Testing for multiplicity of equilibria in a low interest rate environment

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PRELIMINARY AND INCOMPLETE

Abstract

Structural models with multiple solutions are incomplete. Incompleteness depends on the structural parameters and requires restrictions on the support of the distribution of the structural shocks. We develop likelihood-based methods for testing for incompleteness in dynamic macro models with occasionally binding constraints. We provide a general characterization of equilibrium dynamics under incompleteness, and a test for the presence of sunspots.

Keywords: incompleteness, zero lower bound, DSGE, SVAR

JEL codes: C1, C3, C5, E4, E5

1 Introduction

It is well-known that in structural models with occasionally binding constraints, equilibria may not exist (incoherency) or there may be multiple equilibria (incompleteness). It is plausible to assume away incoherency when data is actually observed, but incompleteness may not be ruled out a priori. In fact, several papers in the literature on

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the zero lower bound (ZLB) on interest rates have used models with multiple equilibria and associated steady states, e.g., Benhabib et al. (2001a,b), Fernández-Villaverde et al. (2015) and Aruoba et al. (2017), henceforth ACS.

In this paper, we show that the hypothesis of multiple equilibria (incompleteness) is testable, and develop appropriate tests for it. In doing so, we characterize the set of solutions to the model using arbitrary equilibrium selection mechanisms that may or may not include sunspots, and can depend on the structural shocks of the model. In fact, one of our proposed tests of incompleteness is based on the idea of testing for the presence of sunspots, that must be absent in a complete model. Therefore, rejection of the hypothesis of no sunspots implies multiplicity of equilibria though the converse is not true, as one cannot rule out sunspotless multiplicity.

Our analysis concerns global multiplicity of equilibria, not local indeterminacy as in the literature of Clarida et al. (2000). A well-known example of multiple equilibria in the ZLB literature comes from the Taylor rule. As shown in Benhabib et al. (2001a,b), Fernández-Villaverde et al. (2015) and ACS, active Taylor rules result in incompleteness (two steady states), while passive rules cause completeness. However, the fact that presence of sunspot dynamics can help distinguish between the two situations bears some similarity to the local indeterminacy literature, where (different type of) sunspot dynamics where found to be useful in distinguishing between active and passive policy rules, see Lubik and Schorfheide (2004); Mavroeidis (2010).

On a technical note, we also demonstrate that in incomplete models, the structural shocks cannot be independent and identically distributed over time since the support of their distribution must be bounded, and the bounds will generally be time-varying. In models with multiple shocks, the support of their distribution cannot be rectangular in the incomplete case, so the assumption of orthogonality of structural shocks is incompatible with multiple equilibria. The intuition is that with rectangular support we cannot rule out the possibility of incoherency, i.e., shocks that are sufficiently big to destroy the equilibrium completely. Thus, a further contribution of this paper is the characterization of the requisite restrictions on the support of the distribution of the shocks in the incomplete case. This allows us to derive an accurate representation of the likelihood function over the region of the parameter space that is associated with multiple equilibria.

2 A two-equation example

We start with a two-equation DSGE model based on ACS. The model consists of a consumption Euler equation

$$1 = E_t \left(M_{t+1} \frac{R_t}{\pi_{t+1}} \right) \tag{1}$$

and a forward-looking Taylor rule

$$R_t = \max \left\{ 1, r\pi_* \left(\frac{E_t \pi_{t+1}}{\pi_*} \right)^{\psi} \right\}, \quad \psi \geq 0, \tag{2}$$

where r is the steady-state gross real interest rate, R_t is the gross nominal interest rate, π_t is the gross inflation rate, π_* is the target of the central bank for the gross inflation rate, and M_{t+1} is the stochastic discount factor. Equation (2) differs from the Taylor rule in ACS which has the policy maker react to π_t instead of $E_t \pi_{t+1}$. The reason for considering the forward-looking rule is because it makes the solution of the model analytically tractable. This comes with the unfortunate side-effect of making π_t indeterminate, since the system of equations (1) and (2) can only pin down $E_t \pi_{t+1}$. This restriction will be relaxed in the more general model later. Another difference from ACS is that we do not impose a priori the restriction $\psi > 1$ that leads to incompleteness, because our primary objective is to develop a test of that hypothesis.

To keep the example simple, we follow ACS and model the stochastic discount factor M_{t+1} as an exogenous stationary AR(1) process in logs, i.e.,

$$\log \left(\frac{M_{t+1}}{m} \right) = \rho \log \left(\frac{M_t}{m} \right) + \sigma \epsilon_{t+1}, \quad \epsilon_t \sim iidN\left(0, 1\right),$$

where $m = 1/r$ is the steady state of the stochastic discount factor.

Loglinearizing around $\pi_t = \pi_*$, $M_t = 1/r$, $R_t = r\pi_*$, and using the notation $\hat{X}_t = \log(X_t/x)$, and $\hat{X}_{t+1|t} := E_t\hat{X}_{t+1}$ yields

$$\hat{\pi}_{t+1|t} = \hat{R}_t + \hat{M}_{t+1|t} \tag{3}$$

$$\hat{R}_t = \max\left\{-\log(r\pi_*), \psi\hat{\pi}_{t+1|t}\right\}, \tag{4}$$

$$\hat{M}_{t+1|t} = \rho\hat{M}_{t|t-1} + \rho\sigma\epsilon_t. \tag{5}$$

This model fits in the framework of Mavroeidis (2019), and can be seen as a kinked simultaneous equations model in the two endogenous variables $\hat{\pi}_{t+1|t}$ and $\hat{R}_t$, subject to a lower bound $\hat{R}_t \geq -\log(r\pi_*)$, and driven a single exogenous process $\hat{M}_{t+1|t}$.

2.1 Coherency and completeness

From Mavroeidis (2019, Prop. 1) it follows that, without any restrictions on the support of the exogenous process $\hat{M}_{t+1|t}$, the above model given in equations (3) and (4) is coherent and complete if and only if $\psi < 1$. In other words, there will exist a unique equilibrium if and only if the Taylor rule is passive.

The coherent and complete solution can be obtained from Mavroeidis (2019, Prop. 2), but in this simple case, it is straightforward to derive directly. Substituting for $\hat{\pi}_{t+1|t}$ in (4) using (3), we can write the model as

$$\hat{R}_t = \max\left\{-\log(r\pi_*), \psi\hat{R}_t + \psi\hat{M}_{t+1|t}\right\}, \tag{6}$$

and we can easily verify that the solution will be

$$\hat{R}_t = \max\left\{-\log(r\pi_*), \frac{\psi\hat{M}_{t+1|t}}{1-\psi}\right\}. \tag{7}$$

The univariate model given in equation (7) can be estimated using data on short-term nominal interest rates R_t. To simplify the exposition, define the net nominal interest rate as $y_t := \log R_t$, and let y_t^* be a latent process (which can be thought of as a shadow rate) such that $y_t = y_t^*$ if $y_t > b$, where b is a lower bound on the nominal interest rates that may be different from zero (this is a slight generalization of the previous setup that is useful when we bring the model to the actual data). With these definitions, and using the law of motion for $\hat{M}_{t+1|t}$ in eq. (5), the reduced-form (7) can be written as

$$y_t = \max(b, y_t^*) \tag{8}$$

$$y_t^* = c_0 + \rho y_{t-1}^* + \tau \epsilon_t, \quad \epsilon_t \sim iidN(0,1), \tag{9}$$

where $c_0 = (1-\rho)\log(r\pi_*)$, and $\tau = \frac{\psi\rho\sigma}{1-\psi}$. This is a dynamic Tobit model of the type studied by Lee (1999). It is a univariate version of the models studied in Mavroeidis (2019), where the alternative algorithms for the computation of the likelihood can be found.

2.2 Incompleteness

We now look at what happens when the coherency and completeness condition $\psi < 1$ is violated. First, note that the case $\psi = 1$ is non-generic and it can be ruled out a.s. when there are any unconstrained observations. This is because at $\psi = 1$, the two structural equations (3) and (4) have exactly the same slope above the constraint, so there is either a unique solution at the boundary $\hat{R}_t = -\log(r\pi_*)$, infinite solutions $\hat{R}_t \geq -\log(r\pi_*)$ when the two equations fall on top of each other (an event that occurs with probability zero) or no solution if the two equations do not intersect. The last case is incoherency, which we can rule out with restrictions on the support of the distribution of the errors. These are the same as in the generic case $\psi > 1$, so we do not need to discuss them separately here. See Figure 1 for an illustration.

So, we move directly to the case $\psi > 1$, where the model may be incoherent (nonexistence) or incomplete (two solutions) a.s.[1] First, we need to derive restrictions on the

[1] There is a nongeneric case where the two structural equations intersect exactly at the kink, where, provided $\psi > 1$, the two distinct solutions will coincide, so in that sense the model becomes complete. But this is an event of measure zero when the random variable $\hat{M}_{t+1|t}$ is continuous.

support of the distribution of the exogenous variable to rule out incoherency. We will call this the incompleteness condition. The problem is illustrated in Figure 1, where it can be seen that the necessary condition for incompleteness is

$$\psi > 1, \text{ and } \hat{M}_{t+1|t} < -\frac{1-\psi}{\psi}\log(r\pi_*) \tag{10}$$

When this holds, the two distinct solutions of (6) are given by $-\log(r\pi_*)$ and $\frac{\psi \hat{M}_{t+1|t}}{1-\psi}$. Condition (10) has important implications for the structural model that are sometimes overlooked. It says that $\hat{M}_{t+1|t}$ cannot follow a Gaussian AR(1) process as was originally assumed. In fact, the shock to the stochastic discount factor cannot even be an innovation processes, since, by the fact that the support of its distribution depends on past $\hat{M}_{t+1|t}$, and hence, past ϵ_t, the process ϵ_t must necessarily not be *i.i.d.*, and will therefore be predictable. This finding becomes more pertinent when we move to a model with multiple shocks, such as a monetary policy shock, because it will mean that the shocks cannot be independent of each other, since the support of their distribution cannot be rectangular.

2.3 Characterization of incomplete equilibria – sunspots

When (10) holds, the distribution of the regime indicator D_t is completely unrestricted, i.e., it is not determined by the exogenous variable $\hat{M}_{t+1|t}$. Specifically, the incomplete solutions are given by

$$\hat{R}_t = D_t\left(-\log(r\pi_*)\right) + (1-D_t)\frac{\psi\hat{M}_{t+1|t}}{1-\psi}, \quad \frac{\psi\hat{M}_{t+1|t}}{1-\psi} > -\log(r\pi_*), \tag{11}$$

and an equation determining D_t. Without loss of generality we can model D_t as

$$D_t = 1_{\{s_t<0\}}, \tag{12}$$

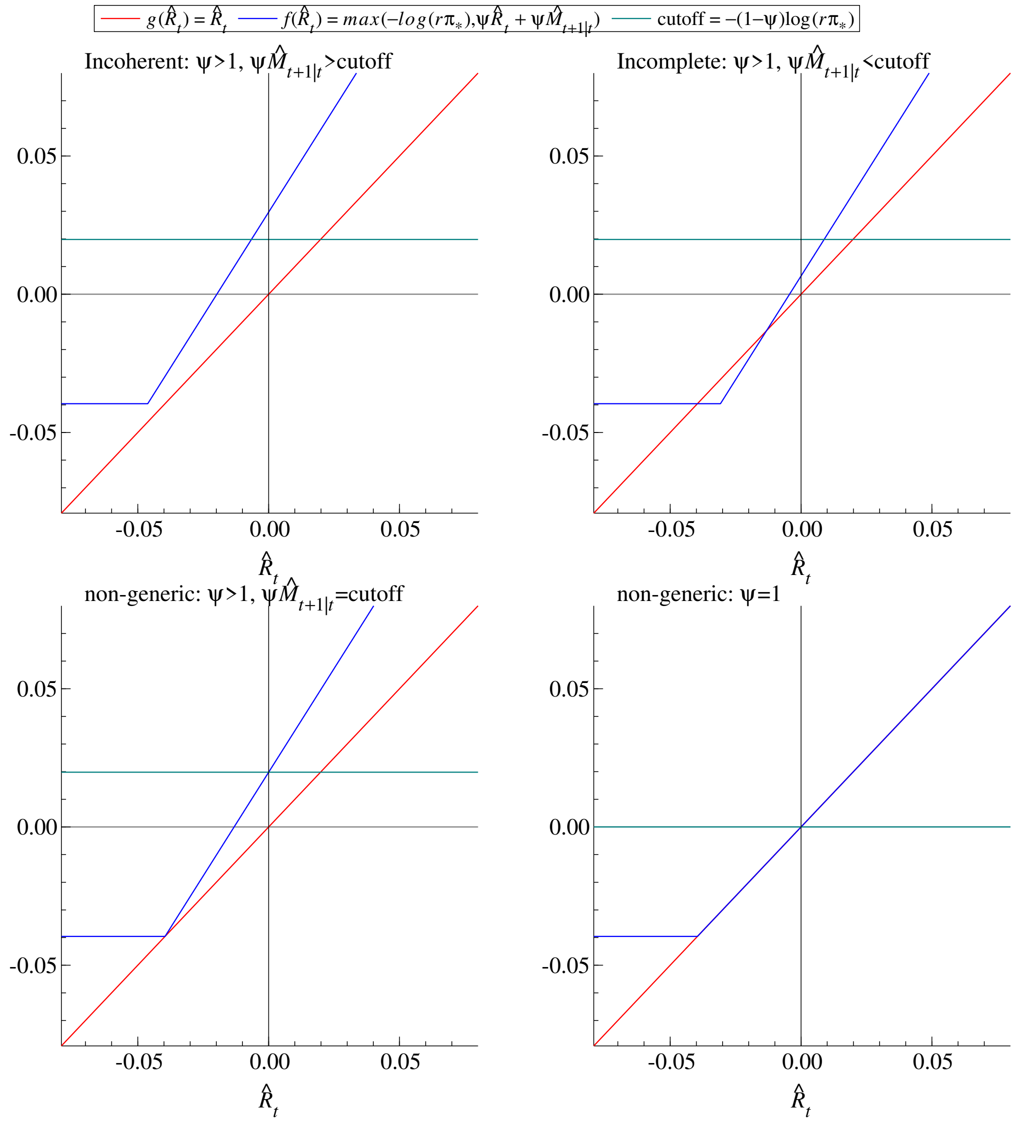

Figure 1: Illustration of the incompleteness restriction on the support of $\hat{M}_{t+1|t}$, $\hat{M}_{t+1|t} < -\frac{1-\psi}{\psi}\log(r\pi_*)$ in the model given by the intersection of $f\left(\hat{R}_t\right) = \max\left\{-\log(r\pi_*), \psi\hat{R}_t + \psi\hat{M}_{t+1|t}\right\}$ and $g\left(\hat{R}_t\right) = \hat{R}_t$. $r = 1.02$, $\pi_* = 1.02$, $\psi = 1.5$ (except for the last plot).

for some new process s_t that may or may not depend on $\hat{M}_{t+1|t}$. For example, the special case that D_t is a purely exogenous Markov chain can be characterized by

$$s_t = \delta_0 + \delta_1 D_{t-1} + \zeta_t, \quad \zeta_t \sim iidN\left(0,1\right), \quad \zeta_t \perp\!\!\!\perp \hat{M}_{t+1|t} \tag{13}$$

so that, e.g., $\Pr\left(D_t = 1|D_{t-1} = 0\right) = \Phi\left(-\delta_0\right)$. One can think of this as a pure sunspot shock in the terminology of ACS. In fact, this is exactly the approach followed by ACS.[2] With this specification, the solution under incompleteness (11) differs from the coherent and complete solution (7), which took the form of a dynamic Tobit model, see (8). The opposite case is arguably a situation in which the two solutions are observationally equivalent. This can be obtained by setting

$$\frac{\psi \hat{M}_{t+1|t}}{1-\psi} = \begin{cases} y_t^* - \log\left(r\pi_*\right), & \text{if } y_t^* > 0 \\ w_t > -\log\left(r\pi_*\right) & \text{otherwise} \end{cases}$$

where y_t^* is the latent process defined in (9), and w_t is an arbitrary random variable (which will not affect the solution), and

$$s_t = y_t^* = c_0 + \rho y_{t-1}^* + \tau \epsilon_t. \tag{14}$$

In that case, the incomplete solution (11) becomes observationally equivalent to (7). Suppose think of the lagged shadow rate y_{t-1}^* and u_t as state variables. Then, since this solution does not involve any additional state variables, such as a sunspot shock, or lagged values of the stochastic discount factor $\hat{M}_{t+1|t}$ in periods when it is not observed, we can think of it as the analog of the 'Minimum State Variable' (msv) solution of linear indeterminate rational expectations models in (Lubik and Schorfheide, 2004, p. 195).

Finally, we can nest the two special cases, the pure sunspot case (13) and the msv

[2] ACS (footnote 6) also commented on the possibility of endogenizing D_t in this simple model, but they did not extend that to the more general implementation of their method for computational reasons.

solution (14) into the nesting model

$$s_t = \delta_0 + \delta_1 D_{t-1} + \delta_2 y_{t-1}^* + \varpi_t, \tag{15}$$

where

$$\varpi_t = \vartheta \zeta_t - \sqrt{1-\vartheta^2}\epsilon_t,$$

so that $var(\varpi_t)$ is normalized to 1 as it is not identified.[3] As in Lubik and Schorfheide (2004), the absence of a sunspot shock per se, $\vartheta = 0$, does not suffice to obtain the msv solution (14).

2.4 Testing for incompleteness

In this univariate model, the parameter ψ is not identified, so we cannot test incompleteness directly. It will be shown below that ψ is identified in more general models. However, even in situations in which the structural model is underidentified, as in the present example, we can still partially test the incompleteness hypothesis. Specifically, we can test whether the dynamics of $\log R_t$ differ significantly from the dynamic Tobit model in (8), which corresponds to the coherent and complete solution. Rejection of the dynamic Tobit model provides unambiguous evidence of incompleteness and multiplicity of equilibria (conditional on the model's assumptions). The converse is not true, because (8) also gives the msv solution under incompleteness.

2.5 Adding a monetary policy shock

Consider a variant of the Taylor rule (2)

$$R_t = \max\left\{1, r\pi_*\left(\frac{E_t\pi_{t+1}}{\pi_*}\right)^{\psi} e^{\nu_t}\right\}, \quad \psi \geq 0, \tag{16}$$

that includes a monetary policy shock ν_t. The coherency and completeness condition is obviously unaffected by the presence of any additional exogenous variables, i.e., it

[3] This is because (15) is a Probit regression which is only identified up to scale.

remains $\psi < 1$. However, the incompleteness condition (10) now becomes

$$\nu_t < -\psi\rho\sigma\epsilon_t - \psi\rho\hat{M}_{t|t-1} - (1-\psi)\log(r\pi_*), \text{ and } \psi > 1. \tag{17}$$

Equation (17) has important implications for the model. It says that the monetary policy shock cannot be too high relative to the discount factor shock ϵ_t, i.e., their support cannot be rectangular. Therefore, the two shocks cannot be independent of each other if we are to rule out incoherency. In fact, in the present example, the support will also be time-varying conditional on the past states,[4] and so not only does the monetary policy shock ν_t need to depend on the demand shock ϵ_t, but it also needs to depend on the past, so it has to be predictable. That is, it cannot be an innovation process.

The solution of the model can be written generically as:

$$\begin{aligned}
\hat{\pi}_{t+1|t} &= \frac{\hat{M}_{t+1|t} + \nu_t}{1-\psi} - D_t\left(\frac{\psi\hat{M}_{t+1|t} + \nu_t}{1-\psi} + \log(r\pi_*)\right) \qquad (18)\\
\hat{R}_t &= D_t(-\log(r\pi_*)) + (1-D_t)\frac{\psi\hat{M}_{t+1|t} + \nu_t}{1-\psi}\\
\hat{M}_{t+1|t} &= \hat{\pi}_{t+1|t} - \hat{R}_t = \rho\hat{M}_{t|t-1} + \rho\sigma\epsilon_t
\end{aligned}$$

where the distributions of D_t and ν_t, ϵ_t depend on the value ψ, as well as any assumptions on the dependence of ν_t over time. In the coherent and complete case $\psi < 1$, the distribution of the shocks is unrestricted, so if we assume ν_t is serially uncorrelated and independent of the real shock ϵ_t, we can write

$$\begin{aligned}
\nu_t &= \sigma_\nu e_t, \quad (e_t, \epsilon_t) \sim iidN(0, I)\\
D_t &= 1_{\left\{\frac{\psi\hat{M}_{t+1|t}+\nu_t}{1-\psi} > -\log(r\pi_*)\right\}},
\end{aligned}$$

If we had data on inflation expectations, $\pi_{t+1|t}$, then the model could be seen as a restricted KSVAR in $Y_t = \left(\pi_{t+1|t}, \log R_t\right)'$ in the terminology of Mavroeidis (2019). From that paper, it follows that the parameter ψ is (over-)identified, so inference on

[4]This wouldn't be the case if we had modelled $\hat{M}_t$ as MA(1) instead of AR(1).

completeness can be obtained by testing the hypothesis $\psi < 1$. In a frequentist approach, this can be done by inverting the Likelihood ratio test for $H_0 : \psi = \psi_0$ to compute a confidence set on ψ and checking whether it includes only values $\psi_0 < 1$. The Bayesian counterpart will involve the odds ratio of the posteriors over the regions $\psi < 1$ and $\psi > 1$.

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