Dynamic Oligopoly and Price Stickiness

Olivier Wang NYU Stern

Iván Werning MIT

June 2020

Abstract

We study oligopolistic competition and its effect on monetary policy. In our model, within each sector, any finite number of firms compete a la Bertrand. Firms can change their posted nominal prices at random intervals a la Calvo. Following an extensive IO literature, we focus on Markov equilibria of the resulting game in each sector. We aggregate up to the macroeconomic level and first consider unexpected monetary shocks. We provide a closed-form formula for the response of aggregate output, highlighting three measurable sufficient statistics: demand elasticities, market concentration, and markups. We show that the empirical heterogeneity in cost pass-through implies that higher market concentration significantly amplifies the real effects of monetary policy. We separate the strategic effects of concentration from its effects on the residual demand system and find that the latter drive most of the amplification.

1 Introduction

The recent rise in product-market concentration in the U.S. has been viewed as a driving force behind several macroeconomic trends. For instance, Gutiérrez and Philippon (2017) document an increase in the mean Herfindahl-Hirschman index since the mid-nineties, and argue that it has weakened investment. Autor, Dorn, Katz, Patterson and Van Reenen (2017) and Barkai (2020) relate the rising concentration of sales over the past 30 years in most US sectors to the fall in the labor share.[1]

What are the implications of trends in concentration or market power for the transmission of monetary policy? Do strategic interactions in pricing between increasingly large firms amplify or dampen the real effects of monetary shocks? The baseline New Keynesian model is not designed to address these questions. Following the recognition that some form of imperfect competition and pricing power is required to model nominal rigidities, the New Keynesian literature has been built on the tractable paradigm of monopolistic competition, pervasive in other areas of macroeconomics and international trade. Under monopolistic competition, markups only depend on tastes, through consumers' elasticity of substitution between competing goods, which leaves no room for changes in concentration to affect markups or monetary policy.

In this paper, we provide a new framework to study the link between market structure and monetary policy. We generalize the standard New Keynesian model by allowing for dynamic oligopolistic competition between any finite number of firms in each sector of the economy, also allowing for heterogeneity across sectors. We use this model to study the aggregate real effects of monetary shocks and highlight the restrictions imposed by monopolistic competition. Departing from monopolistic competition to study oligopoly poses new challenges, because it requires solving a dynamic game with strategic interactions at the sectoral level and embedding it into a general equilibrium macroeconomic model. We focus on Markov equilibria of our dynamic game, where the pricing strategy, or reaction function, of every firm is a function of the prices of its competitors.

Despite these complexities, our first result derives a closed-form formula for the response of aggregate output to small monetary shocks. Our formula inputs the cross-sectoral distribution of three sufficient statistics: market concentration as captured by the effective number of firms within a sector, demand elasticities, and markups. The intuition is based on the link between the steady state markup that can be sustained in an oligopolistic equilibrium and the slope of the reaction function of each firm to the prices

[1] Rossi-Hansberg, Sarte and Trachter (2020) document, however, that diverging trends in national and local measures of concentration. We will discuss how to interpret our results in light of these two views.

of its competitors. All else equal, steeper reaction functions lead to higher equilibrium markups: each firm has little incentives to cut prices when it knows that this would lead its rivals to cut prices as well for some time. Inverting the logic, we can infer from high observed markups that reaction functions are steep and therefore complementarities in pricing are strong, which in turn implies a slow pass-through of monetary shocks into prices and therefore large real effects on output. In this way, our formula encapsulates a tight restriction between endogenous markups and stickiness, conditional on demand elasticities.[2]

While our key sufficient statistics, demand elasticities and markups, can be estimated at any given point in time, they are endogenous objects that change in reaction to shifts in fundamentals. To perform counterfactual experiments, we take a more structural approach and solve numerically the oligopolistic equilibrium in terms of fundamentals. We use a flexible Kimball (1995) demand system that allows us to parametrize separately demand elasticities and superelasticities, as the latter can affect monetary policy transmission through variable markups even under monopolistic competition.

In our main exercise, we vary the number of firms n in each sector while keeping preference parameters fixed. We find that higher concentration (lower n) can significantly amplify or dampen the real effects of monetary policy, depending on how properties of the residual demand vary with n. When preferences are CES, higher concentration amplifies monetary policy transmission, but the maximal effects, attained under duopoly, remain limited: the half-life of the price level in reaction to monetary shocks is around 40% higher than under monopolistic competition. With Kimball preferences and sufficiently high superelasticity, higher concentration dampens monetary policy transmission. Moreover, the dampening can be arbitrarily large.

We use evidence on the heterogeneity in idiosyncratic cost pass-through across small and large firms from Amiti, Itskhoki and Konings (2019) to calibrate how concentration affects the shape of demand functions, and find substantial amplification. The rise in the average Herfindahl index observed in the U.S. since 1990 increases the response of output (and decreases the response of inflation) to monetary shocks by around 15%.

What explains these results? The number of competitors in a market has an effect on firms' strategic incentives, but also on the shape of the residual demand faced by each firm. We disentangle the two ways through which oligopolistic competition differs from monopolistic competition. On the one hand, "feedback effects" make each firm care about

[2]In the standard monopolistic competition model desired markups are constant and only a function of the demand elasticity. However, in a strategic environment the endogenous markup is no longer a simple function of the demand elasticity.

its rivals' current and future prices when setting its price. On the other hand, "strategic effects" arise because each firm realizes its current pricing decision can affect how its rivals will set their prices in the future. We isolate these two effects for each n, by comparing the oligopolistic model with n firms to a "non-strategic" benchmark economy featuring monopolistic competition and Kimball preferences modified to match the elasticity and superelasticity of the residual demand in the finite n model. We find that departures from monopolistic competition are mostly working through feedback effects, that is changes in the shape of residual demand. While strategic effects matter for the level of steady state markups, they only have a modest impact on monetary policy transmission.

It does not follow, however, that oligopoly is isomorphic to monopolistic competition. Besides its improved ability to map micro-evidence on pass-through and market shares to the aggregate effects of monetary policy, the oligopoly model yields a unique link between markups and monetary policy transmission in the cross-section. Under monopolistic competition, demand superelasticities affect cost pass-through and thus monetary policy, but are irrelevant for markups. Under oligopolistic competition, the superelasticity of residual demand has a positive effect on both markups and the pass-through of monetary policy. Therefore, controlling for concentration and demand elasticities, monetary policy is transmitted relatively more through sectors or regions with higher markups, because they are the ones featuring the slowest price adjustment following monetary shocks.

Moreover, the near-equivalence between the oligopoly model and the recalibrated non-strategic economy depends on the specific processes for real and monetary shocks. In order to go beyond the permanent money supply shocks most commonly studied in the literature, we derive a three equations New Keynesian model with an oligopolistic Phillips curve that allows for more general shocks and non-stationary dynamics. We find that strategic effects can be quantitatively important once we allow for richer dynamics. In particular, the oligopolistic Phillips curve features a form of endogenous inflation persistence (or equivalently, endogenous cost-push shocks) that can dampen fluctuations in inflation and output relative to the non-strategic model.

Related literature

An important early exception to the complete domination of monopolistic competition in the macroeconomics literature on firm pricing was Rotemberg and Saloner (1986), who proposed a model of oligopolistic competition to explain the cyclical behavior of markups. Rotemberg and Woodford (1992) later embed their model into a general equilibrium framework with aggregate demand shocks driven by government spending. These two papers assume flexible prices and abstract from monetary policy.[3] Another important difference is that we focus on Markov equilibria, rather than trigger-strategy price-war equilibria.

The first paper to combine non-monopolistic competition and nominal rigidities is Mongey (2018). This paper uses a rich quantitative model with two firms, menu costs, and idiosyncratic shocks to show that duopoly can generate significant non-neutrality relative to the Golosov and Lucas (2007) benchmark. It also finds that duopoly is closer to monopolistic competition under Calvo price-setting than with menu costs. Our analytical approach allows us to explore different questions, in particular by varying the number of firms in each industry and separating strategic complementarities from residual demand effects. Incorporating more than two firms also lets us incorporate recent evidence on cost pass-through and market shares from Amiti, Itskhoki and Konings (2019) to infer the relation between concentration and monetary non-neutrality. We find that even under Calvo pricing, oligopoly leads to significant amplification.[4]

The literature on variable markups in international trade (e.g., Atkeson and Burstein 2008) highlights the importance of market structure and for cost (e.g., exchange rate) pass-through. We provide a dynamic generalization of these models, as is needed to study monetary policy, and we show which properties of residual demand functions matter in this context. In particular, we use the evidence from Amiti et al. (2019) on heterogeneous pass-through behavior across small and large firms to calibrate our oligopolistic model.

Kimball (1995) introduced non-CES aggregators that generate variable markups even under monopolistic competition. As we show in section 6, there is a close connection between this class of models (e.g., Klenow and Willis 2016, Gopinath and Itskhoki 2010) and our oligopolistic model. By expliciting the market structure, our paper microfounds the dynamic pricing complementarities embedded in the monopolistic Kimball aggregator in a way consistent with the data on firm size and long-run pass-through. Relative to this strand of the literature, the oligopolistic model also generates unique predictions on the cross-sectional relation between markups, concentration, and monetary policy transmission.

[3] Rotemberg and Saloner (1987) study a static partial-equilibrium menu-cost model, comparing the incentive to change prices under monopoly and duopoly.

[4] Calvo pricing remains an important benchmark in the literature on price stickiness, due to its tractability, but additionally, recent work on menu costs, such as Gertler and Leahy (2008), Midrigan (2011), Alvarez, Le Bihan and Lippi (2016b) and Alvarez, Lippi and Passadore (2016a), show that certain menu-cost models may actually behave close to Calvo pricing.

2 Model Environment

In this section we first describe the economic environment, preferences, technology, and the market structure. We then define an equilibrium.

Basics. Time is continuous with an infinite horizon $t \in [0, \infty)$.[5] We abstract from aggregate uncertainty. This suffices to study the impact and transitional dynamics induced by an unanticipated shock. Following much of the menu-cost literature, we focus on such a monetary shock, and our goal is to understand the degree of monetary non-neutrality it induces.

There are three types of economic agents: households, firms and the government. Households are described by a continuum of infinitely lived agents that consumes non-durable goods and supplies labor to a competitive labor market.

Firms produce across a continuum of sectors $s \in S$. Each sector is oligopolistic, with a finite number n_s of firms $i \in I_s$, each producing a differentiated variety. Firms can only reset prices at randomly spaced times, so the price vector within a sector is a state variable. By setting $n_s \to \infty$ or $n_s = 1$ we obtain a standard monopolistic setup, where each firm has a negligible effect on competitors. Otherwise, there are strategic interaction across firms within a sector, but not across sectors (due to the continuum assumption). We study the dynamic game within a sector and focus on Markov equilibria.

The government controls the money supply, provides transfers and issues bonds, to satisfy its budget constraint.

Household Preferences. Utility is given by

$$\int_0^\infty e^{-\rho t} U(C(t), \ell(t), m(t)) dt,$$

with real money balances $m(t) = \frac{M(t)}{P(t)}$ and aggregate consumption

$$C(t) = \Psi(\{c_{i,s}(t)\}),$$

where $\{c_{i,s}(t)\}$ describes the consumption of all good varieties across sectors $s \in S$ and firms $i \in I_s$ and where Ψ is an aggregator function that is homogeneous of degree one.

[5]Our analysis translates easily to a discrete-time setup, but continuous time has a few advantages and permits comparisons with the menu-cost literature (e.g. Alvarez and Lippi, 2014).

Following Golosov and Lucas (2007), we adopt the specification

$$U(C,\ell,m) = \frac{C^{1-\sigma}}{1-\sigma} + \alpha \log m - \ell.$$

As is well known, these preferences help simplify the aggregate equilibrium dynamics.

In addition, we adopt a nested CES-Kimball aggregator: across sectors have a CES, while across firms within a sector we have a Kimball (1995) aggregator:[6]

$$\Psi(\{c_{s,i}\}_{s\in S, i\in I}) = \left(\int_S C_s{}^{1-\frac{1}{\omega}} ds\right)^{\frac{1}{1-\frac{1}{\omega}}}$$

where C_s is the unique solution to

$$\frac{1}{n_s}\sum_{i\in I_s} \phi_s\left(\frac{c_{i,s}}{C_s}\right) = 1 \tag{1}$$

for some increasing, concave, function ϕ_s such that $\phi_s(1) = 1$.

An important benchmark is the case where ϕ_s is a power function, in which case we obtain the standard CES aggregator across firms, i.e. $C_s = \left(\frac{1}{n_s}\sum_{i\in I_s} c_{i,s}^{1-\frac{1}{\epsilon}}\right)^{\frac{1}{1-\frac{1}{\epsilon}}}$.

Firms. Each firm $i \in I_s$ in sector $s \in S$ produces linearly from labor according to the production function,[7]

$$y_{s,i}(t) = \ell_{s,i}(t).$$

We assume a linear production function and no sectoral or idiosyncratic differences in productivity for simplicity.

Firms receive opportunities to change their price $p_{i,s}$ at random intervals of time, determined by a Poisson arrival rate $\lambda_s > 0$, the realizations of which are independent across firms and sectors. Between price changes, firms meet demand at their posted prices.

Individual firm nominal profits are

$$\Pi_{i,s}(t) = p_{i,s}(t) y_{i,s}(t) - W(t)\ell_{i,s}(t)$$

and aggregate firm nominal profits $\Pi(t) = \int \sum_{i\in I_s} \Pi_{i,s}(t)\, ds$. Firms seek to maximize the present value of profits,

$$\mathbb{E}_0 \int_0^\infty Q(t)\Pi_{i,s}(t) dt$$

[6]When $\omega = 1$ we set $\Psi(\{c_{s,i}\}_{s\in S,i\in I}) = \exp \int_S \log C_s ds$.

[7]We could introduce productivity shocks, aggregate and idiosyncratic, but abstract from them for the moment to simplify and focus on monetary shocks.

where $Q(t) = e^{-\int_0^t R(s)ds}$ denotes the nominal price deflator between period t and 0.

Although there is no aggregate uncertainty, the expectation averages over the idiosyncratic uncertainty about the dates at which changes are allowed for each firm and its immediate competitors within a sector. (This firm objective can be justified in a number of ways, such as by introducing an asset market for the stock price of firms.)

Household Budget Constraints. The flow budget constraint can be summarized by

$$P(t)C(t) + \dot{B}(t) + \dot{M}(t) = W(t)\ell(t) + \Pi(t) + T(t) + R(t)B(t)$$

for all $t \geq 0$, where $B(t)$ are bonds paying nominal interest rate $R(t)$, $M(t)$ nominal money holdings, $W(t)$ the nominal wage, $T(t)$ nominal lump-sum transfers, and $P(t)$ the (ideal) price index given by

$$P(t) = \mathcal{P}(\{p_{i,s}(t)\}),$$

where $\mathcal{P}(\{p_{i,s}\}) \equiv \min_{\{c_{i,s}\}} \int \sum_{i \in I_s} p_{i,s} c_{i,s} ds$ s.t. $\Psi(\{c_{i,s}\}) = 1$. For $\omega \neq 1$, we can write $\mathcal{P}(\{p_{i,s}\}) \equiv \left(\int P_s^{1-\omega} ds\right)^{\frac{1}{1-\omega}}$ with $P_s = \mathcal{P}_s(p_{1,s}, p_{2,s}, \ldots, p_{n_s,s})$. [8]

Let $A(t) = B(t) + M(t)$ denote total nominal wealth. Households are also subject to the No Ponzi condition $\lim_{t\to\infty} Q(t)A(t) \geq 0$. This leads to the present value condition

$$\int_0^\infty Q(t)(P(t)C(t) + T(t) + R(t)M(t) - W(t)\ell(t) - \Pi(t))dt = A(0) = M(0) + B(0).$$

Demand. Define the vector of prices within a sector s as

$$p_s(t) = (p_{1,s}(t), p_{2,s}(t), \ldots, p_{n_s,s}(t))$$

and let $p_{-i,s}(t) = \left(p_{1,s}(t), \ldots, p_{i-1,s}(t), p_{i+1,s}(t), \ldots, p_{n,s}(t)\right)$ denote the vector that excludes $p_{i,s}(t)$. The demand for firm $i \in I_s$ can be written as

$$c_{i,s}(t) = D_{i,s}(p_{i,s}(t), p_{-i,s}(t); C(t), P(t)).$$

Given symmetry, constant returns and the CES structure across sectors, we obtain

$$D_{i,s}(p_i, p_{-i}; C, P) = d(p_i, p_{-i})CP^{\omega}.$$

The demand faced by firm i is a stable function of the price vector $d^i(p_i, p_{-i})$. This demand captures within-sector substitution as well as across-sector substitution. Firms understand that they can switch expenditure in both ways by changing their price.

[8] We have $\mathcal{P}(\{p_{i,s}\}) \equiv \log \int \exp P_s ds$ when $\omega = 1$.

Nominal profits are then

$$\int_0^\infty e^{-\rho t} C(t) P(t)^\omega \, d(p_{i,s}(t), p_{-i,s}(t)) \, (p_{i,s}(t) - W(t)) \, dt$$

We will focus on cases where $C(t)P(t)^\omega$ is constant along the equilibrium path.

Markov Equilibria. A strategy for firm i specifies its desired reset price at any time t should it have an opportunity to change its price. A Markov equilibrium involves a strategy that is a function only of the price of its rivals and calendar time t,

$$g_{i,s}(p_{-i}; t).$$

Given that sectors are symmetric and firms are symmetric within sectors, we consider strategies $g(p_-, t)$ that are symmetric, except in section 4.2.

Equilibrium Definition. Given initial prices $\{p_{i,s}(0)\}$, an equilibrium is sequence for the aggregate price $P(t)$, wage $W(t)$, interest rate $R(t)$, consumption $C(t)$, labor $\ell(t)$ and money supply $M(t)$, as well as demand functions for consumers $d(p_i, p_{-i}; t)$ and strategy functions for firms $g(p_{-i}; t)$ such that: (a) consumers optimize quantities taking as given the sequence of prices and interest rates; (b) the firm reset price strategy g is optimal, given the path for $P(t), C(t)$ and its rivals' strategies g and demand function of consumers d; (c) consistency: the aggregate price level evolves in accordance with the reset strategy g employed by firms; (d) markets clear: firms meet demand for goods, the supply of labor equals aggregate demand for labor

$$\ell(t) = \int \sum_{i \in I_s} \ell_{i,s}(t) ds$$

and the demand for money equals supply $M(t)$.

3 Stationary Oligopoly Game

We first focus on the dynamics within a sector, assuming all conditions external to the sector are fixed and given: the wage, the nominal discount rate, aggregate consumption and price are assumed constant. These assumptions imply that the oligopoly game within an industry is stationary. This partial equilibrium analysis also characterizes a steady state in general equilibrium.

We shall later explore conditions under which we can use the sectoral dynamics we characterize here to study the aggregate macroeconomic adjustment to a monetary shock.

3.1 Prices, Demands and Profits

We now focus within a sector, suppressing the notation conditioning on $s \in S$ we collect prices within the sector in a vector

$$p = (p_1, \ldots, p_n)$$

and let $p_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ denote competitor prices for firm i. The profit function for firm i is then

$$\Pi^i(p) = d^i(p_i, p_{-i})(p_i - W).$$

Since $R(t) = \rho$ we have $Q(t) = e^{-\rho t}$ and firms maximize

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d^i(p_i, p_{-i})(p_i - W)dt.$$

3.2 Markov Equilibria

In a Markov equilibrium firms follow a strategy specifying the reset price

$$p_j^* = g_j(p_{-j})$$

they will chose in the event that they receive a price change opportunity. Together with an initial price vector and the Poisson arrival rate this fully describes the stochastic dynamics within the sector. We focus on differentiable symmetric Markov equilibria, where

$$g_j = g.$$

Let $V^i(p)$ denote the value function obtained by firm i. The Bellman equation is then

$$(\rho + n\lambda)V^i(p) = \Pi(p) + \lambda \sum_j V^i(g_j(p_{-j}), p_{-j}) \tag{2}$$

where Π is the profit function of firm i and

$$g_j(p_{-j}) = \arg\max_{p'_j} V^j(p'_j, p_{-j}).$$

satisfying the optimality condition

$$V^i_{p_j}(g_j(p_{-j}), p_{-j}) = 0. \tag{3}$$

Remark 1. There could be multiple equilibria even within the Markov class, but our main results apply for any differentiable selection. The differentiability assumption rules out "kinked demand curve" and "Edgeworth cycles" Markov equilibria studied by Maskin and Tirole (1988) in a Bertrand duopoly model with *perfectly substitutable* goods as firms become infinitely patient, which in our setting is equivalent to the flexible prices limit $\lambda \to \infty$ as the model only depends on the ratio ρ/λ. Maskin and Tirole (1988) show that firms can "collude" around the joint monopoly price in this limit. Firms can achieve high profits in steady state, because if not, a firm could deviate to the monopoly price knowing that its rival would follow suit and undercut by a small amount once it gets to reset its price, which eventually ensures some large profit to the deviator once it gets to reset its price again. Figure 17 in Appendix I shows that away from the joint limit of perfect substitution and flexible prices, value function iteration converges to a standard "smooth" (and monotone) MPE that corresponds to the one we study locally. We trace out the locus of existence of equilibria in the (ϵ, λ)-space (where ϵ is the within-sector elasticity of substitution), and find that our smooth equilibrium disappears as ϵ exceeds 9 for λ around 1. While the curse of dimensionality prevents us from solving numerically for the full non-linear MPE with general n, we conjecture that the existence bounds are tightest for $n = 2$, as increases in the number of firms lead to a smaller potential monopoly profit (the case of monopolistic competition $n \to \infty$ being an extreme example). Similarly, a higher outer elasticity ω lowers the joint monopoly profit, which enlarges the region of existence of the smooth equilibrium.

3.3 A Steady State Condition

We now provide a key expression for the slope of the reset price strategy at a steady state. Differentiating the Bellman equation (2) and making use of symmetry, we obtain at the steady state $\bar{p}$ of a symmetric equilibrium:

$$0 = \Pi_{p_i}(\bar{p}) + \lambda \sum_{j \neq i} \left[V^i_{p_j}(\bar{p}) \frac{\partial g_j}{\partial p_i}(\bar{p}) \right]$$

$$V^i_{p_k}(\bar{p}) = \frac{\Pi_{p_k}(\bar{p})}{\rho + \lambda} + \frac{\lambda}{\rho + \lambda} \sum_{j \neq i,k} \left[V^i_{p_j}(\bar{p}) \frac{\partial g_j}{\partial p_k}(\bar{p}) \right] \quad \forall k \neq i$$

Denote $\frac{\partial g_j}{\partial p_k}(\bar{p}) = \beta_n$ for all $k \neq j$. Using $\sum_k \sum_{j \neq i,k} V^i_{p_j}(\bar{p}) = (n-2) \sum_{k \neq i} V^i_{p_k}(\bar{p})$, and the symmetry of Π_{p_k} across $k \neq i$, we obtain

$$0 = \Pi_{p_i}(\bar{p}) + \frac{\lambda\,(n-1)\,\beta_n}{\rho + \lambda\,[1-(n-2)\beta_n]}\Pi_{p_k}(\bar{p}) \tag{4}$$

Without pricing frictions, firms would continuously play the static Nash equilibrium price p^{NE} that solves $0 = \Pi_{p_i}(p^{NE})$. From (4) we see that the steady state price $\bar{p}$ of the dynamic oligopoly game is above the static Nash price p^{NE} if and only if $\beta_n > 0$. Moreover, as the influence of any single rival Π_{p_k} vanishes when n increases, the steady state price converges to the Nash price (i.e., monopolistic competition) as n grows to infinity.

Sufficient statistics: markups and elasticities. The main object of our analysis is the slope $(n-1)\,\beta_n$ of the reaction function, where the term $n-1$ scales the aggregate effect of the rivals. We can further simplify (4) to write $(n-1)\,\beta_n$ in terms of observable sufficient statistics. Use

$$\frac{\Pi^i_j}{-\Pi^i_i} = \frac{\epsilon^i_j\left(\frac{p_i - W}{p_j}\right)}{-\epsilon^i_i\left(\frac{p_i-W}{p_i}\right) - 1}$$

where

$$\epsilon^i_i = \frac{\partial \log d^i}{\partial \log p_i}, \quad \epsilon^i_j = \frac{\partial \log d^i}{\partial \log p_j}$$

to rewrite in terms of demand own- and cross-elasticities

$$(n-1)\,\beta_n = \frac{\rho+\lambda}{\lambda}\frac{1}{\frac{n-2}{n-1} + \frac{\epsilon^i_j}{-\epsilon^i_i - \frac{\bar{p}}{\bar{p}-W}}}$$

Constant returns to scale imply that the cross-elasticity is related to the own-elasticity through $(n-1)\epsilon^i_j = -(1+\epsilon^i_i)$. For any n, we obtain the slope in terms of only two steady state objects, the own-elasticity and the markup:

Proposition 1. *In a sector with n firms, the slope of the reaction function around the steady state* $\beta_n = \frac{\partial g_j}{\partial p_k}(\bar{p})$ *satisfies*

$$(n-1)\beta_n = \frac{\lambda+\rho}{\lambda}\frac{1}{\frac{n-2}{n-1} + \frac{1}{n-1}\left(\frac{-\epsilon_i - 1}{-\epsilon_i - \frac{\bar{\mu}}{\bar{\mu}-1}}\right)} \tag{5}$$

where $\epsilon_i = \frac{\partial \log d^i}{\partial \log p_i}$ *and* $\bar{\mu} = \frac{\bar{p}}{W}$.

Proposition 1 is our first main result, showing how to locally infer unobserved steady state strategies from a small number of potentially observed sufficient statistics. Taking as given market concentration n and the demand elasticity ϵ_i, a higher steady state markup $\bar{\mu}$ is associated with a higher slope β_n. Conversely, for a given observed markup $\bar{\mu}$, a higher elasticity (in absolute value) also reflects a higher slope.

The intuition behind this result is based on reverse causality. Suppose that β_n is high. Then, if firm i decreases its price below the steady state, its rivals will set low prices as well, which undermines firm i's incentives to cut prices. This threat of undercutting allows to sustain a high equilibrium markup. On the other hand, when rivals do not react, for instance in the limit where firm i is an infinitesimal player as in monopolistic competition, then the equilibrium markup is low, equal to the static Nash level.

Turning the argument on its head, for a given elasticity ϵ_i, a high equilibrium markup must then be a consequence of steep reaction functions; we will later analyze the factors that govern these reactions. And conversely, for a given markup, a higher demand elasticity would decrease the Nash markup that arises under monopolistic competition, hence oligopolistic competition would imply a higher "abnormal markup" relative to monopolistic competition, that can again only be sustained through a steep reaction function. In the next section, we will show that strong reaction functions imply a low pass-through of aggregate cost shocks and thus persistent real effects of monetary policy.

4 Permanent Monetary Shocks: Sufficient Statistics

We now study an unanticipated permanent shock to money. In particular, suppose initial prices are all equal, $p_{s,i} = P_-$, and aggregates are at a steady state with constant M_-, C_-, ℓ_-, W_- and $R_- = \rho$. Consider a permanent monetary shock arriving at $t = 0$ so that $M(t) = M_+ = (1+\delta)M_-$ for all $t \geq 0$.

In general, firms would have to forecast the path of macroeconomic variables $P(t)$ and $C(t)$ when choosing their reset price strategies $g_{i,s}$. These strategies would in turn affect the evolution of $P(t)$ and $C(t)$. It is possible to accomodate this fixed-point problem numerically or under additional assumptions, as we do in section (8.1), but for now we want to focus on clear analytical results. In the spirit of Golosov and Lucas (2007), our assumptions on preferences lead to the following simplification:

Proposition 2. *Equilibrium aggregates satisfy*

$$W(t) = (1+\delta)W_-$$

$$P(t)\, C(t)^{\sigma} = \rho M(t) = \rho M_+ \tag{6}$$

$$R(t) = \rho$$

If in addition

$$\omega\sigma = 1 \tag{7}$$

then the game in each sector s along the transition is equivalent to the partial equilibrium one studied earlier.

Proposition 2 is very useful, as it shows when firms can ignore the transitional dynamics of macroeconomic variables following the monetary shock, and therefore allows us to extend results based on the partial equilibrium game in section 3 to general equilibrium. This is an exact result, not an approximation for small monetary shocks as in Alvarez and Lippi (2014). Unless otherwise noted, we set

$$\omega = \sigma = 1$$

which implies condition (7).

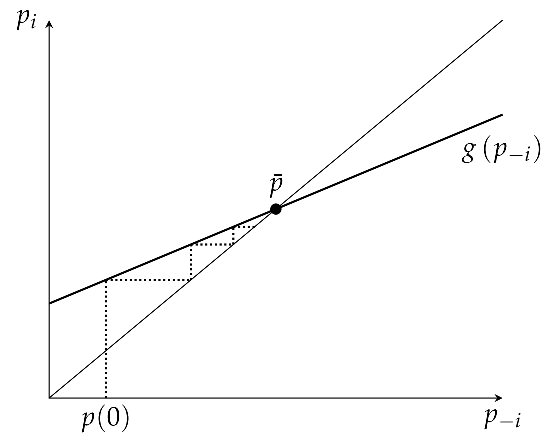
Remark 2. The classic paper by Rotemberg and Saloner (1986) analyzes (non-Markov) trigger strategies that sustain high "collusive" prices in bad times but lead to price wars during booms, because the latter are periods with higher temporary profits to compete over. However, we just showed conditions under which, in general equilibrium, treating the dynamic game as a repeated game can be misleading, as the effect of real interest rates cancels out exactly the effect of higher aggregate demand $C(t)$. Away from this benchmark, the incentives to cut prices could be higher or lower in booms, depending on the elasticity of intertemporal substitution $1/\sigma$.

4.1 Aggregation and Transitional Dynamics

We are interested in the speed of convergence of the aggregate price level to its new steady state $\bar{P} = (1+\delta) P_{-}$. From (6), this speed also tells us the effect of the monetary shock on aggregate consumption.

After the shock, each sector follows stochastic dynamics displayed in Figure 1. When firm i has an opportunity to adjust its price, it does so only when it wasn't the last firm to adjust. The sectoral price level P_s follows a stochastic process, and unlike with monopolistic competition, there is no law of large numbers at the sector level with a finite number of firms. However, there is one once we aggregate across the continuum of (potentially heterogeneous) sectors, yielding a deterministic law of motion for the first-order dynamics of the aggregate price level:

Figure 1: Price dynamics within a sector following an aggregate monetary shock.



Note: Illustration with $n = 2$. Both prices start from $p(0)$ and converge stochastically to $\bar{p}$ on a discrete grid $\{p(0), g(p(0)), g(g(p(0))), \ldots\}$. If a firm was the last one to adjust its price, nothing happens until its rival can adjust. A steeper policy g implies slower convergence in expectation.

Proposition 3. *To first-order in the size of the monetary shock δ, the aggregate price level follows for $t \geq 0$*

$$\log P(t) - \log \bar{P} = -\delta e^{-\lambda\left(1-\sum_n (n-1)\beta_n \omega_n\right)t}, \tag{8}$$

where ω_n is the mass of sectors with n firms. Therefore the cumulative output effect of the shock is (for arbitrary σ)

$$\int_0^\infty \log\left(\frac{C\left(t\right)}{\bar{C}}\right) dt = \frac{\delta}{\sigma\lambda} \times \frac{1}{1 - \sum_n \left(n-1\right)\beta_n \omega_n}. \tag{9}$$

In the standard New Keynesian model with monopolistic competition and CES demand, the half-life of the price level following a monetary shock (up to a factor $\ln 2$) is simply $1/\lambda$ (as in Woodford 2003).[9] The half-life of the price level in the oligopolistic model is

$$hl = \frac{1}{\lambda} \times \frac{1}{1 - \sum_n \left(n-1\right)\beta_n \omega_n}.$$

A higher average slope across sectors $\sum_n \left(n-1\right)\beta_n \omega_n$ implies a slower convergence of the price level $P\left(t\right)$ to its new steady state, and larger real effects of monetary policy. If $\left(n-1\right)\beta_n$ is low on average, then firms in each sector will reset prices close to the new steady state when given a chance, speeding up the convergence.

Combining the results from Propositions 1 and 3, we know the response of the aggregate price level and thus of output to a permanent monetary shock as a function of

[9] With more general demand structures, for instance Kimball demand, the half-life can depart from $1/\lambda$ even under monopolistic competition, see Proposition 7 below.

the distribution of three steady state statistics: markups, demand elasticities and industry concentration. If we can observe or estimate these sufficient statistics and how they evolve over time, for instance following trends in market power, then it is not necessary to solve the full MPE to analyze how the real effects of monetary policy evolve.

For instance, our formula tells us that all else equal, higher observed markups imply higher (unobserved) slopes $(n-1)\,\beta_n$. However, this is only true when fixing the demand elasticity, and if instead higher markups reflect a decline in the elasticity of substitution between competing varieties, then higher markups may be associated with lower slopes instead, as we illustrate in section 5.2. Similarly, an increase in market concentration, captured by a fall in the number of firms n, would also increase monetary non-neutrality holding markups and demand elasticities unchanged. But equilibrium markups and elasticities are likely to be affected by concentration, so our analysis highlights that it is crucial to understand where observed markups come from to understand monetary policy transmission.

4.2 Within-Sector Heterogeneity

We now allow for permanent heterogeneity *within* sectors. Much of the menu-cost literature (e.g., Midrigan 2011, Alvarez and Lippi (2014)) assumes for tractability that there are within-sector demand shocks offsetting perfectly the productivity differences between firms, so as to keep market shares the same. Under this assumption, the model is isomorphic to one with homogeneous firms once we replace prices with markups.

Without these perfectly correlated demand and cost shocks, more productive or demanded firms have a larger market share, and this creates differences in residual demand elasticities as in Atkeson and Burstein (2008), to which we come back in detail in section 7. Given the slopes β_i, we can aggregate the stochastic dynamics in each sector to obtain deterministic aggregate dynamics of the price level as before. While the general case presents no particular difficulty, most of the insights can be gleaned by assuming that there are two firms a and b:

Proposition 4. *Suppose there are two heterogeneous firms a and b in each sector. The aggregate*

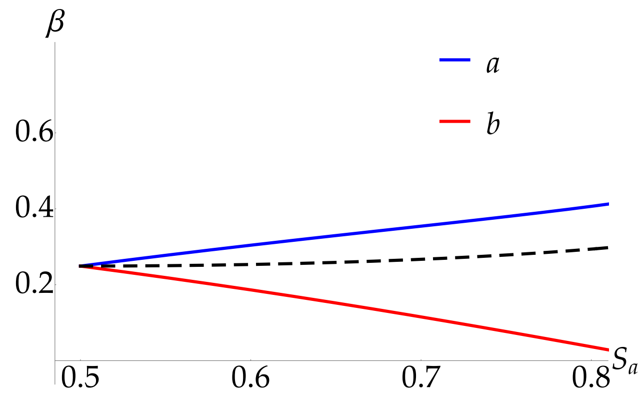


Figure 2: β^a and β^b as a function of firm a's market share S_a. The half-life of the heterogeneous economy is $\frac{1}{\lambda(1-\bar{\beta})}$, where the dashed black line shows $\bar{\beta}$.

price index evolves to first order in δ as

$$\log P(t) - \log \bar{P} = \left[S_a \left(1 + \sqrt{\frac{\beta^a}{\beta^b}} \right) - 1 \right] \left(\frac{1 - \sqrt{\beta^b/\beta^a}}{2} \right) \delta e^{\mu_+ t}$$
$$- \left[1 + S_a \left(\sqrt{\frac{\beta^a}{\beta^b}} - 1 \right) \right] \left(\frac{1 + \sqrt{\beta^b/\beta^a}}{2} \right) \delta e^{\mu_- t}.$$

where

$$\mu_+ = -\lambda \left(1 + \sqrt{\beta^a \beta^b} \right), \quad \mu_- = -\lambda \left(1 - \sqrt{\beta^a \beta^b} \right)$$

and S_a is the steady state market share of firm a.

Figure 2 shows how β^a, β^b respond to permanent multiplicative demand shocks, once we solve the model as in section 5 below. Heterogeneity does not make a substantial difference at the aggregate level, as shown by the relatively flat black dashed line $\bar{\beta}$ that gives the equivalent half-life with symmetric firms. The reason is that there are two offsetting forces. As heterogeneity increases, firm a with a larger market share responds more strongly to firm b's price while firm b becomes less responsive, consistent with the patterns documented by Amiti et al. (2019). This spread in β decreases the dominant eigenvalue μ_- due to the concavity of $\sqrt{\beta^a \beta^b}$. However, the aggregate (sales-weighted) price index also puts more weight on firm a's price, which is "more sticky", as firm a will not adjust by much if it gets to change its price first. Quantitatively, this second effect ends up slightly dominating, and monetary policy becomes more powerful as the two firms become more unequal.

In general, computing the slopes once we allow for heterogeneity requires a more structural approach like the one in section (5). However, in the special case of $n = 2$ firms, our sufficient statistic formula can be adapted to arbitrary heterogeneity stemming from cost or demand differences:

Proposition 5. *Consider a sector with two firms $i = a, b$, that can have different demand functions d^i and different marginal costs MC_i. Then the slopes of the reaction functions $\beta^a = \frac{\partial g^a}{\partial p_b}$ and $\beta^b = \frac{\partial g^b}{\partial p_a}$ around the steady state $(\bar{p}_a, \bar{p}_b)$ are functions of steady state sufficient statistics:*

$$\beta^i = \frac{\lambda + \rho}{\lambda} \frac{-\epsilon^j_j - \frac{\bar{p}_i}{\bar{p}_j - MC_j}}{\epsilon^j_i}$$

where $\epsilon^i_k = \frac{\partial \log d^i}{\partial \log p_k}$.

All else equal, firm j's high price can now be justified by either its rival i's high slope β^i as before, or by its rival's high price. The case of two firms allows us to capture any Herfindahl-Hirschman Index (HHI) between 1/2 and 1; with more symmetric firms we can also obtain HHIs of 1/3, 1/4, and so on. In the case of $n \geq 3$ heterogeneous firms, we cannot back out the slopes from the steady state prices. Intuitively, the system is underdetermined because there are multiple ways to generate the same steady state prices.

5 Comparative Statics

In this section, we take a more structural approach: instead of using the observed equilibrium markup as a sufficient statistic, we propose an approximate solution concept under which we can solve for the equilibrium markup given target elasticities. The main payoff from solving the model is that we can perform counterfactual analyses, and investigate in depth which factors cause the oligopolistic model to depart from the standard monopolistic model. We are particularly interested in the effect of a change in market concentration, captured by the number of firms n, as it is likely to affect both the markup and the residual demand elasticity that enter formula (5).

5.1 Solution Concept

In general, solving for the steady state markup requires solving the full MPE. Since we want a solution for any number of firms, the state space can become very large.[10] We

[10] The IO literature uses approximate solution concepts to alleviate the curse of dimensionality, such as "oblivious equilibria" in Weintraub, Benkard and Van Roy (2008).

avoid this computational burden by approximating consumer's utility in a way that generates an equilibrium that we can solve analytically. Crucially, our approximation leaves enough degrees of freedom to flexibly parametrize the elasticities of the demand system that can be estimated in practice.

Our construction is detailed in Appendix F, and the main idea is as follows. Our earlier sufficient statistic result stems from manipulating the envelope condition applied to the Bellman equation (44) to get rid of derivatives of the value function. The outcome is equation (4) that relates the steady state markup, the elasticity ϵ_i^i, and the first derivative g' of the equilibrium strategy. Differentiating (44) further with respect to all its arguments will generate more such equations, that now relate the derivatives g', g'', and so on, to the steady state markup, demand elasticity ϵ_i^i, superelasticity ϵ_{ii}^i, and so on. If we keep iterating, we obtain an infinite system of equations, and the standard interpretation treats the sequence of derivatives of g as unknowns, and the sequence of higher-order elasticities (all evaluated at the steady state) as parameters. Instead, we take the view that it is empirically impossible to know such fine properties of demand functions, since we can only estimate a finite number of elasticities. Acknowledging this limitation, we take a dual view of the infinite system of envelope equations: we treat higher order elasticities as flexible unknowns that can be perturbed to achieve some desired properties of the derivatives of g. In particular, we seek to simplify the characterization of equilibrium by making g locally polynomial, meaning that all its derivatives higher than an arbitrary order vanish when evaluated at the steady state.

Formally, denote $\epsilon_{(k)}$ is the kth-own-superelasticity evaluated at a symmetric $\bar{p}$, i.e.,

$$\epsilon_{(1)} = \frac{\partial \log d^i(p)}{\partial \log p_i}, \quad \epsilon_{(k)} = \frac{\partial \epsilon_{(k-1)}(p)}{\partial \log p_i} \quad \forall k \geq 2.$$

Proposition 6. *For any order of approximation $m \geq 1$ and target elasticities $\left(\epsilon_{(1)}, \ldots, \epsilon_{(m)}\right)$, there exist Kimball within-sector preferences $\tilde{\Psi}$ such that*

(i) the resulting elasticities up to order m match the target elasticities, and

(ii) any MPE of the game with within-sector preferences $\tilde{\Psi}$, strategy $\tilde{g}$ and steady state $\tilde{p}$ satisfies $\tilde{g}^{(k)}(\tilde{p}) = 0$ for $k \geq m$.

Remark 3. Our approximation relates to the algorithm used in Krusell, Kuruscu and Smith (2002) and later called "Taylor projection" by Levintal (2018). Krusell et al. (2002)'s idea is to fix the parameters and approximate the unknown policy and value functions by polynomials of order m. Instead, we take the view that we lack reliable estimates of higher order elasticities that are taken as inputs to parametrize the game, and show that we can take them as unknowns instead of parameters in the infinite system of equations, while still matching the target elasticities up to order m.

In the remainder of the paper we will apply Proposition 6 in the case $m = 2$, which makes the game linear-quadratic.[11] for given elasticity ϵ_i^i and superelasticity ϵ_{ii}^i, we solve for the steady state price $\bar{p}$ and slope $\beta = \frac{\partial g^i}{\partial p_j}$ given a locally linear equilibrium.

Corollary 1. *In a locally linear equilibrium ($m = 2$), $\bar{p}$ and β solve the system of two equations:*

$$\beta = \frac{(\lambda+\rho)\Pi_i^i(\bar{p})}{\lambda(n-2)\Pi_i^i(\bar{p}) - \lambda(n-1)\Pi_j^i(\bar{p})}$$

$$0 = A_{ii}(\beta)\,\Pi_{ii}^i(\bar{p}) + A_{ij}(\beta)\,\Pi_{ij}^i(\bar{p}) + A_{jj}(\beta)\,\Pi_{jj}^i(\bar{p}) + A_{jk}(\beta)\,\Pi_{jk}^i(\bar{p})$$

where $A_{ii}, A_{ij}, A_{jj}, A_{jk}$ are given by equations (40)-(43) in Appendix G.

Parametrizing the two dimensions of demand. In what follows, we use Klenow and Willis (2016)'s functional form for the Kimball aggregator ϕ_s, which is simpler to define through its derivative

$$\phi_s'(x) = \frac{\eta-1}{\eta}\exp\left(\frac{1-x^{\theta/\eta}}{\theta}\right). \tag{10}$$

η and θ control the elasticity and the superelasticity of demand, respectively: in the limit of monopolistic competition $n \to \infty$, the demand own-elasticity ϵ_i^i converges to $-\eta$ and the ratio $\frac{\epsilon_{ii}^i}{\epsilon_i^i}$, named the "superelasticity" of demand by Klenow and Willis (2016), converges to θ. The limit $\theta \to 0$ corresponds to a standard CES demand with $\phi_s(x) = x^{\frac{\eta-1}{\eta}}$.

With finite n, the perceived elasticities also depend on n because firms face a residual demand that depends on the number of rivals they have, as is well known in the CES case studied by Atkeson and Burstein (2008). We generalize the CES expressions for perceived elasticities as a function of n to any Kimball aggregator in Appendix E, and also derive new expressions for the perceived superelasticities. In particular, with the functional form

[11] In a microeconomic context, Jun and Vives (2004) studied a linear-quadratic dynamic duopoly with Bertrand and Cournot competition and quadratic adjustment costs in prices and quantities, focusing on how dynamics can amplify or reverse static strategic complementarities.

(10) we have:[12]

$$\epsilon_i^i = \frac{\partial \log d^i}{\partial \log p_i} = -\eta + \frac{\eta - 1}{n} \tag{11}$$

$$\epsilon_{ii}^i = \frac{\partial^2 \log d^i}{\partial \log p_i^2} = -\frac{n-1}{n^2}\left[(\eta-1)^2 + (n-2)\theta\eta\right]. \tag{12}$$

These expressions imply a precise dependence on n for the elasticities $\epsilon_i^i, \epsilon_{ii}^i$, but they stem from parametric assumptions made for tractability that have no particular empirical grounding. In section 7 we turn to a more general non-parametric model that controls the superelasticity $\epsilon_{ii}^i(n)$ directly (which is isomorphic to letting θ depend on n in (12)) to match the heterogeneity in idiosyncratic cost pass-through observed in the data.

5.2 Preferences

We first consider changes in steady state markups driven by preferences, holding market concentration (i.e., the number of firms n) fixed.

Changes in the elasticity of substitution η. We first highlight the importance of allowing for more than two firms in each sector. The duopoly model is a knife-edge case, because in sectors with only two firms, the steady state markup and the demand elasticity are related one-to-one, making it sufficient to know a single statistic, the markup, to infer the half-life of monetary shocks. In other words, CES demand systems are without loss of generality within the class of Kimball aggregators in the case $n = 2$, as can be seen from expression (12) (or equation (33) in Appendix E for a non-parametric formulation). When n is above 2, however, CES demand is not without loss, and knowing the markup is not enough to infer the slope: we also need information on demand elasticities.

To illustrate this point, consider Figure 11, which shows the half-life as a function of the steady state markup. Variation in markups is produced through variation in the parameter η that captures the within-sector elasticity of substitution; higher η implies lower markups. When $n = 2$, the value of the superelasticity parameter θ does not matter, and we have a negative relation between the markup and the half-life. This pattern is also present in the duopoly model with menu costs of Mongey (2018). However, as soon as there are at least $n = 3$ firms, there is a crucial interaction between θ and η. When $\theta = 0$ (CES), we have the same negative relation as in the duopoly case, but with a high enough

[12] Recall that we set $\omega = 1$; otherwise residual elasticities would be weighted averages of inner and outer elasticities, for instance $\epsilon_i^i = -\left[\frac{n-1}{n}\eta + \frac{1}{n}\omega\right]$ which specializes to (11) with $\omega = 1$.

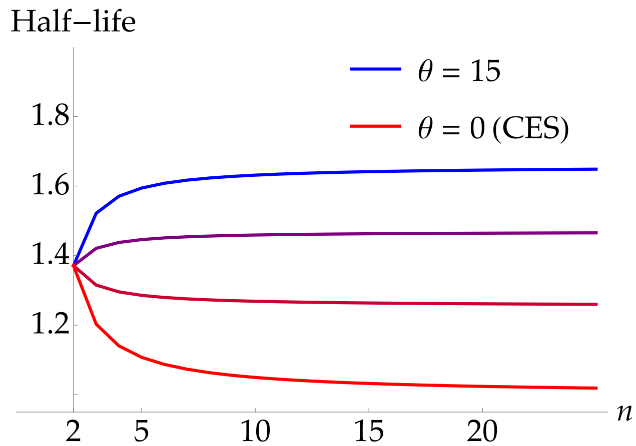


Figure 3: Half-life as a function of n for different values $\theta = 0$ (bottom red line), $5, 10, 15$ (top blue line), with $\eta = 10$.

value of θ, the half-life becomes negatively related to the steady state markup. We will provide an intuition behind this fact in section 6.

Changes in the superelasticity parameter θ. A crucial difference between our framework and a monopolistically competitive economy is that the superelasticity parameter θ can generate variations in the steady state markup $\bar{\mu}$ while keeping η and hence the demand elasticity (11) constant. Note that such an experiment that varies the markup while fixing the demand elasticity is impossible with a duopoly, as θ becomes irrelevant in (12) when $n = 2$.

Figure 12 shows an example with the minimal number of firms $n = 3$ that allows θ to affect the steady state markup. The left panel shows that as θ increases, the markup under dynamic oligopoly rises. Multiple factors determine equilibrium markups, so variation in θ is the most transparent way to apply our formula (5), as in that case a higher markup unambiguously implies a larger half-life, as on the right panel. In a model with monopolistic competition and Kimball (1995) demand, θ would also increase non-neutrality through complementarities in pricing, but would have no effect on the markup, hence markups would be uninformative about the strength of monetary policy. The link between markups and pass-through is a crucial difference between monopolistic models with variable markups and our oligopoly model. In the next sections, we will build on this distinction to calibrate the model to cost pass-through data and then define a precise notion of dynamic strategic complementarities under oligopoly.

Table 1: Parameter values.

Parameter	Description	Value
ρ	Annual discount rate	0.05
λ	Price changes per year	1
ω	Cross-sector elasticity	1
η	Within-sector elasticity	10

5.3 Market Concentration

We now turn to our main counterfactual exercise, in which we study how changes in market concentration (the number of firms n in a sector) affect the transmission of monetary policy. If we knew how our sufficient statistics changed with n, we could just plug them into (5) and it would not be necessary to solve the model further. Absent this information, we need to make assumptions on how these statistics depend on n, for instance by taking a stand on what parameters to keep fixed when changing n. We start by holding "preferences" fixed, interpreting η and θ in the Klenow and Willis (2016) functional form (10) as structural parameters that are robust to shifts in the number of varieties. The remaining parameters are described in Table 1.

Higher market concentration in the sense of lower n increases monetary non-neutrality in the CES case $\theta = 0$. Yet even in the duopoly $n = 2$ case that maximizes the impact of oligopolistic competition, the departure from monopolistic competition remains modest: the half-life under oligopoly is only higher by 37%. But as the blue line in Figure 3 shows, for high values of θ that generate strong demand complementarities and thus large effects of monetary policy under monopolistic competition $n \to \infty$, decreasing the number of firms in each sector can *dampen* monetary policy. In theory, this dampening effect can be arbitrarily large: the half-life under monopolistic competition is unbounded above when θ increases, but the half-life under duopoly is invariant to θ, and thus always the same as with CES demand. This example shows that there is no guarantee that oligopolistic competition generates more non-neutrality than monopolistic competition: the direction of the effect depends on finer properties of demand systems, in particular how concentration affects the superelasticity of demand. We show below how to infer these properties from available pass-through estimates.

5.4 Frequency of Price Changes

In the cross-section, Mongey (2018) shows that price changes are less frequent in more concentrated markets. As we discuss in section 7, this fact is also consistent with Gopinath

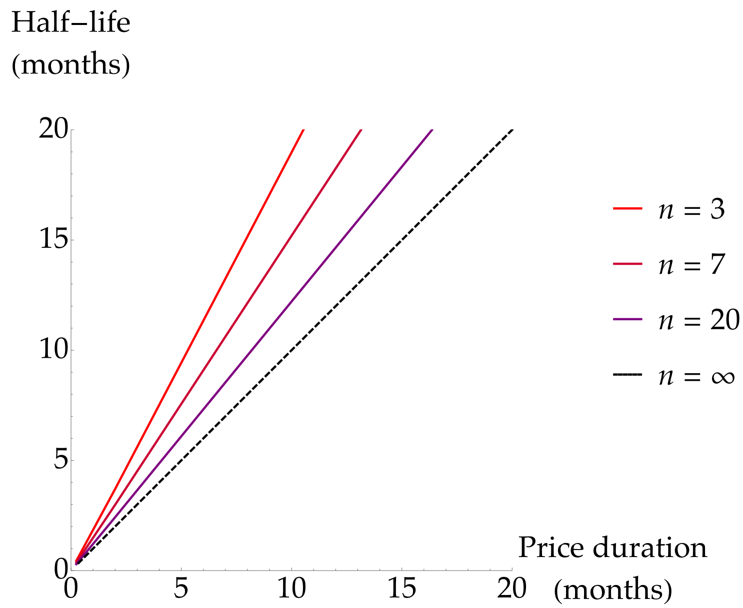


Figure 4: Half-life as a function of a sector's average price duration under the "AIK" calibration.

and Itskhoki (2010) who show price changes are less frequent for goods with a lower long-run exchange rate pass-through, given that market shares and pass-through are negatively correlated. A model with menu costs would be better suited to analyze how trends in concentration may affect price flexibility, but we can still explore in reduced form how the heterogeneity in λ interacts with the market structure within our Calvo pricing framework.

Figure 4 shows how exogenous changes in λ endogenously affect the half-life. Under our baseline calibration "AIK" defined in section 7, higher concentration increases the half-life for a given frequency of price changes. Moreover, the amplification itself is larger when price changes less frequently. For instance, the half-life of the price level in sectors with an effective number of firms $n = 3$ is almost twice the average time between price changes: concentration matters a lot if prices do not change frequently but it makes no difference if prices are very flexible ($\lambda \to \infty$). Suppose then that sectors with n firms are characterized on average by a frequency λ_n. Then the aggregate price level follows

$$\log P(t) - \log \bar{P} = -\delta \exp\left(-\left\{\mathbf{E}\left[\lambda_n\right]\mathbf{E}\left[1-(n-1)\beta_n\right] + \mathbf{Cov}\left(\lambda_n, (n-1)\beta_n\right)\right\} t\right), \quad (13)$$

Figure 4 implies that the term $\mathbf{Cov}\left(\lambda_n, (n-1)\beta_n\right)$ is negative, and thus contributes to decrease aggregate price flexibility further relative to a case with homogeneous frequency of price changes across sectors. As discussed in Carvalho (2006), even under monopolistic competition, the cumulative output effect in a multi-sector Calvo model is convex in the sectoral frequencies of price changes $\{\lambda_s\}$. The heterogeneous economy is equivalent to a homogeneous economy with price duration $\frac{1}{\sum_s \omega_s \lambda_s}$ that is higher than the measured average price duration $\sum \omega_s \frac{1}{\lambda_s}$. We point out an additional effect that is specific to oligopolistic competition.

6 Feedback vs. Strategic Effects

The presence of a finite number of firms has two distinct effects on competition and pricing incentives: "feedback effects" capture the fact that each firm cares about its rivals' current and future prices when setting its price; "strategic effects" capture instead the fact that each firm realizes its current pricing decision can affect how its rivals will set their prices in the future. Feedback effects are what the literature with monopolistic competition calls strategic complementarities in pricing, that could arise from variable markups as in our setting, or other channels such as intermediate inputs or decreasing returns in production. The decomposition we propose is only meaningful under oligopoly, because under monopolistic competition, no single firm can affect the sectoral price index hence strategic effects are nil.

We disentangle the two effects through the lens of a "non-strategic" model. For each n, the associated non-strategic model is an economy with monopolistic competition ($n = \infty$) and modified Kimball preferences $\widetilde{\Psi}(\Psi, n)$ that match the residual demand elasticity and superelasticity of the oligopolistic model with Kimball preferences Ψ and n firms.[13] The non-strategic model captures all the feedback (which in our context only arises from properties of the demand system), while suppressing strategic effects thanks to the monopolistic competition assumption.

We compute the half-life $\widetilde{hl}(n)$ of this non-strategic model, and then define strategic effects in the MPE as the increase in the half-life (relative to $1/\lambda$, the half-life in the standard New Keynesian model with monopolistic competition and CES demand) not explained

[13] This non-strategic model also has a behavioral interpretation. Suppose that all firms are non-strategic in the following sense: when resetting their price, they form correct expectations about the stochastic process governing their competitors' future prices, but incorrectly assume that their own price-setting will have no effect on those competitors' future prices.

by the non-strategic model:

$$\frac{hl\left(n\right)}{1/\lambda}=\underbrace{\frac{\widetilde{hl}\left(n\right)}{1/\lambda}}_{\text{feedback effect}}\times\underbrace{\frac{hl\left(n\right)}{\widetilde{hl}\left(n\right)}}_{\text{strategic effect}}.$$

As n goes to infinity, $hl/\widetilde{hl}$ goes to 1 and the strategic effect disappears; what is left is the standard feedback effect that can stem from a Kimball (1995) demand with positive superelasticity.

6.1 The Non-Strategic Model

The steady state price of the non-strategic model is the static Bertrand-Nash price p^{NE}, that solves $\Pi_i^i\left(p^{NE}\right)=0$. We look for a symmetric equilibrium where, to first order, each resetting firm i sets $p_i^*(t)=\widetilde{\beta}_n\sum_{j\neq i}p_j\left(t\right)$. When it resets, given other firms' strategies β^{NS}, firm i chooses $p_i^*(t)$ to maximize

$$\mathbf{E}_t\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\Pi^i(p_i^*(t),p_{-i}(t+s))ds\right].$$

The key difference with the MPE defined by the Bellman equation (2) is that here, firm i treats the evolution of rivals' prices as exogenous to its choice p_i^*. Define

$$\Gamma_n=\frac{(n-1)\Pi_{ij}}{-\Pi_{ii}}$$

Γ_n is a measure of static feedback effects: it is the slope of the best response of a firm to a simultaneous price change by all its competitors in a static Bertrand-Nash equilibrium. Under static monopolistic competition, $\Gamma_\infty/\left(1-\Gamma_\infty\right)$ is known as the *markup elasticity* (Gopinath and Itskhoki, 2010) (as it measures the elasticity of a firm's desired markup to its own relative price) or *responsiveness* (Berger and Vavra, 2019). In Appendix D we show the following:

Proposition 7. *The half-life of the aggregate price level in the non-strategic equilibrium is*

$$\widetilde{hl}\left(n\right)=\frac{1}{\lambda\left(1-\left(\frac{\rho+2\lambda}{2\lambda}\right)\left[1-\sqrt{1-\frac{4\lambda(\rho+\lambda)}{(\rho+2\lambda)^2}\Gamma_n}\right]\right)}. \tag{14}$$

We can reexpress Γ_n around the Nash markup in terms of the demand elasticities

$\epsilon_i^i = \frac{\partial \log d^i}{d \log p_i}$ and $\epsilon_{ii}^i = \frac{\partial^2 \log d^i}{\partial \log p_i^2}$ as:

$$\Gamma_n = \frac{\frac{\epsilon_{ii}^i(n)}{\epsilon_i^i(n)}}{\frac{\epsilon_{ii}^i(n)}{\epsilon_i^i(n)} - \epsilon_i^i(n) - 1}. \tag{15}$$

In the standard CES case, as n goes to infinity and the model converges to monopolistic competition, $\frac{\epsilon_{ii}^i(n)}{\epsilon_i^i(n)}$ goes to 0 hence $\widetilde{hl}$ converges to $1/\lambda$. Away from CES, Γ can converge to a positive limit. With a finite number of firms, even CES demand implies $\frac{\epsilon_{ii}^i(n)}{\epsilon_i^i(n)} > 0$ and thus $\widetilde{hl} > 1/\lambda$.

Comparative Statics. The effect of oligopoly on monetary policy transmission is transparent in the non-strategic model, as it is entirely captured by Γ_n that we can compute in closed form. When $\Gamma_n > 0$, a higher own price leads to more elastic demand and thus a lower desired markup; this force, known as "Marshall's second law of demand", increases with Γ_n. In turn, (14) shows that higher Γ_n increases the feedback effect.

We can now see that the behavior of Γ_n plays a large part our earlier findings in section 5.2. Recall that in the Klenow and Willis (2016) specification, ϵ_i^i and ϵ_{ii}^i are given by (11) and (12), respectively. Thus Γ_n decreases with the elasticity of substitution η (and thus the observed markup) if and only if

$$\theta < \frac{n}{n-2} \times \frac{(\eta - 1)^2}{1 + (n-1)\,\eta^2},$$

which explains why, in Figure 11, the half-life is decreasing in the markup $\bar{\mu}$ under CES but not when θ is high enough.

Similarly, we can use the non-strategic model to understand how concentration affects the half-life. As shown numerically in Figure 3, this depends again on the value of θ. Indeed, feedback Γ_n is decreasing in n (increasing in concentration) if and only if

$$\theta < \frac{(\eta - 1)^2}{\eta + 1}.$$

In theory, insights based on the non-strategic model could fail to be valid in the full MPE, due to sufficiently strong strategic effects that work in the opposite direction. But as we show next, we find that strategic effects $hl/\widetilde{hl}$ are quantitatively modest.

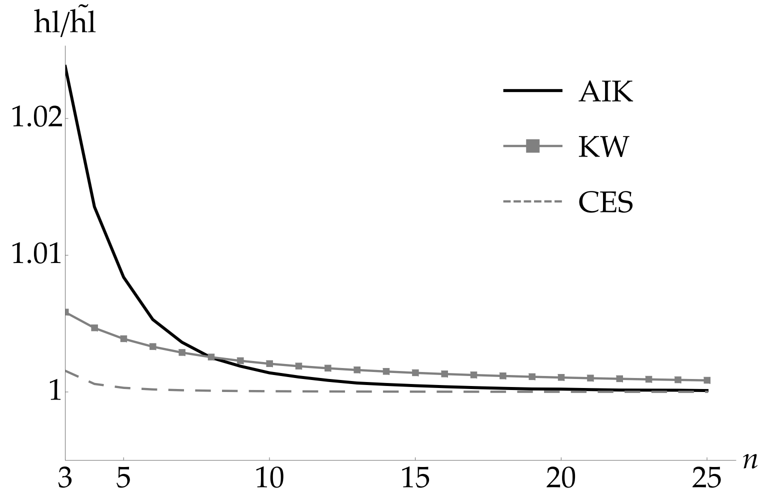


Figure 5: Strategic effect $hl(n)/\widetilde{hl}(n)$ as a function of n. AIK: variable superelasticity to match heterogeneity in pass-through from Amiti et al. (2019). KW: Fixed $\theta = 10$. CES: Fixed $\theta = 0$. In all cases, $\eta = 10$.

6.2 Measuring Strategic Effects

While strategic effects are important determinants of steady state markups, as we saw in Figure 12, we find that quantitatively, they do not explain much of the aggregate response to monetary shocks under oligopoly. Figure 5 displays the strategic effect, defined as $hl(n)/\widetilde{hl}(n)$, as n varies. We contrast our baseline calibration "AIK" with variable superelasticity (defined in section 7) with the CES case and a Kimball demand with fixed $\theta = 10$ "KW"(as in Klenow and Willis 2016). In all specifications, strategic effects are negligible as the half-life is always less than 3% higher than the non-strategic half-life. Strategic effects vanish as n grows and the economy approaches monopolistic competition: they are below 1% already with $n = 5$ firms. Overall, our results suggest that oligopolistic competition can significantly amplify or dampen the real effects of monetary policy, but primarily through "feedback effects", that is changes in residual demand elasticities as measured by Γ_n.

7 A Calibration Based on Pass-Through

We showed that the shape of demand is crucial to understand how market structure impacts the transmission of monetary shocks, which affect all firms at the same time. As Atkeson and Burstein (2008) emphasized in a static setting, changes in residual demand also link market structure and the pass-through of *own cost* shocks, hereafter simply "pass-through". We now argue that the most recent and detailed pass-through estimates imply that market concentration significantly amplifies monetary non-neutrality.

Amiti et al. (2019) find considerable heterogeneity in pass-through.[14] Small firms behave as under a CES monopolistic competition benchmark, passing through own marginal cost shocks fully (and thus maintaining a constant markup) while not reacting to competitors' price changes orthogonal to their own cost. Large firms exhibit substantial strategic complementarities: they only pass through around half of their own cost shocks, thus letting their markup decline to absorb the other half. Amiti et al. (2019) show that this pattern is consistent with a static model of oligopolistic competition that generalizes the duopoly model of Atkeson and Burstein (2008). Importantly, they argue that with nested CES demand, Cournot competition can match the degree of heterogeneity in pass-through but Bertrand competition cannot. As already remarked by Krugman (1986) in his seminal paper on pricing-to-market, under the nested CES assumption, Bertrand and Cournot competition both imply qualitatively that the elasticity of residual demand declines with market share, but quantitatively, Bertrand competition implies only a mild decline relative to Cournot.

We argued in section 5 that an increase in concentration (lower n) can dampen or amplify monetary policy transmission once we depart from nested CES systems. For the same reasons, in a static oligopolistic model with more general demand, an increase in market share holding industry concentration fixed could dampen or amplify pass-through. Reinterpreting Amiti et al. (2019)'s estimates within our dynamic model, we show that the empirical pattern of heterogeneity is consistent with a large superelasticity for large firms, and a small superelasticity for small firms. Our results also highlight that the distinction between Cournot and Bertrand is only meaningful under the CES restriction. With more general preferences, Bertrand models, which are more common to model price-setting in macroeconomics, can also match the sharp decline in pass-through.

Rewrite (2) as

$$(\rho + n\lambda) V^i(p, c) = \Pi^i(p, c_i) + \lambda \sum_j V^i\left(g^j(p_{-j}, c), p_{-j}, c\right)$$

Pass-through, defined in logs as in the empirical literature, is $\alpha = \frac{c_i}{p_i}\frac{\partial g^i}{\partial c_i}$. Following the same envelope arguments as before, we can show that pass-through is given by

$$\alpha = \frac{c\left[\epsilon_i^i + \frac{\lambda(n-1)\beta\epsilon_j^i}{\rho+\lambda[1-(n-2)\beta]}\right]}{(\rho+\lambda)\, p^3 n V_{ii}^i}$$

[14]See also Berman, Martin and Mayer (2012), who show that the pass-through of exchange rates to export prices is lower for larger firms.

where V_{ii}^i is given by the solution to the system in Appendix G.1.[15]

Results. Figure 6 displays pass-through, computed in the dynamic model, under three specifications for within-sector demand. "AIK" is our baseline calibration: the superelasticity varies as a function of n through a variable parameter $\theta(n)$ (defined as in (12)) so as to match the relationship between market share and pass-through in a static Cournot model with $\eta = 10$ which, Amiti et al. (2019) argue, provides a good fit to their Belgian data. In "KW", θ is fixed at 10 as in Klenow and Willis (2016) and in standard DSGE calibration such as Smets and Wouters (2007). In "CES" θ is fixed at 0. In all cases, η equals 10, a common benchmark in the literature since Atkeson and Burstein (2008). Figure 16 in the Appendix shows how the resulting superelasticities $\epsilon_{ii}^i/\epsilon_i^i$ depend on the market share or Herfindahl index $1/n$.

We only vary n and assume symmetry, but allowing for within-sector heterogeneity does not make much difference. Intuitively, under static Bertrand or Cournot competition, market share is a sufficient statistic for the residual elasticity, and so whether a firm is larger or not than its rivals does not matter for pass-through conditional on a given market share. The same insight applies quantitatively in the dynamic model (see also the discussion in section 4.2).

We hold η fixed to focus the discussion on how pass-through and hence the residual superelasticity of demand changes with concentration, but there is no reason for the residual elasticity itself to vary exactly as in (11). Ideally, one would obtain non-parametric estimates of $\epsilon_i^i(n)$ and $\epsilon_{ii}^i(n)$ from matching jointly the relation of markups and pass-through with market shares. However, there is no direct counterpart to Amiti et al. (2019), in part because markups are notably harder to estimate than pass-through. In the model with constant $\eta = 10$, going from $n = 4$ to 5 firms decreases prices by around 2%, which is broadly consistent with the evidence in Atkin, Faber and Gonzalez-Navarro (2018) and Busso and Galiani (2019). Recent work by Burstein, Carvalho and Grassi (2020) examines the relation between market shares and markups at the firm and sectoral levels. They find that a linear regression of the inverse markup against the sectoral HHI yields a coefficient of -0.44. In our dynamic model, the corresponding coefficient is -0.24 and gets closer to their estimate than a CES model, which would yield -0.15. Allowing η to increase with n instead of fixing $\eta = 10$ would improve the fit further.

Figure 8 shows that under the calibration consistent with the micro evidence on pass-through, a rise in national concentration corresponding to an increase in the average

[15] Since that system is linear, it is actually possible to express α in non-parametric closed form as a function of n, the markup and the elasticity, just like in our sufficient statistic formula (5) for β; however the expression is more complex and does not bring particular insight.

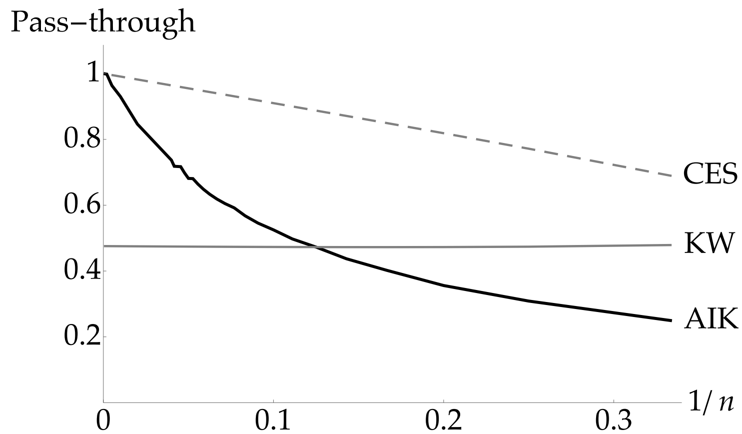


Figure 6: Pass-through as a function of market share $1/n$. AIK: variable superelasticity to match heterogeneity in pass-through from Amiti et al. (2019). KW: Fixed $\theta = 10$. CES: Fixed $\theta = 0$. In all cases, $\eta = 10$.

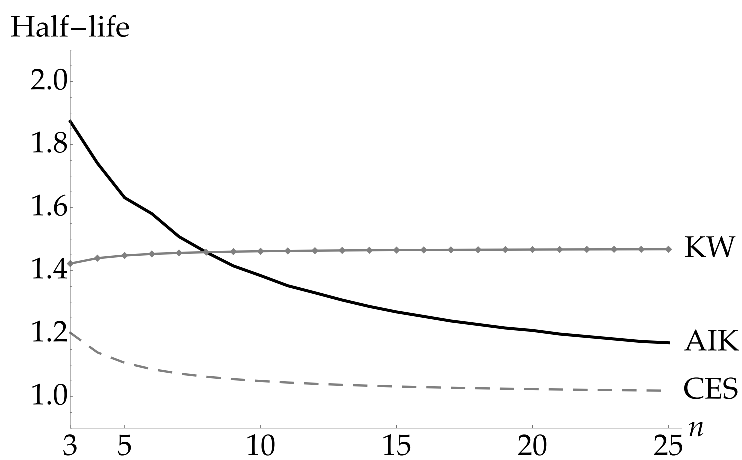


Figure 7: Half-life as a function of number of firms n. AIK: variable superelasticity to match heterogeneity in pass-through from Amiti et al. (2019). KW: Fixed $\theta = 10$. CES: Fixed $\theta = 0$. In all cases, $\eta = 10$.

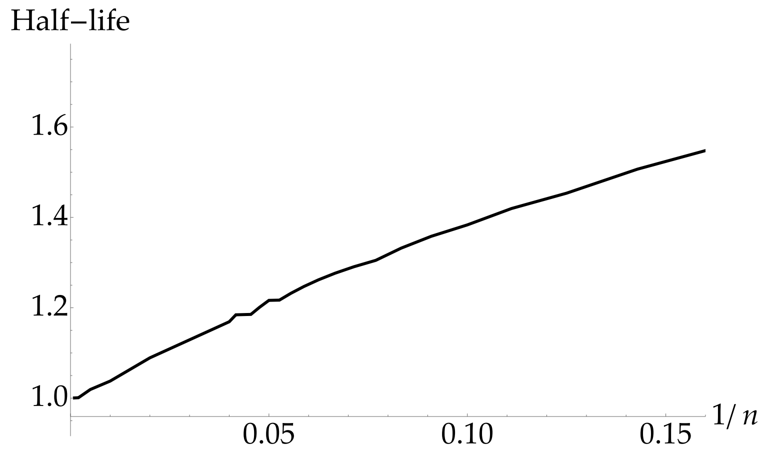


Figure 8: Half-life as a function of average Herfindahl index $1/n$ under the "AIK" calibration.

Herfindahl index $1/n$ from 0.05 to 0.1, reflecting the observed trends since 1990 in e.g. Gutiérrez and Philippon (2017), amplifies the real effects of monetary policy by around 15%. Rossi-Hansberg et al. (2020), however, argue that rising national concentration goes hand in hand with an even stronger decline in local concentration, as the entry of large firms in local markets increases local competition but also these firms' national market share. An interesting open question is then which level of geographic or economic aggregation (what we call "sectors" s) is most relevant for the competition that determines consumer price inflation. If, for instance, competition at the county level matters the most and the local HHI has fallen from 0.15 to 0.05, in line with the evidence from Rossi-Hansberg et al. (2020), then our results would suggest that the half-life of monetary shocks has fallen substantially, by around 25%.

8 A Three-Equation Oligopolistic New Keynesian Model

We focused so far on the dynamics following a permanent monetary shock, under the Golosov and Lucas (2007) assumptions (7). In this section we take a step closer to the New Keynesian framework. We leverage our perturbation argument from section 5.1 further, to allow for general preferences as well as non-stationary dynamics. The main payoff is an oligopolistic Phillips curve that maps any path of future real marginal cost shocks to current inflation, and can be embedded in a standard DSGE model once combined with an Euler equation and a monetary policy rule.

8.1 The Oligopolistic Phillips Curve

Denote $k(t) = \log MC(t) - \log P(t)$ the log real marginal cost. In Appendix H we show the following. In this section we denote $i(t)$ the nominal interest rate.

Proposition 8. *There exists a $q \times q$ matrix $\mathbf{A}$ with $q \leq 7$ that depends on the steady state demand elasticities, markup and slope β (described in Appendix H) such that inflation follows*

$$\pi(t) = \int_0^\infty \gamma^k(s)\, k(t+s)\, ds + \int_0^\infty \gamma^c(s)\, c(t+s)\, ds + \int_0^\infty \gamma^i(s)\, (i(t+s) - \rho)\, ds \quad (16)$$

where for each variable $x \in \{k, c, i\}$, $\gamma^x(s)$ is a linear combination of $\{e^{-\nu_j s}\}_{j=1}^q$ with $\{\nu_j\}_{j=1}^q$ the eigenvalues of $\mathbf{A}$, e.g.,

$$\gamma^k(s) = \sum_{j=1}^q \gamma_j^k e^{-\nu_j s}$$

for some constants $\left\{\gamma_j^k\right\}_{j=1}^q$.

In general $q = 7$ but under condition (78) in Appendix H, which we assume in what follows, q can be reduced to 3. Under monopolistic competition, even with Kimball preferences parametrized by Γ (as in section 6), there is a single eigenvalue $\nu_1 = \rho$ instead of three, and $\gamma^c = \gamma^i = 0$, and the Phillips curve in integral form is simply

$$\pi(t) = \int_0^\infty \underbrace{e^{-\rho s}(1-\Gamma)\lambda(\lambda+\rho)}_{=\gamma^k(s)} k(t+s)\, ds. \quad (17)$$

The slope of the Phillips curve is usually defined as the coefficient $\gamma^k(0)$ that captures how inflation reacts to current marginal cost. It is equal to $\lambda(\lambda+\rho)(1-\Gamma)$ under monopolistic competition: higher feedback effects Γ flatten the Phillips curve, but are isomorphic to a higher degree of stickiness λ.

In the case of oligopolistic competition, inflation is also determined by a weighted average of future marginal costs, with two important differences. First, there are multiple eigenvalues. Second, inflation depends on more than future marginal costs, as the second sum in (16) relates current inflation to future consumption and nominal interest rates. In the standard New Keynesian model, real marginal costs capture all the forces that influence price setting. Here, consumption and interest rates have an independent first-order effect because they alter the strategic complementarities between firms, as in Rotemberg and Saloner (1986). These two differences imply that oligopoly is not equivalent to a higher stickiness parameter λ. As with our earlier permanent money supply shocks, we can compare (16) to a "non-strategic" Phillips curve that corresponds to a mo-

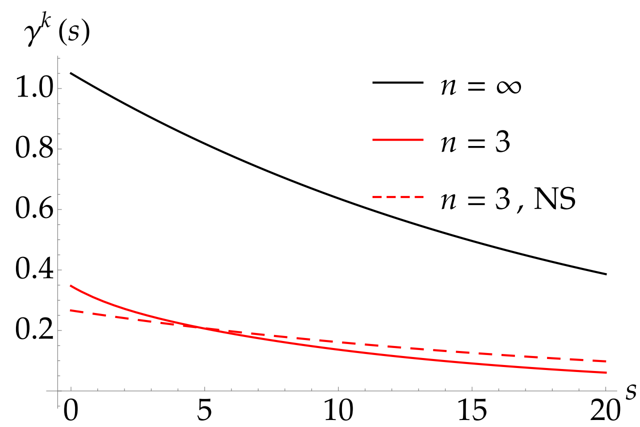


Figure 9: $\gamma^k(s)$ for $n = 3$ under the AIK calibration (red, solid), compared to the associated non-strategic model (red, dashed) and the standard New Keynesian model with CES monopolistic competition (black).

nopolistic competitive economy with Kimball preferences that match the elasticity and superelasticity of the oligopolistic economy, characterized by (17) with $\Gamma = \Gamma_n$ given in (15).

We can also get an equivalent scalar ordinary high-order differential equation for inflation:

Corollary 2. *Inflation π solves a third-order ODE*

$$\sum_{j=0}^{3} \gamma_j^{\pi} \frac{d^j \pi(t)}{dt^j} = \sum_{j=0}^{2} \left(\gamma_j^k \frac{d^j k(t)}{dt^j} + \gamma_j^c \frac{d^j c(t)}{dt^j} + \gamma_j^i \frac{d^j i(t)}{dt^j} \right) \tag{18}$$

with weights $\left\{\gamma_j^{\pi}, \gamma_j^k, \gamma_j^c, \gamma_j^i\right\}$ defined in (79) in Appendix H, and boundary conditions $\frac{d^j \pi(t)}{dt^j} \to 0$ as $t \to \infty$ for all $j = 0, 1, 2$.

Numerically, it turns out that the oligopolistic Phillips curve (18) is essentially a second-order ODE. For instance, for $n = 3$, under the AIK calibration and other parameters as in Table 1, we have

$$\dot{\pi} = 0.07\pi - 0.28k + 0.45\dot{k} + 1.31\ddot{\pi} + 0.03\,(i - \rho)\,. \tag{19}$$

The corresponding non-strategic Phillips curve and the standard CES Phillips curve under the same parameters are respectively

$$\dot{\pi} = 0.05\pi - 0.27k, \tag{20}$$

$$\dot{\pi} = 0.05\pi - 1.05k. \tag{21}$$

Relative to (20), the oligopolistic Phillips curve (19) features more discounting and a term that resembles an endogenous "cost-push" shock

$$u = -\left[0.45\dot{k} + 1.31\ddot{\pi} + 0.03\left(i - \rho\right)\right]. \tag{22}$$

8.2 Three equations model

We can now analyze a three-equation New Keynesian model that combines the oligopolistic Phillips curve (18) with an Euler equation

$$\dot{c} = \sigma^{-1}\left(i - \pi - r^n\right),$$

and a monetary policy rule

$$i = \kappa\rho + (1-\kappa)\, r^n + \phi_\pi \pi + \epsilon^m,$$

where $r^n(t) = \rho + \epsilon^r(t)$ is the natural real interest rate and $\epsilon^m(t)$ is a monetary shock. For simplicity, agents have perfect foresight over the shocks ϵ^r, ϵ^m.

Calibration. Wages are flexible, technology is linear in labor $Y = \ell$ and households have preferences $\frac{C^{1-\sigma}}{1-\sigma} - \frac{\ell^{1+\psi}}{1+\psi}$, hence $k = (\psi + \sigma)\, c$. We set standard values of $\sigma^{-1} = 1$ for the elasticity of intertemporal substitution (as in our monetary shock experiments), $\psi^{-1} = 0.5$ for the Frisch elasticity of labor supply, and $\phi_\pi = 1.5$ for the Taylor rule coefficient on inflation. $1 - \kappa$ measures how well the central bank is able to track the natural rate; κ can be thought of as monetary policy inertia. We set $\kappa = 0.8$.

One-time shocks. Consider first geometrically decaying unanticipated shocks

$$\epsilon^m(t) = \epsilon_0^m e^{-\xi t}, \quad \epsilon^r(t) = \epsilon_0^r e^{-\xi t}$$

with the same decay ξ (a particular case being only one type of shock). It is a standard result in the literature (Woodford, 2003) that under monopolistic competition, all the equilibrium variables are proportional to $e^{-\xi t}$. The same applies to the oligopolistic model, hence all the differences between economies are summarized by the impact effect, e.g. $c(t) = c(0)\, e^{-\xi t}$ and the cumulative output effect is $c(0)/\xi$. This contrasts with the case of permanent money supply shocks, for which impact effects were common to all economies and differences were summarized by the half-life.

Figure 10 displays the impact effect on consumption $c(0)$ for a 100 bps monetary shock $\epsilon_0^m = -0.01$ with $\xi = 1$. The message is consistent with what we found for permanent

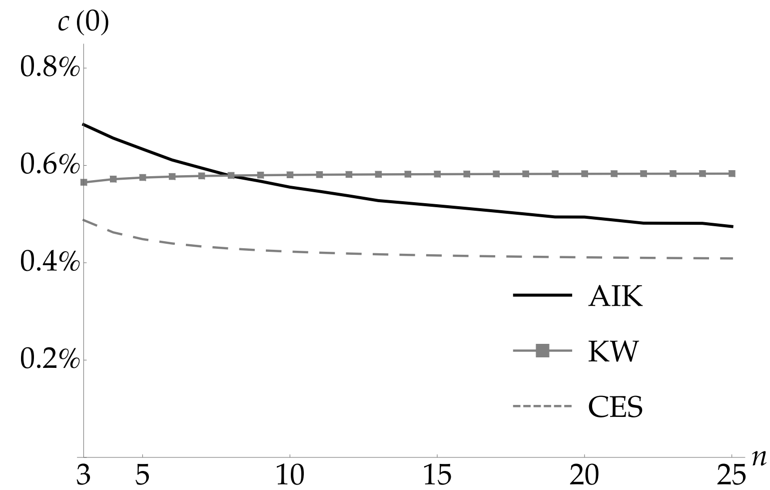


Figure 10: Impact effect of a $\epsilon_0^m = -1\%$ monetary shock on consumption $c(0)$ (log-deviation from steady state) as a function of number of firms n. AIK: variable superelasticity to match heterogeneity in pass-through from Amiti et al. (2019). KW: Fixed $\theta = 10$. CES: Fixed $\theta = 0$. In all cases, $\eta = 10$.

shocks to the money supply: concentration amplifies monetary non-neutrality by a significant amount. As Figure 13 shows, a large part of the amplification can again be explained by feedback effects. Denoting $\tilde{c}(0)$ the initial consumption jump in the monopolistic Kimball economy calibrated to match the parameter Γ_n for each n, we find that $c(0)$ is actually lower than $\tilde{c}(0)$ (so that "strategic effects" are not amplifying) and can deviate from $\tilde{c}(0)$ by around 5% when $n = 3$.

More general shocks. The one-time shocks are not without loss of generality. For instance, the common exponential decay leaves no room for the endogenous cost-push shocks (22) to generate different inflation persistence across models.

Once we allow for a more general process for shocks, there are also meaningful differences between the oligopolistic economy and the non-strategic economy. Consider for instance paths for real and monetary shocks generated from an Ornstein-Uhlenbeck process (a continuous-time version of AR(1) processes)

$$d\epsilon = -a\epsilon + \sigma dZ$$

where Z in a standard Brownian motion, and $a, \sigma^r > 0$ parametrize the speed of mean-reversion and variance of the shocks, respectively.[16] We set $a = 0.3$, $\sigma^r = 0.01$. Note that we are still assuming perfect foresight about the path, as in the case of exponentially

[16] Technically we also multiply ϵ^r by a very slow exponential decay to ensure that the economy converges towards the deterministic steady state as $t \to \infty$.

Table 2: Standard deviations of inflation and consumption.

Number of firms n	Model	Std. dev. of π (%)		Std. dev. of c (%)	
		ϵ^r	ϵ^m	ϵ^r	ϵ^m
∞	Standard NK (CES)	2.2	2.7	0.8	1.0
∞	Klenow-Willis $\theta = 10$	2.0	2.4	1.0	1.3
3	MPE	1.4	1.8	0.8	1.0
	Non-strategic	1.7	2.1	1.1	1.4
10	MPE	2.3	2.8	1.1	1.4
	Non-strategic	2.7	3.3	1.4	1.7
25	MPE	2.7	3.3	1.1	1.4
	Non-strategic	2.8	3.5	1.2	1.5

decaying shocks. Figure 15 shows a sample path for real shocks ϵ^r, and Table 2 shows the results for the two kinds of shocks. Here we see that the standard deviations of inflation and consumption are smaller in the oligopolistic model than in the corresponding non-strategic model. The higher-order terms in the oligopolistic Phillips curve smooth out the path for inflation, which in turn makes the real rate and consumption less volatile. This example demonstrates that the strong equivalence between oligopoly and Kimball economies that we observe in the case of the literature's benchmark shocks (permanent money supply shocks and exponentially decaying interest rate shocks) does not necessarily transpose to more general processes.

9 Conclusion

In this paper, we studied how oligopolistic competition affects monetary policy transmission. We showed a closed-form formula for the response of aggregate output to monetary shocks as a function of three measurable sufficient statistics: demand elasticities, market concentration, and markups. Under our calibration, oligopolistic competition amplifies monetary non-neutrality, but, in the case of the standard shocks to money supply or interest rates studied in the literature, the response approximates a monopolistic competition model with Kimball demand that matches the residual demand elasticity and superelasticity of the oligopolistic model.

This does not imply, however, that oligopoly is isomorphic to monopolistic competition. First, a unique prediction of our model is the link between markups and subtle properties of demand functions such as superelasticities. Under monopolistic competition, superelasticities affect cost pass-through and thus monetary policy, but are irrelevant for markups. Under oligopolistic competition, superelasticities have a positive effect on both markups and cost pass-through. Second, in the context of our three-equations oligopolistic New Keynesian model that allows for more general shocks and non-stationary dynamics, we find that the oligopolistic model can depart significantly from the recalibrated monopolistic model. In particular, the oligopolistic Phillips curve features a form of endogenous inflation persistence (or equivalently, endogenous cost-push shocks) that does not matter with standard shocks, but plays a role once we allow for richer dynamics.

Our calibration relies on estimates of exchange rate pass-through, as we believe they are the most relevant sources of information in the case of strategic interactions. In the menu costs literature, it is more common to target moments of the distribution of price changes. The open economy literature on pass-through and the closed economy monetary literature have thus evolved mostly in parallel, with different conclusions regarding the strength of strategic complementarities in pricing. Our framework provides a natural way to reconcile these two strands: larger firms have more market power, only pass through a fraction of their idiosyncratic shocks, but drive most of the aggregate price stickiness. An interesting avenue for future empirical work would be to analyze how the distribution of price changes depends on firm size and market share.

References

Alvarez, Fernando and Francesco Lippi, "Price Setting With Menu Cost for Multiproduct Firms," *Econometrica*, 2014, *82* (1), 89–135.

__ , __ , and Juan Passadore, "Are State- and Time-Dependent Models Really Different?," in "NBER Macroeconomics Annual 2016, Volume 31" University of Chicago Press November 2016, pp. 379–457.

__ , Hervé Le Bihan, and Francesco Lippi, "The Real Effects of Monetary Shocks in Sticky Price Models: A Sufficient Statistic Approach," *American Economic Review*, October 2016, *106* (10), 2817–51.

Amiti, Mary, Oleg Itskhoki, and Jozef Konings, "International Shocks, Variable Markups, and Domestic Prices," *The Review of Economic Studies*, 02 2019.

Atkeson, Andrew and Ariel Burstein, "Pricing-to-Market, Trade Costs, and International Relative Prices," *American Economic Review*, December 2008, *98* (5), 1998–2031.

Atkin, David, Benjamin Faber, and Marco Gonzalez-Navarro, "Retail Globalization and Household Welfare: Evidence from Mexico," *Journal of Political Economy*, 2018, *126* (1), 1–73.

Autor, David, David Dorn, Lawrence F. Katz, Christina Patterson, and John Van Reenen, "Concentrating on the Fall of the Labor Share," *American Economic Review*, May 2017, *107* (5), 180–85.

Barkai, Simcha, "Declining Labor and Capital Shares," *The Journal of Finance*, 2020.

Berger, David and Joseph Vavra, "Shocks versus Responsiveness: What Drives Time-Varying Dispersion?," *Journal of Political Economy*, 2019, *127* (5), 2104–2142.

Berman, Nicolas, Philippe Martin, and Thierry Mayer, " How do Different Exporters React to Exchange Rate Changes? *," *The Quarterly Journal of Economics*, 01 2012, *127* (1), 437–492.

Burstein, Ariel, Vasco Carvalho, and Basile Grassi, "Bottom-Up Markup Fluctuations," working paper 2020.

Busso, Matias and Sebastian Galiani, "The Causal Effect of Competition on Prices and Quality: Evidence from a Field Experiment," *American Economic Journal: Applied Economics*, January 2019, *11* (1), 33–56.

Carvalho, Carlos, "Heterogeneity in Price Stickiness and the Real Effects of Monetary Shocks," *The B.E. Journal of Macroeconomics*, December 2006, *6* (3), 1–58.

Gertler, Mark and John Leahy, "A Phillips Curve with an Ss Foundation," *Journal of Political Economy*, 2008, *116* (3), 533–572.

Golosov, Mikhail and Robert E. Lucas, "Menu Costs and Phillips Curves," *Journal of Political Economy*, 2007, *115* (2), 171–199.

Gopinath, Gita and Oleg Itskhoki, "Frequency of Price Adjustment and Pass-Through*," *The Quarterly Journal of Economics*, 05 2010, *125* (2), 675–727.

Gutiérrez, Germán and Thomas Philippon, "Declining Competition and Investment in the U.S.," Working Paper 23583, National Bureau of Economic Research July 2017.

Jun, Byoung and Xavier Vives, "Strategic incentives in dynamic duopoly," *Journal of Economic Theory*, 2004, *116* (2), 249 – 281.

Kimball, Miles S., "The Quantitative Analytics of the Basic Neomonetarist Model," *Journal of Money, Credit and Banking*, 1995, *27* (4), 1241–1277.

Klenow, Peter J. and Jonathan L. Willis, "Real Rigidities and Nominal Price Changes," *Economica*, 2016, *83* (331), 443–472.

Krugman, Paul, "Pricing to Market when the Exchange Rate Changes," Working Paper 1926, National Bureau of Economic Research May 1986.

Krusell, Per, Burhanettin Kuruscu, and Anthony A. Smith, "Equilibrium Welfare and Government Policy with Quasi-geometric Discounting," *Journal of Economic Theory*, 2002, *105* (1), 42 – 72.

Levintal, Oren, "Taylor Projection: A New Solution Method for Dynamic General Equilibrium Models," *International Economic Review*, 2018, *59* (3), 1345–1373.

Maskin, Eric and Jean Tirole, "A Theory of Dynamic Oligopoly, II: Price Competition, Kinked Demand Curves, and Edgeworth Cycles," *Econometrica*, 1988, *56* (3), 571–599.

Midrigan, Virgiliu, "Menu Costs, Multiproduct Firms, and Aggregate Fluctuations," *Econometrica*, 2011, *79* (4), 1139–1180.

Mongey, Simon, "Market structure and monetary non-neutrality," working paper 2018.

Rossi-Hansberg, Esteban, Pierre-Daniel Sarte, and Nicholas Trachter, "Diverging Trends in National and Local Concentration," *NBER Macro Annual*, 2020.

Rotemberg, Julio J. and Garth Saloner, "A Supergame-Theoretic Model of Price Wars during Booms," *American Economic Review*, 1986, *76* (3), 390–407.

_ **and** _ , "The Relative Rigidity of Monopoly Pricing," *American Economic Review*, December 1987, *77* (5), 917–926.

_ **and Michael Woodford**, "Oligopolistic Pricing and the Effects of Aggregate Demand on Economic Activity," *Journal of Political Economy*, 1992, *100* (6), 1153–1207.

Smets, Frank and Rafael Wouters, "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, June 2007, *97* (3), 586–606.

Weintraub, Gabriel Y., C. Lanier Benkard, and Benjamin Van Roy, "Markov Perfect Industry Dynamics With Many Firms," *Econometrica*, 2008, *76* (6), 1375–1411.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.

Appendix

A Additional Figures

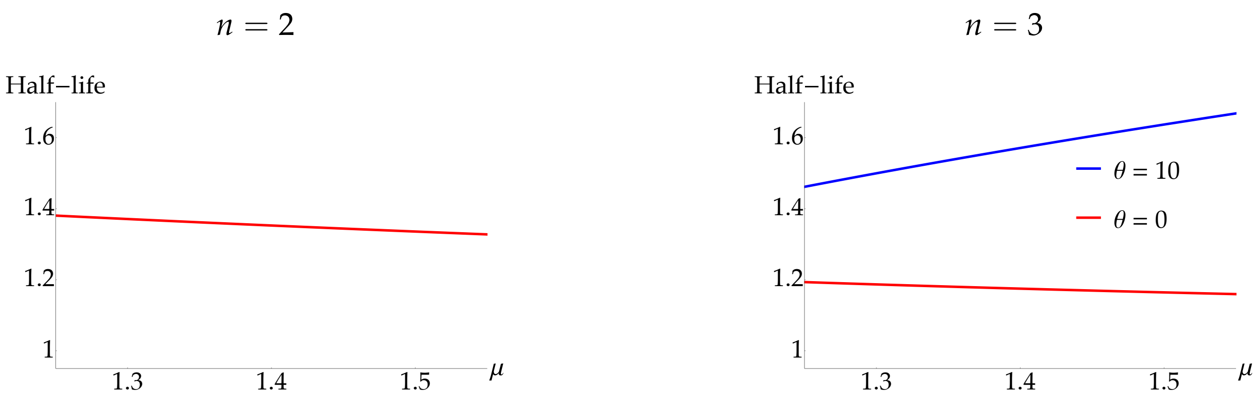


Figure 11: Half-life as a function of resulting steady state markup when η varies.

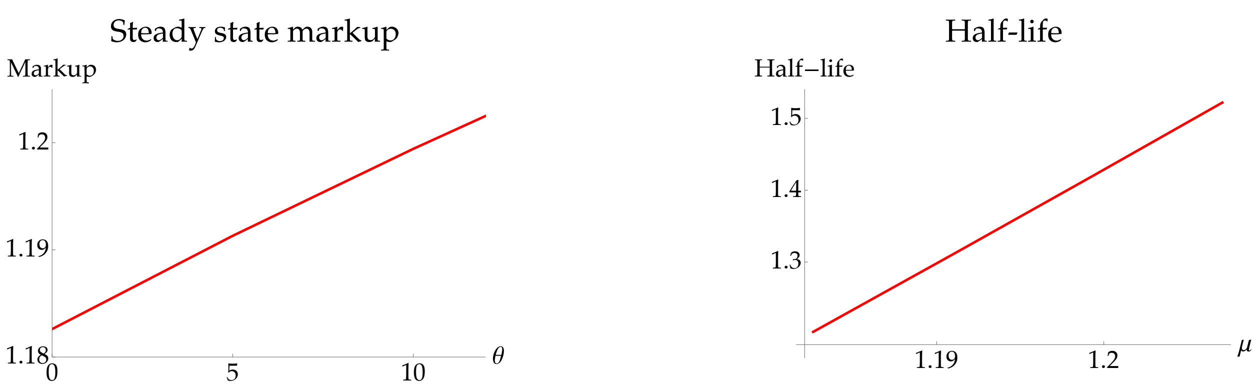


Figure 12: Markup and half-life when θ varies in a model with $n = 3$ and $\eta = 10$.

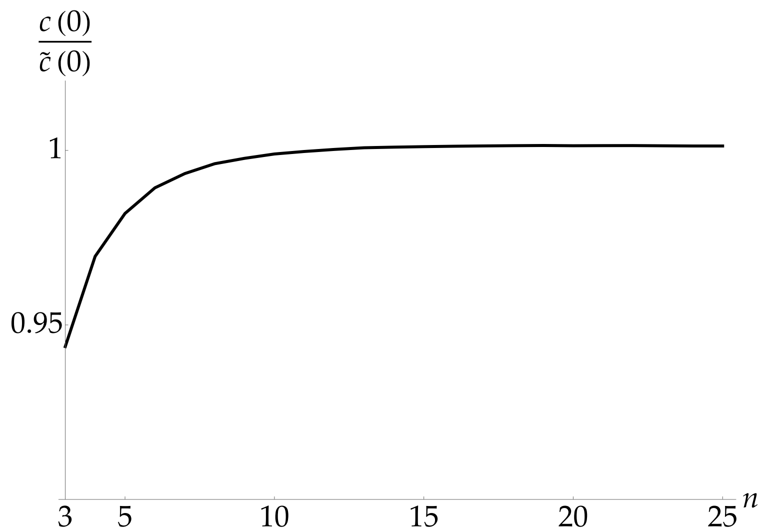


Figure 13: Impact effect of a $\epsilon_0^m = -1\%$ monetary shock on consumption relative to non-strategic model $c(0)/\tilde{c}(0)$ as a function of number of firms n under AIK calibration.

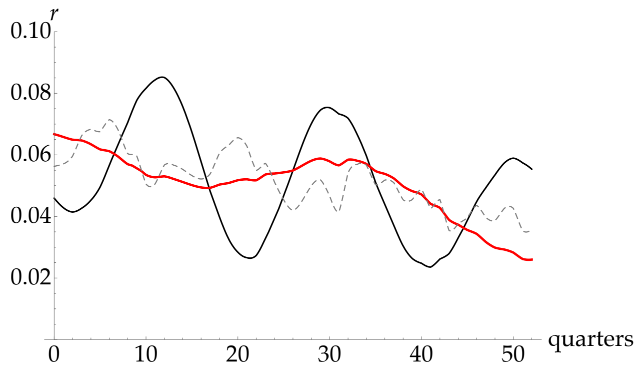


Figure 14: Real interest rate under oligopoly (red), standard New Keynesian model (black) given a path of natural rate r^n (dashed gray).

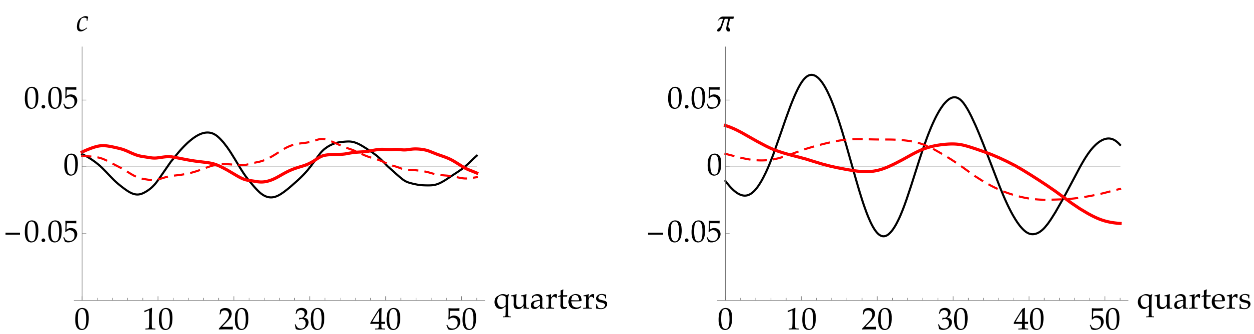


Figure 15: Consumption and inflation under oligopoly (red), non-strategic model (dashed red), and standard New Keynesian model (black).

B Stationary Dynamics after a Permanent M shock

If the consumer maximizes

$$\int e^{-\rho t}\left[\frac{C(t)^{1-\sigma}}{1-\sigma}-\frac{N(t)^{1+\psi}}{1+\psi}+\frac{m(t)^{1-\chi}}{1-\chi}\right]dt$$

we have

$$\begin{aligned}\frac{\dot{C(t)}}{C(t)} &= \frac{1}{\sigma}\left(i(t)-\pi(t)-\rho\right)\\ N(t)^{\psi}C(t)^{\sigma} &= \frac{W(t)}{P(t)} \Rightarrow \psi\frac{\dot{N(t)}}{N(t)} = \frac{\dot{W(t)}}{W(t)} - i(t)+\rho\\ M(t)^{-\chi}P(t)^{\chi}C(t)^{\sigma} &= i(t)\end{aligned}$$

We look for an equilibrium with constant nominal interest rate $i(t)=i$ and nominal wage $W(t)=W$ following a permanent shock to M. Suppose $\psi=0$ then we get

$$\frac{\dot{W(t)}}{W(t)} = i-\rho$$

To get constant wage $W(t)=W$ we need $i=\rho$ (this seems necessary, otherwise we would get permanent wage inflation). The constant wage implies

$$P(t)C(t)^{\sigma}=W$$

Then the third equation gives

$$\rho M^{\chi} = P(t)^{\chi}C(t)^{\sigma}$$

So we need $\chi=1$ for our guess to be indeed an equilibrium.

The representative consumer's expenditure in sector s at time t is

$$E_s(t) = P_s(t)^{1-\omega}\left[C(t)P(t)^{\omega}\right]$$

where $P(t)$ is the aggregate price level $\left(\int_s P_s(t)^{1-\omega}ds\right)^{\frac{1}{1-\omega}}$ hence the real demand vector in sector s is (given our within-sector CRS assumption as in Kimball)

$$d\left(\left\{p_{j,s}(t)\right\}, E_s(t)\right) = d\left(\left\{p_{j,s}(t)\right\},1\right)P_s(t)^{-\omega}C(t)P(t)^{\omega}$$

where we have seen that $P_s = \frac{1}{h_s(d(\{p_s\},1))}$ where $h_s(x)$ is defined by the Kimball aggregator

$$\frac{1}{n}\sum_i \phi\left(\frac{x_i}{h}\right)=1$$

P_s solves

$$\frac{1}{n}\sum_i \phi \circ \left(\phi'\right)^{-1}\left(\phi'(1)\frac{p_{i,s}}{P_s}\right) = 1$$

Denote the function of prices in sector s only

$$D\left(\left\{p_{j,s}\right\}\right) = d\left(\left\{p_{j,s}\right\},1\right) P_s{}^{-\omega}$$

The nominal profit of firm i in sector s given all the other prices in the economy is

$$D\left(\left\{p_{j,s}\right\}\right) C(t)P(t)^{\omega}\left[p_{i,s} - MC^i(t)\right]$$

where $p_{-i,s} = \left\{p_{j,s}\right\}_{j\neq i}$. Thus the real profit is

$$D\left(\left\{p_{j,s}\right\}\right) C(t)P(t)^{\omega-1}\left[p_{i,s} - MC^i(t)\right]$$

Firms maximize the present discounted value of this using Arrow-Debreu SDF, which involves marginal utility, that is

$$\begin{aligned}
&\int e^{-\rho t} C(t)^{-\sigma} D\left(\left\{p_{j,s}\right\}\right) C(t)P(t)^{\omega-1}\left[p_{i,s} - MC^i(t)\right] \\
=&\int e^{-\rho t} D\left(\left\{p_{j,s}\right\}\right) C(t)^{1-\sigma}P(t)^{\omega-1}\left[p_{i,s} - MC^i(t)\right]
\end{aligned}$$

so with $\sigma = 1$ and $\omega = 1$, firms can ignore the behavior of aggregate variables $P(t)$ and $C(t)$.

With general σ (but linear disutility of labor and log-utility of real balances, that are needed to obtain constant nominal interest rate and wage) we have that

$$P(t)C(t)^{\sigma} = W = \text{constant}$$

Therefore the demand shifter becomes

$$C(t)^{1-\sigma}P(t)^{\omega-1} = \frac{C(t)P(t)^{\omega}}{W} = W^{\frac{1}{\sigma}-1}P(t)^{\omega-\frac{1}{\sigma}}$$

so we need

$$\omega\sigma = 1$$

for firms to ignore the behavior of aggregate variables during the transition to the new steady state.

C Aggregation

C.1 Homogeneous Firms

Fix n and a sector $s \in [0,1]$. Define the state $v_s(t)$ as

$$v_s = (z_1, \dots, z_n)'$$

where $z_i = p_i - \bar{p}$ (prices are in log). Denote first-order expansions of best responses by $p_i' = \alpha + \beta \left(\sum_{j \neq i} p_j\right)$ or equivalently $z_i' = \beta \left(\sum_{j \neq i} z_j\right)$. When firm i adjusts its price, the state of sector s changes to $v_s'(t) = M_i v_s(t)$ where M_i is the identity matrix except for row i which is equal to $(\beta, \dots, \beta, \underset{\substack{\uparrow \\ i}}{0}, \beta, \dots, \beta)$.

Define the aggregate state variable

$$V(t) = \int_{s \in [0,1]} v_s(t) ds \in \mathbb{R}^n$$

Between t and $t + \Delta t$, a mass $n\lambda\Delta t$ of firms adjusts prices so V evolves as

$$\begin{aligned} V(t + \Delta t) &= (1 - n\lambda\Delta t)V(t) + \int_{\text{a firm in } s \text{ adjusts}} v_s(t + \Delta t) ds \\ &= (1 - n\lambda\Delta t)V(t) + (\lambda n \Delta t)\frac{\sum_i M_i}{n} V(t) \end{aligned}$$

therefore in the limit $\Delta t \to 0$

$$\dot{V}_t = n\lambda \left(\frac{\sum_i M_i}{n} - I_n\right) V_t$$

where

$$\frac{\sum_i M_i}{n} - I_n = \begin{pmatrix} \frac{-1}{n} & \frac{\beta}{n} & \dots & \frac{\beta}{n} \\ \frac{\beta}{n} & \frac{-1}{n} & \dots & \frac{\beta}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta}{n} & \frac{\beta}{n} & \dots & \frac{-1}{n} \end{pmatrix}$$

The aggregate price level is then, to first order, $\log P(t) = LV_t + \bar{p}$ where $L = \frac{1}{n}(1, \dots, 1)$. The eigenvalues of $n\lambda \left(\frac{\sum_i M_i}{n} - I_n\right)$ are:

- $\mu_1(n) = -\lambda(1 + \beta(n))$ with multiplicity $n - 1$,
- $\mu_2(n) = -\lambda[1 - (n-1)\beta(n)]$ with multiplicity 1.

The vector $(1, \dots, 1)'$ is an eigenvector of $\mu_2(n)$, so if we start from symmetric initial conditions

$$V(0) = (p_0 - \bar{p}, \dots, p_0 - \bar{p})$$

we have

$$V(t) = V(0)e^{\mu_2(n)t}$$

hence, to first order,

$$\log P(t) = \log \bar{P} + (\log P\,(0) - \log \bar{P})\, e^{\mu_2(n)t}.$$

With heterogeneous sectors s differing in the number of firms n_s (and potentially the frequency of price adjustment captured by λ_s) we can just use the previous steps for each positive mass ω_n of sectors with n firms and aggregate to

$$\log P(t) = \log \bar{P} + (\log P\,(0) - \log \bar{P})\, e^{\sum_n \omega_n \mu_2(n)t}.$$

C.2 Heterogeneous Firms

Suppose there are two types of firms a and b with $n_a + n_b = n$. In general, we need to solve for four steady state objects:

$$g^{i,a}_{j_a}, g^{i,a}_{j_b}, g^{i,b}_{j_a}, g^{i,b}_{j_b}$$

Firms of type a's Bellman equation is

$$\begin{aligned}(\rho + n\lambda)\, V^{i,a}\,(p) = \Pi^{i,a}\,(p) + \lambda V^{i,a}\left(g^{i,a}\,(p_{-i})\,, p_{-i}\right) \\ + \lambda \left\{ \sum_{j \in A} V^{i,a}\left(g^{j,a}\,(p_{-j})\,, p_{-j}\right) + \sum_{j \in B} V^{i,a}\left(g^{j,b}\,(p_{-j})\,, p_{-j}\right) \right\}\end{aligned}$$

and similarly for firms of type b. The envelope conditions evaluated at a symmetric steady state p^a, p^b for firms of type a are

$$\begin{aligned}(\rho + n\lambda)\, V^{i,a}_i = \Pi^{i,a}_i + \lambda \sum_{j \neq i} \left[V^i_i(g_j(p_{-j}), p_{-j}) + V^i_j(g_j(p_{-j}), p_{-j}) g^j_i(p_{-j}) \right] \\ 0 = \Pi^{i,a}_i + \lambda\,(n_a - 1) \left[V^{i,a}_i + V^{i,a}_{j_a} g^{j,a}_{i_a} \right] + \lambda n_b \left[V^{i,a}_i + V^{i,a}_{j_b} g^{j,b}_{i_a} \right]\end{aligned}$$

$$
\begin{aligned}
(\rho+n\lambda)V^{i,a}_{k_a} &= \Pi^{i,a}_{k_a} + \lambda \sum_{j\neq k_a} \left[V^{i,a}_{p_j}(g_j(p_{-j}),p_{-j})\frac{\partial g_j}{\partial p_k} + V^{i,a}_{p_k}(g_j(p_{-j}),p_{-j}) \right] \quad \forall k\neq i \\
&= \Pi^{i,a}_{k_a} + \lambda\left(n_a-2\right)\left[V^{i,a}_{j_a} g^{j,a}_{k_a} + V^{i,a}_{k_a}\right] + \lambda\left[V^{i,a}_{i_a} g^{i,a}_{k_a} + V^{i,a}_{k_a}\right] + \lambda n_b \left[V^{i,a}_{j_b} g^{j,b}_{k_a} + V^{i,a}_{k_a}\right] \\
(\rho+n\lambda)V^{i,a}_{k_b} &= \Pi^{i,a}_{k_b} + \lambda \sum_{j\neq k_b} \left[V^{i,a}_{p_j}(g_j(p_{-j}),p_{-j})\frac{\partial g_j}{\partial p_k} + V^{i,a}_{p_k}(g_j(p_{-j}),p_{-j}) \right] \quad \forall k\neq i \\
&= \Pi^{i,a}_{k_b} + \lambda\left(n_a-1\right)\left[V^{i,a}_{j_a} g^{j,a}_{k_b} + V^{i,a}_{k_b}\right] + \lambda\left[V^{i,a}_{i_a} g^{i,a}_{k_b} + V^{i,a}_{k_b}\right] + \lambda\left(n_b-1\right)\left[V^{i,a}_{j_b} g^{j,b}_{k_b} + V^{i,a}_{k_b}\right]
\end{aligned}
$$

hence by symmetry and using the FOC $V^i_i = 0$ we have:

$$
\begin{aligned}
\left(\rho+\lambda\right)V^{i,a}_{j_a} &= \Pi^{i,a}_{j_a} + \lambda\left(n_a-2\right)V^{i,a}_{j_a} g^{i,a}_{j_a} + \lambda n_b V^{i,a}_{j_b} g^{i,b}_{j_a} \\
(\rho+\lambda)V^{i,a}_{j_b} &= \Pi^{i,a}_{j_b} + \lambda\left(n_a-1\right)V^{i,a}_{j_a} g^{i,a}_{j_b} + \lambda\left(n_b-1\right)V^{i,a}_{j_b} g^{i,b}_{j_b}
\end{aligned}
$$

and the equivalent equations for b:

$$
\begin{aligned}
\left(\rho+\lambda\right)V^{i,b}_{j_b} &= \Pi^{i,b}_{j_b} + \lambda\left(n_b-2\right)V^{i,b}_{j_b} g^{i,b}_{j_b} + \lambda n_a V^{i,b}_{j_a} g^{i,a}_{j_b} \\
(\rho+\lambda)V^{i,b}_{j_a} &= \Pi^{i,b}_{j_a} + \lambda\left(n_b-1\right)V^{i,b}_{j_b} g^{i,b}_{j_a} + \lambda\left(n_a-1\right)V^{i,b}_{j_a} g^{i,a}_{j_a}
\end{aligned}
$$

This is a linear system of 4 equations in 4 unknowns $\left\{V^{i,a}_{j_a}, V^{i,a}_{j_b}, V^{i,b}_{j_a}, V^{i,b}_{j_b}\right\}$; we can then inject the solutions into

$$
\begin{aligned}
0 &= \Pi^{i,a}_{i} + \lambda\left(n_a-1\right)V^{i,a}_{j_a} g^{i,a}_{j_a} + \lambda n_b V^{i,a}_{j_b} g^{j,b}_{i_a} \\
0 &= \Pi^{i,b}_{i} + \lambda\left(n_b-1\right)V^{i,b}_{j_b} g^{i,b}_{j_b} + \lambda n_a V^{i,b}_{j_a} g^{j,a}_{i_b}
\end{aligned}
$$

In general, we cannot solve for the slopes as functions of steady state elasticities.

However, when $n_a = n_b = 1$, we obtain the formulas in Proposition 5. Then

$$
0 = \Pi^{i,a}_{i} + \lambda \frac{\Pi^{i,a}_{j_b}}{\rho+\lambda} g^{j,b}_{i_a}
$$

which leads to Proposition 5 after simplifying

$$
\begin{aligned}
\frac{-\Pi^{i,a}_{i}}{\Pi^{i,a}_{j_b}} &= -\frac{d^{i,a}_{i}\left(p_a - MC_a\right) + d^{i,a}}{d^{i,a}_{b}\left(p_a - MC_a\right)} \\
&= \frac{1}{\epsilon^a_b}\left[-\epsilon^a_a - \frac{p_b}{p_a - MC_a}\right]
\end{aligned}
$$

As before, $V(t) = \int_{s\in[0,1]} v_s(t)ds$ follows

$$\dot{V}(t) = \lambda \begin{pmatrix} -1 & \beta^a \\ \beta^b & -1 \end{pmatrix} V(t).$$

The two eigenvalues are $\mu_+ = -\lambda\left(1+\sqrt{\beta^a\beta^b}\right)$ and $\mu_- = -\lambda\left(1-\sqrt{\beta^a\beta^b}\right)$. Hence the solution is

$$\begin{aligned} V(t) = & \frac{\sqrt{\beta^a}\left(p_b(0)-p_b^*\right)-\sqrt{\beta^b}\left(p_a(0)-p_a^*\right)}{2}\begin{pmatrix} \frac{-1}{\sqrt{\beta^b}} \\ \frac{1}{\sqrt{\beta^a}} \end{pmatrix} e^{\mu_+ t} \\ & + \frac{\sqrt{\beta^b}\left(p_a(0)-p_a^*\right)+\sqrt{\beta^a}\left(p_b(0)-p_b^*\right)}{2}\begin{pmatrix} \frac{1}{\sqrt{\beta^b}} \\ \frac{1}{\sqrt{\beta^a}} \end{pmatrix} e^{\mu_- t} \end{aligned}$$

hence (supposing the economy only features such sectors)

$$\begin{aligned} \frac{\log P(t)-\log\bar{P}}{\log P(0)-\log\bar{P}} = & \left[\frac{1-S_a}{\sqrt{\beta^a}}-\frac{S_a}{\sqrt{\beta^b}}\right]\left(\frac{\sqrt{\beta^a}-\sqrt{\beta^b}}{2}\right)e^{\mu_+ t} \\ & + \left[\frac{1-S_a}{\sqrt{\beta^a}}+\frac{S_a}{\sqrt{\beta^b}}\right]\left(\frac{\sqrt{\beta^a}+\sqrt{\beta^b}}{2}\right)e^{\mu_- t}. \end{aligned}$$

where S_a is the steady state market share of type a firms.

D Non-Strategic Model

The quadratic approximation of profit Π^i of firm i around the non-strategic steady state which is the static Nash p^{NE} writes (in log deviations)

$$\pi^i(p_i, Q_i, R_i) = BQ_i + CQ_i^2 + Dp_iQ_i + Ep_i^2 + FR_i$$

where

$$Q_i = \sum_{j\neq i} p_j$$
$$R_i = \sum_{j\neq i} p_j^2$$

There is no term Ap_i because we are approximate around the Nash price $p^{NE}(n)$ where $\Pi_i^i = 0$ for all i. The most important coefficients D and E are

$$D = \Pi_{ij}\left(p^{NE}(n)\right)$$

$$E = \frac{\Pi_{ii}}{2}\left(p^{NE}(n)\right)$$

We look for a symmetric equilibrium where each resetting firm j sets

$$p_j^*(t) = \beta Q_j(t)$$

Then between s and $s + \Delta s$ we have

$$\mathbf{E}_t Q_i(s+\Delta s) = (1-(n-1)\lambda\Delta)\,\mathbf{E}_t Q_i(s) + \lambda\Delta\mathbf{E}_t \sum_{j\neq i}\left[Q_i(s) - p_j(s) + \beta Q_j(s)\right]$$

hence taking the limit $\Delta s \to 0$

$$\frac{d}{ds}\mathbf{E}_t Q_i(s) = \lambda\left\{\beta\sum_{j\neq i}\mathbf{E}_t Q_j(s) - \mathbf{E}_t Q_i(s)\right\}$$

thus the variable $Z(s) = \sum_i \mathbf{E}_t Q_i(s)$ follows

$$\frac{d}{ds}Z(s) = -\lambda\left[1-\beta(n-1)\right]Z(s) \tag{23}$$

Therefore, by symmetry

$$\mathbf{E}_t Q_i(s) = Q_i(t)e^{-\lambda[1-\beta(n-1)](s-t)}$$

When it resets, firm i chooses $p_i^*(t)$ such that

$$\max_{p_i^*(t)} \mathbf{E}_t\left[\int_t^\infty e^{-(\lambda+\rho)(s-t)}\pi^i(p_i^*(t), Q_i(t+s), R_i(t+s))ds\right]$$

The FOC is

$$\begin{aligned}
p_i^*(t) &= -\frac{\int_t^\infty e^{-(\lambda+\rho)(s-t)}D\mathbf{E}_t\left[Q_i(s)\right]ds}{\int_t^\infty e^{-(\lambda+\rho)s}2Eds} \\
&= -\frac{\int_t^\infty e^{-(\lambda+\rho)(s-t)}\left(DQ_i(t)e^{-\lambda(1-(n-1)\beta)(s-t)}\right)ds}{\int_t^\infty e^{-(\lambda+\rho)(s-t)}2Eds} \\
&= -\frac{D(\lambda+\rho)}{2E\left[\lambda+\rho+\lambda(1-(n-1)\beta)\right]}Q_i(t)
\end{aligned}$$

So we need

$$(n-1)\beta = \left(\frac{(n-1)D}{-2E}\right)\frac{1}{1+\frac{\lambda}{\rho+\lambda}\left[1-(n-1)\beta\right]}$$
$$= \left(\frac{(n-1)\Pi_{ij}}{-\Pi_{ii}}\right)\frac{1}{1+\frac{\lambda}{\rho+\lambda}\left[1-(n-1)\beta\right]}$$

Note that in a static model, the ratio $\frac{(n-1)\Pi_{ij}}{-\Pi_{ii}}$ would be the slope of the static best response to a simultaneous price change by all firms $j \neq i$ and we need it to be strictly lower than 1 for a static symmetric Nash equilibrium to exist. The slope of the dynamic non-strategic best response at a stable steady state, if one exists, is always smaller than the slope of the static best response. Thus we already see a form of dynamic complementarity. n affects demand functions and hence the level of the non-strategic steady state, just like it affects the level of the static Nash equilibrium (they are the same). n also affects profit complementarities (potentially in an independent way, away from CES) and thereby the slope of the reaction functions in the static and dynamic (non-strategic) models. But there is a stable relation between the two across n, described by the solution below.

The second-order polynomial

$$X^2 - \left(\frac{\rho+2\lambda}{\lambda}\right)X + \left(\frac{\rho+\lambda}{\lambda}\right)\left(\frac{(n-1)D}{-2E}\right)$$

has a real root if

$$\frac{(n-1)D}{-2E} < \frac{(\rho+2\lambda)^2}{4\lambda(\rho+\lambda)} = 1 + \frac{\rho^2}{4\lambda(\rho+\lambda)}$$

The stable root in $(0,1)$ can only be

$$(n-1)\beta = \left(\frac{\rho+2\lambda}{2\lambda}\right)\left[1-\sqrt{1-4\left(\frac{(n-1)D}{-2E}\right)\frac{\lambda(\rho+\lambda)}{(\rho+2\lambda)^2}}\right]$$

E Kimball Elasticities

In what follows recall that we assume an outer elasticity $\omega = 1$. From budget exhaustion, for any i and p

$$c^i + \sum_j p_j \frac{\partial c^j}{\partial p_i} = 0 \tag{24}$$

Then Slutsky symmetry and constant returns to scale imply

$$\epsilon_i^i + \sum_{j \neq i} \epsilon_j^i = -1 \tag{25}$$

where $\epsilon_j^i = \frac{\partial \log c^i}{\partial \log p_J}$. At a symmetric price, this becomes

$$\epsilon_j^i = -\frac{1 + \epsilon_i^i}{n - 1}$$

so the convergence to Nash holds as long as the own elasticity ϵ_i^i is bounded. Call for any pair j, k

$$\epsilon_{jk}^i = \frac{\partial^2 \log d_i}{\partial \log p_k \partial \log p_j}$$

We can differentiate (25) with respect to $\log p_i$ to get

$$\epsilon_{ii}^i + \sum_{j \neq i} \epsilon_{ij}^i = 0$$

hence at a symmetric price,

$$\epsilon_{ii}^i + (n-1)\epsilon_{ij}^i = 0$$

Differentiating once more the budget constraint with respect to p_i

$$2\frac{\partial c^i}{\partial p_i} + \sum_j \frac{\partial^2 c^j}{\partial p_i^2} = 0 \tag{26}$$

Elasticities and second-derivatives are related by

$$\frac{\partial^2 c^i}{\partial p_k \partial p_j} = \frac{c^i}{p_k p_j}\left[\epsilon_{jk}^i + \epsilon_j^i \epsilon_k^i\right] \text{ for any } j \neq k$$

$$\frac{\partial^2 c^i}{\partial p_j^2} = \frac{c^i}{p_j^2}\left[\epsilon_{jj}^i - \epsilon_j^i + \left(\epsilon_j^i\right)^2\right] \text{ for any } j$$

At a symmetric price (using $\epsilon_{ii}^j = \epsilon_{jj}^i$), we have from (26)

$$\epsilon_{jj}^i = \epsilon_j^i\left(1 - \epsilon_j^i\right) - \frac{1}{n-1}\left[\epsilon_{ii}^i + \epsilon_i^i\left(1 + \epsilon_i^i\right)\right] \tag{27}$$

Finally, differentiating (24) with respect to p_k for some $k \neq i$ gives

$$\frac{\partial c^i}{\partial p_k} + \frac{\partial c^k}{\partial p_i} + \sum_{j \neq i,k} p_j \frac{\partial^2 c^j}{\partial p_k \partial p_i} + p_i \frac{\partial^2 c^i}{\partial p_k \partial p_i} + p_k \frac{\partial^2 c^k}{\partial p_k \partial p_i} = 0$$

and at a symmetric price p

$$\frac{2}{p}\frac{\partial c^i}{\partial p_k} + (n-2)\frac{\partial^2 c^i}{\partial p_k \partial p_j} + 2\frac{\partial^2 c^i}{\partial p_k \partial p_i} = 0$$

Therefore, in elasticities at a symmetric price,

$$2\epsilon^i_j + (n-2)\left[\epsilon^i_{jk} + \left(\epsilon^i_j\right)^2\right] + 2\left[\epsilon^i_{ij} + \epsilon^i_j \epsilon^i_i\right] = 0 \tag{28}$$

for $k \neq j, i$, $j \neq i$. The own-superelasticity is defined as the elasticity of (minus the) elasticity:

$$\Sigma_i = \frac{\partial \log(-\epsilon^i_i)}{\partial \log p_i} = \frac{\epsilon^i_{ii}}{\epsilon^i_i}$$

So in the end we have two degrees of freedom: $\{\epsilon^i_i, \epsilon^i_{ii}\}$ to parametrize a symmetric steady state.

Special case: $n = 2$. If $n = 2$ there is only 1 degree of freedom, so CES is without loss of generality (locally). From (28), the cross-superelasticity ϵ^i_{ij}, hence the own-superelasticity $\epsilon^i_{ii} = -(n-1)\epsilon^i_{ij}$ is determined by elasticities.

E.1 Special case: CES

With CES utility

$$h(x) = \left(\frac{1}{n}\sum_{j=1}^{n} x_j^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}$$

we have only one degree of freedom $\epsilon > 1$ and at any symmetric price

$$\begin{aligned}
\epsilon^i_i &= -\epsilon + \frac{\epsilon - 1}{n} \\
\epsilon^i_{ii} &= -(\epsilon-1)^2 \frac{n-1}{n^2} \\
\epsilon^i_{jj} &= \epsilon^i_{ii}
\end{aligned}$$

which implies from the equalities above

$$\begin{aligned}
\epsilon^i_j &= \frac{\epsilon - 1}{n} \\
\epsilon^i_{jk} &= \frac{(\epsilon-1)^2}{n^2}
\end{aligned}$$

E.2 Special case: Kimball (1995)

Start with a general Kimball aggregator that defines C as

$$\frac{1}{n}\sum_i \Psi\left(\frac{c_i}{C}\right) = 1 \tag{29}$$

where Ψ is increasing, concave, and $\Psi(1) = 1$ which ensures the convention that at a symmetric basket $c_i = c$, we have $C = c$. The consumer's problem is

$$\min_{\{c_i\}} \sum_i p_i c_i \text{ s.t. } \frac{1}{n}\sum_i \Psi\left(\frac{c_i}{C}\right) = 1$$

There exists a Lagrange multiplier $\lambda > 0$ such that for all i

$$p_i = \lambda \Psi'\left(\frac{c_i}{C}\right)\frac{1}{C} \tag{30}$$

If we define the sectoral price index P by

$$\frac{1}{n}\sum_i \phi\left(\Psi'(1)\frac{p_i}{P}\right) = 1$$

where

$$\phi = \Psi \circ (\Psi')^{-1}$$

then at a symmetric price $p_i = p$ we have $P = p$, and $\lambda\Psi'(1) = PC$ so we can rewrite (30) as

$$\frac{p_i}{P}\Psi'(1) = \Psi'\left(\frac{c_i}{C}\right)$$

Taking logs and differentating (30) with respect to $\log p_i$ yields

$$1 = \frac{\partial \log P}{\partial \log p_i} + \frac{\Psi''\left(\frac{c_i}{C}\right)}{\Psi'\left(\frac{c_i}{C}\right)}\frac{c_i}{C}\left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i}\right]$$

Differentiating (29) yields

$$\sum_j \Psi'\left(\frac{c_j}{C}\right)\frac{c_j}{C}\left[\frac{\partial \log c_j}{\partial \log p_i} - \frac{\partial \log C}{\partial \log p_i}\right] = 0$$

hence

$$\frac{\partial \log C}{\partial \log p_i} = \frac{\sum_j \Psi'\left(\frac{c_j}{C}\right)\frac{c_j}{C}\epsilon_i^j}{\sum_j \Psi'\left(\frac{c_j}{C}\right)\frac{c_j}{C}}$$

Using Slutsky symmetry $p_j \epsilon_i^j = p_i \epsilon_j^i$ to express this using demand elasticities for good i only, we can reexpress as

$$\frac{\partial \log C}{\partial \log p_i} = \frac{\sum_j \Psi'\left(\frac{c_j}{C}\right)\frac{c_j}{C}\frac{p_i}{p_j}\epsilon_j^i}{\sum_j \Psi'\left(\frac{c_j}{C}\right)\frac{c_j}{C}}$$

At a symmetric price, budget exhaustion with constant returns implies

$$\frac{\partial \log C}{\partial \log p_i} = \frac{1}{n}\sum_j \epsilon_j^i = \frac{-1}{n}$$

For any $k \neq i$ we can differentiate

$$\log \Psi'\left(\frac{c^i}{C}\right) - \log \Psi'\left(\frac{c^k}{C}\right) = \log p_i - \log p_k \tag{31}$$

with respect to $\log p_i$ to get

$$\frac{\Psi''\left(\frac{c^i}{C}\right)}{\Psi'\left(\frac{c^i}{C}\right)}\left(\frac{c^i}{C}\right)\frac{\partial}{\partial \log p_i}\left[\log c^i - \log C\right] - \frac{\Psi''\left(\frac{c^k}{C}\right)}{\Psi'\left(\frac{c^k}{C}\right)}\left(\frac{c^k}{C}\right)\frac{\partial}{\partial \log p_i}\left[\log c^k - \log C\right] = 1$$

or, defining

$$R(x) = \frac{x\Psi''(x)}{\Psi'(x)}$$

$$R\left(\frac{c^i}{C}\right)\left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i}\right] - R\left(\frac{c^k}{C}\right)\left[\epsilon_i^k - \frac{\partial \log C}{\partial \log p_i}\right] = 1 \tag{32}$$

Hence at a symmetric steady state, using $\epsilon_i^k = \epsilon_i^k = -\frac{1+\epsilon_i^i}{n-1}$ we have

$$\epsilon_i^i = \frac{n-1}{n}\frac{1}{R(1)} - \frac{1}{n}$$

Differentiating once more with respect to $\log p_i$,

$$R'\left(\frac{c^i}{C}\right)\left[\epsilon_i^i - \frac{\partial \log C}{\partial \log p_i}\right]^2 - R'\left(\frac{c^k}{C}\right)\left[\epsilon_i^k - \frac{\partial \log C}{\partial \log p_i}\right]^2 + R\left(\frac{c^i}{C}\right)\left[\epsilon_{ii}^i - \frac{\partial^2 \log C}{\partial^2 \log p_i}\right] - R\left(\frac{c^k}{C}\right)\left[\epsilon_{ii}^k - \frac{\partial^2 \log C}{\partial^2 \log p_i}\right] = 0$$

At a symmetric steady state,

$$R'(1)\left[\epsilon_i^i + \frac{1}{n}\right]^2 - R'(1)\left[\epsilon_i^k + \frac{1}{n}\right]^2 + R(1)\left[\epsilon_{ii}^i - \epsilon_{ii}^k\right] = 0$$

$$R'(1)\left[\epsilon_i^i + \frac{1}{n}\right]^2 - R'(1)\left[\epsilon_i^k + \frac{1}{n}\right]^2 + R(1)\left[\epsilon_{ii}^i - \epsilon_{jj}^i\right] = 0$$

Using (27) we get

$$R'(1)\left[\frac{n-1}{n}\frac{1}{R(1)}+\frac{1}{n}\right]^2 - R'(1)\left[-\frac{1+\epsilon_i^i}{n-1}+\frac{1}{n}\right]^2 + R(1)\left[\epsilon_{ii}^i\frac{n}{n-1}-\epsilon_j^i\left(1-\epsilon_j^i\right)+\frac{1}{n-1}\left[\epsilon_i^i\left(1+\epsilon_i^i\right)\right]\right]=0$$

Now differentiating (32) with respect to $\log p_j$ for some $j \neq i, k$

$$\begin{aligned}
&R'\left(\frac{c^i}{C}\right)\left[\epsilon_j^i-\frac{\partial \log C}{\partial \log p_j}\right]\left[\epsilon_i^i-\frac{\partial \log C}{\partial \log p_i}\right]+R\left(\frac{c^i}{C}\right)\left[\epsilon_{ij}^i-\frac{\partial^2 \log C}{\partial \log p_i \partial \log p_j}\right]\\
-&R'\left(\frac{c^k}{C}\right)\left[\epsilon_i^k-\frac{\partial \log C}{\partial \log p_i}\right]\left[\epsilon_j^k-\frac{\partial \log C}{\partial \log p_j}\right]-R\left(\frac{c^k}{C}\right)\left[\epsilon_{ij}^k-\frac{\partial^2 \log C}{\partial \log p_i \partial \log p_j}\right]=0
\end{aligned}$$

At a symmetric price,

$$R'(1)\left[\epsilon_j^i+\frac{1}{n}\right]\left[\epsilon_i^i+\frac{1}{n}\right]+R(1)\,\epsilon_{ij}^i = R'(1)\left[\epsilon_j^i+\frac{1}{n}\right]^2+R(1)\,\epsilon_{jk}^i$$

Therefore, using (28) we have

$$\begin{aligned}
\epsilon_{ii}^i &= -\frac{n-1}{n^2}\left[\frac{R(1)(1+R(1))^2+R'(1)(n-2)}{R(1)^3}\right] && (33)\\
\epsilon_{jj}^i &= \frac{(n-2)R'(1)-(n-1)R(1)\left[1+R(1)\right]^2}{n^2R(1)^3} \quad (j\neq i)\\
\epsilon_{ij}^i &= \frac{R(1)\left[1+R(1)\right]^2+(n-2)R'(1)}{n^2R(1)^3} \quad (j\neq i)\\
\epsilon_{jk}^i &= \frac{R(1)\left[1+R(1)\right]^2-2R'(1)}{n^2R(1)^3} \quad (j\neq k, n\geq 3)
\end{aligned}$$

Klenow and Willis (2016) use the functional form

$$\Psi'(x)=\frac{\epsilon-1}{\epsilon}\exp\left(\frac{1-x^{\theta/\epsilon}}{\theta}\right)$$

$$\Psi''(x)=-\frac{x^{\frac{\theta}{\epsilon}-1}}{\epsilon}\Psi'(x)$$

$$\Psi'''(x)=\left[\left(\frac{x^{\frac{\theta}{\epsilon}-1}}{\epsilon}\right)^2-\left(\frac{\theta-\epsilon}{\epsilon^2}\right)x^{\frac{\theta}{\epsilon}-2}\right]\Psi'(x)$$

Therefore

$$\begin{aligned}
R(1)&=-\frac{1}{\epsilon}\\
R'(1)&=-\frac{\theta}{\epsilon^2}
\end{aligned}$$

so that this nests CES with $\theta = 0$. We thus have

$$\begin{aligned}
\epsilon_i^i &= -\epsilon + \frac{\epsilon - 1}{n} \\
\epsilon_j^i &= \frac{\epsilon - 1}{n} \\
\epsilon_{ii}^i &= -\frac{n-1}{n^2}\left[(\epsilon-1)^2 + (n-2)\theta\epsilon\right] \\
\epsilon_{ij}^i &= \frac{(\epsilon-1)^2 + \theta\epsilon(n-2)}{n^2} \\
\epsilon_{jj}^i &= \frac{-(n-1)(\epsilon-1)^2 + \theta\epsilon(n-2)}{n^2} \\
\epsilon_{jk}^i &= \frac{(\epsilon-1)^2 - 2\theta\epsilon}{n^2}
\end{aligned}$$

The superelasticity, defined as $\frac{\epsilon_{ii}^i}{\epsilon_i^i}$, satisfies

$$\begin{aligned}
\frac{\epsilon_{ii}^i}{\epsilon_i^i} &= \frac{1}{\frac{S}{1-S} + \eta}\left[\theta\eta + \left((\eta-1)^2 - 2\theta\eta\right)S\right] \\
&\approx \theta + \left[\frac{(\eta-1)^2}{\eta} - 2\theta\right]S
\end{aligned}$$

with $S = 1/n$ denoting the market share. The approximation in the second line holds if S is small relative to $\eta/\left(1+\eta\right)$, as is the case in a calibration with $\eta = 10$. With constant θ and η, the superelasticity is approximately linear in the Herfindahl index, as in Figure 16. If θ is lower than $\frac{(\eta-1)^2}{2\eta}$ which equals 4.05 when $\eta = 10$ (as in the CES case $\theta = 0$) then $\frac{\epsilon_{ii}^i}{\epsilon_i^i}$ increases with S. With high enough θ, it can actually decrease with S, but a high fixed θ is at odds with pass-through being larger for smaller firms.

F Perturbation of utility

Proof of Proposition 6. We start from the system that defines an MPE:

$$(\rho + n\lambda)\, V\left(p\right) = \Pi\left(p\right) + \lambda \sum_j V\left(g\left(p_{-j}\right), p_{-j}\right) \tag{34}$$

$$V_p\left(g\left(p_{-i}\right), p_{-i}\right) = 0 \tag{35}$$

Differentiating k times the Bellman equation (34) gives us for each $k \geq 1$ a linear system in the kth-derivatives $\mathbf{V}^{(k)} = \left(V_{11\ldots11}, V_{11\ldots12}, V_{11\ldots22}, \ldots\right)$ of the value function V (evaluated at the symmetric steady state $\bar{p}$), which we can invert to obtain these derivatives

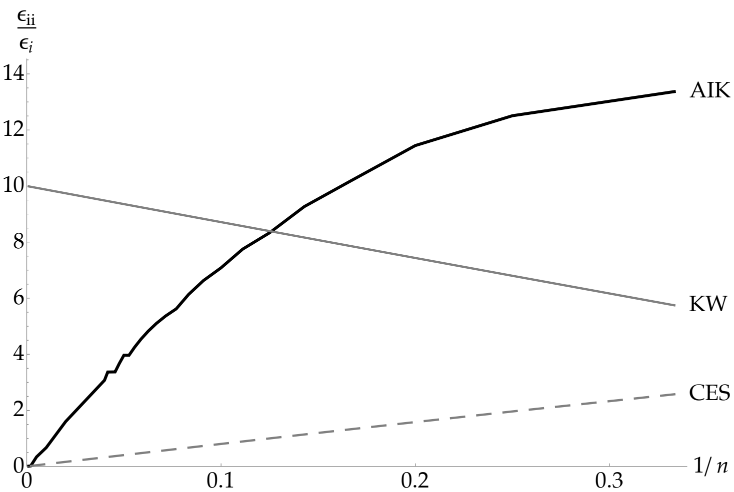


Figure 16: Superelasticities $\epsilon^i_{ii}/\epsilon^i_i$ as a function of market share $1/n$. AIK: variable superelasticity to match heterogeneity in pass-through from Amiti et al. (2019). KW: Fixed $\theta = 10$. CES: Fixed $\theta = 0$. In all cases, $\eta = 10$.

as a function of the profit derivatives $\mathbf{\Pi}^{(k)} = (\Pi_{11\ldots11}, \ldots)$ and derivatives of the policy function (there are $k+1$ such equations in the case of $n = 2$ firms).

We can then compute $\mathbf{\Pi}^{(k)}$ as a function of $\bar{p}$ and own- and cross-superelasticities of the demand function d of order up to k.

Combining the solution $\mathbf{V}^{(k)}$ with the $k-1$th-derivative of the FOC (35) gives us a sequence of equations that must be satisfied at a steady state

$$F^k\left(\bar{p}, g'\left(\bar{p}\right), g''\left(\bar{p}\right), \ldots, g^{(k)}\left(\bar{p}\right); \epsilon_{(0)}, \epsilon_{(1)}, \epsilon_{(2)}, \ldots, \epsilon_{(k)}\right) = 0$$

where F^k is *linear* in $\tilde{\epsilon}_{(k)}$. Thus we can construct recursively a unique sequence $\tilde{\epsilon}_{(k)}$ starting from $k = m+1$, using

$$F^{m+1}\left(\bar{p}, g', \ldots g^{(m-1)}, 0, 0; \epsilon_{(1)}, \epsilon_{(2)}, \ldots, \tilde{\epsilon}_{(m+1)}\right) = 0$$
$$F^{m+2}\left(\bar{p}, g', \ldots g^{(m-1)}, 0, 0, 0; \epsilon_{(1)}, \epsilon_{(2)}, \ldots, \tilde{\epsilon}_{(m+1)}, \tilde{\epsilon}_{(m+2)}\right) = 0$$

and so on. Section F.1 below shows that there are indeed enough degrees of freedom to make the equations $F^m, F^{m+1}, \ldots$ independent.

Define $\tilde{\Psi}$ as

$$\tilde{\Psi}\left(x\right) = \sum_{k=0}^{\infty} \frac{\tilde{\Psi}^{(k)}\left(1\right)}{k!}\left(x-1\right)^k$$

where $\tilde{\Psi}^{(k+1)}(1)$ is characterized by $\left(\epsilon_{(1)}, \ldots, \epsilon_{(m)}, \tilde{\epsilon}_{(m+1)}, \ldots, \tilde{\epsilon}_{(k)}\right)$ through the same computations as in Appendix E.

Given this construction, $\bar{p}, g', \ldots, g^{(m-1)}$ are pinned down by $\left(\epsilon_{(1)}, \ldots, \epsilon_{(m)}\right)$ as the solution to the system of equations F^k for $k = 1, \ldots, m$.

F.1 Counting the degrees of freedom

The main potential impediment to the proof above is that demand integrability (e.g., demand functions being generated by actual utility functions) imposes restrictions on higher-order elasticities that would prevent us from constructing the sequence $\tilde{\epsilon}$. Indeed, in Appendix E we saw that with $n = 2$ firms, general Kimball demand functions cannot generate superelasticities beyond those arising from CES demand. We now show that as long as $n \geq 3$, this is not the case, by proving that the number of elasticities exceeds the number of restrictions.

Formally, we want to compute $\#_n(m)$, the number of cross-elasticities of order m, that is derivatives

$$\frac{\partial^m \log d^1(p)}{\partial^{i_1} \log p_1 \partial^{i_2} \log p_2 \ldots \partial^{i_n} \log p_n}$$

where

$$0 \leq i_1, \ldots, i_n \leq m$$
$$i_1 + \cdots + i_n = m$$

as functions of the own-mth-elasticity $\epsilon^1_{\underbrace{11\ldots1}_{m \text{ times}}}$, and compare $\#_n(m)$ to the number of restrictions imposed by demand integrability and symmetry arguments.

Step 1: Computing $\#_n(m)$. By Schwarz symmetry, in a smooth MPE, we can always invert 2 indices in the derivatives. Moreover, from the viewpoint of firm 1 (whose demand d^1 we're differentiating), firms 2 and 3 are interchangeable. For instance, in the case of $n = 3$ firms and order of differentiation $m = 3$, these symmetries reduce the number of potential elasticities $n^m = 27$ to only 6 elasticities

$$\epsilon^1_{111}, \epsilon^1_{112}, \epsilon^1_{122}, \epsilon^1_{123}, \epsilon^1_{222}, \epsilon^1_{223}.$$

Denote

$$q_n(M)$$

the number of partitions of an integer M into n non-negative integers. For $M \geq n$ we have

$$q_n(M) = p_n(M+n)$$

where $p_n(M)$ is the number of partitions of an integer M into n positive integers. We can see this by writing, starting from a partition of M into n non-negative integers $i_1, \ldots, i_n$:

$$M + n = (i_1 + 1) + \cdots + (i_n + 1)$$

We can then compute $p_j(M)$ using the recurrence formula

$$p_j(M) = \underbrace{p_j(M-j)}_{\text{partitions for which } i_k \geq 2 \text{ for all } k} + \underbrace{p_{j-1}(M-1)}_{\text{partitions for which } i_k = 1 \text{ for some } k}$$

Lemma 1. *For any $n \geq 1$ and $m \geq 1$ the number of elasticities of order m is*

$$\#_n(m) = \sum_{k=0}^{m} q_{n-1}(m-k) \tag{36}$$

hence $\#_n(m+1) = \#_n(m) + q_{n-1}(m+1)$.

Proof. Firm 1 is special, so we need to count the number of times we differentiate with respect to $\log p_1$, which generates the sum over k. Then we get each term in the sum by counting partitions of $m-k$ into $n-1$ non-negative integers. $\square$

Step 2: Computing the number of restrictions arising from demand integrability. Next, we want to use economic restrictions to reduce the number of degrees of freedom, ideally to 1, by having $\#_n(m) - 1$ independent equations. Our restrictions are

$$\Phi(p) = \sum_j p_j d^j(p) = 0 \quad \forall p \tag{37}$$

$$d^i_j(p) = d^j_i(p) \quad \forall p, \forall i, j \tag{38}$$

The first equation is the budget constraint. The second equation is the Slutsky symmetry condition (constant returns to scale allow to go from Hicksian to Marshallian elasticities). Note that Φ defined in (37) is symmetric, unlike the demand function d^1 we are using to compute elasticities. Therefore Φ's derivatives give us fewer restrictions than what we need in (36), leaving room for restrictions to come from the Slutsky equation.

We need to differentiate these two equations to obtain independent equations that relate the mth-cross-elasticities to the mth-own-elasticity. The number of restrictions coming from derivatives of Φ at order m is simply the number of partitions of m into n non-negative integers

$$q_n(m)$$

How many restrictions $\flat_n(m)$ do we have from derivatives of the Slutsky equation? The initial equation

$$d_2^1 = d_1^2$$

is irrelevant at a symmetric steady state; it only starts mattering once we differentiate it. The first terms are (see in next subsection)

$$\begin{aligned}
\flat_n(1) &= 0 \\
\flat_n(2) &= 1 \\
\flat_n(3) &= \begin{cases} 2 & \text{if } n \geq 3 \\ 1 & \text{if } n = 2 \end{cases} \\
\flat_n(4) &= \begin{cases} 5 & \text{if } n \geq 4 \\ 4 & \text{if } n = 3 \\ 3 & \text{if } n = 2 \end{cases}
\end{aligned}$$

Step 3: Comparing the two. We actually do not need to compute $\flat_n(m)$ exactly. The following lemma shows that there are always enough degrees of freedom $\#_n(m)$ to construct the Kimball aggregator in 6:

Lemma 2. *For $n \geq 3$ and any m we have*

$$q_n(m) + \flat_n(m) + 1 \leq \#_n(m) \tag{39}$$

Proof. We know by hand that (39) holds for $m = 1, 2$ so take $m \geq 3$. Then all the Slutsky conditions can be written as starting with

$$d_{12\ldots}^1 = \ldots$$

hence we have

$$\flat_n(m) \leq \#_n(m-2) = \#_n(m) - p_{n-1}(n+m-1) - p_n(n+m-2)$$

hence the number of equations is bounded by

$$q_n(m) + \flat_n(m) \leq p_n(n+m) + \#_n(m) - p_{n-1}(n+m-1) - p_n(n+m-2)$$

Then we have (39) if

$$\begin{aligned}
&p_n\,(n+m) < p_{n-1}\,(n+m-1) + p_n\,(n+m-2) \\
\Leftrightarrow &p_{n-1}\,(n+m-1) + p_n\,(m) < p_{n-1}\,(n+m-1) + p_n\,(n+m-2) \\
\Leftrightarrow &p_n\,(m) < p_n\,(n+m-2)
\end{aligned}$$

which holds for $n \geq 3$. □

Note that so far we have considered general CRS demand functions. Restricting attention to the Kimball class makes the inequality (39) bind, meaning that we can parametrize all the cross-elasticities of order m using the own-elasticity of order m.

What fails in the knife-edge case $n = 2$? Slutsky symmetry imposes too many restrictions: at $m = 2$ we only have 3 elasticities $\epsilon^1_{11}, \epsilon^1_{12}, \epsilon^1_{22}$ and also 3 restrictions, so we can solve out all the superelasticities as functions of ϵ^1_1, which prevents us from constructing the Kimball aggregator in Proposition 6.

F.2 Example with $m = 3$ and any $n \geq 3$

The potential elasticities are

$$\begin{aligned}
m = 1: &\quad \epsilon^1_1, \epsilon^1_2 \\
m = 2: &\quad \epsilon^1_{11}, \epsilon^1_{12}, \epsilon^1_{22}, \epsilon^1_{23} \\
m = 3: &\quad \epsilon^1_{111}, \epsilon^1_{112}, \epsilon^1_{122}, \epsilon^1_{123}, \epsilon^1_{222}, \epsilon^1_{223}, \epsilon^1_{234}
\end{aligned}$$

Differentiating the budget constraint (37) $\Phi\,(p) = 0$ with respect to any i, we get

$$\Phi_i\,(p) = d^i + p_i d^i_i + \sum_{j \neq i} p_j d^j_i = 0$$

Then differentiating with respect to i and any $k \neq i$

$$\begin{aligned}
\Phi_{ii}\,(p) &= 2d^i_i + p_i d^i_{ii} + \sum_{j \neq i} p_j d^j_{ij} = 0 \\
\Phi_{ik}\,(p) &= 2d^i_k + p_i d^i_{ik} + p_k d^k_{ik} + d^k_i + \sum_{j \neq i,k} p_j d^j_{ik} = 0
\end{aligned}$$

Then differentiating the first equation with respect to i and k and the second equation with respect to any $l \neq i, k$

$$\Phi_{iii}(p) = 3d^i_{ii} + p_i d^i_{iii} + \sum_{j \neq i} p_j d^j_{iij} = 0$$
$$\Phi_{iik}(p) = 2d^i_{ik} + p_i d^i_{iik} + d^k_{ik} + p_k d^k_{ikk} + \sum_{j \neq i,k} p_j d^j_{ijk} = 0$$
$$\Phi_{ikl}(p) = 2d^i_{kl} + p_i d^i_{ikl} + p_k d^k_{ikl} + d^k_{il} + d^l_{ik} + p_l d^l_{ikl} + \sum_{j \neq i,k,l} p_j d^j_{ikl} = 0$$

(by symmetry of Φ we have $\Phi_{iik} = \Phi_{ikk}$ and $\Phi_{iii} = \Phi_{kkk}$).

Differentiating the Slutsky equation (38)

$$d^i_{ij} = d^j_{ii} \left(= d^i_{jj}\right)$$

($d^i_{jk} = d^j_{ik}$ is irrelevant) then

$$d^i_{iij} = d^j_{iii} \left(= d^i_{jjj}\right)$$
$$d^i_{ijk} = d^j_{iik} \left(= d^i_{jjk}\right), k \neq i, j$$

($d^i_{ijj} = d^j_{iij}$ is irrelevant)

$$d^i_{iiij} = d^j_{iiii} \left(= d^i_{jjjj}\right)$$
$$d^i_{iijj} = d^j_{iiij} \left(= d^i_{ijjj}\right)$$
$$d^i_{iijk} = d^j_{iiik} \left(= d^i_{jjjk}\right)$$
$$d^i_{ijkk} = d^j_{iikk} \left(= d^i_{jjkk}\right), k \neq i, j$$
$$d^i_{ijkl} = d^j_{iikl} \left(= d^i_{jjkl}\right), k \neq i, j, l \neq k, i, j$$

($d^i_{ijjk} = d^j_{iijk}$ and $d^i_{ijjj} = d^j_{iijj}$ are irrelevant)

So overall we get 1 restriction for $m = 1$, 3 restrictions for $m = 2$, and 5 restrictions for $m = 3$.

G Locally Linear Equilibrium

G.1 Homogeneous Firms

We first solve the linear system in $\left\{V_j^i, V_{ii}^i, V_{ij}^i, V_{jj}^i, V_{jk}^i\right\}$ obtained from envelope conditions

$$\begin{aligned}
(\rho+\lambda)\, V_j^i &= \Pi_j^i + \lambda\,(n-2)\, V_j^i \beta \\
(\rho+\lambda)\, V_{ii}^i &= \Pi_{ii}^i + \lambda\,(n-1)\left(V_{jj}^i \beta^2 + 2V_{ij}^i \beta\right) \\
(\rho+2\lambda)\, V_{ij}^i &= \Pi_{ij}^i + \lambda\,(n-2)\left(V_{jj}^i \beta^2 + V_{ij}^i \beta + V_{jk}^i \beta\right) \\
(\rho+\lambda)\, V_{jj}^i &= \Pi_{jj}^i + \lambda\,(n-2)\left(V_{jj}^i \beta^2 + 2V_{jk}^i \beta\right) + \lambda\left(V_{ii}^i \beta^2 + 2V_{ij}^i \beta\right) \\
(\rho+2\lambda)\, V_{jk}^i &= \Pi_{jk}^i + \lambda\,(n-3)\left(V_{jj}^i \beta^2 + 2V_{jk}^i \beta\right) + \lambda\left(V_{ii}^i \beta^2 + 2V_{ij}^i \beta\right)
\end{aligned}$$

Injecting the solution into the derivative of the first-order condition

$$V_{ii}^i \beta + V_{ij}^i = 0$$

yields

$$0 = A_{ii} \Pi_{ii}^i\,(\bar{p}) + A_{ij} \Pi_{ij}^i\,(\bar{p}) + A_{jj} \Pi_{jj}^i\,(\bar{p}) + A_{jk} \Pi_{jk}^i\,(\bar{p})$$

with coefficients

$$A_{ii} = \beta\left((\beta+1)\lambda^3\left(\beta^2\left(-2n^2+9n-10\right)+\beta^3(n-2)+6\beta(n-2)-4\right)-\lambda^2\rho\left(\beta^3\left(n^2-5n+6\right)+\beta^2\left(2n^2-15n+22\right)+\beta(24-9n)+8\right)+\lambda\rho^2\left(\beta^2(n-2)+\beta(3n-8)-5\right)-\rho^3\right) \tag{40}$$

$$A_{ij} = -\left(2(\beta+1)\lambda^3\left(-2\beta^3\left(n^2-3n+2\right)+\beta^4(n-1)+2\beta^2(n-1)-\beta(n-2)+1\right)+\lambda^2\rho\left(\beta^4\left(-2n^2+7n-5\right)-4\beta^3\left(n^2-4n+3\right)+3\beta^2 n-4\beta(n-3)+5\right)+\lambda\rho^2\left(\beta^2 n-2\beta(n-3)+4\right)+\rho^3\right) \tag{41}$$

$$A_{jj} = \beta^2\lambda\left((\beta+1)\lambda^2\left(2\left(\beta^2+3\beta+2\right)+\beta(\beta+1)n^2-\left(3\beta^2+7\beta+2\right)n\right)+\lambda\rho\left(4\beta^2+10\beta+\beta(\beta+1)n^2-\left(5\beta^2+9\beta+3\right)n+6\right)+\rho^2(\beta-(\beta+1)n+2)\right) \tag{42}$$

$$A_{jk} = -\beta\lambda(n-2)\left((\beta+1)\lambda^2\left(-\beta+\beta^3(n-1)+3\beta^2(n-1)+1\right)+\lambda\rho\left(2\beta^3(n-1)+\beta^2(3n-4)+2\right)+\rho^2\right) \tag{43}$$

G.2 Heterogeneous Firms

Suppose as in section C.2 that there are two types of firms, a and b, with $n = n_a + n_b$. a and b firms can differ permanently in their marginal costs, their demand, or both.

We know need to solve for six unknowns $\left\{\beta_a^a, \beta_b^a, \beta_a^b, \beta_b^b, p_a, p_b\right\}$ where β_j^i is the reaction of a firm of type i to the price change of a firm of type j. The envelope conditions for firms of type a are

$$
\begin{aligned}
(\rho+\lambda)\, V_{i}^{i,a} &= \Pi_{i}^{i,a} + \lambda\,(n_a-1)\, V_{j_a}^{i,a}\beta_a^a + \lambda n_b V_{j_b}^{i,a}\beta_a^b \\
(\rho+\lambda)\, V_{j_a}^{i,a} &= \Pi_{j_a}^{i,a} + \lambda\,(n_a-2)\, V_{j_a}^{i,a}\beta_a^a + \lambda n_b V_{j_b}^{i,a}\beta_a^b \\
(\rho+\lambda)\, V_{j_b}^{i,a} &= \Pi_{j_b}^{i,a} + \lambda\,(n_a-1)\, V_{j_a}^{i,a}\beta_b^a + \lambda\,(n_b-1)\, V_{j_b}^{i,a}\beta_b^b
\end{aligned}
$$

and

$$
\begin{aligned}
(\rho+\lambda)\, V_{ii}^{i,a} &= \Pi_{ii}^{i,a} + \lambda\,(n_a-1)\left[V_{j_aj_a}^{i,a}\,(\beta_a^a)^2 + 2V_{ij_a}^{i,a}\beta_a^a\right] + \lambda n_b\left[V_{j_bj_b}^{i,a}\left(\beta_a^b\right)^2 + 2V_{ij_b}^{i,a}\beta_a^b\right] \\
(\rho+2\lambda)\, V_{ij_a}^{i,a} &= \Pi_{ij_a}^{i,a} + \lambda\,(n_a-2)\left[V_{j_aj_a}^{i,a}\,(\beta_a^a)^2 + V_{j_ak_a}^{i,a}\beta_a^a + V_{ij_a}^{i,a}\beta_a^a\right] + \lambda n_b\left[V_{j_bj_b}^{i,a}\left(\beta_a^b\right)^2 + V_{j_ak_b}^{i,a}\beta_a^b + V_{ij_b}^{i,a}\beta_a^b\right] \\
(\rho+2\lambda)\, V_{ij_b}^{i,a} &= \Pi_{ij_b}^{i,a} + \lambda\,(n_a-1)\left[V_{j_aj_a}^{i,a}\beta_a^a\beta_b^a + V_{j_ak_b}^{i,a}\beta_a^a + V_{ij_a}^{i,a}\beta_b^a\right] + \lambda\,(n_b-1)\left[V_{j_bj_b}^{i,a}\beta_a^b\beta_b^b + V_{j_bk_b}^{i,a}\beta_a^b + V_{ij_b}^{i,a}\beta_b^b\right] \\
(\rho+\lambda)\, V_{j_aj_a}^{i,a} &= \Pi_{j_aj_a}^{i,a} + \lambda\left[V_{ii}^{i,a}\,(\beta_a^a)^2 + 2V_{ij_a}^{i,a}\beta_a^a\right] + \lambda\,(n_a-2)\left[V_{j_aj_a}^{i,a}\,(\beta_a^a)^2 + 2V_{j_ak_a}^{i,a}\beta_a^a\right] + \lambda n_b\left[V_{j_bj_b}^{i,a}\left(\beta_a^b\right)^2 + 2V_{j_ak_b}^{i,a}\beta_a^b\right] \\
(\rho+2\lambda)\, V_{j_ak_b}^{i,a} &= \Pi_{j_ak_b}^{i,a} + \lambda\left[V_{ii}^{i,a}\beta_a^a\beta_b^a + V_{ij_b}^{i,a}\beta_a^a + V_{ij_a}^{i,a}\beta_b^a\right] + \lambda\,(n_a-2)\left[V_{j_aj_a}^{i,a}\beta_a^a\beta_b^a + V_{j_ak_b}^{i,a}\beta_a^a + V_{j_ak_a}^{i,a}\beta_b^a\right] \\
&\quad + \lambda\,(n_b-1)\left[V_{j_bj_b}^{i,a}\beta_a^b\beta_b^b + V_{j_bk_b}^{i,a}\beta_a^b + V_{j_ak_b}^{i,a}\beta_b^b\right] \\
(\rho+\lambda)\, V_{j_ak_a}^{i,a} &= \Pi_{j_ak_a}^{i,a} + \lambda\left[V_{ii}^{i,a}\,(\beta_a^a)^2 + 2V_{ij_a}^{i,a}\beta_a^a\right] + \lambda\,(n_a-2)\left[V_{j_aj_a}^{i,a}\,(\beta_a^a)^2 + 2V_{j_ak_a}^{i,a}\beta_a^a\right] + \lambda n_b\left[V_{j_bj_b}^{i,a}\left(\beta_a^b\right)^2 + 2V_{j_ak_b}^{i,a}\beta_a^b\right] \\
(\rho+\lambda)\, V_{j_bj_b}^{i,a} &= \Pi_{j_bj_b}^{i,a} + \lambda\left[V_{ii}^{i,a}\,(\beta_b^a)^2 + 2V_{ij_b}^{i,a}\beta_b^a\right] + \lambda\,(n_a-1)\left[V_{j_aj_a}^{i,a}\,(\beta_b^a)^2 + 2V_{j_ak_b}^{i,a}\beta_b^a\right] + \lambda\,(n_b-1)\left[V_{j_bj_b}^{i,a}\left(\beta_b^b\right)^2 + 2V_{j_bk_b}^{i,a}\beta_b^b\right] \\
(\rho+2\lambda)\, V_{j_bk_b}^{i,a} &= \Pi_{j_bk_b}^{i,a} + \lambda\left[V_{ii}^{i,a}\,(\beta_b^a)^2 + 2V_{ij_b}^{i,a}\beta_b^a\right] + \lambda\,(n_a-1)\left[V_{j_aj_a}^{i,a}\,(\beta_b^a)^2 + 2V_{j_ak_b}^{i,a}\beta_b^a\right] + \lambda\,(n_b-2)\left[V_{j_bj_b}^{i,a}\left(\beta_b^b\right)^2 + 2V_{j_bk_b}^{i,a}\beta_b^b\right]
\end{aligned}
$$

We can use these 11 envelope conditions to solve linearly for $\left\{V_i^{i,a}, V_{j_a}^{i,a}, V_{j_b}^{i,a}, V_{ii}^{i,a}, \dots\right\}$, and then inject the solution into the first-order conditions

$$
\begin{aligned}
V_i^{i,a} &= 0 \\
V_{ii}^{i,a}\beta_a^a + V_{ij_a}^{i,a} &= 0 \\
V_{ii}^{i,a}\beta_b^a + V_{ij_b}^{i,a} &= 0
\end{aligned}
$$

The same steps for firms of type b give us 3 more equations.

H Oligopolistic Phillips Curve

Consider the general non-stationary versions of the Bellman equation (2) and the first-order condition (3):

$$(i_t + n\lambda)\, V^i(p,t) = V^i_t(p,t) + \Pi^i(p, MC_t, Z_t) + \lambda \sum_j V^i\left(g^j(p_{-j},t), p_{-j}, t\right) \tag{44}$$

$$V^i_i\left(g^i(p_{-i},t), p_{-i}, t\right) = 0 \tag{45}$$

Nominal profits are given by

$$\Pi^i(p, MC, Z) = ZD^i(p)\left[p_i - MC\right]$$

where Z is an aggregate demand shifter that can depend arbitrarily on C_t and P_t.[17]

Define $\alpha(t)$ as

$$g^i(\alpha(t), \alpha(t), \ldots, \alpha(t), t) = \alpha(t)$$

This is the price that each firm would set if all the firms were resetting at the same time. α is the counterpart of the reset price in the standard New Keynesian model.

To obtain the dynamics of α from (44), we start by deriving time-varying envelope conditions evaluated at the symmetric price $p_1 = p_2 = \cdots = p_n = \alpha(t)$. In the non-stationary game, the full non-linear first-order and second-order envelope conditions are:

$$(i_t + n\lambda)\, V^i_i = V^i_{it} + \Pi^i_i + \lambda \sum_{j \neq i} V^i_j g^j_i \tag{46}$$

$$(i_t + n\lambda)\, V^i_j = V^i_{jt} + \Pi^i_j + \lambda \sum_{l \neq j} \left(V^i_l g^l_j + V^i_j\right) \tag{47}$$

$$(i_t + n\lambda)\, V^i_{ii} = V^i_{iit} + \Pi^i_{ii} + \lambda \sum_{j \neq i} \left[\left(V^i_{jj}\left(g^j(p_{-j}), p_{-j}\right) g^j_i + V^i_{ij}\left(g^j(p_{-j}), p_{-j}\right)\right) g^j_i(p_{-j}) + V^i_j g^j_{ii}\right] \tag{48}$$

$$(i_t + n\lambda)\, V^i_{ik} = V^i_{ikt} + \Pi^i_{ik} + \lambda \sum_{j \neq i,k} \left[\left(V^i_{jj} g^j_k + V^i_{jk}\right) g^j_i + V^i_j g^j_{ik}\right] \tag{49}$$

$$(i_t + n\lambda)\, V^i_{jj} = V^i_{jjt} + \Pi^i_{jj} + \lambda \sum_{l \neq j} \left(V^i_{ll}\left(g^l_j\right)^2 + V^i_{jl} g^l_j + V^i_l g^l_{jj} + V^i_{jj}\right) \tag{50}$$

$$(i_t + n\lambda)\, V^i_{jk} = V^i_{jkt} + \Pi^i_{jk} + \lambda \sum_{l \neq j,k} \left(V^i_{ll}\left(g^l_j\right)^2 + V^i_{lk} g^l_j + V^i_l g^l_{jk} + V^i_{jk}\right) \tag{51}$$

[17] In section 4, conditions (7) ensured a constant Z_t.

After applying symmetry and using Proposition 6 to make the strategies approximately linear in the neighborhood of the steady state, we obtain the following system of partial differential equations (52) to (57): After using symmetry we have the following partial differential equations (PDEs):

$$0 = V^i_{it} + \Pi^i_i + \lambda\left(n-1\right)V^i_j\beta \tag{52}$$

$$\left(i_t + \lambda\right)V^i_j = V^i_{jt} + \Pi^i_j + \lambda\left(n-2\right)V^i_j\beta \tag{53}$$

$$\left(i_t + \lambda\right)V^i_{ii} = V^i_{iit} + \Pi^i_{ii} + \lambda\left(n-1\right)\left(V^i_{jj}\beta^2 + 2V^i_{ij}\beta\right) \tag{54}$$

$$\left(i_t + 2\lambda\right)V^i_{ij} = V^i_{ijt} + \Pi^i_{ij} + \lambda\left(n-2\right)\left(V^i_{jj}\beta^2 + V^i_{jk}\beta + \beta V^i_{ij}\right) \tag{55}$$

$$\left(i_t + \lambda\right)V^i_{jj} = V^i_{jjt} + \Pi^i_{jj} + \lambda\left(n-2\right)\left(V^i_{jj}\beta^2 + 2\beta V^i_{jk}\right) + \lambda\left(V^i_{ii}\beta^2 + 2\beta V^i_{ij}\right) \tag{56}$$

$$\left(i_t + 2\lambda\right)V^i_{jk} = V^i_{jkt} + \Pi^i_{jk} + \lambda\left(n-3\right)\left(V^i_{jj}\beta^2 + 2\beta V^i_{jk}\right) + \lambda\left(V^i_{ii}\beta^2 + 2\beta V^i_{ij}\right) \tag{57}$$

Denote the functions

$$W^i_i\left(t\right) = V^i_i\left(\alpha\left(t\right),\ldots,\alpha\left(t\right),t\right),\ W^i_{ii}\left(t\right) = V^i_{ii}\left(\alpha\left(t\right),\ldots,\alpha\left(t\right),t\right)$$

and so on for all derivatives of the value function V^i. We can transform the system (52)-(57) into a system of ordinary differential equations in the functions $W^i_i\left(t\right), W^i_j\left(t\right)$, and so on. The partial derivatives with respect to time such as

$$V^i_{it} = \frac{\partial V^i_i}{\partial t}\left(\alpha\left(t\right),\ldots,\alpha\left(t\right),t\right)$$

in equations (52) to (57) can be mapped to corresponding total derivatives of W functions

$\dot{W}^i_{it} = \frac{dW^i_{it}}{dt}$ using

$$V^i_{it} = \dot{W}^i_i - \left[V^i_{ii} + \sum_{j \neq i} V^i_{ij} \right] \dot{\alpha} \tag{58}$$

$$V^i_{jt} = \dot{W}^i_j - \left[V^i_{ij} + V^i_{jj} + \sum_{k \neq i,j} V^i_{jk} \right] \dot{\alpha} \tag{59}$$

$$V^i_{iit} = \dot{W}^i_{ii} - \left[V^i_{iii} + \sum_{j \neq i} V^i_{iij} \right] \dot{\alpha} \tag{60}$$

$$V^i_{ijt} = \dot{W}^i_{ij} - \left[V^i_{iij} + V^i_{ijj} + \sum_{k \neq i,j} V^i_{ijk} \right] \dot{\alpha} \tag{61}$$

$$V^i_{jjt} = \dot{W}^i_{jj} - \left[V^i_{ijj} + V^i_{jjj} + \sum_{k \neq i,j} V^i_{jjk} \right] \dot{\alpha} \tag{62}$$

$$V^i_{jkt} = \dot{W}^i_{jk} - \left[V^i_{ijk} + V^i_{jjk} + V^i_{jkk} + \sum_{l \neq i,j,k} V^i_{jkl} \right] \dot{\alpha} \tag{63}$$

where the third derivatives of V at the steady state come from the third-order envelope conditions of the stationary game, solving the linear system:

$$\begin{aligned}
(\rho + \lambda)\, V^i_{iii} &= \Pi^i_{iii} + \lambda\,(n-1) \left\{ V^i_{jjj}\beta^3 + 3V^i_{ijj}\beta^2 + 3V^i_{iij}\beta \right\} \\
(\rho + 2\lambda)\, V^i_{iij} &= \Pi^i_{iij} + \lambda\,(n-2) \left\{ V^i_{jjj}\beta^3 + 2V^i_{ijj}\beta^2 + V^i_{jjk}\beta^2 + 2V^i_{ijk}\beta + V^i_{iij}\beta \right\} \\
(\rho + 2\lambda)\, V^i_{ijj} &= \Pi^i_{ijj} + \lambda\,(n-2) \left\{ V^i_{jjj}\beta^3 + 2\beta^2 V^i_{jjk} + \beta^2 V^i_{ijj} + 2\beta V^i_{ijk} + \beta V^i_{jjk} \right\} \\
(\rho + 3\lambda)\, V^i_{ijk} &= \Pi^i_{ijk} + \lambda\,(n-3) \left\{ V^i_{jjj}\beta^3 + 2\beta^2 V^i_{jjk} + \beta^2 V^i_{ijj} + 2\beta V^i_{ijk} + \beta V^i_{jkl} \right\} \\
(\rho + \lambda)\, V^i_{jjj} &= \Pi^i_{jjj} + \lambda\,(n-2) \left\{ \beta^3 V^i_{jjj} + 3\beta^2 V^i_{jjk} + 3\beta V^i_{jjk} \right\} \\
&\quad + \lambda \left\{ \beta^3 V^i_{iii} + 3\beta^2 V^i_{iij} + 3\beta V^i_{ijj} \right\} \\
(\rho + 2\lambda)\, V^i_{jjk} &= \Pi^i_{jjk} + \lambda\,(n-3) \left\{ \beta^3 V^i_{jjj} + 3\beta^2 V^i_{jjk} + \beta V^i_{jjk} + 2\beta V^i_{jkl} \right\} \\
&\quad + \lambda \left\{ \beta^3 V^i_{iii} + 3\beta^2 V^i_{iij} + \beta V^i_{ijj} + 2\beta V^i_{ijk} \right\} \\
(\rho + 3\lambda)\, V^i_{jkl} &= \Pi^i_{jkl} + \lambda\,(n-4) \left\{ \beta^3 V^i_{jjj} + 3\beta^2 V^i_{jjk} + 3\beta V^i_{jkl} \right\} \\
&\quad + \lambda \left\{ \beta^3 V^i_{iii} + 3\beta^2 V^i_{iij} + 3\beta V^i_{ijk} \right\}
\end{aligned} \tag{64}$$

Importantly, to approximate the second derivatives of V^i, we need to solve for the third derivatives of V^i around the steady state by applying the envelope theorem one more time.

Imposing symmetry again, the following non-linear system of ODEs in the 8 functions $\left(\alpha, \beta, W^i_j, W^i_j, W^i_{ii}, W^i_{ij}, W^i_{jj}, W^i_{jk}\right)$ holds exactly (omitting the time argument):

$$0 = -\left[W^i_{ii} + (n-1) W^i_{ij}\right] \dot{\alpha} + \Pi^i_i + \lambda (n-1) W^i_j \beta \tag{65}$$
$$(i_t + \lambda) W^i_j = \dot{W}^i_j - \left[W^i_{ij} + W^i_{jj} + (n-2) W^i_{jk}\right] \dot{\alpha} + \Pi^i_j + \lambda (n-2) W^i_j \beta \tag{66}$$
$$0 = W^i_{ii}\beta + W^i_{ij} \tag{67}$$
$$(i_t + \lambda) W^i_{ii} = \dot{W}^i_{ii} - \left[V^i_{iii} + (n-1) V^i_{iij}\right] \dot{\alpha} + \Pi^i_{ii} + \lambda (n-1) \left(W^i_{jj}\beta^2 + 2W^i_{ij}\beta\right) \tag{68}$$
$$(i_t + 2\lambda) W^i_{ij} = \dot{W}^i_{ij} - \left[V^i_{iij} + V^i_{ijj} + (n-2) V^i_{ijk}\right] \dot{\alpha} + \Pi^i_{ij} + \lambda (n-2) \left(W^i_{jj}\beta^2 + W^i_{jk}\beta + W^i_{ij}\beta\right) \tag{69}$$
$$(i_t + \lambda) W^i_{jj} = \dot{W}^i_{jj} - \left[V^i_{ijj} + V^i_{jjj} + (n-2) V^i_{jjk}\right] \dot{\alpha} + \Pi^i_{jj} + \lambda (n-2) \left(W^i_{jj}\beta^2 + 2\beta W^i_{jk}\right) + \lambda \left(W^i_{ii}\beta^2 + 2\beta W^i_{ij}\right) \tag{70}$$
$$(i_t + 2\lambda) W^i_{jk} = \dot{W}^i_{jk} - \left[V^i_{ijk} + V^i_{jjk} + V^i_{jkk} + (n-3) V^i_{jkl}\right] \dot{\alpha} + \Pi^i_{jk} + \lambda (n-3) \left(W^i_{jj}\beta^2 + 2\beta W^i_{jk}\right) + \lambda \left(W^i_{ii}\beta^2 + 2\beta W^i_{ij}\right) \tag{71}$$

Next, we linearize this system (**??**)-(**??**) around a symmetric steady state $\bar{\alpha} = \alpha(\infty)$ with zero inflation (and steady state values of aggregate variables $\bar{C}, \bar{P}$). Let lower case variables denote log-deviations, e.g., $a(t) = \log \alpha(t) - \log \bar{\alpha}$, and write marginal cost as

$$mc(t) = p(t) + k(t)$$

where $k(t)$ is the log-deviation of the real marginal cost. Profit derivatives such as $\Pi^i_i(t)$ in (**??**) are evaluated at the moving price $\alpha(t)$, hence become once log-linearized

$$\pi^i_i(t) = \bar{\alpha}\left[\Pi^i_{ii} + (n-1)\Pi^i_{ij}\right] a(t) + \bar{MC}\Pi^i_{i,MC}\left(p(t) + k(t)\right) + \Pi^i_i\left(z_c c(t) + z_p p(t)\right)$$
$$\pi^i_j(t) = \bar{\alpha}\left[\Pi^i_{ij} + \Pi^i_{jj} + (n-2)\Pi^i_{jk}\right] a(t) + \bar{MC}\Pi^i_{j,MC}\left(p(t) + k(t)\right) + \Pi^i_j\left(z_c c(t) + z_p p(t)\right)$$
$$\pi^i_{ii}(t) = \bar{\alpha}\left[\Pi^i_{iii} + (n-1)\Pi^i_{iij}\right] a(t) + \bar{MC}\Pi^i_{ii,MC}\left(p(t) + k(t)\right) + \Pi^i_{ii}\left(z_c c(t) + z_p p(t)\right)$$
$$\pi^i_{ij}(t) = \bar{\alpha}\left[\Pi^i_{iij} + \Pi^i_{ijj} + (n-2)\Pi^i_{ijk}\right] a(t) + \bar{MC}\Pi^i_{ij,MC}\left(p(t) + k(t)\right) + \Pi^i_{ij}\left(z_c c(t) + z_p p(t)\right)$$
$$\pi^i_{jj}(t) = \bar{\alpha}\left[\Pi^i_{ijj} + \Pi^i_{jjj} + (n-2)\Pi^i_{jjk}\right] a(t) + \bar{MC}\Pi^i_{jj,MC}\left(p(t) + k(t)\right) + \Pi^i_{jj}\left(z_c c(t) + z_p p(t)\right)$$
$$\pi^i_{jk}(t) = \bar{\alpha}\left[\Pi^i_{ijk} + 2\Pi^i_{jjk} + (n-3)\Pi^i_{jkl}\right] a(t) + \bar{MC}\Pi^i_{jk,MC}\left(p(t) + k(t)\right) + \Pi^i_{jk}\left(z_c c(t) + z_p p(t)\right)$$

where $\bar{\Pi}^i_i, \bar{\Pi}^i_{ii}$ etc. denote steady state values.

This yields the system of 6 linear ODEs in $\left(a(t), w^i_j(t), w^i_{ii}(t), w^i_{ij}(t), w^i_{jj}(t), w^i_{jk}(t)\right)$

(72)-(77):

$$\left[V_{ii}^{i}+(n-1)\,V_{ij}^{i}\right]\dot{a}\,(t)=\frac{1}{\bar{\alpha}}\pi_{i}^{i}\,(t)+\lambda\,(n-1)\,\frac{V_{j}^{i}\beta}{\bar{\alpha}}\left[w_{j}^{i}\,(t)+b\,(t)\right] \tag{72}$$

$$(\rho+\lambda)\,w_{j}^{i}\,(t)+i_{t}-\rho=\dot{w}_{j}^{i}\,(t)-\bar{\alpha}\left[\frac{V_{ij}^{i}+V_{jj}^{i}+(n-2)\,V_{jk}^{i}}{V_{j}^{i}}\right]\dot{a}\,(t)+\frac{1}{V_{j}^{i}}\pi_{j}^{i}\,(t)+\lambda\,(n-2)\,\beta\left[w_{j}^{i}\,(t)+b\,(t)\right] \tag{73}$$

$$\begin{aligned}(\rho+\lambda)\,w_{ii}^{i}\,(t)+i_{t}-\rho=\dot{w}_{ii}^{i}\,(t)&-\frac{\bar{\alpha}}{V_{ii}^{i}}\left[V_{iii}^{i}+(n-1)\,V_{iij}^{i}\right]\dot{a}\,(t)+\frac{1}{V_{ii}^{i}}\pi_{ii}^{i}\,(t)\\&+\lambda\,(n-1)\left\{\frac{V_{jj}^{i}\beta^{2}}{V_{ii}^{i}}\left[w_{jj}^{i}\,(t)+2b\,(t)\right]+\frac{2V_{ij}^{i}\beta}{V_{ii}^{i}}\left[w_{ij}^{i}\,(t)+b\,(t)\right]\right\}\end{aligned} \tag{74}$$

$$\begin{aligned}(\rho+2\lambda)\,w_{ij}^{i}\,(t)+i_{t}-\rho=\dot{w}_{ij}^{i}\,(t)&-\frac{\bar{\alpha}}{V_{ij}^{i}}\left[V_{iij}^{i}+V_{ijj}^{i}+(n-2)\,V_{ijk}^{i}\right]\dot{a}\,(t)+\frac{1}{V_{ij}^{i}}\pi_{ij}^{i}\,(t)\\&+\lambda\,(n-2)\left\{\frac{V_{jj}^{i}\beta^{2}}{V_{ij}^{i}}\left[w_{jj}^{i}\,(t)+2b\,(t)\right]+\frac{V_{jk}^{i}\beta}{V_{ij}^{i}}\left[w_{jk}^{i}\,(t)+b\,(t)\right]+\beta\left[w_{ij}^{i}\,(t)+b\,(t)\right]\right\}\end{aligned} \tag{75}$$

$$\begin{aligned}(\rho+\lambda)\,w_{jj}^{i}\,(t)+i_{t}-\rho=\dot{w}_{jj}^{i}&-\frac{\bar{\alpha}}{V_{jj}^{i}}\left[V_{ijj}^{i}+V_{jjj}^{i}+(n-2)\,V_{jjk}^{i}\right]\dot{a}\,(t)+\frac{1}{V_{jj}^{i}}\pi_{jj}^{i}\,(t)\\&+\lambda\,(n-2)\left\{\frac{V_{jj}^{i}\beta^{2}}{V_{jj}^{i}}\left[w_{jj}^{i}\,(t)+2b\,(t)\right]+\frac{2V_{jk}^{i}\beta}{V_{jj}^{i}}\left[w_{jk}^{i}\,(t)+b\,(t)\right]\right\}\\&+\lambda\left\{\frac{V_{ii}^{i}\beta^{2}}{V_{jj}^{i}}\left[w_{ii}^{i}\,(t)+2b\,(t)\right]+\frac{2V_{ij}^{i}\beta}{V_{jj}^{i}}\left[w_{ij}^{i}\,(t)+b\,(t)\right]\right\}\end{aligned} \tag{76}$$

$$\begin{aligned}(\rho+2\lambda)\,w_{jk}^{i}\,(t)+i_{t}-\rho=\dot{w}_{jk}^{i}&-\frac{\bar{\alpha}}{V_{jk}^{i}}\left[V_{ijk}^{i}+V_{jjk}^{i}+V_{jkk}^{i}+(n-3)\,V_{jkl}^{i}\right]\dot{a}\,(t)+\frac{1}{V_{jk}^{i}}\pi_{jk}^{i}\,(t)\\&+\lambda\,(n-3)\left\{\frac{V_{jj}^{i}\beta^{2}}{V_{jk}^{i}}\left[w_{jj}^{i}\,(t)+2b\,(t)\right]+\frac{2V_{jk}^{i}\beta}{V_{jj}^{i}}\left[w_{jk}^{i}\,(t)+b\,(t)\right]\right\}\\&+\lambda\left\{\frac{V_{ii}^{i}\beta^{2}}{V_{jk}^{i}}\left[w_{ii}^{i}\,(t)+2b\,(t)\right]+\frac{2V_{jk}^{i}\beta}{V_{jj}^{i}}\left[w_{jk}^{i}\,(t)+b\,(t)\right]\right\}\end{aligned} \tag{77}$$

In general there are thus 6 ODEs because β may be time-dependent hence $b\,(t)\neq 0$. But note that if $b\,(t)=0$ then the system becomes block-recursive and we can solve separately the first two equations in a and w_{j}^{i}. From the optimality conditions we have

$$\dot{\beta}=-\dot{\alpha}\left[W_{iij}^{i}\left[1-(n-1)\,\beta\right]+(n-1)\,W_{ijj}^{i}-\beta W_{iii}\right]$$

Using our perturbation argument we can show that there exists a third-order cross-elasticity ϵ_{iij}^{i} such that at the steady state

$$V_{iij}^{i}\left[1-(n-1)\,\beta\right]+(n-1)\,V_{ijj}^{i}-\beta V_{iii}=0 \tag{78}$$

where $V_{iij}, V_{ijj}, V_{iii}$ are solutions to the system (64). Thus in what follows we consider β

as constant for the first-order dynamics to simplify expressions, although we could solve the larger system without this assumption.

The last step is to replace the single "reset price" variable $a(t)$ with two variables, the aggregate price level $p(t)$ and inflation $\pi(t) = \dot{p}(t)$ using our aggregation result that inflation follows

$$\pi(t) = \lambda\left[1-(n-1)\beta(t)\right]\left[\log\alpha(t) - \log P(t)\right].$$

After log-linearization we have

$$a(t) = \frac{\pi(t)}{\lambda\left[1-(n-1)\beta\right]} + p(t).$$

Therefore, we obtain

$$
\begin{aligned}
\dot{\pi}(t) &= \left[\frac{\Pi^i_{ii} + (n-1)\Pi^i_{ij}}{V^i_{ii} + (n-1)V^i_{ij}} - \lambda\left[1-(n-1)\beta\right]\right]\pi(t) \\
&\quad + \lambda\left[1-(n-1)\beta\right]\left[\frac{\bar{M}C\Pi^i_{i,MC} + \Pi^i_i z_p}{\bar{\alpha}\left[V^i_{ii} + (n-1)V^i_{ij}\right]} + \frac{\Pi^i_{ii} + (n-1)\Pi^i_{ij}}{V^i_{ii} + (n-1)V^i_{ij}}\right] p(t) \\
&\quad + \frac{\lambda(n-1)V^i_j\beta}{\bar{\alpha}\left[V^i_{ii} + (n-1)V^i_{ij}\right]} w^i_j(t) \\
&\quad + \frac{\lambda\left[1-(n-1)\beta\right]\bar{M}C\Pi^i_{i,MC}}{\bar{\alpha}\left[V^i_{ii} + (n-1)V^i_{ij}\right]} k(t) \\
&\quad + \frac{\lambda\left[1-(n-1)\beta\right]\Pi^i_i z_c}{\bar{\alpha}\left[V^i_{ii} + (n-1)V^i_{ij}\right]} c(t) \\
\dot{p}(t) &= \pi(t) \\
\dot{w}^i_j(t) &= \frac{\bar{\alpha}}{\lambda\left[1-(n-1)\beta\right]}\left\{\left[\frac{V^i_{ij} + V^i_{jj} + (n-2)V^i_{jk}}{V^i_j}\right]\left[\frac{\Pi^i_{ii} + (n-1)\Pi^i_{ij}}{V^i_{ii} + (n-1)V^i_{ij}}\right] - \frac{\Pi^i_{ij} + \Pi^i_{jj} + (n-2)\Pi^i_{jk}}{V^i_j}\right\}\pi(t) \\
&\quad + \Bigg[\left[\frac{V^i_{jj} - V^i_{ii} + (n-2)\left(V^i_{jk} - V^i_{ij}\right)}{V^i_{ii} + (n-1)V^i_{ij}}\right]\frac{\bar{M}C\Pi^i_{i,MC} + \Pi^i_i z_p}{V^i_j} \\
&\quad + \bar{\alpha}\left\{\left[\frac{V^i_{ij} + V^i_{jj} + (n-2)V^i_{jk}}{V^i_j}\right]\left[\frac{\Pi^i_{ii} + (n-1)\Pi^i_{ij}}{V^i_{ii} + (n-1)V^i_{ij}}\right] - \frac{\Pi^i_{ij} + \Pi^i_{jj} + (n-2)\Pi^i_{jk}}{V^i_j}\right\}\Bigg] p(t) \\
&\quad + \left[\rho + \lambda\left(1-(n-2)\beta\right)\right] w^i_j(t) \\
&\quad + \left[\frac{V^i_{jj} - V^i_{ii} + (n-2)\left(V^i_{jk} - V^i_{ij}\right)}{V^i_{ii} + (n-1)V^i_{ij}}\right]\frac{\Pi^i_i z_c}{V^i_j} c(t) \\
&\quad + \left[\frac{V^i_{jj} - V^i_{ii} + (n-2)\left(V^i_{jk} - V^i_{ij}\right)}{V^i_{ii} + (n-1)V^i_{ij}}\right]\frac{\bar{M}C\Pi^i_{i,MC}}{V^i_j} k(t) \\
&\quad + i_t - \rho
\end{aligned}
$$

or in matrix form that the vector

$$\mathbf{Y}(t) = \left(\pi(t), p(t), w^i_j(t)\right)'$$

solves the linear differential equation

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{Z}_k k(t) + \mathbf{Z}_c c(t) + \mathbf{Z}_i\left[i(t) - \rho\right]$$

where $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{Z}_k, \mathbf{Z}_c, \mathbf{Z}_i \in \mathbb{R}^3$ collect the terms above (evaluated at the steady state), with boundary conditions $\lim_{t\to\infty} \mathbf{Y}(t) = 0$. The solution is given by

$$\mathbf{Y}(t) = -\int_0^{\infty} e^{s\mathbf{A}} \left\{ \mathbf{Z}_k k(t+s) + \mathbf{Z}_c c(t+s) + \mathbf{Z}_i \left[i(t+s) - \rho \right] \right\} ds$$

where $e^{s\mathbf{A}} = \sum_{k=0}^{\infty} \frac{s^k \mathbf{A}^k}{k!}$ denotes the matrix exponential of $s\mathbf{A}$. Proposition 8 then follows by taking the first component of $\mathbf{Y}$.

Proof of Corollary 2. Let $[\mathbf{M}]_i$ and $[\mathbf{M}]_{xy}$ denote the ith line and the (x, y) element of a generic matrix $\mathbf{M}$ respectively. Let $\mathbf{B}(t) = \mathbf{Z}_k k(t) + \mathbf{Z}_c c(t) + \mathbf{Z}_r [r(t) - \rho]$. Iterating $\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}(t)$, we have for all $k \geq 1$

$$\mathbf{Y}^{(k)}(t) = \mathbf{A}^k \mathbf{Y}(t) + \sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t).$$

Taking the first line for each $k = 1, \ldots, n = 3$, we have n equations

$$\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 = \left[\mathbf{A}^k \right]_1 \mathbf{Y}(t)$$

which we can each rewrite as

$$\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 - \left[\mathbf{A}^k \right]_{11} \pi(t) = \sum_{i=2}^{n} \left[\mathbf{A}^k \right]_{1i} y_i(t)$$

Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_{12} & \ldots & \mathbf{A}_{1n} \\ [\mathbf{A}^2]_{12} & & [\mathbf{A}^2]_{1n} \\ \vdots & & \vdots \\ [\mathbf{A}^n]_{12} & \ldots & [\mathbf{A}^n]_{1n} \end{pmatrix} \in \mathbb{R}^{n \times (n-1)}$$

Take any vector $\gamma^\pi = \left(\gamma_j^\pi \right)_{j=1}^n$ in $\ker \mathbf{M}'$ (whose dimension is at least 1), i.e., such that $\mathbf{M}' \gamma^\pi = 0 \in \mathbb{R}^{n-1}$. Then

$$\sum_{k=1}^{n} \gamma_k^\pi \left(\frac{d^k \pi(t)}{dt^k} - \left[\sum_{j=0}^{k-1} \mathbf{A}^j \mathbf{B}^{(k-1-j)}(t) \right]_1 - \left[\mathbf{A}^k \right]_{11} \pi(t) \right) = 0.$$

and we can define $\gamma_0^\pi = -\sum_{k=1}^{n} \gamma_k^\pi \left[\mathbf{A}^k\right]_{11}$. This simplifies to

$$\begin{aligned}\dddot{\pi} &= \left(\mathbf{A}_{\pi\pi} + \mathbf{A}_{ww}\right)\ddot{\pi} \\ &\quad + \left(\mathbf{A}_{\pi p} + \mathbf{A}_{\pi w}\mathbf{A}_{w\pi} - \mathbf{A}_{\pi\pi}\mathbf{A}_{ww}\right)\dot{\pi} \\ &\quad + \left(\mathbf{A}_{\pi w}\mathbf{A}_{wp} - \mathbf{A}_{\pi p}\mathbf{A}_{ww}\right)\pi \\ &\quad + \mathbf{A}_{\pi w}\dot{\mathbf{B}}_w + \ddot{\mathbf{B}}_\pi - \mathbf{A}_{ww}\dot{\mathbf{B}}_\pi\end{aligned} \tag{79}$$

H.1 One-time shocks

Given (16) we can guess and verify that $x = \psi_x e^{-\xi t}$ for all variables $x \in \{\pi, k, c, r-\rho, i-\rho\}$ and solve for the coefficients ψ_x using the system

$$\begin{aligned}\psi_\pi\left(\gamma_0^\pi - \gamma_1^\pi\xi + \gamma_2^\pi\xi^2 - \gamma_3^\pi\xi^3\right) &= \psi_k\left(\gamma_0^k - \gamma_1^k\xi + \gamma_2^k\xi^2\right) \\ &\quad + \psi_c\left(\gamma_0^c - \gamma_1^c\xi + \gamma_2^c\xi^2\right) \\ &\quad + \left(\psi_i - \psi_\pi\right)\left(\gamma_0^r - \gamma_1^r\xi + \gamma_2^r\xi^2\right) \\ -\xi\psi_c &= \sigma^{-1}\left(\psi_i - \psi_\pi - \epsilon_0^r\right) \\ \psi_i &= \phi_\pi\psi_\pi + \epsilon_0^m + (1-\kappa)\,\epsilon_0^r\end{aligned}$$

Thus

$$\begin{aligned}\psi_c &= \frac{1}{\sigma\xi}\left(\psi_\pi\left(1-\phi_\pi\right) + \kappa\epsilon_0^r - \epsilon_0^m\right) \\ \psi_k &= \psi_c\left(\chi + \sigma\right)\end{aligned}$$

and

$$\begin{aligned}\psi_\pi\left(\gamma_0^\pi - \gamma_1^\pi\xi + \gamma_2^\pi\xi^2 - \gamma_3^\pi\xi^3\right) &= \frac{1}{\sigma\xi}\left(\kappa\epsilon_0^r - \epsilon_0^m - \psi_\pi\left(\phi_\pi - 1\right)\right)\left[\left(\chi+\sigma\right)\left(\gamma_0^k - \gamma_1^k\xi + \gamma_2^k\xi^2\right) + \left(\gamma_0^c - \gamma_1^c\xi + \gamma_2^c\xi^2\right)\right] \\ &\quad + \left(\epsilon_0^m + (1-\kappa)\,\epsilon_0^r + \psi_\pi\left(\phi_\pi - 1\right)\right)\left(\gamma_0^r - \gamma_1^r\xi + \gamma_2^r\xi^2\right)\end{aligned}$$

which yields

$$\psi_\pi = \frac{\frac{\kappa\epsilon_0^r - \epsilon_0^m}{\sigma\xi}\left[\left(\chi+\sigma\right)\left(\gamma_0^k - \gamma_1^k\xi + \gamma_2^k\xi^2\right) + \left(\gamma_0^c - \gamma_1^c\xi + \gamma_2^c\xi^2\right)\right] + \left(\epsilon_0^m + (1-\kappa)\,\epsilon_0^r\right)\left(\gamma_0^r - \gamma_1^r\xi + \gamma_2^r\xi^2\right)}{\gamma_0^\pi - \gamma_1^\pi\xi + \gamma_2^\pi\xi^2 - \gamma_3^\pi\xi^3 + \left(\phi_\pi - 1\right)\left[\frac{\left(\chi+\sigma\right)\left(\gamma_0^k - \gamma_1^k\xi + \gamma_2^k\xi^2\right) + \left(\gamma_0^c - \gamma_1^c\xi + \gamma_2^c\xi^2\right)}{\sigma\xi} - \left(\gamma_0^r - \gamma_1^r\xi + \gamma_2^r\xi^2\right)\right]}$$

I Non-linear Duopoly

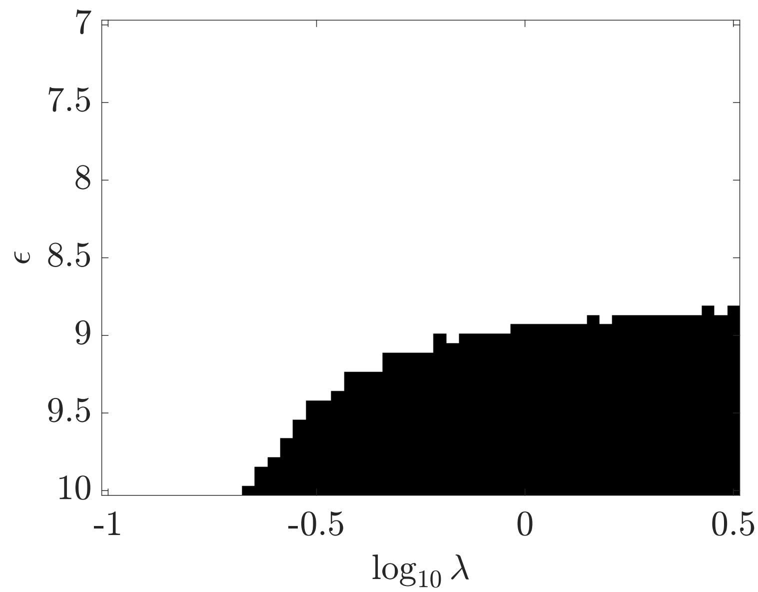


Figure 17: In white: convergence of value function iteration algorithm towards a monotone MPE in (λ, ϵ) space.