

# Macro skewness and conditional second moments: evidence and theories

Ian Dew-Becker, Alireza Tahbaz-Salehi, and Andrea Vedolin\*

October 14, 2019

## Abstract

We establish four facts about skewness and conditional volatility in the economy: (1) aggregate activity is negatively skewed; (2) sector activity is negatively skewed, but less than aggregate; (3) the cross-sectional variance of output growth is countercyclical; (4) when a sector shrinks, it subsequently covaries more with other sectors. Those facts can all be generated qualitatively and quantitatively by a multisector equilibrium model with the key feature that production inputs are gross complements. Three alternative models that have been proposed to generate skewness and stochastic volatility are unable to simultaneously match all four facts even qualitatively.

## 1 Introduction

### Background and empirical facts

A defining feature of the business cycle is the existence of recessions as distinct episodes. Rather than simply experiencing symmetric random fluctuations around a trend, output, unemployment, and other measures of the state of the economy display sharp declines and relatively smooth expansions: levels and growth in real activity are skewed left.<sup>1</sup> At the same time, many measures of volatility, both in the aggregate and the cross-section, are countercyclical. As a mathematical matter there is a mechanical link between skewness and countercyclical volatility – high volatility in bad times leads to a long left tail of outcomes – but few models capture that feature of the economy.

---

\*Dew-Becker: Northwestern University and NBER; Tahbaz-Salehi: Northwestern University; Vedolin: Boston University, NBER, and CEPR.

<sup>1</sup>See Sichel (1993), McKay and Reis (2008), Morley and Piger (2012), Berger, Dew-Becker, and Giglio (2019), and Dupraz, Nakamura, and Steinsson (2019) among others.

It is also not just aggregate output that is skewed left: sector- and firm-level activity display the same behavior. Similarly, while aggregate output features countercyclical volatility, the variance of the *cross-sectional* distribution is also countercyclical.

This paper begins by systematically documenting a set of key facts about aggregate and cross-sectional variance and skewness. The third moment of some variable  $x$  is the same as the covariance of  $x$  with  $x^2$ , so skewness itself can be thought of as a conditional second moment. In that sense, our four facts are all about time-variation in second moments:

1. Aggregate economic activity is skewed left (its second moment is countercyclical)
2. Sector activity is skewed left, and skewness decreases at lower levels of aggregation
3. The variance of the cross-sectional distribution of sector activity is countercyclical
4. When activity in a given sector declines, that sector covaries relatively more with other sectors.

The unifying theme behind the four facts is that they are all about second moments being high when aggregate or sector activity is low. Negative time-series skewness in some series is a well known feature of business cycles,<sup>2</sup> but the fact that it monotonically grows with the level of aggregation has not been previously noted. The cyclical nature of cross-sectional variance is also well known and appears in a number of contexts,<sup>3</sup> but the dependence of conditional covariance on sector shocks is also novel, and we will argue directly points to the paper's proposed model.

### **Proposed model**

We show that the four facts are naturally generated in a multisector model – where individual sectors of the economy are explicitly modeled and aggregate to produce total GDP (e.g. Long and Plosser (1983)) – and in which production at the sector and aggregate level is complementary in inputs – the elasticity of substitution is less than 1, as advocated by Baqaee and Farhi (2019). To build intuition, consider an extreme case in which aggregate output is Leontief in a range of inputs (consistent with the estimates of Barrot and Sauvagnat (2016), Atalay (2017), and Carvalho et al. (2016)), and that production of the inputs depends only on sector-specific productivity shocks. Then the distribution of output is the distribution of the minimum of the technology shocks, which is in general skewed left. With network effects, that minimum – the productivity of the worst-performing sector – becomes a common component that affects all sectors.

Formally, the solution to the model yields a simple linear factor model for sector output

---

<sup>2</sup>Most relevant to this paper, see Baqaee and Farhi (2019), Berger, Dew-Becker, and Giglio (2019), and Sichel (1993).

<sup>3</sup>For recent work, see Bloom et al. (2018) and Guvenen, Ozkan, and Song (2014). Salgado, Guvenen, and Bloom (2019) go further and show that cross-sectional skewness is also countercyclical. We show that their result is also consistent with our model.

with a skewed common factor. When, consistent with the data, the sector-specific shocks are symmetrical, sector output is naturally less skewed than aggregate output.

Countercyclical cross-sectional variance is generated simply because the minimum of the sector-level shocks tends to be lower in periods when the sample standard deviation happens to be higher. That is, the model generates countercyclical volatility – which is sometimes taken as evidence for countercyclical uncertainty – even though all shocks are homoskedastic, thus calling into question empirical analyses of uncertainty shocks based on cross-sectional volatilities (e.g. see the discussion in Bloom (2014)).

We include the fourth fact precisely because it directly addresses the key mechanism in the model. In a model with complementarities, when the supply of one input shrinks, that input becomes more important, in the sense that output becomes relatively more sensitive to it and less sensitive to other inputs. Because the sectors that shrink become more relevant, they also covary more strongly with any other sector that buys their output (including final production).

A single simple idea, then, that production features complementarities, explains aggregate and sector skewness, the time-series patterns of cross-sectional volatility, and the response of the cross-sector covariance matrix to shocks. Another way to put it is that the model is overidentified relative to the three empirical facts. One parameter – the elasticity of substitution across inputs – determines the signs of the time-series skewness of output, the cyclicity of cross-sectional variance, and the relationship between a sector’s centrality and its past shocks.

### **Alternatives**

There are other models that have been proposed to match some of the facts that we study. What we find, though, is that none of the alternative models we examine can match all of the facts simultaneously. We study three basic specifications:

- a. Skewed aggregate shocks (or perhaps skewness in a universal input, such as finance, as in Brunnermeier and Sannikov (2014));<sup>4</sup>
- b. Stochastic volatility and skewness in aggregate and idiosyncratic shocks (Bloom (2009));
- c. Concave responses to shocks (Ilut, Kehrig, and Schneider (2018)).

Each of the models, under suitable restrictions, is able to generate a subset of the facts that we document, but they all fail on at least one. The comparison with the alternative models also helps illustrate precisely why the aggregative model is able to fit the data. In brief, facts 1 and 2 point to the existence of a skewed common component in output,

---

<sup>4</sup>The literature on rare disasters, e.g. Barro (2006), Gourio (2012, 2013), and Wachter (2013)

while facts 3 and 4 imply that the common component is endogenous to sector-level shocks. Explicit aggregation is therefore central because it creates that endogeneity. There always exists some sufficiently rich purely statistical specification that can match any set of empirical facts, and the facts could also likely be matched by some combination of the alternative models we study, but the aggregative model is notable for fitting the facts parsimoniously, relying essentially just on complementarity.

### **Implications**

The results in this paper have important implications for how to think about volatility and skewness in the economy.<sup>5</sup> A large literature studies stochastic volatility (and, more recently, stochastic skewness), often implicitly assuming that because the dispersion in observed distributions changes over time, that means that there are “uncertainty shocks” (e.g. Bloom et al. (2018)). The results here show that time-variation in sample moments – and their cyclicity – need not have anything to do with uncertainty. Rather, increases in the dispersion of realized shocks may simply be associated with declines in output because of strong concavity in the production function.

The model and empirical results also demonstrate that the economy in an important sense does not have a single fixed network structure – in terms of the covariance of output across sectors – and in fact that the variation is important in causing asymmetries in outcomes. Baqaee and Farhi (2019) study this point extensively, and our results provide further empirical support. They show that complementarity can generate aggregate skewness and variation in the centrality of sectors. The second, third, and fourth empirical facts – on sector level output, cross-sectional volatility, and conditional covariances – are all steps beyond their work, and necessary for distinguishing other models.

Output is skewed left and stochastically volatile in the model precisely because a single sector – or set of sectors – will occasionally receive a negative shock and become (approximately) a limiting factor in production. It is exactly that change in the network – that the negatively shocked industry becomes central – that creates a recession, negative skewness, and volatility. So while it is common for studies of economic networks to use snapshots of the inter-sector trade structure from the input-output tables, such an analysis can miss important variation in linkages, and this paper shows that the variation, which produces the time-varying second moments – is empirically relevant.

Finally, and perhaps most importantly, the analysis helps understand why recessions

---

<sup>5</sup>For work on unconditional skewness, see Sichel (1993), McKay and Reis (2008), Morley and Piger (2012), and Berger, Dew-Becker, and Giglio (2019). For recent work on time-varying volatility in the real economy, see Justiniano and Primiceri (2008), Clark and Ravazzolo (2015), and Schorfheide, Song, and Yaron (2018). Work on time-varying cross-sectional moments includes Guvenen, Ozkan, and Song (2014), Salgado et al. (2018), and Dew-Becker and Giglio (2019).

exist as discrete events. Hypotheses like skewness in aggregate shocks or concavity in firm responses to shocks have been proposed to match the steepness and deepness of recessions, but they turn out to fail on other dimensions. The necessary concavity arises naturally when there is complementarity in production.

### **Related work**

The paper is related to a number of active literatures. First, as described above, is work on time-variation in time-series and cross-sectional moments of output. Second is an emerging literature on production networks with complementarities in production. Atalay (2017) and Atalay, Drautzburg, and Wang (2018) estimate the elasticity of substitution using data on sector production and input-output linkages, finding extremely strong complementarities (up to the point of a Leontief production function). Barrot and Sauvagnat (2016) find similar results in firm-level responses to natural disasters. Baqaee and Farhi (2019) theoretically study models with complementarities and show in a calibration that they can generate a realistic level of skewness at the aggregate level, in addition to studying how covariances and the network structure change over time in general production frameworks. Carvalho et al. (2016) study the economy of Japan following the 2011 earthquake in a network model with non-unitary elasticities of substitution. Our contribution relative to this literature is in showing that an aggregative model is able to match the four stylized facts on skewness and to compare its performance to other leading potential explanations.

The remainder of the paper is organized as follows. Section 2 documents the three empirical facts. Section 3 presents a simple and stylized aggregative model with complementarities and formally shows that it can match the three empirical facts. In section 4 we examine the predictions of three other models of economic skewness. Section 5 examines a model with complementarities in a more quantitatively realistic calibration, and section 6 concludes.

## **2 Empirical facts**

This section establishes four facts about variance, covariance, and skewness in the economy:

1. Aggregate economic activity is skewed left.
2. Sector-level activity is skewed left, but less so than aggregate skewness.
3. The variance of the cross-sectional distribution of activity is countercyclical.
4. When activity in sector  $i$  declines, sector  $i$  becomes more central in the covariance matrix of activity, in the sense that elements of the covariance matrix in its row and column rise compared to other elements.

## 2.1 Data

We focus on measures of activity that have data at monthly frequency and are measured at a high level of sectoral detail. The two-main series are industrial production (from the Federal Reserve), which is measured to the five-digit NAICS level in manufacturing industries, and employment (from the Current Employment Survey of the BLS), which is measured at up to the six-digit NAICS level and covers the entire economy. For industrial production, we follow Foerster, Sarte, and Watson (2011) and study data since 1972. For employment, the sample with detailed NAICS coverage begins in 1990.

## 2.2 Time-series skewness

The fact that aggregate output and employment, in both levels and growth rates, are skewed left has been established in previous work.<sup>6</sup> Not surprisingly, sector-level measures of activity are also generally skewed left. We now show, though, that they are less negatively skewed than aggregate growth rates, and that the magnitude of the negative skewness increases with the level of aggregation.<sup>7</sup>

Table 1 reports the average level of skewness at different levels of aggregation for levels and growth rates of industrial production and employment growth. For each sector we calculate the coefficient of skewness for the time series of its, say, industrial production growth. The table then reports the (unweighted) mean of those skewness coefficients at each level of aggregation. Standard errors are reported in brackets.<sup>8</sup>

The top panel reports results for industrial production. The skewness of total industrial production growth is -1.18. At the two-digit level – just three sectors: durable and non-durable manufacturing and mining – average skewness is -0.96. At the three- and four-digit levels, where there are 43 and 81 total sectors, respectively, skewness falls to -0.55 then -0.45. Finally, at the five-digit level skewness is only -0.41. Skewness at the aggregate level is therefore three times higher than at the most disaggregated sector level. A similar pattern

---

<sup>6</sup>Berger, Dew-Becker, and Giglio (2019), show that growth rates of employment, capacity utilization, industrial production, GDP, durable and non-durable consumption, and residential and nonresidential investment are all skewed left. Furthermore, returns on the S&P 500 are skewed left, as is their option-implied distribution. Morley and Piger (2012) provide a much more thorough analysis of asymmetry in the output gap – that is, on skewness in levels, rather than growth rates – and finding similar results – while Sichel (1993) provides an earlier analysis distinguishing asymmetry in levels from growth rates. See also references therein for the literature on business cycle asymmetry.

<sup>7</sup>Ilut, Kehrig, and Schneider (2018) examine employment and find skewness at both the firm and aggregate levels, but, as discussed below, do not emphasize the difference between the two.

<sup>8</sup>The standard errors are calculated with a blockwise jackknife that clusters by date. Specifically, each jackknife replication removes 50 consecutive months of data from the sample – the same 50 months for all sectors – and we iterate over all possible starting months for the excluded dates.

holds using quarterly instead of monthly growth rates. In both cases the differences between the skewness at the various levels of aggregation are themselves statistically significant. In other words, at every level, as aggregation increases, skewness becomes more negative, and the differences across levels of aggregation are both statistically and economically significant.

The right-hand panel reports results for levels of IP minus an exponentially weighted moving average trend. The results are similar, though the increase in skewness is no longer monotone. Panel B table 1 shows that similar results hold for employment growth, with highly similar magnitudes for the coefficients.

To help identify the source of skewness, we examine the properties of residuals from a time-series regression of each sector’s growth rate on aggregate growth. Specifically, if  $x_{i,t}$  is growth in sector  $i$  in month  $t$  and  $\bar{x}_t$  is growth in the aggregate in month  $t$ , we estimate the regression

$$x_{i,t} = a_i + b_i \bar{x}_t + \nu_{i,t}, \tag{1}$$

and table 1 reports skewness for the residuals  $\nu_{i,t}$  in the columns labeled “residuals”.

The results are very different from what is observed for the raw growth rates. The residuals are still negatively skewed, but by much less than the raw growth rates, and with surprising stability across levels of aggregation. In other words, most of the negative skewness – and all of the increase with aggregation – appears to come from each sector’s exposure to aggregate growth – the sector-specific component contributes little. While that is natural from the perspective of some models, it will turn out to rule out others.

To summarize, then, we have two basic results regarding time series skewness of raw growth rates: measures of real activity, both in levels and growth rates, are skewed to the left, and the magnitude of that skewness increases with the level of aggregation. After controlling for the aggregate component, though, there is little skewness remaining in the residual growth rates,  $\nu_{i,t}$ , and it does not increase with aggregation.

### 2.3 Realized cross-sectional variance

Our second empirical fact is that cross-sectional variance is countercyclical.<sup>9</sup> In each month  $t$ , we calculate the standard deviations of monthly growth rates of industrial production and employment at the four-digit NAICS level. It is important to emphasize that these

---

<sup>9</sup>A similar fact is well known from the literature individual earnings (Storesletten, Telmer, and Yaron (2004); Guvenen, Ozkan, and Song (2014)). Salgado, Guvenen, and Bloom (2018) provide evidence both within and across countries on the cyclical nature of the variance and skewness of cross-sectional outcomes. We contribute to that literature by providing novel evidence on the cyclical nature of skewness in the cross-section of sectoral industrial production and employment growth, including looking at the sector-specific component, as above.

are *realized* sample moments – they do not measure a conditional distribution. That is, the conditional probability density from which the sector growth rates are drawn could be constant over time. We are simply measuring sample moments – which are random variables – and examining their contemporaneous relationship with the state of the business cycle.<sup>10</sup>

Table 2 reports results from univariate regressions of the cross-sectional standard deviation on an NBER recession indicator and aggregate employment growth as two different measures of the state of the business cycle (i.e. in separate regressions). The cross-sectional standard deviation and aggregate employment growth are normalized to have unit variance to help in interpreting the coefficients.

The results confirm previous findings that cross-sectional volatility is countercyclical, and the magnitude of the variation is economically significant. It rises by up to a full standard deviation during NBER-dated recessions, and its correlation with aggregate employment growth is between -0.25 and -0.35.

As in table 1, we also estimate the regressions using residuals from a regression of sector growth rates on aggregate growth (the  $\nu_{i,t}$  from equation (1)). Contrary to table 1, in this case the results are essentially unchanged looking at the cross-sectional distribution of residuals compared to raw growth rates. Unlike what is observed in table 1, then, the behavior of the cross-sectional moments is not driven purely by the behavior of a common component. Even with a common component taken out of the growth rates, the cyclicity of the cross-sectional moments is unchanged. The sector residuals,  $\nu_{i,t}$ , are, by construction, uncorrelated with aggregate activity. The regressions in table 2 show, though, that they are not statistically independent of aggregate activity – their higher moments are correlated with the state of the business cycle. In other words, the common component of output is not independent of the sector-specific shocks.

## 2.4 Conditional covariances

The final fact relates most directly to the key mechanisms in the structural model. When a sector contracts, it becomes more central in the economy, in the sense that its covariances with other sectors rise.

---

<sup>10</sup>The distinction between the conditional distribution and a realized moment is emphasized by Berger, Dew-Becker, and Giglio (2019). As an example, consider some mean-zero shock  $\varepsilon$  with a constant conditional distribution. One could look at the realization of  $\varepsilon_t^2$  – essentially a sample variance – and ask how it correlates with other variables. For example, one might find that  $\varepsilon_t^2$  is negatively correlated with  $\varepsilon_t$ . That does not mean that the conditional variance of  $\varepsilon_t$  (i.e. uncertainty) is countercyclical. Rather, it is just an indication that the distribution of  $\varepsilon_t$  is left skewed. So nothing in this subsection should be interpreted as measuring variation in conditional distributions. In the example in this footnote, doing so would require *forecasting*  $\varepsilon_t^2$ , not just looking at contemporaneous correlations.



### 2.4.1 Empirical method

Define  $\Sigma_t$  to be the (unobservable) conditional covariance matrix of sector-level growth rates on date  $t$ . Define  $\Sigma_{t,i}$  to be the average of the  $i$ 'th column of  $\Sigma_t$ , excluding the  $i, i$  element.  $\Sigma_{t,i}$  is the average of the covariances of sector  $i$  with all other sectors (or, equivalently, the covariance of growth in sector  $i$  with average growth in all other sectors). When we say that sector  $i$  covaries more strongly with other sectors, we mean  $\Sigma_{t,i}$  rises.

The goal is to estimate a relationship of the form

$$\Sigma_{t,i} = a_{i,1} + b_1(L) \varepsilon_{i,t-1} + c_{1,t} + \tilde{\eta}_{i,t}^1 \quad (2)$$

where  $\varepsilon_{i,t}$  measures the innovation to the level of activity in sector  $i$  on date  $t$ ,  $b_1$  is a polynomial in the lag operator,  $L$  ( $L^j x_{i,t} = x_{i,t-j}$ ), and  $\tilde{\eta}_{i,t}^1$  is a residual. The problem is that  $\Sigma_{t,i}$  is not observable. We therefore proxy for it with a simple date- $t$  product, similar to the literature on heteroskedasticity and feasible GLS.

More specifically, define  $\varepsilon_{i,t}$  to be the residual from an AR(p) model for growth in activity in sector  $i$ . We then proxy for  $\Sigma_{t,i}$  with  $\sum_{j \neq i} \varepsilon_{i,t} \varepsilon_{j,t}$ . That product is a single-observation sample moment, with the property that  $E \left[ \sum_{j \neq i} \varepsilon_{i,t} \varepsilon_{j,t} \right] = \Sigma_{t,i}$ . The regression that we actually estimate is then

$$\sum_{j \neq i} \varepsilon_{i,t} \varepsilon_{j,t} = a_{i,1} + b_1(L) \varepsilon_{i,t-1} + c_{1,t} + \eta_{i,t}^1 \quad (3)$$

$\eta_{i,t}^1$  captures both the true residual,  $\tilde{\eta}_{i,t}^1$ , and also the measurement error in the dependent variable,  $E \left[ \sum_{j \neq i} \varepsilon_{i,t} \varepsilon_{j,t} \right] - \Sigma_{t,i}$ . The errors are therefore in general non-Gaussian.

Because there may be common components across sectors in the innovations,  $\varepsilon_{i,t}$ , we include time fixed effects,  $c_{1,t}$ , in the estimation and cluster the standard errors by date. Similarly, some sectors will covary with others more strongly on average, so the constant  $a_{i,1}$  is allowed to vary across sectors (i.e. the estimation includes sector fixed effects).

The inclusion of time fixed effects means that changes in  $\Sigma_{t,i}$  must be interpreted as changes relative to  $\Sigma_{t,j}$  for  $j \neq i$ . Since the date- $t$  mean of  $\Sigma_{t,i}$  is equal to the mean of all pairwise covariances, a positive value for  $b_1(L)$  means that a positive shock to sector  $i$  raises its covariances *relative to those between other sectors*.

Because  $\varepsilon_{i,t}$  appears on the left-hand side, we include only its lags on the right hand side, so the regression can be interpreted as a forecasting exercise. Since the  $\varepsilon_{i,t}$  are constructed to be uncorrelated over time, their estimated coefficients trace out a nonparametric impulse response function, similar to the method of Jordá (2005). In that spirit, we include 36

monthly lags, to trace out the full response.

### 2.4.2 Results

The top panels of figure 1 report the estimated coefficients along with 90-percent confidence bands (instead of 95-percent because of the large number of coefficients and hence low power) for IP and employment. In both cases, the coefficients are consistently negative, with an upward trend as the lag grows. Because the individual coefficient estimates are somewhat noisy, the bottom panels plot the average value of the coefficients over five-lag windows. That pair of plots emphasizes the point more clearly, showing that there is a negative response of the covariances,  $\Sigma_{t,i}$ , to  $\varepsilon_i$  at lags of up to 1–2 years.

To get a sense of the magnitude, the mean of the left-hand side variable in the IP regressions is  $7.1 \times 10^{-5}$ , while the standard deviation of  $\varepsilon_{i,t}$  after controlling for the dummies is  $4.0 \times 10^{-2}$ . A coefficient of  $-2 \times 10^{-4}$  means that a unit standard deviation decline in  $\varepsilon_{i,t}$  is associated with a relative increase in the covariance of 11 percent of its mean, which the estimates imply lasts for 1–2 years. The combined effect of a number of past shocks to  $\varepsilon$  on the covariance can thus be substantial.

## 3 Aggregative model with complementarities

This section presents our benchmark multi-sector production model that can match the four facts presented above. In order to obtain analytic results, the model is highly stylized, with very strong forms of symmetry that yield simple solutions. It is not the only solvable specification, though. Furthermore, there are related versions of the model in which the results can be qualitatively different. The model should be viewed as giving an example of a specification that can qualitatively match the data – this section does not claim that *all* input-output networks with complementarities have the same characteristics as what is observed in our special case. To confirm that the theoretical results hold in a setting that is empirically more realistic, section 5 examines results from a numerical solution of a richer specification.

### 3.1 Economic structure

Output is produced in sector  $i$  according to the function

$$Y_i = \exp(\varepsilon_i) N_i^{1-\alpha} \left( \sum_j a_j x_{i,j}^\gamma \right)^{\alpha/\gamma} \quad (4)$$

where  $\varepsilon_i$  is sector  $i$ 's productivity and is distributed symmetrically and independently across sectors,  $N_i$  is sector  $i$ 's use of labor,  $x_{i,j}$  is sector  $i$ 's use of input  $j$ ,  $1/(1-\gamma)$  is the elasticity of substitution (so  $\gamma < 1$ ), and  $a_j$  a coefficient that determines the relative importance of good  $j$  in production.

Aggregate consumption is

$$C = \left( \sum_j a_j x_{C,j}^\gamma \right)^{1/\gamma} \quad (5)$$

where  $x_{C,j}$  is consumption of good  $j$ . The parameters of the final good production function are the same as for the individual sectors except that no labor is used ( $\alpha = 1$ ) and productivity is constant (in other words, (5) can also be a feature of preferences, rather than a physical production function). There is no investment and the economy is closed, so gross domestic product is equal to  $C$ .

The resource constraint for each sector is

$$\sum_i x_{i,j} + x_{C,j} = Y_j \quad (6)$$

There are a number of major restrictions in this setup that allow for tractability. First, the assumption that production takes the constant elasticity form, though standard, is obviously highly restrictive. Second, and more specific to this model, the coefficients determining the importance of each input,  $a_j$ , are the same across sectors. While this assumption will yield tractability, it is obviously counterfactual. Third, the elasticities of substitution are the same in all sectors.

There are at least three ways labor can be modeled: the aggregate supply can be fixed and it can adjust endogenously across sectors (e.g. Long and Plosser (1983)); it can be fixed within each sector (Baqae and Farhi (2019)); or the real wage can be fixed and the quantity allowed to vary freely (e.g. Blanchard and Gali (2010)). We focus on the case where  $N_i$  is fixed, as in Baqae and Farhi (2019), (and we simply normalize it to 1), but we also examine results in the other two cases.

## 3.2 Solution

The appendix shows that the model's solution for sector and aggregate output is (up to constant factors)

$$Y_i \propto \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} \exp(\varepsilon_j)^{\frac{\gamma}{1-\alpha\gamma}} \right)^{\frac{1-\gamma}{\gamma} \frac{\alpha}{1-\alpha}} a_i^{\frac{\alpha}{1-\alpha\gamma}} \exp(\varepsilon_i)^{\frac{1}{1-\alpha\gamma}} \quad (7)$$

$$C \propto \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} \exp(\varepsilon_j)^{\frac{\gamma}{1-\alpha\gamma}} \right)^{\frac{1-\alpha\gamma}{\gamma} \frac{\alpha}{1-\alpha}} \quad (8)$$

Aggregate output is a CES aggregate over the sector productivities, with exponent  $\gamma/(1-\alpha\gamma)$ . Since  $(1-\alpha\gamma) > 0$  for  $\gamma < 1$ , the sign of that exponent depends just on the sign of  $\gamma$ . When the inputs are gross complements,  $\gamma < 0$ , aggregate output is a complementary function of the sector productivities.

Sector-level output can be decomposed into two terms. The first is a non-negative power of aggregate output (positive for  $\gamma < 1$ ). The second is a positive power of  $z_i$ . So each sector's output has in essence a common component and a sector-specific component. In that respect, it is similar to many reduced-form statistical specifications, since in logs it is a linear single-factor model. Ignoring constants,

$$\log Y_i = \frac{1-\gamma}{1-\alpha\gamma} \log C + \frac{1}{1-\alpha\gamma} \varepsilon_i \quad (9)$$

Unlike here, though, statistical models almost always specify the common component as being independent of (or at least orthogonal to) the sector-specific shocks. In what follows, the centrality of endogeneity of the common component to the sector shocks will become clear.

## 3.3 Implications for observables

We now examine the model's ability to match the three empirical facts developed above. We focus on the behavior of output since the model assumes labor is fixed. We leave the more difficult task of incorporating a realistic specification for labor markets into the model to future work.

The empirical analysis is almost entirely in terms of growth rates due to the fact that measures of real activity are non-stationary (and not even cointegrated across sectors). The model is in general very difficult to analyze in terms of growth rates, however. In linear

Gaussian models, the distinction between levels and growth rates is nearly immaterial, so typical theoretical analyses have little trouble with the distinction. Ilut, Kehrig, and Schneider (2018), who also study a nonlinear model, simply specify the entire structure in terms of growth rates from the beginning, but that does not work in the present setting because aggregation happens in terms of levels.

We therefore take the following approach. For the first two facts – regarding time-series skewness and the cyclical nature of cross-sectional moments – we analyze the model purely in terms of levels. Appendix C shows that in the continuous limit, when the innovations to an AR(1) process are fat-tailed (but not when they are Gaussian) the first differences inherit the skewness of the levels, implying that the theoretical results derived for levels here also apply to growth rates.

The assumption that the shocks have fat tails is empirically valid. The average kurtosis of sector-level growth rates of IP and employment is between 8 and 11, depending on the level of aggregation.<sup>11</sup> Such behavior could arise from shocks drawn from a  $t$  distribution with 5 degrees of freedom, for example.

To evaluate whether the theoretical results derived in terms of levels are revealing for growth rates in a more quantitatively realistic setting, we examine a calibration in section 5, which confirms that the analytic results that we obtain for the model in levels also apply in terms of growth rates.

Finally, for the fourth fact – time-variation in conditional covariances – we are able to derive results for growth rates in a continuous-time version of the model, which we also show hold in the calibration.

### 3.3.1 Unconditional skewness

The log of aggregate output is

$$\log C = \frac{1 - \alpha\gamma}{\gamma} \frac{\alpha}{1 - \alpha} \log \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} \exp \left( \frac{\gamma}{1 - \alpha\gamma} \varepsilon_j \right) \right) \quad (10)$$

$\log C$  is a concave function of the  $\varepsilon$  when  $\gamma < 0$  and convex when  $\gamma > 0$ . So when  $\gamma < 0$  – when the inputs are gross complements – aggregate output is skewed left. When inputs are gross substitutes, on the other hand, aggregate output is skewed right. So in order for this model to generate negative skewness,  $\gamma$  must be negative.

---

<sup>11</sup>Under the theoretical model, the sector-level technology shocks can be identified from the residuals from a regression of sector output on aggregate output. The kurtosis of those residual growth rates is the same as the kurtosis of the raw growth rates – 8 to 11.

The log of sector output is

$$\log Y_i = \frac{1 - \gamma}{1 - \alpha\gamma} \log C + \frac{1}{1 - \alpha\gamma} \varepsilon_i + \text{constants} \quad (11)$$

Because sector output is equal to aggregate output plus a symmetrically distributed shock, it inherits the skewness of aggregate output, but with a smaller magnitude. The model therefore matches the result, when  $\gamma < 0$ , that output is skewed left, with the magnitude increasing as the level of aggregation increases.

In addition to the results on skewness of overall growth rates, table 1 also reports skewness for residuals, showing that once the aggregate component is controlled for, sector-level growth rates are substantially less skewed and that there is no increase in skewness with aggregation. Here when there are many sectors, the residual component of sector output, after controlling for aggregate output, converges to  $\frac{1}{1 - \alpha\gamma} \varepsilon_i$ . We have assumed that there is no skewness in the  $\varepsilon_i$ , so there is then no skewness in the residuals. That is more extreme than what we find empirically, but it is qualitatively consistent in that skewness is greater for total output,  $Y_i$ , than the residuals,  $\varepsilon_i$ .

While the analysis here applies to levels, as noted above, section C in the appendix analyzes the link between skewness of levels and growth rates in a continuous limit. It shows that skewness will also arise in growth rates as long as the technology shocks have fat tails (but still under the assumption that they are symmetrically distributed).

### 3.3.2 Realized cross-sectional variance and skewness

The model generates countercyclical volatility and procyclical skewness because aggregate output is a concave function of the shocks  $\varepsilon_i$ . All else equal, an increase in the dispersion in the  $\varepsilon_i$  yields a decline in output. To see why, denote the sample cumulants, weighted by  $\left\{a_j^{1/(1-\alpha\gamma)}\right\}$ , by  $\hat{\kappa}_{a,m}$ . The first three sample cumulants are equal to the first three central moments. That is,  $\hat{\kappa}_{a,1}(\{\varepsilon_j\})$  is the sample mean of the  $\{\varepsilon_j\}$ ,  $\hat{\kappa}_{a,2}(\{\varepsilon_j\})$  is the sample variance, and  $\hat{\kappa}_{a,3}(\{\varepsilon_j\})$  is the sample third moment, all weighted by  $\left\{a_j^{1/(1-\alpha\gamma)}\right\}$ .<sup>12</sup> We then

---

<sup>12</sup>Specifically, define the weighted sample mean to be  $E_a[\{x_j\}] = n^{-1} \sum_j \frac{a_j^{1/(1-\alpha\gamma)}}{\sum_k a_k^{1/(1-\alpha\gamma)}} x_j$ . Then  $\hat{\kappa}_{a,1} = E_a[\{\varepsilon_j\}]$ ,  $\hat{\kappa}_{a,2} = E_a\left[\left\{(\varepsilon_j - E_a[\{\varepsilon_k\}])^2\right\}\right]$ , and  $\hat{\kappa}_{a,3} = E_a\left[\left\{(\varepsilon_j - E_a[\{\varepsilon_k\}])^3\right\}\right]$ .

More generally, the  $n$ th weighted sample cumulant is the  $n$ th derivative with respect to  $t$ , evaluated at zero, of the sample cumulant generating function,

$$K_a(t; \{\varepsilon_j\}) \equiv \log \sum_j \frac{a_j^{1/(1-\alpha\gamma)}}{\sum_k a_k^{1/(1-\alpha\gamma)}} \exp(t\varepsilon_j) \quad (12)$$

have the following simple result, which follows directly from the definition of the cumulant generating function.

**Proposition 1** *Log output is*

$$\log C = \frac{\alpha}{1 - \alpha} \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{\gamma}{1 - \alpha\gamma} \right)^{m-1} \hat{\kappa}_{a,m}(\{\varepsilon_j\}) + \text{constants} \quad (13)$$

That is, output is linear in the sample cumulants of the set of realized productivity shocks,  $\{\varepsilon_j\}$ , weighted by the importance of each sector in production. When  $\gamma < 0$ , the coefficients on the even cumulants are negative, while the coefficients on the odd cumulants are positive. So an increase in the cross-sectional variance,  $\hat{\kappa}_{a,2}(\{\varepsilon_j\})$ , holding all other cumulants fixed, mechanically reduces output. Similarly, an increase in the third central moment,  $\hat{\kappa}_{a,3}(\{\varepsilon_j\})$  – an increase in skewness, with all other cumulants fixed – increases output.

Conversely, when  $\gamma > 0$ , cross-sectional variance becomes procyclical, while skewness remains procyclical, since the sign of the coefficients on the odd cumulants is always positive, while the sign of the coefficients on the negative cumulants is equal to the sign of  $\gamma$ . So in this model, the sign of the cyclicity of cross-sectional variance identifies the sign of  $\gamma$ .

As with the empirical results, a key observation is that the cyclicity of cross-sectional variance and skewness – whether they are positive or negative – has nothing to do with changes in conditional distributions or “time-varying uncertainty”. The cross-sectional distribution is a random variable that is correlated with output. All the fundamental shocks have constant volatility, and there is no sense in which there are uncertainty shocks here. The proposition simply shows that there is a mechanical relationship between cross-sectional variance and output. For  $\gamma < 0$ , output is a concave function of the (log) productivity shocks, so increases in dispersion reduce output.

Finally, recall that table 2 also shows that the cross-sectional moments of the residuals in sector growth rates, after controlling for the aggregate component, display the same cyclicity as the cross-sectional moments of overall sector growth. The model generates the same result. The reason is simple: the cross-sectional distribution of the sector growth rates is identical to the cross-sectional distribution of the sector residuals,  $\varepsilon_i$ , up to a shift in location. That is, the second and third (and higher) moments of the cross-sectional distribution of growth rates are identical to those for the cross-sectional distribution of residuals, so they also have the same cyclicity. The model thus matches the findings in table 2 both for overall growth rates and for residuals.

There are two notable features of the results in this section. First, they continue to rely on  $\gamma < 0$ , showing that the model is overidentified. Second, they rely on the fact that the

common component is a function of the sector shocks. If aggregate output were independent of the sector-specific shocks – i.e. if the sector shocks in some sense “washed out” – then proposition 1 would not hold. The explicit aggregation in the model is therefore critical to its ability to match the data.

### 3.3.3 Conditional covariances

The model description above does not say anything about the time series dependence of the levels of productivity,  $\varepsilon_i$ . In order to analyze the conditional covariances we examined empirically, we consider here a version of the model in which the  $\varepsilon_i$  follow AR(1) processes (though the logic extends to much more general processes).

**Proposition 2** *Sector  $i$ 's covariance with other sectors increases relative to covariances between other sectors when  $\varepsilon_i$  falls for  $\gamma < 0$  (the result is reversed when  $\gamma > 0$ ). Specifically,*

$$\text{sign} \left( \frac{d}{d\varepsilon_{i,t}} \left[ \begin{array}{c} \sum_{k \neq i} \text{cov}_t (\log Y_{i,t+1}, \log Y_{k,t+1}) \\ - \sum_{k \neq j} \text{cov}_t (\log Y_{j,t+1}, \log Y_{k,t+1}) \end{array} \right] \right) = \text{sign} (\gamma) \quad (14)$$

Intuitively, proposition 2 follows from the fact that when a sector receives a negative shock, it becomes relatively more important in determining variation in aggregate output when  $\gamma < 0$ . The importance of a sector  $i$  is quantified by

$$\frac{n^{-1} a_i^{\gamma/(1-\alpha\gamma)} \exp(\varepsilon_i)^{\gamma/(1-\alpha\gamma)}}{n^{-1} \sum_j a_j^{\gamma/(1-\alpha\gamma)} \exp(\varepsilon_j)^{\gamma/(1-\alpha\gamma)}} \quad (15)$$

The sign of the derivative of that ratio with respect to  $\varepsilon_i$  is the sign of  $\gamma$ . Since all covariation between sectors comes through variation in aggregate output, the relative increase in the importance of the given sector increases its covariances with all other sectors and reduces the covariances of other sectors with each other.

Proposition 2 formalizes the idea that under complementarity, when a sector receives a negative shock it becomes more central. That result is not universal – there are production networks for which it can fail to hold. However, it is natural in the baseline symmetric case, and it is apparently a reasonable description of the data. As with proposition 1, proposition 2 follows from the fact that the common component of output is endogenous to the sector-specific shocks, again emphasizing the importance of explicit aggregation.



### 3.3.4 Overidentification

All of the results derived in this section depend on the parameter  $\gamma$ . The sign of the skewness of output growth, the sign of the cyclical variance of the cross-sectional variance of output growth (both total growth and residuals with respect to aggregate output), and the sign of the response of a sector's covariances to its past growth rate all depend on the sign of  $\gamma$ . In that sense, the model is overidentified: all four of the empirical results depend on a single parameter.

That is the formal sense in which the model is overidentified or parsimonious.  $\gamma$  cannot be chosen completely freely. As soon as it is selected to match one moment, that predicts the signs of three others. That parsimony will not be shared by all of the alternatives below.

## 4 Alternative models of skewness

This section examines alternative models of skewness. We examine relatively stylized forms of models meant to capture different potential economic mechanisms that could be driving aggregate or cross-sectional skewness. The analysis shows that none of the alternatives are able to match the three sets of facts about volatility and skewness presented above, which the model with complementarity is able to match.

### 4.1 Skewed aggregate shocks

Consider a simple empirical model for sector output,  $Y_{i,t}$ , which could be generated by a number of different structural models:<sup>13</sup>

$$Y_{i,t} = b_i Y_t + \mu_{i,t} \tag{16}$$

where  $b_i$  is sector  $i$ 's loading on the aggregate shock  $\varepsilon_t$ , and  $\mu_{i,t}$  is a sector specific shock.  $\varepsilon_t$  is skewed left, while  $\mu_{i,t}$  has a symmetrical distribution and is independent of  $\varepsilon_t$ , and both shocks have zero mean. This model is a natural simple reduced-form specification that one might consider to match the result in table 1 that time-series skewness comes from

---

<sup>13</sup>One possibility is that there are fundamental shocks to the economy, e.g. technology or policy shocks, that are skewed left and can be represented by  $\varepsilon_t$  (e.g. rare disasters models (Rietz (1988), Barro (2006)) or smaller business-cycle frequency shocks as in Berger, Dew-Becker, and Giglio (2019)). Another microfoundation would be that some input to production used by all sectors, e.g. the output of the financial sector (financial intermediation) is skewed to the left. For example, the financial sector might face occasionally binding constraints, causing its ability to provide funding to the rest of the economy to face occasional crashes (e.g. Brunnermeier and Sannikov (2014), Kocherlakota (2000)).

a common component rather than sector-specific residuals. We now examine the model's ability to match the three empirical facts.

#### 4.1.1 Fact 1: Increasing negative skewness with aggregation

In the limit as the number of sectors grows large, if they aggregate linearly then aggregate output is simply proportional to  $\varepsilon_t$ . So both  $Y_t$  and  $Y_{i,t}$  are negatively skewed. Furthermore, it follows from the fact the symmetry of  $\mu_{i,t}$  and independence from  $\varepsilon_t$  that  $Y_t$  has a more negative skewness coefficient than  $Y_{i,t}$ . In addition, since the  $\mu_{i,t}$  are the residuals of sector output after controlling for the common component, the residuals are unskewed. This model can thus generate the time series facts, both for overall growth rates and for residuals without requiring nonlinear aggregation.

#### 4.1.2 Fact 2: Countercyclical cross-sectional variance

On any date, the cross-sectional variance of sector output is

$$\text{var}_t(Y_{i,t}) = \varepsilon_t^2 \text{var}(b_i) + \text{var}_t(\mu_{i,t}) \quad (17)$$

When the number of sectors is large,  $\varepsilon_t^2 \text{var}(b_i)$  dominates  $\text{var}_t(\mu_{i,t})$  as long as  $\text{var}(b_i) > 0$ . Since left skewness means that  $E[\varepsilon_t^3] < 0 \Rightarrow \text{corr}(\varepsilon_t, \varepsilon_t^2) < 0$ , we immediately have

$$\text{corr}(\text{var}_t(Y_{i,t}), Y_t) < 0 \quad (18)$$

So this model can generate countercyclical volatility as long as  $b_i$  differs across sectors (and without stochastic volatility or uncertainty shocks). Recall, though, that the empirical results in table 2 apply not just to sector growth rates, but also to residuals from regressions of sector growth rates on aggregate growth. In this case, that regression identifies  $\mu_{i,t}$ . If  $\varepsilon_t$  and  $\mu_{i,t}$  are independent, then the cross-sectional moments of the  $\mu_{i,t}$  are acyclical.

In other words, then, a model with skewed aggregate shocks can generate countercyclical volatility in sector output,  $Y_{i,t}$ , but it cannot replicate the empirical result that the cross-sectional distribution of residuals from regressions of sector output on aggregate output has the same cyclical properties. So at best the model only get half-way to matching our second empirical fact.

The failure of this model to match the cyclicity of the cross-sectional distribution of the residuals is instructive. The basic structure of this model is superficially similar to the equilibrium of the aggregative model, but it has a critical difference: the common component

here is exogenous and independent of the sector-specific shocks, whereas in the aggregative model the common component is endogenous to the sector-specific shocks.

### 4.1.3 Fact 3: Sector covariances rise following negative shocks

Under this model, the covariance between any pair of sectors is

$$\text{cov}(Y_{i,t}, Y_{j,t}) = b_i b_j \text{var}(\varepsilon_t) + \text{cov}(\mu_{i,t}, \mu_{j,t}) \quad (19)$$

Nothing about the model therefore implies that when a sector shrinks its covariance with other sectors should rise relative to their covariances with each other. Obviously the model could be augmented so that the covariances of the idiosyncratic shocks, or the loadings  $b$ , change over time, but there is nothing fundamental about the hypothesis that aggregate shocks are skewed to the right that would require such a scenario.

The failure of the model on this dimension is again a result of the independence of the aggregate and sector-specific components.

## 4.2 Sector output is a concave function of symmetric shocks

Ilut, Kehrig, and Schneider (2018) study aggregate and firm-level skewness and argue that skewness can be generated by a model in which firms have concave responses to economic shocks, such that firm output or employment takes the form

$$Y_{i,t} = f(\varepsilon_t + \mu_{i,t}) \quad (20)$$

where  $\varepsilon_t$  is again an aggregate shock and  $\mu_{i,t}$  is an idiosyncratic shock. The shocks are mean-zero and independent with symmetrical distributions. The function  $f$  is assumed to be smooth, strictly increasing, and strictly concave.

Ilut, Kehrig, and Schneider (2018) argue that this is a convenient reduced-form representation of a few different possible models of firm behavior that could generate asymmetry, including irreversible investment (George and Kuhn (2014)), learning (Senga (2016)), or matching frictions (Ferraro (2018)).

### 4.2.1 Unconditional skewness

It is straightforward to show that sector (or firm) output,  $Y_{i,t}$ , is skewed left, simply due to the concavity of  $f$ . However, skewness in this model decreases with the level of aggregation. Ilut, Kehrig, and Schneider's (2018) find this in their simulation (see their table 9). To see

why, consider a second-order approximation to sector output, and treat aggregate output as the mean across sectors,

$$Y_{i,t} \approx f(0) + f'(0) (\varepsilon_t + \mu_{i,t}) + \frac{1}{2} f''(0) (\varepsilon_t + \mu_{i,t})^2 \quad (21)$$

$$Y_t \approx f(0) + f'(0) \varepsilon_t + \frac{1}{2} f''(0) (\varepsilon_t^2 + \text{var}(\mu_{i,t})) \quad (22)$$

The quadratic term in  $Y_{i,t}$  is  $\frac{1}{2} f''(0) (\varepsilon_t + \mu_{i,t})^2$ , while in  $Y_t$  it is  $\frac{1}{2} f''(0) (\varepsilon_t^2 + \text{var}(\mu_{i,t}))$ . The skewness of  $Y_{i,t}$  is larger than the skewness of  $Y_t$  essentially because there is more variability in what is being squared at the sector than at the aggregate level –  $\varepsilon_t + \mu_{i,t}$  instead of just  $\varepsilon_t$ .

These results would also obtain in a granular model, as in Gabaix (2011), in which sector shocks are skewed left. That is, suppose there is effectively a small number of sectors, so that the sector shocks have nontrivial effects on aggregate output. Then even if they are skewed, after (linear) aggregation, aggregate output will be less skewed than sector output, due to simple averaging. In other words, the fact that skewness increases with aggregation is inconsistent with a simple form of micro granularity.

#### 4.2.2 Cyclicity of cross-sectional volatility

This model naturally generates countercyclical volatility. Consider a simple linear approximation to sector output around different levels of the aggregate shock,

$$Y_{i,t} \approx f(\varepsilon_t) + f'(\varepsilon_t) \mu_{i,t} \quad (23)$$

$$\text{var}_t(Y_{i,t}) \approx f'(\varepsilon_t)^2 \text{var}(\mu_{i,t}) \quad (24)$$

By assumption,  $f'(\varepsilon_t)$  strictly increases as  $\varepsilon_t$  declines, making cross-sectional variance is countercyclical.

Finally, the model also generates countercyclicity for the variance of the sector-specific component of output, as in the data. In the first-order approximation above, the sector-specific component, after removing aggregate output, is  $f'(\varepsilon_t) \mu_{i,t}$ . The cross-sectional variance of those residuals is then the same as the cross-sectional variance of output itself, and thus has the same cyclicity.

### 4.2.3 Sector covariances following negative shocks

Since all covariance across sectors comes through the  $\varepsilon_t$  term, consider the following approximation

$$Y_{i,t} \approx f(\mu_{i,t}) + f'(\mu_{i,t}) \varepsilon_t \quad (25)$$

$$\text{cov}(Y_{i,t}, Y_{j,t}) \approx f'(\mu_{i,t}) f'(\mu_{j,t}) \text{var}(\varepsilon_t) \quad (26)$$

When a sector receives a negative shock,  $f'(\mu_{i,t})$  rises. That raises the covariance of sector  $i$ 's output with all other sectors, without having any effect on covariances between other sectors. In that sense, sector  $i$  becomes more central. This result is not noted by Ilut, Kehrig, and Schneider (2018), but it represents an additional empirical fact that the model matches.

### 4.2.4 Summary

The model of concave decision rules generates negative skewness, and can match the cross-sectional facts, but it fails to generate the result that skewness increases with the level of aggregation. The key difference between this model and the model of complementarity in production is where skewness arises. With concave decision rules, the skewness arises at the firm or sector level. Under linear aggregation, that skewness washes out (as in the central limit theorem) at the aggregate level. In the aggregative model with complementarity, it is fundamentally *created by* aggregation, leading to a better fit to the data.

## 4.3 Idiosyncratic and aggregate shocks with time-varying distributions

A large literature studies models in which shock volatilities change over time. There can be changes in volatility at the aggregate level (Gourio (2012)), idiosyncratic level (Christiano, Motto, and Rostagno (2014)), or both (Bloom (2009)). Such a model, suitably enriched, can potentially also generate skewness. To see why, consider the following specification,

$$Y_t = \varepsilon_t - k\sigma_{t-1}^2 \quad (27)$$

$$Y_{i,t} = Y_t + \mu_{i,t} \quad (28)$$

$$\varepsilon_t \sim N(0, \sigma_{t-1}^2) \quad (29)$$

$$\mu_{i,t} \sim N(0, m\sigma_{t-1}^2) \quad (30)$$

where  $Y_t$ ,  $Y_{i,t}$ ,  $\varepsilon_t$ , and  $\mu_{i,t}$  continue to represent aggregate and sector output and innovations,  $k$  is a coefficient determining how output responds to variation in cross-sectional volatility relates to the level of output, and  $m$  determines the relative volatility of the two shocks.

### 4.3.1 Unconditional skewness

Aggregate and sectoral output are naturally skewed left because there is a mechanical relationship between variance and the level of output – the variance is higher when output is low. The appendix derives this result formally. Furthermore, skewness is greater at the aggregate than at the sector level because sector output has the additional (symmetric) component  $\mu_{i,t}$ , which drives its skewness towards zero.

The residual component of sector output is  $\mu_{i,t}$ , which is, by assumption, unskewed, roughly consistent with the data.

### 4.3.2 Time-varying cross-sectional moments

Cross-sectional variance is time-varying by assumption, since the distribution of  $\mu_{i,t}$  has variance  $\sigma_{t-1}^2$ . Cross-sectional variance is also countercyclical for  $k > 0$ . This result holds both for total sector output and for the sector specific component,  $\mu_{i,t}$ . This again emphasizes that the key feature of the model to match the cyclicity of cross-sectional moments is that the common component of output cannot be independent of the sector-level residuals.

It is worth noting here, though, that for both of the facts, there is a different parameter in the model. Aggregate skewness is determined by the parameter  $k$ , and cyclicity of cross-sectional volatility depends on  $k$  and  $m$ , whereas the aggregative model depends on a single microfounded parameter.

### 4.3.3 Sector covariances following negative shocks

The covariance of output between sectors is

$$\text{cov}(Y_{i,t}, Y_{j,t}) = \text{var}(\varepsilon_t) = \sigma_{t-1}^2 \tag{31}$$

In other words, the covariances are all identical. They change over time due to  $\sigma_t^2$ , and they are all countercyclical, but there is no variation across sectors. Certainly nothing in the model requires that when a firm receives a negative shock it will covary more strongly with other sectors, unlike the aggregative model or the one with concave decision rules. In both of those cases, the source of the increased covariance is that a sector loads more heavily on the common component following a negative sector-specific shock. This again illustrates the

value of the common component being endogenous to the sector shocks, unlike here, where it is purely exogenous.

### 4.3.4 Summary

In a model where aggregate output responds negatively to uncertainty, skewness arises naturally and increases with aggregation, as in the data. The model also, by assumption, generates countercyclical cross-sectional volatility. However, it has no prediction for differences in covariances across sectors. The analysis in this section is based on a reduced-form representation, but it is possible that fully nonlinear solutions of structural models, like that of Bloom et al. (2018), might yield different results. We examined simulations of that model, however (helpfully provided by the authors), and find that output and employment, in both levels and growth rates, are strongly positively skewed, implying that the at least the baseline calibration of Bloom et al. (2018) does not generate the simplest of our empirical results, that levels and growth rates of output and employment are skewed left.

## 4.4 Implications

The table below summarizes the facts and the ability of the models to qualitatively match them.

Fact:	Model:			
	Aggregative	Skewed shocks	Concave responses	Uncertainty shocks
Increasing skewness with aggregation	✓	✓	×	✓
No skewness for residuals	✓	✓	×	✓
Cyclicity of cross-sectional variance	✓	✓	✓	✓
Cyclicity of cross-sectional resid. var.	✓	×	✓	✓
Centrality rises after negative shocks	✓	×	✓	×

The equilibrium of the aggregative model has two key features that allow it to match the empirical facts: there is skewness in the common but not sector-specific components, and the common component is endogenous to the sector shocks.

## 5 Quantitative illustration

The analysis of the models so far has been purely qualitative. This section briefly asks whether a quantitative version of the aggregative model can match the actual numbers reported in section 2. It is not a full estimation, but rather a simple calibration, showing

that the quantitative results are plausibly produced by the model. We leave the formidable task of full estimation of a large-scale multisector model to future work.

## 5.1 Model and calibration

The specification follows that of Baqaee and Farhi (2019) closely (drawing on their posted replication files). The structure of the economy is a general CES setup, where production at the sector level,  $Y_i$ , relative to its steady-state,  $\bar{Y}_i$ , is

$$\frac{Y_i}{\bar{Y}_i} = A_i \left( \omega_{iL} + (1 - \omega_{iL}) \left( \frac{\hat{X}_i}{\bar{X}_i} \right)^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}} \quad (32)$$

where  $A_i$  is a technology shock,  $\omega_{iL}$  determines the importance of labor in production (where labor input is assumed to be fixed and normalized to 1) and  $\hat{X}_i$  measures the use of intermediates, with

$$\frac{\hat{X}_i}{\bar{X}_i} = \left( \sum_{j=1}^N \omega_{ij} \left( \frac{x_{ij}}{\bar{x}_{ij}} \right)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad (33)$$

where variables with bars again represent steady-state values. Finally, aggregate output (consumption) is

$$\frac{C}{\bar{C}} = \left( \sum_{i=1}^N \omega_{0i} \left( \frac{c_i}{\bar{c}} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad (34)$$

The model is both more general than that studied above and at the same time still highly restricted. The specification here allows for different elasticities of substitution at the aggregate and sector levels, for an arbitrary elasticity of substitution between inputs and labor, and for arbitrary weights on the inputs in each sector. At the same time, all sectors have the same elasticity parameters,  $\theta$  and  $\varepsilon$ , labor cannot be adjusted (which can be thought of as essentially a short-run restriction), and the production weights,  $\omega_{i,j}$ , are fixed over time. In reality, all of those assumptions are likely violated.

Our calibration follows that of Baqaee and Farhi (2019) for the most part. The production weights are chosen to match the 1982 input-output table (results are similar for other years). TFP shocks are calibrated to match the relative variance of TFP by industry along with the observed correlations. Their overall scale is chosen to match the volatility of industrial production growth (and the moments we will examine will be matched to the IP data). We draw the shocks from a  $t$  distribution with four degrees of freedom, which generates fat tails consistent with observed sector-specific IP growth. The autocorrelation of sector



productivity is set to 0.85 (at the monthly frequency) to match the dynamics of sector-level IP growth.

The elasticities of substitution are set to  $(\sigma, \theta, \varepsilon) = (0.9, 0.5, 0.1)$ . Aggregate output is thus relatively substitutable across sectors – close to a log-linear production function. The strong complementarity arises in production at the sector level, where the assumption is that the mix of material inputs is not amenable to adjustment. These values are similar to those estimated empirically by Atalay (2017).

## 5.2 Results

Table 3 reports moments of the model corresponding to the results from tables 1 and 2 along with the associated empirical results for industrial production. The top section shows that, as in the data, skewness is higher at the aggregate than the sector level, though the magnitudes are somewhat larger.

The bottom section examines the cyclicity of cross-sectional variance and skewness. NBER recessions in the model are defined as periods when aggregate output growth is in the bottom 15 percent of the unconditional distribution, to replicate the empirical frequency of recessions. The signs and magnitudes of the coefficients are highly similar between the model and the data.

Finally, the lines with squares in figure 1 plot the coefficients from the covariance forecasting regression. They show that the model generates regression coefficients that are negative, as in the data, and revert to zero at same general rate, but are larger by a factor of 5 on average.

Overall, the quantitative model performs reasonably well in matching the time-series, cross-sectional, and conditional moments, given that we made few choices in the calibration. Table 3 and figure 1 therefore show that a richer version of the model, which is designed to be closer to quantitative realism than the highly restricted setup analyzed theoretically above, is able to broadly match the empirical behavior of the economy documented in tables 1–2 and figure 1.

## 6 Conclusion

This paper develops a set of facts regarding aggregate and sectoral variances, covariances, and skewness. We then examine a model in which production displays complementarity across inputs and show that it can match all four facts: skewness is negative and increases in magnitude with the level of aggregation, cross-sectional variance is countercyclical, and a

decline in output in a sector is associated with an increase in the sector’s relative covariance with other sectors. The idea of complementarity, advanced most recently by Baqaee and Farhi (2019), is powerful in understanding both the aggregate and cross-sectional behavior of the economy.

That idea has important implications for how to think about skewness and time-varying volatility. The model implies that second and third moments change over time and are cyclical. In the past, it has sometimes been argued that the observed cyclical behavior of those moments implies that there are exogenous shocks to *uncertainty*, and that uncertainty then has negative effects on the economy. The model advanced here, though, is one in which changes in volatility are a result of fundamental productivity shocks and have no independent effect on the level of output.

A second important implication, which is supported by our empirical contributions, is that the centrality of sectors changes over time. In some models, recessions have common causes, e.g. technology shocks. Here, though, every episode is different. When a sector receives a negative shock, it becomes relatively more important. So in a period where oil stocks are low, shocks to the oil sector become a major driving force (e.g. Hamilton (2003), Kilian (2008)), whereas in periods when the financial sector is highly constrained, financial shocks become most relevant (e.g. Brunnermeier and Sannikov (2014)). A key insight of this paper is that complementarity means that the aggregate effects of shocks change in important ways over time, those changes can be measured from the covariances of sector growth rates, and many models fail to match them.

## References

- Atalay, Englin, “How important are sectoral shocks?,” *American Economic Journal: Macroeconomics*, 2017, 9 (4), 254–80.
- , Thorsten Drautzburg, and Zhenting Wang, “Accounting for the sources of macroeconomic tail risks,” *Economics Letters*, 2018, 165, 65–69.
- Baqaee, David Rezza and Emmanuel Farhi, “The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten’s Theorem,” 2019. Forthcoming, *Econometrica*.
- Barro, Robert J., “Rare Disasters and Asset Markets in the Twentieth Century,” *Quarterly Journal of Economics*, 2006, 121(3), 823–866.

- Barrot, Jean-Noël and Julien Sauvagnat**, “Input specificity and the propagation of idiosyncratic shocks in production networks,” *The Quarterly Journal of Economics*, 2016, *131* (3), 1543–1592.
- Berger, David, Ian Dew-Becker, and Stefano Giglio**, “Uncertainty shocks as second-moment news shocks,” 2019. Forthcoming, *Review of Economic Studies*.
- Blanchard, Olivier and Jordi Gali**, “Labor Markets and Monetary Policy: A New Keynesian Model with Unemployment,” *American Economic Journal: Macroeconomics*, 2010, *2* (2), 1–30.
- Bloom, Nicholas**, “The Impact of Uncertainty Shocks,” *Econometrica*, 2009, *77*(3), 623–685.
- , “Fluctuations in uncertainty,” *Journal of Economic Perspectives*, 2014, *28* (2), 153–76.
- , **Max Floetotto, Nir Jaimovich, Itay Saporta-Eksten, and Stephen J Terry**, “Really uncertain business cycles,” *Econometrica*, 2018, *86* (3), 1031–1065.
- Brunnermeier, Markus K and Yuliy Sannikov**, “A macroeconomic model with a financial sector,” *American Economic Review*, 2014, *104* (2), 379–421.
- Carvalho, Vasco M, Makoto Nirei, Yukiko Saito, and Alireza Tahbaz-Salehi**, “Supply chain disruptions: Evidence from the great east japan earthquake,” 2016. Working paper.
- Christiano, Lawrence J., Roberto Motto, and Massimio Rostagno**, “Risk Shocks,” *American Economic Review*, 2014, *104*(1), 27–65.
- Clark, Todd E and Francesco Ravazzolo**, “Macroeconomic forecasting performance under alternative specifications of time-varying volatility,” *Journal of Applied Econometrics*, 2015, *30* (4), 551–575.
- Dew-Becker, Ian and Stefano Giglio**, “Measuring the effects of cross-sectional uncertainty: evidence from implied volatilities, 1980–2017,” 2019. Working paper.
- Dupraz, Stéphane, Emi Nakamura, and Jon Steinsson**, “A plucking model of business cycles,” 2019. Working paper.
- Ferraro, Domenico**, “The asymmetric cyclical behavior of the us labor market,” *Review of Economic Dynamics*, 2018, *30*, 145–162.

- Foerster, Andrew T, Pierre-Daniel G Sarte, and Mark W Watson**, “Sectoral versus aggregate shocks: A structural factor analysis of industrial production,” *Journal of Political Economy*, 2011, *119* (1), 1–38.
- Gabaix, Xavier**, “The granular origins of aggregate fluctuations,” *Econometrica*, 2011, *79* (3), 733–772.
- George, Chacko and Florian Kuhn**, “Aggregate Implications of Capacity Constraints under Demand Shocks,” 2014. Working paper.
- Gourio, Francois**, “Disaster Risk and Business Cycles,” *American Economic Review*, 2012, *102*(6), 2734–2766.
- , “Credit Risk and Disaster Risk,” *American Economic Journal: Macroeconomics*, 2013, *5*(3), 1–34. Working paper.
- Guvenen, Fatih, Serdar Ozkan, and Jae Song**, “The Nature of Countercyclical Income Risk,” *Journal of Political Economy*, 2014, *122* (3), 621–660.
- Hamilton, James D**, “What is an oil shock?,” *Journal of econometrics*, 2003, *113* (2), 363–398.
- Ilut, Cosmin, Matthias Kehrig, and Martin Schneider**, “Slow to Hire, Quick to Fire: Employment Dynamics with Asymmetric Responses to News,” *Journal of Political Economy*, 2018, *126* (5), 2011–2071.
- Jordà, Òscar**, “Estimation and inference of impulse responses by local projections,” *American economic review*, 2005, pp. 161–182.
- Jr, John B Long and Charles I Plosser**, “Real business cycles,” *Journal of political Economy*, 1983, *91* (1), 39–69.
- Justiniano, Alejandro and Giorgio E Primiceri**, “The time-varying volatility of macroeconomic fluctuations,” *American Economic Review*, 2008, *98* (3), 604–41.
- Kilian, Lutz**, “The economic effects of energy price shocks,” *Journal of Economic Literature*, 2008, *46* (4), 871–909.
- Kocherlakota, Narayana**, “Creating business cycles through credit constraints,” *Federal Reserve Bank of Minneapolis Quarterly Review*, 2000, *24* (3), 2–10.

- McKay, Alisdair and Ricardo Reis**, “The brevity and violence of contractions and expansions,” *Journal of Monetary Economics*, 2008, 55 (4), 738–751.
- Morley, James and Jeremy Piger**, “The asymmetric business cycle,” *Review of Economics and Statistics*, 2012, 94 (1), 208–221.
- Rietz, Thomas A.**, “The Equity Risk Premium: A Solution,” *Journal of Monetary Economics*, 1988, 22(1), 117–131.
- Salgado, Sergio, Fatih Guvenen, and Nicholas Bloom**, “Skewed Business Cycles,” 2018. Working paper.
- Schorfheide, Frank, Dongho Song, and Amir Yaron**, “Identifying Long-run Risks: A Bayesian Mixed-Frequency Approach,” *Econometrica*, 2018, 86 (2), 617–654.
- Senga, Tatsuro**, “A New Look at Uncertainty Shocks: Imperfect Information and Misallocation,” 2016. Working paper.
- Sichel, Daniel E.**, “Business cycle asymmetry: a deeper look,” *Economic inquiry*, 1993, 31 (2), 224–236.
- Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron**, “Cyclical Dynamics in Idiosyncratic Labor Market Risk,” *Journal of Political Economy*, 2004, 112 (3), 695–717.
- Wachter, Jessica A.**, “Can time-varying risk of rare disasters explain aggregate stock market volatility?,” *Journal of Finance*, 2013, 68(3), 987–1035.

## A Solution of network model

The structure is

$$C = \left( n^{-1} \sum_i a_i x_{C,i}^\gamma \right)^{1/\gamma} \quad (35)$$

$$Y_i = z_i \left( n^{-1} \sum_j a_j x_{i,j}^\gamma \right)^{\alpha/\gamma} \quad (36)$$

$$Y_i = x_{C,i} + n^{-1} \sum_j x_{j,i} \quad (37)$$

The first equation is the production function for the consumption sector, the second for the intermediate sectors, and the third the resource constraint.

Now we guess that every sector uses the same mix of inputs, which means that  $x_{i,j} = \bar{x}_i x_{C,j}$  for some  $\bar{x}_i$  (this is simple to prove). The resource constraint and sector production functions become

$$Y_i = x_{C,i} \left( 1 + n^{-1} \sum_j \bar{x}_j \right) \quad (38)$$

$$Y_i = z_i \bar{x}_i^\alpha \left( n^{-1} \sum_j a_j x_{C,j}^\gamma \right)^{\alpha/\gamma} = z_i \bar{x}_i^\alpha C^\alpha \quad (39)$$

The aggregate optimization problem and FOC are

$$\max \left( n^{-1} \sum_i a_i x_{C,i}^\gamma \right)^{1/\gamma} - \sum_i x_{C,i} p_i \quad (40)$$

$$p_i = C^{1-\gamma} n^{-1} a_i x_{C,i}^{\gamma-1} \quad (41)$$

The sector optimization problem and FOC are

$$\max p_i z_i \bar{x}_i^\alpha \left( n^{-1} \sum_j a_j x_{C,j}^\gamma \right)^{\alpha/\gamma} - \bar{x}_i \sum_j p_j x_{C,j} \quad (42)$$

$$\sum_i p_j x_{C,j} = \alpha p_i z_i \bar{x}_i^{\alpha-1} C \quad (43)$$

## A.1 Proportionality results and solution

Equations (41) and (43), respectively, imply

$$x_{C,i} \propto (p_i/a_i)^{\frac{1}{\gamma-1}} \quad (44)$$

$$\bar{x}_i \propto (p_i z_i)^{\frac{1}{1-\alpha}} \quad (45)$$

where the factors of proportionality do not depend on any  $i$ -indexed variables. The resource constraint combined with the production function implies

$$x_{C,i} \propto z_i \bar{x}_i^\alpha \quad (46)$$

which, combined with the above, yields

$$a_i^{\frac{1-\alpha}{1-\alpha\gamma}} z_i^{\frac{\gamma-1}{1-\alpha\gamma}} \propto p_i \quad (47)$$

So there exist  $\bar{X}$  and  $\bar{X}_C$  such that

$$\bar{x}_i = \bar{X} a_i^{\frac{1}{1-\alpha\gamma}} z_i^{\gamma \frac{1}{1-\alpha\gamma}} \quad (48)$$

$$x_{C,i} = \bar{X}_C a_i^{\alpha \frac{1}{1-\alpha\gamma}} z_i^{\frac{1}{1-\alpha\gamma}} \quad (49)$$

Inserting those into the resource constraint, we have

$$\bar{X}_C a_i^{\alpha \frac{1}{1-\alpha\gamma}} z_i^{\frac{1}{1-\alpha\gamma}} \left( 1 + \bar{X} n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\gamma \frac{1}{1-\alpha\gamma}} \right) = z_i \bar{X}^\alpha a_i^{\frac{1-\alpha}{1-\alpha\gamma}} z_i^{\gamma \frac{1-\alpha}{1-\alpha\gamma}} \bar{X}_C^\alpha \left( n^{-1} \sum_j a_j a_j^{\alpha \frac{1}{1-\alpha\gamma}} z_j^{\frac{1-\alpha}{1-\alpha\gamma} \gamma} \right)^{\alpha/\gamma} \quad (50)$$

$$\bar{X}_C^{1-\alpha} + \bar{X}_C^{1-\alpha} \bar{X} \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\gamma \frac{1}{1-\alpha\gamma}} \right) = \bar{X}^\alpha \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\gamma \frac{1}{1-\alpha\gamma}} \right)^{\alpha/\gamma} \quad (51)$$

The FOC for  $\bar{x}_i$  becomes

$$\bar{X}_C \bar{X} = (n^{-1} \alpha)^{\frac{1}{1-\alpha}} \left( n^{-1} \sum_i a_i^{\frac{1}{1-\alpha\gamma}} z_i^{\gamma \frac{1}{1-\alpha\gamma}} \right)^{\frac{(\alpha-\gamma)/\gamma}{1-\alpha}} \quad (52)$$

Inserting that into (51) yields

$$\frac{(n^{-1} \alpha)}{(1 - n^{-1} \alpha)} \left( n^{-1} \sum_i a_i^{\frac{1}{1-\alpha\gamma}} z_i^{\gamma \frac{1}{1-\alpha\gamma}} \right)^{-1} = \bar{X} \quad (53)$$

which then implies

$$\bar{X}_C = (1 - n^{-1} \alpha) (n^{-1} \alpha)^{\frac{\alpha}{1-\alpha}} \left( n^{-1} \sum_i a_i^{\frac{1}{1-\alpha\gamma}} z_i^{\gamma \frac{1}{1-\alpha\gamma}} \right)^{\frac{\alpha}{1-\alpha} \frac{1-\gamma}{\gamma}} \quad (54)$$

Finally, then, aggregate output is

$$C = (1 - n^{-1} \alpha) (n^{-1} \alpha)^{\frac{\alpha}{1-\alpha}} \left( n^{-1} \sum_i a_i^{\frac{1}{1-\alpha\gamma}} z_i^{\gamma \frac{1}{1-\alpha\gamma}} \right)^{\frac{1-\alpha\gamma}{\gamma} \frac{1}{1-\alpha}} \quad (55)$$

## A.2 Results

Aggregate output is (up to a factor of proportionality)

$$C \propto \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\frac{\gamma}{1-\alpha\gamma}} \right)^{\frac{1-\alpha\gamma}{\gamma} \frac{1}{1-\alpha}} \quad (56)$$

Sector output is

$$Y_i \propto \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\frac{\gamma}{1-\alpha\gamma}} \right)^{\frac{1-\gamma}{1-\alpha} \frac{\alpha}{\gamma}} a_i^{\frac{\alpha}{1-\alpha\gamma}} z_i^{\frac{1}{1-\alpha\gamma}} \quad (57)$$

## B Proposition 2

$$Y_i = \frac{1-\gamma}{\gamma} \frac{\alpha}{1-\alpha} \log \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\frac{\gamma}{1-\alpha\gamma}} \right) + \frac{1}{1-\alpha\gamma} \varepsilon_i \quad (58)$$

$$= \frac{1-\gamma}{1-\alpha\gamma} \log C + \frac{1}{1-\alpha\gamma} \varepsilon_i \quad (59)$$

$$\text{where } \log C = \frac{1-\alpha\gamma}{\gamma} \frac{\alpha}{1-\alpha} \log \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} z_j^{\frac{\gamma}{1-\alpha\gamma}} \right) \quad (60)$$

This section considers a specification in which the  $\varepsilon$  follow AR(1) processes, but the logic applies in much more general settings. Formally, for all  $i$ ,  $\varepsilon_{i,t} = \phi \varepsilon_{i,t-1} + \mu_{i,t}$ .

We want to know the conditional covariance,

$$\begin{aligned} \text{cov}_t(\log Y_{i,t+1}, \log Y_{j,t+1}) &= \text{cov}_t \left( \frac{1-\gamma}{1-\alpha\gamma} \log C_{t+1} + \frac{1}{1-\alpha\gamma} \varepsilon_{i,t+1}, \frac{1-\gamma}{1-\alpha\gamma} \log C_{t+1} + \frac{1}{1-\alpha\gamma} \varepsilon_{j,t+1} \right) \\ &= \left( \frac{1-\gamma}{1-\alpha\gamma} \right)^2 \text{var}_t(\log C_{t+1}) + \frac{1-\gamma}{(1-\alpha\gamma)^2} \text{cov}_t(\log C_{t+1}, \varepsilon_{i,t+1}) \end{aligned} \quad (61)$$

$$+ \frac{1-\gamma}{(1-\alpha\gamma)^2} \text{cov}_t(\log C_{t+1}, \varepsilon_{j,t+1}) \quad (62)$$



We then consider the sum across all other sectors,

$$\begin{aligned}
\sum_{j \neq i} \text{cov}_t(\log Y_{i,t+1}, \log Y_{j,t+1}) &= (n-1) \left( \frac{1-\gamma}{1-\alpha\gamma} \right)^2 \text{var}_t(\log C_{t+1}) + \frac{1-\gamma}{(1-\alpha\gamma)^2} (n-1) \text{cov}_t(\log C_{t+1}, \varepsilon_{i,t+1}) \\
&\quad + \frac{1-\gamma}{(1-\alpha\gamma)^2} \text{cov}_t \left( \log C_{t+1}, \sum_{j \neq i} \varepsilon_{j,t+1} \right) \\
&= (n-1) \left( \frac{1-\gamma}{1-\alpha\gamma} \right)^2 \text{var}_t(\log C_{t+1}) + \frac{1-\gamma}{(1-\alpha\gamma)^2} (n-2) \text{cov}_t(\log C_{t+1}, \varepsilon_{i,t+1}) \\
&\quad + \frac{1-\gamma}{(1-\alpha\gamma)^2} \text{cov}_t \left( \log C_{t+1}, \sum_j \varepsilon_{j,t+1} \right)
\end{aligned} \tag{68}$$

Since the empirical regressions include time fixed effects, all that matters is how this sum of covariances differs across  $i$ , so we can drop the first and third terms, since they are identical for all  $i$ . We are then left with

$$\sum_{j \neq i} \text{cov}_t(\log Y_{i,t+1}, \log Y_{j,t+1}) = \frac{1-\gamma}{(1-\alpha\gamma)^2} (n-2) \text{cov}_t(\log C_{t+1}, \varepsilon_{i,t+1}) + \text{terms independent of } i \tag{68}$$

Now consider the derivative with respect to  $\varepsilon_{i,t}$ ,

$$\frac{d}{d\varepsilon_{i,t}} \sum_{j \neq i} \text{cov}_t(\log Y_{i,t+1}, \log Y_{j,t+1}) = \frac{1-\gamma}{(1-\alpha\gamma)^2} (n-2) \frac{d}{d\varepsilon_{i,t}} \text{cov}_t(\log C_{t+1}, \varepsilon_{i,t+1}) \tag{69}$$

$$= \frac{1-\gamma}{(1-\alpha\gamma)^2} (n-2) \text{cov}_t \left( \frac{d}{d\varepsilon_{i,t}} \log C_{t+1}, \varepsilon_{i,t+1} \right) \tag{70}$$

We have

$$\log C_{t+1} = \frac{1-\alpha\gamma}{\gamma} \frac{\alpha}{1-\alpha} \log \left( n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} \exp \left( \frac{\gamma}{1-\alpha\gamma} (\phi\varepsilon_{j,t} + \mu_{j,t+1}) \right) \right) \tag{71}$$

$$\frac{d}{d\varepsilon_{i,t}} \log C_{t+1} = \phi \frac{\alpha}{1-\alpha} \frac{a_j^{\frac{1}{1-\alpha\gamma}} \exp \left( \frac{\gamma}{1-\alpha\gamma} (\phi\varepsilon_{j,t} + \mu_{j,t+1}) \right)}{n^{-1} \sum_j a_j^{\frac{1}{1-\alpha\gamma}} \exp \left( \frac{\gamma}{1-\alpha\gamma} (\phi\varepsilon_{j,t} + \mu_{j,t+1}) \right)} \tag{72}$$

It is then straightforward to show that the derivative of  $\frac{d}{d\varepsilon_{i,t}} \log C_{t+1}$  with respect to  $\mu_{i,t+1}$  has the same sign as  $\gamma$  globally. So when  $\gamma < 0$ , the covariance is negative. Furthermore,  $\frac{1-\gamma}{(1-\alpha\gamma)^2} > 0$  for  $\gamma < 1$ , so the sign of the sum of covariances is the sign of  $\gamma$ .

## C Results in levels and growth rates

We analyze in this section a continuous limit of an AR(1) process. We show that changes in a concave function of a set of those processes,  $df_t \equiv f(\dots, \varepsilon_{i,t}, \dots) - f(\dots, \varepsilon_{i,t-dt}, \dots)$ , are skewed left when the innovations to the underlying  $\varepsilon$  have fat tails, but not when they are Gaussian (purely diffusive).

We consider an underlying process, which can be thought of as the productivity process analyzed in the paper, that follows an Ornstein–Uhlenbeck process augmented with compound Poisson jumps. Specifically, consider an  $\varepsilon_{i,t}$  that follows

$$d\varepsilon_{i,t} = -\phi\varepsilon_{i,t} + \sigma dW_t + k_t dN_t \quad (73)$$

where  $W_t$  is a standard Wiener process,  $N_t$  a Poisson counting process with intensity  $\lambda$ , and  $k_t$  a random variable with a symmetrical distribution.

The solution of the model is such that aggregate output is a function of the sector productivities with the characteristics that  $f_i > 0$  and  $f_{ii} < 0 \forall i$  and  $f_{ij} > 0 \forall i \neq j$ . The question here is under what circumstances  $df_t$  is skewed left. That is, when do our results on skewness in levels also apply to first differences?

Now first suppose there are no jumps. Then we can write, somewhat informally,

$$df_t \equiv f(\dots, \varepsilon_{i,t}, \dots) - f(\dots, \varepsilon_{i,t-dt}, \dots) \quad (74)$$

$$\varepsilon_{i,t} = (1 - \phi dt) \varepsilon_{i,t-dt} + \sigma dt^{1/2} \mu_{i,t} \quad (75)$$

where  $\mu_{i,t}$  is a standard Normal random variable.

$$df_t = f(\dots, (1 - \phi dt) \varepsilon_{i,t-dt} + \sigma dt^{1/2} \mu_{i,t}, \dots) - f(x_{t-dt}) \quad (76)$$

$$= \sum_i f_{i,t-dt} \sigma dt^{1/2} \mu_{i,t} + o(\sigma dt^{1/2} \varepsilon_t) \quad (77)$$

where  $f_{i,t} = \frac{df_t}{d\varepsilon_{i,t}}$ . In the limit as  $dt \rightarrow 0$ , we then have

$$E[(f(x_t) - f(x_{t-dt}))^3] = 0 \quad (78)$$

so that the skewness of the changes in  $f(x)$  is zero. This follows simply from the smoothness, or local linearity, of  $f$ .

Now suppose there are jumps. We have

$$df_t = f(\dots, (1 - \phi dt) \varepsilon_{i,t-dt} + \sigma dt^{1/2} \mu_{i,t} + k_{i,t} dN_{i,t}, \dots) - f(x_{t-dt}) \quad (79)$$

$$= \sum_i f_{i,t-dt} (\sigma dt^{1/2} \mu_{i,t} + k_{i,t} dN_{i,t}) \quad (80)$$

$$+ \frac{1}{2} \sum_i \sum_j f_{i,j,t-dt} (\sigma dt^{1/2} \mu_{i,t} + k_{i,t} dN_{i,t}) (\sigma dt^{1/2} \mu_{j,t} + k_{j,t} dN_{j,t}) + o(k_t^2) \quad (81)$$

$$\sum_i f_{i,t-dt} (\sigma dt^{1/2} \mu_{i,t} + k_{i,t} dN_{i,t}) + \frac{1}{2} f_{ii} (\sigma^2 dt \mu_{i,t}^2 + k_{i,t}^2 dN_{i,t}^2) \quad (82)$$

The third moment of  $df_t$  is the expectation of its cube. That involves taking all third-order combinations of the various terms, such that the expectations are of order  $dt$ . It is straightforward to show that all terms involving interactions either between  $dN$  or  $\mu$  terms,  $\mu^j$  for  $j > 2$ , or between different  $i$  indexes, are of smaller order than  $dt$ .

We then have

$$E[df_t^2] = \sum_i f_i^2 (\sigma^2 dt + \text{var}(k) \lambda dt) + \frac{1}{4} f_{ii}^2 \kappa_4 \lambda dt \quad (83)$$

$$E[df_t^3] = \sum_i \left( \frac{3}{2} f_i^2 f_{ii} \kappa_4 + \frac{1}{8} f_{ii}^3 \kappa_6 \right) \lambda dt \quad (84)$$

where  $\kappa_j = E[k_t^j]$ .

The skewness is then

$$\frac{E[df_t^3]}{E[df_t^2]^{3/2}} = \frac{\sum_i \left( \frac{3}{2} f_i^2 f_{ii} \kappa_4 + \frac{1}{8} f_{ii}^3 \kappa_6 \right) \lambda dt}{\left( \sum_i f_i^2 (\sigma^2 dt + \text{var}(k) \lambda dt) + \frac{1}{4} f_{ii}^2 \kappa_4 \lambda dt \right)^{3/2}} < 0 \quad (85)$$

where the inequality follows from the concavity of  $f$ .

We thus have the claimed result that skewness in output growth is zero when the innovations are purely Gaussian but negative when they have fat tails due to jumps.

## D Results on model of concave responses

Suppose sector and aggregate output are

$$Y_{i,t} = f(\varepsilon_t + \mu_{i,t}) \quad (86)$$

$$Y_t = \int_i Y_{i,t} \quad (87)$$

We can approximate these, assuming symmetric fundamental shocks, as

$$Y_{i,t} \approx f(0) + f'(0) (\varepsilon_t + \mu_{i,t}) + \frac{1}{2} f''(0) (\varepsilon_t + \mu_{i,t})^2 \quad (88)$$

$$\begin{aligned} Y_t &\approx f(0) + f'(0) \varepsilon_t + \frac{1}{2} f''(0) \varepsilon_t^2 + f'(0) \int_i \mu_{i,t} + \frac{1}{2} f''(0) 2\varepsilon_t \int_i \mu_{i,t} + \frac{1}{2} f''(0) \int_i \mu_{i,t}^2 \\ &= f(0) + f'(0) \varepsilon_t + \frac{1}{2} f''(0) (\varepsilon_t^2 + \sigma_\mu^2) \end{aligned} \quad (89)$$

### D.0.1 Lemma for skewness

First, a lemma. We are going to examine random variables of the form

$$x = \frac{a\sigma^2}{2} \left(\frac{\varepsilon}{\sigma}\right)^2 + b\sigma \frac{\varepsilon}{\sigma} + c \quad (91)$$

This has the form of a non-central  $\chi^2$ . Specifically,

$$x = \frac{a}{2} \sigma^2 \left(\frac{\varepsilon}{\sigma_\varepsilon} + \frac{b}{a\sigma}\right)^2 - \frac{1}{2} \frac{b^2}{a} + c \quad (92)$$

$$\sim -\frac{1}{2} \frac{b^2}{a} + c + \frac{a}{2} \sigma^2 \chi^2(1, \lambda) \quad (93)$$

where  $\chi^2(1, \lambda)$  is a non-central  $\chi^2$  with one degree of freedom and noncentrality parameter  $\lambda$ , where in this case  $\lambda = \left(\frac{b}{a\sigma}\right)^2$ . The skewness of  $x$  is then

$$skew(x) = sign\left(\frac{a}{2}\sigma^2\right) 2^{3/2} \frac{k + 3\lambda^2}{(k + 2\lambda^2)^{3/2}} \quad (94)$$

$$\frac{dskew(x)}{d\lambda} = -sign\left(\frac{a}{2}\sigma^2\right) \frac{3\lambda}{(1 + 2\lambda)^{5/2}} \quad (95)$$

Finally,

$$\frac{dskew(x)}{d\sigma} = \frac{dskew(x)}{d\lambda} \frac{d\lambda}{d\sigma} \quad (96)$$

$$= -2 \frac{dskew(x)}{d\lambda} \left(\frac{b}{a}\right)^2 \sigma^{-3} \quad (97)$$

which implies

$$\text{sign} \left( \frac{d\text{skew}(x)}{d\sigma} \right) = -\text{sign} \left( \frac{d\text{skew}(x)}{d\lambda} \right) \quad (98)$$

$$\text{sign} \left( \frac{d\text{skew}(x)}{d\sigma} \right) = \text{sign}(a) \quad (99)$$

## D.1 Aggregate and sector skewness

Both aggregate and sector output take the form

$$x = \frac{a\sigma^2}{2} \left( \frac{\varepsilon}{\sigma} \right)^2 + b\sigma_\varepsilon \frac{\varepsilon}{\sigma} \quad (100)$$

from above. Specifically, for aggregate output

$$Y_t = \frac{1}{2}\sigma_\varepsilon^2 f''(0) \frac{\varepsilon_t^2}{\sigma_\varepsilon^2} + f'(0) \sigma_\varepsilon \frac{\varepsilon_t}{\sigma_\varepsilon} + f(0) + \frac{1}{2}f''(0) \sigma_\mu^2 \quad (101)$$

$$a = f''(0), b = f'(0), \text{ and } \sigma^2 = \sigma_\varepsilon^2 \quad (102)$$

and for sector output

$$Y_{i,t} = \frac{1}{2}f''(0) (\sigma_\varepsilon^2 + \sigma_\mu^2) \frac{(\varepsilon_t + \mu_{i,t})^2}{(\sigma_\varepsilon^2 + \sigma_\mu^2)} + f'(0) \sqrt{\sigma_\varepsilon^2 + \sigma_\mu^2} \frac{(\varepsilon_t + \mu_{i,t})}{\sqrt{\sigma_\varepsilon^2 + \sigma_\mu^2}} + f(0) \quad (103)$$

$$a = f''(0), b = f'(0), \text{ and } \sigma^2 = \sigma_\varepsilon^2 + \sigma_\mu^2 \quad (104)$$

so they have the same form except for differences in the variance –  $\sigma_\varepsilon^2$  versus  $\sigma_\varepsilon^2 + \sigma_\mu^2$ . We can therefore apply the result from above, Since the  $\sigma^2$  for sector output is greater than the  $\sigma^2$  for aggregate output, and since  $\text{sign}(a) < 0$ ,  $\text{skew}(Y_{i,t}) < \text{skew}(Y_t)$ . That is, sector skewness is more negative than aggregate skewness. Intuitively, the curvature in  $f$  has greater relevance when the shocks are larger – for very small shocks,  $f$  is effectively linear and induces no asymmetry in the distribution. So in general this model predicts aggregate skewness is smaller than sector skewness.

**Table 1. Means and medians of skewness of sector-level growth rates**

## Panel A: Industrial production

Aggregation level	<i>Monthly growth rates</i>		<i>Quarterly growth rates</i>		<i>Levels</i>	<i># of sectors</i>
	Raw	Residuals	Raw	Residuals		
Aggregate	-1.183 [.135]	N/A	-1.868 [.129]	N/A	-1.078 [.229]	1
2-digit	-0.961 [.142]	-0.224 [.059]	-1.007 [.103]	0.131 [.048]	-0.412 [.147]	3
3-digit	-0.545 [.057]	-0.207 [.033]	-0.792 [.07]	-0.054 [.036]	-0.98 [.095]	43
4-digit	-0.454 [.057]	-0.24 [.033]	-0.589 [.07]	-0.111 [.036]	-0.835 [.076]	81
5-digit	-0.409 [.037]	-0.21 [.024]	-0.515 [.044]	-0.125 [.017]	-0.877 [.06]	62

## Panel B: Employment

Aggregation level	<i>Monthly growth rates</i>		<i>Quarterly growth rates</i>		<i>Levels</i>	<i># of sectors</i>
	Raw	Residuals	Raw	Residuals		
Aggregate	-1.488 [.311]	N/A	-1.894 [.339]	N/A	-0.788 [.172]	1
1-digit	-0.905 [.192]	-0.133 [.032]	-1.179 [.224]	0.118 [.046]	-0.751 [.1]	7
2-digit	-0.64 [.149]	-0.119 [.054]	-0.987 [.16]	-0.138 [.057]	-0.534 [.091]	18
3-digit	-0.559 [.097]	-0.143 [.033]	-0.771 [.125]	-0.106 [.025]	-0.367 [.07]	78
4-digit	-0.362 [.097]	-0.085 [.033]	-0.579 [.125]	-0.081 [.025]	-0.444 [.07]	237
5-digit	-0.291 [.054]	-0.093 [.017]	-0.467 [.081]	-0.083 [.011]	-0.394 [.06]	344

Notes: Average time series skewness across sectors at various levels of aggregation. Jackknife standard errors (with a block length of 50 months) are reported in brackets. The “raw” columns are for the raw data, while the columns labeled “residuals” report skewness for residuals from regressions of sector on aggregate growth rates. In levels, we report skewness for IP and employment minus an exponentially weighted moving average with a decay of 5 percent per month.

**Table 2. Correlation of cross-sectional variance with the business cycle**

	IP growth	IP growth residuals	Employment growth	Employment growth residuals
NBER rec. ind.	0.69 *** [0.19]	0.66 *** [0.18]	0.218 ** [0.091]	0.510 ** [0.24]
Agg. empl. growth	-0.24 ** [0.10]	-0.21 ** [0.09]	-0.311 *** [0.083]	-0.250 *** [0.065]
# of obs.	566	566	351	351

Notes: Employment growth and cross-sectional variance are standardized to have unit over time. Each cell is the coefficient from a univariate regression. Standard errors, reported in brackets, are calculated by Newey–West with 12 monthly lags. The columns labeled residuals use the cross-sectional variance of residuals from regressions of sector growth rates on aggregate growth. \* indicates significance at the 10-percent level, \*\* 5 percent, and \*\*\* 1 percent.

**Table 3. Model simulation results***Skewness*

	Model	Data	Data std. err.
Aggregate	-1.72	-1.18	[0.135]
Sector	-0.94	-0.45	[0.057]
Residual	-0.17	-0.24	[0.033]

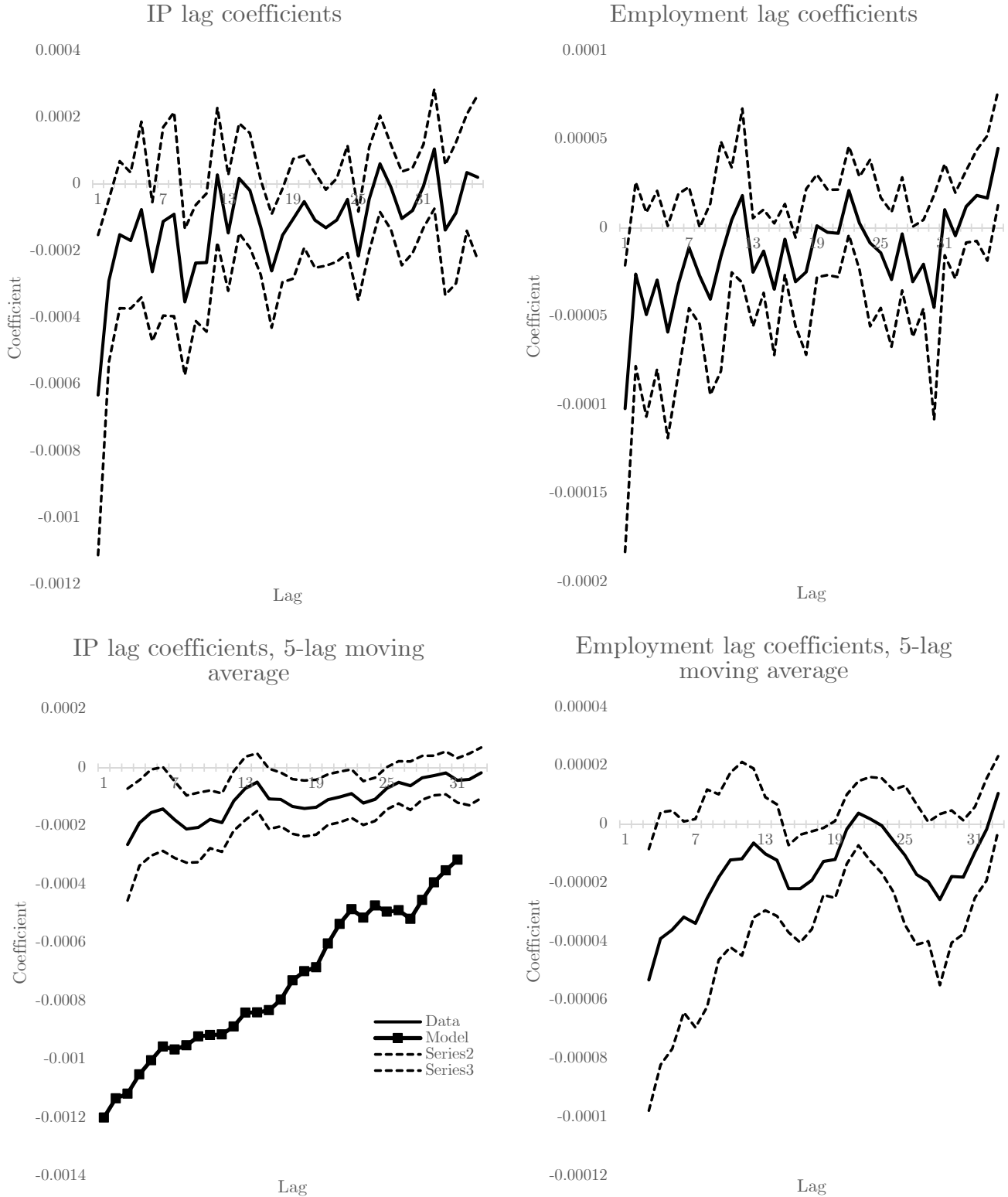
*Cyclicalilty of cross-sectional moments*

	Growth rates:			Residuals:		
	Model	Data	Data std. err.	Model	Data	Data std. err.
Variance:						
Rec. ind.	0.61	0.32	[0.09]	0.40	0.31	[0.09]
$\Delta$ IP	-0.24	-0.35	[0.12]	-0.17	-0.33	[0.11]

Notes: The left-hand column reports moments from a simulation of 10,000 periods from the numerical solution to the model. The middle column reports corresponding empirical estimates and the right-hand column has standard errors. The sector-level estimate in the top-section is for IP for 4-digit industries. The cyclicalilty results are different from table 2 in two respects: they report pairwise correlations instead of regression coefficients, and they use IP as the measure of aggregate activity instead of employment.



Figure 1. Response of sector covariances to sector shocks



Notes: the top two panels report coefficients from regressions of the sum of sector I's covariances with all other sectors on lagged innovations to activity in sector  $i$ . The regressions include time and sector fixed effects and standard errors are clustered by date. Dotted lines represent 90-percent confidence intervals. The bottom panels report moving averages of the coefficients.