilibria in pure strategies: $\sigma^* = \sigma_1^* = \sigma_2^* = 0$, $\sigma^* = \sigma_1^* = \sigma_2^* = 0.069$, and $\sigma^* = \sigma_1^* = \sigma_2^* = 0.181$. These equilibria are Pareto-ranked: consumption (a welfare measure in our environment) is 0.5 in the first equilibrium, 0.535 in the second equilibrium, and 0.598 in the third equilibrium. When $\alpha = 0.063$, there are two Nash equilibria in pure strategies: $\sigma^* = \sigma_1^* = \sigma_2^* = 0$, and $\sigma^* = \sigma_1^* = \sigma_2^* = 0.126$. Again, the equilibria are Pareto-ranked, with consumption in the active equilibrium equal to 0.565. When $\alpha = 0.07$, the only Nash equilibrium in pure strategies is passive, $\sigma^* = \sigma_1^* = \sigma_2^* = 0$.

2.3 Stochastic shocks

To generate additional results beyond the multiplicity of equilibria, we introduce stochastic shocks in the production function of matched firms. Instead of jointly producing 2 units of output, as in the baseline case, we now assume that firms produce $2z_t$, where z_t is a productivity shock in period t (we must start indexing variables by t, but because of symmetry, there is no need to index them by the island). Productivity shocks will induce movements in the economy along one Nash equilibrium and, sometimes, changes among the Nash equilibria firms play.

The new expected profit function of firm 1 is:

$$J\left(\sigma_{1,t}, \overline{\sigma}_{2,t}, z_t\right) = z_t \frac{1 + \sigma_{1,t} + \overline{\sigma}_{2,t} + \sigma_{1,t}\overline{\sigma}_{2,t}}{4} - \frac{1+\alpha}{4}\sigma_{1,t} - \frac{\sigma_{1,t}^3}{3}.$$

Following the same reasoning as in the deterministic case, the best response function $\Pi(\sigma_{2,t}, z_t)$ for firm 1 is:

$$\sigma_{1,t}^{*} = \begin{cases} 0 & \text{if } z_t\left(1+\overline{\sigma}_{2,t}\right) \leq (1+\alpha) \\ \frac{1}{2}\sqrt{z_t\left(1+\sigma_{2,t}\right)-(1+\alpha)} & \text{if } z_t\left(1+\overline{\sigma}_{2,t}\right) > (1+\alpha), \end{cases} \tag{4}$$

and the best response function $\Pi(\sigma_{1,t}, z_t)$ for firm 2 is:

$$\sigma_{2,t}^{*} = \begin{cases} 0 & \text{if } z_t\left(1+\overline{\sigma}_{1,t}\right) \leq (1+\alpha) \\ \frac{1}{2}\sqrt{z_t\left(1+\sigma_{1,t}\right)-(1+\alpha)} & \text{if } z_t\left(1+\overline{\sigma}_{1,t}\right) > (1+\alpha). \end{cases} \tag{5}$$

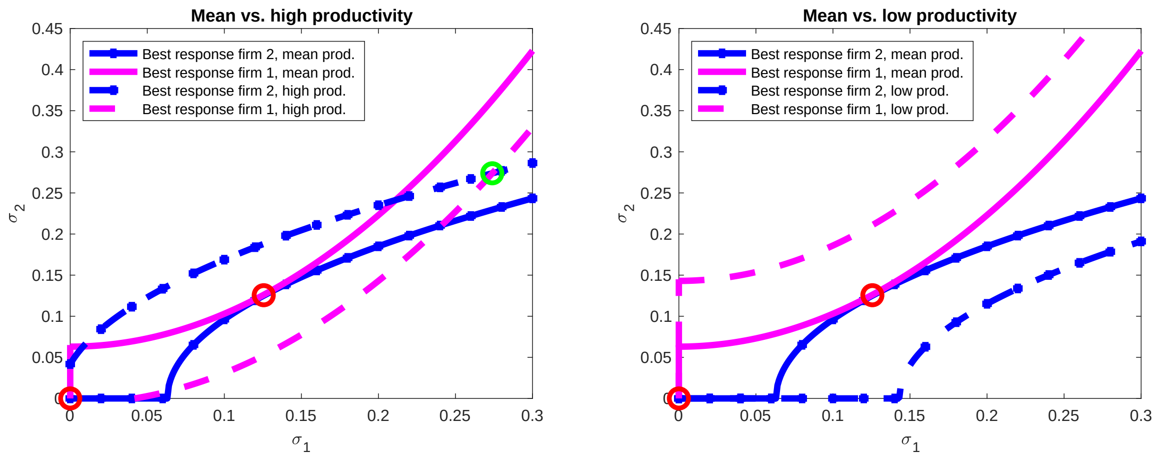
When $z_t = 1$, equations (4) and (5) collapse to equations (2) and (3).

A tuple $\{\sigma_1^*, \sigma_2^*\}$ is a static pure Nash equilibrium if it is a fixed point of the product of the best response functions (4) and (5). As before, we can have one, two, or three Nash equilibria with matching probability given by $\frac{1+2\sigma_t^*+(\sigma_t^*)^2}{4}$, gross aggregate output y_t by $z_t\frac{1+2\sigma_t^*+(\sigma_t^*)^2}{2}$, and consumption c_t by $z_t\frac{1+2\sigma_t^*+(\sigma_t^*)^2}{2} - \frac{1+\alpha}{2}\sigma_t^* - \frac{2}{3}\left(\sigma_t^*\right)^3$.

To illustrate the behavior of our economy, we fix $\alpha = 0.063$ and assume that z_t follows a Markov chain with support $\{0.93, 1, 1.07\}$. Since the values of the transition matrix for this chain will not matter for the next few paragraphs, we momentarily defer its specification. We pick the average value of z_t to be 1 to make the stochastic model coincide, for that realization, with the deterministic environment. The value of $\alpha = 0.063$ ensures that, when $z_t = 1$, there is only one active Nash equilibrium. We pick the high realization of z_t to be 1.07 to get $z_t > 1+\alpha$. When this condition holds, zero search effort is not a Nash equilibrium. We pick a low realization of 0.93 for symmetry.

Figure 2 plots the best response functions under each realization of productivity. The left panel shows in solid lines the best responses for $z_t = 1$ (with crosses for the best response of firm 2). These are the same as those drawn in the central panel of Figure 1 and show two fixed points: one with $\sigma_t^* = \sigma_{1,t}^* = \sigma_{2,t}^* = 0$, and one with $\sigma_t^* = \sigma_{1,t}^* = \sigma_{2,t}^* = 0.126$. Consumption in the first equilibrium is 0.5 and 0.565 in the second equilibrium, even if productivity remains the same. The dashed lines in the same panel are the best responses when $z_t = 1.07$ (with crosses for the best response of firm 2). Now we have a unique Nash equilibrium at $\sigma_t^* = \sigma_{1,t}^* = \sigma_{2,t}^* = 0.274$ (the

Figure 2: Changing productivity z_t



green circle), with consumption at 0.709. The right panel plots in solid lines the best responses for $z_t = 1$, with the same explanation as above. The dashed lines now draw the best responses for $z_t = 0.93$, with a unique Nash equilibrium at $\sigma_t^* = \sigma_{1,t}^* = \sigma_{2,t}^* = 0$ and consumption at 0.465.

Figure 2 illustrates how consumption usually moves more than productivity. For example, consumption increases 27% when the economy starts at the passive equilibrium and z_t moves from 1.0 to 1.07. This amplification mechanism comes from search complementarities: when firm 1 searches more because productivity is higher, firm 2 increases its search effort in response to the higher search effort of firm 1 (and vice versa).

Indeed, the multiplier $|\frac{\Delta c_t/c_t}{\Delta z_t/z_t}|$ of consumption to a productivity shock is state-dependent: the same productivity shock leads to different changes in consumption depending on the state of the economy. Table 1 reports the multiplier in six relevant cases (and where subindexes denote the productivity level and type of equilibria). The multiplier ranges from as low as 1 –when the economy moves from low productivity to mean productivity, as search effort is zero in both cases– to nearly 6 –when the economy moves from mean productivity and zero search effort to high productivity.

Our last task is to specify a transition matrix Π for productivity shocks. We select a standard business cycle parameterization with symmetry and medium persistence:

$$\Pi = \begin{pmatrix} 0.90 & 0.08 & 0.02 \\ 0.05 & 0.90 & 0.05 \\ 0.02 & 0.08 & 0.90 \end{pmatrix}.$$

Table 1: Multiplier

Productivity shock	$\lvert\frac{\Delta c_t/c_t}{\Delta z_t/z_t}\rvert$
$z_{low} \rightarrow z_{mean,passive}$	1
$z_{low} \rightarrow z_{high}$	3.485
$z_{mean,passive} \rightarrow z_{high}$	5.969
$z_{mean,active} \rightarrow z_{high}$	3.627
$z_{high} \rightarrow z_{low}$	4.009
$z_{high} \rightarrow z_{mean,active}$	3.095

When z_t is high or low, the Nash equilibrium is unique. When $z_t = 1$, there are two Nash equilibria, and we select between them through history dependence following Cooper (1994). More concretely, if the economy was in a passive equilibrium in the previous period, we stay in such an equilibrium today. Conversely, if the economy was in an active equilibrium in the previous period, firms continue searching with positive effort today.

Our equilibrium selection has two implications. First, the effects of a productivity shock persist longer than the shock. In particular, the economy cannot move directly from z_{low} to $z_{mean,active}$ or from z_{high} to $z_{mean,passive}$ (this explains why Table 1 does not report these cases). Instead, to switch equilibria, the economy must transition through an intermediate stage of high productivity (when we start from $z_t = 0.93$) or low productivity (when we start from $z_t = 1.07$). Second, we do not generate fluctuations through sunspots. Changes among Nash equilibria in our economy always derive from the movement in fundamentals.

Figure 3: Simulation of aggregate consumption

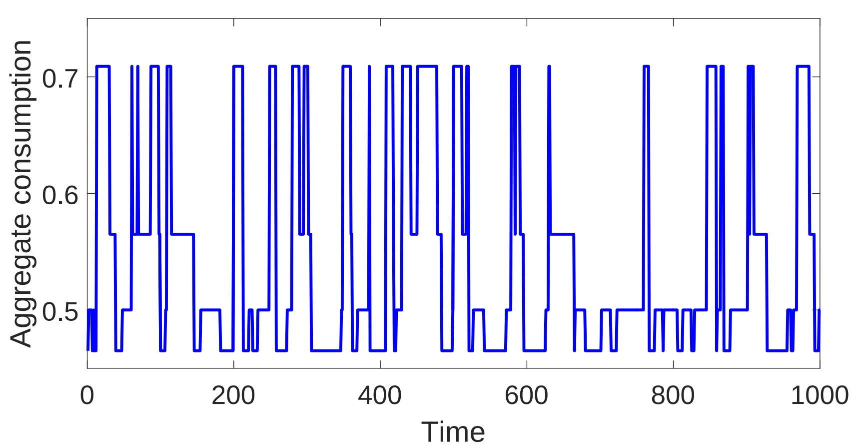


Figure 3 displays a typical realization of consumption for 1,000 periods. Consumption takes four different values: 0.465 ($z_t = 0.93$), 0.5 ($z_t = 1.0$, passive equilibrium), 0.565 ($z_t = 1.0$, active equilibrium), and 0.709 ($z_t = 1.07$). Given Π, the stationary distribution of productivity is $(0.278, 0.444, 0.278)$. Since our simulations start from $z_t = 1.0$ (and an active equilibrium), we have a slightly higher level of mean realizations of productivity, with a count of $(233, 490, 277)$. Consumption is 0.465 in 233 periods and 0.552 in 277 periods. More interesting is the breakdown of the 490 periods when $z_t = 1.0$: 180 happen in a passive equilibrium and 310 in an active equilibrium. Asymptotically, due to the symmetry of Π, the realizations of z_{mean} will split evenly between both levels of consumption.

The simple model has illustrated four points. First, search complementarities create multiple Nash equilibria. Second, the interaction of search complementarities with stochastic shocks amplifies the impact of the latter. Third, the multiplier of consumption to a productivity shock is a highly non-linear function of the state of the economy and the size of the shock. Fourth, history dependence enhances the persistence of aggregate variables to shocks. Next, we show how these four key points appear in a quantitative business cycle model with search complementaries.

3 A model with search complementarities

We work with a search and matching model where time is discrete and infinite. The economy is composed of households, firms in the intermediate-goods production sector (I), and firms in the final-goods production sector (F).

3.1 Households

There is a continuum of households of size 1. Households are risk neutral and discount the future by $\beta\xi_t$ per period. This term is the product of a constant $\beta < 1$ and a discount factor shock ξ_t. Innovations to ξ_t may encapsulate demographic shifts, movements in financial frictions, or fluctuations in risk tolerance that we abstract from. Cochrane (2011) and Hall (2016, 2017) provide evidence for the importance of those shocks as a central source of aggregate fluctuations. Since households own the firms, firms also employ $\beta\xi_t$ to discount future profits.

Households can either work one unit of time per period for a wage w or be unemployed and receive h utils of home production and leisure. Households do not have preferences for working –or searching for a job– in either sector $i \in \{I, F\}$ and receive the aggregate firms' profits.

3.2 Labor matching

At the beginning of each period t, any willing new firm can post a vacancy in either sector at the cost of χ per period to hire job-seeking households. Each firm posts a vacancy for one worker. Vacancies and job seekers meet in a DMP frictional labor market.

Given $u_{i,t}$ unemployed households and $v_{i,t}$ posted vacancies in sector i, a constant-returns-to-scale matching technology $m(u_{i,t}, v_{i,t})$ determines the number of hires and vacancies filled in period t. The new hires start working in period $t+1$. The job-finding rate for unemployed households, $\mu_{i,t} = m(u_{i,t}, v_{i,t})/u_{i,t} = \mu(\theta_{i,t})$, and the probability of filling a vacancy, $q_{i,t} = m(u_{i,t}, v_{i,t})/v_{i,t} = q(\theta_{i,t})$, are functions of each sector's labor market tightness ratio $\theta_{i,t} = v_{i,t}/u_{i,t}$. Then, $\mu'(\theta_{i,t}) > 0$ and $q'(\theta_{i,t}) < 0$: in a tighter labor market, unemployed households are more likely to find a job, and firms are less likely to fill vacancies.

At the end of each period t, already existing jobs terminate at a rate δ and unfilled vacancies expire. We assume that 50% of the newly unemployed workers are assigned to search in each sector. To simplify, once an unemployed worker is assigned to search in one sector, she is not allowed to move to search in another sector (given the symmetry of our model across sectors and our calibration below, workers do not mind this restriction). Appealing to a law of large numbers, unemployment is set by changes in job creation that depend on $\theta_{i,t}$:

$$u_{t+1} = u_t - \underbrace{\left[\mu_I\left(\theta_{I,t}\right)u_{I,t} + \mu_F\left(\theta_{F,t}\right)u_{F,t}\right]}_{\text{Job creation}} + \underbrace{\delta\left(1-u_t\right)}_{\text{Job destruction}} \tag{6}$$

where $u_t = u_{I,t} + u_{F,t}$.

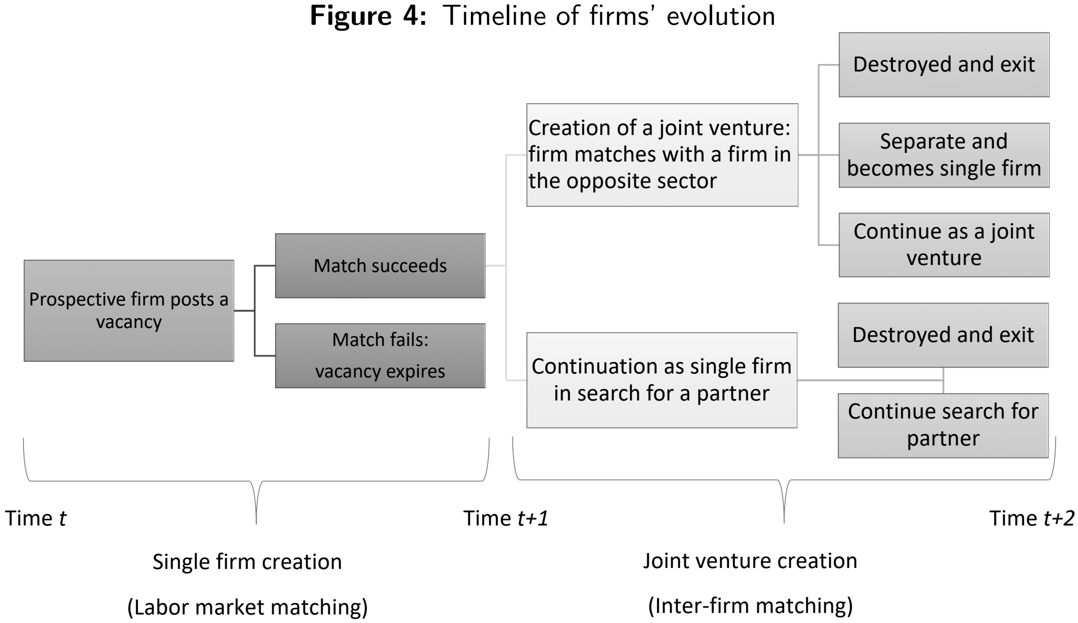
Note that our DMP block is standard. More concretely, we do not include search complementarities on it. The only role of the DMP block is to provide us with a natural framework to discuss unemployment and vacancies without unduly complicating the rest of the model.

3.3 Inter-firm matching

Once job vacancies are filled, a final-goods firm must form a joint venture with an intermediate-goods firm to manufacture together, starting in $t+1$, the final good sold to households. The final good is also the numeraire in the economy. If a firm fails to form a joint venture in period t, it produces no output and continues searching for a partner in $t+1$. This stylized matching problem summarizes more sophisticated inter-firm network structures such as those in Jones (2013) and Acemoglu et al. (2012) and that we motivated in the introduction.

A technology with variable search effort governs inter-firm matching. Search effort is costly, but it reduces the expected duration of remaining unable to produce. At the end of each period, a constant fraction of already existing joint ventures are destroyed because either the job is destroyed with probability δ, or the joint venture fails at a rate $\widetilde{\delta}$.[2] In the former case, the firms dissolve. In the latter case, the firms revert to their status as single firms, but the jobs survive.

Figure 4: Timeline of firms' evolution



The actions of these firms, summarized in Figure 4, require more explanation. In a joint venture, the intermediate-goods firm uses its worker to produce $y_{I,t} = z_t$, where z_t is the stochastic productivity in the intermediate-goods sector. The final-goods firm takes this $y_{I,t}$ and, employing its worker, transforms it one-to-one into the final good, $y_{F,t} = y_{I,t} = z_t$.

Extending the search effort model in Burdett and Mortensen (1980), we assume that the number of inter-firm matches is $M(\widetilde{n}_{F,t}, \widetilde{n}_{I,t}, \eta_{F,t}, \eta_{I,t}) = (\phi + \eta_{F,t}\eta_{I,t}) H(\widetilde{n}_{F,t}, \widetilde{n}_{I,t})$, where $\widetilde{n}_{F,t}$ is the number of single firms in sector F with search effort, $\eta_{F,t}$; $\widetilde{n}_{I,t}$ and $\eta_{I,t}$ are the analogous variables for the I sector. The parameter $\phi > 0$ represents the efficiency in matching unrelated to search effort and it will help us to replicate the average inter-firm matching probabilities in

[2] To simplify the algebra, we assume that, in a joint venture, the jobs in the intermediate-goods firm and the final-goods firm terminate simultaneously with probability δ or survive simultaneously with probability $1 - \delta$. In single firms, the job destruction rate is also δ. Also, we assume that $\delta + \widetilde{\delta} < 1$, and that the separation of job matches and joint ventures is a mutually exclusive event.

the data. The function $H(\cdot)$ has constant returns to scale and it is strictly increasing in both search efforts. We set up its units by choosing $H(1,1) = 1$. Variable search effort generates *strategic complementarities* in the sense of Bulow et al. (1985) since the degree of optimal search effort by one firm will be (weakly) increasing in the number of firms searching in the opposite sector and their search effort.

Given the inter-firm market tightness ratio $\widetilde{n}_F/\widetilde{n}_I$, the probability that a sector I firm will form a joint venture with a sector F firm is:

$$\pi_I = \frac{M(\widetilde{n}_F, \widetilde{n}_I, \eta_F, \eta_I)}{\widetilde{n}_I} = (\phi + \eta_F \eta_I) H\left(\frac{\widetilde{n}_{F,t}}{\widetilde{n}_{I,t}}, 1\right), \tag{7}$$

and the probability that a sector F firm will form a joint venture with a sector I firm is:

$$\pi_F = \frac{M(\widetilde{n}_F, \widetilde{n}_I, \eta_F, \eta_I)}{\widetilde{n}_F} = (\phi + \eta_F \eta_I) H\left(1, \frac{\widetilde{n}_{I,t}}{\widetilde{n}_{F,t}}\right). \tag{8}$$

Search effort in sector i, $\eta_{i,t} = \psi + \widetilde{\sigma}_{i,t}^{0.5}$, has a fixed component, $\psi > 0$, and a variable component, $\widetilde{\sigma}_{i,t} \geq 0$. The fixed component ψ bounds the marginal return to searching from below when prospective partners search with zero effort. Each firm optimally chooses $\widetilde{\sigma}_{i,t} \geq 0$ to trade off search cost and the profits from matching success.

We will focus on symmetric equilibria where $\widetilde{n}_{F,t} = \widetilde{n}_{I,t}$ and $\widetilde{\sigma}_{F,t} = \widetilde{\sigma}_{I,t}$ is the same across all firms. Thus, the inter-firm matching probability is:

$$\pi_{F,t} = \pi_{I,t} = \phi + \eta_{F,t}\eta_{I,t} = \phi + \left(\psi + \widetilde{\sigma}_{F,t}^{0.5}\right)\left(\psi + \widetilde{\sigma}_{I,t}^{0.5}\right). \tag{9}$$

If we had $\psi = 0$, $\pi_{F,t} = \pi_{I,t} = \phi + \widetilde{\sigma}_{F,t}^{0.5}\widetilde{\sigma}_{I,t}^{0.5}$, and we could have placed the power 0.5 in equations (7) and (8) and written $\phi + \eta_F^{0.5}\eta_I^{0.5}$ instead, placing the power on the $\widetilde{\sigma}_{i,t}$'s. The case $\psi = 0$ delivers constant returns to scale in the search efforts, always a natural benchmark. However, our parameterization with two separate constants is more convenient. In equation (9), ψ determines the impact for the matching probability of the variable search effort in the opposite sector, while ϕ does not. This will give us, in our calibration in Section 5, identification for $\pi_{F,t}$ and $\pi_{I,t}$ and further flexibility matching the data.[3]

[3] With $\psi > 0$, equation (9) has decreasing returns to scale on $\widetilde{\sigma}_{F,t}$ and $\widetilde{\sigma}_{I,t}$. Nonetheless, $\widetilde{\sigma}_{F,t}^{0.5}\widetilde{\sigma}_{I,t}^{0.5}$, the most relevant term for the quantitative analysis, is homogeneous of degree 1. Since we are looking for a microfoundation to the increasing returns to scale assumption in Diamond (1982) through the endogenous choice of search effort,

The cost of $\widetilde{\sigma}_{i,t}$ is:

$$c\left(\widetilde{\sigma}_{i,t}\right)=c_0\widetilde{\sigma}_{i,t}^{0.5}+c_1\frac{\widetilde{\sigma}_{i,t}^{(1+\nu)/2}}{1+\nu}, \tag{10}$$

where $c_0>0$ creates a quadratic cost tranche and $c_1>0$, where $\nu>1$, a convex cost tranche.

To simplify the algebra, we define $\sigma_{i,t}=\widetilde{\sigma}_{i,t}^{0.5}$. Then, equations (10)-(9) become:

$$c\left(\sigma_{i,t}\right)=c_0\sigma_{i,t}+c_1\frac{\sigma_{i,t}^{1+\nu}}{1+\nu}, \tag{11}$$

and

$$\pi_{F,t}=\pi_{I,t}=\phi+\left(\psi+\sigma_{F,t}\right)\left(\psi+\sigma_{I,t}\right). \tag{12}$$

The first term in equation (11) implies that the net gain from searching can be negative, in which case the firm chooses $\sigma_i=0$. If $c_0=0$, the benefit from an additional unit of search effort is always positive, and the firm chooses $\sigma_i>0$ in all states. Instead, $c_0>0$ generates the non-convexity that triggers, as we will see, multiple equilibria.[4]

The number of joint ventures in period $t+1$ comprises firms that survive job separation and joint venture destruction plus newly formed joint ventures:

$$n_{t+1}=(1-\delta-\widetilde{\delta})n_t+\left(\phi+\left(\psi+\sigma_{F,t}\right)\left(\psi+\sigma_{I,t}\right)\right)\widetilde{n}_{I,t}. \tag{13}$$

The number of single firms in sector i in period $t+1$ includes firms that survive job separation ($(1-\delta)\,\widetilde{n}_{i,t}$), newly created single firms whose vacancies are filled by job-seekers ($\mu_i\left(\theta_{i,t}\right)\cdot u_{i,t}$), and firms whose joint ventures exogenously terminate ($\widetilde{\delta}n_{i,t}$), net of the number of single firms that form joint ventures ($\pi_{i,t}\widetilde{n}_{i,t}$):

$$\widetilde{n}_{i,t+1}=(1-\delta)\,\widetilde{n}_{i,t}+\mu_i\left(\theta_{i,t}\right)u_{i,t}+\widetilde{\delta}n_{i,t}-\pi_{i,t}\widetilde{n}_{i,t}. \tag{14}$$

We will prove below that search complementarities beget multiple static equilibria. As in Section 2, one of these equilibria is *passive*, with $\sigma_{I,t}=\sigma_{F,t}=0$, low production, and high

homogeneity of degree 1 is, indeed, a natural benchmark.

[4]The cost function in Section 2 follows equation (10) when $c_0=\frac{1+\alpha}{4}$, $c_1=1$, and $\nu=2$. The matching probability (1) is nearly the same as the one in equation (9) when $\phi=\frac{3}{16}$ and $\psi=\frac{1}{4}$, except for a term $\frac{1}{4}$ missing in front of $\sigma_{I,t}\sigma_{F,t}$, which we introduced to ensure that the matching probability was always between $(0,1)$. Also, in the simple model, we have a strategic complementarity between two firms in each island; in the complete model, the strategic complementarity is among a continuum of firms.

unemployment. The other equilibria are *active*, with $(\sigma_{I,t}, \sigma_{F,t}) > 0$, high production, and low unemployment. Also, as in Section 2, the selection of static equilibria is history-dependent. Sufficiently large shocks to productivity or the discount factor induce firms to adjust search effort, and the economy shifts from one equilibrium to the other. Otherwise, the economy stays in the same equilibrium as in the previous period.

An indicator function, ι_t, with value 0 if the static equilibrium is passive and 1 if active, keeps track of those equilibria. This indicator function is taken as given by all agents.[5]

3.4 Values of households and firms

We can now define the Bellman equations that determine the value, for each sector i, of an unemployed household $(U_{i,t})$, of an employed household in a single firm $(\widetilde{W}_{i,t})$ and in a joint venture $(W_{i,t})$, of a filled job in a single firm $(\widetilde{J}_{i,t})$ and in a joint venture $(J_{i,t})$, and of a vacant job $(V_{i,t})$. We index all of these value functions by ι_t since they depend on the type of equilibrium at t, which affects the future path of the equilibrium and the match value.

The value of an unemployed household in sector i and equilibrium ι is:

$$U_{i,t|\iota_t} = h + \beta\xi_t \mathbb{E}_t \left[\mu_{i,t} \widetilde{W}_{i,t+1} + (1 - \mu_{i,t}) \, U_{i,t+1} \mid \iota_t \right]. \tag{15}$$

In the current period, the unemployed household receives a payment h. The household finds a job with probability $\mu_{i,t}$ and circulates into employment during the next period, or it fails to find employment with probability $1 - \mu_{i,t}$ and remains unemployed. To save space, we ignore the state variables when presenting the equations, but they are described in Appendix E.

The value of a household with a job in a single firm in sector i is:

$$\widetilde{W}_{i,t|\iota_t} = \widetilde{w}_{i,t} + \beta\xi_t \mathbb{E}_t \left\{ (1-\delta) \left[\pi_{i,t} W_{i,t+1} + (1 - \pi_{i,t}) \, \widetilde{W}_{i,t+1} \right] + \delta U_{i,t+1} \mid \iota_t \right\}. \tag{16}$$

The first term on the right-hand side (RHS) is the period wage $\widetilde{w}_{i,t}$ (to be determined below by Nash bargaining). In period $t + 1$, the match that survives job destruction may either form a

[5] There might exist mixed-strategy Nash equilibria in which firms search with positive variable effort with a certain probability. We ignore those equilibria because Appendix F shows the mixed-strategy is not robust: when one sector changes the probability slightly due to a trembling-hand perturbation, the opposite sector would immediately set the probability to either zero or one. We will leave non-Markov strategies, limit cycles, and alternative equilibria selection devices for future investigation.

joint venture with a firm in the opposite sector with probability $\pi_{i,t}$, gaining the value $W_{i,t+1}$, or otherwise remain a single firm with probability $1-\pi_{i,t}$, with value $\widetilde{W}_{i,t+1}$. With probability δ, the job is destroyed, and the household transitions into unemployment.

The value of a household with a job in a joint venture in each sector i is:

$$W_{i,t|\iota_t} = w_{i,t} + \beta\xi_t\mathbb{E}_t\left[(1-\delta-\widetilde{\delta})W_{i,t+1} + \widetilde{\delta}\widetilde{W}_{i,t+1} + \delta U_{i,t+1} \mid \iota_t\right]. \tag{17}$$

A worker in a joint venture receives the wage $w_{i,t}$. In period $t+1$, the worker becomes unemployed with probability δ, gaining the value $U_{i,t+1}$. With probability $\widetilde{\delta}$, the joint venture is terminated, and the value becomes $\widetilde{W}_{i,t+1}$. Otherwise, the match continues, gaining the value $W_{i,t+1}$.

The value of a single firm in sector i is:

$$\widetilde{J}_{i,t|\iota_t} = \max_{\sigma_{i,t}\geq 0}\left\{-\widetilde{w}_{i,t} - c\left(\sigma_{i,t}\right) + \beta\left(1-\delta\right)\xi_t\mathbb{E}_t\left[\pi_{i,t}J_{i,t+1} + \left(1-\pi_{i,t}\right)\widetilde{J}_{i,t+1} \mid \iota_t\right]\right\}. \tag{18}$$

Equation (18) tells us that single firms have zero revenues until they form a joint venture with a firm in the opposite sector. Despite zero production, the firm pays the wage ($\widetilde{w}_{i,t}$) and incurs search costs $c\left(\sigma_{i,t}\right)$, as described in equation (11). In period $t+1$, conditional on surviving job destruction with probability $1-\delta$, the firm forms a joint venture with probability $\pi_{i,t}$ given by equation (12), gaining the flow value $J_{i,t+1}$. Otherwise, the firm remains single with flow value $\widetilde{J}_{i,t+1}$. If the job is destroyed, the firm exits the market with zero value.

The value of a joint venture for a sector I firm is:

$$J_{I,t|\iota_t} = z_t p_t - w_{I,t} + \beta\xi_t\mathbb{E}_t\left[(1-\delta-\widetilde{\delta})J_{I,t+1} + \widetilde{\delta}\widetilde{J}_{I,t+1} \mid \iota_t\right]. \tag{19}$$

This profit comprises revenues $z_t p_t$ from selling intermediate goods to the final-goods firm, net of the wage $w_{I,t}$. Both p_t and $w_{I,t}$ are determined by Nash bargaining. In period $t+1$, with probability $\widetilde{\delta}$, the firm is separated from its partner and becomes a single firm, gaining a value of $\widetilde{J}_{I,t+1}$; with probability δ, the job match is destroyed, and the firm exits the market with zero value. Otherwise the joint venture continues with flow value $J_{i,t+1}$.

The value of a joint venture for a sector F firm is:

$$J_{F,t|\iota_t} = z_t(1-p_t) - w_{F,t} + \beta\xi_t\mathbb{E}_t\left[(1-\delta-\widetilde{\delta})J_{F,t+1} + \widetilde{\delta}\widetilde{J}_{F,t+1} \mid \iota_t\right]. \tag{20}$$

The profit for the joint venture in the final-goods sector comprises revenues from selling z_t units of final goods at price 1, net of the costs of purchasing intermediate goods ($z_t p_t$) and paying the wage ($w_{F,t}$). The rest of the equation follows the same interpretation as equation (19).

The value of a vacant job in sector i is:

$$V_{i,t|\iota_t} = -\chi + \beta \xi_t \mathbb{E}_t \left[q\left(\theta_{i,t}\right) \widetilde{J}_{i,t+1} + \left(1 - q\left(\theta_{i,t}\right)\right) \max\left(0, V_{I,t+1}, V_{F,t+1}\right) \mid \iota_t \right]. \tag{21}$$

Equation (21) shows that the value of a vacant job comprises the fixed cost of posting a vacancy χ in period t. With probability $q\left(\theta_{i,t|\iota_t}\right)$, the vacancy is filled, and a single firm with flow value $\widetilde{J}_{i,t+1}$ is created. Otherwise, the vacancy remains open, generating the flow value of $V_{i,t+1}$. The last term in the equation shows that firms that fail to recruit a worker may choose to be inactive or post a vacancy in either sector in the next period $t+1$.

By free-entry, we have $V_{i,t} = 0$ and the condition that pins down labor market tightness:

$$\chi = \beta \xi_t \mathbb{E}_t \left[q\left(\theta_{i,t}\right) \widetilde{J}_{i,t+1} \mid \iota_t \right]. \tag{22}$$

3.5 Wages and prices

We can now define the Nash bargaining rules that determine wages and prices. During each period t, wages are pinned down by Nash bargaining between firms in joint ventures and workers:

$$\max_{w_{i,t}} (W_{i,t} - U_{i,t})^{1-\tau} J_{i,t}^{\tau} \tag{23}$$

and between single firms and workers:

$$\max_{\widetilde{w}_{i,t}} (\widetilde{W}_{i,t} - U_{i,t})^{1-\tau} \widetilde{J}_{i,t}^{\tau}, \tag{24}$$

where the parameter $\tau \in [0,1]$ is the firm's bargaining power.

The price for goods manufactured in the intermediate-goods sector is determined by Nash bargaining between the final-goods producer and the intermediate-goods producer within the joint venture:

$$\max_{p_t} (J_{F,t} - \widetilde{J}_{F,t})^{1-\widetilde{\tau}} (J_{I,t} - \widetilde{J}_{I,t})^{\widetilde{\tau}}, \tag{25}$$

where the parameter $\widetilde{\tau} \in [0,1]$ is the intermediate-goods producer's bargaining power.

3.6 Stochastic processes and aggregate resource constraint

The discount factor shock, ξ_t, has a log-normal *i.i.d.* distribution, $\log(\xi_t) \sim \mathcal{N}\left(0, \sigma_\xi^2\right)$. This shock is not persistent over time. In this way, we can show that the propagation mechanism created by ξ_t is, in its entirety, a combination of endogenous forces and history dependence (although introducing persistence in the shock would be straightforward). Productivity follows $\log(z_{t+1}) = \rho_z \log(z_t) + \sigma_z \epsilon_{z,t+1}$, where $\rho_z \leq 1$.

The total resources of the economy, equal to $z_t n_t$ (i.e., production per joint venture times the number of existing joint ventures; h is in util terms and, thus, fails to appear here), are used for aggregate consumption by households, c_t, and to pay for vacancies and inter-firm search:

$$c_t + \sum_{i=I,F} \chi v_{i,t} + \sum_{i=I,F} \widetilde{n}_{i,t} \left(c_0 \sigma_{i,t} + c_1 \frac{\sigma_{i,t}^{1+\nu}}{1+\nu} \right) = z_t n_t. \tag{26}$$

4 Characterizing the equilibrium

The equilibrium definition for our model is standard and we include it in Appendix A. We can use this definition to characterize the optimal search strategy of firms and show the existence of multiple static equilibria.

4.1 Optimal search effort

Following condition 3 above, the optimal $\sigma_{i,t}$ maximizes the value of the single firm, $\widetilde{J}_{i,t}$. We can express this value function as a response function to $\sigma_{j,t}$ given an equilibrium ι_t:

$$\Pi_i\left(\sigma_{i,t} \mid \sigma_{j,t}, \iota_t\right) = -\widetilde{w}_{i,t} - c\left(\sigma_{i,t}\right) + \beta \xi_t \left(1-\delta\right) \mathbb{E}_t \left[\pi_{i,t}(J_{i,t+1} - \widetilde{J}_{i,t+1}) + \widetilde{J}_{i,t+1} \mid \iota_t \right]. \tag{27}$$

A single firm i chooses $\sigma_{i,t}$ to maximize $\Pi_i\left(\sigma_{i,t} \mid \sigma_{j,t}, \iota_t\right)$. The interior solution $\sigma_{i,t} > 0$ satisfies:

$$c_0 + c_1 \sigma_{i,t}^{\nu} = \widetilde{\beta} \underbrace{\left(\psi + \sigma_{j,t}\right)}_{\text{Search effort in sector } j} \underbrace{\xi_t}_{\text{discount factor shock}} \underbrace{\mathbb{E}_t\left(J_{i,t+1} - \widetilde{J}_{i,t+1} \mid \iota_t\right)}_{\text{Expected capital gain}} \tag{28}$$

where $\widetilde{\beta} = \beta(1-\delta)/\tau$ (the wage Nash bargaining implies that the firm bears τ fraction of the search cost). The left-hand side (LHS) of equation (28) is the marginal cost of exerting $\sigma_{i,t}$ to build a joint venture in sector i, while the RHS is the expected benefit of searching for a partner, which increases with $\sigma_{j,t}$, and the expected capital gain from entering into a joint venture, $\mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1} \mid \iota_t)$ times ξ_t. The expected capital gains depend positively on z_t. Hence, condition (27) shows how higher ξ_t or z_t (fundamentals) and higher $\sigma_{i,t}$ (search complementarities) lead to higher $\sigma_{i,t}$.

Because the optimization problem is non-convex, we also have a corner solution $\sigma_{i,t} = 0$, either because the firms in the other sector search too little or because the discounted expected gains from matching are too small. The next proposition summarizes this argument.

Proposition 1. *The optimal $\sigma_{i,t}$ is equal to:*

$$\sigma_{i,t} = \begin{cases} \left[\frac{\widetilde{\beta}(\psi+\sigma_{j,t})\xi_t\mathbb{E}_t\left(J_{i,t+1}-\widetilde{J}_{i,t+1}\mid\iota_t\right)-c_0}{c_1}\right]^{\frac{1}{\nu}} & \text{if } \widetilde{\beta}\left(\psi+\sigma_{j,t}\right)\xi_t\mathbb{E}_t\left(J_{i,t+1}-\widetilde{J}_{i,t+1}\mid\iota_t\right) > c_0 \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Proposition 1 establishes why search complementarities beget a multiplicity of equilibria (this proposition follows directly from equation (28); the proofs of the other propositions and lemmas in this subsection appear in Appendix D). Sufficiently large shocks to either ξ_t or z_t move the system between interior and corner solutions, generating alternate business cycle phases with robust search effort, a large number of joint ventures, and low unemployment with phases marked by no search effort, few joint ventures, and high unemployment. The parameter c_0 determines whether $\sigma_{i,t} > 0$ while c_1 controls the marginal cost of search.

4.2 The deterministic steady states of the model

We study now the existence and stability properties of the deterministic steady states (DSSs) of the model that appear when we shut down the shocks ξ_t and z_t by making them constant and equal to their mean values (both equal to 1). The model encompasses two types of DSSs: a passive DSS with zero search effort ($\sigma_I = \sigma_F = 0$) and active DSSs with positive search effort ($\sigma_I > 0$, $\sigma_F > 0$). The level of economic activity is different across DSSs.

Proposition 2. *The level of output is strictly lower and the unemployment rate is strictly higher in a passive DSS than in an active DSS.*

Intuitively, zero search effort in the passive DSS implies few joint ventures and low production. A small probability of forming a joint venture reduces the value of a single firm and generates a fall in posted vacancies and an increase in unemployment.

The next two propositions establish conditions for the existence of the different DSSs.

Proposition 3. *The passive DSS exists if and only if*

$$\frac{\tilde{\beta}\psi}{2-2\beta\left[\left(1-\delta-\tilde{\delta}\right)-(1-\delta)\left(\phi+\psi^{2}\right)\right]}<c_{0}. \tag{30}$$

Proposition 3 states that the passive DSS exists for any sufficiently large value of c_0—that is, when the benefit from an additional unit of search effort is lower than the cost associated with it. In such a case, $\sigma_I=\sigma_F=0$. The critical cost for the existence of the passive DSS is c_0. In comparison, c_1 does not appear in Proposition 3.

Proposition 4. *An active DSS exists if and only if there exists* $\sigma\in\left(0,\sqrt{1-\phi}-\psi\right)$ *that solves*

$$\tilde{\beta}\left(\psi+\sigma\right)\frac{1+\left(c_{0}\sigma+c_{1}\frac{\sigma^{1+\nu}}{1+\nu}\right)}{2-2\beta\left[\left(1-\delta-\tilde{\delta}\right)-(1-\delta)\left(\phi+(\sigma+\psi)^{2}\right)\right]}=c_{0}+c_{1}\sigma^{\nu}. \tag{31}$$

The LHS of equation (31) captures the marginal gain of searching with positive effort in the active equilibrium. The RHS reflects the marginal cost of searching. In the active DSS, both quantities must be equal. Proposition 4 defines the parameter values that guarantee the existence of the active DSS. The restriction $\sigma\in\left(0,\sqrt{1-\phi}-\psi\right)$ ensures that the matching probability $\phi+(\psi+\sigma_I)(\psi+\sigma_F)$ is within (0,1).

Proposition 5. *The active and passive DSSs coexist if and only if equations (30) and (31) hold simultaneously.*

Equations (30) and (31) can hold simultaneously, since they depend on different parameter combinations. The passive DSS characterized by equation (30) is uniquely pinned down when $\sigma_I=\sigma_F=0$. In comparison, the system allows for multiple active DSSs, since equation (31) can hold for different symmetric $(\sigma_{I,t}, \sigma_{F,t}) > 0$. When the best response function is strictly concave (i.e., $\nu > 1$), the system admits, at most, two DSSs (if $\nu < 1$, we would only have one active and unstable equilibrium). The argument is formalized next.

Lemma 1. *The system has a unique passive DSS and at most two active DSSs.*

Figure 17 in Appendix C illustrates these two lemmas by drawing regions of values of c_0 and c_1 for which there exists a unique passive DSS, a unique active DSS, and where both coexist.

The next proposition establishes the stability of the DSSs. This stability guarantees that a slight deviation of a subset of firms from their best response will fail to cause the system to deviate from the initial DSS permanently.

Proposition 6. *Suppose the active and passive DSSs coexist. The passive DSS is stable. When two active DSSs coexist, one DSS is stable and the other DSS is unstable. When only one active DSS exists, it is unstable.*

For the remainder of the analysis, we mainly focus on stable DSSs. Also, we can study the transition path from an arbitrary point in the state space of the system to the DSS. The endogenous state variables of the system are the unemployment rates $(u_{I,t}, u_{F,t})$, the measure of single firms $(\tilde{n}_{I,t}, \tilde{n}_{F,t})$, the measure of firms in joint ventures $(n_{I,t}, n_{F,t})$, and the current equilibrium (ι_t). Knowledge of $\tilde{n}_{i,t}$ and $u_{i,t}$ gives us $n_{i,t} = 1 - \tilde{n}_{i,t} - u_{i,t}$.

Figure 5: Transition path to the DSS

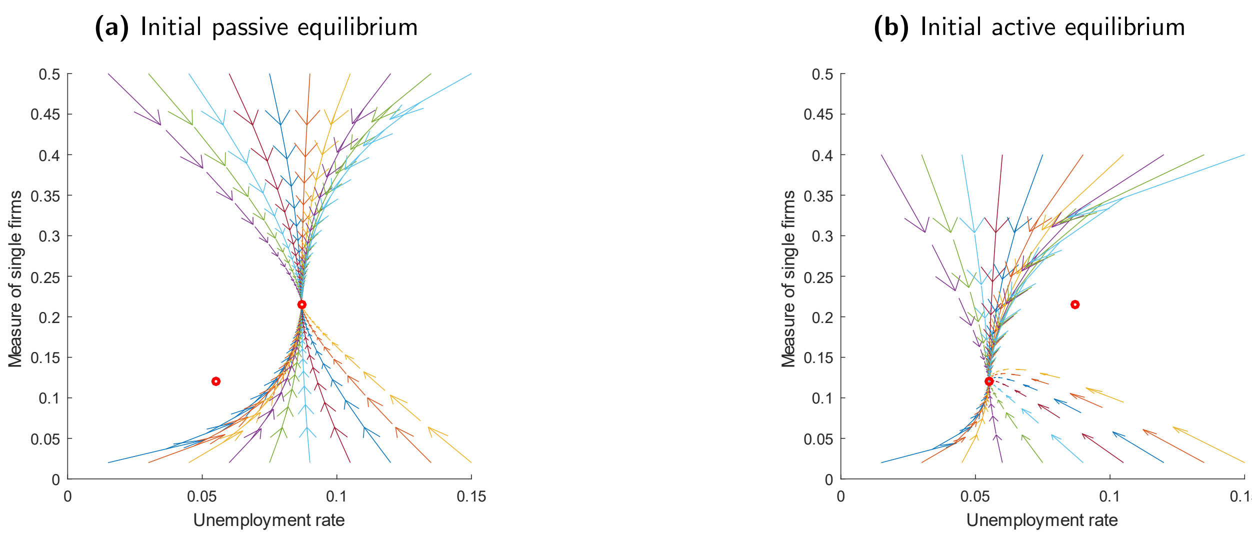


Figure 5 shows the transition path of the system to the DSS for different initial values of the unemployment rate (x-axes) and the measure of single firms (y-axes) and the calibration in

Section 5. Since we consider the case of a symmetric economy, the analysis is representative of each sector. Panel (a) shows the transition path to the DSS when the system starts from a passive equilibrium (with each red dot representing a DSS of the system). Given the history dependence of the equilibrium selection, the system remains in the passive equilibrium and converges to the passive DSS indicated by the higher red circle, where the unemployment rate is 8.7% and the measure of single firms is 22%. Analogously, panel (b) shows that the system converges to the active and stable DSS, when it starts from an active equilibrium. In the active DSS (the lower red dot), the unemployment rate is 5.5%, and the measure of single firms is 12%.

4.3 Existence of two (stochastic) equilibria

Once we have characterized the DSSs of the model, we can reintroduce the shocks to the discount factor and productivity. The following propositions characterize the conditions for the existence of (stochastic) passive and active equilibria and their coexistence.

Proposition 7. *The passive equilibrium exists if and only if*

$$\frac{\partial \Pi_i \left(0|0, \iota_t = 0\right)}{\partial \sigma_{i,t}} \leq 0 \quad \text{for } i = I, F \tag{32}$$

or equivalently

$$c_0 > \widetilde{\beta} \psi \xi_t \mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1} \mid \iota_t = 0). \tag{33}$$

Proposition 7 states that the passive equilibrium exists when the marginal benefit from increasing search effort is negative. Equation (33) highlights that the existence of the passive equilibrium requires either a low ξ_t or a small z_{t+1} (and, hence, a low $\mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1} \mid \iota_t = 0)$).

Proposition 8. *The active equilibrium exists if and only if there exists a pair of positive search efforts ($\{\sigma_{I,t}\sigma_{F,t}\} > 0$) that satisfies:*

$$\frac{\partial \Pi_i \left(\sigma_{i,t} \mid \sigma_{j,t}, \iota_t = 1\right)}{\partial \sigma_{I,t}} = 0 \quad \text{for } i = \{I, F\} \tag{34}$$

or, equivalently,

$$c_0 + c_1 \sigma_{i,t}^{\nu} = \widetilde{\beta} \left(\psi + \sigma_{j,t}\right) \xi_t \mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1} \mid \iota_t = 1), \tag{35}$$

with $(\sigma_{I,t}, \sigma_{F,t}) > 0$ and the second derivatives of Π_i are negative.

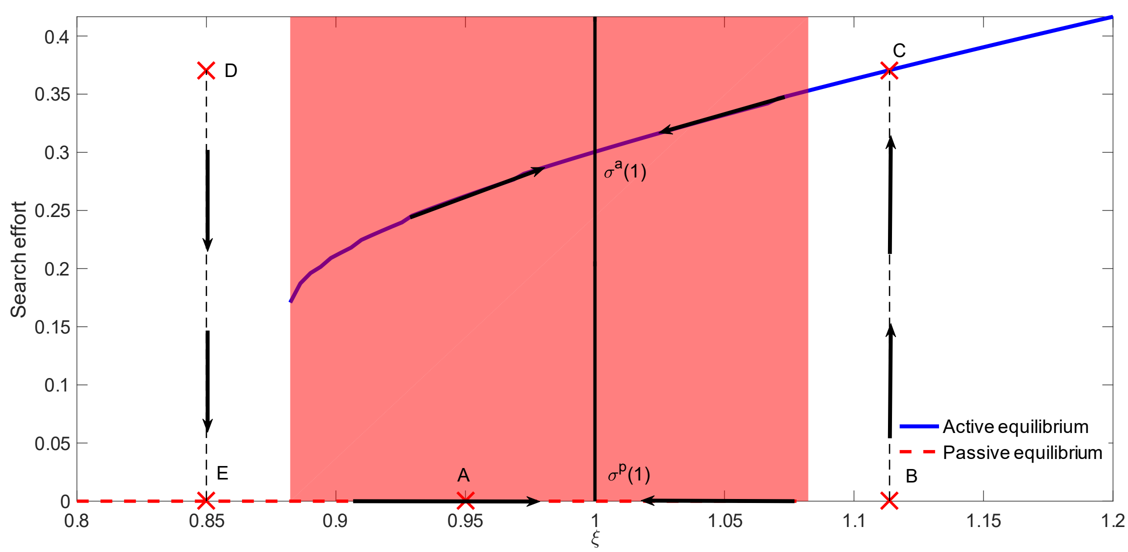
Proposition 8 states that an active equilibrium exists when the optimal response of the firm is to choose $\sigma_{i,t} > 0$ that satisfies equation (35). The next proposition states the condition for the coexistence of the two static equilibria. History dependence selects between them.

Proposition 9. *The active and passive equilibria coexist if and only if Propositions 7 and 8 hold simultaneously.*

4.4 Transitional dynamics and ξ_t

To illustrate the deterministic transitional dynamics of the model, Figure 6 illustrates movements in search effort as a function of ξ_t (a similar figure could be drawn for z_t), again for the calibration in Section 5. The dashed line plots the passive equilibrium path with low search effort and the solid line the active equilibrium path with high search effort. The arrows indicate the direction of the transition dynamics for the endogenous variable to reach the basins of attraction of the system, represented by point $\sigma^p(1)$ for the passive DSS and $\sigma^a(1)$ for the active DSS. The shaded area indicates the range of values of ξ_t that support multiple static equilibria. The passive equilibrium fails to exist for sufficiently large values of ξ_t and, conversely, the active equilibrium fails to exist for sufficiently small values of ξ_t. In the absence of innovations to ξ_t, the system converges and remains on the original basins of attraction in the passive equilibrium, $\sigma^p(1)$, and the active equilibrium, $\sigma^a(1)$, depending on the starting equilibrium.

Figure 6: Phase diagram for search effort



Temporary shifts to ξ_t, which are sufficiently strong to change search effort, move the system to a new static equilibrium. For example, if the system starts in the passive equilibrium at point A and a large and positive innovation to ξ_t moves the system to point B, the passive equilibrium disappears, and the equilibrium of the system becomes active. The economy moves to the new active equilibrium at point C, converging to the stationary basin of attraction $\sigma^a(1)$ in the long run. The system remains in the active equilibrium until a sufficiently negative innovation to ξ_t returns the system to the passive equilibrium. For instance, a large negative innovation to ξ_t, which moves the system from point C to point D, triggers the new passive equilibrium at point E, converging to the stationary basin of attraction $\sigma^p(1)$. In comparison, innovations to ξ_t that move the equilibrium of the system within the shaded area, where both static equilibria coexist, fail to shift the equilibrium because of history dependence.

5 Calibration

We calibrate the model at a monthly frequency for U.S. data over the post-WWII period. Table 2 summarizes the value and the source or target for each parameter.

Table 2: Parameter calibration

Parameter	Value	Source or Target
β	0.996	5% annual risk-free rate
α	0.4	Shimer (2005)
τ	0.4	Hosios condition
χ	0.28	0.45 monthly job-finding rate
κ	1.25	den Haan et al. (2000)
h	0.3	Thomas and Zanetti (2009)
$\widetilde{\tau}$	0.5	Sectoral symmetry
δ	0.027	5.5% unemployment rate in active DSS
$\widetilde{\delta}$	0.021	4 years' duration of joint venture
ϕ	0.135	22% rate of idleness in recessions
ψ	0.114	Condition of Propositions 3 and 4 and 15% recession periods
c_0	0.33	Condition of Propositions 3 and 4 and 15% recession periods
c_1	5	12% rate of idleness in booms
ν	2	Ensure concavity of best response function
σ_ξ	0.05	Justiniano and Primiceri (2008)
ρ_z	$0.95^{1/3}$	BLS
σ_z	0.008	BLS

The constant β is set to 0.996 (equivalent to 0.99 at a quarterly frequency) to replicate an average annual interest rate of 5% over the sample period. In keeping the DMP block of the model as standard as possible, we assume a Cobb-Douglas matching function $m(u, v) = u^{1-\alpha}v^{\alpha}$ in the labor market and calibrate the elasticity of vacancies in the matching function $\alpha = 0.4$, the average value estimated in the literature (see Petrongolo and Pissarides, 2001). We set the wage bargaining power equal to $\tau = \alpha = 0.4$, which satisfies the Hosios (1990) condition. We follow den Haan et al. (2000) in selecting the inter-firm matching function:

$$H\left(\widetilde{n}_F, \widetilde{n}_I\right) = \frac{\widetilde{n}_F \cdot \widetilde{n}_I}{\left(\widetilde{n}_F^{\kappa}/2 + \widetilde{n}_I^{\kappa}/2\right)^{1/\kappa}}. \tag{36}$$

This functional form ensures that matching probabilities are between 0 and 1 without introducing truncations (as often happens with Cobb-Douglas matching functions). After den Haan et al. (2000), we set $\kappa = 1.25$.

We pick the cost of posting a vacancy $\chi = 0.28$ to match the monthly job-finding rate in the active DSS, $\mu(\theta) = 0.45$, as in Shimer (2005). Conditional on $\chi = 0.28$, we select a job-separation rate $\delta = 0.027$ to match an unemployment rate of 5.5% in the active DSS. The flow value of unemployment h is set at 0.3, which consists of the value of leisure and home production and the unemployment benefit, as in Thomas and Zanetti (2009). In this calibration, the flow value of unemployment is about 61% of the average wage in the active DSS, which is in the range of replacement rates documented by Hall and Milgrom (2008).

Compared to a standard DMP economy, our model includes seven new parameters: $\widetilde{\tau}$, $\widetilde{\delta}$, ϕ, ψ, c_0, c_1, and ν. The bargaining share of the intermediate-goods firm $\widetilde{\tau}$ is set to 0.5, to evenly split the total surplus from matching between firms and make the workers indifferent between working in either sector. The rate of termination of inter-firm matches $\widetilde{\delta}$ is 0.021 to replicate the 4 years' average duration of a match. The median and the mean of the duration of inter-firm matches are around 4 years in the Compustat Customer Segment data, which report the major customers for a subset of U.S. listed companies on a yearly basis.

Once we have set values for the previous parameters, c_1 and ϕ pin down the measure of single firms in the active DSS and passive DSS, respectively. The ratio of the measure of single firms to employment corresponds to the rate of idleness, indicating the share of time when employed workers are idle due to a lack of activity (Michaillat and Saez, 2015). The Institute for Supply Management constructs the operating rates (one minus the rate of idleness) in the U.S. According to its measurements, the rate of idleness is about 30% for the non-manufacturing sector and 20% for the manufacturing sector during the Great Recession, and 12% for both sectors before this event. Thus, we set $\phi = 0.135$ and $c_1 = 5$ to yield a rate of idleness equal to

0.22 and 0.12 in the passive DSS and the active DSS, respectively. Finally, $\nu = 2$ ensures the concavity of the best response function of search effort.

There is no direct empirical guidance for the calibration of c_0 and ψ. We calibrate them as 0.33 and 0.114, respectively, to satisfy the conditions for the coexistence of passive and active DSSs in Proposition 5. Our calibration of c_0 and c_1 implies that search cost is about 2% of output. This value is consistent with the fact that 2.5% of workers are employed in search-related occupations (see Subsection 7.1 for our measure of search-related employment).

We set σ_ξ to 0.05.[6] Such a value, given the rest of the calibration, generates a passive equilibrium with 15% probability, consistent with the frequency of recessions in the post-WWII U.S. The persistence of the productivity shock, ρ_z, is set to $0.88^{1/3}$ to match the observed quarterly autocorrelation of 0.88, and the standard deviation, σ_z, is set to 0.0057 to match the quarterly standard deviation of 0.02, as in Shimer (2005).

Once the model is calibrated, we compute the different value functions using value function iteration and exploit the equilibrium conditions of the model to find all variables of interest. See Appendices B and E for technical details.

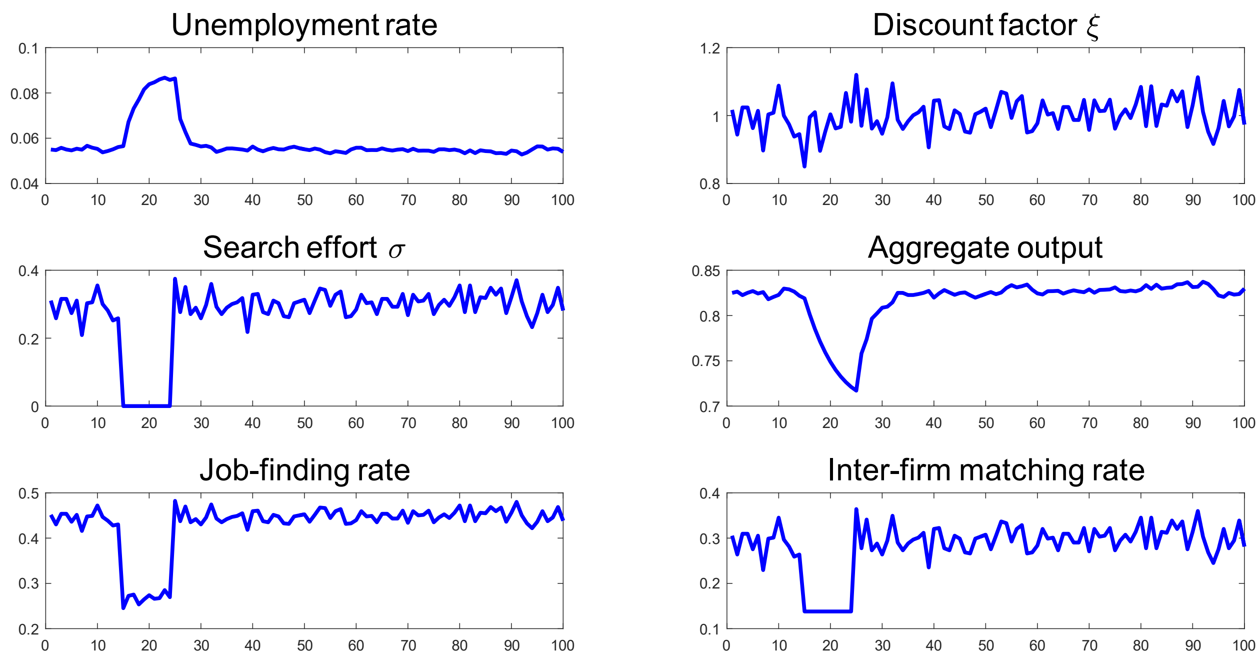
6 Quantitative analysis

To study the dynamic properties of the model, we simulate it for 3,000,000 months and time-average the resulting variables to generate quarterly data. We start the simulation from the active DSS, focusing on the case when only discount factor shocks are present. Appendix G provides a quantitative analysis of properties of the model with productivity shocks. We relegate that case to the appendix because productivity shocks of plausible magnitude are unable to move the system between different equilibria, unless those shocks are permanent.

Figure 7 reports the responses of key variables to shocks to ξ_t for the first 100 periods. The economy begins at a positive search effort with high output, low unemployment, and a high job-finding rate. Then, in period 15, a sufficiently large shock to the discount factor pushes the economy to the low search equilibrium until period 25, with a prolonged drop in output (as joint ventures terminate faster than they are replaced), high unemployment, and a low job-finding

[6] Justiniano and Primiceri (2008, Table 1) find a quarterly $\sigma_{\epsilon_\xi} = 3.13\%$, with a persistence of 0.84. This implies that $\sigma_\xi = 0.0313/\sqrt{1-0.84^2} = 0.0577$. If we extrapolate the quarterly AR(1) process to a monthly AR(1) process, the implied standard deviation is about 0.056. To be cautious, we round down to 0.05.

Figure 7: Simulated variables for the first 100 periods with shocks to ξ_t



rate. In that period, a large positive discount factor shock shifts the economy back to the active equilibrium with positive search effort.

Figure 8: Ergodic distribution with *i.i.d.* shocks to ξ_t

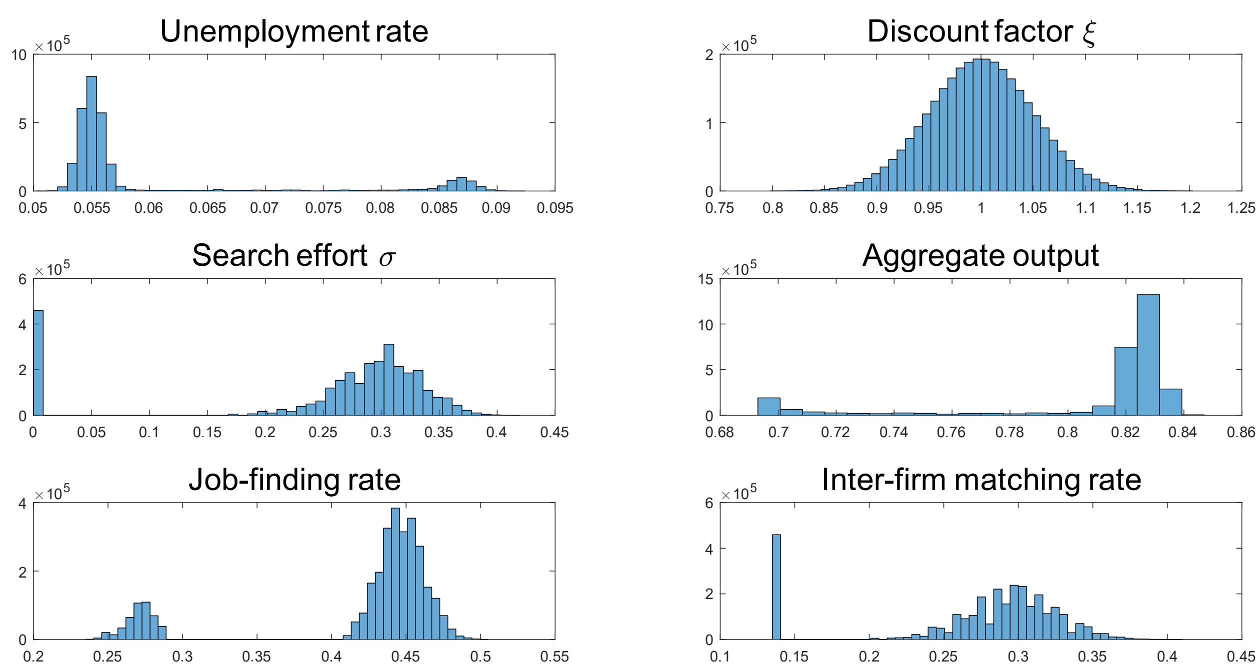
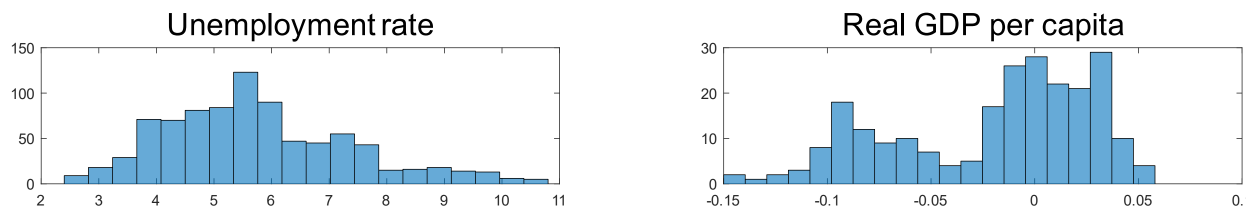


Figure 8 plots the ergodic distribution of selected variables implied by the entire simulation. Endogenous switches between the two equilibria generate a distinctive bimodal distribution of aggregate variables resembling those documented in Adrian et al. (2019) or the ones that you would get from models with increasing returns to scale to search in the tradition of Diamond (1982).[7] As required by our calibration, the figure implies that the economy spends about 85% of the time in the active equilibrium and 15% in the passive equilibrium. In the active equilibrium, the unemployment rate fluctuates around 5.5%. In the passive equilibrium with zero search effort, unemployment fluctuates around 8.7%. Similarly, the job-finding rate moves around 45% in the active equilibrium and 27% in the passive equilibrium.

Figure 9: Distribution of unemployment rate and output growth in the data



To compare our results to the data, Figure 9 plots the empirical distribution for the unemployment rate and real GDP per capita (unemployment rate is monthly from 1960 to 2018; real GDP per capita is quarterly from 1960 to 2018 and is linearly detrended in logs). The distribution of both variables shows skewness and bimodality that is consistent with the prediction of our model.[8] Recall, when comparing the simulated and real data, that our Figure 8 is generated –for parsimony– only with shocks to the discount factor, while the data in Figure 9 are driven by a combination of different shocks. Nonetheless, the behavior of the model is commendable. We will revisit this issue in more detail in Subsection 8.2.

Panel (a) of Table 3 reports various second moments of observed business cycle statistics following the same structure as in Shimer (2005, Table 1). Panel (b) reports second moments of the benchmark model with two DSSs. Finally, panel (c) reports second moments of a version of the model without search complementarities and calibrated on the active equilibrium. Each entry presents the autocorrelation coefficient, the standard deviation, and the correlation matrix for the variables listed across the first row of the table.

Several lessons arise from Table 3. First, our model generates a robust internal propagation:

[7]See Pizzinelli et al. (2020) and Taschereau-Dumouchel and Schaal (2015) for additional evidence on skewness and bimodality in macroeconomic variables.

[8]The Hartigan's dip test for unimodality (Hartigan and Hartigan, 1985) rejects unimodality for the unemployment rate and real GDP per capita with 5% and 1% significance levels, respectively.

Table 3: Second moments

		u	v	v/u	lp	ξ
(a) Quarterly U.S. data, 1951-2016						
Autocorrelation coefficient		0.95	0.95	0.95	0.90	–
Standard deviation		0.20	0.21	0.40	0.02	–
	u	1	-0.92	-0.98	-0.25	–
Correlation matrix	v		1	0.98	0.29	–
	v/u			1	0.27	–
	lp				1	–
(b) Benchmark model						
Autocorrelation coefficient		0.82	0.55	0.71	0.88	0
Standard deviation		0.10	0.21	0.28	0.02	0.03
	u	1	-0.71	-0.85	-0.94	-0.06
Correlation matrix	v		1	0.97	0.54	0.39
	v/u			1	0.72	0.30
	lp				1	0.00
	ξ					1
(c) Model without search complementarities						
Autocorrelation coefficient		0.06	-0.27	-0.08	1	0
Standard deviation		0.02	0.04	0.05	0	0.03
	u	1	-0.27	-0.56	0	-0.56
Correlation matrix	v		1	0.95	0	0.95
	v/u			1	0	1.00
	lp				1	0
	ξ					1

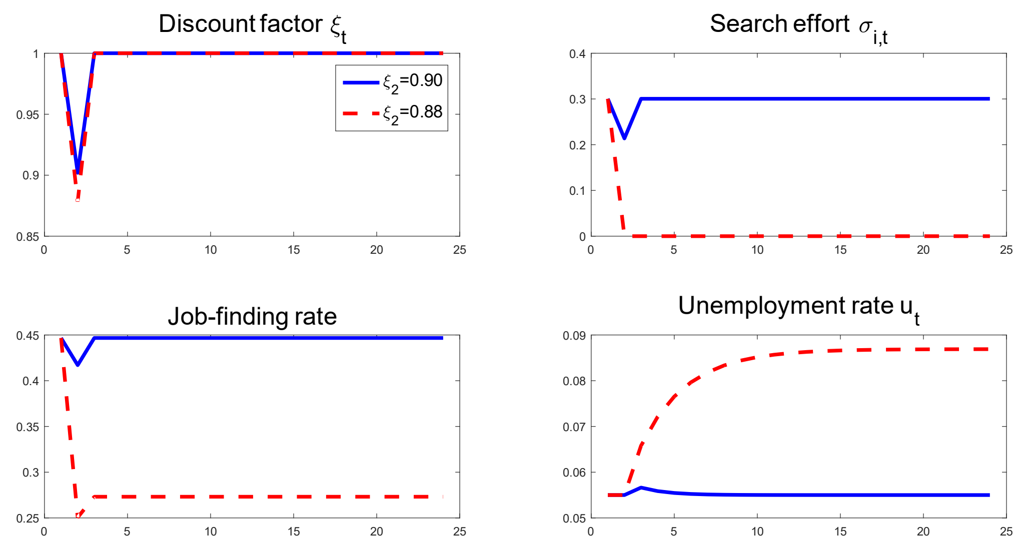
the autocorrelation coefficients of the aggregate variables are significantly larger than in the model without complementarities and much closer to the observed ones. Complementarities in search effort plus history dependence amplify and prolong the effect of shocks.

Second, our model generates empirically plausible standard deviations for the selected variables that are much larger than those in the model without complementarities. This property of the model comes from the amplification of shocks created by the shift between equilibria.

Third, our model produces endogenous movements in labor productivity ("lp" in the table) that would be otherwise absent. The model assumes that firms manufacture goods after matching with a partner. Hence, measured labor productivity depends on the endogenous fraction of the joint ventures over the total number of firms, $n_{i,t}/\left(\widetilde{n}_{i,t}+n_{i,t}\right)$. In comparison, without inter-firm matches, labor productivity is exogenous. Table 3 shows that business cycle statistics for labor productivity generated by our benchmark model are close to those in the data.

Fourth, the benchmark model generates a correlation between unemployment and vacancies (i.e., the Beveridge curve) equal to -0.71, which is close to the value of -0.92 in the data and much larger than the correlation of -0.27 in the model without complementarities. The large negative correlation between vacancies and unemployment is a direct consequence of strategic complementarities in search effort. In the active equilibrium, there is robust vacancy posting and low unemployment, while the relationship is reversed in the passive equilibrium. The switching between equilibria results in periods with a consistently negative relationship between vacancies and unemployment that generates the downward sloping Beveridge curve.

Figure 10: GIRFs to a negative discount factor shock



Note: Each panel shows the response of a variable to a negative discount factor shock (ξ_t) with magnitudes of 0.10 (solid line) and 0.12 (dashed line).

Finally, Figure 10 shows generalized impulse response functions (GIRF) of selected variables to a 12% (solid line) and 10% (dashed line) shock to ξ_t, respectively (we are not dealing with a linear model; thus, we use the adjective "generalized"). In period $t = 1$, the economy starts from the active DSS. In period $t = 2$, an exogenous and one-period disturbance to the discount factor hits the economy. When the contractionary shock to ξ_t is 10%, the firm's search effort temporarily declines in response to the fall in the stream of benefits in forming a joint venture, generating a temporary fall in labor market tightness and a rise in the unemployment rate. This shock is too small to move the system to the passive equilibria and the variables return to the original DSS. However, when the fall in ξ_t is sufficiently large, the system moves to the equilibrium with zero search effort, low output, and high unemployment. While the shock is only 2% larger (12% instead of 10%), its effects are quite different: search complementarities induce large non-linearities in the model.

7 Evidence on the theoretical mechanism

The mechanism in our model builds on three legs: the existence of search complementarities among firms that lead to a co-movement of output and intermediate inputs, the shocks to the discount factor, and history dependence. We will not discuss the last leg. As argued in the introduction, history dependence is an intuitive selection device that has shown considerable empirical success in experiments (Van Huyck et al., 1990, 1991). We focus, instead, on the existence of search complementarities and the shocks to the discount factor.

7.1 Evidence for search complementarities

Through the lenses of our model, search effort can be measured by the inputs employed by final (*customer*) and intermediate (*supplier*) industries in forming value-added chains. Thus, to document the existence of search complementarities, we can check in the data how these inputs co-move among partners.

To understand this idea better, consider the linearized best response curves:

$$\sigma_{F,t} = \beta_1 \sigma_{I,t} + \xi_{F,t} \tag{37}$$

$$\sigma_{I,t} = \beta_2 \sigma_{F,t} + \xi_{I,t}, \tag{38}$$

where $\sigma_{F,t}$ and $\sigma_{I,t}$ are the observed search efforts of industry F and I, and $\xi_{F,t}$ and $\xi_{I,t}$ are the respective unobserved exogenous shocks due to unspecified industry-specific shocks such as a regulatory change. To simplify the exposition, we will assume momentarily that $corr\left(\xi_{F,t}, \xi_{I,t}\right) = 0$ (we remove this assumption below). From (37) and (38), we get:

$$cov\left(\sigma_{F,t}, \sigma_{I,t}\right) = \left(\frac{1}{1-\beta_1\beta_2}\right)^2 \left[\beta_1 var\left(\xi_{F,t}\right) + \beta_2 var\left(\xi_{I,t}\right)\right]. \tag{39}$$

Search complementarities require that β_1 and β_2 be positive: a higher search effort in one sector increases search effort in the other and, thus, $cov\left(\sigma_{F,t}, \sigma_{I,t}\right) > 0$. Symmetrically, if both β's are negative and we have strong search substitutabilities, $cov\left(\sigma_{F,t}, \sigma_{I,t}\right) < 0$. The ambiguous case where $\operatorname{sgn} \beta_1 \neq \beta_2$ (weak substitutabilities) is of less interest, since it violates the symmetry assumptions underlying our analysis. In conclusion, looking at the regression coefficient of the search effort in one sector into the other sector's search effort, which provides us with an estimate of $cov\left(\sigma_{F,t}, \sigma_{I,t}\right)$, is a sharp test for the existence of search complementarities.

To implement this idea (and to generalize it to the case with correlation of errors), we identify each industry's supplier industries from the BEA input-output tables, which report the use of intermediate input for 66 private industries in 3-digit NAICS. Following Michaillat and Saez (2015), we approximate customer industries' search efforts as the number of workers whose occupation is ordering, buying, purchasing, and procurement. The data come from the OES database, constructed by the BLS, which reports yearly employment and wage at the 3-digit-NAICS industry levels, with detailed occupation levels between 2003 and 2018. Analogously, we approximate supplier industries' search efforts as the number of workers whose occupation is advertising, marketing, sales, demonstration, and promotion. We measure search effort in terms of log-linearly detrended employment levels.

Since co-movements of search effort can reflect both search complementarities and time effects (i.e., the correlation of shocks that we ignored to build intuition), we take advantage of industry- and firm-specific shocks in the data to exploit cross-sectional variation and estimate:

$$\sigma_{i,t} = \omega \sigma_{i,t}^{connect} + \upsilon_i + \gamma_t + \epsilon_{i,t}, \tag{40}$$

where $\sigma_{i,t}$ is the search effort of industry i as a customer industry at period t, $\sigma_{i,t}^{connect}$ is the search effort of industry i's supplier industries, and υ_i and γ_t are industry and time fixed effects, respectively. Since each industry has multiple supplier industries, we measure the average search effort for industry i's supplier industries as the mean of these supplier industries' search effort weighted by the value of intermediate goods that industry i purchases from them.

Our point estimate $\widehat{\omega} = 0.45$ (column (1) in Table 4) and its significance at the 1% level is strong support for the central mechanism in our paper: the search efforts of supplier industries are positively correlated with the search efforts of customer industries.

A possible complication for our finding could be the presence of shocks that are specific to each pair of connected industries and that cannot be removed by time fixed effects. To address this concern, we use a two-stage regression procedure to purge the observed search efforts from the influence of common shocks. In the first stage, we regress the search effort in the customer

Table 4: Search efforts are positively correlated between connected industries

	(1)	(2)	(3)	(4)
Measure of search efforts	Search-related employment		Signaling cost	
	Level	Residual	Level	Residual
$\sigma_{i,t}^{connect}$	0.45***	0.16*	1.04***	2.39***
	(0.08)	(0.09)	(0.16)	(0.08)
Time FE	Yes	Yes	Yes	Yes
Industry FE	Yes	Yes	Yes	Yes
R^2	0.16	0.08	0.32	0.29
Observations	15×47	14×47	21×66	20×66

Note: Data are yearly from 2003 to 2016 and 1998 to 2017 for columns (1) (2) and (3) (4), respectively. The dependent variables are the search effort of industry i. $\sigma_{i,t}^{connect}$ is the search effort of industry i's supplier industries and connected industries, respectively. Standard errors, in parentheses, are clustered at the industry level. * and *** denote significance at the 10% and 1% level, respectively.

industry and supplier industries on measures of industry-level economic activity:

$$\sigma_{i,t} = \alpha_i + \beta_i y_{i,t} + \gamma_i y_{i,t}^{connect} + \widehat{\sigma}_{i,t}, \tag{41}$$

$$\sigma_{i,t}^{connect} = \widetilde{\alpha}_i + \widetilde{\beta}_i y_{i,t} + \widetilde{\gamma}_i y_{i,t}^{connect} + \widehat{\sigma}_{i,t}^{connect}, \tag{42}$$

where $y_{i,t}$ and $y_{i,t}^{connect}$ are loglinearly detrended employment of industry i and industry i's supplier industries, respectively.[9] Analogously to $\sigma_{i,t}^{connect}$, $y_{i,t}^{connect}$ is the weighted average of the employment of industry i's supplier industries. The terms $\widehat{\sigma}_{i,t}$ and $\widehat{\sigma}_{i,t}^{connect}$ in equations (41) and (42) are residuals that encapsulate the part of the search effort that is orthogonal to measures of industry-level economic activity. Presumably, common shocks would shift industry-level economic activity, so the residuals exclude the influence of common shocks.

In the second stage, we estimate equation (40) using residuals in equations (41) and (42):

$$\widehat{\sigma}_{i,t} = \omega \widehat{\sigma}_{i,t}^{connect} + \upsilon_i + \gamma_t + \epsilon_{i,t}.$$

The rationale for the second stage regression is to study the co-movement in residual search efforts that exclude the influence of common shocks obtained from the first stage.[10] Column

[9] We use industry employment instead of industry output since it entails a greater sample size by avoiding the merging of the OES data for search effort and the BEA data for output.

[10] Residual search efforts might reflect exogenous shocks to the search effort. In our model, exogenous shocks to search effort would not affect output in the same period, as joint venture formation is time-consuming. Thus, these shocks are not filtered out in the first stage. Although exogenous shocks to search effort would not affect

(2) in Table 4 shows that $\widehat{\omega} = 0.16$, supporting a positive correlation between search effort in connected industries even after excluding common shocks.

As an alternative exercise, we approximate the search effort by the signaling costs that make firms more visible to potential trading partners. Following Hall (2014), we measure an industry i's signaling cost as the value of its intermediate input from the four industries of publishing, motion picture/sound recording, broadcasting/telecommunications, and data processing/internet publishing/other information services, obtained from the BEA input-output tables. The difference between this measurement and our previous measurement is that the former gauges the search effort outsourced from the other industries, while the latter focuses on the search effort exerted within the industry.

More precisely, we measure search costs as the intermediate input from the industries above by deriving a measure of the signaling cost for industry i's connected industries by weighting signaling costs by the value of input-output intermediate goods traded with industry i. Then, we estimate $\sigma_{i,t} = \omega \sigma_{i,t}^{connect} + \nu_i + \gamma_t + \epsilon_{i,t}$, where $\sigma_{i,t}$ is the measure of the signaling cost of industry i; $\sigma_{i,t}^{connect}$ is the signaling cost of industry i's connected industries, which include both industry i's customer and supplier industries; and υ_i and γ_t are the industry and the year fixed effects, respectively. Column (3) in Table 4 shows that signaling costs are positively correlated between connected industries, which again supports the existence of search complementarities.

To ensure that results are not driven by common shocks, we also implement the two-stage regression approach described above. Table 4 in column (4) shows that signaling costs are positively correlated between connected industries, supporting the existence of search complementarities.

Once we have ascertained the existence of search complementarities, we provide evidence for the positive relationship between inter-firm matches and firm growth measures. We use the list of major customers for U.S. publicly listed firms from Compustat Customer Segment data (publicly listed firms in the U.S. are required to disclose the identity of customers that account for at least 10% of annual sales). We measure match creation by using a dummy variable ($par_{i,t}$) equal to one if firm i reports at least one new major customer in year t. Columns (a) and (b)

output in the same month (our calibration period), they might still affect output in the same year (our estimation period). However, as long as some shocks to search effort do not shift output in the same year, these shocks can be captured by the residuals in the first stage and help to identify the sign of β_1 and β_2. We run Monte Carlo simulations to check that our two-stage test is valid with yearly data. Detailed results are available upon request.

in Table 5 show a significant and positive relationship between match creation and the firm's market value and sales growth. When we control for year fixed effects (columns (c) and (d)), the effect of match creation on sales growth becomes insignificant, while its effect on the growth rate of market value remains statistically significant and economically large.

Table 5: Match creation improves firm growth

	(a) Market Return	(b) Sales Growth	(b) Market Return	(d) Sales Growth
$par_{i,t}$	0.144**	0.026**	0.119*	0.008
	(0.065)	(0.012)	(0.067)	(0.012)
Time FE	No	No	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes
R^2	0.00	0.00	0.06	0.08
Observations	2,456	2,219	2,456	2,219

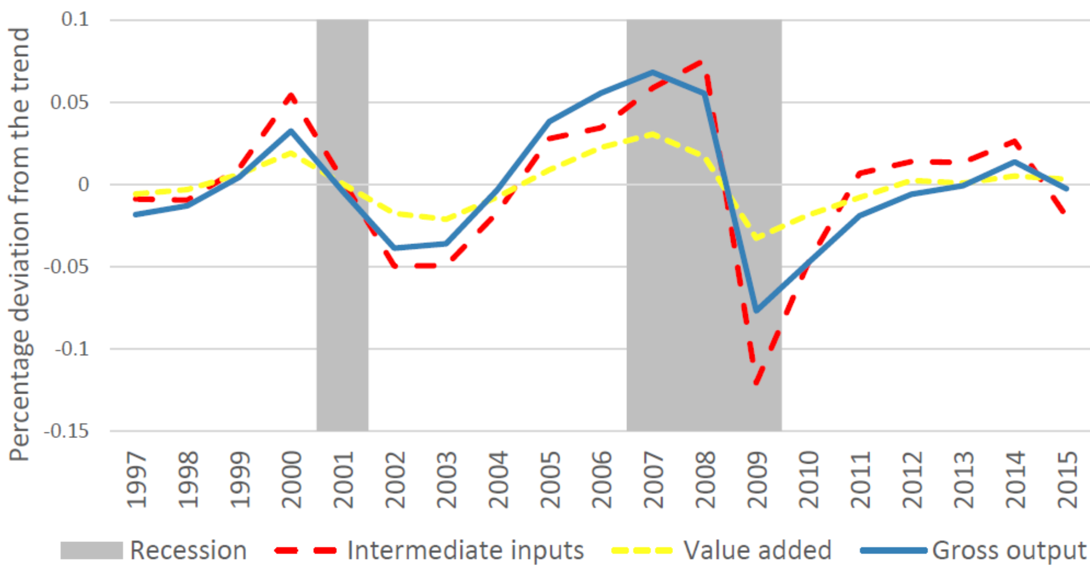
Note: Data are yearly from 1999 to 2014. The dependent variables are the yearly growth rates of market value and sales, obtained from CRSP and Compustat Fundamentals Annual data, respectively. $par_{i,t}$ is a dummy variable equal to one if firm i reports at least one new major customer in year t, according to the major customer records constructed by Compustat Customer Segment data. We restrict the analysis to firms with continuous records of major customers between 1999 and 2014. Standard errors are in parentheses. * and ** denote significance at the 10% and 5% level, respectively.

We close by showing that output and intermediate inputs co-move in the fashion predicted by search complementarities. The BEA compiles a measure of gross output (O) equal to the sum of an industry's value added (VA) and intermediate inputs (II), i.e., $O = VA + II$. BEA data are annual over the period 1997-2015. Figure 11 plots the cyclical component of gross output (blue line), intermediate inputs (red line), and industry value added (yellow line) together with NBER-dated recession periods (grey bands). We extract the cyclical component of the variable using an HP filter. The figure reveals that fluctuations in intermediate inputs are more procyclical than those in output. The Great Recession witnessed a sharp fall in intermediate input and gross production across industries, while the value added remained more stable.

To establish the relative contribution of value added and industry input to the overall volatility of gross output, we decompose the variance of the gross industrial output in terms of its covariance terms: $\text{Var}(O) = \text{Cov}(VA, O) + \text{Cov}(II, O)$. Using this identity, together with the definition $O = VA + II$, and plugging in observed data, we find that the contribution of industry inputs to movements in industrial gross output is:

$$\frac{\text{Cov}(II, VA + II)}{\text{Var}(VA + II)} = 0.71.$$

Figure 11: Intermediate inputs, value added, and gross output



Thus, fluctuations in intermediate input account for 71% of the movements in gross industry output. This average contribution increases during recessions. For instance, in 2008, industry intermediate input decreased by 1.9 trillion, making up 84% of the decline in gross industrial output (2.3 trillion).

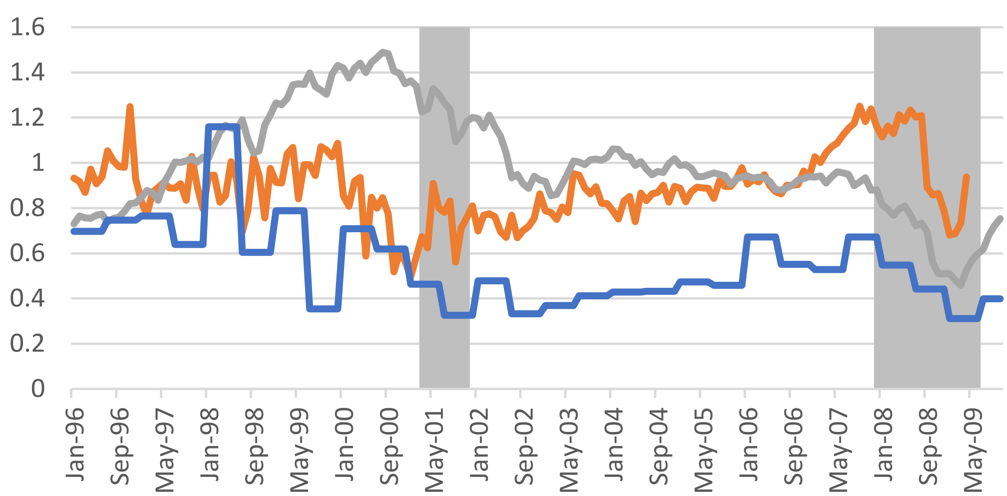
7.2 Discount factor shocks

The second leg in the model is the relevance of discount factor shocks for fluctuations in intermediate inputs and aggregate fluctuations. The presence of discount factor shocks has been documented in a long list of papers. See, among many others, Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2015), Cochrane (2011), and Hall (2016, 2017). These authors have argued that, beyond pure shocks to preferences, discount factor shocks can also represent demographic shifts, movements in financial frictions, fluctuations in risk tolerance, and changes in fiscal and monetary policy that we abstract from in the model. Discount factor shocks can also capture heightened risk aversion caused by a health crisis.

Our task is to relate measures of the discount factor to changes in aggregate output, unemployment, and inter-firm matching. To do so, we use the standard definition of the discount factor as the ratio of the current market price of a future cash receipt to the expected value of the payment (our households are risk neutral and, hence, we do not need to adjust for risk).

There are three popular measures of the discount factor. In measure 1, we follow Hall (2017) and construct the series for the market discount rate for dividends payable from one year (12 months) to two years (24 months) as: $\xi_t = p_t / (\mathbb{E}_t \sum_{\tau=13}^{24} d_{t+\tau})$, where p_t is the market price in month t of the claim of future dividends inferred from option prices and the stock price, and d_t is the dividend paid in month t. The data on p_t are from Binsbergen et al. (2012). In measure 2, we proxy the discount factor using the price-dividend ratio (p/d) of the stock market, as described in Cochrane (2011). Finally, in measure 3, we proxy the discount rate r_t using the measure of expected returns from the S&P stock price index. We obtain the median 12-months-ahead forecast of the stock market index (mnemonics: SPIF, Forecast12month) from the Livingston Survey. Then, we divide by the index of the base period to calculate the expected gross return $1 + r_t$ and compute the discount factor as $\xi_t = 1/(1 + r_t)$.

Figure 12: Alternative measures of the discount factor



Note: Alternative measures of the discount factor from dividend strip (red line), the price-to-dividend ratio (green line), and the Livingston Survey (blue line).

Figure 12 plots the three measures of the discount factor for the period between January 1996 and May 2009. All three measures agree that i) there was a sizable decline in the discount factor during the Great Recession (as our theory requires) and ii) the series display high variance (reflecting the large sensitivity of the discount factor over the business cycle, also required by our theory). The low correlation across the three measures is not surprising, since each of these series reflects discounting from different financial players and assets (see Hall, 2017).

Table 6 shows that the three measures of the discount factor are positively correlated with GDP and input of intermediate goods (columns (a) and (c)), and negatively correlated with unemployment (column (b)). The discount factor positively correlates with the rate of match

Table 6: Correlation between discount rates and aggregate variables

Correlation coefficient	(a) Unemployment rate	(b) GDP	(c) Intermediate input	(d) Match creation
Livingston Survey	-0.55	0.53	0.42	0.16
S&P dividend strip p/d ratio	-0.33	0.50	0.21	0.32
P/d ratio	-0.75	0.80	0.53	0.79

Note: Discount rates and unemployment: monthly data from January 1996 to May 2009. GDP: quarterly data from 1996Q1 to 2009Q1. Intermediate input: annual data from 1997 to 2009. Rate of match creation: annual data from 1996 to 2009. Series are HP filtered.

creation (column (d)), measured from Compustat Customer Segment data. These patterns corroborate the important relation between shocks to the discount factor and movements in production, unemployment, and inter-firm matching highlighted by our model.

8 The volatility of shocks

We now gauge how the volatility of the shocks to the economy determines the dynamic properties of the model and the likelihood and duration of each equilibrium.

8.1 Analytical illustration with a simplified model

To gain intuition, and before we report the quantitative results from the full model, we derive an analytical characterization of the effect of volatility on the likelihood and duration of each static equilibrium by simplifying the model in Section 3. First, we assume that firms produce their output without labor. Thus, we can drop the whole DMP module of the model and set a constant measure of size 1 of firms in each sector. Second, we assume that $\widetilde{\delta} = 1$, i.e., all joint ventures terminate after one period. Also, joint ventures start producing in the same period in which firms match. Hence, the firm's problem is equivalent to a sequence of static maximization problems and we do not need to specify a discount factor. To ease the algebra, we also set $\rho_z = 0$, and as in the calibration in Section 5, $\widetilde{\tau} = 0.5$ and $\nu = 2$. This simplified model is nearly identical to the model in Section 2 except for a slightly different matching function.

Under these simplifications, each firm optimally chooses the level of its search effort, $\sigma_{i,t}$, given the search effort of the firms in the opposite sector, $\sigma_{-i,t}$, and productivity, z_t, by maximizing:

$$J_{i,t}(\sigma_{i,t}, \sigma_{-i,t}, z_t) = (\phi + (\psi + \sigma_{i,t})(\psi + \sigma_{-i,t}))\frac{z_t}{2} - c_0\sigma_{i,t} - c_1\frac{\sigma_{i,t}^3}{3}.$$

The first term of the RHS is the inter-firm matching probability defined in equation (12) multiplied by half the expected production, $\pi_{i,t} z_t$ (recall the equal split of output between the firms given $\widetilde{\tau} = 0.5$) minus the cost of searching.

The interior solution $\sigma_{i,t} > 0$ satisfies:

$$c_0 + c_1 \sigma_{i,t}^2 = (\psi + \sigma_{-i,t}) \frac{z_t}{2}. \tag{43}$$

Otherwise, $\sigma_{i,t} = 0$. Hence, as in the benchmark model, the simplified model entails passive and active equilibria. The passive equilibrium with zero search effort exists if and only if $c_0 > \psi \frac{z_t}{2}$. Thus, we can define a threshold of productivity $\bar{z} = \frac{2c_0}{\psi}$ that determines whether the passive equilibrium exists.

Lemma 2. *The passive equilibrium exists if and only if $z_t < \bar{z}$.*

Recall that we assumed that $\psi > 0$. If $\psi = 0$, a passive equilibrium always exists regardless of the value of z_t.

In an active equilibrium, firms in each sector optimally choose a positive search effort that comes from finding the fixed point of the product of equation (43) for each sector:

$$\sigma_{F,t} = \sigma_{I,t} = \frac{z_t + \sqrt{z_t^2 + 8\psi z_t - 16 c_0 c_1}}{4 c_1}. \tag{44}$$

This optimal search effort is increasing in z_t.[11]

From equation (44), the threshold for the active equilibrium is $\underline{z} = 4\left(\sqrt{\psi^2 c_1^2 + c_1 c_0} - \psi c_1\right)$, and we get the following lemma.[12]

Lemma 3. *An active equilibrium exists if and only if $z_t \geq \underline{z}$.*

Proposition 10 merges lemmas 2 and 3 to characterize the range of values z_t compatible with multiple equilibria.

Proposition 10. *The economy retains multiple equilibria if $z_t \in (\underline{z}, \bar{z})$. The passive equilibrium is the unique equilibrium if $z_t \leq \underline{z}$. The active equilibrium is the unique equilibrium if $z_t \geq \bar{z}$.*

[11] There is a second fixed point, $\sigma_{i,t} = \frac{z_t - \sqrt{z_t^2 + 8\psi z_t - 16 c_0 c_1}}{4 c_1}$. However, this solution is locally unstable.

[12] To prevent the marginal search cost from converging to zero when $\sigma_{i,t}$ is zero, the term c_0 must be positive. If $c_0 = 0$, it yields $\underline{z} = 0$. In such an instance, the active equilibrium exists for any positive value of z_t.

Proposition 10 establishes that if economic fundamentals are sufficiently weak or strong, the static equilibrium is unique, either passive or active; otherwise, we have two static equilibria. Sufficiently large shocks to z_t move the system between the two alternative static equilibria. Proposition 10 is empirically relevant because we can calibrate ψ to a small number so that $\bar{z}$ is low and c_1 to a large number so that $\underline{z}$ is high. In that way, the model will allow multiple static equilibria for a wide range of productivity $\underline{z} < 1 < \bar{z}$.

Since we have set $\rho_z = 0$, $\log(z_t) \sim \mathcal{N}(0, \sigma_z^2)$. Using the distribution for z_t and the thresholds $\underline{z}$ and $\bar{z}$, we derive the transition matrix between equilibria:

	Active	Passive
Active	$1 - \Phi\left[\log\left(\underline{z}\right)/\sigma_z\right]$	$\Phi\left[\log\left(\underline{z}\right)/\sigma_z\right]$
Passive	$1 - \Phi\left[\log\left(\bar{z}\right)/\sigma_z\right]$	$\Phi\left[\log\left(\bar{z}\right)/\sigma_z\right]$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. The next proposition establishes that aggregate volatility plays a critical role in the selection and duration of each static equilibrium.

Proposition 11. *The expected duration of a passive equilibrium spell is* $\frac{1}{\Phi[\log(\underline{z})/\sigma_z]}$, *and the expected duration of an active equilibrium spell is* $\frac{1}{1-\Phi[\log(\bar{z})/\sigma_z]}$. *The duration of each equilibrium is inversely related to the volatility of* z_t.

Proposition 11 shows that a reduction in volatility induces the system to remain for a prolonged spell in one static equilibrium, with a decreased probability for the system to move to the alternative static equilibrium. However, if a sufficiently large change in fundamentals triggers a change in the static equilibrium, the economy would move to the alternative static equilibrium and stay there for a long time.

The dynamics in the simple model are consistent with the large and ongoing low employment-to-population ratio in the aftermath of the financial crisis of 2007-2009 (even if the headline unemployment rate recovered by early 2017). The financial crisis was preceded by a long spell of stable economic conditions during the Great Moderation that started in the mid-1980s, which the model identifies as a prerequisite for the unprecedented persistence in the low employment-to-population ratio (although, see Fernald et al., 2017, for an alternative interpretation based on a change in long-run growth trends).

8.2 Bimodality and volatility

With the intuition from the simplified model, we return to our benchmark model to gauge the changes in the volatility of shocks. Table 7 reports business cycle statistics for a low (column (a)) and a high (column (b)) variance of shocks to the discount factor (σ_ξ). As before, we simulate the model for 3,000,000 months and time average to obtain quarterly data. The first and second rows report the number of periods and the average duration of the passive equilibrium, respectively, and the third row reports the transition matrix between equilibria. We calibrate high and low volatility by following Justiniano and Primiceri (2008), who estimate that the volatility of the discount factor shocks is equal to 0.07 before 1984 and 0.04 after that date.

Table 7: Variance of shocks and duration of equilibria

	(a) $\sigma_\xi = 0.04$		(b) $\sigma_\xi = 0.07$	
Fraction of periods in passive equilibrium	0.11		0.27	
Average number of quarters in passive equilibrium	11		3.4	
Transition matrix				
	Active	Passive	Active	Passive
Active	0.98	0.02	0.89	0.11
Passive	0.09	0.91	0.29	0.71

The passive equilibrium materializes with a probability of around 11% in the low-volatility economy, in contrast with a 27% probability in the high-volatility economy. Despite the lower chance of moving to a passive equilibrium, the low-volatility economy stays longer on average in a passive equilibrium, 11 quarters, than the high-volatility economy, 3.4 quarters. Low volatility induces less frequent but long-lasting periods of low output and high unemployment.

The last two rows in Table 7 report the transition matrix between equilibria. The low-volatility economy transitions between equilibria infrequently. The probability of moving from active equilibrium to passive equilibrium is equal to 2%, and the probability of a reverse move from passive equilibrium to active equilibrium is equal to 9%. The rotation among equilibria gets much higher in the high-volatility economy, as the probability of moving from an active to a passive equilibrium is 11%, and the probability of a reverse move is 29%.

While Figures 8 and 9 demonstrate the bimodality of output's unconditional distribution, our model predicts that the bimodality of output's conditional distribution is significant only in periods of high volatility. When volatility is low, the system mostly stays in one of the equilibria, and the bimodality in the distribution of output is very mild.[13]

To show that this implication of the model is consistent with the data, we estimate –for each quarter– the one-quarter-ahead conditional distribution of real output growth with the non-parametric approach proposed by Adrian et al. (2019). Then, we calculate the p-value of Hartigan's dip test for each quarter. A lower p-value indicates a higher probability of rejecting unimodality. We find that the correlation between the Hartigan's dip test's p-value and the VIX index is -0.30, which is statistically significant at the 1% level. In words: unimodality is more likely to be rejected when VIX is high.

Figure 13: Conditional distribution of output growth

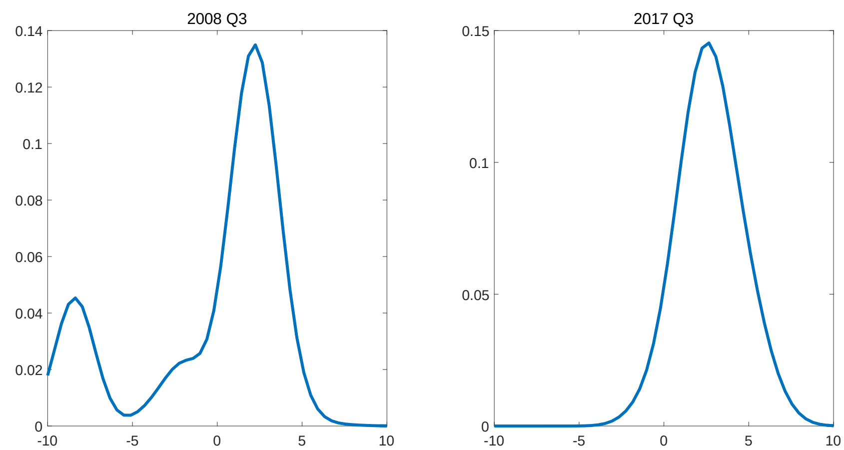


Figure 13 illustrates this point by plotting output growth's conditional distribution in 2008 Q3 and 2017 Q3, respectively. Bimodality was pronounced in 2008 Q3 (left panel) when the volatility was extraordinarily high ($VIX = 58.6$, compare with the sample mean of 19.2). In contrast (right panel), there was little bimodality in 2017 Q3 when volatility was low ($VIX = 11.0$).

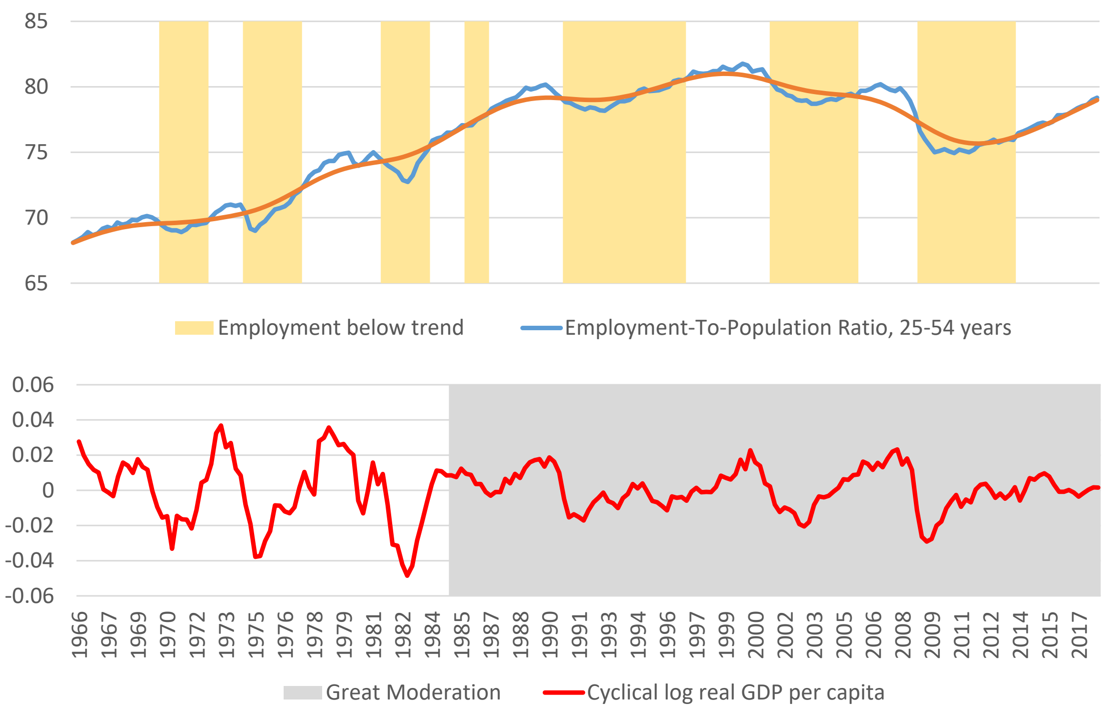
8.3 The Great Moderation and the persistence of business cycles

Our model predicts that a lower volatility of fundamentals is associated with more prolonged equilibrium spells. This prediction is consistent with the empirical pattern in the U.S. data.

In Figure 14, the upper panel plots the U.S. employment rate (blue curve) and its trend (orange curve) estimated from an HP filter with $\lambda = 1600$ from 1996 to 2017. The light-orange bars indicate labor market downturns. Inspired by the NBER's methodology, we define a labor

[13] See, in Appendix H, the histograms of the model's endogenous variables when volatility is high and low.

Figure 14: The Great Moderation and labor market downturns



market downturn as starting when the employment rate falls below the trend for two quarters and ending when the employment rate rises above the trend for two quarters. As noted by many researchers (see Jaimovich and Siu 2012 and references therein), the figure shows how the three labor market downturns that occurred after 1984 were longer than the previous ones. Precisely after 1984, the U.S. economy experienced a substantial reduction in aggregate volatility, which Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2015) attribute, in part, to a lower volatility of shocks to fundamentals. To illustrate this point, the bottom panel in Figure 14 plots the cyclical component of real GDP per capita, with a grey area to indicate the Great Moderation that started around 1984.

Our model suggests an intrinsic link between the Great Moderation and the increasing persistence in labor market downturns. While the Great Moderation improves macroeconomic stability and reduces the occurrences of recessions, it makes these recessions and the associated labor market downturns more durable.

9 The role of fiscal policy

In our model, government spending that stimulates joint venture formation may permanently move the system from a passive to an active equilibrium, inducing a large fiscal multiplier. To study this hypothesis, we embed government spending in the economy and derive the analytical conditions for fiscal policy to move the system from a passive to an active equilibrium. We then investigate the effect of public spending on the DSSs in the model. Finally, we evaluate the size and state dependence of the impact of government spending.

9.1 Government spending as a set of final-goods producers

We focus our investigation on government spending (government consumption expenditures and gross investment). We ignore transfers because our model abstracts from aggregate demand considerations. We model government spending as an exogenous increase in the number of single firms in the final-goods sector, where these additional firms can be interpreted as new public projects such as building a new school. Thus, we have government-owned single final-goods firms, $\widetilde{n}^G_{F,t}$, that operate together with private single firms in both sectors. The formation of private firms remains endogenous, as described by equation (14). We assume that government spending is financed by lump-sum taxes.

The law of motion for government single final-goods firms is $\widetilde{n}^G_{F,t+1} = (1-\delta-\pi_F)\,\widetilde{n}^G_{F,t} + \epsilon^G_t$, where ϵ^G_t are the new government-owned single firms created in period t.[14] Like the private firms, government-owned firms must form a joint venture with firms in the intermediate-goods sector to manufacture goods (for example, a public school requires CFRPs produced by private firms). Joint ventures with government-owned firms follow $n^G_{F,t+1} = \left(1-\delta-\widetilde{\delta}\right) n^G_{F,t} + \pi_F \widetilde{n}^G_{F,t}$. A government firm exits the market when its job match or joint venture is terminated.

The inflow ϵ^G_t changes the matching probabilities in the inter-firm matching market:

$$\pi_{I,t} = [\phi + (\psi+\sigma_I)(\psi+\sigma_F)]\, H\left(1, \widetilde{\theta}_t\right), \tag{45}$$

and $\pi_F = [\phi + (\psi+\sigma_I)(\psi+\sigma_F)]\, H\left(\frac{1}{\widetilde{\theta}_t}, 1\right)$, where $\widetilde{\theta}_t = (\widetilde{n}_{F,t} + \widetilde{n}^G_{F,t})/\widetilde{n}_{I,t}$ is the inter-firm

[14] We assume that government spending shocks hit once per year. With probability $1/12$, ϵ^G_t is drawn from the uniform distribution with the support $[0, \widetilde{n}_{F,t}/2]$. Otherwise, $\epsilon^G_t = 1$. This specification ensures a non-negative measure of government firms and that the inter-firm matching market tightness ratio does not explode.

matching market tightness ratio in the presence of government single firms.

Since H is increasing in both arguments, $\epsilon_t^G > 0$ increases the matching probability for intermediate-goods firms (more potential partners) and decreases the matching probability for final-goods firms (stiffer competition for partners). These changes in matching probabilities, in turn, move search effort and, potentially, the equilibrium of the economy.

Total government spending is equal to the output produced by government-owned firms in joint ventures and the single government-owned firms' search cost $g_t = z_t n_F^G + \widetilde{n}_F^G \left(c_0 \sigma_F + c_1 \frac{\sigma_F^{1+\nu}}{1+\nu} \right)$. Gross aggregate output comprises government and private production: $y_t = z_t (n_{F,t}^G + n_{F,t})$, and it is used for private consumption, government spending, and search costs. The aggregate resource constraint is $y_t = c_t + g_t + \sum_{i=I,F} \chi v_i + \sum_{i=I,F} \widetilde{n}_i \left(c_0 \sigma_i + c_1 \frac{\sigma_i^{1+\nu}}{1+\nu} \right)$.

9.2 Shocks to government spending and equilibria switches

We assume that the economy is in the passive equilibrium (i.e., $\sigma_I = \sigma_F = 0$) before the arrival of a positive government spending shock, ϵ_t^G.

Upon the realization of the shock, the passive equilibrium continues to exist if and only if:

$$\widetilde{\beta} \xi_t \psi H \left(1, \widetilde{\theta}_t \right) \mathbb{E}_t \left(J_{I,t+1} - \widetilde{J}_{I,t+1} \mid \iota = 0 \right) < c_0, \tag{46}$$

and

$$\widetilde{\beta} \xi_t \psi H \left(\widetilde{\theta}_t^{-1}, 1 \right) \mathbb{E}_t \left(J_{F,t+1} - \widetilde{J}_{F,t+1} \mid \iota = 0 \right) < c_0. \tag{47}$$

where, recall, $\widetilde{\beta} = \beta (1 - \delta) / \tau$. Equation (46) shows that the passive equilibrium disappears if the increase of a government-owned single firm tightens the inter-firm matching market enough and makes the expected capital gain of intermediate-goods firms so high that these firms search with positive effort even if the final-goods firms search with zero effort.

Proposition 12. *Starting from the passive equilibrium, the size of government spending needed to move the system to the active equilibrium is:*

$$\frac{\widetilde{n}_{F,t}^G}{\widetilde{n}_{I,t}} > \Psi \left[\frac{c_0}{\widetilde{\beta} \xi \psi \mathbb{E}_t \left(J_{I,t+1} - \widetilde{J}_{I,t+1} \mid \iota = 0 \right)} \right] - \frac{\widetilde{n}_{F,t}}{\widetilde{n}_{I,t}}, \tag{48}$$

with $\Psi' > 0$.[15]

Equation (48) shows that the magnitude of the policy intervention that moves the economy to an active equilibrium is proportional to the cost-to-benefit ratio of forming a joint venture, and it decreases with the measure of private firms in the final-goods sector relative to intermediate-goods firms. A large quantity of private final-goods firms improves the joint venture prospects for intermediate-goods firms, decreasing the magnitude of government spending needed to move to the active equilibrium.

9.3 The fiscal multiplier

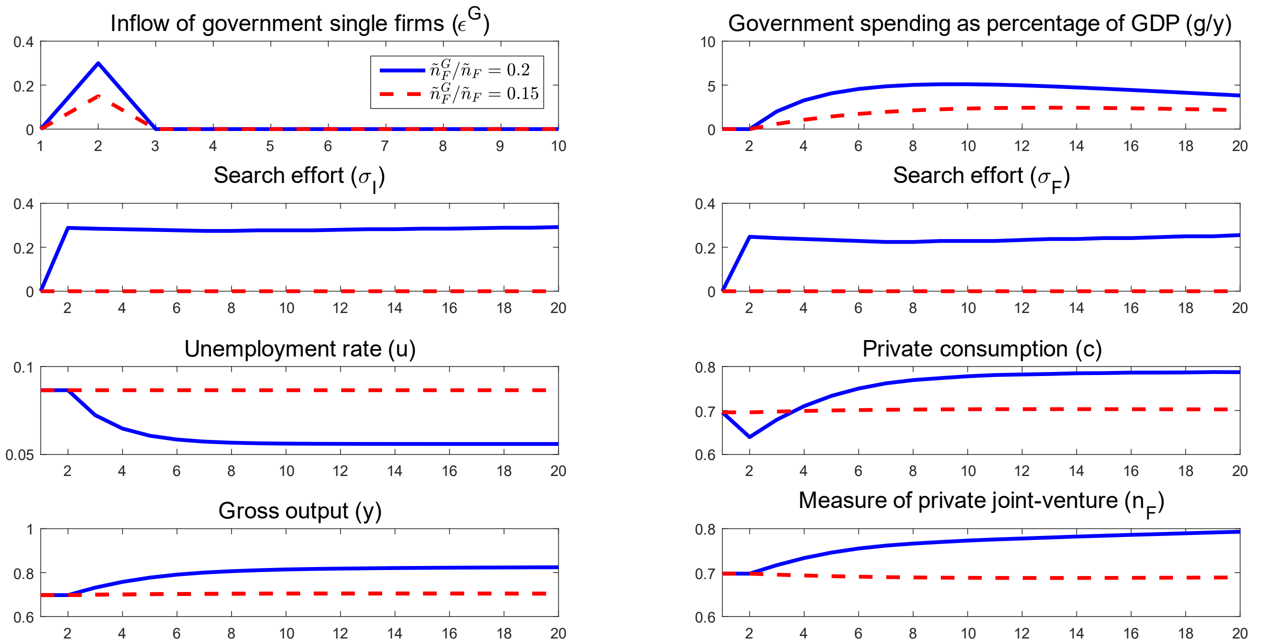
We provide now quantitative results regarding the dynamic response of the economy to expansionary fiscal policy shocks and the size of the fiscal multiplier. See Appendix E.2 for details of the computation of this case. Once we introduce government spending, we have 12 state variables. Due to this large number of state variables, we implement a dimensionality reduction algorithm inspired by Krusell and Smith (1998) that is of interest in itself and potentially applicable to similar problems.

Figure 15 shows the dynamic reaction of selected variables to the same 15% (dotted line) and 20% (solid line) shocks to the relative size of the final-goods sector that we just described when the economy starts at the passive equilibrium DSS (Appendix I shows the responses for the system that starts from the active equilibrium). Since the 20% fiscal expansion satisfies Proposition 12, it produces a significant and persistent increase in output and a fall in unemployment. Nevertheless, this fiscal expansion crowds out private consumption upon impact. This reaction is due to two mechanisms. First, a rise in government-owned firms reduces, in the short run, the formation of joint ventures that produce goods for private consumption. Second, the shift of equilibrium triggers an increase in the cost associated with vacancy posting and joint venture formation, which further reduces private consumption. The first mechanism still exists in the 15% fiscal expansion, inducing a small drop in private consumption.

We also calculate the fiscal multiplier for our economy, defined as the ratio of the cumulative change in output over one quarter and one year, generated by the one-period change in government

[15] Denote $h\left(\widetilde{\theta}\right) = H\left(1, \widetilde{\theta}\right)$. Ψ is the inverse function of $h(\cdot)$. As $h(\cdot)$ is strictly increasing in $\widetilde{\theta}$ by assumption, Ψ is also a strictly increasing function. In our calibration: $h(\theta) = 2^{\frac{1}{\kappa}}\left(1 + \widetilde{\theta}_t^{-\kappa}\right)^{-\frac{1}{\kappa}}$, $\Psi(x) = \left(2x^{-\kappa} - 1\right)^{-\frac{1}{\kappa}}$.

Figure 15: GIRFs to positive government spending shock

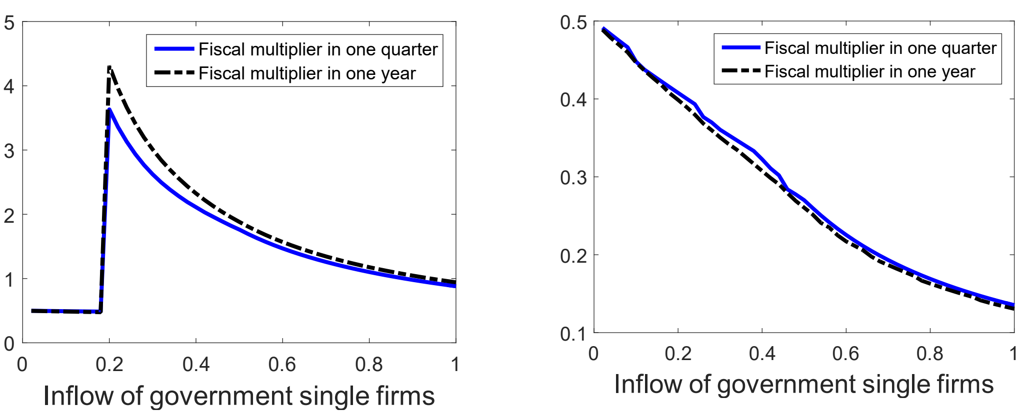


Note: Each panel shows the response of a variable to a one-period, 15% (dashed line) and 20% (solid line) increase in government spending.

spending triggered by the inflow of government-owned single firms in the final-goods sector (we could compute the fiscal multiplier at other horizons if desired). Panel (a) in Figure 16 shows the fiscal multiplier as a function of the inflow of government-owned single firms when the economy is in the passive equilibrium at the start of the fiscal expansion. Panel (b) replicates the exercise for the case when the economy is in the active equilibrium.

Figure 16: Fiscal multiplier

(a) Passive equilibrium **(b)** Active equilibrium



In the passive equilibrium, a sufficiently large fiscal expansion generates a multiplier larger than 1 since it triggers a rise in search effort. The fiscal multiplier peaks at the threshold where we shift from the passive to the active equilibrium. In our calibration, the peak quarterly fiscal multiplier, 3.5, is at a 19% increase in the number of government-owned firms, which is equivalent to a 3.8% increase in government spending relative to output in the first quarter (since the increase in government spending is persistent, the overall size of the fiscal intervention is larger than the impact change of 3.8%). Any stimulus beyond this level reduces the fiscal multiplier because the crowding out of private consumption outweighs the increase in output from the fiscal expansion. A doubling in the number of government-owned final-goods firms generates a fiscal multiplier of around 1 over a quarter. Similarly, a fiscal expansion below the threshold generates a less than unitary fiscal multiplier since it creates a large crowding-out effect and no equilibrium switch.

Panel (b) in Figure 16 shows that the fiscal multiplier is substantially lower in the active equilibrium. The increased costs of forming joint ventures for private firms in the final-goods sector reduce private output, and we have a less than unitary fiscal multiplier for any size of the fiscal stimulus. The multiplier declines with the size of government spending for a crowding-out effect across a wide range of time horizons.

Our results in Figure 16 agree with the recent empirical literature that has documented the acute state dependence of fiscal multipliers. See, for example, Auerbach and Gorodnichenko (2012), Owyang et al. (2013), and Ghassibe and Zanetti (2020). Our model accounts for such state dependence of fiscal multipliers parsimoniously.

10 Conclusion

This paper shows that search complementarities in the formation of inter-firm joint ventures have broad implications for the magnitude and persistence of business cycle fluctuations and the effect of fiscal policy. The optimal degree of search effort is either zero or positive, and the system entails two static equilibria: an active one with large economic activity and a passive one with high economic activity. Sufficiently large changes in fundamentals that change search effort move the system between the two static equilibria.

The dynamic properties of our economy are unlike those of standard models. The model generates bimodal ergodic distributions of variables and protracted slumps after large shocks.

Macroeconomic volatility plays an essential role in the selection and duration of each static equilibrium. In particular, large negative shocks during spells of low volatility generate a persistent shift to the passive equilibrium, which is consistent with the large and persistent deviation of economic variables from trend following the financial crisis in the aftermath of the Great Moderation. Fiscal policy operates markedly different than in standard models, and it is powerful in stimulating the economy in the passive equilibrium, with a non-monotonic effect on economic activity, while its effectiveness significantly declines in the active equilibrium.

The analysis opens exciting avenues for additional research. A direct extension would be to embed strategic complementarities in richer models of the business cycle such as those including money, nominal rigidities, and financial frictions. Furthermore, the role of agent and spatial heterogeneity deserves further exploration. We will pursue some of those ideas in future work.

References

Acemoglu, D., Carvalho, V. M., Ozdaglar, A., and Tahbaz-Salehi, A. (2012). The network origins of aggregate fluctuations. *Econometrica*, 80(5):1977–2016.

Adrian, T., Boyarchenko, N., and Giannone, D. (2019). Multimodality in macro-financial dynamics. Staff Report 903, Federal Reserve Bank of New York.

Auerbach, A. J. and Gorodnichenko, Y. (2012). Measuring the output responses to fiscal policy. *American Economic Journal: Economic Policy*, 4(2):1–27.

Binsbergen, J. v., Brandt, M., and Koijen, R. (2012). On the timing and pricing of dividends. *American Economic Review*, 102(4):1596–1618.

Bulow, J. I., Geanakoplos, J. D., and Klemperer, P. D. (1985). Multimarket oligopoly: Strategic substitutes and complements. *Journal of Political Economy*, 93(3):488–511.

Burdett, K. and Mortensen, D. (1980). Search, layoffs, and labor market equilibrium. *Journal of Political Economy*, 88(4):652–72.

Chatterjee, S., Cooper, R., and Ravikumar, B. (1993). Strategic complementarity in business formation: Aggregate fluctuations and sunspot equilibria. *Review of Economic Studies*, 60(4):795–811.

Cochrane, J. H. (2011). Discount rates. *Journal of Finance*, 66(4):1047–1108.

Cooper, R. (1994). Equilibrium selection in imperfectly competitive economies with multiple equilibria. *Economic Journal*, 104(426):1106–1122.

Cooper, R. and John, A. (1988). Coordinating coordination failures in Keynesian models. *Quarterly Journal of Economics*, 103(3):441–463.

den Haan, W. J., Ramey, G., and Watson, J. (2000). Job destruction and propagation of shocks. *American Economic Review*, 90(3):482–498.

Diamond, P. (1982). Aggregate demand management in search equilibrium. *Journal of Political Economy*, 90(5):881–894.

Diamond, P. and Fudenberg, D. (1989). Rational expectations business cycles in search equilibrium. *Journal of Political Economy*, 97(3):606–619.

Eeckhout, J. and Lindenlaub, I. (2018). Unemployment cycles. Mimeo, UCL.

Fernald, J. G., Hall, R. E., Stock, J. H., and Watson, M. W. (2017). The disappointing recovery of output after 2009. *Brookings Papers on Economic Activity*, Spring:1–58.

Fernández-Villaverde, J., Guerrón, P., and Rubio-Ramírez, J. F. (2015). Estimating dynamic equilibrium models with stochastic volatility. *Journal of Econometrics*, 185(1):216–229.

Ghassibe, M. and Zanetti, F. (2020). State dependence of fiscal multipliers: the source of fluctuations matters. Mimeo, University of Oxford.

Hall, R. E. (2014). What the cyclical response of advertising reveals about markups and other macroeconomic wedges. Mimeo, Stanford University.

Hall, R. E. (2016). *Macroeconomics of Persistent Slumps*, volume 2 of *Handbook of Macroeconomics*, pages 2131–2181. Elsevier.

Hall, R. E. (2017). High discounts and high unemployment. *American Economic Review*, 107(2):305–330.

Hall, R. E. and Milgrom, P. (2008). The limited influence of unemployment on the wage bargain. *American Economic Review*, 98(4):1653–1674.

Hartigan, J. A. and Hartigan, P. M. (1985). The dip test of unimodality. *The Annals of Statistics*, 13(1):70–84.

Heizer, J., Render, B., and Munson, C. (2016). *Operations Management: Sustainability and Supply Chain Management*. Pearson Education, 12th edition.

Hosios, A. J. (1990). On the efficiency of matching and related models of search and unemployment. *Review of Economic Studies*, 57(2):279–298.

Howitt, P. (1985). Transaction costs in the theory of unemployment. *American Economic Review*, 75(1):88–100.

Huo, Z. and Ríos-Rull, J.-V. (2013). Paradox of thrift recessions. Working Paper 19443, National Bureau of Economic Research.

Jaimovich, N. and Siu, H. E. (2012). The trend is the cycle: Job polarization and jobless recoveries. Working Paper 18334, National Bureau of Economic Research.

Jones, C. I. (2013). Misallocation, economic growth, and input-output economics. In Acemoglu, D., Arellano, M., and Dekel, E., editors, *Advances in Economics and Econometrics: Tenth World Congress*, volume 2, pages 419–456. Cambridge University Press.

Justiniano, A. and Primiceri, G. E. (2008). The time-varying volatility of macroeconomic fluctuations. *American Economic Review*, 98(3):604–641.

Kaplan, G. and Menzio, G. (2016). Shopping externalities and self-fulfilling unemployment fluctuations. *Journal of Political Economy*, 124(3):771–825.

Kehoe, P. J., Lopez, P., Midrigan, V., and Pastorino, E. (2019). Asset prices and unemployment fluctuations. Working Paper 26580, National Bureau of Economic Research.

Krusell, P. and Smith, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Leduc, S. and Liu, Z. (2020). The weak job recovery in a macro model of search and recruiting intensity. *American Economic Journal: Macroeconomics*, 12(1):310–43.

Liu, P., Theodoridis, K., Mumtaz, H., and Zanetti, F. (2019). Changing macroeconomic dynamics at the zero lower bound. *Journal of Business & Economic Statistics*, 37(3):391–404.

Michaillat, P. and Saez, E. (2015). Aggregate demand, idle time, and unemployment. *Quarterly Journal of Economics*, 130(2):507–569.

Owyang, M. T., Ramey, V. A., and Zubairy, S. (2013). Are government spending multipliers greater during periods of slack? Evidence from twentieth-century historical data. *American Economic Review*, 103(3):129–134.

Petrongolo, B. and Pissarides, C. A. (2001). Looking into the black box: A survey of the matching function. *Journal of Economic Literature*, 39(2):390–431.

Pizzinelli, C., Theodoridis, K., and Zanetti, F. (2020). State Dependence in Labor Market Fluctuations. *International Economic Review*, page forthcoming.

Shimer, R. (2005). The cyclical behavior of equilibrium unemployment and vacancies. *American Economic Review*, 95(1):25–49.

Sterk, V. (2016). The Dark Corners of the Labor Market. Discussion Papers 1603, Centre for Macroeconomics (CFM).

Stevenson, W. J. (2018). *Operations Management*. McGraw-Hill/Irwin, 13th edition.

Taschereau-Dumouchel, M. and Schaal, E. (2015). Coordinating Business Cycles. 2015 Meeting Papers 178, Society for Economic Dynamics.

Thomas, C. and Zanetti, F. (2009). Labor market reform and price stability: An application to the Euro Area. *Journal of Monetary Economics*, 56(6):885–899.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1990). Tacit coordination games, strategic uncertainty, and coordination failure. *American Economic Review*, 80(1):234–248.

Van Huyck, J. B., Battalio, R. C., and Beil, R. O. (1991). Strategic uncertainty, equilibrium selection, and coordination failure in average opinion games. *Quarterly Journal of Economics*, 106(3):885–910.

Weitzman, M. (1982). Increasing returns and the foundations of unemployment theory. *Economic Journal*, 92(368):787–804.

Appendix

We include a series of appendices. Appendix A formally defines an equilibrium for our economy. Appendix B shows the derivation of the total surplus of a filled job and the capital gain from forming a joint venture. Appendix C describes how we compute the DSSs of the model. Appendix D presents the proofs of several propositions in the main text. Appendix E outlines how to compute the model. Appendix F discusses the role of mixed-strategy Nash equilibria. Appendix G completes our discussion of the effects of technology shocks. Appendix H looks at the ergodic distribution of variables of interest in cases of high and low volatility of the shocks to ξ_t. Last, Appendix I, reports the GIRFs to government spending shocks in the active equilibrium.

A Equilibrium

A recursive, symmetric equilibrium of type ι_t for our economy is a collection of Bellman equations $U_{i,t}$, $\widetilde{W}_{i,t}$, $W_{i,t}$, $\widetilde{J}_{i,t}$, $J_{i,t}$, and $V_{i,t}$, a variable search effort $\sigma_{i,t}$, and sequences for unemployment u_t, single firms $\widetilde{n}_{i,t}$, joint ventures n_t, the price of the intermediate good p_t, and wages $\widetilde{w}_{i,t}$ and $w_{i,t}$, all for $i \in \{I, F\}$, such that:

1. $U_{i,t}$, $\widetilde{W}_{i,t}$, $W_{i,t}$, $\widetilde{J}_{i,t}$, $J_{i,t}$, and $V_{i,t}$ satisfy equations (15)-(21).

2. The free-entry condition $V_{i,t} = 0$ holds.

3. $\sigma_{i,t}$ maximizes the asset value of the single firm $\widetilde{J}_{i,t}$.

4. The sequences of unemployment u_t, single firms $\widetilde{n}_{i,t}$, and joint ventures n_t follow the laws of motion in equations (6), (14), and (13), respectively.

5. The intermediate-goods price p_t and the wage for single and joint ventures, $\widetilde{w}_{i,t}$ and $w_{i,t}$, respectively, are determined by the Nash bargaining equations (23)-(25).

6. The type of equilibrium ι_t is consistent with $\sigma_{i,t}$.

7. ξ_t and z_t follow their stochastic processes.

8. The aggregate resource constraint (26) is satisfied.

B Total surplus

The total surplus of a labor market match at time t in a joint venture in either sector $i \in \{I, F\}$ of the economy is $TS_{i,t} = W_{i,t} - U_{i,t} + J_{i,t}$. Analogously, the total surplus of a filled job in a single firm is $\widetilde{TS}_{i,t} = \widetilde{W}_{i,t} - U_{i,t} + \widetilde{J}_{i,t}$.

Given the bargaining weight, τ, common across sectors, Nash bargaining for wages implies:

$$J_{i,t} = \tau TS_{i,t}, \tag{49}$$

$$W_{i,t} - U_{i,t} = (1-\tau)\, TS_{i,t}, \tag{50}$$

$$\widetilde{J}_{i,t} = \tau \widetilde{TS}_{i,t}, \tag{51}$$

$$\widetilde{W}_{i,t} - U_{i,t} = (1-\tau)\, \widetilde{TS}_{i,t}. \tag{52}$$

The free-entry condition of the labor market is:

$$\chi = \beta \xi_t \tau H\left(\widetilde{\theta}_t, 1\right) \mathbb{E}_t\left(\widetilde{TS}_{I,t+1}\right) = \beta \xi_t \tau H\left(1, 1/\widetilde{\theta}_t\right) \mathbb{E}_t\left(\widetilde{TS}_{F,t+1}\right). \tag{53}$$

The total surplus of establishing a joint venture is the sum of the capital gain from matching for the firms in the intermediate-goods sector, $J_{I,t} - \widetilde{J}_{I,t}$, and final-goods sector, $J_{F,t} - \widetilde{J}_{F,t}$ is $TSJV_t = J_{I,t} - \widetilde{J}_{I,t} + J_{F,t} - \widetilde{J}_{F,t}$. The price for intermediate goods, p_t, is set according to the Nash bargaining rules $J_{I,t} - \widetilde{J}_{I,t} = \widetilde{\tau} TSJV_t$ and $J_{F,t} - \widetilde{J}_{F,t} = (1-\widetilde{\tau})\, TSJV_t$, where $\widetilde{\tau}$ is the intermediate-goods producer's bargaining power.

We derive now the total surplus of a filled job in a joint venture, $TS_{i,t}$. Using the equations for $W_{I,t}$, $J_{I,t}$, and $U_{I,t}$ in the definition of $TS_{I,t}$, we get:

$$\begin{aligned} W_{I,t} + J_{I,t} - U_{I,t} &= z_t p_t - h \\ &+ \beta \xi_t \mathbb{E}_t \begin{bmatrix} \left(1-\delta-\tilde{\delta}\right)\left(W_{I,t+1} + J_{I,t+1} - U_{I,t+1}\right) \\ +\tilde{\delta}\left(\widetilde{W}_{I,t+1} + \widetilde{J}_{I,t+1} - U_{I,t+1}\right) - \mu_{I,t}\left(\widetilde{W}_{I,t+1} - U_{I,t+1}\right) \end{bmatrix}, \end{aligned} \tag{54}$$

or, equivalently,

$$TS_{I,t} = z_t p_t - h + \beta \xi_t \mathbb{E}_t\left[\left(1-\delta-\tilde{\delta}\right) TS_{I,t} + \left(\tilde{\delta} - \mu_{I,t}\,(1-\tau)\right) \widetilde{TS}_{I,t}\right], \tag{55}$$

where, in the interest of space, we omit the variable ι_t.

Analogously, the total surplus of a filled job in a joint venture for the firm in the final-goods sector F is:

$$TS_{F,t} = z_t\left(1 - p_t\right) - h + \beta\xi_t\mathbb{E}_t\left[\left(1 - \delta - \tilde{\delta}\right)TS_{F,t} + \left(\tilde{\delta} - \mu_{F,t}\left(1 - \tau\right)\right)\widetilde{TS}_{F,t}\right]. \tag{56}$$

Next, we derive the total surplus of a filled job in a single firm, $\widetilde{TS}_{i,t}$. The equations for $\widetilde{W}_{I,t}$, $\widetilde{J}_{I,t}$, and $U_{I,t}$ yield:

$$\begin{aligned}\widetilde{J}_{I,t} + \widetilde{W}_{I,t} - U_{I,t} = {}& -h - c\left(\sigma^*_{I,t}\right) + \\ & \beta\xi_t\mathbb{E}_t\begin{bmatrix} \left(1-\delta\right)\left(1-\pi^*_{I,t}\right)\left(\widetilde{J}_{I,t+1} + \widetilde{W}_{I,t+1} - U_{I,t+1}\right) + \\ \left(1-\delta\right)\pi^*_{I,t}\left(W_{I,t+1} + J_{I,t+1} - U_{I,t+1}\right) - \mu_{I,t}\left(\widetilde{W}_{I,t+1} - U_{I,t+1}\right)\end{bmatrix},\end{aligned} \tag{57}$$

where $\sigma^*_{I,t}$ is the search effort that maximizes $\widetilde{J}_{I,t}$ and $\pi^*_{I,t}$ is the matching probability induced by $\sigma^*_{I,t}$. By using the definition of $\widetilde{TS}_{i,t}$ above, we re-arrange the previous equation as:

$$\begin{aligned}\widetilde{TS}_{I,t} = {}& -h - c\left(\sigma^*_{I,t}\right) \\ & + \beta\xi_t\mathbb{E}_t\left[\left(1-\delta\right)\pi_{I,t}TS_{I,t+1} + \left(\left(1-\delta\right)\left(1-\pi_{I,t}\right) - \left(1-\tau\right)\mu_{I,t}\right)\widetilde{TS}_{I,t+1}\right].\end{aligned} \tag{58}$$

Nash bargaining for wages, as shown by equation (51), indicates that firm and worker choose search effort to maximize their joint surplus $\widetilde{TS}_{I,t}$. Specifically, since $\sigma^*_{I,t}$ maximizes $\widetilde{J}_{I,t}$, it also maximizes $\widetilde{TS}_{I,t}$. Thus, equation (58) becomes:

$$\begin{aligned}\widetilde{TS}_{I,t} = \max_{\sigma_{I,t}\geq 0}\Big\{ & -h - c\left(\sigma_{I,t}\right) \\ & + \beta\xi_t\mathbb{E}_t\left[\left(1-\delta\right)\pi_{I,t}TS_{I,t+1} + \left(\left(1-\delta\right)\left(1-\pi_{I,t}\right) - \left(1-\tau\right)\mu_{I,t}\right)\widetilde{TS}_{I,t+1}\right]\Big\},\end{aligned} \tag{59}$$

and where $\pi_{I,t}$ is an increasing function of $\sigma_{I,t}$.

We denote the gain for total surplus from forming a joint venture as $\Delta TS_{i,t} = TS_{i,t} - \widetilde{TS}_{i,t}$, and rewrite equation (59) as:

$$\begin{aligned}\widetilde{TS}_{I,t} = \max_{\sigma_{I,t}\geq 0}\Big\{ & -h - c\left(\sigma_{I,t}\right) \\ & + \beta\xi_t\mathbb{E}_t\left[\left(1-\delta\right)\pi_{I,t}\Delta TS_{I,t+1} + \left(\left(1-\delta\right) - \left(1-\tau\right)\mu_{I,t}\right)\widetilde{TS}_{I,t+1}\right]\Big\}.\end{aligned} \tag{60}$$

Similarly, we write the total surplus for single firms in the final-goods sector as:

$$\begin{aligned}\widetilde{TS}_{F,t} = \max_{\sigma_{F,t}\geq 0} \Big\{ & -h - c\left(\sigma_{F,t}\right) \\ & + \beta\xi_t\mathbb{E}_t\left[\left(1-\delta\right)\pi_{F,t}TS_{F,t+1} + \left(\left(1-\delta\right)\left(1-\pi_{F,t}\right) - \left(1-\tau\right)\mu_{F,t}\right)\widetilde{TS}_{F,t+1}\right]\Big\},\end{aligned} \tag{61}$$

or, equivalently,

$$\begin{aligned}\widetilde{TS}_{F,t} = \max_{\sigma_{F,t}\geq 0} \Big\{ & -h - c\left(\sigma_{F,t}\right) \\ & + \beta\xi_t\mathbb{E}_t\left[\left(1-\delta\right)\pi_{F,t}\Delta TS_{F,t+1} + \left(\left(1-\delta\right) - \left(1-\tau\right)\mu_{F,t}\right)\widetilde{TS}_{F,t+1}\right]\Big\}.\end{aligned} \tag{62}$$

Finally, we derive the total surplus of a joint venture, $TSJV_{i,t}$. The Nash bargaining for the intermediate goods price and wage yields $\Delta TS_{I,t} = \frac{\widetilde{\tau}}{\tau}TSJV_t$ and $\Delta TS_{F,t} = \left(\frac{1-\widetilde{\tau}}{\tau}\right)TSJV_t$. Using equations (55) and (59) in the definition of $\Delta TS_{i,t}$ produces:

$$\Delta TS_{I,t} = \min_{\sigma_{I,t}}\Big\{z_t p_t + c\left(\sigma_{I,t}\right) + \beta\left[\left(1-\delta-\tilde{\delta}\right) - \left(1-\delta\right)\pi_{I,t}\right]\xi_t\mathbb{E}_t\left(\Delta TS_{I,t+1}\right)\Big\}, \tag{63}$$

or after using the Nash bargaining condition $\Delta TS_{I,t} = \frac{\tau}{\widetilde{\tau}}TSJV_t$:

$$TSJV_t = \min_{\sigma_{I,t}}\Big\{\frac{\tau}{\widetilde{\tau}}\left[z_t p_t + c\left(\sigma_{I,t}\right)\right] + \beta\left[\left(1-\delta-\tilde{\delta}\right) - \left(1-\delta\right)\pi_{I,t}\right]\xi_t\mathbb{E}_t\left(TSJV_{t+1}\right)\Big\}. \tag{64}$$

Analogously, the total surplus of a joint venture from sector F's optimization problem is:

$$\begin{aligned}TSJV_t = \min_{\sigma_{F,t}}\Big\{ & \frac{\tau}{1-\widetilde{\tau}}\left[z_t\left(1-p_t\right) + c\left(\sigma_{F,t}\right)\right] \\ & + \beta\left[\left(1-\delta-\tilde{\delta}\right) - \left(1-\delta\right)\pi_{F,t}\right]\xi_t\mathbb{E}_t\left(TSJV_{t+1}\right)\Big\}.\end{aligned} \tag{65}$$

Combining Equation (64)$\times\widetilde{\tau}$+Equation (65)$\times\left(1-\widetilde{\tau}\right)$, p_t cancels out and we find:

$$\begin{aligned}TSJV_t = {} & \tau\cdot z_t + \beta\left(1-\delta-\tilde{\delta}\right)\xi_t\mathbb{E}_t\left(TSJV_{t+1}\right) \\ & + \min_{\sigma_{I,t}}\Big\{\tau\cdot c\left(\sigma_{I,t}\right) - \beta\left(1-\delta\right)\pi_{I,t}\xi_t\mathbb{E}_t\left(\widetilde{\tau}\cdot TSJV_{t+1}\right)\Big\} \\ & + \min_{\sigma_{F,t}}\Big\{\tau\cdot c\left(\sigma_{F,t}\right) - \beta\left(1-\delta\right)\pi_{F,t}\xi_t\mathbb{E}_t\left[\left(1-\widetilde{\tau}\right)\cdot TSJV_{t+1}\right]\Big\}.\end{aligned} \tag{66}$$

The first-order conditions for $\{\sigma_{I,t}, \sigma_{F,t}\}$ in equation (66) are:

$$\beta(1-\delta)(\psi+\sigma_{F,t}) H\left(\widetilde{\theta}_t, 1\right) \widetilde{\tau} \xi_t \mathbb{E}_t (TSJV_{t+1}) = \tau \left[c_0 + c_1 (\sigma_{I,t})^\nu\right], \tag{67}$$

$$\beta(1-\delta)(\psi+\sigma_{I,t}) H\left(1, \widetilde{\theta}_t^{-1}\right)(1-\widetilde{\tau}) \xi_t \mathbb{E}_t (TSJV_{t+1}) = \tau \left[c_0 + c_1 (\sigma_{F,t})^\nu\right]. \tag{68}$$

The active equilibrium exists if and only if there exists a pair $(\sigma_{I,t}, \sigma_{F,t}) > 0$ that jointly solves equations (67) and (68). In the symmetric equilibrium for which $\widetilde{\tau} = 1/2$ and $\widetilde{\theta}_t = 1$, equations (67) and (68) become:

$$\widetilde{\beta}(\psi+\sigma_{F,t}) \xi_t \mathbb{E}_t \left(J_{I,t+1} - \widetilde{J}_{I,t+1}\right) = c_0 + c_1 (\sigma_{I,t})^\nu, \tag{69}$$

$$\widetilde{\beta}(\psi+\sigma_{I,t}) \xi_t \mathbb{E}_t \left(J_{F,t+1} - \widetilde{J}_{F,t+1}\right) = c_0 + c_1 (\sigma_{F,t})^\nu, \tag{70}$$

where $\widetilde{\beta} = \beta(1-\delta)/\tau$. Equivalently, we can express the first-order conditions as:

$$\beta(1-\delta)(\psi+\sigma_{F,t}) \xi_t \mathbb{E}_t (\Delta TS_{I,t+1}) = c_0 + c_1 (\sigma_{I,t})^\nu \tag{71}$$

$$\beta(1-\delta)(\psi+\sigma_{I,t}) \xi_t \mathbb{E}_t (\Delta TS_{F,t+1}) = c_0 + c_1 (\sigma_{F,t})^\nu. \tag{72}$$

C Solving for the DSSs

To solve for the DSSs, we evaluate the equilibrium conditions of the model when the variables are constant over time and the exogenous shocks take their average value. The model entails a passive and an active (stable) DSS. We disregard the active and unstable DSS in our analysis. We denote the variables referring to the passive and active DSS with superscript "pas" and "act," respectively.

Using equation (66), the total surplus of a joint venture in the passive DSS is:

$$\begin{aligned} TSJV^{pas} &= \tau \cdot z^{ss} + 2\tau \cdot c(0) \\ &\quad + \beta \xi^{ss} \left[\left(1-\delta-\tilde{\delta}\right) - \widetilde{\tau}(1-\delta)\pi_I^{pas} - (1-\widetilde{\tau})(1-\delta)\pi_F^{pas}\right] TSJV^{pas} \end{aligned} \tag{73}$$

where $c(0) = 0$, $\pi_I^{pas} = (\phi+\psi^2) H\left(1, \widetilde{\theta}^{pas}\right)$, and,

$$\pi_F^{pas} = \left(\phi+\psi^2\right) H\left(1, 1/\widetilde{\theta}^{pas}\right). \tag{74}$$

As in our baseline calibration, we set $\widetilde{\tau} = 0.5$ and assume a symmetric equilibrium so that $\widetilde{\theta}^{pas} = 1$. Applying these conditions in equation (73) yields:

$$TSJV^{pas} = \frac{\tau \cdot z^{ss}}{1 - \beta \xi^{ss} \left[\left(1 - \delta - \tilde{\delta} \right) - (1 - \delta) \left(\phi + \psi^2 \right) \right]}. \tag{75}$$

The gain of total surplus from forming a joint venture in the passive DSS is determined by $\Delta TS_I^{pas} = \frac{\widetilde{\tau}}{\tau} TSJV^{pas}$ and $\Delta TS_F^{pas} = \left(\frac{1 - \widetilde{\tau}}{\tau} \right) TSJV^{pas}$, which are useful in deriving the total surplus of a filled job in the passive DSS.

Analogously, the total surplus of a joint venture in the active DSS is:

$$TSJV^{act} = \frac{\tau \cdot \left[z^{ss} + \left(c_0 \sigma_I^{act} + c_1 \frac{\left(\sigma_I^{act} \right)^{\nu + 1}}{1 + \nu} \right) + \left(c_0 \sigma_F^{act} + c_1 \frac{\left(\sigma_F^{act} \right)^{\nu + 1}}{1 + \nu} \right) \right]}{1 - \beta \xi^{ss} \left[\left(1 - \delta - \tilde{\delta} \right) - \widetilde{\tau} (1 - \delta) \pi_I^{act} - (1 - \widetilde{\tau}) (1 - \delta) \pi_F^{act} \right]}, \tag{76}$$

where:

$$\pi_I^{act} = \left[\phi + \left(\sigma_F^{act} + \psi \right) \left(\sigma_I^{act} + \psi \right) \right] H \left(1, \widetilde{\theta}^{act} \right),$$

and

$$\pi_F^{act} = \left[\phi + \left(\sigma_F^{act} + \psi \right) \left(\sigma_I^{act} + \psi \right) \right] H \left(1, 1/\widetilde{\theta}^{act} \right).$$

By imposing the symmetry conditions $\widetilde{\tau} = 1/2$, $\widetilde{\theta}^{act} = 1$ and $\sigma_F^{act} = \sigma_I^{act} = \sigma^{act}$, equation (76) becomes:

$$TSJV^{act} = \frac{\tau \cdot \left[z^{ss} + 2 \left(c_0 \sigma^{act} + c_1 \frac{\left(\sigma^{act} \right)^{\nu + 1}}{1 + \nu} \right) \right]}{1 - \beta \xi^{ss} \left[\left(1 - \delta - \tilde{\delta} \right) - (1 - \delta) \left[\phi + \left(\sigma^{act} + \psi \right)^2 \right] \right]}. \tag{77}$$

In the active DSS, the first-order condition for $\{\sigma_{I,t}, \sigma_{F,t}\}$ described by equations (67) and (68) is:

$$\frac{\beta (1 - \delta) \left(\psi + \sigma^{act} \right) \xi^{ss} TSJV^{act}}{2} = \tau \left[c_0 + c_1 \left(\sigma^{act} \right)^{\nu} \right]. \tag{78}$$

Equations (77) and (78) can be used to solve numerically for σ^{act} and $TSJV^{act}$.

The gain of total surplus from forming a joint venture in the active DSS is determined by $\Delta TS_I^{act} = \frac{\widetilde{\tau}}{\tau} TSJV^{act}$ and $\Delta TS_F^{act} = \left(\frac{1 - \widetilde{\tau}}{\tau} \right) TSJV^{act}$. Next, we derive the total surplus of a filled job in a single firm and the job-finding rate in the DSS. Using equation (58), the total surplus of a filled job for a single firm in sector I in the passive DSS is:

$$\widetilde{TS}_I^{pas} = -h + \beta\left\{(1-\delta)\,\pi_I^{pas}\cdot\Delta TS_I^{pas} + \left[(1-\delta) - \mu_I^{pas}(1-\tau)\right]\widetilde{TS}_I^{pas}\right\}, \tag{79}$$

where ΔTS_I^{pas} and π_I^{pas} were solved analytically as in equations (74) and (75). Using the matching function and free-entry condition in the labor market, the job-finding rate in the passive DSS is:

$$\mu_I^{pas} = \left(\frac{\beta\tau\widetilde{TS}_I^{pas}}{\chi}\right)^{\frac{\alpha}{1-\alpha}}. \tag{80}$$

Equations (79) and (80) are solved numerically for $\widetilde{TS}_I^{pas}$ and μ_I^{pas}.

Applying the same approach, we solve for $\widetilde{TS}_F^{pas}$ and μ_F^{pas}. Analogously, the total surplus of a filled job in a single firm and the job-finding rate in the active DSS solves:

$$\begin{aligned}\widetilde{TS}_i^{act} &= -h + \left[c_0\sigma_i^{act} + c_1\frac{\left(\sigma_i^{act}\right)^{\nu+1}}{1+\nu}\right]\\ &\quad + \beta\left\{(1-\delta)\,\pi_i^{act}\cdot\Delta TS_i^{act} + \left[(1-\delta) - \mu_i^{act}(1-\tau)\right]\widetilde{TS}_i^{act}\right\}, i\in F, I\end{aligned} \tag{81}$$

and

$$\mu_i^{act} = \left(\frac{\beta\tau\widetilde{TS}_i^{act}}{\chi}\right)^{\frac{\alpha}{1-\alpha}}, i\in F, I. \tag{82}$$

The total surplus of a filled job in a joint venture in the DSS is $TS_i^l = \widetilde{TS}_i^l + \Delta TS_i^l,\ i\in\{I,F\}, l\in\{act, pas\}$. The firm's asset value in the DSS is $J_i^l = \tau TS_i^l,\ \widetilde{J}_i^l = \tau\widetilde{TS}_i^l,\ i\in\{I,F\}, l\in\{act, pas\}$. Finally, we can derive the DSS value for the remaining variables. Substituting the job-finding rate into the matching function of the labor market, we get $\theta^{pas} = (\mu^{pas})^{\frac{1}{\alpha}}$ and $\theta^{act} = (\mu^{act})^{\frac{1}{\alpha}}$.

The value for the unemployment rate, the measure of single firms, and the measure of joint ventures in the passive and active DSS are:

$$\begin{aligned}u^{pas} &= \frac{\delta}{\delta+\mu^{pas}}\\ u^{act} &= \frac{\delta}{\delta+\mu^{act}}\\ \tilde{n}^{pas} &= \frac{\tilde{\delta} + \left(\mu^{pas}-\tilde{\delta}\right)u^{pas}}{\delta+\pi^{pas}+\tilde{\delta}}\end{aligned}$$

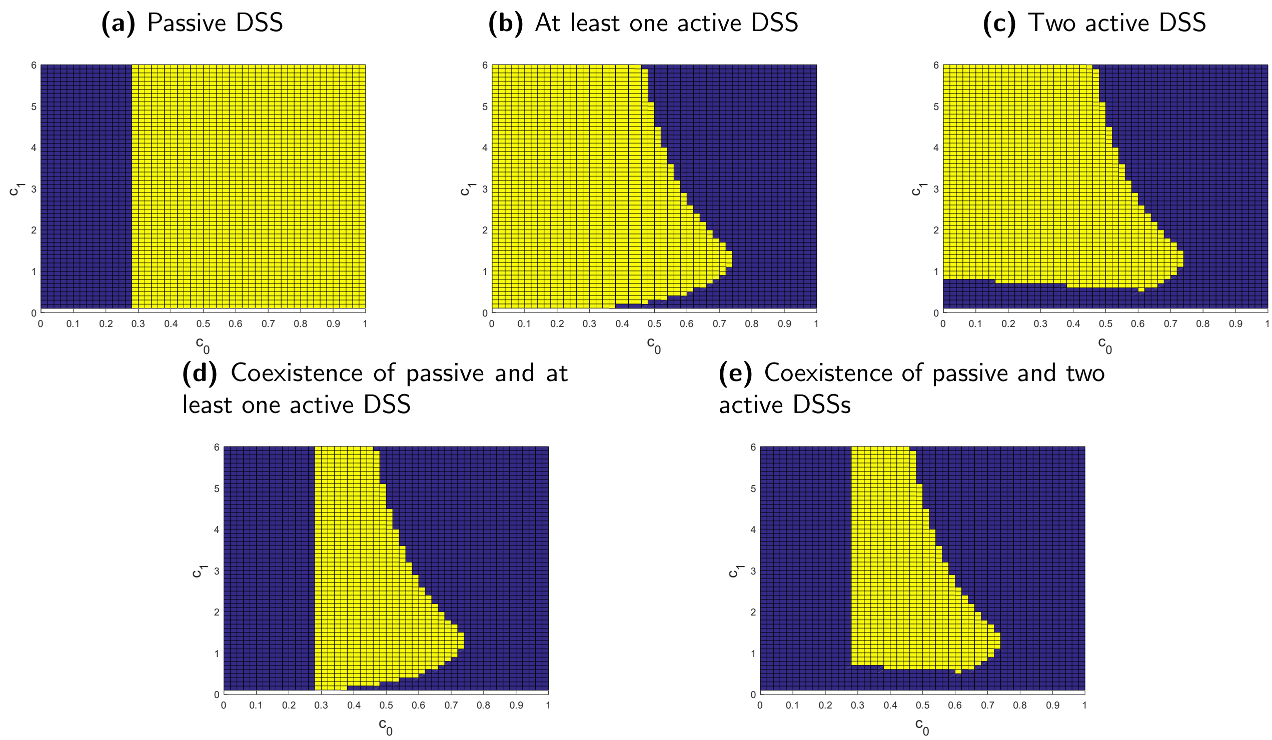
$$\tilde{n}^{act} = \frac{\tilde{\delta} + \left(\mu^{act} - \tilde{\delta}\right) u^{act}}{\delta + \pi^{act} + \tilde{\delta}}$$

$$n^{pas} = 1 - u^{pas} - \tilde{n}^{pas}$$

$$n^{act} = 1 - u^{act} - \tilde{n}^{act}.$$

The value for total final output in the passive and active DSSs is $y^{pas} = z^{ss} n^{pas}$ and $y^{act} = z^{ss} n^{act}$.

Figure 17: Existence of DSSs



(a) Passive DSS **(b)** At least one active DSS **(c)** Two active DSS **(d)** Coexistence of passive and at least one active DSS **(e)** Coexistence of passive and two active DSSs

With the previous solution, Figure 17 illustrates, for a range of values of c_0 (x-axes) and c_1 (y-axes), the conditions for the existence of a passive DSS, an active DSS, and the coexistence of DSSs stated in the main text when the model is calibrated using the parameter values described in Section 5. The yellow-shaded area shows the values that guarantee the existence of such a DSS, while the blue area shows the non-existence region. Panel (a) shows that for sufficiently large values of c_0 that the passive DSS exists irrespective of c_1. Panel (b) demonstrates that the active DSS exists for sufficiently low values of c_0. An increase in the value of c_1 has two opposing effects. On the one hand, it increases the cost of $\sigma_{i,t}$ and, on the other hand, it decreases the value of remaining a single firm, which raises the relative value of forming a joint venture. If the second effect dominates, a large c_1 expands the range of values of c_0 that satisfy Proposition 4. Panel (c) shows that two active DSSs exist when c_1 is sufficiently large. Panel (d) combines panels (a) and (b) to draw the values for c_0 and c_1 that support the coexistence of passive and active DSSs. Lastly, panel (e) plots the values of c_0 and c_1 where a passive and two active DSSs coexist.

D Proof of propositions

Proof of Proposition 2

Proof. We consider the case of symmetric sectors, so we drop the sector subscripts. We first show that the labor market tightness ratio is strictly lower in the passive DSS, i.e., $\theta^{pas} < \theta^{act}$, or, equivalently $\widetilde{TS}^{pas} < \widetilde{TS}^{act}$, as implied by the free-entry condition of the labor market.

We start with

$$\widetilde{TS}^{act} = -h + \left[c_0\sigma^{act} + c_1\frac{\left(\sigma^{act}\right)^{\nu+1}}{1+\nu}\right] \\ + \beta\left\{(1-\delta)\,\pi^{act}\cdot\Delta TS^{act} + \left[(1-\delta) - \mu^{act}\,(1-\tau)\right]\widetilde{TS}^{act}\right\}, \tag{83}$$

and

$$\mu^{act} = \left(\frac{\beta\tau\widetilde{TS}^{act}}{\chi}\right)^{\frac{\alpha}{1-\alpha}}, \tag{84}$$

which are equivalent to equations (81) and (82) except that here we drop the sector subscripts.

The values for $\widetilde{TS}^{act}$ and θ^{act} solve equations (83) and (84). We rewrite equation (84) as

$$\chi = \beta\tau q^{act}\widetilde{TS}^{act} = \beta\tau q^{pas}\widetilde{TS}^{pas}. \tag{85}$$

Given the Cobb-Douglas matching function for the labor market, equation (85) is equivalent to:

$$\theta^{act} = \left(\frac{\beta\tau\widetilde{TS}^{act}}{\chi}\right)^{\frac{1}{1-\alpha}}. \tag{86}$$

Applying equation (85) to equation (83), delivers:

$$\left(\frac{1-\tau}{\tau}\right)\chi\theta^{act} = \Bigg\{ -h - \left[c_0\sigma^{act} + c_1\frac{\left(\sigma^{act}\right)^{\nu+1}}{1+\nu}\right] \\ + \beta\,(1-\delta)\,\pi^{act}\cdot\Delta TS^{act} - \left[1-\beta\,(1-\delta)\right]\widetilde{TS}^{act}\Bigg\}, \tag{87}$$

where we used $\mu^{act} = \theta^{act} q^{act}$. In equation (87), θ is linear and strictly decreasing in the total surplus for a single firm, $\widetilde{TS}$. In equation (85), θ is strictly increasing in $\widetilde{TS}$. Since σ^{act} and ΔTS^{act} were solved in equations (77) and (78), they are treated as constant terms here.

Hence, values for θ^{act} and $\widetilde{TS}^{act}$ solve:

$$\left(\frac{1-\tau}{\tau}\right)\chi\theta = \Bigg\{ -h - \left[c_0\sigma^{act} + c_1\frac{\left(\sigma^{act}\right)^{\nu+1}}{1+\nu}\right] + \beta\left(1-\delta\right)\pi^{act}\cdot\Delta TS^{act} - \left[1-\beta\left(1-\delta\right)\right]\widetilde{TS}\Bigg\} \tag{88}$$

$$\theta = \left(\frac{\beta\tau\widetilde{TS}}{\chi}\right)^{\frac{1}{1-\alpha}}. \tag{89}$$

Similarly, values for $\widetilde{TS}^{pas}$ and θ^{pas} solve:

$$\left(\frac{1-\tau}{\tau}\right)\chi\theta = \left[-h + \beta\left(1-\delta\right)\pi^{pas}\cdot\Delta TS^{pas}\right] - \left[1-\beta\left(1-\delta\right)\right]\widetilde{TS}, \tag{90}$$

$$\theta = \left(\frac{\beta\tau\widetilde{TS}}{\chi}\right)^{\frac{1}{1-\alpha}}. \tag{91}$$

In equations (88) and (90), θ is linear and strictly decreasing in $\widetilde{TS}$. In equations (89) and (91), θ is strictly increasing in $\widetilde{TS}$.

For $\theta^{act} > \theta^{pas}$, it must be that the intercept term in equation (87) is greater than the intercept term in equation (90), which occurs if:

$$-h - \left[c_0\sigma^{act} + c_1\frac{\left(\sigma^{act}\right)^{\nu+1}}{1+\nu}\right] + \beta\left(1-\delta\right)\pi^{act}\cdot\Delta TS^{act} > -h + \beta\left(1-\delta\right)\pi^{pas}\cdot\Delta TS^{pas}. \tag{92}$$

To simplify notation, denote $W\left(\sigma\right) = -h + \frac{W_1(\sigma)}{W_2(\sigma)}$, where

$$W_1\left(\sigma\right) = \left[\beta\left(1-\delta-\tilde{\delta}\right)-1\right]\left[c_0\sigma + c_1\frac{\sigma^{\nu+1}}{1+\nu}\right] + \beta\left(1-\delta\right)\left[\phi + \left(\psi+\sigma\right)^2\right],$$
$$W_2\left(\sigma\right) = 1 - \beta\left\{\left(1-\delta-\tilde{\delta}\right) - \left(1-\delta\right)\left[\phi + \left(\psi+\sigma\right)^2\right]\right\}.$$

It can be shown that equation (92) is equivalent to $W\left(\sigma^{act}\right) > W\left(0\right)$. We verify that, for $\sigma \in \left(0, \sqrt{1-\phi}-\psi\right)$, $\frac{dW_1}{d\sigma}/W_1 > \frac{dW_2}{d\sigma}/W_2$, which implies $dW/d\sigma > 0$. Consequently, equation (92) holds, and $\theta^{act} > \theta^{pas}$.

Since the job-finding rate is strictly increasing in labor market tightness, $\mu^{act} > \mu^{pas}$. Since $u = \delta / (\delta + \mu)$ in the DSS, $u^{act} < u^{pas}$ holds.

Finally, we show that $y^{act} > y^{pas}$. Since $y = n$ and $n = 1 - \tilde{n} - u$, $y^{act} > y^{pas}$ is equivalent to showing that $\tilde{n}^{act} + u^{act} < \tilde{n}^{pas} + u^{pas}$.

In the DSS, it holds that:

$$\tilde{n} + u = \frac{\tilde{\delta} + (\pi + \mu + \delta) \frac{\delta}{\delta + \mu}}{\delta + \pi + \tilde{\delta}}. \tag{93}$$

The RHS of equation (93) is strictly decreasing in both μ and π. Given that $\mu^{pas} < \mu^{act}$ and $\pi^{pas} < \pi^{act}$, it holds that $\tilde{n}^{act} + u^{act} < \tilde{n}^{pas} + u^{pas}$, or, equivalently, $y^{act} > y^{pas}$. □

Proof of Proposition 3

Proof. Proposition 3 holds if it is optimal for firms in one sector to search with zero effort when firms in the opposite sector search with zero effort. In such a case, the Nash equilibrium with zero search effort exists in the passive DSS.

The firm's maximization problem in the passive DSS is:

$$\widetilde{TS}^{pas} = \max_{\sigma \geq 0} -h - \left(c_0 \sigma + c_1 \frac{\sigma^{\nu+1}}{1+\nu} \right) + \beta \begin{Bmatrix} (1-\delta) \left[\phi + \psi \left(\psi + \sigma \right) \right] \cdot \Delta TS^{pas} \\ + \left[(1-\delta) - \mu^{pas} (1-\tau) \right] \widetilde{TS}^{pas} \end{Bmatrix}. \tag{94}$$

The total surplus of a single firm $\widetilde{TS}^{pas}$ is strictly concave in σ, for $\sigma > 0$. Hence, the corner solution $\sigma = 0$ is optimal if and only if the first-order derivative is non-positive at $\sigma = 0$:

$$c_0 + c_1 0^\nu \geq \beta (1-\delta) \psi \Delta TS^{pas}, \tag{95}$$

where ΔTS^{pas} is given by equation (75), or, equivalently:

$$c_0 > \frac{\beta (1-\delta) \psi}{2 - 2\beta \left[\left(1 - \delta - \tilde{\delta} \right) - (1-\delta) (\phi + \psi^2) \right]}, \tag{96}$$

where we assume $z^{ss} = 1$, $\xi^{ss} = 1$, and $\widetilde{\tau} = 0.5$. □

Proof of Proposition 4

Proof. Proposition 4 holds if there exist $\sigma \in \left(0, \sqrt{1-\phi}-\psi\right)$ (to guarantee that the matching probability is bounded by one) and $\Delta TS \in \mathbb{R}$ that solve equations (78) and (77).

By substituting equation (78) into equation (77), we get:

$$\frac{1+\left(c_0\sigma + c_1\frac{\sigma^{\nu+1}}{1+\nu}\right)}{2-2\beta\left[\left(1-\delta-\tilde{\delta}\right)-(1-\delta)\left[\phi+(\sigma+\psi)^2\right]\right]} = \frac{c_0 + c_1\sigma^{\nu}}{\beta(1-\delta)(\psi+\sigma)}, \tag{97}$$

where we assume $\widetilde{\tau} = 1/2$, $\xi^{ss} = 1$, $z^{ss} = 1$. □

Proof of Proposition 6

Proof. We first show that the Nash equilibrium in the passive DSS is stable. To do so, we demonstrate that there exists an $\epsilon > 0$, such that when a firm in sector j deviates from the passive DSS by searching with a small and positive effort bounded by ϵ, it remains optimal for the firm in the opposite sector i to search with zero effort:

$$c_0 + c_1 0^{\nu} > \beta(1-\delta)(\psi+\sigma_j)\,\mathbb{E}(\Delta TS_i), \tag{98}$$

where $\sigma_j \in (0,\epsilon)$. The RHS of equation (98) is a function of σ_j, which is continuous at $\sigma_j = 0$ (note that $\mathbb{E}(\Delta TS_i)$ is a continuous function of σ_j). Given the existence of the passive DSS, we know that $c_0 + c_1 0^{\nu} > \beta(1-\delta)\psi\Delta TS^{pas}$. Because of continuity, there exists $\epsilon > 0$, so that equation (98) holds when $\sigma_j < \epsilon$.

Next, we show that one Nash equilibrium in the active DSS is stable when two active DSSs exist. The best response function of sector i implied by equations (71) and (72) in the active DSS is:

$$\sigma_i = \begin{cases} \left[\frac{\beta(1-\delta)(\psi+\sigma_j)\Delta TS^{act}-c_0}{c_1}\right]^{\frac{1}{\nu}} & \text{if } \beta(1-\delta)(\psi+\sigma_j)\Delta TS^{act} \geq c_0 \\ 0 & \text{if } \beta(1-\delta)(\psi+\sigma_j)\Delta TS^{act} < c_0, \end{cases} \tag{99}$$

which is strictly increasing and concave in σ_j since $c_1 > 0$ and $\nu > 1$. When two active DSSs exist, the best response curve (99) intersects with the 45-degree line at $\sigma_F = \sigma_I = \sigma^*$ and $\sigma_F = \sigma_I = \sigma^{**}$ with $0 < \sigma^* < \sigma^{**} < \sqrt{1-\phi}-\psi$. Due to strict concavity, we have $\frac{d\sigma_i}{d\sigma_j}\mid_{\sigma_i=\sigma_j=\sigma^*} > 1$ and $\frac{d\sigma_i}{d\sigma_j}\mid_{\sigma_i=\sigma_j=\sigma^{**}} < 1$. Therefore, the active Nash equilibrium at $\sigma_F = \sigma_I = \sigma^*$ is unstable, while the one at $\sigma_F = \sigma_I = \sigma^{**}$ is stable.

Finally, consider the case when the passive DSS and one active DSS exist, where $\sigma_F = \sigma_I = \sigma^*$

and $0 < \sigma^* < \sqrt{1-\phi} - \psi$. Since the passive DSS exists, the inequality $c_0 > \beta(1-\delta)\psi\Delta TS^{pas}$ holds. In addition, we have that $\Delta TS^{act} < \Delta TS^{pas}$, which results from equations (75) and (76). We also have that $c_0 > \beta(1-\delta)\psi\Delta TS^{act}$. So $\sigma_i(\sigma_j) = 0$ in the active DSS for $\sigma_j \in [0, \hat{\sigma}]$ with $\hat{\sigma} = \frac{c_0}{\beta(1-\delta)\Delta TS^{act}} - \psi$. Since $\sigma_F = \sigma_I = \sigma^*$ is the only intersection between $\sigma_i(\sigma_j)$ and the 45-degree line in the range $\sigma_j \in [\hat{\sigma}, \sigma^*]$ with $\sigma_i(\hat{\sigma}) = 0$, we must have $\frac{d\sigma_i}{d\sigma_j}|_{\sigma_i=\sigma_j=\sigma^*} \geq 1$. When the derivative is equal to one, the best response curves are tangent to the 45-degree line; when the derivative is greater than one, the best response curve may have two intersections with the 45-degree line, in which case we have $0 < \sigma^* < \sqrt{1-\phi} - \psi < \sigma^{**}$ which ensures that only one intersection (σ^*) is the active equilibrium. Since the derivative is greater than or equal to one, the active static Nash equilibrium at $\sigma_F = \sigma_I = \sigma^*$ is unstable. □

E Model solution

In this appendix, we outline the algorithm to solve the model numerically.

E.1 Solution without government spending

We first discuss the solution to the benchmark case without government spending. The vector of state variables is $S_t = (z_t, \xi_t, \iota_{t-1}, u_t, n_t, \widetilde{n}_t)$, where we omit the sector subscripts. At the beginning of period t, S_t is taken as given. The states z_t and ξ_t are exogenous, and the states ι_{t-1}, u_t, n_t, and $\widetilde{n}_t$ are endogenous and predetermined. To derive the solution of the system, we require the value functions $TSJV(S_t)$, and $\widetilde{TS}(S_t)$; two policy functions $\sigma(S_t)$, and $\theta(S_t)$; and the transition rule of $\iota_t = \iota(\iota_{t-1}, S_t)$. The transition rule for the other endogenous states (u_t, n_t and $\widetilde{n}_t$) is directly given by the model once the other functions have been found.

Because of sectoral symmetry, $\widetilde{\theta}_t = \widetilde{n}_{F,t}/\widetilde{n}_{I,t} = 1$. As we will show below, a fixed $\widetilde{\theta}$ implies that the value functions, policy functions, and the transition rule for ι_t depend on $(z_t, \xi_t, \iota_{t-1})$ only.

Step 1: Solve for $TSJV$, σ, and ι. Equation (66) can be rewritten as:

$$TSJV(z_t, \xi_t, \iota_{t-1}) = \min_{\sigma_t \geq 0} \tau \cdot [z_t + 2c(\sigma_t)] + \beta \left\{ \left(1 - \delta - \tilde{\delta}\right) - (1-\delta)\left[\phi + (\psi + \sigma_t)(\psi + \overline{\sigma_t})\right] \right\}$$
$$* \xi_t \mathbb{E}_t \left[TSJV(z_{t+1}, \xi_{t+1}, \iota_t)\right], \tag{100}$$

where $\overline{\sigma_t}$ is the search effort in the opposite sector, taken as given by the firms. In the symmetric equilibrium, $\sigma_t = \overline{\sigma_t}$.

The equilibrium type ι_t is determined by the best response functions implied by equation (100) and the history dependence of equilibrium selection. Specifically, if $\iota_{t-1} = 0$ (passive equilibrium in period $t-1$), we first verify whether the passive equilibrium continues to exist in period t by checking whether:

$$\arg\min_{\sigma_t \geq 0} 2c(\sigma_t) - \widetilde{\beta}\left[\phi + (\psi + \sigma_t)\psi\right]\xi_t \mathbb{E}_t\left[TSJV(z_{t+1}, \xi_{t+1}, \iota_t = 0)\right] = 0 \tag{101}$$

holds. If it does, the passive equilibrium exists and persists (i.e., $\iota_t = \iota_{t-1} = 0$). Otherwise, the passive equilibrium fails to exist and the active equilibrium is selected (i.e., $\iota_t = 1$).

Analogously, if $\iota_{t-1} = 1$ (active equilibrium in period $t-1$), we verify whether the active equilibrium continues to exist in period t by checking whether:

$$\arg\min_{\sigma_t \geq 0} 2c(\sigma_t) - \widetilde{\beta}\left[\phi + (\psi + \sigma_t)(\psi + \sigma^*)\right]\xi_t \mathbb{E}_t\left[TSJV(z_{t+1}, \xi_{t+1}, \iota_t = 1)\right] > 0 \tag{102}$$

holds. If it does, the active equilibrium exists and persists (i.e., $\iota_t = \iota_{t-1} = 1$). Otherwise, the active equilibrium fails to exist and the passive equilibrium is selected (i.e., $\iota_t = 0$).

We use value function iteration methods to solve for the value function $TSJV$, the policy function σ, and the transition rule of ι using equation (100) and conditions (101) and (102).

Step 2: Solve for $\widetilde{TS}$ and θ. Equation (58) can be rewritten as:

$$\widetilde{TS}(z_t, \xi_t, \iota_{t-1}) = -h - c(\sigma_t) + \beta\xi_t \mathbb{E}_t \begin{bmatrix} (1-\delta)\pi_t \Delta TS(z_{t+1}, \xi_{t+1}, \iota_t) + \\ ((1-\delta) - (1-\tau)\theta^{\alpha}(z_t, \xi_t, \iota_{t-1}))\widetilde{TS}(z_{t+1}, \xi_{t+1}, \iota_t) \end{bmatrix}, \tag{103}$$

where we used $\Delta TS_{t+1} = TS_{t+1} - \widetilde{TS}_{t+1}$ and $\mu_t = \theta_t^{\alpha}$.

The free-entry condition of the labor market (equation 53) can be rewritten as:

$$\chi = \beta\xi_t \tau \theta^{\alpha-1}(z_t, \xi_t, \iota_{t-1}) \mathbb{E}_t\left[\widetilde{TS}(z_{t+1}, \xi_{t+1}, \iota_t)\right]. \tag{104}$$

With $\Delta TS_t = \widetilde{\tau} TSJV_t / \tau$, σ_t, and ι_t being solved in step 1, we find the value function $\widetilde{TS}$ and the policy function θ with equations (103) and (104) by using value function iteration.

E.2 Solution with government spending

We consider now the case with government spending. This case is challenging to solve since, in general, it implies sectoral asymmetry. The model's vector of state variables is: $S_t = \left(z_t, \xi_t, \epsilon_t^G, \iota_{t-1}, u_t^F, u_t^I, n_t^F, n_t^I, n_t^G, \widetilde{n}_t^F, \widetilde{n}_t^I, \widetilde{n}_t^G\right)$. States z_t, ξ_t, and ϵ_t^G are exogenous, and states $\iota_{t-1}, u_t^F, u_t^I, n_t^F, n_t^I, n_t^G, \widetilde{n}_t^F, \widetilde{n}_t^I$, and $\widetilde{n}_t^G$ are endogenous. To derive the solution of the system, we need the solution for the value functions $TSJV(S_t)$, $\widetilde{TS}_F(S_t)$, and $\widetilde{TS}_I(S_t)$ (the other value functions can be derived from these three value functions), the four policy functions $\sigma_I(S_t)$, $\sigma_F(S_t)$, $\theta_I(S_t)$, and $\theta_F(S_t)$, and the transition rule of $\iota_t = \iota(\iota_{t-1}, S_t)$. The transition rule of the other endogenous states is directly given once the other functions have been found.

In the asymmetric case, the value functions, the policy functions, and the transition rule of ι_t depend on the entire vector of states S_t rather than a subset of S_t as in Appendix E.1. The reason is that the measure of single firms $\left(\widetilde{n}_t^F, \widetilde{n}_t^I, \widetilde{n}_t^G\right)$ determines the inter-firm market tightness ratio $\widetilde{\theta}_t$, which affects firms' value and policy. In addition, the transition rule of $\left(\widetilde{n}_t^F, \widetilde{n}_t^I, \widetilde{n}_t^G\right)$ depends on the $\left(u_t^F, u_t^I, n_t^F, n_t^I, n_t^G\right)$.

Given the high dimension of the state space, we simplify the model solution with a forecast rule for $\tilde{\theta}$ that only depends on a small number of state variables. This approach is inspired by similar ideas in Krusell and Smith (1998). Intuitively, firms do not need to know $\left(u_t^F, u_t^I, n_t^F, n_t^I, n_t^G, \widetilde{n}_t^F, \widetilde{n}_t^I, \widetilde{n}_t^G\right)$ to make decisions if the forecast rule is accurate, which greatly reduces the dimension of the state space when solving the value and policy functions.

We choose the forecast rule:

$$\begin{aligned}\log\left(\widetilde{\theta}_{t+1}\right) &= \left(a_{\tilde{\theta}} + a_{\tilde{\theta},\iota}\iota_{t-1}\right)\log\left(\widetilde{\theta}_t\right) + \left(a_z + a_{z,\iota}\iota_{t-1}\right)\log(z_t) \\ &\quad + \left(a_\xi + a_{\xi,\iota}\iota_{t-1}\right)\log(\xi_t) + \left(a_G + a_{G,\iota}\iota_{t-1}\right)\epsilon_t^G, \end{aligned} \tag{105}$$

where $A = \left(a_{\tilde{\theta}}, a_{\tilde{\theta},\iota}, a_z, a_{z,\iota}, a_\xi, a_{\xi,\iota}, a_G, a_{G,\iota}\right)$ is the vector of coefficients to be determined.

To do so, we proceed as follows:

Step 1: Initialize the algorithm. We initialize the forecast rule with some initial guess:

$$A^{(0)} = \left(a_{\tilde{\theta}}^{(0)}, a_{\tilde{\theta},\iota}^{(0)}, a_z^{(0)}, a_{z,\iota}^{(0)}, a_\xi^{(0)}, a_{\xi,\iota}^{(0)}, a_G^{(0)}, a_{G,\iota}^{(0)}\right). \tag{106}$$

Step 2: Solve for $TSJV$, σ_F, σ_I, and ι. Equation (66) can be rewritten as:

$$
\begin{aligned}
TSJV\left(z_t, \xi_t, \epsilon_t^G, \iota_{t-1}, \widetilde{\theta}_t\right) = {} & \tau \cdot z_t + \beta\left(1-\delta-\tilde{\delta}\right)\xi_t \mathbb{E}_t\left[TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t, \widetilde{\theta}_{t+1}\right)\right] \\
& + \min_{\sigma_{I,t}} \tau \cdot c\left(\sigma_{I,t}\right) - \beta\left(1-\delta\right)\pi_{I,t}\xi_t \mathbb{E}_t\left[\widetilde{\tau} TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t, \widetilde{\theta}_{t+1}\right)\right] \\
& + \min_{\sigma_{F,t}} \tau \cdot c\left(\sigma_{F,t}\right) - \beta\left(1-\delta\right)\pi_{F,t}\xi_t \mathbb{E}_t\left[\left(1-\widetilde{\tau}\right) TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t, \widetilde{\theta}_{t+1}\right)\right],
\end{aligned} \tag{107}
$$

where $\pi_{I,t} = \left[\phi + \left(\psi + \sigma_{F,t}\right)\left(\psi + \sigma_{I,t}\right)\right] H\left(\widetilde{\theta}_t, 1\right)$, $\pi_{F,t} = \left[\phi + \left(\psi + \sigma_{F,t}\right)\left(\psi + \sigma_{I,t}\right)\right] H\left(1, 1/\widetilde{\theta}_t\right)$,

$$
\begin{aligned}
\log\left(\widetilde{\theta}_{t+1}\right) = {} & \left(a_{\tilde{\theta}}^{(q)} + a_{\tilde{\theta},\iota}^{(q)}\iota_{t-1}\right)\log\left(\widetilde{\theta}_t\right) + \left(a_z^{(q)} + a_{z,\iota}^{(q)}\iota_{t-1}\right)\log\left(z_t\right) \\
& + \left(a_\xi^{(q)} + a_{\xi,\iota}^{(q)}\iota_{t-1}\right)\log\left(\xi_t\right) + \left(a_G^{(q)} + a_{G,\iota}^{(q)}\iota_{t-1}\right)\epsilon_t^G,
\end{aligned}
$$

and $A^{(q)} = \left(a_{\tilde{\theta}}^{(q)}, a_{\tilde{\theta},\iota}^{(q)}, a_z^{(q)}, a_{z,\iota}^{(q)}, a_\xi^{(q)}, a_{\xi,\iota}^{(q)}, a_G^{(q)}, a_{G,\iota}^{(q)}\right)$ is the vector of coefficients of the forecast rule in the q-th iteration.

The equilibrium type ι_t is determined by the best response functions implied by equation (100) and the history dependence of equilibrium selection. If $\iota_{t-1} = 0$ (passive equilibrium in period $t-1$), we verify whether the passive equilibrium still exists in the current period t, i.e., $\iota_t = 0$, by checking whether:

$$
\arg\min_{\sigma_{I,t}\geq 0} c\left(\sigma_{I,t}\right) - \widetilde{\beta}\left[\phi + \left(\psi + \sigma_{I,t}\right)\psi\right]\xi_t \mathbb{E}_t\left[\widetilde{\tau} TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t = 0, \widetilde{\theta}_{t+1}\right)\right] = 0 \tag{108}
$$

$$
\arg\min_{\sigma_{F,t}\geq 0} c\left(\sigma_{F,t}\right) - \widetilde{\beta}\left[\phi + \left(\psi + \sigma_{F,t}\right)\psi\right]\xi_t \mathbb{E}_t\left[\left(1-\widetilde{\tau}\right) TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t = 0, \widetilde{\theta}_{t+1}\right)\right] = 0 \tag{109}
$$

hold. If these conditions hold, $\iota_t = \iota_{t-1} = 0$. Otherwise, $\iota_t = 1$.

Analogously, if $\iota_{t-1} = 1$ (active equilibrium in period $t-1$), we verify whether the active equilibrium still exists in the current period, i.e., $\iota_t = 1$, by checking whether:

$$
\arg\min_{\sigma_{I,t}\geq 0} c\left(\sigma_{I,t}\right) - \widetilde{\beta}\left[\phi + \left(\psi + \sigma_{I,t}\right)\left(\psi + \sigma_{F,t}\right)\right]\xi_t \mathbb{E}_t\left[\widetilde{\tau} TSJV\left(z_{t+1}, \xi_{t+1}, \epsilon_{t+1}^G, \iota_t = 1, \widetilde{\theta}_{t+1}\right)\right] > 0 \tag{110}
$$

and

$$\arg\min_{\sigma_{F,t}\geq 0} c\left(\sigma_{F,t}\right)-\widetilde{\beta}\left[\phi+\left(\psi+\sigma_{I,t}\right)\left(\psi+\sigma_{F,t}\right)\right]\xi_t\mathbb{E}_t\left[\left(1-\widetilde{\tau}\right)TSJV\left(z_{t+1},\xi_{t+1},\epsilon_{t+1}^G,\iota_t=1,\widetilde{\theta}_{t+1}\right)\right]>0 \tag{111}$$

hold. If these conditions hold, $\iota_t=\iota_{t-1}=1$. Otherwise, $\iota_t=0$.

Given the forecast rule with $A^{(q)}$, we can solve for the value function $TSJV$, the policy function σ, and the transition rule ι with equation (107) and conditions (108)-(111) using value function iteration.

Step 3: Solve for $\widetilde{TS}$ and θ. Equation (58) can be rewritten, for $i\in\{I,F\}$, as:

$$\begin{aligned}\widetilde{TS}\left(z_t,\xi_t,\epsilon_t^G,\iota_{t-1},\widetilde{\theta}_t\right)&=-h-c\left(\sigma_{i,t}\right)\\&+\beta\xi_t\mathbb{E}_t\left[\begin{array}{c}\left(1-\delta\right)\pi_{i,t}\Delta TS\left(z_{t+1},\xi_{t+1},\epsilon_{t+1}^G,\iota_t,\widetilde{\theta}_{t+1}\right)+\\\left(\left(1-\delta\right)-\left(1-\tau\right)\theta_i^\alpha\left(z_t,\xi_t,\epsilon_t^G,\iota_{t-1},\widetilde{\theta}_t\right)\right)\\*\widetilde{TS}\left(z_{t+1},\xi_{t+1},\epsilon_{t+1}^G,\iota_t,\widetilde{\theta}_{t+1}\right)\end{array}\right],\end{aligned} \tag{112}$$

where we have used the fact that $\Delta TS_{i,t+1}=TS_{i,t+1}-\widetilde{TS}_{i,t+1}$ and $\mu_{i,t}=\theta_{i,t}^\alpha$.

We also have, for $i\in\{I,F\}$, the free-entry condition implied by equation (53):

$$\chi=\beta\xi_t\tau\theta_{i,t}^{\alpha-1}\left(z_t,\xi_t,\epsilon_t^G,\iota_{t-1},\widetilde{\theta}_t\right)\mathbb{E}_t\left[\widetilde{TS}\left(z_{t+1},\xi_{t+1},\epsilon_{t+1}^G,\iota_t,\widetilde{\theta}_{t+1}\right)\right]. \tag{113}$$

With $\Delta TS_{i,t}$, $\sigma_{I,t}$, $\sigma_{F,t}$ and ι_t being solved in step 2 (in particular, $\Delta TS_t=\widetilde{\tau}TSJV_t/\tau$), we can solve for the value function $\widetilde{TS}_{i,t}$ and the policy function $\theta_{i,t}$ approximately with equations (112) and (113) using value function iteration.

Step 4: Simulate the model. We simulate the model for 10,000 periods (disregarding the first 2,000 as a burn-in) with random draws of $\left\{z_t,\xi_t,\epsilon_t^G\right\}$. Then, we compute the realized equilibrium inter-firm market tightness ratio $\widetilde{\theta}_t$.

Step 5: Update the forecast rule. Based on the simulated data, we update the coefficient of the forecast rule $A^{(q)}$ with $A^{(q+1)}$ using ordinary least squares. If $A^{(q)}$ and $A^{(q+1)}$ are sufficiently close to each other, we stop the iteration. Otherwise, we return to step 2. The converged forecasting rule explains the fluctuations of $\widetilde{\theta}_t$ well, with an R^2 of 0.91.

F Mixed-strategy Nash equilibria

This appendix discusses the role of mixed-strategy Nash equilibria in our model. We first establish the condition for the existence of a mixed-strategy Nash equilibrium in the DSS (the case with stochastic shocks is similar, but more cumbersome to derive). Then, we argue that such a mixed-strategy Nash equilibrium exists and is unique for the calibration in Section 5. However, this mixed-strategy Nash equilibrium is unstable: a small deviation from the mixed-strategy makes the system converge to the pure-strategy Nash equilibrium.

In a mixed-strategy setting, firms randomize their search effort by choosing $\sigma = 0$ with probability q and choosing $\sigma = \hat{\sigma}$ with probability $(1-q)$. We numerically verify that the solution to equation (71) is unique in the range of $0 < \hat{\sigma} < \sqrt{1-\phi} - \psi$. So firms cannot randomize their search effort by choosing between multiple positive efforts. The random choice is independent across firms. Due to the law of large numbers, the average search effort in both sectors is $\overline{\sigma} = q \cdot 0 + (1-q)\,\hat{\sigma}$. For a single firm, the inter-firm matching probability is given by $\pi(\sigma) = \phi + \psi(\psi + \sigma)(\psi + \overline{\sigma})$. In the mixed-strategy Nash equilibrium, the inter-firm matching probability takes two values: $\pi(0) = \phi + \psi(\psi + \overline{\sigma})$ and $\pi(\hat{\sigma}) = \phi + (\psi + \hat{\sigma})(\psi + \overline{\sigma})$.

A mixed-strategy Nash equilibrium consists of a tuple $\{q, \hat{\sigma}\}$ with $\hat{\sigma} \in \left(0, \sqrt{1-\phi} - \psi\right)$ and $q \in (0,1)$. The tuple $\{q, \hat{\sigma}\}$ implies that single firms are indifferent between choosing $\sigma = 0$ and $\sigma = \hat{\sigma}$, i.e., $\widetilde{TS}(0) = \widetilde{TS}(\hat{\sigma})$. Since $\Delta TS(0) = TS - \widetilde{TS}(0)$ and $\Delta TS(\hat{\sigma}) = TS - \widetilde{TS}(\hat{\sigma})$, it holds that $\Delta TS(0) = \Delta TS(\hat{\sigma})$. We denote $\Delta TS(0) = \Delta TS(\hat{\sigma}) = \Delta TS$.

According to equation (63):

$$\Delta TS = z^{ss} p^{ss} + \beta \left[\left(1 - \delta - \tilde{\delta}\right) - (1-\delta)\,\pi(0) \right] \Delta TS, \tag{114}$$

where $c(0) = 0$. From equation (60), the single firm's total surplus with zero search effort is:

$$\widetilde{TS}(0) = -h + \beta \left[(1-\delta)\,\pi(0)\,\Delta TS + ((1-\delta) - (1-\tau)\,\theta^{\alpha})\,\widetilde{TS}(0) \right]. \tag{115}$$

Analogously, the single firm's total surplus by choosing $\hat{\sigma}$ search effort satisfies:

$$\widetilde{TS}(\hat{\sigma}) = -h - c(\hat{\sigma}) + \beta \left[(1-\delta)\,\pi(\hat{\sigma})\,\Delta TS + ((1-\delta) - (1-\tau)\,\theta^{\alpha})\,\widetilde{TS}(\hat{\sigma}) \right]. \tag{116}$$

Since $\widetilde{TS}(0) = \widetilde{TS}(\hat{\sigma})$ in the mixed-strategy Nash equilibrium, combining equations (115) and (116) delivers:

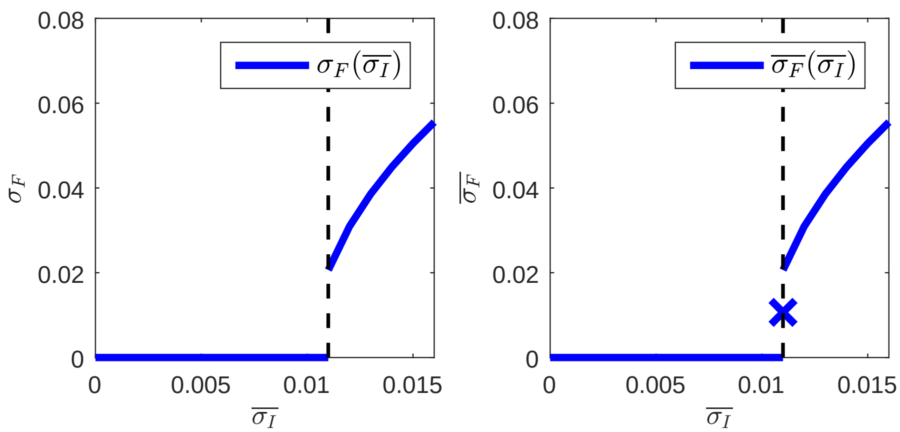
$$c(\hat{\sigma}) = \beta(1-\delta)(\pi(\hat{\sigma}) - \pi(0))\Delta TS. \tag{117}$$

Finally, according to the first-order condition for $\{\sigma_{I,t}, \sigma_{F,t}\}$ in equation (71):

$$\beta(1-\delta)(\psi + \overline{\sigma})\Delta TS = c_0 + c_1\hat{\sigma}^{\nu}. \tag{118}$$

In sum, we have the three equations (114), (117), and (118) and three unknowns (i.e., $\hat{\sigma}$, q, ΔTS). The mixed-strategy Nash equilibrium exists if the system of equations has a solution for the three unknowns. Using the calibration in Section 5, the mixed-strategy Nash equilibrium is $q = 0.3425$, $\hat{\sigma} = 0.0164$, and $\Delta TS = 2.7417$. The average search effort $\overline{\sigma}$ is $(1-q) \times \hat{\sigma} = 0.0107$.

Figure 18: Best response in the mixed-strategy Nash equilibrium



The left panel in Figure 18 displays the firm's optimal search effort in sector F as a function of $\overline{\sigma}_I$. The firm chooses a positive search effort if $\overline{\sigma}_I > 0.0107$ (i.e., for values to the right of the vertical dashed line). The firm chooses a zero search effort if $\overline{\sigma}_I < 0.0107$ (i.e., for values to the left of the vertical dashed line). The firm is indifferent between choosing $\sigma = 0.0164$ and $\sigma = 0$ if $\overline{\sigma}_I = 0.0107$ (i.e., if sector I uses the mixed-strategy $q_I = 0.3425$, $\hat{\sigma}_I = 0.0164$).

The right panel in Figure 18 plots $\overline{\sigma}_F$ as a function of $\overline{\sigma}_I$. The firm chooses a positive search effort if $\overline{\sigma}_I > 0.0107$ (i.e., for values to the right of the vertical dashed line). The firm would choose a zero search effort if $\overline{\sigma}_I < 0.0107$ (i.e., for values to the left of the vertical dashed line). If $\overline{\sigma}_I < 0.0107$ (i.e., if sector I uses the mixed-strategy $q_I = 0.3425$, $\hat{\sigma}_I = 0.0164$), a fraction 0.3425 of firms chooses $\sigma = 0$, while the rest of the firms choose $\sigma = 0.0164$, which implies $\overline{\sigma}_F = 0.0107$ (i.e., the cross marker).

Figure 18 shows that the mixed-strategy Nash equilibrium is unstable: a decrease in $\overline{\sigma}_I$ induces all firms in sector F to search with zero effort and the system converges to the pure-strategy Nash equilibrium with zero search effort (i.e., passive equilibrium). Similarly, an increase in $\overline{\sigma}_I$ induces all firms in sector F to search with positive effort; hence, the system converges to the pure-strategy Nash equilibrium with positive search effort (i.e., active equilibrium).

G Simulations based on shocks to productivity

In this appendix, we complete our discussion of the effects of technology shocks in the model. Figure 19 plots the ergodic distribution of selected variables for the case where we only have AR(1) shocks to technology, z_t (for transparency, we eliminate the discount factor shocks). As outlined in the paper, persistent exogenous disturbances to the technological process fail to move the system to a different equilibrium, the equilibrium is always active, and the ergodic distributions of the variables of interest are unimodal.

In Figure 20, we plot the ergodic distribution of selected variables for the case where we have shocks both to technology, z_t and to the discount factor, ξ_t. We recover bimodality, but this feature is induced by the shocks to ξ_t and their ability to switch equilibria. The main effect of the shocks to productivity is to spread out the ergodic distribution in Figure 8 in the main text (only shocks to ξ_t) around its two modes.

Figure 21 shows the GIRFs to a range of persistent negative productivity shocks when the economy starts from the active DSS. Negative productivity shocks are unable to generate a shift in equilibrium even when their magnitude gets very large. In each case, the costly search effort falls after the productivity shock, and then gradually recovers. The effect of a productivity shock on the labor market tightness ratio and the unemployment rate is also transitory. The mechanism is that the gain of matching with a partner ($TS - \widetilde{TS}$) in the active equilibrium is inelastic with the change in productivity. This result is similar to the one in Shimer (2005), who points out that the gain of matching with a worker, $\widetilde{TS}$ and TS, is inelastic with the change in productivity in a canonical DMP model. Since $\widetilde{TS}$ and TS move in the same direction in reaction to productivity shocks, the response of $TS - \widetilde{TS}$ to productivity shocks is even weaker. As a result, the existence condition for the active equilibrium in equation (35) keeps holding: if we start at the active DSS, firms find it desirable to search actively for a partner even when productivity is low.

Figure 19: Ergodic distribution with AR(1) shocks to z_t

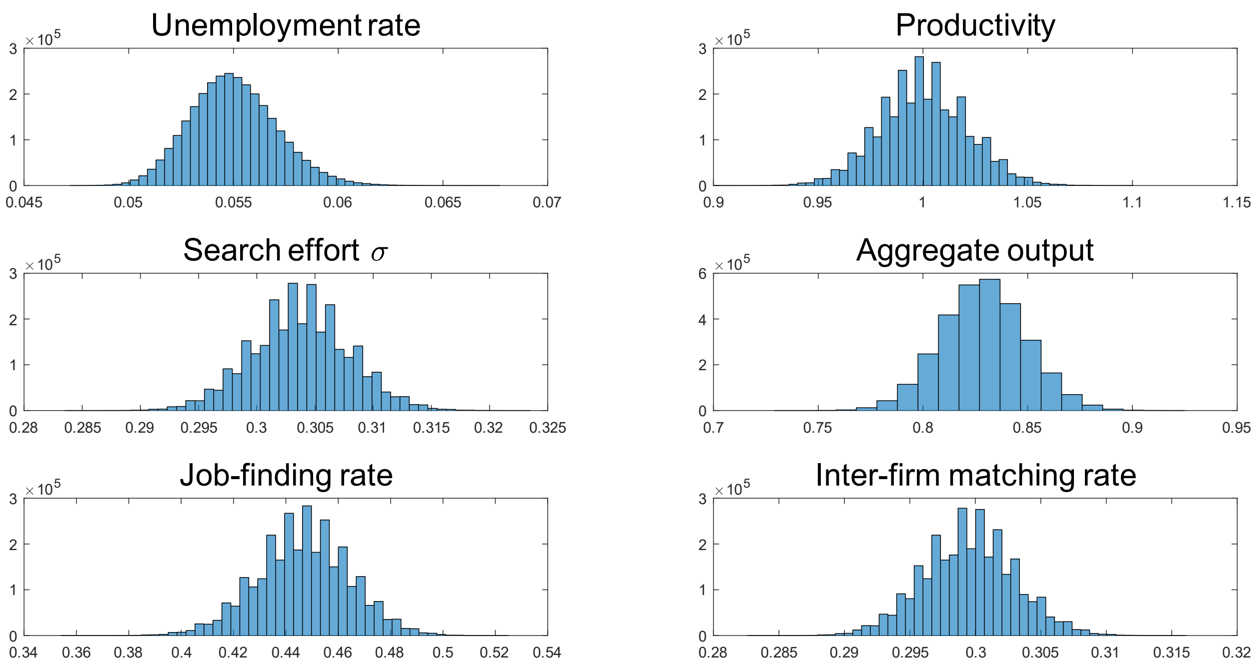
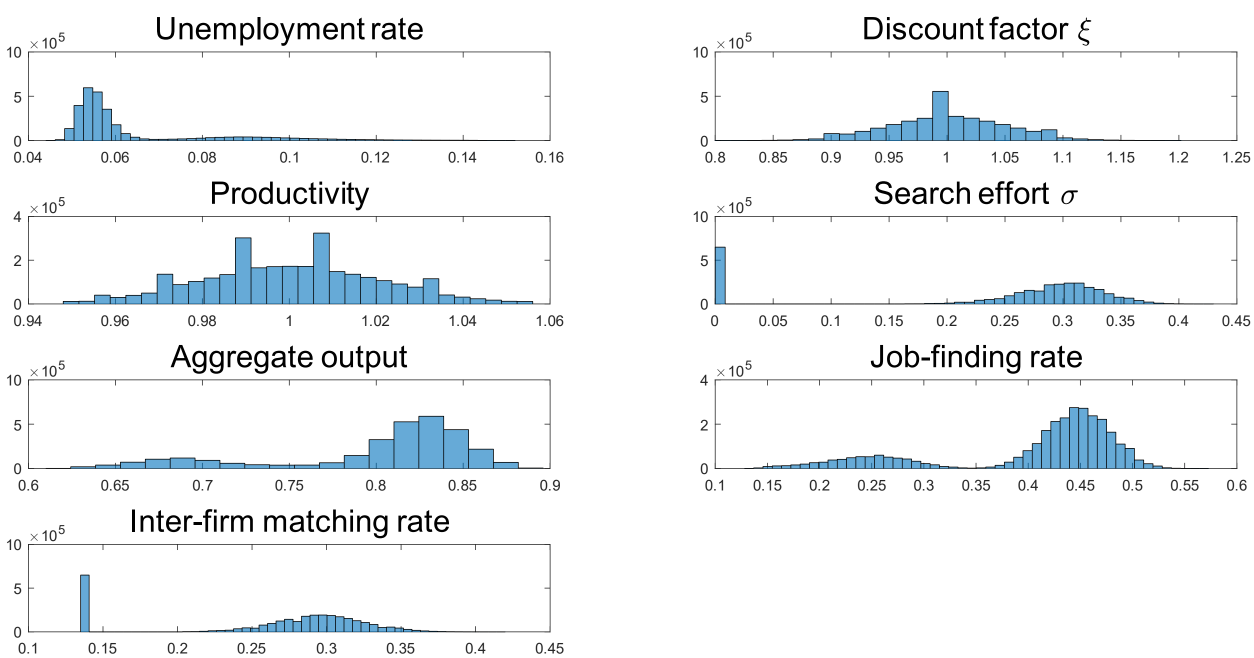
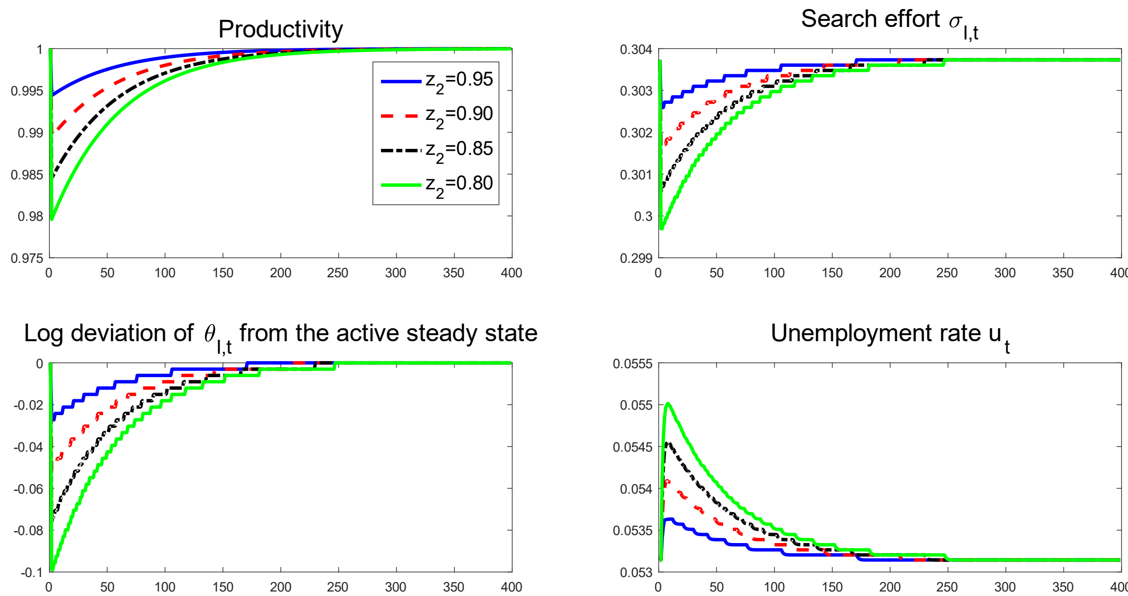


Figure 20: Ergodic distribution with *i.i.d.* shocks to ξ_t and AR(1) shocks to z_t



We also experiment with permanent changes in productivity. In period $t = 1$, the economy starts from the active DSS with positive search effort, and in period $t = 2$ a permanent fall in productivity hits the economy. This permanent shock may shift the equilibrium of the system

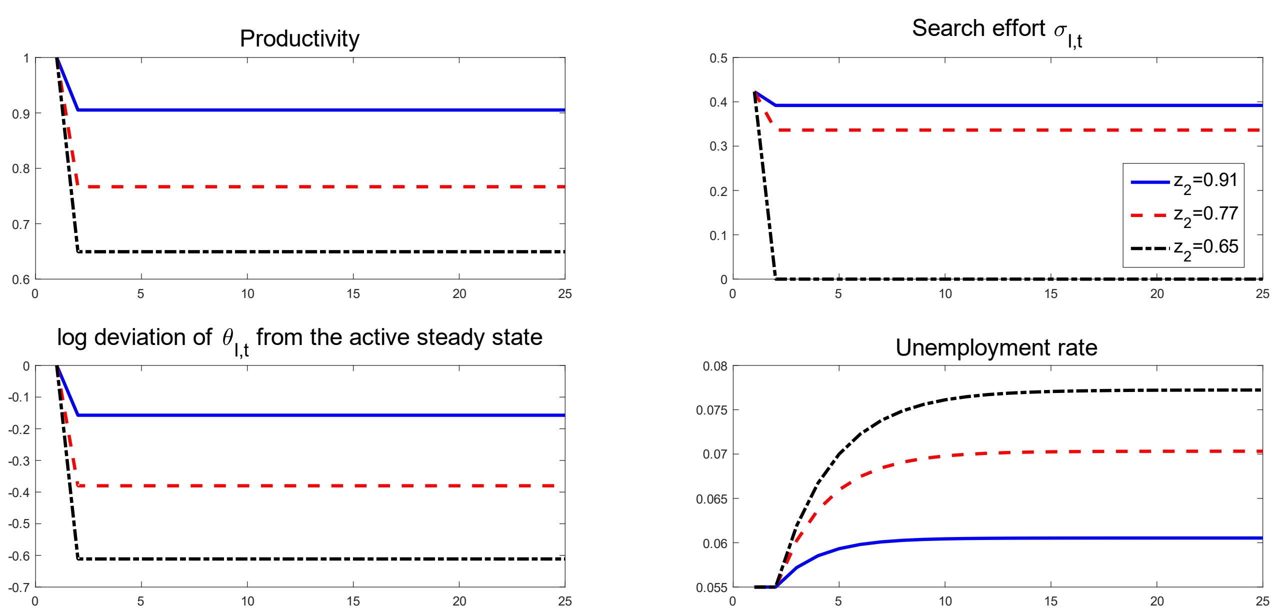
Figure 21: GIRFs to a negative productivity shock



Note: Each panel shows the response of a variable to a negative productivity shock (z) with magnitudes of 0.05 (blue line), 0.10 (red line), 0.15 (black line), and 0.20 (green line).

by affecting the expected gain of match $\mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1})$. For example, in an economy in the active equilibrium, a sufficiently large fall in z_t decreases the expected gain from joint venture formation and moves the system to the passive equilibrium.

Figure 22: GIRFs to a negative permanent productivity shock



Note: Each panel shows the response of a variable to a permanent negative productivity shock (z) with magnitudes of 0.19 (solid line), 0.23 (dashed line), and 0.35 (dot-dashed line).

We use the model to assess the magnitude of the fall in z_t needed to move the system from the active to the passive equilibrium. Figure 22 shows the GIRFs of selected variables to a 19% (solid line), 23% (dashed line), and 35% (dot-dashed line) permanent decline in productivity (z_t). The first two shocks are unable to move the system to the active equilibrium because the expected gain from inter-firm matching is relatively inelastic to permanent changes in productivity. Productivity shocks induce $\widetilde{J}_{i,t+1}$ and $J_{i,t+1}$ to comove, leading to a weak response of $\mathbb{E}_t(J_{i,t+1} - \widetilde{J}_{i,t+1})$ to the shock. As we mentioned above, this finding is consistent with Shimer (2005). In comparison, a sufficiently large productivity shock of 35% pushes the economy to the passive equilibrium. This analysis suggests that a permanent productivity shock is unlikely to move the system between equilibria unless the shock is massive.

H Volatility of shocks

Figure 23 plots the ergodic distribution of endogenous variables with *i.i.d.* shocks to ξ in the case of high volatility. Figure 24 repeats the same exercise, but in the case of low volatility. In both figures, we see the bimodal distributions that we discussed in the main text and the long left tail of output when the volatility of ξ_t is high.

Figure 23: Ergodic distribution with *i.i.d.* shocks to ξ, high volatility

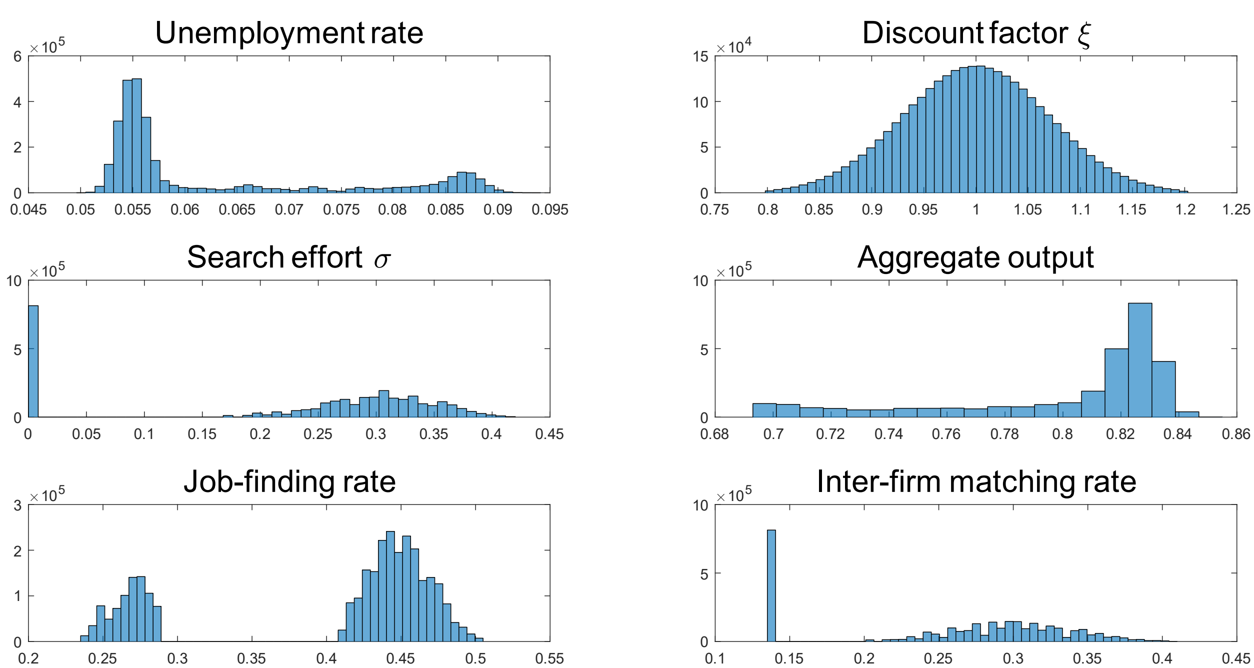


Figure 25 lowers volatility of ξ_t even further, to 0.02. Now, all traces of bimodality disappear.

Figure 24: Ergodic distribution with *i.i.d.* shocks to ξ, low volatility

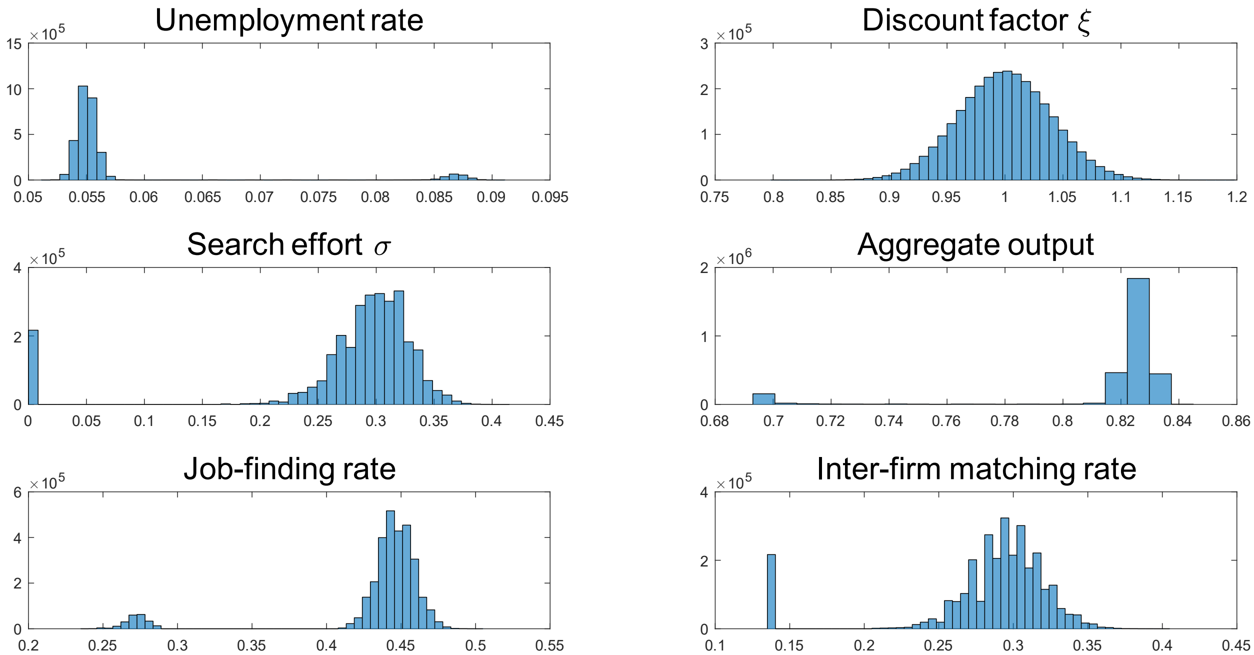
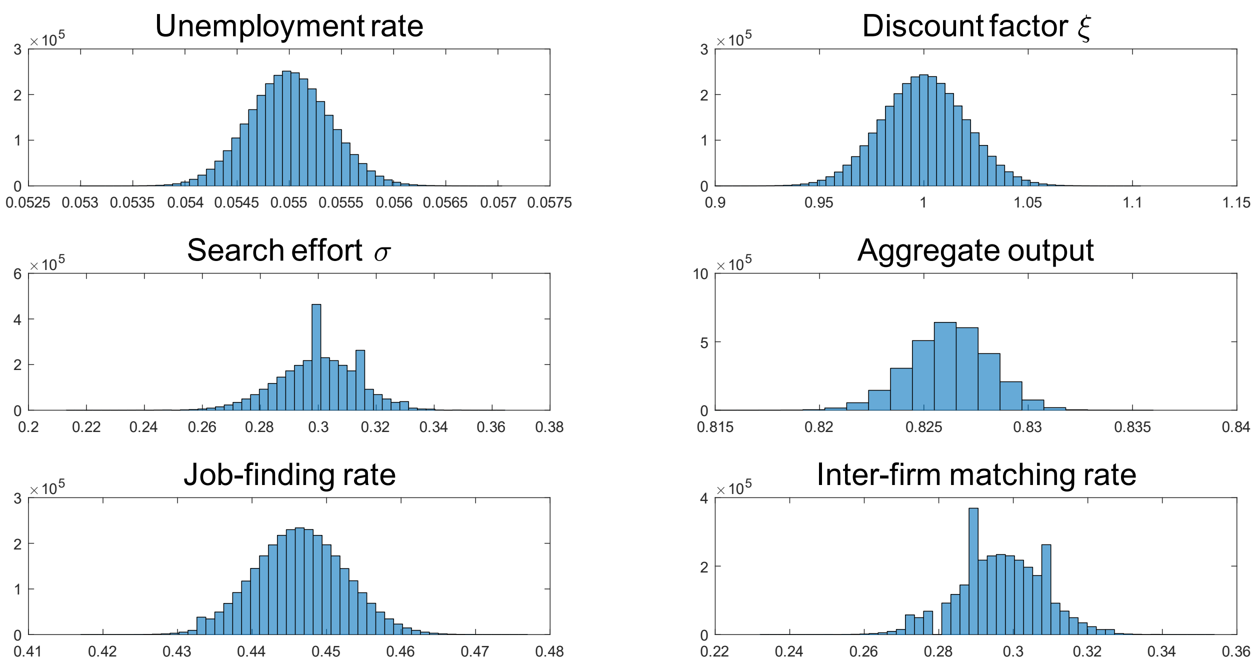


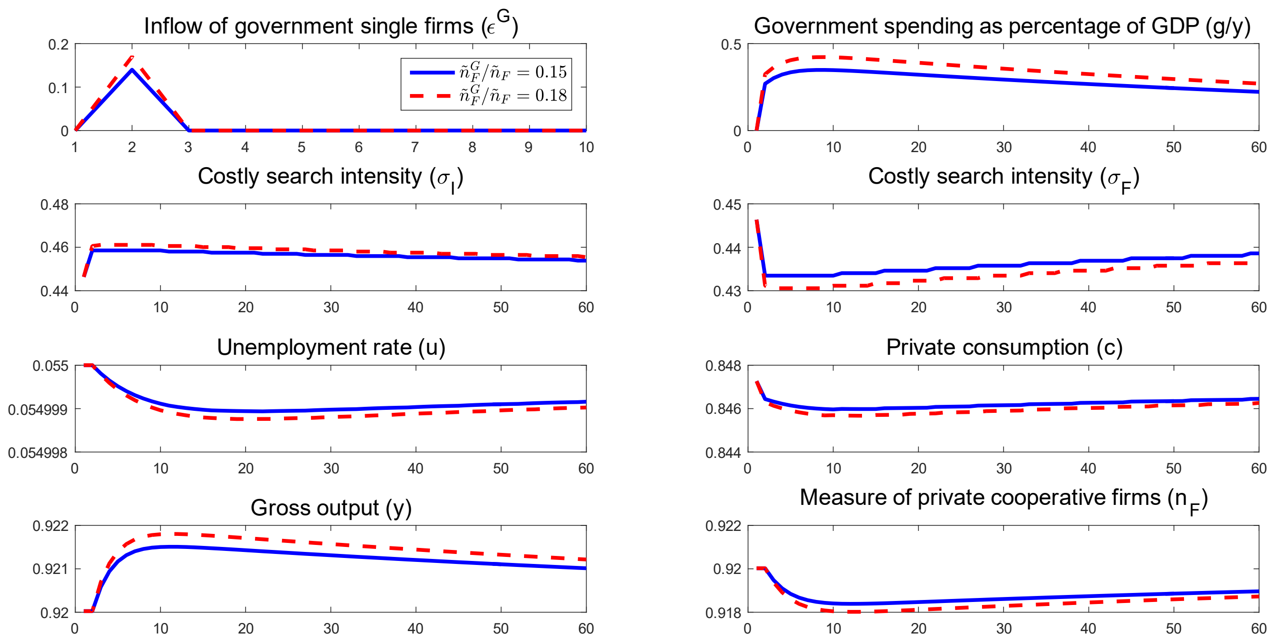
Figure 25: Ergodic distribution with *i.i.d.* shocks to ξ, extra-low volatility



I GIRFs to government spending shock in the active equilibrium

This appendix studies the effect of government spending shocks when the economy starts from the active equilibrium. Figure 26 shows the response in the level of selected variables to a 15% (the solid line) and an 18% (the dashed line) government spending shock. Since the economy is already in the active equilibrium, the effects of the fiscal expansion are limited and transitory.

Figure 26: GIRFs to positive government spending shock in the active equilibrium



Note: Each panel shows the response of a variable to a one-period 15% (solid line) and 18% (dashed line) increase in government spending.