Testing for multiplicity of equilibria in a low interest rate environment

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Motivation

- In structural models with occasionally binding constraints
    - an equilibrium may not exist (incoherency); or
    - there may be multiple equilibria (incompleteness)
- Literature on the Zero Lower Bound (ZLB) often focuses on two equilibria: targeted inflation and liquidity trap, but there can be many more
- Coherency and Completeness (CC) requires restrictions on the structural parameters
- Incompleteness requires restrictions on the support of the shocks for coherency
- No theoretical results on CC for DSGE models
- We attempt to fill that gap

Preview of results

- (It appears that) DSGE models with the ZLB are *generically* incomplete and they require restrictions on the support of the shocks to avoid incoherency
- We provide a solution method that is exact when the model is linear (except for the ZLB) and the distribution of the shocks is discrete with bounded support
- But it is computationally infeasible when the number of states is large
- So far have been unable to generalize it to continuous distributions
- In the special case of a SVAR, we can fully characterize incompleteness and develop likelihood tests for it

Outline

1 A simple example

2 Discrete shocks

3 Continuous shocks

4 References

A simple two-equation example

- From Aruoba et al. (2018) – henceforth ACS
- Original ACS example consists of a consumption Euler equation

$$1 = E_t \left(M_{t+1} \frac{R_t}{\pi_{t+1}} \right) \tag{1}$$

and a (contemporaneous) Taylor rule

$$R_t = \max \left\{ 1, r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi} \right\}. \tag{2}$$

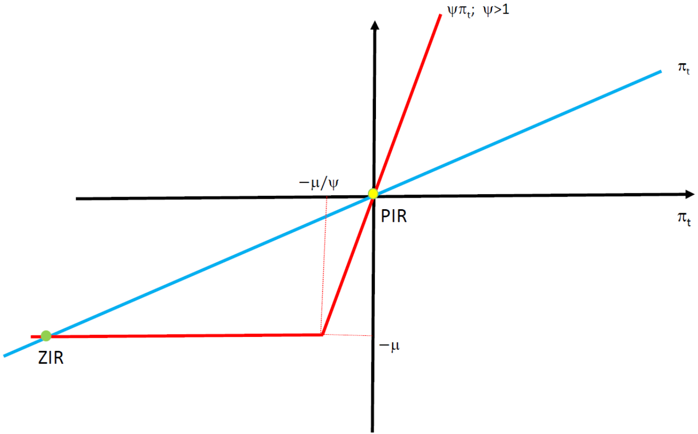
Steady states

- When $\psi > 1$ (active Taylor rule) the model

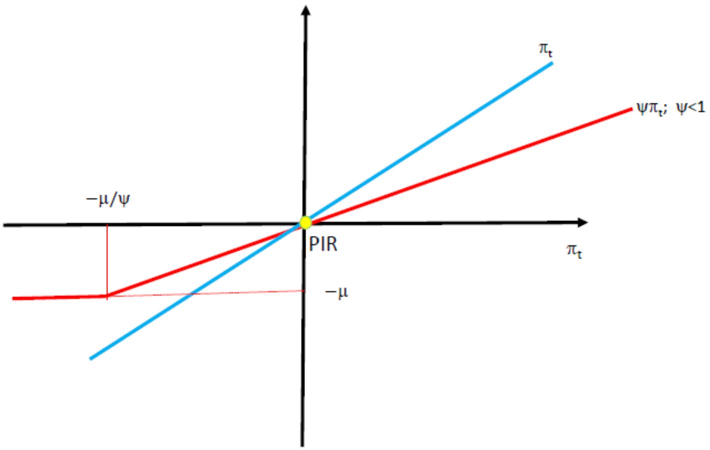
$$1 = E_t \left(M_{t+1} \frac{R_t}{\pi_{t+1}} \right)$$
$$R_t = \max \left\{ 1, r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi} \right\}$$

  has two steady states (Benhabib et al. (2001)):
  - Targeted-inflation steady state: $\pi = \pi_*$, $R = r\pi_*$
  - Deflation steady state: $\pi = 1/r$, $R = 1$
  - (steady state of stoch disc factor M_t is $1/r$)
- If $\psi < 1$, there is only the targeted-inflation steady state

Graphical illustration: two steady states



Graphical illustration: unique steady state



Steady states and (dynamic) equilibria

- Multiplicity of steady states often guides the characterization of equilibria
- E.g., ACS looked for two equilibria when $\psi > 1$
- In fact, there are typically many more than two equilibria
- And even when $\psi < 1$ there are multiple dynamic equilibria even though steady state is unique
- So, thinking about multiplicity of equilibria via multiplicity of steady states can be misleading

Characterizing multiplicity

- We want to find the conditions for a unique equilibrium (CC)
- They depend both on ψ and on the distribution of the shock
- If the CC condition is violated, we want to find the set of multiple equilibria
- We start with discrete distribution of shock
- Then move to continuous distribution which is *a lot* harder!

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2. Discrete shocks
3. Continuous shocks
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Solving log-linearized model

- Log-linearizing around target inflation steady state π_* the model becomes

$$\hat{\pi}_{t+1|t} = \hat{R}_t + \hat{M}_{t+1|t} \tag{3}$$

$$\hat{R}_t = \max\{-\mu, \psi\hat{\pi}_t\}, \quad \mu := \log(r\pi_*). \tag{4}$$

- Combine (3)-(4) to get:

$$\hat{\pi}_{t+1|t} = \max\{-\mu, \psi\hat{\pi}_t\} + \hat{M}_{t+1|t}$$

- Assume $\hat{M}_t$ is first-order Markovian as in ACS
- We now derive the CC condition and find multiple equilibria when the model is incomplete
- We first analyze a simple special case graphically

2-state Markov chain with absorbing state

- Assume $\hat{M}_t = -r^L > 0$ (transitory) and $\hat{M}_t = 0$ (absorbing) with transition Kernel

$$K = \begin{pmatrix} p & 1-p \\ 0 & 1 \end{pmatrix}$$

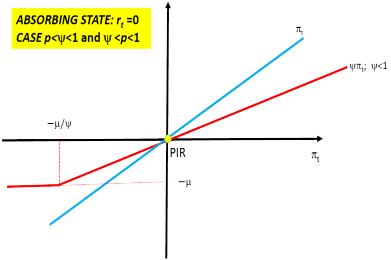
($r^L < 0$ can be interpreted as negative real interest rate shock)

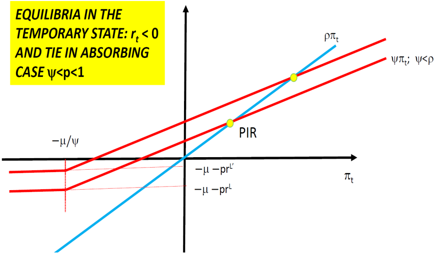
- We illustrate graphically that the CC condition holds iff

$$\psi < p$$

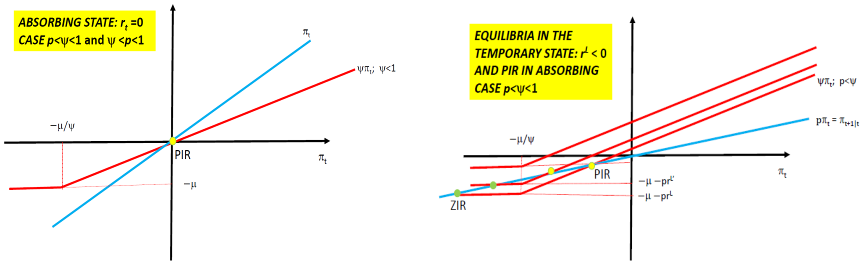
- CC does not depend on the support of $\hat{M}_t$, only on transition matrix K

Coherent and complete case: unique solution



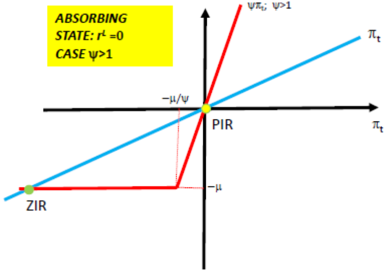


Incomplete case p $<$ psi $<$ 1: two equilibria

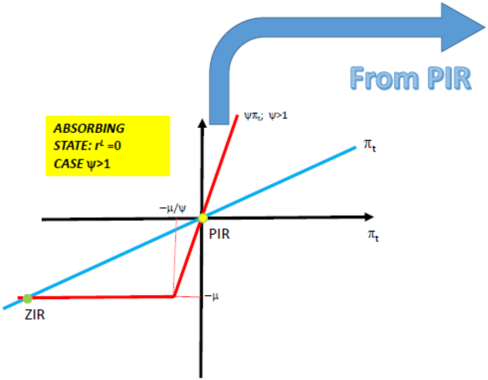


Equilibrium	Absorbing state	Transitory state
1	PIR (on target)	PIR (below target)
2	PIR (on target)	ZIR

Incomplete case psi > 1: up to four equilibria



Incomplete case psi > 1: up to four equilibria



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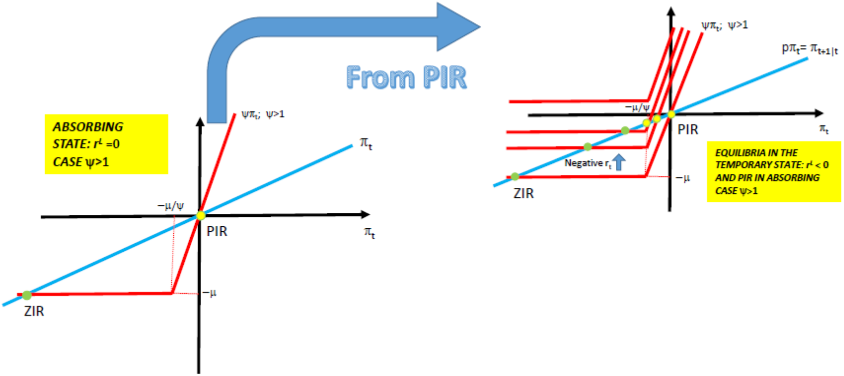

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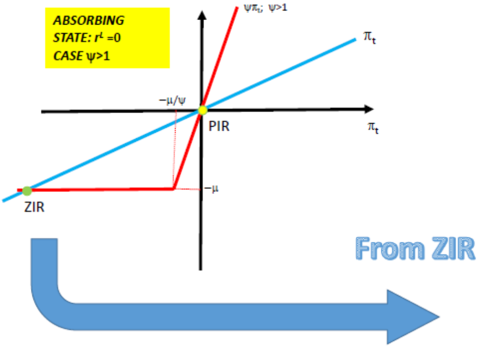

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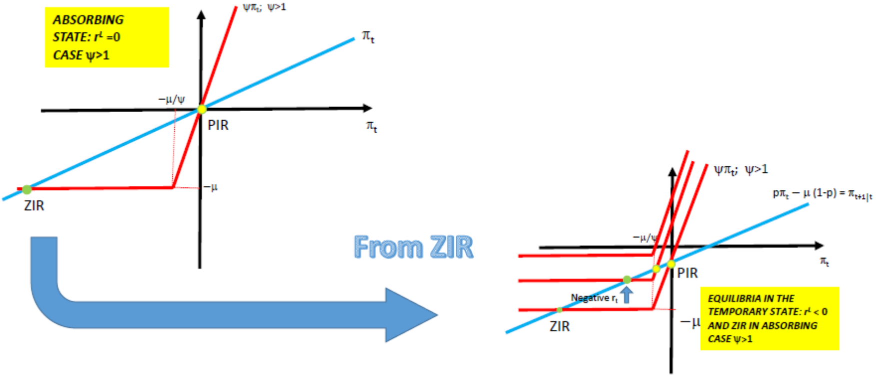

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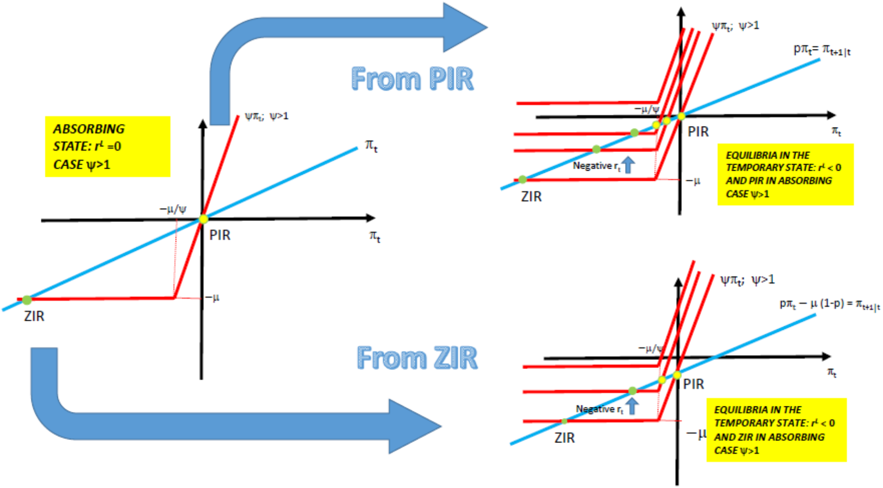


Analytical description of multiple equilibria

If $\psi > p$, we have

$$\hat{\pi}_t = \begin{cases} r^L \frac{p}{\psi - p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi - p}{\psi p}\right) \quad \text{(PIR)} \\ 0, & \text{if } \hat{M}_t = 0, \quad \text{(PIR)} \end{cases} \tag{1}$$

$$\hat{\pi}_t = \begin{cases} -r^L - \frac{\mu}{p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi - p}{\psi p}\right) \quad \text{(ZIR)} \\ 0, & \text{if } \hat{M}_t = 0. \quad \text{(PIR)} \end{cases} \tag{2}$$

If $\psi > 1$, we also get

$$\hat{\pi}_t = \begin{cases} \frac{pr^L - (1-p)\mu}{\psi - p}, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi - 1}{\psi}\right) \quad \text{(PIR)} \\ -\mu, & \text{if } \hat{M}_t = 0, \quad \text{(ZIR)} \end{cases} \tag{3}$$

$$\hat{\pi}_t = \begin{cases} -r^L - \mu, & \text{if } \hat{M}_t = -r^L \in \left(0, \mu \frac{\psi - 1}{\psi}\right) \quad \text{(ZIR)} \\ -\mu, & \text{if } \hat{M}_t = 0. \quad \text{(ZIR)} \end{cases} \tag{4}$$

Sunspot equilibria

- The previous equilibria were solutions of the form $\hat{\pi}_t = g\left(\hat{M}_t\right)$
- There are possibly other equilibria
- E.g., ACS propose $\hat{\pi}_t = g\left(\hat{M}_t, s_t\right)$, where $s_t \in \{0, 1\}$ is sunspot
- These are *in addition* to the four equilibria above

Distinguishing between complete and incomplete models

- In the present example, the CC condition is $\psi < p$
- Suppose we observed data on π_t and R_t
- The transition probability p is identified
- But ψ is not identified
- So how can we tell whether the CC condition is satisfied in the data?
- In the specific example above, this is trivial because the distribution of the data (the support points) differs for each equilibrium
- But this is not the case in general

Observationally equivalent models

Complete model

- $\hat{M}_t \in \{m_1, m_2\}$, with transition kernel $K = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$ and $\psi < p + q - 1$ (CC condition)
- If you choose $m_1 < m_2$ appropriately, the unique equilibrium oscillates between $\hat{R}_t > 0$ if $\hat{M}_t = m_1$ ('good' state) and $\hat{R}_t = 0$ if $\hat{M}_t = m_2$ ('bad' state)

Incomplete model with sunspot

- $\hat{M}_t = m$ always, $\psi > 1$ and sunspot process $s_t \in \{0, 1\}$, with transition kernel K.
- There exists sunspot equilibrium such $\hat{R}_t > 0$ if $s_t = 0$ and $\hat{R}_t = 0$ if $s_t = 1$ (ZLB state)
- We can find m_1, m_2 and m such that the distribution of the data is identical for both models

Moving beyond 2 states

- Our goal is to characterize CC and incompleteness in a model with continuous support
- We don't know how to do that
- So we first look at k states...

Three states

- With two states the CC condition was found to be $\psi < p$
- Suppose there are two transitory and one absorbing state, with

$$K = \begin{pmatrix} p_{11} & p_{12} & 1 - p_{11} - p_{12} \\ p_{21} & p_{22} & 1 - p_{21} - p_{22} \\ 0 & 0 & 1 \end{pmatrix}$$

- Now CC condition becomes more stringent: if $p_{11}p_{22} > p_{12}p_{21}$,

$$\psi < \min \begin{pmatrix} p_{11} - \frac{p_{12}p_{21}}{p_{22}}, \\ p_{22} - \frac{p_{12}p_{21}}{p_{11}}, \\ \frac{p_{11}+p_{22}-\sqrt{(p_{11}-p_{22})^2+4p_{12}p_{21}}}{2} \end{pmatrix}$$

Intuition

- CC requires RHS (red line) to be flatter than LHS (blue line)
- When we go from 2 to 3 states, the LHS of

$$\hat{\pi}_{t+1|t} = \max\left\{-\mu, \psi\hat{\pi}_t\right\} + \hat{M}_{t+1|t}$$

  becomes flatter, hence CC condition tightens
- This suggests CC condition tightens as # of states increases
- We explore that numerically using a very useful result from Gourieroux et al. (1980) – henceforth GLM

A general solution method with k states

- The model to solve is

$$\hat{\pi}_{t+1|t} = \hat{M}_{t+1|t} + \max\left(-\mu, \psi\hat{\pi}_t\right) \quad (1)$$

- If $\hat{M}_t$ is MC with support $m \in \Re^k$ and transition kernel $K \in \Re^{k\times k}$ and if solution is of the form $\hat{\pi}_t = f\left(\hat{M}_t\right)$ (msv)
- Then $\hat{\pi}_t$ is MC with support π and kernel K and (1) becomes

$$Ky = b + \max\left(0, \psi y\right) \quad (2)$$

with $y := \pi + \frac{\mu}{\psi}\iota$, $b := Km + \frac{(1-\psi)\mu}{\psi}\iota$, ι is k-vector of ones

- (2) is a piecewise linear function in each of the orthants of $\Re^k$

CC condition using GLM

- Let C_i denote the ith orthant of $\Re^k$, $i = 1, ..., 2^k$
- $Ky = b + \max(0, \psi y)$ can be written as

$$f(y) = b, \quad f := \sum_{i=1}^{2^k} \left(A_i 1_{\{y \in C_i\}}\right) y$$

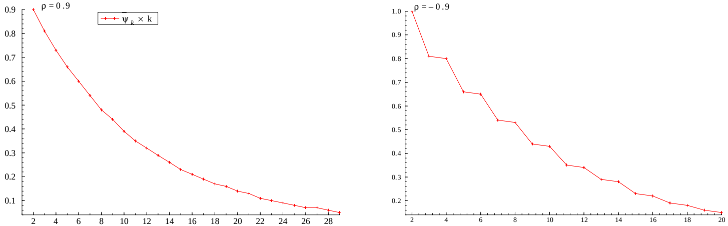
  - E.g., if $C_1 = \{y \in \Re^k : y_j \geq 0 \text{ for all } j\}$, then $A_1 = K - \psi I_k$
  - if $C_2 = \{y \in \Re^k : y_j < 0 \text{ for all } j\}$, then $A_2 = K$, etc.
- Using GLM Theorem 1, we find that CC holds (f is invertible) iff

$$\det A_i \text{ has the same sign } \forall\ i = 1, ..., 2^k$$

- Straightforward to program, but quickly becomes infeasible
  - finding CC with $k = 29$ took 9 hours, $k = 50$ would take 2 millennia

Specific example: M follows AR(1)

- Use Rouwenhorst method to obtain k-state MC with $\hat{M}_{t+1|t} = \rho \hat{M}_t$
- Let $\overline{\psi}_k$ be cutoff such that CC satisfied for $\psi < \overline{\psi}_k$
    - e.g., $\overline{\psi}_2 = \rho$ (analytically)
- We establish numerically that $\overline{\psi}_k \searrow$ as $k \nearrow$

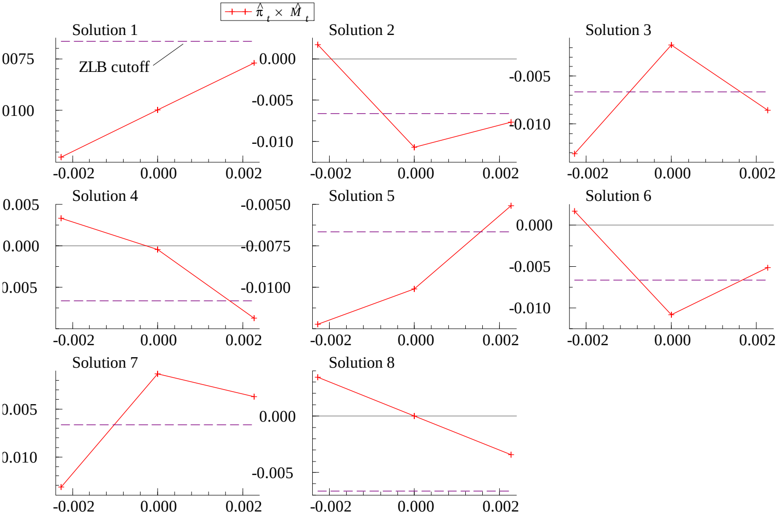


- So we *conjecture* that $\overline{\psi}_k \to 0$ as $k \to \infty$

Incompleteness: many many equilibria

- Rouwenhorst AR(1): $\hat{M}_t \in \{-\overline{m}, ..., \overline{m}\}$, K such that $Km = \rho m$
- If $\psi > \overline{\psi}_k$ (violation of CC) and $\overline{m}$ sufficiently small (support restriction), we have incompleteness
- There are $n \leq 2^k$ equilibria of the form $\hat{\pi}_t = g_i\left(\hat{M}_t\right)$, $i = 1, ..., n$.
- Next slide plots equilibria in specific example $k = 3$, $\overline{m} = \sigma\sqrt{\frac{k-1}{1-\rho^2}}$, matching ACS's calibration ($\sigma = 0.0007$, $\rho = 0.9$, $\psi = 1.5$)
- With these values, there are exactly 8 equilibria without sunspots

Various equilibria



Continuous support

- We have not been able to find a general solution of this model when the distribution of $\hat{M}_t$ is continuous
- If we treat the k-state MC as an approximation to a continuous distribution, then we *conjecture* the CC condition is $\psi \leq 0$
  - (model is trivially complete at $\psi = 0$)
- In the case where $\hat{M}_{t+1} = \rho \hat{M}_t + \varepsilon_{t+1}$ and ε_{t+1} is continuously distributed, we can show that the model is incomplete when $\psi > \rho$
  - We can find two specific equilibria, one always at ZLB and one always above ZLB, provided $\varepsilon_{t+1} < \frac{\psi - 1}{\psi}\mu - \rho \hat{M}_t$
  - So $\psi < \rho$ is necessary but not sufficient for CC
- Note:
  - Support of $\hat{M}_t$ depends on past shocks, so not Markovian
  - Would depend on current value of other shocks if there were any, so shocks cannot be orthogonal (more on this later)

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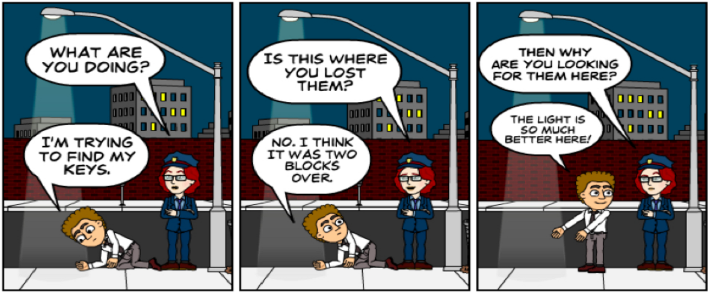
Looking under the street light



Forward looking Taylor rule

- Consider a modified version of the ACS model with a FL Taylor rule

$$\hat{\pi}_{t+1|t} = \hat{R}_t + \hat{M}_{t+1|t} \quad (1)$$

$$\hat{R}_t = \max\left\{-\mu, \psi\hat{\pi}_{t+1|t}\right\}, \quad (2)$$

$$\hat{M}_t = \rho\hat{M}_{t-1} + \sigma\epsilon_t, \quad E_{t-1}(\epsilon_t) = 0 \quad (3)$$

- Substituting for $\hat{\pi}_{t+1|t}$ in (2) using (1) yields

$$\hat{R}_t = \max\left\{-\mu, \psi\hat{R}_t + \psi\rho\hat{M}_t\right\} \quad (4)$$

- The CC condition for this model is $\psi < 1$
- (Incidentally, steady state and dynamic CC conditions coincide)

Coherent and Complete solution

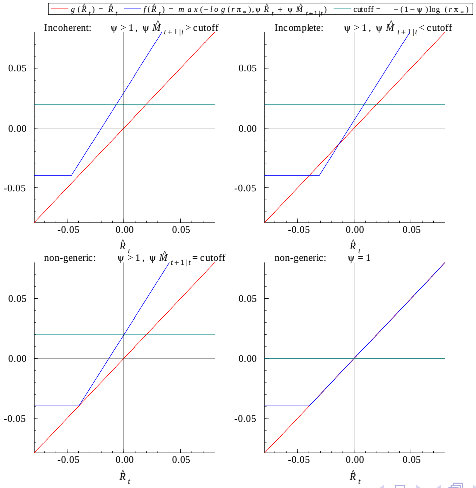
- We need to solve $\hat{R}_t = \max\left\{-\mu, \psi\hat{R}_t + \psi\rho\hat{M}_t\right\}$
- Notice there are *no expectations* of the endogenous variables in this
- This makes the solution *a lot* simpler
- The problem fits into the framework of "Censored and Kinked SVAR" model introduced by Mavroeidis (2019) "Identification at the ZLB"
- There I show that if the CC condition holds, and the error ϵ_t is Gaussian, the reduced form can be written as a Tobit

$$y_t = \max\left(0, y_t^*\right) \tag{1}$$
$$y_t^* = \left(1-\rho\right)\mu + \rho y_{t-1}^* + \tau\epsilon_t, \tag{2}$$

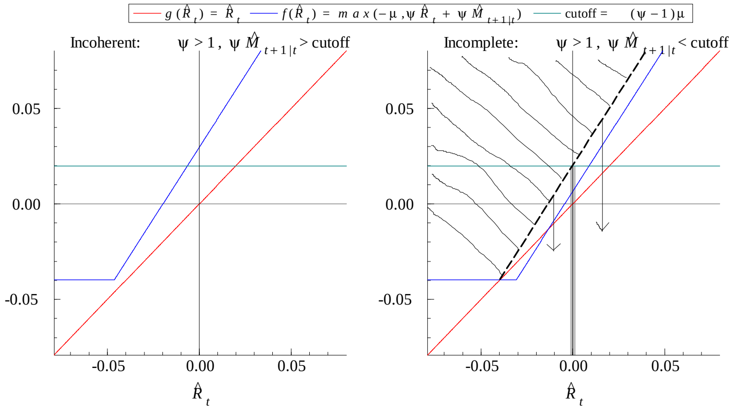
where $y_t = \log R_t = \hat{R}_t + \mu$, and $\tau = \frac{\psi\rho\sigma}{1-\psi}$

Incomplete solutions

When CC fails, we have generically 0 or 2 solutions



Incompleteness condition



Incomplete iff $\psi > 1$, and $\hat{M}_{t+1|t} < \frac{\psi - 1}{\psi}\mu$

Sunspot equilibria

- The set of equilibria is (recall $\mu = \log(r\pi_*)$, net nominal rate in TI steady state)

$$\hat{R}_t = -\mu D_t + (1 - D_t)\frac{\psi \hat{M}_{t+1|t}}{1-\psi}, \quad \frac{\psi \hat{M}_{t+1|t}}{1-\psi} > -\mu, \qquad (1)$$

  where D_t is an indicator ($D_t = 1$ means at ZLB) with arbitrary distribution

- Wlog, we can set

$$D_t = 1_{\{s_t > 0\}}$$

  where s_t is a 'sunspot' process that may or may not depend on $\hat{M}_{t+1|t}$

- E.g., D_t is a purely exogenous Markov chain can be characterized by

$$s_t = \delta_0 + \delta_1 D_{t-1} + \zeta_t, \quad \zeta_t \sim iidN(0,1), \quad \zeta_t \perp\!\!\!\perp \hat{M}_{t+1|t} \qquad (2)$$

Observational equivalence with CC model

- We can set the distribution of the sunspot such that the equilibrium is observationally equivalent to the CC model

$$\hat{R}_t = -\mu + \max\left(0, y_t^*\right)$$
$$y_t^* = (1-\rho)\,\mu + \rho y_{t-1}^* + \tau\epsilon_t$$

- This is achieved if we set:

$$s_t = y_t^* \tag{3}$$

Testing incompleteness

- We can nest 'pure sunspot' and OE-to-CC cases using

$$s_t = \delta_0 + \delta_1 D_{t-1} + \delta_2 y^*_{t-1} + \vartheta \zeta_t - \sqrt{1-\vartheta^2}\epsilon_t \qquad (1)$$

  - (this bears some similarity to Lubik and Schorfheide (2004) characterization of indeterminacy)
- Even though ψ is not identified, we can test for multiplicity by testing if data comes from a dynamic Tobit model
- This corresponds to the parametric restriction on (1)

$$H_0 : \delta_1 = \vartheta = 0 \quad \text{against } H_1 : \text{not } H_0$$

- Rejection of the H_0 provides unambiguous evidence of incompleteness
- The converse is not true because data consistent with both CC and incomplete models
  - (bears some similarity to Mavroeidis (2010))

Adding a monetary policy shock

- The above analysis showed that, in the case $\psi > 1$, existence of an equilibrium requires strange restrictions on the distribution of the real interest rate shock (path-dependent support, non-Markovian dynamics)
- If we add a monetary policy shock, we can relax the above restrictions on the real shock
- But existence (coherency) requires that the policy shock *cannot be independent* of the real shock
- The math: if Taylor rule becomes

$$\hat{R}_t = \max\left\{-\mu, \psi\hat{\pi}_{t+1|t} + \nu_t\right\}, \quad \psi > 1$$

- then support restriction becomes

$$\nu_t < -\psi\rho\sigma\epsilon_t - \psi\rho^2\hat{M}_{t-1} - (1-\psi)\mu \qquad (2)$$

CKSVAR

- The above approach generalizes to the CKSVAR of Mavroeidis (2019)
- Mavroeidis (2019) showed that in a completely unrestricted CKSVAR (that incorporates various forms of unconventional monetary policy, including forward guidance), the CC condition depends on a parameter κ that is not identified
  - CC condition is $\kappa > 0$, where κ is a function of the SVAR coefficients
  - in the present example, $\kappa = 1 - \psi$, so CC requires $\psi < 1$
- Mavroeidis (2019) focused on the CC case
- Here we show how to handle the incomplete case and test for incompleteness

Conclusions

- DSGE models subject to a ZLB constraint have many equilibria (even if we rule out sunspots)
- But existence of multiple equilibria requires 'strange' restrictions on the distribution of the shocks
  - Support cannot be unbounded
  - Shocks cannot be orthogonal
- Simple method to characterize existence and uniqueness of equilibria with discrete shocks
- But computationally too expensive to be useful for empirical application
- In SVAR models, we can characterize and test for multiplicity of equilibria

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