City Characteristics, Land Prices and Volatility

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Abstract

We develop a model that describes the evolution of rental rates in a monocentric city. The model explores how differences in urban characteristics and the agglomeration externalities associated with their industrial mix affect both the level and volatility of rents. The volatilities of both commercial and residential rental rates are amplified when agglomeration externalities are stronger. The volatility of commercial rents is dampened when constraints on development and the transportation infrastructure inhibit growth, but these same constraints on growth can dampen the volatility of residential rents when agglomeration externalities are strong. An implication of the model is that productivity shocks can have a larger immediate effect on rents in large cities but a larger long-term effect in smaller cities, suggesting that prices may initially respond more in small cities even when their rents respond less.

1 Introduction

According to a research report by Savills, a UK real estate consultant, at the end of 2017, the value of the world's real estate reached US $280 trillion, which is about 3 times the world's GDP. Real estate is clearly the most important capital asset in the world economy, but as illustrated by the recent financial crisis, our understanding of the determinants of real estate valuation, and in particular, the volatility of real estate prices, is still incomplete.

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This paper explores the determinants of the volatility of rents and property values in a setting where cities vary along a number of important dimensions. Following Lucas and Rossi-Hansberg (2002), we explicitly consider agglomeration externalities that make firms more productive in cities with greater populations. Offsetting these agglomeration benefits are congestion costs that make it costlier for workers to commute to their jobs in cities with greater populations. These commuting costs, which effectively constrain the availability of residential land, are determined by city characteristics such as the citys size, its transportation technology, and the proportion of its land that cannot be developed. As we show, these city characteristics constrain a city's growth, which in term determines the magnitude and timing of the response of its population, wages and rents to exogenous productivity shocks.

Our model builds on the literature that explores the internal structure of cities, exemplified by Lucas and Rossi-Hansberg (2002), Ahlfeldt et al. (2015) and Chatterjee and Eyigungor (2017). This literature is part of the broader literature of spatial quantitative economics that provides insights into the spatial distribution of economic activities, as reviewed by Redding and Rossi-Hansberg (2017).[1] For simplicity and tractability, we focus on the special case of monocentric city structure, which allows us to derive analytical expressions for the elasticity of land rent with respect to productivity changes. As shown in Lucas and Rossi-Hansberg (2002) and Chatterjee and Eyigungor (2017), the monocentric city structure arises endogenously when the transportation cost gradient, i.e. the rate at which transportation cost increases with distance, is small relative to the strength of agglomeration externalities and the rate at which the externality decays with distance between firms.

Following the Rosen (1979) and Roback (1982) system of city framework, we assume that individuals are mobile, and thus enjoy the same reservation utility in each city. However, because there exists cross-city differences in productivity and urban characteristics, cities will have different populations and their workers will earn different wages and pay different rents. Because we are mainly interested in the effects of various forms of land supply constraints on land rent volatility, we go beyond the assumptions of inelastic land supply and no-congestion in transportation, allowing the city to expand in response to productivity increases.

The population of cities in our model is determined by three different channels. The first is the productivity channel; cities that host more productive industries will be larger. The second is the transit channel; cities with transit technology that exhibits less congestion will be larger. And the third is land constraints; cities with more undevelopable land will have lower populations.

Commercial and residential rents, as well as their sensitivity to exogenous shocks to productivity, are also determined by these three channels. Both commercial and residential rents are

[1]This literature, in turn, builds on the seminal monocentric city models of Alonso (1964), Mills (1967), and Muth (1969).

higher in cities where productivity is higher, however, only commercial rents are always higher in cities with fewer residential land constraints and less costly transit. The relationship between land constraints and transit costs and residential rents are somewhat more complicated because residential rents are affected by two offsetting channels. The first is a scarcity channel; ceteris paribus, reducing access to residential land increases rents. The second is the agglomeration channel. When less land is available for development, the city's population will be lower, which reduces total factor productivity because the reduced agglomeration effect. Wages are thus lower, which in turn reduces rents. As we show, when the agglomeration parameter in the model is sufficiently large, the agglomeration channel dominates the scarcity channel. When this is the case, both undevelopable land and increased transit costs reduce rather than increase residential rents.

The volatility of rent in our model is amplified because of the feedback between productivity growth and population growth. Specifically, a positive exogenous shock to a city's productivity increases wages, which in turn attracts new migrants to the city. Because of the agglomeration externalities, the increased population makes the workers even more productive, attracting more migrants, further increasing productivity, and so on, until the city reaches a new steady state. In the absence of frictions, this process will be instantaneous, and cities will reach a new steady-state immediately. As we discuss below, if the agglomeration externalities take time to materialize, it will take time for the city to reach a new steady-state.

As we show, because of this feedback process, the volatility of both population growth and rents are amplified more when agglomeration externalities are higher. In contrast, the presence of undevelopable land and transit costs always dampen the volatility of both population growth and commercial rents. However, because of the offsetting effect of agglomeration, development constraints and transit costs do not necessarily increase the volatility of residential rents. Indeed, a key insight of our model is that when the agglomeration channel is stronger than the scarcity channel, supply constraints and transit congestion can actually dampen the effect of productivity shocks on residential rents.

When the agglomeration effect takes time to evolve there can be important distinctions between the initial response and the steady state response to productivity shocks. As we show, the difference between the immediate and steady-state response depends on the exogenous characteristics of cities. We are particularly interested in contrasting the effect of productivity shocks on small (relatively unconstrained) and large (more constrained) cities that host industries that exhibit high agglomeration externalities. As we show, an equivalent productivity shock initially affects rents and wages more in larger cities, because of constraints on expansion, but in the long run, rents in smaller cities increase more, because of the future agglomeration benefits that arise as the city grows. Given that property prices anticipate future rent increases, they initially respond more to productivity shocks in smaller cities even though wages and rents initially respond very little.

Our analysis contributes to a growing literature that examines the relationship between city structures and the sensitivity of property prices to exogenous shocks. The closest to our paper is Chatterjee and Eyigungor (2017), which considers the effect of physically constraining the size of a city.[2] We extend the Chatterjee and Eyigungor (2017) analysis by considering a number of urban attributes that effectively constrain how cities grow in response to productivity increases. Specifically, we identify a simple measure of the effective land supply constraint, which combines the effect of transportation technology and both the physical size and the population of the city. This notion that the effective land supply constraint depends on transportation technology is related to Miles and Sefton (2020), which shows that changes in urban house prices over time and across locations are affected by changes in commuting speeds. The focus in this paper is on how long-term trends in house price growth depend on transportation technology improvements and productivity increases, assuming no agglomeration effect in production. In contrast, our study focuses on how fluctuations in land rents depend on the interaction of agglomeration effects and exogenous city characteristics, which include land supply constraints as well as transportation technology. We also consider how these exogenous characteristics affect the population and physical size of a city, and discuss how these relationships imply important differences in the land rent dynamics in large and small cities.

Our paper also addresses issues raised by Davidoff (2013), Gao et al. (2020) and Nathanson and Zwick (2018), which introduce combinations of behavioral biases and market frictions to explain why relatively unconstrained cities, like Las Vegas, experienced large price run ups in the early 2000s.[3] We contribute to this debate by showing that large price run ups can be generated with rational and unconstrained agents. Indeed, we show that productivity shocks are likely to be amplified the most in relatively unconstrained cities that host, or hope to host, industries that exhibit strong agglomeration externalities. In such cities, we may observe large increases in prices, even though rents and wages may initially respond only modestly.

Our study is also related to the recent literature that explores how housing supply constraints have contributed to the increase in the cross-city dispersion in housing prices. In particular, Nieuwerburgh and Weill (2010) develop a dynamic general equilibrium model that illustrates how housing supply constraints can amplify relatively small differences in productivity and create relatively large differences in house prices, and Gyourko and Sinai (2013) provide a model

[2]Chatterjee and Eyigungor (2017) endogenously allows a city to be either monocentric or decentralized. Our analysis is particularly close to the analysis in their setting that results in monocentric cities.

[3]Glaeser and Nathanson (2017) also consider behavioral biases in a dynamic model of the housing market. They assume that home buyers naively use past house prices to estimate housing demand and show that this can create persistent house price changes. The dynamic version of our model generates serially correlated land rents, because we assume that agglomeration externalities take time to materialize. However, because we assume that individuals are rational, actual land prices follow a random walk.

that describes how supply constraints can further increase dispersion in housing prices if cities attract a heterogeneous mix of residents with different tastes for amenities, i.e., certain cities will have amenities that cater to a wealthier clientele who are willing to pay higher housing prices. Hseih and Moretti (2017) also consider supply constraints and develop a model that illustrates how inelastic housing supply, perhaps caused by restrictive zoning, dampens economic growth by implicitly limiting migration from less productive to more productive cities.

These more recent models extend the Rosen (1979) and Roback (1982) framework, which takes the supply elasticities and productivity shocks as given, but ignore within-city spatial characteristics and agglomeration externalities that can amplify and dampen exogenous productivity shocks. By including these elements in a monocentric city model, we provide the micro-foundations of the cross-city differences in land supply constraints and productivity differences considered in these recent papers. The implications of these micro-foundations are not completely obvious. For example, Hseih and Moretti (2017) suggest that a policy of developing public transportation to relax housing supply constraints in high productivity cities can reduce spatial misallocations of labor. Our model explicitly addresses the role of transportation and shows that although better transportation increases migration to high productivity cities, it does not necessarily dampen the effect of exogenous productivity shocks on housing prices.[4]

The rest of the paper is organized as follows. Section 2 introduces the benchmark model and shows, that in general, the model exhibits multiple equilibria. Section 3 focuses on what we think is the most plausible equilibrium and examines the elasticities of wages, population and land rents with respect to changes in exogenous shocks to productivity. This section also shows how the elasticities depend on the city size in the short run and the long run. Section 4 discusses how the presence of undevelopable land affects city configuration and real estate volatility. Section 5 considers these same elasticities in alternative settings that allow us to explore the implications of CBD land flexibility and capital mobility. Section 6 concludes and provides a discussion of potential future studies.

2 The Benchmark Model

In this section we develop our benchmark model. As we describe below, relative to existing monocentric urban models, the main contribution of the benchmark model lies in the linkage between city characteristics and what is effectively the elasticity of the supply of land. Specifically, we consider the fraction of residential land that cannot be developed and the transportation technology. In the extended model, we further consider the flexibility of city boundaries. Both

[4]This observation is also related to the transportation literature (e.g. Duranton and Turner (2012)) which observes that increases in transportation capacity does not always reduce congestion. In our model, the gain from better transportation comes from the agglomeration benefits associated with the rising city population.

capital and labour are assumed to be perfectly mobile in the benchmark model and the size of the CBD is exogenous. However, in extended models we consider endogenous CBD size and immobile capital.

2.1 City Geometry and Transportation

The circular city consists of a commercial CBD of size S, implying a CBD radius of $\sqrt{S}/\sqrt{\pi}$, surrounded by discrete rings of residential land indexed by i, with the ring nearest to the CBD being $i = 0$. The land area in each ring is normalized to unity and includes both developable and undevelopable land. Specifically, Λ percent of the land in each ring cannot be developed, because of either geographical constraints, such as lakes or oceans, or regulatory constraints, such as green areas that are used for parks or drainage. We will assume that these areas are evenly distributed throughout the city and have no inherent amenity values.

The distance from a ring to the CBD is measured by the distance between its inner circle to the nearest boundary of the CBD. Thus, for the i^{th} ring, the distance is the difference between the radius of its inner circle and the radius of the CBD. Since the inner circle of the i^{th} ring encompasses a land area of $S + i$, its radius is $\sqrt{S+i}/\sqrt{\pi}$, hence its distance is

$$j = \frac{\sqrt{S+i} - \sqrt{S}}{\sqrt{\pi}} \tag{1}$$

The distance j is simply a non-linear transformation of the location index i, so without loss of generality, we use the distance j to denote location, with $j = 0$ representing the inner-most ring with a zero distance. The outer-most ring, denoted by $j = J$, is endogenously determined by equating its rent with the exogenous agricultural rent.

We use w to denote the wage for all workers, and wage net of transportation costs for workers living at location j is $w \times e^{-f(j,N)}$ where N is the city population. The function $f(j, N)$ assumes that transportation costs increase with distance, since transportation takes time, and also increases with population, since larger cities are more congested.[5] Specifically, the transportation cost function $f(j, N)$ is assumed to have the following form:

$$f(j, N) = \beta_0 + \beta_1 j + \beta_2 jN \tag{2}$$

where $\beta_1 > 0$ is the distance gradient of transportation, and $\beta_2 > 0$ captures the congestion effect. The congestion effect increases with distance since

$$\frac{\partial f(j, N)}{\partial N} = \beta_2 j. \tag{3}$$

[5] The exact transportation cost as a fraction of wage is $1 - e^{-f(j,N)}$. We refer to $f(j, N)$ as the transportation cost function, noting that $1 - e^{-f(j,N)} \approx f(j, N)$ when $f(j, N)$ is small.

2.2 Firms and Workers

The city is populated by a continuum of firms and a continuum of workers. Both are price takers and they produce tradable goods which serve as the numeraire in the model. Following the standard practice in the urban literature, we assume the land and capital are owned by absentee owners who collect rent from either land or capital but do not live in the city.

Workers Workers are perfectly mobile both within and across cities, which implies that they realize a reservation level of utility wherever they live, denoted $\underline{u}$. Each worker is endowed with one unit of labour and allocates the wage to land rent, transportation cost, and the consumption good. Workers have the option to live adjacent to the CBD and have zero commuting costs, or alternatively, they can live farther-out and spend resources to commute to the CBD.

Workers at location j take their wage and the land rent as given and choose their consumption of land, h, and the tradable good, c to solve the following optimization problem:

$$\begin{aligned} &\max{}_{c,h} = c^{1-\theta} h^{\theta} \\ s.t. \quad & c + p_r(j)h = w \times e^{-f(j,N)} \end{aligned} \tag{4}$$

where $p_r(j)$ is the rental rate of residential land in location j.[6]

It is straightforward to show that the optimal allocation between land and the tradable good satisfies:

$$p_r(j) = \frac{\partial u(c,h)/\partial h}{\partial u(c,h)/\partial c} = \frac{\theta}{1-\theta}\frac{c}{h} \tag{5}$$

where the right side is the marginal rate of substitution between land and the tradable good.

From equation (5), we get $c = \frac{1-\theta}{\theta} p_r(j)h$. Substituting this into the budget constraint yields the optimal tradable good and land demand functions

$$c(j) = (1-\theta)w \times e^{-f(j,N)}, \tag{6}$$

$$h(j) = \theta \frac{we^{-f(j,N)}}{p_r(j)}. \tag{7}$$

Since workers are identical, the rents, in equilibrium, make workers indifferent about where they live. Because rents decrease with distance to the CBD, workers that live near the CBD consume less land but more of the tradable good.

[6] The above maximization problem implicitly assumes that there are no cross-city differences in amenities. We can easily account for differences in amenities by multiplying the worker's utility levels by an amount that differs across locations, i.e. modifying the objective to $mc^{1-\theta}h^{\theta}$ where m denotes amenity. This will change a city's reservation utility and thereby changes the city population. But all the propositions that follow are unchanged.

Firms Firms are also perfectly mobile, and for the baseline model there is no cost associated with adjusting capital. There exists a unit measure of identical firms located in the CBD which has a fixed size of S in the baseline model. Firms use the CBD land long with capital and labour to produce the tradable good using a constant returns to scale Cobb-Douglas production technology:

$$F(\ell, k, n) = A\ell^{\sigma}k^{\xi}n^{1-\sigma-\xi} \tag{8}$$

where ℓ, k and n are land, capital and labour input respectively, the relative importance of which is determined by the parameters σ, ξ, and $1-\sigma-\xi$, respectively, and A is the total factor productivity (TFP) of this city. To simplify our notations, we assume that $A = 1$ in all other cities.

The firms take A, land rent p_c, the price of capital r, and wage w as given, and solve the following optimization problem:

$$\max_{\ell,k,n} F(\ell, k, n) - wn - rk - p_c, \ell$$

subject to equation (8). From the first-order conditions, we obtain the usual allocation rules as follows:

$$\frac{n}{\ell} = \left(\frac{1-\sigma-\xi}{\sigma}\right)\left(\frac{p_c}{w}\right), \tag{9}$$

$$\frac{n}{k} = \left(\frac{1-\sigma-\xi}{\xi}\right)\left(\frac{r}{w}\right), \tag{10}$$

$$\frac{\ell}{k} = \left(\frac{\sigma}{\xi}\right)\left(\frac{r}{p_c}\right). \tag{11}$$

In other words, production inputs are determined by their relative prices and their marginal contributions to the production.

Agglomeration The agglomeration effect in our model is modelled as production externalities in the form of TFP being an increasing function of city population. Specifically, the city level TFP is given by

$$A = \tilde{A}N^{\lambda}, \tag{12}$$

where $\tilde{A}$ is the exogenous productivity of a city, and λ is the agglomeration parameter that determines how the city TFP increases with the total number of workers in the city.

2.3 Bid-rent Functions

We separately describe bid-rent functions for the residential and commercial land markets. The residential bid-rent function, as in Fujita (1989), describes the rent of residential land as a function of the wage and the distance from the CBD. Following Lucas and Rossi-Hansberg (2002), the commercial bid-rent function describes the rent of commercial land as a function of the wage and the price of capital.

Commercial Bid-rent Function Because firms are competitive and enter and exit the city freely, owners of commercial land capture all the economic benefits from production. Thus, commercial land rent equals the maximum revenue from one unit of land after paying for labour and capital. Production per unit of land is $f(\ell) = Ak^{\xi}n^{1-\sigma-\xi}$, which implies that the commercial rent is the solution of the following maximization problem:

$$p_c = \max_{n,k} Ak^{\xi}n^{1-\sigma-\xi} - wn - rk.$$

The first-order conditions with respect to labour and capital are:

$$\begin{aligned} wn &= (1-\sigma-\xi)Ak^{\xi}n^{1-\sigma-\xi}, && (13) \\ rk &= \xi Ak^{\xi}n^{1-\sigma-\xi}. && (14) \end{aligned}$$

From the above three equations we obtain the commercial bid-rent function shown below.[7]

$$p_c = \left[\frac{A\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}} . \quad (15)$$

We use $A = \tilde{A}N^{\lambda}$ to substitute out A in the above equation to obtain

$$p_c = \left[\frac{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}} N^{\frac{\lambda}{\sigma}} . \quad (16)$$

Residential Bid-rent Functions By substituting equation (6)-(7) into the Cobb-Douglas utility function, we can express the worker's reservation utility as a function of rent, the wage rate and transportation costs:

$$\underline{u} = \frac{(1-\theta)^{1-\theta}\theta^{\theta}}{p_r(j)^{\theta}} we^{-f(j,N)} \quad (17)$$

which can be rearranged into the following, which is also shown in equation (18)

$$\begin{aligned} p_r(j) &= \left[\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right]^{1/\theta} \times \left[we^{-f(j,N)}\right]^{1/\theta} \\ &= B_0 \times \left[we^{-f(j,N)}\right]^{1/\theta}, && (18) \end{aligned}$$

[7]Substituting the first-order conditions back to the objective function, we obtain $p_c = \sigma Ak^{\xi}n^{1-\sigma-\xi}$, hence $p_c = \frac{\sigma}{1-\sigma-\xi}wn$ using equation (13). Also we get $n = (1-\sigma-\xi)^{(1-\xi)/\sigma}(\xi/r)^{\xi/\sigma}A^{1/\sigma}w^{-(1-\xi)/\sigma}$ using equations (13)-(14), which substitutes out n in $p_c = \frac{\sigma}{1-\sigma-\xi}wn$ to reach equation (15).

where we have defined $B_0 = \left[\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right]^{1/\theta}$ for simplicity of notation. It is noteworthy that $p_r(j)$ increases with w, while the commercial land rent p_c decreases with w. Ceteris paribus, higher wages allow workers to pay more for rent, but reduce the rent that firms can pay and still earn zero profits.

From equation (18) we have the following lemma.

Lemma 1 *The elasticity of residential land rent with respect to wage decreases with θ, the land share in workers' preference. Specifically,*

$$\frac{\partial log[p_r(j)]}{\partial log(w)} = \frac{1}{\theta}.$$

The above lemma implies that a 1% increase in the wage rate leads to a $1/\theta$ percent increase in rent.[8] Thus, when θ is smaller, residential land rent is more sensitive to wage. Our intuition is that on the margin, given a higher wager, workers increase non-land consumption more than land consumption when θ is smaller, and as a result, land owners can charge higher rents while ensuring renters the reservation utility. As we will show, this intuition is also key to understanding how residential land rent responds to productivity shocks.

The city boundary is determined by J, the distance from the city's border and the CBD. Equating the residential bid-rent function at location J and the exogenous agricultural rent $\underline{p}$, the equilibrium boundary satisfies:

$$\underline{p} = p_{r(j=J)} = B_0 \left[we^{-f(J,N)}\right]^{1/\theta}. \tag{19}$$

Parameters and Variables To help readers keep track of the notation, we list parameters and the exogenous variables of the model in Table 1.

2.4 Aggregate Labour Supply and Demand

The model has seven endogenous variables, $\{p_r, p_c, w, N, K, J, A\}$, and as we show in Appendix A.1 the general equilibrium is determined by seven equations. Furthermore, as shown in Appendix A.2, the system of seven equations can be reduced to two equations that describe the relation between wages and population:

$$log(N) = \frac{1}{\lambda - \sigma} log\left(\frac{r^{\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}S^{\sigma}}\right) + \frac{1-\xi}{\lambda-\sigma} log(w) \tag{20}$$

[8]This is in the partial equilibrium sense. In a general equilibrium, both wage and population respond endogenously to productivity shocks.

Table 1: Parameters and Exogenous Variables

λ	agglomeration parameter
ξ	capital share in production
σ	land share in production
θ	land share in preference
β_0	fixed transportation cost
β_1	distance gradient of transportation cost
β_2	congestion parameter of transportation cost
$\underline{u}$	reservation utility
$\underline{p}$	agricultural land rent
r	interest rate
S	CBD size
$\tilde{A}$	exogenous productivity
Λ	fraction of undevelopable residential land

$$log(N) = log\left(\frac{(1-\Lambda)B_0}{\theta}\right) + \frac{1-\theta}{\theta}log(w) + log\left(\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj\right).^9 \tag{21}$$

We refer to the above two equations as the aggregate labour demand function and the aggregate labour supply function respectively, similar to the "population supply function" and "population demand function" used in Fujita (1989). Equation (20) describes the total labour demand by firms in a city as a function of the wage rate.[10] The equation represents a linear relationship between $log(w)$ and $log(N)$, with a slope of $\frac{1-\xi}{\lambda-\sigma}$. When the agglomeration parameter λ is larger than the land share in production (σ), labour demand is more elastic. This is intuitive; with a large λ, a small increase in the wage leads to more productivity gains through the agglomeration effect, which enables firms to hire more workers. Equation (21) shows the number of workers that choose to live in the city, as a function of the wage rate, thus it describes the effective labour supply. The upper bound of the integral in the last term is the city boundary J, which is itself a function of the wage and population.

As we show in Appendix A.3, the slope of the labour supply curve has the following expression:

$$\frac{dlog(N)}{dlog(w)} = \frac{1}{F}, \tag{22}$$

where F captures the effect of adding an additional worker on equilibrium wages. It has the

[9] If we modify the worker's objective function from $c^{1-\theta}h^\theta$ to $mc^{1-\theta}h^\theta$ where m stands for amenity, then equation (21) becomes $log(N) = log\left(\frac{(1-\Lambda)m^{1/\theta}B_0}{\theta}\right) + \frac{1-\theta}{\theta}log(w) + log\left(\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj\right)$. From this equation, we can see that cross-city differences in amenity has the same effct on labour supply function as cross-city differences in the proportion of developable land.

[10] When making hiring decisions, firms take as given the land rent which itself is a function of wage.

following expression:

$$F = \frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right)\left(\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\right). \tag{23}$$

$$\approx (\beta_1 + 2\beta_2 N)J$$

$$= \frac{\partial f(J,N)}{\partial J} \times J + \frac{\partial f(J,N)}{\partial N} \times N \tag{24}$$

To understand the above expression, note that for a worker that lives on the city boundary, the change in the wage rate must compensate for the increase in the cost of traveling to the CBD, which has two components: (i) the increase in distance from the boundary to the CBD, which is captured by the first term in equation (24) and (ii) the increase in the cost of congestion, which is captured by the second term. An analysis of equation (23) reveals that the change in the wage rate also depends on θ, which determines the importance of land in the worker's preference, since workers substitute out some land consumption when land rent increases with wage.

We will refer to F as the effective supply constraint. Since the boundary of the city in our baseline model can expand indefinitely, it is the cost of transporting workers to the CBD that effectively constrains the physical size of the city. F is a function of population and the city boundary, and it captures two aspects of the cost of transporting workers from the boundary to the CBD: the effect of congestion that increases with population, and the effect arising from the distance between the CBD and the boundary. As we will discuss, because F increases with city population, the growth of large cities are more constrained than small cities, which has important implications on how city size affects the sensitivity of rents to productivity shocks.

Appendix A.3 proves the following lemma.

Lemma 2 *The aggregate labour supply curve is concave. Equivalently, the effective land supply constraint F increases with city population, i.e.,*

$$\frac{dF}{dN} > 0.$$

As we will discuss below, F plays a key role in determining the elasticity of population, wages and rents with respect to changes in total factor productivity (TFP). Various factors that contribute to the effective land supply constraint, such as the proportion of undevelopable land and the CBD size, do not show up explicitly in these elasticities, however, they affect the elasticities through their effect on F.

2.5 General Equilibrium

The supply and demand of labour, as expressed in Equations (20) and (21), determine the population and the wage in equilibrium. As we show in the appendix, these variables in turn determine the city level TFP, the capital stock, the city boundary, and residential and commercial land rents. Formally, we define equilibrium in this economy as follows:

Equilibrium Definition Given the parameters and exogenous variables listed in Table 1, an equilibrium is represented by the prices w, p_c and $p_r(j)$, the quantities N, K and J, and the city level TFP A, such that:

(i) the wage and population satisfies equations (20)-(21)*;*

(ii) K is determined by equating the marginal product of capital to the interest rate r;

(iii) J is determined by equating $p_r(J)$ to the agricultural land rent $\underline{p}$;

(iv) the city level TFP satisfies equation (12)*;*

(v) commercial land rent satisfies the bid-rent function (16)*;*

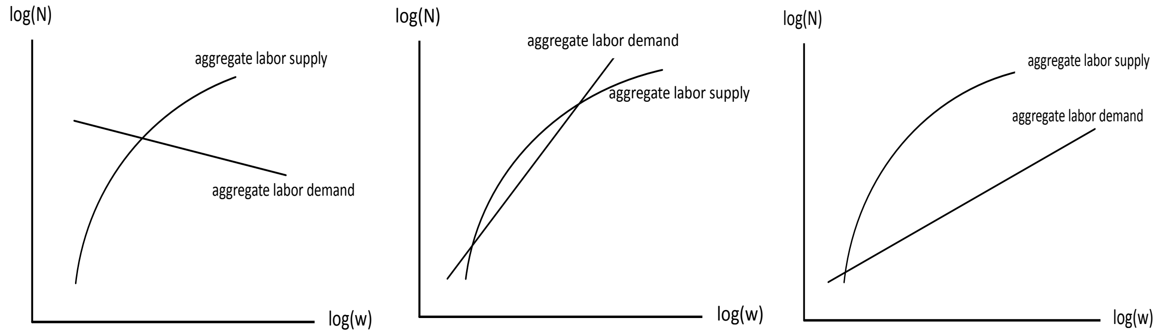
(vi) residential land rent satisfies the bid-rent function (18)*.*

Is the Equilibrium Unique? When the congestion effect is absent (i.e. $\beta_2 = 0$) and the city boundary is fixed, F as in equation (23) is a constant, thus the slope of aggregate labour supply curve $1/F$ is also a constant and equation (21) represents a linear relationship between the wage and population. In this case our model is a special case of Lucas and Rossi-Hansberg (2002). It has a unique equilibrium as both the aggregate labour demand function and the aggregate labour supply function are linear equations with different slopes.

However, the equilibrium is not necessarily unique in our more general setting that allows the physical size of the city to be endogenous and assumes that greater population causes more congestion. The number of equilibria depends on the slope of the aggregate labour demand curve, which is $\frac{1-\xi}{\lambda-\sigma}$, relative to the slope of the aggregate labour supply curve, $\frac{1}{F}$. Ultimately the equilibrium depends on the agglomeration strength, the land and capital share in production, and the effective supply constraint.

Figure 1 illustrates three possibilities. The first panel illustrates a case where the agglomeration effect is relatively weak with $\lambda \leq \sigma$; in this case there is a unique equilibrium. As illustrated in the second panel, when the agglomeration effect is stronger, there can be two equilibria. In the small city equilibrium, firms are less productive and thus pay lower wages, but workers are able to achieve their reservation utility levels because rents and congestion are lower in smaller cities. In the large city equilibrium, cities are more congested and rents are higher, but firms are able to pay a higher wage due to the higher TFP; which comes from the agglomeration externality. With multiple equilibria, workers in small and large cities achieve the same reservation utility and firms all make zero profits, but landlords receive more rents in large cities. Finally, the third panel illustrates the case where the agglomeration externalities are very high. In this case, we still get a small city equilibrium. However, a sufficiently large city will generate a level of utility for workers that is greater than the reservation utility, and the level of utility will grow without bounds as the size of the city increases.[11]

Figure 1: Equilibrium



(a) $\sigma \geq \lambda$ (b) $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$ (c) $\lambda \geq \sigma + (1-\xi)\frac{2\theta}{1-\theta}$

Note: The number of equilibrium (equilibria) is determined by the slopes of aggregate labour supply curve and aggregate labour demand curve.

These three possibilities are summarized in the following proposition. A formal proof is given in Appendix A.4.

Proposition 1 *Given the parameters and the exogenous variables listed in Table 1, the model*

(i) has a unique equilibrium if $\lambda \leq \sigma$;

(ii) has two equilibria if $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$;

(iii) has two possibilities if $\lambda > \sigma + (1-\xi)\frac{2\theta}{1-\theta}$: (1) an equilibrium with a small population, and (2) a situation where there is no steady-state, and the city grows without a bound.

Regularity Conditions In our analysis below, we impose the following two regularity conditions that rule out some perverse outcomes, with a formal proof provided in Appendix A.5:

$$F > \frac{\lambda - \sigma}{1-\xi}; \tag{25}$$

$$F > \beta_2 JN. \tag{26}$$

The first condition rules out the situation where the city grows without a bound, illustrated in panel (c). This is essentially the "no-black-hole condition" in Fujita et al. (1999). Intuitively, for a city not to grow without a bound, its agglomeration effect needs to be balanced out by the cost of commuting between the CBD and the city boundary.

The second regulatory condition rules out the possibility that a rise in productivity leads to a smaller geographical size of the city. As we will show in the next section, without the second

[11]Technically this happens when the slope of aggregate labour demand curve ($\frac{1-\xi}{\lambda-\sigma}$) is flatter than the slope of aggregate labour supply curve as N goes to infinity so that the two curves do not cross twice. As shown in Appendix A.4, the slope of aggregate labour supply curve is $\frac{1-\theta}{2\theta}$ when N goes to infinity.

regulatory condition, land rent near the city edge may fall in response to a positive productivity shock.

3 Comparative Statics

This subsection examines how land rent, the wage rate and population are affected by exogenous productivity changes. Specifically, we will analyze what we refer to as elasticities, which is the rate of change of an endogenous variable in response to an exogenous productivity shock. Our comparative statics initially examine changes in the steady state value of these endogenous variables. We later consider a case where the city does not move to the steady state immediately, and contrast the short-term elasticities with the long-run steady state elasticities.

We will pay particular attention to how these elasticities are affected by the production technology, the transportation technology and the amount of undevelopable land. The variables $\zeta_w = \frac{dw/w}{d\tilde{A}/\tilde{A}}$, $\zeta_N = \frac{dN/N}{d\tilde{A}/\tilde{A}}$, $\zeta_{p_c} = \frac{dp_c/p_c}{d\tilde{A}/\tilde{A}}$ and $\zeta_{p_r(j)} = \frac{dp_r(j)/p_r(j)}{d\tilde{A}/\tilde{A}}$ denote the productivity elasticities of the wage rate, population, the commercial land rent, and the residential land rent in location j respectively. These elasticities are analogous to the volatilities of these variables in a dynamic model.

3.1 The Elasticity of the Wage, Population and the City Boundary

We start by examining the elasticity of the wage rate and the population. As discussed in Glaeser et al. (2006) and others, a positive shock to productivity is likely to result in a large increase in population and a small increase in wages in a city that can easily expand, but a small increase in population and a large increase in wages in a city whose growth is constrained. In our benchmark model, the size of the city is not explicitly constrained, but workers bear higher transportation costs when the population increases, which effectively constrains the size of the city.

To explore the tradeoff between population growth and wage growth in our model, we differentiate the aggregate labour supply function (equation 21) with respect to the productivity shock, $log(\tilde{A})$, to obtain the following relationship between the elasticity of the wage rate and the elasticity of the population.[12]

$$\frac{\zeta_w}{\zeta_N} = F \tag{27}$$

where F, defined in equation (23), captures various land supply constraints. The above equation indicates that a productivity shock affects wages relatively more than population when land supply is effectively more constrained.

[12]Equation (27) can be easily derived from equation (22).

By differentiating equation (20), the aggregate labour demand equation, with respect to $\tilde{A}$, we obtain

$$\zeta_N = -\frac{1}{\lambda - \sigma} + \frac{1-\xi}{\lambda - \sigma}\zeta_w. \tag{28}$$

The above equation, together with equation (27), leads to the following expression for the elasticity of population:

$$\zeta_N = \frac{1}{-\lambda + \sigma + (1-\xi)F} \tag{29}$$

Since we have ruled out the small city equilibrium using inequality (25), the denominator in the above equation is positive, which implies that $\zeta_N > 0$, i.e., a positive productivity shock always increases city population, but the amount of the increase is dampened by the effective land supply constraint F.

3.2 Elasticity of Commercial Land Rent

We differentiate the commercial land rent equation, i.e., equation (16), with respect to $\tilde{A}$ to obtain

$$\zeta_{p_c} = \frac{1}{\sigma} + \frac{\lambda}{\sigma}\zeta_N - \frac{1-\sigma-\xi}{\sigma}\zeta_w, \tag{30}$$

which shows that the commercial land rent elasticity decreases with ζ_w but increases with ζ_N. Commercial land rent elasticity is dampened if the wage rate elasticity is higher, because a high wage decreases the economic benefits available to commercial land owners under the zero-profit condition of firms. Commercial land rent elasticity is amplified if the population elasticity is higher, because a larger population is associated with more agglomeration benefits.

Using equations (27)-(28) to substitute out ζ_w and ζ_N in equation (30), we obtain the following expression for the elasticity of commercial land rent:

$$\zeta_{p_c} = \frac{1+F}{-\lambda + \sigma + (1-\xi)F} \tag{31}$$

where the denominator is positive, again due to the regularity condition represented by inequality (25).

It should be noted that this elasticity is not explicitly a function of the population or the share of undevelopable land. These variables are important, but are subsumed by the effective supply constraint F. The derivative of ζ_{p_c} with respect to F is $\frac{-\lambda-(1-\sigma-\xi)}{[-\lambda+\sigma+(1-\xi)F]^2}$, which is always negative, indicating that the effective land supply constraint always dampens the commercial land rent elasticity.

One can also consider the effect of a fixed boundary, which is an explicit land supply constraint. In Appendix B.2 we consider the implications of such a constraint by comparing two cities with the same initial population, but one has a fixed boundary and the other has a flexible boundary. We find that the elasticity of commercial land rent is lower in cities with a fixed boundary.

The following proposition summarizes our results about the elasticity of commercial land rent:

Proposition 2 *In the benchmark model, the elasticity of commercial land rent is*

(i) always positive

(ii) decreasing in F

(iii) is lower if the city boundary is fixed.

To understand the effect of F on commercial land rent elasticity, recall that F captures the cost of commuting between the CBD and the city boundary. When a city experiences an exogenous shock to productivity, firms hire more workers, which expands the boundary of the city and generates more congestion, thus increases the commuting costs of workers at the boundary and rents in the interior. To maintain the workers' reservation utility level, wages must also increase. When F is higher, the effect of a productivity shock on both wages and rental rates are higher because an increase in population results in a greater increase in the cost of commuting from the boundary. This in turn implies that the effect of the productivity shock on firm revenues, and thus commercial rents, is dampened. The increased commuting costs also reduce the number of new workers that are hired, so the agglomeration effect is also dampened. If the city boundary is fixed, the residential rents and wages increase more, which also reduces the elasticity of commercial land rents.

Equation (31) also reveals how the commercial land rent elasticity depends on production-related parameters. Given the same F, commercial land rent elasticity increases with λ, the agglomeration parameter. This reflects the fact that agglomeration externalities amplify productivity shocks. Commercial land rent elasticity increases with ξ, the share of capital in the production function, but decreases with σ, the share of land. This follows from the fact that capital is adjusted in response to a productivity shock but commercial land is fixed. What this means is that the inelastic supply of commercial land will be less constraining when capital is more important in the production function, e.g., when it is not too expensive to build taller buildings. As we will show in Section 5, when capital supply is fixed, ζ_{p_c} decreases with both ξ and σ.

3.3 Elasticity of Residential Land Rent

We now turn to the elasticity of residential land rent. From the residential bid-rent function (equation 18), we derive the following equation:

$$\zeta_{p_r(j)} = \frac{1}{\theta}\zeta_w - \frac{\beta_2 jN}{\theta}\zeta_N, \tag{32}$$

which shows that residential land rent elasticity increases with ζ_w but decreases with ζ_N. In other words, a positive productivity shock leads to an increase in the wage rate, which has the effect of increasing rent. However, this effect is partially offset by the fact that an increase in productivity also increases population, and thus transportation congestion, which dampens the increase in residential land rent.

Using equations (27) and (29) to substitute for ζ_w and ζ_N in (32), we derive the following expression for the elasticity of residential land rent in the close-in location with $j = 0$:

$$\zeta_{p_r(j=0)} = \frac{1}{\theta} \times \frac{F}{-\lambda + \sigma + (1-\xi)F}, \tag{33}$$

which shows that the effective land supply constraint F plays a key role. As we discussed earlier, more explicit forms of land supply constraints, such as the presence of undevelopable land, affect land rent elasticity indirectly through F.

More generally, equation (32) implies that residential land rent elasticity for all the locations can be expressed as:

$$\zeta_{p_r(j)} = \frac{1}{\theta} \times \frac{F - \beta_2 jN}{-\lambda + \sigma + (1-\xi)F} \quad . \tag{34}$$

This equation, compared with equation (33), includes the additional term $\beta_2 jN$ which captures the congestion effect.[13] Due to our assumption that farther-out locations are more affected by congestion, the elasticity decreases with j, the distance to CBD. Indeed, land rent elasticity is the same in each location if the congestion has the same effect on commuting costs in each location.[14]

Although F captures various forms of land supply constraints, its effect on residential land rent elasticity is not necessarily positive, it depends on the magnitude of the agglomeration externalities. Specifically, taking the partial derivative of $\zeta_{p_r}(j)$ with respect to F, we find that land rent elasticity decreases with F if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$.

Fixing the city boundary has a similar effect. It decreases residential land rent elasticity if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$. In other words, fixing the boundary can decrease the land rent elasticity when the agglomeration parameter is sufficiently large. This is proved in Appendix B.2.

From equation (34) we can also see that residential land rent elasticity depends on the characteristics of production in a city in a similar way as commercial land rent elasticity. It increases with λ and ξ, but decreases with σ.

[13] The regulatory condition (26) ensures that $F - \beta_2 jN > 0$.

[14] Specifically, with the alternative transportation cost function of $f(j, N) = \beta_0 + \beta_1 j + \beta_2 N$, residential land rent elasticity is $\zeta_{p_r^\star} = \frac{1}{\theta} \times \frac{F^\star - \beta_2 N}{-\lambda + \sigma + (1-\xi)F^\star}$ which does not depend on location. Appendix B.3 provides the proof as well as the expression of $F^\star$.

In addition, residential land rent elasticity decreases with θ, the land share in workers' utility function, because when θ is smaller the rental rate can increase more with an increase in wages, following Lemma 1 which shows the role of θ in determining the changes of residential land rent with respect to the changes in wage.

We summarize our analysis of residential land rent elasticities in the following proposition:

Proposition 3 *In the benchmark model, residential land rent elasticity is*

(i) always positive.

(ii) decreasing in θ, the share of land consumption in the worker's utility function.

(iii) decreasing in distance to the CBD.

(iv) decreasing in F if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$.

(v) lowered by keeping the city boundary fixed if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$.

The last two points of the above proposition describe one of our central results: constraints that limit a city's population increases residential land rent elasticities only if the agglomeration parameter is small relative to other parameters. To understand this result, note that $\lambda - \sigma$ captures the net agglomeration externality with σ measuring the dampening effect of CBD land on productivity. In contrast, the term $(1-\xi)\beta_2 jN$ captures the net negative congestion externality that arises from having a higher population. The parameter β_2 determines the strength of the congestion effect, and $\beta_2 jN$ describes the location-specific congestion. This negative externality is mitigated if the capital share in production is large, as captured by the term $1-\xi$, since a larger ξ implies fewer workers per unit of output must be transported to the CBD.

3.4 Rent Elasticity of Commercial Land Relative to Residential Land

Existing empirical evidence suggests that the rent of commercial land tends to be more volatile than residential land.[15] To describe the conditions under which our model can generate such a result we combine equation (34) with equation (31) to obtain the following proposition:

Proposition 4 *Commercial land rent is more volatile than residential land rent next to the CBD, i.e. $\zeta_{p_c} > \zeta_{p_r(j=0)}$, if and only if*

$$F < \frac{\theta}{1-\theta} \tag{35}$$

To understand condition (44) it is useful to consider the right and left hand sides of the inequality separately. The left hand side effectively captures the supply elasticity of residential land. When F is small, residential land can more easily expand in response to productivity

[15]See Leung and Kwong (2000).

shocks, which means that residential rents increase less. However, if residential land rents increase less, wages also increase less, which in turn makes commercial land in the CBD more valuable. Hence, when F is small, CBD commercial rents are more volatile than residential rents. The right hand side of the inequality is increasing in θ, and we have shown that a larger θ dampens residential land rent elasticity, Hence, with sufficiently high θ, residential land rent is less volatile than commercial land rent.

It is noteworthy that the importance of land in production, σ, does not appear in condition 44, which is in contrast to the role played by θ, the parameter that determines the importance of land in the utility of workers. As it turns out, σ has the same dampening effect on both residential and commercial land rent elasticities as shown in equation (32) and equation (31).

3.5 Discussion

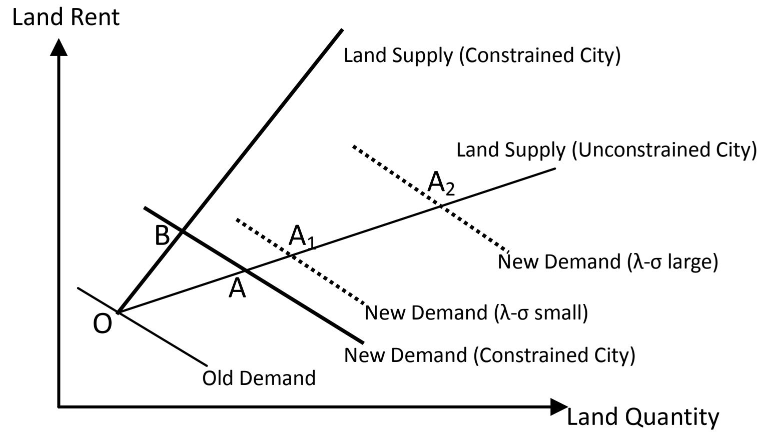
As we mentioned in the introduction, we are not the first to ask how land supply constraints affect the responsiveness of land prices to shocks that increase the demand for both commercial and residential land. For example, Glaeser et al. (2006) and Saiz (2010) talk in terms of the elasticity of housing supply, and provide models where an increase in the elasticity of housing supply decreases the effect of a demand shock on prices. Our model contributes to this literature by providing micro foundations that allow us to link urban characteristics to the supply elasticity of residential land.

To study the effect of supply constraints on residential rent elasticities we identify a variable, F, that captures the effective land supply constraint. A higher F steepens the residential land supply curve. We then show in Proposition 3 that the effective supply constraint does not always increase the sensitivity of residential rents to exogenous productivity shocks. To understand this result, it should be noted that there are two offsetting channels that determine how supply constraints affect residential rents. The first is what we will call the scarcity channel — when it is costly to increase supply, rents increase more. The second is what we will call the agglomeration channel. When agglomeration externalities are strong, the supply constraints reduce the feedback effect that amplifies the exogenous productivity shock.

As we discussed earlier, $\lambda - \sigma$ determines the strength of agglomeration externalities, and thus the importance of the agglomeration channel. When $\lambda - \sigma$ is quite small, the scarcity channel dominates and the resulting elasticity is consistent with the findings in Glaeser et al. (2006) and Saiz (2010). Intuitively, a positive productivity shock increases demand for workers, and residential land rent needs to increase to accommodate these additional workers and maintain land market equilibrium. For supply constrained cities, residential land rent needs to increase more due to the low price elasticity of supply. This argument reflects the traditional discussion of how the elasticity of housing supply affects the sensitivity of the cost of housing to demand shocks.

However, when the agglomeration channel is more important, i.e., when $\lambda - \sigma$ is large, the effect that supply constraints has on the amplification of exogenous productivity shock more than offsets the scarcity effect. Recall that an exogenous increase in productivity is amplified because the resulting increase in population makes the workers more productive. The magnitude of this amplification depends on the growth in population. Hence constraints that dampen the increase in population also dampen the growth in productivity and wages. As a result, if the agglomeration effect is sufficiently large, an exogenous productivity shock increases land rents less when supply is more constrained.

Figure 2: Land Supply Constraints and Land Rent Elasticity



Note: The responsiveness of land rent to a positive productivity shock depends on both the supply elasticity of land (i.e. the steepness of land supply curve) and the extent to which land demand curve shifts in response to a productivity shock. The old demand curve represents the land demand before the productivity shock and point O represents the original equilibrium. Given a positive productivity shock, point B is the new equilibrium for supply constrained cities; and for cities with less supply constraints, point A_1 and A_2 represent the new equilibrium for the case of weak and strong agglomeration effect, respectively.

We illustrate these two effects in Figure 2, which describes the supply and demand curves for land, and is similar to Figure 1 in Glaeser et al. (2006). Here constrained cities are represented by the steeper land supply curve, and unconstrained cites are represented by the flatter supply curve. Starting from the original equilibrium (point O), traditional theory posits that a positive productivity shock shifts the demand curve to the right, crossing the supply curve at point A for unconstrained cities and point B for constrained cities, thus land rent should rise more in constrained cities. However, this analysis ignores the agglomeration channel: given the same exogenous productivity shock, the amount the demand curve shifts also depends on the extent to which residential land can expand. For supply constrained cities that are costlier to expand, the agglomeration effect is dampened, and thus the demand curve shifts less. In Figure 2, for constrained cities, the new equilibrium is determined by the crossing of the two thick lines at point B. For unconstrained cities, the demand curve shifts further to the right, crossing the supply curve at point A_1 when $\lambda - \sigma$ is small, and at point A_2 when $\lambda - \sigma$ is large. Therefore, compared with constrained cities, land rent in unconstrained cities may rise less ($A_1 < B$) or more ($A_2 > B$), depending on the strength of the agglomeration effect.

3.6 Dynamics of Land Rent

Up to this point, we have considered a setting where a shock to productivity causes residential land rents to instantaneously move to a new steady state. In this subsection, we consider a setting where the agglomeration effect takes one period to materialize, which implies that it takes time for rents to fully respond to productivity shocks. When this is the case, there will be a distinction between short-run and long-run elasticities.

To understand this, it is useful to divide the evolution of a productivity shock into three phases. In phase one, the shock to productivity affects population, wages and rents, but the agglomeration benefits from the additional population has not yet materialized. In phase two, the agglomeration benefits from the additional population provides an additional boost to productivity, which further increases wages and attracts additional population. The feedback between population growth and productivity growth effectively feeds on itself, leading to a persistent increase in city population, physical size, wage and rents. In phase three, the process converges and the city is in a new steady state equilibrium.

To understand the distinction between the short-run and long-run responses to an exogenous productivity shock, we compare land rent elasticities in phase one (the short-run) and phase three (the long-run), using $\zeta^0_{p_r}$ and $\zeta^0_{p_c}$ to denote phase-one rent elasticities of residential land and commercial land, respectively. Recall that phase-three rent elasticities are ζ_{p_r} and ζ_{p_c}, which were previously shown in equation (31) and equation (34). As we show in Appendix B.1, the phase-one rent elasticities, $\zeta^0_{p_r}$ and $\zeta^0_{p_c}$ are expressed as,

$$\zeta^0_{p_c} = \frac{1+F}{\sigma + (1-\xi)F} \tag{36}$$

$$\zeta^0_{p_r(j)} = \frac{1}{\theta} \times \frac{F - \beta_2 j N}{\sigma + (1-\xi)F} \quad . \tag{37}$$

It is noteworthy that if we set λ in phase-three elasticities to zero, we get phase-one elasticities as shown in the above two equations. This is no coincidence – in phase one, the increased population does not feed back into productivity, and the phase-one agglomeration depends on the past population only. As a result, λ in the elasticities, which captures the feedback effect, is dropped out of the elasticity equations.

Given the phase-one and phase-three elasticities equations, the normalized distances between elasticities in phase one and phase three are

$$\frac{\zeta_{p_c} - \zeta^0_{p_c}}{\zeta_{p_c}} = \frac{\lambda}{\sigma + (1-\xi)F} \tag{38}$$

$$\frac{\zeta_{p_r} - \zeta^0_{p_r}}{\zeta_{p_r}} = \frac{\lambda}{\sigma + (1-\xi)F} \tag{39}$$

Because the right sides of equations (38)-(39) are always positive, land rent elasticities are always larger in the long run than in the short run, and the difference depends on parameters governing the effective land supply constraint, the agglomeration externality and land use intensity. These properties are summarized in the following proposition.

Proposition 5 *Given a productivity increase, land rent elasticity is larger in the long run than in the short run for both residential land and commercial land, and the long-run versus short-run difference is greater if a city features:*
(i) a larger agglomeration parameter λ,
(ii) a smaller effective land supply constraint F,
(iii) a smaller land use intensity parameter σ.

To provide additional intuition that might be useful for future empirical tests we explore how the differences between short and long-run elasticities relate to city size. To do this, we examine the partial derivatives of land rent elasticities with respect to our two measures of city size: population N and the physical size, as summarized by the effective land supply constraint F. For simplicity, we only consider residential land near the CBD with $j = 0$. Using equation (37), we derive the following derivatives for phase-one elasticities:

$$\frac{d\zeta^0_{p_r}}{dF} = \frac{1}{\theta} \times \frac{\sigma}{[\sigma + (1-\xi)F]^2} \qquad \textit{and} \tag{40}$$

$$\frac{d\zeta^0_{p_r}}{dN} = \frac{1}{\theta} \times \frac{dF}{dN} \times \frac{\sigma}{[\sigma + (1-\xi)F]^2} \qquad , \tag{41}$$

where $\frac{dF}{dN}$ in the right side of equation (41) is positive as shown in lemma 2. It follows from equations (40)-(41) that $\frac{d\zeta^0_{p_r}}{dF} > 0$ and $\frac{d\zeta^0_{p_r}}{dN} > 0$, which implies that in the short-run, residential land rents respond more to productivity increases in larger or more effectively constrained cities.

To understand relationship between land rent elasticity and city size in the long run, we take the partial derivative of the phase-three elasticity, ζ_{p_r}, with respect to population and the effective land supply constraint using equation (33):

$$\frac{d\zeta_{p_r}}{dF} = \frac{1}{\theta} \times \frac{-\lambda + \sigma}{[-\lambda + \sigma + (1-\xi)F]^2} \qquad \textit{and} \tag{42}$$

$$\frac{d\zeta_{p_r}}{dN} = \frac{1}{\theta} \times \frac{dF}{dN} \times \frac{-\lambda + \sigma}{[-\lambda + \sigma + (1-\xi)F]^2} \qquad . \tag{43}$$

As shown in the above expressions, the signs of these derivatives depend on the relative magnitude of the agglomeration externality, i.e., whether or not $\lambda > \sigma$. Smaller or less effectively constrained cities respond more to a productivity increase if and only if $\lambda > \sigma$. That is, in the long run, $\frac{d\zeta_{pr}}{dF} < 0$ and $\frac{d\zeta_{pr}}{dN} < 0$ if $\lambda > \sigma$. If $\lambda < \sigma$, we still have $\frac{d\zeta_{pr}}{dF} > 0$ and $\frac{d\zeta_{pr}}{dN} > 0$ as in the short run.

The above results are summarized in the following proposition:

Proposition 6 *Consider cities with different population sizes or effective land supply constraints. Given a productivity increase,*

(i) in the short run before the agglomeration takes effect, land rent increases more in cities that have larger populations or more effective supply constraints;

(ii) in the long run when the economy reach the new steady state equilibrium,

(a) land rent increases ***less*** *in larger or more constrained cities if $\lambda > \sigma$,*

(b) land rent increases ***more*** *in larger or more constrained cities if $\lambda < \sigma$.*

It should be noted that our analysis of long-run elasticities is exactly the same as what we discussed in the previous subsection. Specifically, in the long-run, less constrained cities may experience greater rental rate elasticity if agglomeration externalities are large. An important implication of proposition 6 is that this is not the case in the short run. Given a positive productivity shock, rental rates initially increase less in smaller or less constrained cities. We believe that this distinction between the long-run and short-run effects may explain the rapid increase in prices, relative to rents, in relatively unconstrained cities, like Las Vegas, prior to the financial crisis. Real estate prices in these cities may have initially increased more than rents because investors anticipated the long-run increases that can be experienced by relatively unconstrained cities with strong agglomeration externalities.

4 Undevelopable Land

Up to this point we have established that Λ, the proportion of land that cannot be developed, indirectly affects land rent elasticities through the effective land supply constraint F. Because the literature emphasizes land supply constraints that arise from this channel, i.e., Saiz (2010) and others, this section examines the various effects of Λ in more detail. As we will show, all else equal, cities will have lower populations if Λ is higher. This follows directly from the fact that for any population, it is costlier to transport workers from the city boundary to the CBD when Λ is higher. This in turn implies that a higher TFP is needed to sustain a given population. Hence, if we consider the elasticities of cities with identical exogenous parameters other than Λ, we are necessarily comparing cities with different populations.

It may also be of interest to compare the elasticities of cities with similar populations, but with different Λs. For example, empiricists may want to measure the effect of Λs on elasticities in regressions that control for population. However, such an analysis would implicitly assume that there are other sources of cross-city variation that allows cities with different Λs to have similar populations. For example, as we discuss below, the TFP of the high Λ city may be sufficiently higher than the TFP of the low Λ city so that their steady state populations are the same.[16]

4.1 Undevelopable Land and City Configuration

In Appendix B.4.1, we prove that the following Proposition.

Proposition 7 *Given the exogenous variables and parameters in Table 1, the proportion of undevelopable land Λ has the following effect on the equilibrium population N, city boundary J, wage w, commercial land rent p_c, and residential land rent $p_r(j)$:*

(i) N decreases with Λ;

(ii) w decreases with Λ if and only if $\lambda > \sigma$;

(iii) J increases with Λ if and only if $\lambda - \sigma > (1 - \xi)\beta_2 JN$;

(iv) p_c decreases with Λ;

(v) $p_r(j)$ decreases with Λ if and only if $\lambda - \sigma > (1 - \xi)\beta_2 jN$.

The result that undevelopable land decreases the city population is straightforward, and the fact that commercial rents decline with undevelopable land follows directly from the drop in population. As we discuss below, the other results are somewhat subtler and depend on the strength of the agglomeration effect.

The effect of undevelopable land on the wage rate depends on the strength of the agglomeration effect relative to the land share parameter in the production function, i.e. whether $\lambda > \sigma$. The intuition is that the wage rate equals the marginal product of labour, which increases with population, as reflected by λ and decreases with the ratio of CBD land to labour, which is captured by σ. When $\lambda > \sigma$, the agglomeration effect dominates, which implies that the wage rate decreases with the share of undevelopable land.

The effect of undevelopable land on both the city boundary and residential rent also depends on λ and σ. On one hand, a larger share of undevelopable land means less residential land is available in close-in locations, causing the city boundary to expand and causing rents to increase.

[16] As indicated in footnote to equation (21), it is also possible to change Λ and hold population constant by simultaneously changing the level of amenities. In some ways this is a more appealing approach. For example, a lake in the middle of a city, which reduces the amount of developable land, also makes the city more appealing and attracts more residents. Our analysis that allows the proportion of developable land to change while holding population constant implicitly assumes that these two effects exactly offset.

However, since more undevelopable land reduces population, it lowers the marginal product of labour and the wage rate as long as $\lambda - \sigma$ is positive. This effect reduces the costs that workers are willing to pay for rent or to bear to travel to the CBD, which causes the boundary to shrink. When $\lambda - \sigma$ is sufficiently large, the latter channel dominates, so undevelopable land causes the city to have a smaller residential area as well as lower rents. These effects are also influenced by congestion as captured by the parameter β_2. When the congestion effect is stronger, the boundary is grows less with Λ and rents decrease less.

4.2 Holding the Exogenous Productivity $\tilde{A}$ Constant

In this subsection we will examine the effect of changing the proportion of undevelopable land in a setting where $\tilde{A}$ is fixed, which means the population changes with Λ. When this is the case Λ does not appear in the population and land rent elasticities, as indicated in equations (29), (31) and (34), but the undevelopable land affect these elasticities indirectly through F.

In Appendix B.4.2, we prove the following lemma regarding how Λ affects F.

Lemma 3 *F decreases with Λ if and only if $\lambda > \sigma$.*

The lemma indicates that because of its effect on population, increases in undevelopable land can either increase or decrease the effective land supply constraint depending on the agglomeration effect. When the agglomeration effect is strong, the dampening effect of undevelopable land is also strong, so a larger Λ causes both N and J to be smaller, and hence F to be smaller as well.

Since commercial land rent elasticity ζ_{p_c} always decreases with F, it follows from Lemma 3 that a larger Λ is associated with a larger ζ_{p_c} if and only if $\lambda > \sigma$. In addition, residential land rent elasticity $\zeta_{p_r(j)}$ is larger when F is smaller if and only if $\lambda - \sigma > (1 - \xi)\beta_2 jN$ as stated in Proposition 3. We summarize these results in the following proposition.

Proposition 8 *If we hold all the other exogenous parameters fixed, the following describes how commercial land rent elasticity ζ_{p_c} and residential land rent elasticity ζ_{p_r} depend on the proportion of developable land Λ:*

(i) both ζ_{p_c} and $\zeta_{p_r(j=0)}$ are increasing in Λ if and only if $\lambda - \sigma > 0$.

(ii) for $j > 0$, $\zeta_{p_r(j)}$ is increasing in Λ if $\lambda - \sigma > (1 - \xi)\beta_2 jN$.

See Appendix B.4.3 for a proof.

The above result is driven by the size effect: everything else equal, cities with more undevelopable land have a smaller population, and are thus more responsive to productivity shocks.

Note that in the second point we only give a sufficient condition for $\zeta_{p_r}(j)$ to decrease with population. The necessary condition is given in Appendix B.4.3. The intuition is that a larger population leads to more congestion, thus it has a similar effect on $\zeta_{p_r}(j)$ as F. However, since population increases productivity, there are some combinations of population and the agglomeration externality where $\zeta_{p_r}(j)$ increases with population.

4.3 Holding Population Constant

In this subsection we consider the effect of undeveloped land in a case that holds population constant. As we mentioned previously, if we hold productivity fixed, then cities with more undevelopable land will have fewer people, so if we hold the population of the city fixed, we must assume that the workers are more productive in the city with more undevelopable land.[17] Under this assumption, the city with more undevelopable land will have a larger J, the distance between the boundary and CBD, and hence a larger F, the effective land supply constraint. The following Proposition follows directly from Proposition 2 and Proposition 3:

Proposition 9 *Suppose the effect of a larger (smaller) Λ on city population is exactly offset by the higher (lower) productivity $\tilde{A}$, then the following describes how commercial land rent elasticity ζ_{p_c} and residential land rent elasticity ζ_{p_r} depend on the proportion of developable land Λ:*

(i) ζ_{p_c} is decreasing in Λ.

(ii) $\zeta_{p_r(j=0)}$ is decreasing in Λ if and only if $\lambda - \sigma > 0$.

(iii) for $j > 0$, $\zeta_{p_r(j)}$ is decreasing in Λ if and only if $\lambda - \sigma > (1 - \xi)\beta_2 jN$.

The above result indicates that a city with a given population but more undevelopable land will exhibit lower commercial land rent volatility. The effect of undevelopable land on residential land rent elasticity depends on λ and σ. For locations close to the CBD, the elasticity is decreasing in the amount of undevelopable land when $\lambda > \sigma$. For locations farther from the CBD, the proposition indicates that land rent elasticity is more likely to increase with Λ due to the congestion effect captured by the term $\beta_2 jN$. This result is again due to the dampening effect that land supply constraints have on agglomeration; an effect that is stronger when λ is larger.

4.4 Summarizing The Effect of Λ on Land Rent Elasticities

Table 2 summarizes the effect of the fraction of undevelopable land (Λ) on land rent elasticities. As is evident in the table, the result presented in Glaeser et al. (2006) and Saiz (2010) are special cases in our model, corresponding to the positive signs in the table.

For example, the left panel of Table 2 indicates that Λ increases residential land rent elasticity under two scenarios. First, among cities with the same exogenous productivity, land rent

[17] Saiz (2010) also mentions that cities with lots of undevelopable land and large populations must have either more productive workers or superior amenities.

Table 2: The Effect of Λ on Land Rent Elasticities

	residential land ($\zeta_{p_r(j=0)}$)		commercial land (ζ_{p_c})	
	fixed $\tilde{A}$	fixed N	fixed $\tilde{A}$	fixed N
$\lambda > \sigma$	$+$	$-$	$+$	$-$
$\lambda < \sigma$	$-$	$+$	$-$	$-$

is more volatile if the agglomeration parameter λ is small, which is due to the size effect – a smaller λ implies a smaller city which is more responsive to productivity shocks. Second, among cities with the same population size, land rent is more volatile if the agglomeration parameter λ is small, which is because the undevelopable land is less dampening to agglomeration when λ itself is small.

5 Extensions

This section discusses various extensions of the benchmark model. In these extensions we consider the effect of having a flexible CBD boundary and also consider a case where the amount of capital in the city is fixed.

5.1 Flexible CBD

Up to this point we have assumed that the physical size of the CBD is fixed. In this section we consider the case where the CBD expands or contracts depending on the relative demand for commercial and residential space. Specifically, we assume that the CBD will expand or contract up to the point where the rent on commercial land and residential land at the boundary are equal.

To understand how productivity shocks affect the CBD boundary one should first note that a positive productivity shock increases the demand for both commercial land and residential land, as reflected by their positive rent elasticities. We combine equation (34) with equation (31) in the baseline model to obtain the following condition for commercial land rent to be more responsive to a productivity shock than residential land rent next to the CBD. Namely, $\zeta_{p_c} > \zeta_{p_r(j=0)}$ if and only if

$$F < \frac{\theta}{1-\theta}, \tag{44}$$

which leads to the following Lemma. The proof is given in Appendix C.1.3.

Lemma 4 *The CBD expands in response to a positive productivity shock if and only if condition* (44) *holds.*

Recall that θ is the households' preference for land, and F is the effective supply constraint of residential land. To understand condition (44) it is useful to consider the right and left hand sides of the inequality separately. The left hand side effectively captures the supply elasticity of residential land. When F is small, residential land can more easily expand in response to productivity shocks, which means that residential rents increase less. However, if residential land rents increase less, wages also increase less, which in turn makes commercial land in the CBD more valuable. Hence, when F is small, CBD commercial rents are more volatile than residential rents. The right hand side of the inequality is increasing in θ, and we have shown that a larger θ dampens residential land rent elasticity, Hence, with sufficiently high θ, residential land rent is less volatile than commercial land rent.[18]

We are interested in how the flexible supply of commercial land affects land rent elasticities. The following proposition describes the comparison between the flexible CBD model and the benchmark model:

Proposition 10 *Relative to the benchmark model, for a given population, the following is true when the size of the CBD is flexible,*

(i) residential land rent elasticity is higher if and only if $F < \frac{\theta}{1-\theta}$,

(ii) if $F < \frac{\theta}{1-\theta}$, *then commercial land rent elasticity is higher if and only if* $\lambda > (1-\sigma-\xi)F$.

The proof is given in Appendix C.1.

The first part of the proposition states residential land rent elasticity is higher if the CBD is flexible when a positive shock increases the size of the CBD. When this is the case, the marginal product of labour, and hence the wage rate and residential rents, is more sensitive to an exogenous productivity shock because more land becomes available for production. The second part of the proposition shows that the linkage between agglomeration externalities and elasticities continues to hold. The flexible CBD model relaxes commercial land supply constraint, which strengthens the agglomeration effect. When the agglomeration is important, i.e., when λ is large, the relaxation of the land supply constraint leads to a larger elasticity of commercial land rent.

5.2 Immobility of Capital

In our benchmark model we assume that capital is perfectly mobile, which means that the owners of capital capture none of the benefits of a positive productivity shock. In this section we consider the polar opposite case, where the amount of capital in the city is fixed, which

[18] It is noteworthy that the importance of land in production, σ, does not appear in condition 44, which is in contrast to the role played by θ, the parameter that determines the importance of land in the utility of workers. As it turns out, σ has the same dampening effect on both residential and commercial land rent elasticities as shown in equation (32) and equation (31).

implies that capital costs fluctuate. As we show in Appendix C.2, the main comparison between this extension and the benchmark model is the following:

Proposition 11 *The following is true when the city-level capital stock is fixed:*

(i) land rent elasticity (both commercial and residential) is lower relative to the benchmark model,

(ii) land rent elasticity decreases with ξ, the share of capital in production.

The lower elasticities in the model with fixed capital arise for two reasons. The first is that when capital is fixed, the city grows less in response to a productivity shock, which reduces the agglomeration externalities. The second is that capital owners capture some of the benefits associated with higher rates of productivity when the amount of capital is fixed.

Based on equations (84)-(85) in Appendix C.2, it is straightforward to show that in this extended model, commercial land rent elasticity always decreases with F, the effective land supply constraint, as in the benchmark model. For residential land near the CBD, the elasticity decreases with F if and only if the agglomeration parameter is large relative to the threshold, i.e., $\lambda > \sigma + \xi$. Thus our intuition from the benchmark model still holds. However, compared with the benchmark model, the model with immobile capital requires a stronger agglomeration effect for F to dampen residential land rent elasticity, as the threshold condition in the benchmark model is $\lambda > \sigma$, which is easier to satisfy than the condition $\lambda > \sigma + \xi$.

6 Conclusion

Although the 2007-2009 global financial crisis had a number of causes, an important contributor was the perception that real estate is a relatively low risk investment. This misperception created an overly levered property sector as well as overly exposed financial institutions, some of which failed.

The model developed in this paper provides a framework for thinking about how the design of a city and the firms that inhabit it affect rents, land values, and fluctuations in these values. The key elements in our model include the magnitude of agglomeration externalities and various urban attributes that can effectively constrain the extent to which a city receiving an exogenous productivity shock will grow. Stronger agglomeration externalities always amplify exogenous productivity shocks. Populations grow more and rents increase more in response to productivity shocks when agglomeration externalities are stronger. By definition, growth constraints suppress the effect of productivity shocks on population growth, and in some cases that depend on the magnitude of the agglomeration externality, the constraints amplify the effect of productivity shocks on rents, and in some cases they suppress the effect.

When agglomeration externalities take time to materialize, rents may take time to fully respond to a shock to productivity. To illustrate the distinction between short and long run responses, it is useful to contrast the responses of a smaller city, like Las Vegas, that is likely to have less stringent growth constraints, to that of a larger city, like San Francisco, that is likely to be much more constrained. Our analysis suggests that because of the growth constraints, land rents in cities like San Francisco are likely to initially increase more than they will in a less constrained city, like Las Vegas, that will experience a much greater increase in population. However, if we believe the population growth in Las Vegas generates agglomeration benefits that materialize over time, a single productivity shock can generate persistent increases in land rents that will result in a long-run steady state increase in rent that exceeds the long run increase in the larger city. Because land prices are forward-looking, the initial land price response in the smaller city may be substantially greater than the land price response in the more constrained larger cities even though the initial rent response is weaker.

It should be noted that we have limited our analysis to changes in rents that are generated solely from city specific shocks to productivity. Our analysis thus implicitly assumes that other city attributes, like zoning and the transportation technology, affects volatility only by amplifying or suppress the effect of these productivity shocks. However, it is likely, that in reality, these city attributes are also uncertain, and thus contribute to uncertainty about future rents and land values.

Examples would include self-driving cars that are likely to reduce transit congestion, and improved telecommunication technology, which allow workers to work at least part of the time from home. These changes are likely to have an effect on the internal structure of cities, and as our model suggests, they can also influence the volatility of rents and property values.

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Appendices

A Equilibrium in the Benchmark Model

An equilibrium in our benchmark model is described by seven equations that determine the seven endogenous variables: $\{p_r, p_c, w, N, K, J, A\}$. In this appendix we derive the seven equations and show that they can be reduced to two equations: the aggregate labour supply equation and aggregate labour demand equation. We also prove Proposition 1 in section A.4 of this appendix.

A.1 A System of Seven Equations

In the system of seven equations, the first one is equation (12) which describes how the city level TFP depends on the city population. The second and third equations are the bid-rent functions as given by Equations (16)-(18). Equations (45)-(48) below represent the remaining four equations.

A.1.1 City Level Quantities

The relative inputs of labour, land, and capital in production at the city level are:

$$\frac{N}{S} = \frac{1-\sigma-\xi}{\sigma} \times \frac{p_c}{w}, \tag{45}$$

$$\frac{N}{K} = \frac{1-\sigma-\xi}{\xi} \times \frac{r}{w}. \tag{46}$$

These equations show the number of workers per unit of commercial land and per unit of capital in a city. They can be easily derived from equations (9)-(11) which describe the optimal relative inputs at the firm level.

Total number of workers that can be housed in a city is the integral of the number of workers in each location in the city which is $(1-\Lambda)/h(j)$. Here $1-\Lambda$ is the share of developable land and $h(j)$ is land demand per worker. Therefore, total number of workers in a city as the function of wage and land rent is:

$$N = \int_{j=0}^{J} \frac{1-\Lambda}{h(j)} dj = \int_{j=0}^{J} \frac{(1-\Lambda)p_r(j)}{\theta w e^{-f(j,N)}} dj \tag{47}$$

where we have used equation (7) to substitute out $h(j)$.[19] It should be noted that although workers are assumed to be perfectly mobile, the supply of labour in a city is constrained by residential land – given any wage and rent, there are limited amount of residential land available and only a limited number of workers can be housed on the land.

The last equation in the system of seven equations determines the city boundary. Equating the residential bid-rent function at location J with the exogenous agricultural rent $\underline{p}$, the equilibrium boundary satisfies:

$$\underline{p} = p_{r(j=J)} = B_0 \left[w e^{-f(J,N)} \right]^{1/\theta},$$

which is equivalent to $f(J,N) = log(w) + \theta log\left(\frac{B_0}{\underline{p}}\right)$. Using the explicit form of the transportation cost function (equation 2), we obtain

$$J = \frac{log(w) + \theta log\left(\frac{B_0}{\underline{p}}\right) - \beta_0}{\beta_1 + \beta_2 N}, \tag{48}$$

[19] On the right hand side of this equation, the number of workers at each location is determined in part by congestion, as expressed by the transportation cost function $f(j,N)$. Since N is on both sides of the equation, it must be solved as the fixed point that satisfies both sides of the equation.

which indicates a positive relationship between the city boundary and wage, and a negative relationship between the city boundary and population due to the congestion effect.

A.2 Aggregate Labour Supply/Demand Functions

Aggregate Labour Supply Aggregate labour supply in a city is defined as the total number of workers that can be housed in a city as a function of wage. Substituting out the land rent in equation (47) with the residential bid-rent function, we obtain the following:

$$N = \frac{(1-\Lambda)B_0}{\theta} w^{\frac{1-\theta}{\theta}} \int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj. \tag{49}$$

Taking logarithm of the above equation leads to the aggregate labour supply equation which is equation (21).

Aggregate Labour Demand We obtain the following expression which describes the total labour input relative to land by substituting out land rent in equations (45) with the commercial bid-rent function:

$$\frac{N}{S} = \left[\frac{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}}{r^{\xi}}\right]^{\frac{1}{\sigma}} \frac{N^{\frac{\lambda}{\sigma}}}{w^{\frac{1-\xi}{\sigma}}}. \tag{50}$$

Rearranging terms, we re-write equation (50) as

$$N = \left[\frac{r^{\xi} w^{1-\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi} S^{\sigma}}\right]^{\frac{1}{\lambda-\sigma}}. \tag{51}$$

Taking logarithm of the above equation leads to equation (20), the aggregate labour demand function.

Solving Other Variables We solve the equilibrium wage (w) and population (N) from the aggregate labour demand and supply functions. Using the market clearing $\{w, N\}$, we can solve for A, the city level TFP, from the agglomeration function $A = \tilde{A}N^{\lambda}$. The remaining endogenous variables, namely p_r, p_c, K, and J, are solved from equation (16), equation (18), equation (46), and equation (48) respectively. ∎

A.3 Slope of Aggregate Labour Supply Curve

Here we show that the slope of the aggregate labour supply function is $\frac{1}{F}$, with the definition of F given in equation (23). In addition, we show that the aggregate labour supply curve is upward sloping and concave.

A.3.1 Expression of the Slope

Given the transportation cost function $f(j,N) = \beta_0 + \beta_1 j + \beta_2 jN$, the transportation gradient is $\frac{\partial f(j,N)}{\partial j} = \beta_1 + \beta_2 N$, thus the term $\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj$ in equation (21) can be re-written as

$$\begin{aligned}\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj &= \int_0^J \frac{1}{-\frac{1-\theta}{\theta}\frac{\partial f(j,N)}{\partial j}} de^{-\frac{1-\theta}{\theta}f(j,N)} \\ &= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)}\int_0^J de^{-\frac{1-\theta}{\theta}f(j,N)} \\ &= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)}\left(e^{-\frac{1-\theta}{\theta}f(J,N)} - e^{-\frac{1-\theta}{\theta}f(0,N)}\right) \\ &= \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)}\left(e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}f(J,N)}\right). \qquad (52)\end{aligned}$$

where we have used the condition that $f(0,N) = \beta_0$.

We have shown that $f(J,N) = log(w) - \theta\left(log\frac{p}{B_0}\right)$ based on equation (48). Substituting this boundary condition into equation (52), we obtain:

$$\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj = \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)}\left(e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta\left(log\frac{p}{B_0}\right)\right]}\right).$$

With the above equation, the aggregate labour supply function (equation 21) can be rewritten into:

$$\begin{aligned}log(N) &= log\left[\frac{(1-\Lambda)B_0}{1-\theta}\right] + \frac{1-\theta}{\theta}log(w) - log(\beta_1+\beta_2 N) + log\left(\frac{\theta}{1-\theta}\right) \\ &\quad + log\left[e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}\right]. \qquad (53)\end{aligned}$$

From equation (53), the derivative of $log(N)$ with respect to $log(w)$ is:

$$\frac{dlog(N)}{dlog(w)} = -\frac{\beta_2 N}{\beta_1+\beta_2 N}\times\frac{dlog(N)}{dlog(w)} + \frac{1-\theta}{\theta}\left(1 + \frac{e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}{e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right).$$

After rearranging terms, we have

$$\begin{aligned}\frac{dlog(N)}{dlog(w)} &= \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}\beta_0}}{e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right) \\ &= \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}f(0,N)}}{e^{-\frac{1-\theta}{\theta}f(0,N)} - e^{-\frac{1-\theta}{\theta}f(J,N)}}\right) \qquad (54) \\ &= \frac{1}{F}.\end{aligned}$$

where the last equality comes from equation (23) by using $f(0,N) = \beta_0$ and $f(J,N) = \beta_0 + \beta_1 J + \beta_2 JN$. ∎

A.3.2 Shape of the Slope

First, we show that the aggregate labour supply curve is upward sloping. The term $e^{-\frac{1-\theta}{\theta}f(0,N)} - e^{-\frac{1-\theta}{\theta}f(J,N)}$ in equation (54) is positive. because transportation cost increases in distance, i.e., $f(0,N) < f(J,N)$. Thus, $\frac{1}{F} = \frac{dlog(N)}{dlog(w)} > 0$, i.e., the aggregate labour supply curve is upward sloping.

Second, we show the aggregate labour supply curve is concave, i.e. $d\left[\frac{dlog(N)}{dlog(w)}\right]/d[log(w)] < 0$, which is equivalent to showing $d\left[\frac{dlog(w)}{dlog(N)}\right]/dN = dF/dN > 0$ because $dw/dN > 0$. Using equation (54), we have

$$F = \frac{\theta}{1-\theta}\left(1 + \frac{\beta_2 N}{\beta_1 + \beta_2 N}\right)\left(1 - e^{-\frac{1-\theta}{\theta}[f(J,N)-f(0,N)]}\right) \tag{55}$$

Since each term in equation (55) is positive, to prove $dF/dN > 0$, it suffices to prove that each term has a positive derivative. It is straightforward to show:

$$\begin{aligned}
\frac{d\left(1 + \frac{\beta_2 N}{\beta_1 + \beta_2 N}\right)}{dN} &= \frac{\beta_1 \beta_2}{(\beta_1 + \beta_1 N)^2} > 0 \\
\frac{d\left(1 - e^{-\frac{1-\theta}{\theta}[f(J,N)-f(0,N)]}\right)}{dN} &= \beta_2 J\left(\frac{1-\theta}{\theta}\right) e^{-\frac{1-\theta}{\theta}[f(J,N)-f(0,N)]} > 0
\end{aligned}$$

Therefore,

$$dF/dN > 0, \tag{56}$$

which leads to the conclusion that $d\left[\frac{dlog(N)}{dlog(w)}\right]/d[log(w)] < 0$. This also proves lemma 2.

A.4 Proof of Proposition 1

To prove Proposition 1, we need to show that the number of crossings of the aggregate labour demand and supply curves depends on their slopes which in turn depend on the strength of agglomeration, the importance of land and capital in production, and the cost of transportation.

We have shown in equation (54) that the aggregate labour supply curve is upward sloping and concave. In addition:

1. when $log(N)$ and $log(w)$ are small, distance from the CBD to the boundary J is near zero, thus the slope given by equation (54) converges to infinity as the term $e^{-\frac{1-\theta}{\theta}f(J,N)}$ converges to $e^{-\frac{1-\theta}{\theta}f(0,N)}$, and the term $\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}$ converges to one.

2. when $log(N)$ and $log(w)$ approach infinity, the slope given by equation (54) converges to $\frac{1-\theta}{2\theta}$ because the term $e^{-\frac{1-\theta}{\theta}f(J,N)}$ converges to zero and the term $\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}$ converges to $\frac{1}{2}$ as N converges to infinity.

The slope of aggregate labour demand curve, as given by equation (20), is $\frac{1-\xi}{\lambda-\sigma}$. When $\lambda < \sigma$, $\frac{1-\xi}{\lambda-\sigma} < 0$ and the curve is downward sloping. In this case the curve has a single crossing with the aggregate labour supply curve, and the equilibrium is unique.

If $\lambda > \sigma$, then the aggregate labour demand curve is upward sloping. It crosses the aggregate labour supply curve at least once because: (i) the latter (i.e. the aggregate labour supply curve) has a near-infinity slope when wage and population are small; and (ii) the latter goes to negative infinity more quickly than the aggregate labour demand curve when wage tends toward zero.

If λ is larger than σ but not too large so that the slope $\frac{1-\xi}{\lambda-\sigma}$ is larger than $\frac{2\theta}{1-\theta}$ which is the slope of aggregate labour supply curve when wage and population tend toward infinity, then the aggregate labour demand and supply curves will cross twice, leading to two equilibria.

Thus the necessary and sufficient condition for the existence of two equilibria is that the aggregate labour demand curve is steeper than the aggregate labour supply curve when wage and population converge to infinity, i.e. $\frac{1-\xi}{\lambda-\sigma} > \frac{2\theta}{1-\theta}$, which is equivalent to $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$.

Finally, if

$$\lambda \geq \sigma + (1-\xi)\frac{2\theta}{1-\theta},$$

then the aggregate labour demand curve is flatter than the aggregate labour supply curve, and the city keeps expanding with population and wage converging to infinity. ∎

A.5 Regularity Conditions

In this subsection we prove that the model has stable equilibria given the two regularity conditions: (i) the no-black-hole condition as stated in (25), and (ii) condition (26) which ensures a positive elasticity of the city boundary with respect to productivity.

A.5.1 No-black-hole Condition

Since the equilibrium is unique with a finite population when $\lambda \leq \sigma$, we just need to consider the case of $\lambda > \sigma$ to understand the no-black-hole condition. When $\lambda > \sigma$, $F > \frac{\lambda-\sigma}{1-\xi}$ is equivalent to $\frac{1}{F} < \frac{1-\xi}{\lambda-\sigma}$, which means the aggregate labour supply curve is flatter than the aggregate labour demand curve at the point they intersect, and only the large city equilibrium in panel (b) of Figure 1 satisfies this condition. Since $F < \frac{2\theta}{1-\theta}$, condition (25) implies $\frac{\lambda-\sigma}{1-\xi} < \frac{2\theta}{1-\theta}$, or equivalently,

$$\lambda - \sigma < (1-\xi)\frac{2\theta}{1-\theta}, \tag{57}$$

which prevents the city from growing without a bound as discussed in Proposition 1.

A.5.2 Positive Elasticity of City Boundary w.r.t. Productivity

Here we prove that regulatory condition (26) is a necessary and sufficient condition for a city NOT to contract in response to a positive productivity shock.

Since $J = \frac{log(w)+\theta log\left(B_0/\underline{p}\right)-\beta_0}{\beta_1+\beta_2 N}$ (equation 48), we see that a positive productivity shock affects the city boundary through two channels: (i) transportation costs increase as more population leads to more congestion; and (ii) a higher wage allows workers to spend more on transportation, while keeping their utility at the reservation utility level.

Using equation (48) and taking the derivative of J with respect to $log(\tilde{A})$, we obtain

$$\begin{aligned} \frac{dJ}{dlog(\tilde{A})} &= \frac{\frac{dlog(w)}{dlog(\tilde{A})}(\beta_1+\beta_2 N) - \beta_2 \frac{dN}{dlog(\tilde{A})}\left[log(w) + \theta log\left(B_0/\underline{p}\right) - \beta_0\right]}{(\beta_1+\beta_2 N)^2} \\ &= \frac{\frac{dlog(w)}{dlog(\tilde{A})}(\beta_1+\beta_2 N) - \beta_2 \frac{dN}{dlog(\tilde{A})}(\beta_1+\beta_2 N)J}{(\beta_1+\beta_2 N)^2} \\ &= \frac{\frac{dlog(w)}{dlog(\tilde{A})} - \beta_2 \frac{dlog(N)}{dlog(\tilde{A})}JN}{\beta_1+\beta_2 N} \\ &= \frac{\zeta_w - \zeta_N \beta_2 JN}{\beta_1+\beta_2 N}. \end{aligned}$$

Therefore, the elasticity of city boundary with respect to $\tilde{A}$ is

$$\zeta_J = \frac{dlog(J)}{dlog(\tilde{A})} = \frac{\zeta_w - \beta_2 JN\zeta_N}{(\beta_1+\beta_2 N)J} = \frac{F-\beta_2 JN}{(\beta_1+\beta_2 N)J}\zeta_N, \tag{58}$$

where we have used equation (27) to substitute out ζ_w. Thus, a positive productivity shock expands the size of the city if and only if $F > \beta_2 JN$ which is regulatory condition (26). ∎

B Elasticities

This appendix provides some technical details about how land rent elasticities depends on the timing of production externality and city characteristics.

B.1 Elasticities Before Feedback

Here we prove equations (36)-(37) which are land rent elasticities in phase one before the feedback from increased population is materialized.

We revisit the aggregate labor supply/demand equation. The aggregate labor demand equation captures the number of workers that can be housed in a city, which is not affected by the phase-one assumption that the feedback from increased population is NOT materialized. Therefore we only need to revise the aggregate labor demand equation. Let N_0 denote city population before the productivity shock occurs. Substituting out land rent in equations (45) with the commercial bid-rent function, we obtain

$$N = \left[\frac{r^{\xi} w^{1-\xi}}{\tilde{A} N_0^{\lambda} \xi^{\xi} (1-\sigma-\xi)^{1-\xi} S^{\sigma}} \right]^{-\frac{1}{\sigma}}, \tag{59}$$

which is comparable to equation (51), except that the city level TFP here is $\tilde{A}N_0$ rather than $\tilde{A}N$. We take logarithm of the above equation, then differentiate it with respect to $log(\tilde{A})$ to obtain the following:

$$\zeta_N = \frac{1}{\sigma} - \frac{1-\xi}{\sigma}\zeta_w, \tag{60}$$

which is identical to equation (28) if λ is set to zero.

Recall that the aggregated labor supply equation is not affected by our phase-one assumption, so equations (27) still holds true, which, along with equation (60), is used to substitute out ζ_w and ζ_N in equation (30) and equation (32), leading to equations (36)-(37). ∎

B.2 Fixed City Boundary

This subsection considers cities where the boundaries are fixed rather than endogenously determined, and proves the third point of Proposition 2 and the fourth point of Proposition 3 which states how land rent elasticities are affected by the fixed boundaries.

We differentiate equation (21), the aggregate labour supply function, with respect to $\tilde{A}$, keeping in mind that J is now treated as a constant.[20] The following is obtained:

$$\frac{\zeta_w}{\zeta_N} = \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N} - \frac{\beta_2 J N}{e^{\frac{1-\theta}{\theta}(\beta_1 J + \beta_2 J N)} - 1} = \bar{F}. \tag{61}$$

As in the benchmark model, the inverse of $\frac{\zeta_w}{\zeta_N}$ describes the slope of the aggregate labour supply curve. i.e.,

$$\frac{dlog(N)}{dlog(w)} = \frac{1}{\bar{F}}.$$

Note the new effective land supply constraint, $\bar{F}$, is also increasing in N. It is straightforward to show:

$$\lim_{N\to 0} \bar{F} = \frac{\theta}{1-\theta},$$
$$\lim_{N\to \infty} \bar{F} = \frac{2\theta}{1-\theta}.$$

Therefore the shape of the aggregate labour supply curve is similar to the one shown in Figure 1. Consequently, all the results in Proposition 1 hold true.

[20]Before differentiating, we use equation (52) to substitute out the term $\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj$.

Since fixing the city boundaries does not affect the aggregate labour demand function, equation (28) from the benchmark model is still valid. We combine it with equation (61) to substitute out ζ_w and ζ_N in (32), and derive the following expression for elasticity of land rent in a city with fixed boundaries:

$$\bar{\zeta}_{p_r} = \frac{1}{\theta} \times \frac{\bar{F} - \beta_2 jN}{-\lambda + \sigma + (1-\xi)\bar{F}}. \tag{62}$$

As we will show below, $\bar{F} > F$. In addition, given the same exogenous productivity and other parameters, population is always smaller in cities with fixed boundaries, thus it is guaranteed that $\bar{\zeta}_{p_r} > 0$ because $\bar{F} > F > \beta_2 jN$.

Using equation (16), the elasticity of commercial land rent is

$$\bar{\zeta}_{p_c} = \frac{1 + \bar{F}}{-\lambda + \sigma + (1-\xi)\bar{F}}. \tag{63}$$

Obviously these elasticities of commercial and residential land rents are identical to those in the benchmark model, except that F is replaced by $\bar{F}$. Therefore the results in Propositions 2-3 also hold true.

We now show that $\bar{F} > F$ given the same population. Using regulatory condition (26), we have:

$$\beta_2 JN < F = \frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right)\left(\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\right).$$

thus,

$$\begin{aligned}
\bar{F} &= \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N} - \frac{\beta_2 JN}{e^{\frac{1-\theta}{\theta}(\beta_1 J + \beta_2 JN)} - 1} \\
&> \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N} - \frac{\frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right)\left(\frac{\beta_1+2\beta_2 N}{\beta_1+\beta_2 N}\right)}{e^{\frac{1-\theta}{\theta}(\beta_1 J + \beta_2 JN)} - 1} \\
&= \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\left(1 - \frac{1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}}{e^{\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J} - 1}\right) \\
&= \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\left(1 - \frac{e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}(e^{\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J} - 1)}{e^{\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J} - 1}\right) \\
&= \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right) \\
&= F.
\end{aligned}$$

The result in Proposition 3 follows directly from $\bar{F} > F$. Given the same population, the effective land supply constraint is more stringent when city boundary is fixed, which dampens residential land rent elasticity if and only if agglomeration effect is strong, i.e., when $\lambda - \sigma > (1-\xi)\beta_2 jN$.

In addition, by comparing equation (63) with equation (31), provided that $\bar{F} > F$, it is straightforward to show that the commercial land rent elasticity is lower when the boundary is fixed, as stated in Proposition 2.

B.3 Residential Land Rent Elasticity When Congestion Effect Is Not Location-Specific

Here we prove that the residential land rent elasticity is

$$\zeta_{p_r^\star} = \frac{1}{\theta} \times \frac{F^\star - \beta_2 N}{-\lambda + \sigma + (1-\xi)F^\star}, \tag{64}$$

when the the transportation cost function is $f = \beta_0 + \beta_1 j + \beta_2 N$, which is a special case of the more general function given in equation (34) where congestion is location-specific.

With the new transportation cost function, $\frac{\partial f(j,N)}{\partial j} = \beta_1$, and equation (52) becomes:

$$\int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj = \frac{\theta}{(1-\theta)\beta_1} \left[e^{-\frac{1-\theta}{\theta}(\beta_0 + \beta_2 N)} - e^{-\frac{1-\theta}{\theta} f(J,N)} \right].$$

Substituting this into equation (21), the aggregate supply equation becomes:

$$\begin{aligned} log(N) &= log\left[\frac{(1-\Lambda)B_0}{(1-\theta)\beta_1}\right] + \frac{1-\theta}{\theta} log(w) \\ &\quad + log\left[e^{-\frac{1-\theta}{\theta}(\beta_0+\beta_2 N)} - e^{-\frac{1-\theta}{\theta}\left[log(w) - \theta log\left(\frac{p}{B_0}\right)\right]}\right]. \end{aligned}$$

Note that $\beta_0 + \beta_2 N = f(0, N)$ and $log(w) - \theta log\left(\frac{p}{B_0}\right) = f(J, N)$. Differentiating both sides of the above equation with respect to $log(w)$, we obtain the following:

$$\frac{dlog(N)}{dlog(w)} = \frac{1-\theta}{\theta} + \frac{-\frac{(1-\theta)\beta_2 N}{\theta} e^{-\frac{1-\theta}{\theta} f(0,N)} \frac{dlog(N)}{dlog(w)} + \frac{1-\theta}{\theta} e^{-\frac{1-\theta}{\theta} f(J,N)}}{e^{-\frac{1-\theta}{\theta} f(0,N)} - e^{-\frac{1-\theta}{\theta} f(J,N)}},$$

which leads to the following slope of the aggregate labour supply curve:

$$\frac{dlog(N)}{dlog(w)} = \frac{\frac{1-\theta}{\theta}}{1 - e^{-\frac{1-\theta}{\theta}\beta_1 J} + \frac{1-\theta}{\theta}\beta_2 N} := \frac{1}{F^\star}. \tag{65}$$

Here $F^\star$ is analogous to F for the case of location-specific congestion in the benchmark model. The explicit expression of $F^\star$ is

$$F^\star = \frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}\beta_1 J}\right) + \beta_2 N \approx \beta_1 J + \beta_2 N.$$

Equation (65) leads to the following:

$$\frac{\zeta_w}{\zeta_N} = \frac{dlog(w)/dlog(\tilde{A})}{dlog(N)/dlog(\tilde{A})} = \frac{dlog(w)}{dlog(N)} = F^\star, \tag{66}$$

which is analogous to equation (27) for the case of location-specific congestion.

Note that equation (28) still holds true because the aggregate labour demand is not affected by the new transportation cost function, thus we substitute equation (66) into equation (28) to obtain:

$$\zeta_N = \frac{1}{-\lambda + \sigma + (1-\xi)F^{\star}}. \tag{67}$$

Differentiating the residential bid-rent function with respect to $log(\tilde{A})$, we obtain

$$\zeta_{p_r^{\star}} = \frac{1}{\theta}(\zeta_w - \beta_2 N \zeta_N), \tag{68}$$

where we have used $\frac{df(j,N)}{dlog(\tilde{A})} = \beta_2 N \zeta_N$ based on the transportation cost function specification of $f(j,N) = \beta_0 + \beta_1 j + \beta_2 N$. Substituting equations (66)-(67) into equation (68), we obtain $\zeta_{p_r^{\star}}$ as in equation (64). ∎

B.4 Undevelopable Land

This subsection shows the effects of Λ, the fraction undevelopable land. We start with showing how Λ affects city configuration, including the population size and geographical size. Next we show how land rent elasticity depends on Λ as discussed in Section **??**, as well as on population and CBD size.

B.4.1 Undevelopable Land and City Configuration

Here we show the effects of Λ on the city configuration as summarized in Proposition 7.

Wage and Population More undevelopable land is reflected in the downward shift of the aggregate labour supply curve, leading to a smaller equilibrium population. As one can see from panel (b) of figure 1, when $\lambda > \sigma$, the shift causes wage to fall if we exclude the small city equilibrium; while the shift causes wage to rise when $\lambda < \sigma$, which can be seen from panel (a) of the figure. This proves point 1 of Proposition 7.

Commercial Land Rent To see how Λ affects commercial land rent p_c, we rewrite the commercial bid-rent function as follows:

$$log(p_c) = \frac{1}{\sigma} log\left(\frac{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}}\right) + \frac{\lambda}{\sigma} log(N) - \frac{1-\sigma-\xi}{\sigma} log(w).$$

The derivative of $log(p_c)$ with respect to Λ is:

$$\frac{dlog(p_c)}{d\Lambda} = \frac{\lambda}{\sigma} \times \frac{dlog(N)}{dlog(\Lambda)} - \frac{1-\sigma-\xi}{\sigma} \times \frac{dlog(w)}{dlog(\Lambda)}. \tag{69}$$

Based on the aggregate labour demand function, the following relationship exists between $\frac{dlog(N)}{dlog(\Lambda)}$ and $\frac{dlog(w)}{dlog(\Lambda)}$:

$$\frac{dlog(N)}{dlog(\Lambda)} = \frac{1-\xi}{\lambda-\sigma} \times \frac{dlog(w)}{dlog(\Lambda)}. \tag{70}$$

Substituting this relationship into equation (69), we obtain

$$\begin{aligned} \frac{dlog(p_c)}{d\Lambda} &= \frac{\lambda}{\sigma} \times \frac{dlog(N)}{dlog(\Lambda)} - \frac{1-\sigma-\xi}{\sigma} \times \frac{\lambda-\sigma}{1-\xi} \times \frac{dlog(N)}{dlog(\Lambda)} \\ &= \frac{\lambda+1-\sigma-\xi}{1-\xi} \times \frac{dlog(N)}{dlog(\Lambda)} \\ &< 0, \end{aligned}$$

where the inequality holds because $\frac{dlog(N)}{dlog(\Lambda)} < 0$ which is true because population falls with Λ. Thus cities with larger shares of undevelopable land always have lower commercial land rents.

Residential Land Rent Based on the residential bid-rent function, the logarithm of residential rent is:

$$log(p_{r(j)}) = log(B_0) + \frac{1}{\theta}[log(w) - f(j,N)].$$

The derivative with respect to Λ is:

$$\begin{aligned} \frac{dlog(p_{r(j)})}{d\Lambda} &= \frac{1}{\theta}\left[\frac{dlog(w)}{d\Lambda} - \beta_2 jN\frac{dlog(N)}{d\Lambda}\right] \\ &= \frac{1}{\theta}\left[\frac{\lambda-\sigma}{1-\xi} \times \frac{dlog(N)}{d\Lambda} - \beta_2 jN\frac{dlog(N)}{d\Lambda}\right] \\ &= \frac{\lambda-\sigma-(1-\xi)\beta_2 jN}{\theta(1-\xi)} \times \frac{dlog(N)}{d\Lambda}. \end{aligned}$$

where we have used equation (70) to substitute out $\frac{dlog(w)}{d\Lambda}$ in the second equality. Since population falls with Λ so $\frac{dlog(N)}{d\Lambda} < 0$. From the above equation, we conclude that

$$\frac{dlog(p_{r(j)})}{d\Lambda} < 0 \quad iff \quad \lambda - \sigma > (1-\xi)\beta_2 jN.$$

City Boundatry We use the expression of $J = \frac{log(w)+\theta log\left(B_0/\underline{p}\right)-\beta_0}{\beta_1+\beta_2 N}$ (equation 48). The boundary J changes with population and wage which in turn change when Λ changes. The derivative of J with respect to Λ is:

$$\begin{aligned}
\frac{dJ}{d\Lambda} &= \frac{(\beta_1+\beta_2 N)\times\frac{dlog(w)}{d\Lambda} - \beta_2[log(w)+\theta log\left(B_0/\underline{p}\right)-\beta_0]N\times\frac{dlog(N)}{d\Lambda}}{(\beta_1+\beta_2 N)^2} \\
&= \frac{(\beta_1+\beta_2 N)\times\frac{dlog(w)}{d\Lambda} - (\beta_1+\beta_2 N)\beta_2 JN\times\frac{dlog(N)}{d\Lambda}}{(\beta_1+\beta_2 N)^2} \\
&= \frac{\frac{dlog(w)}{d\Lambda} - \beta_2 JN\times\frac{dlog(N)}{d\Lambda}}{\beta_1+\beta_2 N} \\
&= \frac{\left(\frac{\lambda-\sigma}{1-\xi} - \beta_2 JN\right)\times\frac{dlog(N)}{d\Lambda}}{\beta_1+\beta_2 N},
\end{aligned}$$

where we have used equation (70) in the last equality. Therefore, $\frac{dJ}{d\Lambda}<0$ if and only if

$$\lambda > \sigma + (1-\xi)\beta_2 JN > 0.$$

B.4.2 How Does F Depend on CBD Size and Undevelopalbe Land?

This subsection proves Lemma 3 which states that F decreases with the share of undevelopable land Λ if and only if $\lambda>\sigma$. We also prove that F increases with the CBD size S, i.e., $\frac{dF}{dS}>0$ if and only if $\lambda>\sigma$. In other words, we show the following:

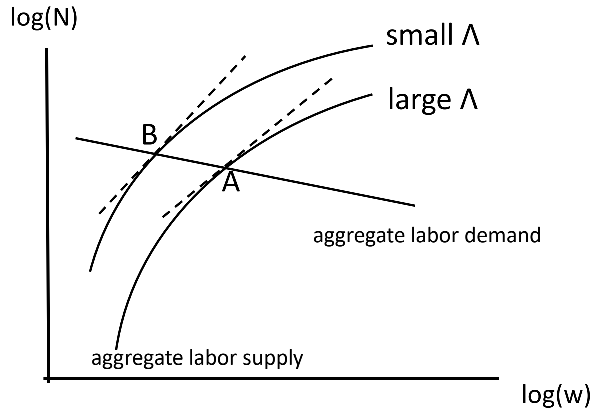
$$\frac{dF}{dS} > 0 \quad iff \quad \lambda>\sigma; \tag{71}$$

$$\frac{dF}{d\Lambda} < 0 \quad iff \quad \lambda>\sigma, \tag{72}$$

We revisit equations (20)-(21), the aggregate labour supply and labour demand functions. A smaller Λ shifts the aggregate labour supply curve up. When $\lambda-\sigma\leq 0$, this leads to a larger population but a smaller wage. As shown in Figure 3, the equilibrium moves from point A to point B. As indicated by equation (21), $log(N)$ and $log(1-\Lambda)$ have a linear relationship, so the two curves are parallel to each other, implying that the slope is steeper at point B than at point A. Recall that the slope is $1/F$, so F is smaller at point B than at point A. Therefore, the smaller Λ leads to smaller F if $\lambda-\sigma\leq 0$, i.e., $\frac{dF}{d\Lambda}>0$. On the other hand, if $\lambda-\sigma>0$, more developable land increases F by increasing both wage and population, hence $\frac{dF}{d\Lambda}<0$. This proves Lemma 3.

Based on equation (20), a larger CBD leads to a downward shift of the aggregate labour demand curve because commercial land is relatively cheaper. If $\lambda-\sigma\leq 0$, the situation corresponds to panel (a) of Figure 1, and this shift decreases the equilibrium wage and population, which lowers F, i.e. $\frac{dF}{dS}<0$. However if $\lambda-\sigma>0$, the situation correspond to panel (b) of Figure 1, and the shift of aggregate labour demand curve increases equilibrium wage and population, thus $\frac{dF}{dS}>0$. In sum, $\frac{dF}{dS}>0$ if and only if $\lambda>\sigma$

Figure 3: Effects of More Developable Land



Note: More developable land (i.e. smaller Λ) shifts the aggregate labour supply curve up, and the equilibrium moves from point A to B, resulting in a steeper tangent of aggregate labour supply curve at the point of equilibrium.

B.4.3 Effects of Undevelopable Land, Population, and CBD size on Land Rent Elasticity

This appendix shows that land rent elasticity depends on the share of undevelopable land Λ, as stated in Proposition 8. We also show that land rent elasticity depends on the existing city population N and the size of CBD S in the following way:

Proposition 12 *Given the exogenous variables and parameters in Table 1, the following is true regarding how land rent elasticity depends on population N and CBD size S:*

(i) In location $j = 0$, ζ_{p_r} is decreasing in N and S if and only if $\lambda - \sigma > 0$.

(ii) In locations $j > 0$, ζ_{p_r} is decreasing in N and S if $\lambda - \sigma > (1 - \xi)\beta_2 jN$.

(iii) ζ_{p_c} is decreasing in the population N

(iv) ζ_{p_c} is decreasing in the CBD size S if and only if $\lambda - \sigma > 0$.

Residential Land Rent We start from the effects of population on land rent elasticity. From equation (34), the derivative of residential land rent elasticity to city population is:

$$\begin{aligned} \frac{d\zeta_{p_r}}{dN} &= \frac{\left(\frac{dF}{dN} - \beta_2 j\right)[-\lambda + \sigma + (1-\xi)F] - \frac{dF}{dN}(1-\xi)(F - \beta_2 jN)}{\theta[-\lambda + \sigma + (1-\xi)F]^2} \\ &= \frac{(-\lambda + \sigma)\left[\frac{dF}{dN} - \beta_2 j\right] + (1-\xi)\beta_2 j\left[\frac{dF}{dN}N - F\right]}{\theta[-\lambda + \sigma + (1-\xi)F]^2} \\ &= \frac{[-\lambda + \sigma + (1-\xi)\beta_2 jN]\frac{dF}{dN} + [\lambda - \sigma - (1-\xi)F]\beta_2 j}{\theta[-\lambda + \sigma + (1-\xi)F]^2}, \end{aligned} \tag{73}$$

where the second term in the numerator is negative due to the regulatory condition (25). When $\lambda - \sigma > (1 - \xi)\beta_2 jN$, the first term in the numerator is also negative. Therefore $\lambda - \sigma > (1 - \xi)\beta_2 jN$ is a sufficient condition for $\frac{d\zeta_{p_r}}{dN}$ to be negative.

For location $j = 0$, equation (73) implies:

$$\frac{d\zeta_{p_r}}{dN} = \frac{[-\lambda + \sigma]\frac{dF}{dN}}{\theta[-\lambda + \sigma + (1-\xi)F]^2},$$

which is negative if and only if $\lambda - \sigma > 0$.

The necessary and sufficient condition for $\frac{d\zeta_{p_r}}{dN} < 0$ is $\lambda - \sigma > -\chi$ where $\chi = \frac{F - \frac{dF}{dlog(N)}}{\frac{dF}{dlog(N)} - \beta_2 jN}(1-\xi)\beta_2 jN$. Note the term $\frac{dF}{dlog(N)}$ in χ is equivalent to $\frac{dlog(w)}{d^2log(N)}$, i.e. the second derivative of the inverse aggregate labour supply function, because $F = \frac{dlog(w)}{dlog(N)}$.

To derive the condition $\lambda - \sigma > -\chi$, we use equation (73) to obtain

$$\frac{d\zeta_{p_r}}{dN} = \frac{(-\lambda + \sigma)\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] + (1-\xi)\beta_2 jN\left[\frac{dF}{dlog(N)} - F\right]}{\theta N[-\lambda + \sigma + (1-\xi)F]^2}, \tag{74}$$

thus $\frac{d\zeta_{p_r}}{dN} < 0$ is equivalent to

$$(-\lambda + \sigma)\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] < (1-\xi)\beta_2 jN\left[F - \frac{dF}{dlog(N)}\right].$$

Since $\left[\frac{dF}{dlog(N)} - \beta_2 jN\right]$, this is equivalent to

$$\lambda - \sigma > -\left[\frac{F - \frac{dF}{dlog(N)}}{\frac{dF}{dlog(N)} - \beta_2 jN}\right](1-\xi)\beta_2 jN := -\chi.$$

That is, $\frac{d\zeta_{p_r}}{dN} < 0$ if and only if $\lambda - \sigma > -\chi$.

Next, we derive the expression for $\frac{d\zeta_{p_r}}{dS}$ and $\frac{d\zeta_{p_r}}{d(1-\Lambda)}$. Similar to equation (73), for the effects of CBD size S and the share of developable land $1 - \Lambda$, we derive the following:

$$\begin{aligned}
\frac{d\zeta_{p_r}}{dS} &= \frac{[-\lambda + \sigma + (1-\xi)\beta_2 jN]\frac{dF}{dS} + [\lambda - \sigma - (1-\xi)F]\beta_2 j\frac{dN}{dS}}{\theta[-\lambda + \sigma + (1-\xi)F]^2}; \\
\frac{d\zeta_{p_r}}{d(1-\Lambda)} &= \frac{[-\lambda + \sigma + (1-\xi)\beta_2 jN]\frac{dF}{d(1-\Lambda)} + [\lambda - \sigma - (1-\xi)F]\beta_2 j\frac{dN}{d(1-\Lambda)}}{\theta[-\lambda + \sigma + (1-\xi)F]^2}.
\end{aligned}$$

Using (71)-(72), it is straightforward to see that, for the case of $j = 0$, $\frac{d\zeta_{p_r}}{dS} < 0$ and $\frac{d\zeta_{p_r}}{d(1-\Lambda)} < 0$ if and only if $\lambda - \sigma > 0$. Further, if $\lambda - \sigma > (1-\xi)\beta_2 jN$, then the first term in each of the numerators is negative, which implies $\frac{d\zeta_{p_r}}{dS} < 0$ and $\frac{d\zeta_{p_r}}{d(1-\Lambda)} < 0$ since $\lambda - \sigma - (1-\xi)F < 0$ by the regulatory condition (25).

If $\lambda - \sigma < (1-\xi)\beta_2 jN$, it is possible for ζ_{p_r} to increase with S. This occurs when $\frac{dF}{dS}$ is very large and $\beta_2 j\frac{dN}{dS}$ is very small. Similar analysis applies to $\frac{d\zeta_{p_r}}{d(1-\Lambda)}$.

Commercial Land Rent Elasticity From equation (31) it is straightforward to derive the following:

$$\begin{aligned}
\frac{d\zeta_{p_c}}{dN} &= -\frac{\lambda + (1-\sigma-\xi)}{[-\lambda+\sigma+(1-\xi)F]^2} \times \frac{dF}{dN} \\
\frac{d\zeta_{p_c}}{dS} &= -\frac{\lambda + (1-\sigma-\xi)}{[-\lambda+\sigma+(1-\xi)F]^2} \times \frac{dF}{dS} \\
\frac{d\zeta_{p_c}}{d(1-\Lambda)} &= -\frac{\lambda + (1-\sigma-\xi)}{[-\lambda+\sigma+(1-\xi)F]^2} \times \frac{dF}{d(1-\Lambda)}
\end{aligned}$$

From the definition of F, it is clear that $\frac{dF}{dN} > 0$, therefore $\frac{d\zeta_{p_c}}{dN} < 0$. Using (71)-(72), it is straightforward to see that $\frac{d\zeta_{p_c}}{dS} < 0$ and $\frac{d\zeta_{p_c}}{d(1-\Lambda)} < 0$ if and only if $\lambda - \sigma > 0$. ∎

C More Details on Extended Models

This Appendix provides details about the extended model and proves Propositions 10 and 11.

C.1 Expandable CBD

C.1.1 Aggregate Labour Demand and Multiple Equilibria

When the CBD is expandable, land use competition ensures that commercial land rent equals residential rent on the border of the CBD, i.e. $p_{r(j=0)} = p_c$. Using the commercial and residential bid rent functions (equation 16 and equation 18), we have

$$B_0(e^{-\beta_0}w)^{1/\theta} = \left[\frac{\tilde{A}N^{\lambda}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}}.$$

This is equivalent to

$$log(N) = \frac{1}{\lambda}log\left(\frac{r^{\xi}(B_0e^{-\beta_0/\theta})^{\sigma}}{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}\right) + \frac{1}{\lambda}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)log(w), \tag{75}$$

which is the aggregate labour demand function when the CBD is expandable.

Clearly this new aggregate labour demand function is upward sloping unless $\lambda = 0$. Therefore with $\lambda > 0$, this extended model always has two equilibria. In contrast, two equilibria arise in the benchmark model only when the agglomeration effect is strong enough, i.e. $\lambda > \sigma$.

The extended model here has the same aggregate labour supply function as equation(21), since the function is derived from the equilibrium of the residential land market, and it is not affected by the expandable CBD. The slope of aggregate labour supply curve is still $\frac{1}{F}$ with F defined in equation (23).

As in the benchmark model, we rule out the situation where city grows without a bound by assuming that the slope of aggregate labour demand curve is steeper than the slope of aggregate labour supply curve $-\frac{1}{\lambda}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)>\frac{1}{F}$, which is equivalent to:

$$\lambda<\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F \quad . \tag{76}$$

This is the "no-black-hole condition" when the CBD is flexible.

C.1.2 Elasticities

We use $\tilde{\zeta}_N$, $\tilde{\zeta}_w$, $\tilde{\zeta}_{p_r}$, $\tilde{\zeta}_{p_c}$ to denote elasticities in the model with expandable CBD. Differentiating equation (75) with respect to $log(\tilde{A})$, we obtain obtain:

$$\tilde{\zeta}_N=\frac{1}{\lambda}\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)\tilde{\zeta}_w-\frac{1}{\lambda}. \tag{77}$$

This, combined with equation (27), yields the following population elasticity:

$$\tilde{\zeta}_N = \frac{1}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}, \tag{78}$$

which is positive given the "no-black-hole condition" given by (76).

The elasticity of residential land rent as in equation (32) is still valid in the extended model because is derived from residential bid-rent function. We substitute out ζ_N and ζ_w using equations (27) and (78) to obtain:

$$\tilde{\zeta}_{p_r(j)} = \frac{1}{\theta}\times\frac{F-\beta_2 jN}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}. \tag{79}$$

This elasticity has the same properties as the residential land rent elasticity in the benchmark model, except that $\tilde{\zeta}_{p_r(j)}$ decreasing in F if and only if $\lambda>\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)\beta_2 jN$.

Compared with $\zeta_{p_r(j)}$ in the benchmark model as given by equation (34), it is straightforward to see that the elasticity in equation (79) is larger if and only if $F<\frac{\theta}{1-\theta}$, i.e.,

$$\tilde{\zeta}_{p_r}>\zeta_{p_r} \quad iff \quad F<\frac{\theta}{1-\theta}.$$

This is point 2(a) of Proposition 10.

Since commercial land rent always equals the residential land rent in location $j=0$, we have $\tilde{\zeta}_{p_r(j=0)}=\tilde{\zeta}_{p_c}$, i.e.,

$$\tilde{\zeta}_{p_c}=\frac{1}{\theta}\times\frac{F}{\left(\frac{\sigma}{\theta}+1-\sigma-\xi\right)F-\lambda}.$$

Now we study whether commercial rent elasticity is larger or smaller when CBD is expandable by comparing $\tilde{\zeta}_{p_c}$ with ζ_{p_c} in equation (31). Given the same population, the necessary and sufficient condition for $\tilde{\zeta}_{p_c} < \zeta_{p_c}$ is

$$\begin{aligned}
\frac{1}{\theta} \times \frac{F}{\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) F - \lambda} &< \frac{1+F}{-\lambda + \sigma + (1-\xi)F} \\
\Leftrightarrow \quad -\lambda F + (1-\xi)F^2 &< -\theta\lambda + \theta(1-\xi-\sigma-\lambda)F + \sigma(1-\theta)F^2 + \theta(1-\xi)F^2 \\
\Leftrightarrow \quad [(1-\xi)(1-\theta) - \sigma(1-\theta)]F^2 &< -\theta\lambda + [(1-\theta)\lambda + \theta(1-\xi-\sigma)F \\
\Leftrightarrow \quad \frac{\theta}{1-\theta}\lambda + (1-\sigma-\xi)F^2 &< \left[\lambda + \frac{\theta}{1-\theta}(1-\sigma-\xi)\right] F \\
\Leftrightarrow \quad \frac{\theta}{1-\theta}\lambda + (1-\sigma-\xi)F^2 &< \lambda F + \frac{\theta}{1-\theta}(1-\sigma-\xi)F \\
\Leftrightarrow \quad [(1-\sigma-\xi)F - \lambda]F &< [(1-\sigma-\xi)F - \lambda]\frac{\theta}{1-\theta}.
\end{aligned}$$

Thus if we impose the condition that $F < \frac{\theta}{1-\theta}$, then

$$\tilde{\zeta}_{p_c} < \zeta_{p_c} \quad iff \quad \lambda < (1-\sigma-\xi)F.$$

This proves point 2(b) of Proposition 10. ▮

C.1.3 Proof of Lemma 4

Here we prove that the CBD size increases in response to a positive productivity shock if and only if $F < \frac{\theta}{1-\theta}$. Based on equation (45), total residential land demand is $S = \frac{\sigma w N}{(1-\sigma-\xi)p_c}$ where p_c can be substituted by the commercial bid-rent function (equation 16). Thus we have:

$$\begin{aligned}
S &= \frac{\sigma w N}{(1-\sigma-\xi)p_c} \\
&= \frac{\sigma w N}{(1-\sigma-\xi)} \left[\frac{\tilde{A}\sigma^\sigma \xi^\xi (1-\sigma-\xi)^{1-\sigma-\xi}}{r^\xi w^{1-\sigma-\xi}}\right]^{-\frac{1}{\sigma}} N^{-\frac{\lambda}{\sigma}} \\
&= \left[\frac{r^\xi}{(1-\sigma-\xi)^{1-\xi}\xi^\xi \tilde{A}}\right]^{\frac{1}{\sigma}} w^{\frac{1-\xi}{\sigma}} N^{\frac{\sigma-\lambda}{\sigma}} \quad . \qquad (80)
\end{aligned}$$

Let ζ_S denote the elasticity of CBD size with respect to the exogenous productivity $\tilde{A}$. We take logarithm of equation (80) and differentiate it with respect to $log(\tilde{A})$ to obtain:

$$\begin{aligned}
\zeta_S &= \frac{\sigma-\lambda}{\sigma}\tilde{\zeta}_N + \frac{1-\xi}{\sigma}\tilde{\zeta}_w - \frac{1}{\sigma} \\
&= \tilde{\zeta}_N \left(\frac{\sigma-\lambda}{\sigma} + \frac{1-\xi}{\sigma}\frac{\tilde{\zeta}_w}{\tilde{\zeta}_N}\right) - \frac{1}{\sigma} \\
&= \frac{\left(\frac{\sigma-\lambda}{\sigma} + \frac{1-\xi}{\sigma}F\right)}{\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) F - \lambda} - \frac{1}{\sigma}
\end{aligned}$$

where we have used $F = \frac{\tilde{\zeta}_w}{\tilde{\zeta}_N}$ and used equation 78) to substitute out $\tilde{\zeta}_N$. After some algebra, the above equation becomes

$$\zeta_S = \frac{\sigma}{\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) F - \lambda} \times \left(1 - \frac{1-\theta}{\theta} F\right) \quad . \tag{81}$$

The first term in equation 81 is positive due to the "no-black-hole condition". Thus $\zeta_S > 0$, i.e., the CBD expands in response to a positive productivity shock, if and only if $1 - \frac{1-\theta}{\theta} F > 0$ which is equivalent to $F < \frac{\theta}{1-\theta}$, the condition given in Lemma 4. $\blacksquare$

C.2 Immobile Capital

In this subsection, we consider the alternative assumption of immobile capital. This is partly motivated by the observations in Glaeser and Gyourko (2005) that the depreciation of urban buildings is slow which causes the slow decline of cities that experience negative productivity shocks.

Endogenous Capital Price We assume the city has a fixed $\bar{K}$ stock of capital, and the price of capital r is endogenously determined by the capital market clearing condition. From the firm's problem we have shown that $\frac{n}{k} = \left(\frac{1-\sigma-\xi}{\xi}\right)\left(\frac{r}{w}\right)$ (equation 10) for each firm. Aggregating over all the firms we have:

$$r = \frac{\xi}{1-\sigma-\xi} \frac{N}{\bar{K}} w. \tag{82}$$

Given a higher productivity, both wage and total number of workers rise, thus equation (82) predicts that r should rise.[21]

Elasticities We use ζ^* to denote elasticities in the model with fixed capital stock. Substituting out r in the aggregate labour demand function, we rewrite equation (20) as:

$$log(N) = \frac{1}{\lambda - \sigma - \xi} log\left(\frac{1}{\tilde{A}(1-\sigma-\xi)\bar{K}^{\xi} S^{\sigma}}\right) + \frac{1}{\lambda - \sigma - \xi} log(w).$$

Differentiating both sides the above with respect to $log(\tilde{A})$ leads to $\zeta^*_N = \frac{1-\zeta^*_w}{\sigma+\xi-\lambda}$ where ζ^*_w can be substituted out using $\frac{\zeta^*_w}{\zeta^*_N} = F$ (equation 27).[22] This leads to the following population elasticity:

$$\zeta^*_N = \frac{1}{-\lambda + \sigma + \xi + F}. \tag{83}$$

[21]Capital owners share part of the economic benefits (costs) from the rising (falling) productivity in this case.

[22]Equation (27) is still valid in the case of immobile capital because it is derived from the aggregate labour supply function from the benchmark model which is not affected by capital immobility assumption.

Using equation (83), we substitutes out ζ_w and ζ_N in equation (32) to reach:

$$\zeta^*_{p_r(j)} = \frac{1}{\theta} \times \frac{F - \beta_2 j N}{-\lambda + \sigma + \xi + F}, \tag{84}$$

where F is defined in equation (23). It is straightforward to show $\zeta^*_{p_r(j)} > 0$ given the same regularity conditions in the benchmark model.

For commercial land rent, we substitute out r in commercial bid-rent function (equation 16) to obtain:

$$p_c = \left[\frac{\tilde{A}\sigma^\sigma (1-\sigma-\xi)^{1-\sigma} \bar{K}^\xi}{w^{1-\sigma}} \right]^{\frac{1}{\sigma}} N^{\frac{\lambda-\xi}{\sigma}}.$$

Thus the elasticity of commercial land rent is

$$\begin{aligned} \zeta^*_{p_c} &= \frac{1}{\sigma} + \frac{\lambda-\xi}{\sigma}\zeta^*_N - \frac{1-\sigma}{\sigma}\zeta^*_w \\ &= \frac{1+F}{-\lambda+\sigma+\xi+F}. \end{aligned} \tag{85}$$

Point (i) of Proposition 11 Note that $\zeta^*_{p_c}$ and ζ_{p_c} have the same numerators, so do $\zeta^*_{p_r}$ and ζ_{p_r}. Thus we just need to compare the denominators. Obviously $-\lambda + \sigma + \xi + F > -\lambda + \sigma + (1-\xi)F$ holds true, where $-\lambda + \sigma + (1-\xi)F > 0$ is the denominator of ζ_{p_r} and ζ_{p_c} in the bench mark model. In other words, in the case of immobile capital, the denominator of land rent elasticities is larger, which implies

$$\begin{aligned} \zeta^*_{p_r} &< \zeta_{p_r}; \\ \zeta^*_{p_c} &< \zeta_{p_c} \quad . \end{aligned}$$

Point (ii) of Proposition 11 Equations (84)-(85) indicate that, when capital is immobile, land rent elasticities decrease with ξ, the share of capital in production, exactly the opposite of the results in the benchmark model. ∎