# Firm Dynamics, On-the-Job Search and Labor Market Fluctuations

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**Abstract**

We devise a tractable model of firm dynamics with on-the-job search. The model admits analytical solutions for equilibrium outcomes, including quit, layoff, hiring and vacancy-filling rates, as well as the distributions of job values, a fundamental challenge posed by the environment. Optimal labor demand takes a novel form whereby hiring firms allow their marginal product to diffuse over an interval. The evolution of the marginal product over this interval endogenously exhibits gradual mean reversion, evoking a notion of imperfect labor market competition. This in turn contributes to dispersion in marginal products, giving rise to endogenous misallocation. Mirroring establishment microdata, quit and layoff rates fall, while hiring and vacancy-filling rates rise with firm growth in the model. We further show how it is possible to solve for the dynamic equilibrium path of model outcomes—including the distribution of job values—out of steady state.

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The labor market is in a perpetual state of flux. In any given period, some unemployed workers find new jobs while other employed workers lose them (Blanchard and Diamond 1990). Some firms grow through job creation while others shrink through job destruction (Davis and Haltiwanger 1992). And, in tandem, some employed workers move directly from one employer to another (Fallick and Fleischman 2004). These worker and job flows are substantial in magnitude, vary considerably over the business cycle, and exhibit clear cross-sectional correlations (Davis, Faberman and Haltiwanger 2012, 2013).

The purpose of this paper is to understand the economics underlying this rich array of empirical regularities. To do so we devise a model that integrates firm dynamics with on-the-job search. Firms subject to hiring costs face idiosyncratic shocks that drive changes in their desired employment, and thereby job creation and destruction. Workers search for jobs across firms while both unemployed and employed, driving worker flows. Direct employer-to-employer transitions emerge naturally from the heterogeneity across firms induced by idiosyncratic shocks. And we show how the model can be extended to accommodate aggregate shocks, and thereby business cycles. The result is a framework in which an understanding of the economics of the foregoing stylized facts is feasible.

Attaining this goal is easier said than done, however. The interplay of firm dynamics with on-the-job search poses a seemingly daunting analytical challenge. In general, the rate of worker turnover faced by a firm will depend on the firm's position in the hierarchy of job values in the economy. Firms further up in the hierarchy will face lower turnover. Steady-state labor market equilibrium thus involves finding a fixed point of an entire *distribution* of job values, one that both sustains firms' labor demand decisions and is implied by aggregation of those same decisions. Out of steady state, equilibrium further involves finding a fixed point of the *dynamic path* of the distribution.

An important contribution of the paper is that we are able to provide an analytical characterization of this distribution. In section 1, we study an environment that gives rise to a normalization in which the value of jobs to workers and firms are monotone functions of a single idiosyncratic state variable, the marginal product of labor. The distribution of job values can thus be summarized by the distribution of marginal products. Furthermore, optimal labor demand can be decoupled into two regions for the marginal product. Mirroring canonical models of firm dynamics (Bentolila and Bertola 1990; Hopenhayn and Rogerson 1993; Abel and Eberly 1996), there is a *natural wastage region*. At its lower boundary, firms shed workers into unemployment. On its interior, firms neither hire nor fire, and turnover occurs at a maximal constant quit rate.

A novel implication of the presence of on-the-job search, however, is the addition of a nondegenerate *hiring region*. Importantly, this emerges even in the absence of heterogeneity in marginal hiring costs. The key intuition is that hiring firms face a novel trade off in the presence of on-the-job search. On the one hand, they value the additional output generated by new hires. On the other, they value reductions in turnover associated with a higher marginal product. We show that this tradeoff is resolved by a novel solution: Firms allow their marginal products to diffuse across an interval, a strategy that is supported by a quit rate that declines with the marginal product at an appropriate rate. We show that the latter force is captured by a simple differential equation that gives rise to a closed-form solution for the quit rate. Crucially, this in turn gives rise to a closed-form solution for the distribution of marginal products offered to new hires—a key result in the light of the analytical challenge noted above.

The hiring region varies interestingly with the structure of wage determination. We explore two wage protocols. The first is a model of *ex post* wage bargaining in the absence of offer matching. This synthesizes the insights of credible bargaining (Binmore et al. 1986) and multilateral bargaining (Bruegemann et al. 2018) in the presence of on-the-job search (Gottfries 2019). Interestingly, it provides a rationale for the absence of offer matching if job offers are private information, since it is not credible to elicit them through the use of layoff lotteries (Moore 1985). The second protocol extends the sequential auctions approach of Postel-Vinay and Robin (2002) to allow for multi-worker firms and (partial) offer matching. A revealing implication is that firms' expected costs of turnover, and thereby the size of the hiring region, are declining in firms' ability to match offers. In the limit in which firms can respond perfectly to the idiosyncratic outside offers of each of their workers, firms become *indifferent* to turnover, and the hiring region converges to a point. Away from that limit, turnover is costly to firms, and a nondegenerate hiring region emerges.

The implications of the preceding behavior for aggregate labor market equilibrium are not obvious: Optimal labor demand and turnover are heterogeneous across firms, and evolve in a nonlinear fashion with idiosyncratic shocks. Nonetheless, we show in section 2 how it is possible to derive an analytical characterization of steady-state labor market equilibrium. We begin by aggregating microeconomic behavior, obtaining expressions for the separation rate into unemployment, as well as the hiring rate, the vacancy-filling rate, and the distribution of workers at each marginal product. These in turn imply two conditions for aggregate steady-state equilibrium that mirror those in the canonical

Mortensen and Pissarides (1994) model: a *Beveridge curve* implied by steady-state unemployment flows; and a *job creation curve* that summarizes aggregate labor demand.

A host of insights follow on the nature of labor market behavior induced by the model. A first insight emerges from the fact that hiring rates are increasing in the marginal product. Coupled with decreasing quit rates, this gives rise to *endogenous mean reversion* in marginal products. Positive innovations raise a firm's hiring rate and reduce its quit rate. Firms thus accumulate more workers and the marginal product reverts back in expectation. An appealing interpretation is that the latter is a manifestation of *imperfect labor market competition*; perfect competition would imply infinite mean reversion.

Second, the model reveals a novel paradox in the interplay between on-the-job search and misallocation. As in canonical models of on-the-job search (Burdett and Mortensen 1998), equilibrium in our model involves dispersion in marginal products across workers, and thereby misallocation. In stark contrast to canonical models, however, on-the-job search *contributes to*, rather than resolves, such misallocation by inducing the presence of a nondegenerate hiring region. The model thus captures a novel notion of *endogenous misallocation*, driven by the interaction of firm dynamics and on-the-job search.

Third, the model naturally generates cross-sectional relationships between worker flows and firm growth that mirror those documented in recent empirical work by Davis, Faberman and Haltiwanger (2012, 2013). Firm growth in the model is monotone in the marginal product. It follows that faster-growing firms in the model are less likely to lay off workers, more likely to hire and post vacancies and, most notably, will face lower quit rates and higher vacancy-filling rates. The latter in particular are highlighted by Davis et al. as important channels missing from conventional models. To the contrary, these appear to be natural implications of labor demand and turnover decisions in the presence of on-the-job search.

Finally, we explore the aggregate dynamics implied by the model out of steady state. Recall that, in general, this involves a fixed point in the dynamic path of the entire distribution of job offers, a formidable prospect. Note that this problem is distinctly harder than those that arise in standard heterogeneous agent models in which agents must forecast a market price. Here the analogue of the market price is a whole function, the offer distribution. Nonetheless, we are able to make progress by generalizing our earlier results. In particular, the same forces that give rise to a closed-form solution for the offer distribution of marginal products in steady state allow us to infer the functional form of the offer distribution out of steady state. Doing so reduces the problem to one of inferring the dynamic path of a single scalar, labor market tightness. We show that this is feasible by presenting an illustrative simulation of the transition dynamics implied by the model. Future work will explore this (and other model outcomes) from a quantitative perspective.

**Related literature**. The model set out in this paper provides a new theory of firm dynamics with (random) job search, both off- and on-the-job. In addition to the work already cited, it relates to three further strands of literature.

First, our model builds on recent work that has developed so-called "large-firm" search models that fuse firm dynamics with off-the-job search. These have been used to study firm growth (Acemoglu and Hawkins 2014), worker flows over the business cycle (Elsby and Michaels 2013), the role of wage posting and directed search in recruitment (Kaas and Kircher 2015), and cyclical recruitment intensity (Gavazza, Mongey and Violante 2016). None of these papers incorporates on-the-job search, however.

Second, a further strand of related literature has incorporated a business cycle into models of on-the-job search (Moscarini and Postel-Vinay 2013; Coles and Mortensen 2016; Lise and Robin 2017). As in our model, these papers have addressed a related challenge posed by the presence of on-the-job search of solving for the dynamics of distributions of job values. In contrast to our model, however, all such work has maintained the assumption of linear production technologies.

Third, and most closely related to our work, a handful of recent papers has sought to integrate firm dynamics with on-the-job search. Lentz and Mortensen (2012) focus on firm lifecycles and steady-state wage and productivity dispersion in a model without idiosyncratic or aggregate shocks. Fujita and Nakajima (2016) study the relation between worker and job flows over the business cycle, but assume that workers have no bargaining power and firms cannot respond to outside offers. Schaal (2017) studies the effects of time-varying idiosyncratic risk in a related model that incorporates job-to-job flows, but where search is directed, and firms can commit to complete state-contingent contracts. Elsby, Michaels and Ratner (2019) focus on the interaction between replacement hiring and on-the-job search across firms in amplifying labor market responses through vacancy chains. Bilal, Engbom, Mongey and Violante (2019) make two important contributions relative to our analysis. First, they provide sufficient conditions based on limited commitment and mutual consent that distil the firm's problem into one of surplus maximization. Second, they explore a model with a convex vacancy cost, and firm entry and exit, enabling a quantitative study of worker flows and employment dynamics over firms' lifecycles.

Importantly, relative to all these, a key contribution of the present paper is that it provides several novel analytical results. The identification and characterization of the hiring region and the associated equilibrium quit rate, the role of wage determination in shaping these, the analytics of aggregation, and the use of all of these in simplifying and solving for aggregate dynamics are new results of this paper.

# 1. Turnover, wages and labor demand

Our point of departure is a canonical model of firm dynamics in the presence of frictions, mirroring that in Bentolila and Bertola (1990). The labor market is comprised by a mass of firms, normalized to one, and a mass of potential workers, equal to the labor force $L$. Firms use labor $n$ to produce output $y$ using an isoelastic production technology $y = xn^{\alpha}$, where $\alpha \in (0,1)$. $x$ is an idiosyncratic shock that is the source of uncertainty to the firm, and of heterogeneity across firms. It evolves over time according to the geometric Brownian motion

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}z, \tag{1}$$

where $\mathrm{d}z$ is the increment to a standard Brownian motion.

Firms hire workers subject to a per-worker hiring cost $c$. Denoting the cumulative sum of a firm's hires by $H$, and its increment over the time interval $\mathrm{d}t$ by $\mathrm{d}H$, the firm faces flow hiring costs of $c \cdot \mathrm{d}H$. Separations occur through two channels. First, the firm's employees quit at rate $\delta$. Second, additional separations may be implemented at zero cost; we denote their cumulative sum by $S$, and its increment $\mathrm{d}S$.[1] It follows that the firm's employment evolves according to the law of motion

$$\mathrm{d}n = \mathrm{d}H - \mathrm{d}S - \delta n \mathrm{d}t. \tag{2}$$

Given this environment, standard methods (see, for example, Dixit 1993, and Stokey 2009) imply that the Bellman equation for the value of the firm $\Pi$ can be written as

$$r\Pi \mathrm{d}t = \max_{\mathrm{d}H \geq 0, \mathrm{d}S \geq 0} \left\{ \left[ xn^{\alpha} - wn - \delta n \Pi_n + \mu x \Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx} \right] \mathrm{d}t - (c - \Pi_n)\mathrm{d}H - \Pi_n \mathrm{d}S \right\}, \tag{3}$$

where $r$ is the firm's discount rate. The firm chooses its hires $\mathrm{d}H$ and separations $\mathrm{d}S$ to maximize the expected present discounted value of its profit stream. Its flow profits are given by the flow revenue $xn^{\alpha}$, less wage payments $wn$ and hiring costs $c \cdot \mathrm{d}H$. The firm

[1] We use this notation to allow for the possibility that a firm may choose a continuous, but non-differentiable path for cumulative hires and separations.

faces capital gains from two sources. First, the firm's employment $n$ evolves according to the law of motion (2). Each incremental change $\mathrm{d}n$ is valued by the firm according to the marginal value $\Pi_n$. The second source of capital gains to the firm arises from the idiosyncratic shocks $x$, which evolve according to the stochastic law of motion (1). Application of Ito's lemma yields the form in (3).

Our key innovation is to incorporate endogenous turnover into this otherwise-canonical environment, by integrating it with a model of on-the-job search and associated theories of wage setting. The next two subsections describe in turn each of these elements of the environment.

## 1.1 Turnover

Consider first turnover. Workers can search while unemployed, or while employed with relative search intensity $s$. Accordingly, unemployed searchers receive job offers at rate $\lambda$, and employed workers at rate $s\lambda$. An important consequence of on-the-job search for the latter is that not all such offers will be accepted: an offer will be accepted only if its associated worker surplus exceeds that in the worker's current employment state. The quit rate $\delta$ faced by the firm in (3) thus becomes endogenous in the presence of on-the-job search. Each of the firm's employees receives an offer from another firm at rate $s\lambda$. And each contacted employee will choose to quit if the outside contact offers a worker surplus that exceeds that at the current firm, $W$. Denoting the offer distribution of worker surpluses by $\Phi(\cdot)$, we can thus write the quit rate faced by the firm as

$$\delta(W) = s\lambda[1 - \Phi(W)]. \tag{4}$$

A fundamental, and analytically challenging implication of the interaction of firm dynamics with on-the-job search is that the quit rate $\delta(\cdot)$—or, equivalently, the offer distribution of worker surpluses $\Phi(\cdot)$—is in general a state variable for the firm. Firms must know this distribution in order to make labor demand decisions; and the distribution in turn is determined by aggregation of those same decisions. We will see that steady-state equilibrium thus involves a fixed point in this distribution. And out of steady state equilibrium further involves a fixed point in the *dynamic path* of the distribution.

Note that this challenge is distinct from that posed in standard models of aggregate equilibrium in heterogeneous agent economies (as in, for example, Krusell and Smith 1998). As we shall see, in our environment the latter involves firms having to forecast the path of the distribution of employment as a means to forecast the path of equilibrium wages—a *scalar*. The presence of on-the-job search overlays on top of this the higher-dimensional challenge of firms having to forecast the path of the distribution $\Phi(\cdot)$ as a means of forecasting the *function* $\delta(\cdot)$. In what follows, we show how progress can be made on this challenge.

## 1.2 Wage setting

To complete our description of the firm's problem, it remains to specify how wages are determined. Wages are a key determinant not only of firms' labor costs, but also of the worker surplus, and thereby turnover. The interaction of multi-worker firms and the presence of employees with outside offers renders wage determination challenging in this environment. We will consider two protocols for wage determination that can be accommodated by variations on the preceding framework.

We first present a benchmark case in which wages are determined entirely *ex post*—that is, after all search decisions have been completed—according to a simple model of bargaining between a firm and its many workers. A corollary of this bargaining protocol is that all workers in a given firm are paid a common wage. Firms do not engage in offer matching in response to their employees' outside offers. As we will discuss at further length, potential motivations for such a protocol include non-verifiability of outside offers, and the presence of equal treatment constraints across employees within a firm. For these reasons, in addition to the relative simplicity of the implied wage outcomes, we study this benchmark case for the remainder of the present section.

Later, we extend this wage determination protocol to accommodate the possibility that firms may respond to their employees' outside offers with some degree of offer matching, generalizing the sequential auctions approach of Postel-Vinay and Robin (2002) to a multi-worker firm context with partial offer matching. Although the presence of offer matching gives rise to more complicated wage outcomes that differ across workers within firms, we will see that it is nonetheless tractable, and provides a useful point of contrast that elucidates the role of wage determination in shaping the effects of on-the-job search on equilibrium firm dynamics.

**Bargaining in the absence of offer matching**. For now, though, we begin by describing a simple model of *ex post* bargaining between a firm and its many workers in the absence of offer matching. To clarify our meaning of *ex post*, it is helpful first to return to the firm's problem in (3) and consider the order of events within each $\mathrm{d}t$ period. At the beginning of the period, productivity is realized, and hiring and separation decisions are made. Upon completion, a bargaining stage then begins in which wages are negotiated between the firm and its many workers—it is in this sense that bargaining is *ex post*. Once bargaining is complete, production takes place, agreed wages are paid, and the period concludes.

The bargaining stage takes the following form. The firm and its workers bargain over the flow wage for the current period, $wdt$, according to the bargaining game proposed by Bruegemann, Gautier and Menzio (2018). The firm engages in a sequence of bilateral bargaining sessions with each of its workers subject to breakdown risk. The sequence of play is devised such that the strategic position of each worker within the firm is symmetric. They characterize an equilibrium[2] of the game in which all workers within the same firm receive the same wage, and this wage coincides with that implied by a marginal surplus-sharing rule proposed by Stole and Zwiebel (1996).

The relevant marginal surplus that the firm and its workers share is determined by the threats that the firm and each of its workers can credibly issue in the event of a breakdown of negotiations. Binmore et al. (1986) and, more recently, Hall and Milgrom (2008) emphasize that threats of *permanent* suspension of negotiations are not plausibly credible in this setting: Regardless of a breakdown in the current period, the firm will wish to resume negotiations with the same workers in the subsequent period. Instead, breakdown is credibly associated only with a *temporary* disruption of production due to delayed agreement. Since wages are renegotiated every period, turnover and wages in subsequent periods will be independent of the current wage, and the effective surplus that the firm and its workers share will be the marginal *flow* surplus.

This approach to wage bargaining has several appealing properties. First, wage outcomes take a particularly simple form. Following Hall and Milgrom, suppose that, in the event of breakdown, workers receive a flow payoff $\omega_w$, and a firm incurs a per-worker flow cost $\omega_f$. Then, marginal flow surplus sharing implies

$$\beta\left(x\alpha n^{\alpha-1} - w - w_n n + \omega_f\right) = (1-\beta)(w - \omega_w), \tag{5}$$

where $\beta \in (0,1)$ indexes worker bargaining power. It is straightforward to verify that the wage solution takes the following simple form,

[2] Specifically, the no-delay subgame perfect equilibrium in the limit as the probability of breakdown goes to zero. In their static setting, they show that this no-delay equilibrium is unique. A sufficient condition for this equilibrium to hold in our dynamic setting is the presence of non-history-dependent strategies.

$$w = \frac{\beta}{1-\beta(1-\alpha)} x\alpha n^{\alpha-1} + \omega_0, \tag{6}$$

where $\omega_0 \equiv \beta\omega_f + (1-\beta)\omega_w$.[3]

The wage equation captures some familiar forces: Wages are increasing in the marginal product $x\alpha n^{\alpha-1}$, and the flow payoffs from breakdown, summarized by $\omega_0$. The wage equation also captures standard "large-firm" effects: Due to decreasing returns in production, $\alpha \in (0,1)$, failure to agree with an individual worker will result in higher bargained wages for all remaining workers. Using these threats, workers are able to capture some of the inframarginal product, giving rise to the leading coefficient. Because breakdown of negotiations does not involve permanent severance of a match, the option values to search (both off- and on-the-job) do not play a role in wage outcomes. In this respect, the wage bargain resembles that proposed by Hall and Milgrom, extended to accommodate multi-worker firms and continual renegotiation.

A further virtue of this approach to wage bargaining is that it can be reconciled with the presence of on-the-job search, in two important senses. First, it is not subject to the concern noted in Shimer (2006) that the effects of bargained wages on turnover will render the bargaining set nonconvex. Since bargaining pertains only to the current flow wage, which in turn is re-bargained each period, current wages have no effect on future wages, and thereby turnover (see Nagypal 2007, and Gottfries 2019). Second, this approach to wage bargaining also suggests a natural rationale for the absence of offer matching. Suppose job offers are privately observed by workers and unverifiable. A firm would be able to elicit the value of such offers if it were able to confront its (potential) workers with a set of appropriately-devised layoff lotteries (Moore 1985). But, echoing our earlier discussion of the bargaining stage, such layoff lotteries will not be credible *ex post*: the firm will wish to resume its relationship with a worker after any such layoff realization. Thus, inability to commit to permanent severance provides a simple reconciliation of wage bargaining, on-the-job search, and absence of offer matching.

**Worker values**. A key implication of the wage solution in (6) for what follows is that it determines the worker surplus $W$, and thereby worker turnover decisions. To see how,

[3] Strictly, the wage equation holds in the event of agreement, which occurs provided the marginal flow surplus is positive, $\frac{1}{1-\beta(1-\alpha)} x\alpha n^{\alpha-1} + \omega_f - \omega_w > 0$. We assume this holds in what follows.

consider the value of employment $\Omega$ to a worker currently employed in a firm offering worker surplus $W$. This satisfies the Bellman equation

$$r\Omega \mathrm{d}t = \max\left\{\left[w + s\lambda \int_W \left(\widetilde{W} - W\right)\mathrm{d}\Phi\left(\widetilde{W}\right) - \delta n \Omega_n + \mu x \Omega_x + \frac{1}{2}\sigma^2 x^2 \Omega_{xx}\right]\mathrm{d}t \right. \\ \left. + \Omega_n(\mathrm{d}H^* - \mathrm{d}S^*) - W\frac{\mathrm{d}S^*}{n}, \Upsilon \mathrm{d}t\right\}. \tag{7}$$

An employed worker receives a flow wage $w$ given by (6), and faces capital gains from three sources. First, at rate $s\lambda$ she contacts an outside firm with worker surplus $\widetilde{W}$ drawn from the offer distribution of worker surpluses $\Phi(\cdot)$. She accepts the outside job only if it offers a larger worker surplus, $\widetilde{W} > W$. Second, employment at her current firm will evolve according to the law of motion (2). If the worker remains employed by the firm, she values each incremental change $\mathrm{d}n$ by $\Omega_n$. If the firm implements layoffs, $\mathrm{d}S^* > 0$, the worker faces a uniform risk of being laid off and realizing a capital loss equal to the worker surplus $W$. Since the flows of hires and fires are chosen by the firm, they are evaluated at the equilibrium values that maximize the firm's problem in (3), $\mathrm{d}H^*$ and $\mathrm{d}S^*$. Third, her current firm's idiosyncratic productivity evolves according to the stochastic law of motion (1) and, by Ito's lemma, gives rise to the remaining capital gain terms.

Finally, note that the worker retains an option to quit employment at the firm, which she will exercise whenever $\Omega$ falls below the value of unemployment $\Upsilon$ to a worker. This in turn satisfies the Bellman equation

$$r\Upsilon = b + \lambda \int \widetilde{W} d\Phi\left(\widetilde{W}\right). \tag{8}$$

While unemployed, a worker receives a flow payoff $b$. At rate $\lambda$ she receives an offer with worker surplus $\widetilde{W}$ drawn from the offer distribution of worker surpluses $\Phi(\cdot)$. Since it is never optimal for a firm to make an offer that would not be accepted by an unemployed searcher, the worker accepts with certainty.

Recalling that the worker surplus is the additional value to a worker of employment over unemployment, $W \equiv \Omega - \Upsilon$, and noting that the value of unemployment $\Upsilon$ is independent of any firm's idiosyncratic employment or productivity state, we obtain the following recursion for the worker surplus,

$$
\begin{aligned}
rW\mathrm{d}t = \max\Bigg\{\Bigg[& w - b - \lambda \int \widetilde{W} d\Phi(\widetilde{W}) + s\lambda \int_{W} (\widetilde{W} - W)\mathrm{d}\Phi(\widetilde{W}) - \delta n W_n + \mu x W_x \\
&+ \frac{1}{2}\sigma^2 x^2 W_{xx}\Bigg]\mathrm{d}t + W_n(\mathrm{d}H^* - \mathrm{d}S^*) - W\frac{\mathrm{d}S^*}{n}, 0\Bigg\}.
\end{aligned} \tag{9}
$$

In what follows we assume that the worker's reservation wage is sufficiently low such that the firm (weakly) initiates all separations into unemployment, and optimality decisions over hires and fires can be inferred from solving the firm's problem. Empirically, only a small fraction of flows from employment to unemployment are reported as quits (Elsby, Michaels and Solon 2009). However, the alternative case can be accommodated by a similar analysis.

## 1.3 Optimal labor demand and equilibrium turnover

Our description of turnover and wage setting completes the environment faced by firms and workers. We can now proceed to consider optimal labor demand and turnover decisions. Recall that the latter is a key challenge that arises from the interaction of firm dynamics and on-the-job search, as labor demand decisions and turnover rates are intertwined in this environment. In this subsection, we provide a solution in which the joint determination of labor demand and turnover takes a surprisingly simple and tractable form.

We begin by returning to the firm's problem in (3), and noting that optimality conditions for hires and separations imply that[4]

$$
(-c + \Pi_n)\mathrm{d}H = 0, \text{ and, } \Pi_n \mathrm{d}S = 0. \tag{10}
$$

Optimality requires that the marginal value of labor $\Pi_n$ be set equal to the marginal hiring cost $c$ in the event of hiring, $\mathrm{d}H > 0$, and to zero in the event of firing, $\mathrm{d}S > 0$. It follows that the maximized value of the firm satisfies

$$
r\Pi = xn^{\alpha} - wn - \delta n \Pi_n + \mu x \Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx}. \tag{11}
$$

The proximate effects of on-the-job search on the firm are thus distilled in the *turnover costs* $\delta n \Pi_n$. Intuitively, each of the firm's $n$ employees quits at rate $\delta$, and is valued on

[4] The reader may wonder whether the firm's optimality conditions also should include terms that capture potential effects of the firm's choice of hires $\mathrm{d}H$ and separations $\mathrm{d}S$ on turnover, via effects on the worker surplus in (9). Note, however, that the terms in $\mathrm{d}H^*$ and $\mathrm{d}S^*$ in (9) capture the present discounted value of the effects of the firm adhering to its *optimal* hiring and separation policy *in the future*.

the margin by the firm at $\Pi_n$. The magnitude of these turnover costs, and the firm's response to them, will play a central role in the model.

Optimal labor demand in (10) is determined by the marginal value of labor to the firm $\Pi_n$. For brevity, in what follows we shall denote the latter by $J \equiv \Pi_n$. Differentiating the firm value in (11) in turn implies that

$$rJ = x\alpha n^{\alpha-1} - \frac{\partial(wn)}{\partial n} - \frac{\partial(\delta nJ)}{\partial n} + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx}, \tag{12}$$

The marginal value of labor to the firm is determined by the flow marginal product $x\alpha n^{\alpha-1}$ net of the marginal cost of labor $\partial(wn)/\partial n$ and the marginal turnover costs $\partial(\delta nJ)/\partial n$, together with the capital gains associated with shocks to the firm's idiosyncratic productivity.

**A proposed solution**. Together, the wage equation (6), the worker surplus (9), the firm's optimality conditions for hires and separations (10), and the firm's marginal value of labor (12) provide a recursive system that jointly determines optimal firm labor demand and optimal worker turnover.

To solve this system, we propose a simplification. As stated, the worker surplus in (9) and the firm's marginal value in (12) require two idiosyncratic state variables: the firm's (endogenous) employment $n$, and the firm's (exogenous) productivity $x$. In what follows, we show how the structure of the problem admits a normalization that allows one to distil these forces into a single idiosyncratic state, namely the firm's flow marginal product, which we shall hereafter denote $m \equiv x\alpha n^{\alpha-1}$. Thus, we propose a solution in which the marginal product $m$ is a sufficient statistic for worker and firm behavior. We gather this together with a regularity condition on the evolution of firm employment in the following definition.

**Definition** *An $m$-*solution *is a solution to (9), (10) and (12) such that, for any aggregate state, the worker surplus $W$, the firm marginal value $J$, and the firm hiring and firing rates, $dH^*/n$ and $dS^*/n$, are uniquely determined by the marginal product $m$.*

In what follows, we verify that a unique $m$-solution exists. With a slight abuse of notation, we shall henceforth write the worker surplus as $W(m)$ and the firm's marginal value as $J(m)$.

**Optimal worker turnover**. Consider first worker turnover. Confronted with an outside offer, workers optimally choose the firm that offers the higher worker surplus. The following result establishes that such decisions take a particularly simple form under the proposed $m$-solution.

**Lemma 1** *Under an $m$-solution, the worker surplus $W(m)$ is monotonically increasing in the marginal product $m$.*

The intuition for this result comes from two channels. First, the wage in (6) is an increasing function of the marginal product. Thus a direct benefit of being employed in a firm with a higher marginal product is a higher flow wage. Second, under the proposed $m$-solution, a higher marginal product in the current period also implies a weakly higher path of future marginal products for any sequence of realizations of idiosyncratic productivity shocks in (1).

The upshot of Lemma 1 for what follows is that optimal turnover decisions take a simple form, as orderings of worker surpluses coincide with orderings of marginal products. Thus, all job-to-job switches involve worker transitions from low-$m$ firms to high-$m$ firms. The marginal product becomes a sufficient statistic for worker turnover. Recall from (4) that the quit rate $\delta$ depends on the offer distribution of worker surpluses $\Phi(W)$. With another slight abuse of notation, it follows that we can rewrite this as

$$\delta(m) = s\lambda[1 - F(m)], \tag{13}$$

where $F(m) = \Phi[W(m)]$ is the offer distribution of marginal products.

**Optimal labor demand**. Now consider the determination of the firm's marginal value of labor. Applying the proposed $m$-solution, and the wage equation (6), the marginal value in (12) can be rewritten in normalized form,

$$\begin{aligned} rJ(m) = (1-\omega_1)m - \omega_0 - [\delta(m) - (1-\alpha)m\delta'(m)]J(m) \\ + [\mu + (1-\alpha)\delta(m)]mJ'(m) + \frac{1}{2}\sigma^2 m^2 J''(m), \end{aligned} \tag{14}$$

where $1 - \omega_1 \equiv (1-\beta)/[1-\beta(1-\alpha)]$ is the firm's share of the marginal product implied by the wage bargaining solution.

The optimality conditions for hires and separations (10) provide boundary conditions for the firm's marginal value in (14). We will show that these are resolved by a labor demand policy with three thresholds for the marginal product, $m_l < m_m < m_u$. Optimal hires and separations are zero whenever the firm's marginal value $J$ lies in the interval $(0, c)$. Because the presence of quits will induce employment to decline over time in this region, we shall refer to it as the *natural wastage region.* The firm will undertake non-zero separations $\mathrm{d}S > 0$ whenever the firm's marginal value $J$ reaches the lower boundary $0$, where the marginal product is $m_l$. Likewise, the firm will undertake non-zero hires $\mathrm{d}H > 0$ as soon as the firm's marginal value $J$ reaches the boundary $c$, where the marginal product is $m_m$.

We shall see, however, that a distinctive implication of the interaction of on-the-job search with firm dynamics is the additional presence of a *hiring region* in which optimal hires $\mathrm{d}H$ are positive for all $m \in (m_m, m_u)$ such that the firm's marginal value $J$ is equal to the marginal hiring cost $c$. That this interval may be nondegenerate is a novel and surprising feature of this environment. It also provides a key solution to the challenge of solving for the equilibrium distributions that, as we have discussed, are fundamental to models of on-the-job search. We now characterize each of these two regions.

**The natural wastage region**. The natural wastage region is the more straightforward of the two. Under the proposed $m$-solution, the lowest-value hiring firm has marginal product $m_m$, which exceeds that for any firm in the natural wastage region where $m \in (m_l, m_m)$. Firms thus face the maximal quit rate $\delta(m) = s\lambda$, and thus $\delta'(m) = 0$, for all $m$ on the interior of this region—hence *natural wastage.* This considerably simplifies the recursion for the firm's marginal value (14) and, together with value-matching and smooth-pasting conditions implied by optimality, admits the following solution.

**Proposition 1** *In the natural wastage region, the firm's marginal value is given by*

$$J(m) = \frac{(1-\omega_1)m}{\rho(1)} - \frac{\omega_0}{\rho(0)} + J_1 m^{\gamma_1} + J_2 m^{\gamma_2}, \tag{15}$$

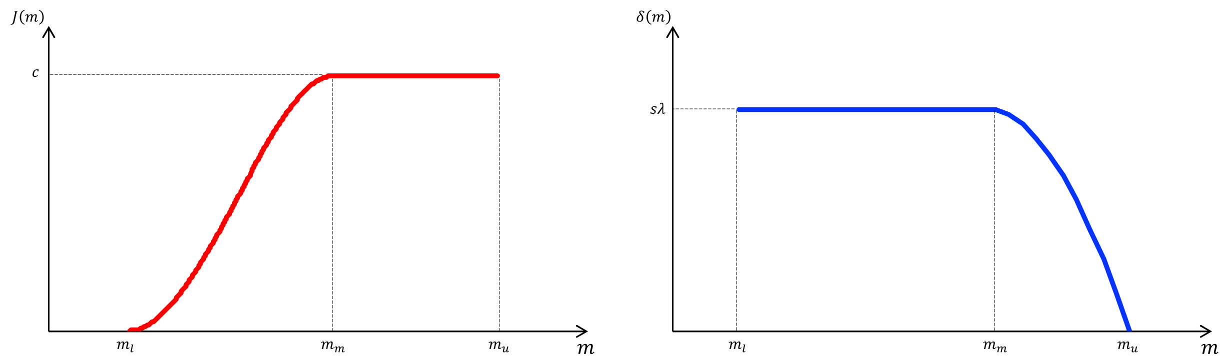
*for all* $m \in (m_l, m_m)$*. The coefficients* $J_1$ *and* $J_2$*, and the boundaries* $m_l$ *and* $m_m$*, are known implicit functions (provided in the appendix) of the parameters of the firm's problem, and* $\gamma_1 < 0$ *and* $\gamma_2 > 1$ *are roots of the fundamental quadratic,*

$$\rho(\gamma) = -\frac{1}{2}\sigma^2\gamma^2 - \left[\mu - \frac{1}{2}\sigma^2 + (1-\alpha)s\lambda\right]\gamma + r + s\lambda = 0. \tag{16}$$

Constancy of the quit rate in the natural wastage region transforms the firm's labor demand decision into a canonical firm dynamics problem. An extension of the approach

Figure 1. Optimal labor demand and the equilibrium quit rate

A. Marginal value of labor $J(m)$ B. Quit rate $\delta(m)$




devised by Abel and Eberly (1996) yields the solution for the firm's marginal value in Proposition 1.

The first two terms in (15) characterize the certainty equivalent value to the firm of a marginal employee. The final two terms in (15) capture, respectively, the value to the firm of the option to separate from employees in adverse future states, and the value of the option to hire employees in favorable future states. In combination, these forces give rise to a marginal value to the firm that is shaped like a slide in the natural wastage region, a shape that is characteristic of firm dynamics models with constant depreciation and infinitesimal control (Dixit 1993). Figure 1 illustrates.

Optimal labor demand in the natural wastage region thus corresponds closely to that in existing models of firm dynamics. We will see, however, that firm behavior differs importantly from this benchmark in the hiring region, to which we now turn.

**The hiring region and the equilibrium quit rate**. Recall that a distinctive feature of the interaction of firm dynamics and on-the-job search is that labor demand and turnover are jointly determined among hiring firms. Formally, we seek solution for the firm's marginal value $J(m)$ and the quit rate $\delta(m)$ that are mutually consistent in the hiring region.

The model offers a considerable simplification, however. The optimality condition for hiring in (10) stipulates that a hiring firm's marginal surplus be equal to the marginal hiring cost. Thus, $J(m) = c$, and $J'(m) = J''(m) = 0$, for all $m$ in the hiring region $(m_m, m_u)$. This observation transforms the recursion for the firm's marginal value in (14) into a differential equation for the quit rate $\delta(m)$. This in turn gives rise to a simple solution.

**Proposition 2** *In the hiring region, the quit rate is given by*

$$\delta(m) = s\lambda + \frac{1}{c}\left\{\frac{(1-\omega_1)(m-m_m)}{\alpha} - \left[\frac{(1-\omega_1)m_m}{\alpha} - \omega_0 - (r+s\lambda)c\right]\left[\left(\frac{m}{m_m}\right)^{\frac{1}{1-\alpha}} - 1\right]\right\}, \tag{17}$$

*for all* $m \in (m_m, m_u)$*, where the upper boundary solves* $\delta(m_u) = 0$ *and is unique. Furthermore,* $\delta(m)$ *is strictly decreasing and concave for all* $m \in (m_m, m_u)$*.*

Proposition 2 is an important result. By establishing the equilibrium quit rate $\delta(m)$, it in turn implies a solution for the equilibrium offer distribution of marginal products, $F(m)$ in (13). Proposition 2 thus provides a key part of the solution to the challenge of how to determine equilibrium turnover, and thereby the equilibrium distributions of marginal products, in this environment. We will see in later sections that this in turn provides a key building block to the determination of steady-state aggregate equilibrium, as well as out-of-steady-state aggregate dynamics.

Proposition 2 also has a surprising implication: Hiring firms that face a *homogeneous* per-worker hiring cost $c$ nonetheless allow their marginal products to *vary* over an interval, giving rise to a *non-degenerate* distribution of worker values across hiring firms. Proposition 2 reveals that this surprising property is fundamentally linked to the interaction of on-the-job search with firm dynamics. It is straightforward to verify from (17) that eliminating on-the-job search ($s \to 0$), or a notion of firm size ($\alpha \to 1$), implies that the hiring region collapses to a point, $m_u \to m_m$.

The intuition for why is as follows. Consider a firm at the middle boundary $m_m$. Following a positive innovation to its productivity $x$, and thereby its marginal product $m$, the firm faces a tension in the presence of on-the-job search. On the one hand, it is optimal for the firm to hire whenever $m$ rises above $m_m$, a force which lowers the marginal product back toward the middle boundary. On the other hand, a higher marginal product is valuable to the firm as it reduces turnover costs. This tension is resolved by the firm diffusing its hires across an interval of marginal products $(m_m, m_u)$, a policy which in turn is supported by the quit rate in (17). Furthermore, because $\delta(m)$ is declining throughout the hiring region, the implied offer distribution $F(m)$ is rising in $m$. By Lemma 1, it follows that the quit rate in (17) is consistent with optimal worker turnover.

## 1.4 Discussion

**The role of wage setting: Offer matching**. Thus far, we have characterized optimal labor demand and turnover for a case in which wages are bargained *ex post* and firms do not engage in offer matching. Recall that potential justifications for the latter include lack of verifiability of job offers, and the presence of equal treatment constraints across workers within a firm.

We now consider an alternative wage determination protocol that accommodates some degree of offer matching via a generalization of the sequential auctions approach of Postel-Vinay and Robin (2002). As in their model, firms are assumed to have all the bargaining power. In a simple extension of their model, we allow for a variable propensity for offer matching among competing firms, indexed by a parameter $\xi$. Mirroring the preceding discussion, one interpretation of $\xi$ is the probability that both firms are credibly informed over the presence of both job offers (with $1-\xi$ the probability neither firm is informed). An alternative interpretation is that the firm and its workers will tolerate unequal treatment up to some limit, expressed for convenience as a fraction $\xi$ of the firm's marginal value of labor. We will see that these interpretations are analytically equivalent. Under either interpretation, the special case of $\xi = 1$ then corresponds to the model of Postel-Vinay and Robin.

To map the worker and firm values implied by this protocol to a path of flow wage payments, firms are assumed to be able to commit to payments to workers only in the current $\mathrm{d}t$ period (as in Moscarini 2005). This aids comparability of this case with the preceding sections, and simplifies the contract structure as we will see that workers within a firm are almost always paid the same flow wage.

The resultant equilibrium then takes a simple form. Consider a worker employed in a firm with marginal value $\Pi_n$. Upon realization of an outside offer from a firm with marginal value $\widetilde{\Pi}_n$, the worker chooses the firm with the higher marginal value. If she quits from her current firm (at rate $\delta$), she receives (in expectation) a lump-sum recruitment bonus equal to $\xi\Pi_n$. If she stays with her current firm (at rate $s\lambda-\delta$), she receives (in expectation) a lump-sum retention bonus equal to $\xi\widetilde{\Pi}_n$. In the absence of an outside offer, the worker receives a flow wage payment such that she is indifferent to unemployment and the worker surplus is zero, $W = 0$. The option value to search while unemployed is thus also zero, and the firm's hiring and firing behavior has no effect on worker values.

Applying similar arguments to those underlying (9) and (11) above,[5] we can write the firm and worker values implied by this environment as follows:

$$r\Pi = xn^{\alpha} - wn - \delta n\Pi_n - (s\lambda - \delta)n\xi\mathbb{E}\left[\tilde{\Pi}_n \middle| \tilde{\Pi}_n < \Pi_n\right] + \mu x\Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx}, \text{ and}$$
$$rW = w - b + \delta\xi\Pi_n + (s\lambda - \delta)\xi\mathbb{E}\left[\tilde{\Pi}_n \middle| \tilde{\Pi}_n < \Pi_n\right] - \delta n W_n + \mu x W_x + \frac{1}{2}\sigma^2 x^2 W_{xx}. \tag{18}$$

The flow wage paid in the absence of outside offers solves $W = 0$, and takes the form

$$w = b - \delta\xi\Pi_n - (s\lambda - \delta)\xi\mathbb{E}\left[\tilde{\Pi}_n \middle| \tilde{\Pi}_n < \Pi_n\right]. \tag{19}$$

The wage is equal to the flow payoff from unemployment $b$ less the expected capital gains from recruitment and retention bonuses associated with future outside offers. Inserting the wage solution into the firm's value yields the simple result

$$r\Pi = xn^{\alpha} - bn - (1 - \xi)\delta n\Pi_n + \mu x\Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx}. \tag{20}$$

Equation (20) yields an important insight. Recall that the key channel through which on-the-job search interacts with firm decisions is through turnover costs; these are now given by $(1 - \xi)\delta n\Pi_n$. The upshot of (20), then, is that the presence of offer matching implicitly reduces the turnover costs faced by the firm, and does so in proportion to the firm's propensity to match offers, $\xi$. The intuition stems from the wage equation (19). The prospect of future recruitment and retention bonuses leads the worker to accept lower flow wages. The firm implicitly recoups the entirety of the cost of its retention bonuses in this way. To the extent that firms' propensity to match offers is incomplete ($\xi < 1$), the wage reductions implied by prospective recruitment bonuses only partially offset the firm's turnover costs. Thus, through the degree of offer matching, the nature of wage setting plays an important role in shaping the effective costs of turnover to the firm, and thereby the nature of labor market equilibrium.

In the limit case of complete offer matching ($\xi = 1$), the firm recoups its turnover costs entirely, and thereby becomes *indifferent* to turnover. This outcome suggests further intuition. Absent an ability to match offers, a firm faces a quandary in the presence of on-the-job search: it has one instrument—the marginal product $m$—to respond to a *continuum* of outside offers. As $\xi$ approaches one, the firm is able to tailor its recruitment

[5] Note that the effective cost of hiring will now include both the base hiring cost as well as the recruitment bonuses firms expect to pay. By the same logic as before, these will cancel from the firm's maximized value by optimality. It is important in this case that the base hiring cost is incurred prior to meeting a searcher.

and retention strategy to the idiosyncratic circumstances of all of its contacted employees. In this way, the ability to engage in offer matching has a nonlinear pricing interpretation. By the same token, absent an unconstrained ability to set a continuum of such nonlinear prices, the firm will face costs associated with turnover, and the insights of the preceding sections will apply.

Indeed, the qualitative resemblance between the firm's problem with offer matching (20) and its counterpart with *ex post* wage bargaining and no offer matching (11) makes it clear that optimal labor demand and equilibrium turnover will have the same qualitative form. The following Lemma confirms this for a case analogous[6] to that in Proposition 2.

**Lemma 2** *In the preceding model of (partial) offer matching, Proposition 1 holds* mutatis mutandis *with* $\omega_0 = b$, $\omega_1 = 0$, *and* $s\lambda$ *exchanged with* $(1-\xi)s\lambda$. *Furthermore, the quit rate in the hiring region becomes*

$$\delta(m) = s\lambda + \frac{1}{(1-\xi)c}\left\{\frac{m - m_m}{\alpha} - \left(\frac{m_m}{\alpha} - b - [r + (1-\xi)s\lambda]c\right)\left[\left(\frac{m}{m_m}\right)^{\frac{1}{1-\alpha}} - 1\right]\right\}. \quad (21)$$

The degree of offer matching is thus a further channel that shapes the presence of a hiring region, in addition to the extent of on-the-job search (mediated through $s$), and a notion of firm size ($\alpha < 1$). As anticipated by the preceding logic, the hiring region collapses to a point, $m_u \to m_m$, in the limit case of complete offer matching ($\xi = 1$). Interestingly, a novel implication of the disappearance of a hiring region in this case is that optimal labor demand will share the same qualitative analytical properties as a model without on-the-job search (similar to that in Elsby and Michaels 2013).

More generally, though, the message of Lemma 2 is that this framework can accommodate a wide class of wage setting protocols which in turn play an important role in the nature of labor market equilibrium.

**The role of the structure of frictions**. The preceding analysis has maintained a particular assumption on the structure of frictions in the labor market—specifically, that they take the form of a constant per-worker hiring cost $c$. This is a compelling baseline case to study because, as we have seen, it yields the stark prediction of heterogeneity in worker values across hiring firms, *despite* a lack of heterogeneity in hiring costs.

[6] Specifically, for comparability with the preceding analysis, we assume that the effective hiring cost is a constant, equal to $c$ (for example, by appropriate scaling of a pure vacancy cost).

The model can accommodate deviations from this baseline case, however. One natural example is the case of a *training cost*. Suppose that the cost of training a flow $\mathrm{d}H$ of new hires is captured by the lost output of $\tau \cdot \mathrm{d}H$ existing employees of the firm. Because the flow of new hires is small relative to the stock of employees, the effective per-worker hiring cost is thus equal to $\tau \cdot m$. The qualitative properties of labor demand and turnover implied by this alternative structure of frictions will resemble those in the preceding results. Its quantitative implications, however, will differ, and in interesting ways summarized by the following Lemma.

**Lemma 3** *In the preceding model of training costs, Proposition 1 holds up to a change in coefficients. Furthermore, the quit rate in the hiring region becomes*

$$\delta(m) = s\lambda + \frac{1}{\tau m_m}\left\{\omega_0\left(1 - \frac{m_m}{m}\right) - \left[\frac{1 - \omega_1 - (r - \mu + \alpha s\lambda)\tau}{\alpha} m_m - \omega_0\right]\left[\left(\frac{m}{m_m}\right)^{\frac{\alpha}{1-\alpha}} - 1\right]\right\}. \tag{22}$$

Although the nonlinear nature of the model makes comparisons difficult, notice that the presence of training costs has the effect of reducing the power in the final term of the quit rate—the term that dominates its decline as $m$ rises. Intuitively, training costs imply that hiring becomes costlier as firms become more productive on the margin. The result is that, relative to the case of a constant per-worker hiring cost, firms become even less inclined to hire aggressively in response to positive innovations to productivity in the hiring region. Consequently, there is a force toward widening the hiring region.

Clearly, the framework can accommodate a broader class of alternative structures of frictions—for example, a hybrid of constant per-worker hiring costs and training costs—provided that the requisite normalization that underlies the proposed $m$-solution is preserved. We return to this point in section 4, where we show how the model can be extended to accommodate convex hiring costs.

# 2. Aggregation and steady-state equilibrium

In this section, we take on the task of inferring the implications of the preceding microeconomic structure for equilibrium labor market dynamics. An important first step toward this end is to aggregate individual firm and worker behavior for a given aggregate state. Notice that the latter is complicated in this environment by the fact that optimal labor demand and turnover are nonlinear in the source of heterogeneity in the economy, firms' idiosyncratic productivity $x$. Given a solution to this aggregation problem, a further task is to characterize the conditions for aggregate steady-state labor market equilibrium.

We make two further assumptions to aid these steps. First, we restrict the drift of the stochastic process for idiosyncratic shocks in (1) to ensure that aggregate labor demand is stationary. This obtains when frictionless employment, which is proportional to $x^{1/(1-\alpha)}$, has no drift. Applying Ito's lemma, this requires that

$$\mu + \frac{1}{2}\frac{\alpha}{1-\alpha}\sigma^2 = 0. \tag{23}$$

This assumption is made purely to simplify the analysis by abstracting from growth.

Second, toward the end of deriving aggregate labor market equilibrium, we endogenize the job offer arrival rate $\lambda$. We do so by invoking a conventional, constant-returns-to-scale matching function $M(U + s(L-U), V)$ that regulates the total flow of contacts $M$ arising from $V$ vacancies, $U$ unemployed searchers, and $s(L-U)$ effective employed searchers. An implication is that the ratio of vacancies to searchers $\theta \equiv V/[U + s(L-U)]$ is a sufficient statistic for contact rates: Workers contact a vacancy at rate $\lambda(\theta) = M/[U + s(L-U)] = M(1,\theta)$ while unemployed, and at rate $s\lambda(\theta)$ while employed.

A further implication of the matching process is that hires are in turn mediated through vacancies: Each vacancy contacts a searcher at rate $\chi(\theta) = M/V = M(1/\theta, 1)$. With probability $\psi = U/[U + s(L-U)]$ the searcher is unemployed and therefore hired with certainty. With probability $1-\psi$, the searcher is employed, and is hired only if the worker surplus $W$—or, equivalently, the marginal product $m$—associated with the vacancy exceeds that of the employed searcher at her current firm. Denoting the distribution of marginal products among *employees* by $G(\cdot)$, the vacancy-filling rate faced by the firm can thus be written as

$$q(m) = \chi[\psi + (1-\psi)G(m)]. \tag{24}$$

Under this interpretation, then, the flow of hires $\mathrm{d}H$ in (3) can be written as the vacancy-filling rate $q(m)$ multiplied by the firm's flow of vacancies.

## 2.1 Aggregation

We are now in a position to infer the steady-state aggregate labor market stocks and flows implied by the model. These are summarized by solutions for the separation rate into unemployment (denoted $\varsigma$), the hiring rate (denoted $\eta$), and the density of the stock of employees $g$, at each marginal product $m$.

**Proposition 3** *In steady state, (i) the separation rate into unemployment is given by*

$$\varsigma = \frac{\sigma^2/2}{1-\alpha} m_l g(m_l). \tag{25}$$

*(ii) The hiring rate is given by*

$$\eta(m) = -\frac{\sigma^2/2}{1-\alpha} \frac{m\delta'(m)}{\delta(m)}. \tag{26}$$

*(iii) The vacancy-filling rate is given by*

$$q(m) = \chi\psi \exp\left[\frac{1-\alpha}{\sigma^2/2} \int_{m_l}^{m} \frac{\delta(\widetilde{m})}{\widetilde{m}} \mathrm{d}\widetilde{m}\right], \tag{27}$$

*which, using (24), yields the worker distribution* $G(m)$*.*

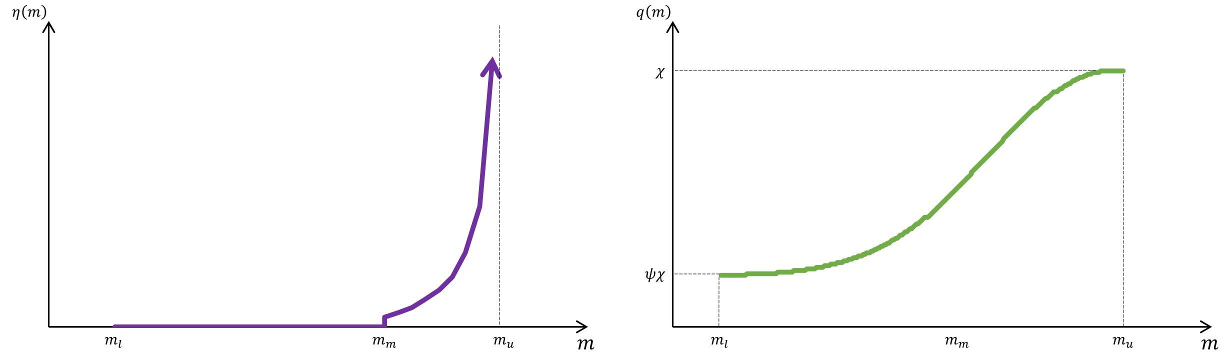
We now explain the intuition behind each element of Proposition 3. The most standard is the solution for the separation rate into unemployment $\varsigma$. Given the structure of optimal labor demand, all such separations arise at the lower boundary for the marginal product $m_l$. There, a density of $g(m_l)$ employees receives shocks of instantaneous variance $\sigma^2$ to their log marginal product. Following negative shocks, employees are shed into unemployment until the marginal product is replenished, with more employees shed the greater is the elasticity of labor demand, $1/(1-\alpha)$.

The remaining results in Proposition 3 are novel features of this environment, however. We explain these in more detail in what follows.

**Hiring, mean reversion and labor market competition**. Consider first the solution for the hiring rate at each marginal product. Proposition 3 reveals that, in steady state, $\eta(m)$ is proportional to minus the elasticity of the quit rate. Equivalently, it is proportional to the hazard function of the offer distribution of log marginal products, $mf(m)/[1-F(m)]$. This interpretation in turn reveals some useful intuition. A firm's ability to hire is determined by the intensity of offers at $m$, $f(m)$, *relative to* the intensity of offers at *higher* $m$s, as captured by $1-F(m)$. The intensity of offers at *lower* $m$s is not directly relevant, since all such offers are dominated by those issued by firms at $m$.

Figure 2. Steady-state hiring and vacancy-filling rates

A. Hiring rate $\eta(m)$ B. Vacancy-filling rate $q(m)$




In combination with Proposition 2, this result yields further intuitive insights. First, and most simply, since the quit rate is a constant (equal to $s\lambda$) in the natural wastage region, (26) confirms that the hiring rate is zero for $m < m_m$. Second, because $\delta(m)$ is strictly decreasing and concave in $m$ in the hiring region, it follows that the hiring rate $\eta(m)$ is strictly positive and increasing in $m$ for $m > m_m$. Third, since the quit rate is equal to zero at the upper boundary, a further implication is that the hiring rate asymptotes to infinity at $m_u$.

These properties of the hiring rate yield further insights on the behavior of the labor market. A key implication is that the marginal product $m$ becomes endogenously *mean reverting* in the presence of on-the-job search. To see how, note that the stochastic law of motion for $m$ in the hiring region takes the form

$$\mathrm{d}m = \{\mu - (1-\alpha)[\eta(m) - \delta(m)]\}m\mathrm{d}t + \sigma m\mathrm{d}z. \tag{28}$$

Positive innovations to the marginal product $m$ induce increases in the hiring rate $\eta(m)$, and declines in the quit rate $\delta(m)$, such that the firm accumulates more employees, and the marginal product reverts back down in expectation. This is a distinctive consequence of the interaction of on-the-job search with firm dynamics. As we have emphasized, absent on-the-job search, the hiring region is degenerate on $m_m$.

This observation in turn suggests further intuition for the behavior of the labor market. One interpretation of mean reversion in the marginal product is that it is a manifestation of *imperfect labor market competition*. Perfect competition would induce infinite mean reversion in marginal products—the law of one wage (or marginal product) restored. The interaction of on-the-job search and labor market frictions weakens this mean reversion, and thereby labor market competition. Thus, one interpretation of the nondegenerate interval of marginal products among hiring firms uncovered in Proposition 2 is that it is a manifestation of imperfect labor market competition.

Viewed from this perspective, the analysis of the preceding sections further highlights the economic forces that shape the competitiveness of the labor market in the model. Lemma 2 underscores the role of wage determination. Quite intuitively, the greater the propensity of offer matching (indexed by $\xi$ in Lemma 2), the smaller the hiring region, the greater the degree of mean reversion in marginal products, and the greater the degree of labor market competition. The nonlinear pricing interpretation of offer matching is also intuitive: To the extent that the firm can tailor wages to the idiosyncratic outside offers of its workers, competitive outcomes can be achieved.

Lemma 3, by contrast, emphasizes the role of the structure of frictions. Again, the implication is intuitive: In the presence of training costs, positive shocks raise the opportunity cost in terms of lost output of training new recruits, and so there is a force that pushes firms to hire less aggressively in response to such shocks. To the extent that this force dominates, mean reversion in the marginal product, and hence labor market competition, weaken.

A final implication of endogenous mean reversion is that it also shapes the steady-state distribution of employees $G(m)$ or, equivalently, the vacancy-filling rate $q(m)$. Absent such mean-reversion, a canonical implication of geometric Brownian motion is that it induces stationary distributions that obey a power law. And, indeed, it can be verified that this holds in the natural wastage region, where constancy of the quit rate implies that the marginal product evolves according to the geometric Brownian motion,

$$\mathrm{d}m = [\mu + (1-\alpha)s\lambda]m\mathrm{d}t + \sigma m\mathrm{d}z, \tag{29}$$

and, from (24) and (27), the worker distribution takes the form

$$G(m) = \frac{\psi}{1-\psi}\left[\left(\frac{m}{m_l}\right)^{\frac{1-\alpha}{\sigma^2/2}s\lambda} - 1\right], \text{ for all } m \in (m_l, m_m). \tag{30}$$

By contrast, the presence of mean reversion in the hiring region thins the tail of the steady-state worker distribution in that region. Formally, because the quit rate is strictly declining in $m$ in the hiring region, it follows from Proposition 3 that the vacancy-filling rate $q(m)$ in (27), and thereby the worker distribution $G(m)$, rise ever more slowly in $m$ relative to the power law in the natural wastage region (30). In the limit, because the quit rate is zero (and the hiring rate explodes) at $m_u$, mean reversion is so extreme at that point that there can in fact be no density of employees at the upper boundary.

**On-the-job search and misallocation**. A key feature of the aggregation results in Proposition 3 is the presence of dispersion in marginal products across workers, as summarized by $G(m)$, and thereby the presence of *misallocation.* A natural intuition suggests that on-the-job search might alleviate such misallocation, by allowing employees to transition faster to more productive jobs. Paradoxically, it turns out that the preceding model cautions against this intuition. The following Lemma provides a stark example of this paradox.

**Lemma 4** *Suppose there are no idiosyncratic shocks, $\mu = \sigma = 0$, and separations into unemployment occur at exogenous rate $\varsigma$. Then, (i) the hiring region and quit rate in Proposition 2 hold* mutatis mutandis *with $r$ exchanged with $r + \varsigma$; (ii) the boundary $m_m$ is such that $(1 - \omega_1)m_m - \omega_0 = (r + \varsigma + s\lambda)c$ and $\delta'(m_m) = 0$; (iii) the natural wastage region is never entered; and (iv) the vacancy-filling rate takes the form*

$$q(m) = \frac{\chi\varsigma}{\varsigma + \delta(m)}. \tag{31}$$

Lemma 4 reveals that the hiring region induced by the interaction of firm dynamics and on-the-job search is present even in the absence of idiosyncratic shocks and endogenous job destruction. Importantly, by extension of Proposition 2, this hiring region, and the accompanying dispersion in marginal products, would not emerge in the absence of on-the-job search ($s = 0$). A striking implication, then, is that on-the-job search in fact *gives rise to* equilibrium misallocation in this case. We argue in what follows that Lemma 4 presents a stark point of contrast to existing canonical models of on-the-job search.

On one hand, the models have much in common. The hiring region shares interesting parallels with a large literature inspired by Burdett and Mortensen (1998). These models emphasize the interplay of *ex ante* wage posting and firms' turnover concerns in generating "residual" *wage* dispersion among identical workers. By contrast, in our model the interplay between *ex post* wage bargaining and firms' turnover concerns gives rise instead to "residual" dispersion in *marginal products*, and thereby in wages. Both results can be traced to notions of imperfect labor market competition associated with on-the-job search and labor market frictions, as well as to the nature of wage setting. In Burdett and Mortensen (1998) and its descendants, wage dispersion is a consequence of firms having to commit *ex ante* to wage payments that cannot respond to workers' future outside offers. In the model of the preceding sections, marginal product dispersion is a consequence of firms being unable to tailor wages perfectly to the idiosyncratic outside offers of each of their workers.

Further aspects of the special case in Lemma 4 also mirror the implications of canonical models of on-the-job search with wage posting. In particular, given the offer distribution $F(m)$ that emerges from the hiring region, equations (13), (24) and (31) confirm that the worker distribution in this special case takes the form

$$G(m) = \frac{\psi}{1-\psi}\frac{s\lambda F(m)}{\varsigma + s\lambda[1-F(m)]}. \tag{32}$$

This outcome is again reminiscent of Burdett and Mortensen (1998): The model gives rise to a job ladder whereby workers move towards higher-wage, more productive firms. And the worker distribution $G(m)$ that emerges from this process mirrors the form in Burdett and Mortensen (1998).

On the other hand, a crucial message of Lemma 4 is that these models have fundamentally different implications for misallocation. Models of wage posting in the mold of Burdett and Mortensen (1998) invariably invoke linear technologies. When extended to accommodate productive heterogeneity (for example, as in Bontemps et al. 2000), an extreme implication is that allocative efficiency requires all workers to be employed in the most-productive firm. A corollary is that on-the-job search is a force toward *resolution* of misallocation in these models, since it accelerates worker transitions toward more productive firms.

The paradox of Lemma 4 is that this last implication is turned on its head. In Lemma 4, heterogeneity in marginal products emerges as an *equilibrium outcome*, rather than by assumption. And the presence of on-the-job search is the primitive force that *gives rise to* equilibrium misallocation, rather than solely being an equilibrium response to it.

The key difference is the presence of diminishing returns. This provides an economic margin by which differences in firm marginal productivity can be resolved. Indeed, in the absence of on-the-job search, marginal products are equalized: $s = 0$ implies $m_u = m_m$. Instead, in the presence of on-the-job search, firms allow their marginal products to vary as a means to manage turnover, generating equilibrium misallocation. Thus, integrating the allocative consequences of firms' desire to manage turnover with neoclassical forces that militate toward equality of marginal products greatly alters the economic role of on-the-job search in misallocation.

Lemma 4 makes this point particularly starkly, by abstracting from idiosyncratic shocks and endogenous job destruction. Returning to the general case, though, Proposition 2 implies that on-the-job search will at least give rise to greater misallocation *among hiring firms*. It follows that there must be configurations of the parameters of the model such that this effect dominates, and on-the-job search can raise misallocation *overall*. Lemma 4 describes one particularly instructive such example.

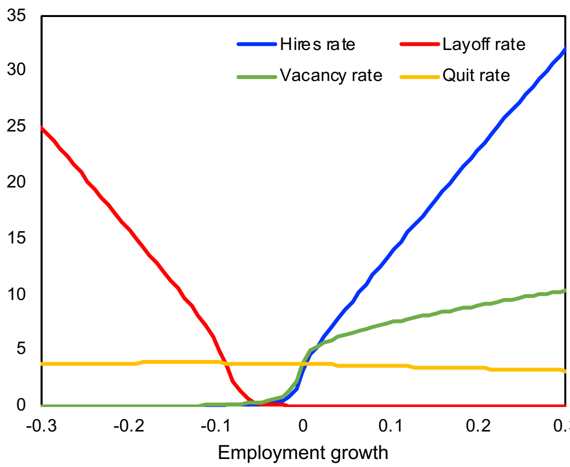
**Worker flows, job flows and vacancy yields**. The aggregation results in Proposition 3 also provide a novel perspective on the relationship between worker flows and job flows. In a pair of influential papers, Davis, Faberman and Haltiwanger (2012, 2013) document a set of stylized facts on the relationships between gross flows and net employment growth at the establishment level. Gross layoff rates rise in tandem with job destruction in shrinking establishments, and flatten out at a minimal level in growing establishments. Symmetrically, gross hiring rates display a tight positive link to job creation, and are minimal and roughly invariant in the presence of job destruction.

As Davis et al. note, these so-called "hockey stick" relations for gross hires and layoffs are a natural outcome of firm dynamics models in which hires and separations are governed by an "iron link" between employment growth in an establishment and its gross worker flows. But they also note two stark empirical deviations from such an iron link. The first, noted in Davis et al. (2012), is that the presence of quits drives a wedge between job flows and gross worker flows, and that these quits vary negatively with establishment growth. The second, noted in Davis et al. (2013), is that vacancy-filling rates vary positively with establishment growth, thereby driving a wedge between gross hires and vacancies.

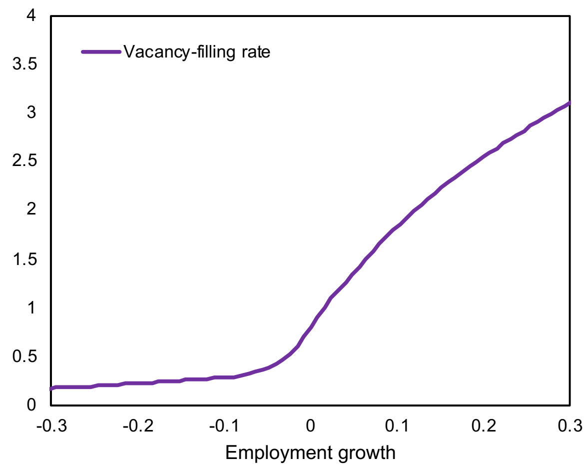
An important implication of Proposition 3 is that the model of the preceding sections accommodates all of these stylized facts, in particular those that deviate from an iron link between worker flows and job flows. The key observation is that (away from the lower and upper boundaries) the marginal product $m$ is a sufficient statistic for a firm's net employment growth $\eta(m) - \delta(m)$ in the model: Higher marginal products are associated with faster firm growth. It is then immediate from Propositions 2 and 3 that expanding firms will face lower quit rates, and higher vacancy-filling rates, as documented by Davis et al.

Figure 3. Gross worker and job flows implied by the model

A. Hires, layoffs, quits and vacancies




B. Vacancy-filling rate




Notes. Based on the model calibrated as described in section 3 simulated over one month, applying the Davis et al. and JOLTS survey methodologies to model-generated data.

To illustrate this feature, Figure 3 reports the results of applying the methods of Davis et al. to data simulated from the model. Mirroring the JOLTS methodology, vacancies in the model are measured at a point in time and are scaled to match an aggregate vacancy rate of 3 percent. Hires, layoffs and quits are cumulated over the subsequent month. The model is otherwise parameterized according to the calibration described later in section 3.

Figure 3 reveals that the hockey sticks implied by the model qualitatively resemble those documented by Davis et al. This contrasts interestingly with recent work by Kaas and Kircher (2015). In a model without on-the-job search, they instead invoke the presence of convex vacancy costs and directed search to explain the same patterns. Vacancies and wages are imperfect substitutes in recruiting. Growing firms thus use increased wage offers to attract workers, and the vacancy-filling rate rises with firm growth. By contrast, the present model suggests that the interaction of on-the-job search with firm dynamics can generate similar patterns, without invoking convexity in the hiring technology.

In terms of magnitudes, the outcomes in Figure 3 differ from the empirical results of Davis et al. (2013) in two related dimensions. First, the rise in the vacancy-filling rate with firm growth is around half as steep as its empirical analogue. Second, the rise in the vacancy rate with firm growth is steeper than in the data. The latter goes some way to explaining the former: *Ceteris paribus*, a shallower rise in the vacancy rate would induce the vacancy-filling rate to rise more steeply in firm growth.

The reason for this discrepancy is that, in the model, all vacancies are assigned to hiring firms that, in turn, are unlikely to shrink substantially. In the data, however, a nontrivial fraction of aggregate vacancies is accounted for by establishments that are shrinking, often at substantial rates. This may reflect a form of replacement hiring, or mismeasurement of vacancies, neither of which is captured by the model. For a fixed aggregate vacancy rate, assigning more vacancies to shrinking establishments would lower the gradient of the vacancy rate in firm growth, and thereby increase the gradient of the vacancy yield toward its empirical counterpart.

## 2.2 Steady-state equilibrium

We now complete the model by characterizing its steady-state equilibrium. Toward that end, note that the matching structure implies that all endogenous outcomes of the model described thus far—the marginal value in Proposition 1, the quit rate in Proposition 2, and the separation, hiring and vacancy-filling rates in Proposition 3—depend on a single endogenous aggregate state, labor market tightness $\theta$, which determines the contact rates $\lambda(\theta)$ and $\chi(\theta)$. Given this, we can summarize the steady state in terms of two equilibrium conditions reminiscent of those that characterize the standard search and matching model of Mortensen and Pissarides (1994).

The first emerges from the law of motion that governs the evolution of unemployment. Making explicit the dependence of the separation rate on tightness, we can write this as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \varsigma(\theta)(L - U) - \lambda(\theta)U. \tag{33}$$

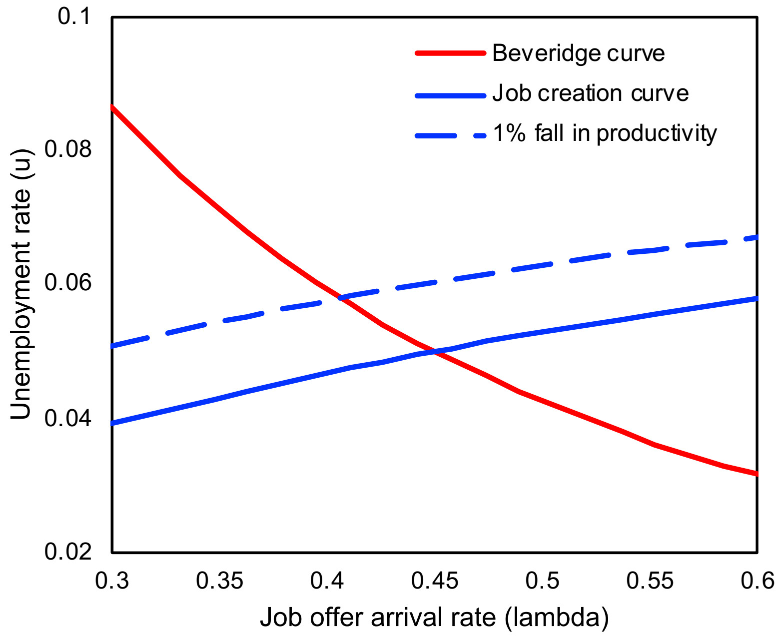
In steady state, aggregate unemployment is stationary, and thus we obtain the Beveridge curve condition,

$$U_{BC}(\theta) = \frac{\varsigma(\theta)}{\varsigma(\theta) + \lambda(\theta)} L. \tag{34}$$

The second steady-state condition is implied by aggregation of firms' labor demand. Aggregate employment is the mean of employment across firms, $N = \mathbb{E}\left[(\alpha x/m)^{1/(1-\alpha)}\right]$. Observing that the latter is equal to the ratio of the mean of $(\alpha x)^{1/(1-\alpha)}$ across firms and the employment-weighted mean of $m^{1/(1-\alpha)}$ gives rise to the job creation condition,

$$U_{JC}(\theta) = L - \left\{\mathbb{E}\left[(\alpha x)^{\frac{1}{1-\alpha}}\right] \Big/ \int m^{\frac{1}{1-\alpha}} g(m;\theta)\mathrm{d}m\right\}. \tag{35}$$

Figure 4. Steady-state equilibrium and comparative statics




Notes. Based on simulation of the model calibrated as described in Table 1. The figure illustrates the steady-state response to a one-percent decline in aggregate labor productivity.

Steady-state equilibrium unemployment and labor market tightness are then jointly determined by (34) and (35). Figure 4 illustrates the steady-state job creation and Beveridge curve, and depicts the upward shift of the job creation curve induced by a decline in aggregate labor productivity. Specifically, it plots the effect of modifying the production function to $pxn^{\alpha}$, such that $p$ falls by one percent.

# 3. Aggregate dynamics

The analysis thus far has addressed the first part of the analytical challenge posed by the interaction of firm dynamics with on-the-job search—namely that of inferring the steady-state fixed points of the offer and worker distributions, $F(m)$ and $G(m)$. We now show how these results also inform the solution to the second part of the analytical challenge—that of inferring out of steady state equilibrium dynamics, which involves a fixed point in the dynamic path of the distributions.

The key insight is that the form of the quit rate in Proposition 2 also will hold out of steady state, subject to the modification that the middle boundary $m_m$ and the job offer arrival rate $\lambda$ will vary over time. The intuition is simple. Out of steady state dynamics give rise to additional capital gains in the firm's marginal value relative to its steady-state form in (14). Optimality in the hiring region, however, requires that the firm's marginal value of labor is a constant, equal to the marginal hiring cost $c$. It follows that any such out-of-steady-state capital gains are zero in the hiring region, and that the quit rate shares the same form as in Proposition 2. This is a considerable simplification, as the solution for the dynamic path of the quit rate—or, equivalently, the offer distribution $F(m)$—is thus known up to the path of two scalars, $m_m$ and $\lambda$, a much simpler prospect.

This in turn aids the solution for the time path of the worker distribution $G(m)$. In the same way that the quit rate informs the steady-state vacancy-filling rate in (27), and thereby the steady-state worker distribution, its time path induces the dynamics of $G(m)$ via the out-of-steady-state Fokker-Planck (Kolmogorov Forward) Equation. Thus, the dynamic path of the two scalars $m_m$ and $\lambda$ also implies the time path of the worker distribution $G(m)$.

Finally, consider first the natural wastage region. Here, the quit rate is maximal and equal to $s\lambda$. The job offer arrival rate $\lambda$ is thus the sole aggregate state in this region. Given a time path for $\lambda$, the firm's marginal value $J(m)$, and the boundaries $m_l$ and $m_m$, can then be inferred out of steady state. This implies a further simplification: the path of $\lambda$ is also sufficient to determine the paths of $m_l$ and $m_m$.

The upshot is that the dimensionality of the problem of inferring the model's transition dynamics is greatly reduced by the analytical results developed earlier in the paper. Absent these results, solving the model out of steady state would involve forecasts of a *sequence of unknown functions*, a daunting prospect. With these results, we can distil the problem to one which requires a forecast of the dynamic path of just one scalar, $\lambda$.

## 3.1 Quantitative illustration

To demonstrate the feasibility of the latter, we now illustrate the quantitative implications of a calibrated version of the model. In what follows, we provide an account of the empirical moments that are most relevant for the calibration of each parameter. Of course, in cases for which direct calibration is infeasible, in principle all target moments inform all parameters.

We begin with a normalization. Note that, in the limit in which the hiring cost $c$ is zero, optimal labor demand implies that marginal products are equalized across firms at a level $m^* \equiv \omega_0/(1-\omega_1)$. It follows from (6) that there is a common wage in this case equal to $w^* \equiv \omega_0/(1-\beta)$. We normalize $w^* \equiv 1$ or, equivalently, $\omega_0 \equiv 1-\beta$. It follows that all flow parameters are thus expressed in terms of monthly frictionless wages.

We begin by setting the discount rate $r$ to replicate an annual real interest rate of 5 percent. We then set the curvature of the production function $\alpha$ to equal 0.64 based on the estimates of Cooper, Haltiwanger and Willis (2007, 2015).

Idiosyncratic shocks $x$ in the model drive changes in firms' desired labor demand. Accordingly, we choose the standard deviation of idiosyncratic shocks $\sigma$ such that, when simulated over a quarter, the model replicates the empirical standard deviation of quarterly employment growth of 0.287.[7]

The monthly job offer arrival rate for unemployed searchers is set to 0.45 to mirror empirical unemployment outflow rates, as in Shimer (2005). Relatedly, the search intensity of employed searchers $s$ is chosen such that the model replicates a monthly job-to-job transition rate of 0.032, as in Moscarini and Thomsson (2007).

Turning now to wages, we seek to accommodate two aspects of recent empirical work on wage determination: First, the degree of rent sharing between firms and their workers; and, second, the degree of procyclicality in real wages. It is not possible to do justice to both of these outcomes with the single remaining wage parameter $\beta$. For this reason, we use worker bargaining power $\beta$ to replicate an elasticity between wages and firm-specific productivity of 0.15. As noted by Manning (2011) and Card et al. (2018), estimates of such rent-sharing parameters can vary depending on the methods used, but this is within the plausible range of such estimates. To capture the procyclicality of real wages, we modify slightly the model of preceding sections to allow the flow breakdown payoff to vary with the cycle. Specifically, we modify the wage equation (6) to an out-of-steady-state counterpart,

$$w = \frac{\beta}{1-\beta(1-\alpha)} m + p^{\varpi}\omega_0, \tag{36}$$

where $\varpi$ is the elasticity of the flow breakdown payoff to aggregate labor productivity $p$, which is normalized to one in the initial steady state. We then choose $\varpi$ such that the semi-elasticity of average wages with respect to the unemployment rate is equal to -0.7. This is in line with the procyclicality of estimates of worker composition-adjusted real wages in Elsby, Shin and Solon (2016).

---

[7] This is inferred from Figure 5 in Davis, Faberman and Haltiwanger (2012), which reports employment-weighted kernel density functions of establishment growth rates from Business Employment Dynamics data.

Table 1. Calibrated parameters (monthly frequency)

| **Parameter** | **Meaning** | **Value** | **Reason** |
|---|---|---|---|
| $r$ | Discount rate | 0.004 | Annual real interest rate = 0.05 |
| $\alpha$ | Returns to scale | 0.64 | Cooper et al. (2007, 2015) |
| $\sigma$ | Std. dev. of $x$ shocks | 0.075 | Std. dev. quarterly emp. growth = 0.287 |
| $\lambda$ | Unemployed contact rate | 0.45 | Job-finding rate = 0.45 |
| $s$ | Employed search intensity | 0.074 | E-to-E transition rate = 0.032 |
| $\beta$ | Worker bargaining power | 0.147 | Avg. elasticity $\{w$, firm $y\}$ = 0.15 |
| $\omega_0$ | Flow breakdown payoff | 0.853 | Normalization |
| $\varpi$ | Elasticity of $\omega_0$ to $p$ | 0.525 | Semi-elasticity {avg. $w$, $U/L$} = -0.7 |
| $\varsigma_0$ | Exogenous separation rate | 0.0053 | Quit share of $U$ inflows = 0.225 |
| $c$ | Per-worker hiring cost | 0.238 | Unemployment rate = 0.05 |
| $L$ | Labor force | 21.05 | Avg. firm size = 20 |

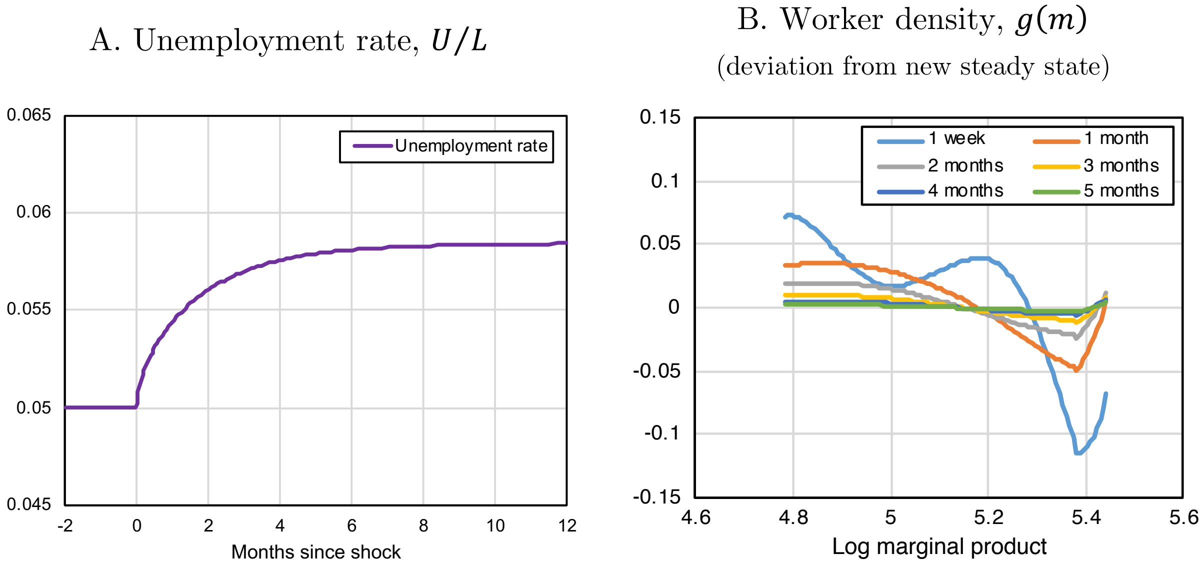
We next consider inflows into unemployment. We use two parameters to target two moments of related data. First, we augment the model of the preceding sections to incorporate a portion of separations into unemployment that are exogenous. Specifically, we allow such exogenous separations to occur at rate $\varsigma_0$. We calibrate $\varsigma_0$ to replicate the fraction of unemployment inflows that are recorded as "job leavers" in the Current Population Survey, which is 22.5 percent. Second, we use the hiring cost $c$ to target the extent of the remaining (endogenous) separations such that the initial steady-state unemployment rate is 5 percent. Intuitively, the hiring cost determines the irreversibility of firms' endogenous separation decisions, and thereby their incidence.

Finally, we set the labor force $L$ such that the average size of a firm is 20 employees, consistent with data from the Small Business Administration.

Table 1 summarizes the parameters that emerge from this calibration exercise, expressed at a monthly frequency.

Given this calibration, we solve for the transition path of model outcomes in response to a permanent unanticipated decline in aggregate labor productivity $p$ (similar to Boppart, Krusell, and Mitman 2018). Our analytical results provide us with solutions for the initial and final steady states. Given a (conjectured) path for $\lambda$, we first solve backwards from the final steady state for the implied sequence of firm marginal value functions (in the natural wastage region). This implies sequences for the boundaries $m_l$ and $m_m$, and thereby for the quit rate $\delta(m)$. Given these, we can then use the Fokker

Figure 5. Transition dynamics of calibrated model




Notes. Based on simulation of the model calibrated as described in Table 1. The figure illustrates the dynamic response to an unanticipated, permanent one-percent decline in aggregate labor productivity.

Planck (Kolmogorov Forward) Equation to solve forward for the implied sequence of distributions of marginal products across employees $G(m)$ and thereby the vacancy-filling rate $q(m)$. We then iterate over the path of $\lambda$ until a measure of excess labor demand at each point in time is reduced to zero (up to numerical error).

Figure 5 depicts some preliminary results from this exercise. It illustrates the evolution of the unemployment rate, and the distribution of employees across marginal products following a permanent, unanticipated decrease in aggregate productivity of one percent.

# 4. Extensions

We noted earlier that the methods applied thus far are amenable to being extended to richer environments, provided the requisite normalization that underlies the $m$-solution is preserved. One notable example is the case in which the adjustment friction takes a convex form. As emphasized by Bilal, Engbom, Mongey and Violante (2019) in a related model of firm dynamics and on-the-job search, such convex adjustment costs have rich implications for firm *growth* dynamics, as it becomes optimal to adjust employment *incrementally*. It turns out that the analytical results described in the preceding sections can be used to simplify greatly the analysis of such environments. In what follows, we show how.

**Convex hiring costs.** We begin by extending our previous results to a case in which there is convexity in the hiring cost. A commonly-used functional form has the cost of generating a flow $h\mathrm{d}t$ of hires[8] equal to

$$C(h;n)\mathrm{d}t = \frac{ch}{1+\gamma}\left(\frac{h}{n}\right)^{\gamma}\mathrm{d}t, \tag{37}$$

where $\gamma > 0$ regulates the degree of convexity. This isoelastic form combines simplicity with the property that firms do not "grow out" of their hiring costs, in the sense that the marginal hiring cost is a function of the firm's hiring *rate*, $C'(h;n) = c(h/n)^{\gamma}$. This in turn preserves the homogeneity that gives rise to an $m$-solution.

Since the hiring cost is continuously differentiable in $h$ for all $h \geq 0$, and convex, optimal labor demand under an $m$-solution involves just one boundary in this case, $m_l$. As before, whenever the firm's marginal product reaches $m_l$, the firm implements fires—$m_l$ is a reflecting barrier. For all $m > m_l$, the firm implements positive hires $h > 0$. There is no natural wastage region induced by inaction. The following Lemma then summarizes the results of following steps analogous to those in the preceding sections.

**Lemma 5** *Under the convex hiring cost (37), the steady-state marginal value satisfies*

$$\begin{aligned} rJ(m) = (1-\omega_1)m - \omega_0 + \gamma\left[\frac{1}{1+\gamma}\eta(m) - (1-\alpha)m\eta'(m)\right]J(m) \\ - [\delta(m) - (1-\alpha)m\delta'(m)]J(m) + [\mu + (1-\alpha)\delta(m)]mJ'(m) \\ + \frac{1}{2}\sigma^2 m^2 J''(m), \end{aligned} \tag{38}$$

*subject to the first-order condition for optimal hiring*

$$J(m) = c[\eta(m)]^{\gamma}, \tag{39}$$

*the steady-state aggregation identity*

$$\eta(m) = -\frac{\sigma^2/2}{1-\alpha}\frac{m\delta'(m)}{\delta(m)}, \tag{40}$$

*and the boundary conditions*

$$J(m_l) = 0, \;\; J'(m_l) = 0, \;\; \delta(m_l) = s\lambda, \;\; \text{and} \lim_{m\to\infty}\delta(m) = 0. \tag{41}$$

[8] In contrast to the case of a per-worker hiring cost, convexity of the hiring cost implies that it is optimal to hire incrementally, such that the flow of hires is differentiable in time.

Relative to the case of a per-worker hiring cost ($\gamma = 0$), the presence of a convex hiring cost ($\gamma > 0$) induces the firm's marginal valuation of labor in (38) to depend additionally on the firm's hiring rate $\eta(m)$. Intuitively, the firm's marginal hiring costs are determined by its hiring rate under the convex hiring cost in (37). In turn, optimal hiring requires that the firm equate the marginal value of labor to the marginal hiring cost in (39).

What is particularly useful about Lemma 5 is that it distils a complicated dynamic problem into one of solving a more manageable system of nonlinear differential equations: in the marginal value $J(m)$, the hiring rate $\eta(m)$, and the quit rate $\delta(m)$. Alternatively, after substitution the ingredients of Lemma 5 comprise a *third-order* differential equation in the quit rate $\delta(m)$. This contrasts with the simpler environment underlying Proposition 2, in which the hiring cost is linear, and the quit rate is summarized by a *first-order* differential equation. The implication is that the presence of a convex hiring cost gives rise to richer dynamics of worker retention.

The key insight is that all of the aggregation results in Proposition 3 apply more generally to richer structures of adjustment frictions, provided the simplification of an $m$-solution holds. Thus, the hiring rate $\eta(m)$ can be linked both to the marginal value $J(m)$ through the first-order condition (39), and to the quit rate via the aggregation identity (40). Although analytical solution of the system of differential equations in (38) to (41) remains a significant challenge (at least, it has so far eluded us), Lemma 5 greatly simplifies numerical solution of the model.

**Convex vacancy costs**. Another alternative structure of frictions posits that adjustment costs have their origins in the posting of vacancies—as opposed to gross hires *per se*. A key distinction is that vacancies need not always generate hires, a difference mediated by the vacancy-filling rate $q(m)$. We will see that this adds a further layer of economics to the model, as the effective hiring cost becomes endogenous.

Nonetheless, the techniques developed in the preceding sections again aid progress for this adjustment cost structure. Mirroring our discussion of convex hiring costs above, we consider a functional form for vacancy costs in which the cost of sustaining a flow $v\mathrm{d}t$ of vacancies is

$$C(v;n)\mathrm{d}t = \frac{cv}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma} \mathrm{d}t, \tag{42}$$

where, again, $\gamma > 0$ indexes the degree of convexity. As above, this form sustains an $m$-solution and, for the same reasons, induces optimal labor demand with a single boundary $m_l$. For $m > m_l$, the firm now posts vacancies $v > 0$, and its hires are given by the flow of filled vacancies. The following Lemma summarizes.

**Lemma 6** *Under the convex vacancy cost (42), the steady-state marginal value satisfies*

$$\begin{aligned} rJ(m) = (1-\omega_1)m - \omega_0 + \gamma\left[\frac{1}{1+\gamma}v(m) - (1-\alpha)mv'(m)\right]q(m)J(m) \\ - [\delta(m) - (1-\alpha)m\delta'(m)]J(m) + [\mu + (1-\alpha)\delta(m)]mJ'(m) \\ + \frac{1}{2}\sigma^2 m^2 J''(m), \end{aligned} \tag{43}$$

*subject to the definition of the firm's vacancy rate*

$$v(m) \equiv \frac{\eta(m)}{q(m)}, \tag{44}$$

*the first-order condition for optimal hiring*

$$q(m)J(m) = c[v(m)]^\gamma, \tag{45}$$

*the steady-state aggregation identities*

$$\eta(m) = -\frac{\sigma^2/2}{1-\alpha}\frac{m\delta'(m)}{\delta(m)}, \text{ and } q(m) = \chi\psi \exp\left[\frac{1-\alpha}{\sigma^2/2}\int_{m_l}^{m}\frac{\delta(\tilde{m})}{\tilde{m}}\mathrm{d}\tilde{m}\right], \tag{46}$$

*and the boundary conditions*

$$J(m_l) = 0,\ \ J'(m_l) = 0,\ \ \delta(m_l) = s\lambda,\ \text{ and } \lim_{m\to\infty}\delta(m) = 0. \tag{47}$$

Relative to the case of a convex *hiring* cost in Lemma 5, the presence of a convex *vacancy* cost in Lemma 6 adds further layers of complexity through the additional dependence of the firm's labor demand problem on the vacancy-filling rate $q(m)$. This emerges through two channels. First, the recursion for the firm's marginal valuation of labor in (43) now includes terms in the vacancy rate $v(m)$, which in turn is the ratio of the hiring rate $\eta(m)$ to the vacancy-filling rate $q(m)$, as in (44). Second, optimal hiring now requires that the firm sets its marginal value $J(m)$ equal to its marginal vacancy costs divided by the vacancy-filling rate $q(m)$, as in (45). In this way, the firm's effective marginal hiring cost is mediated by the firm's (endogenous) ability to fill its vacancies.

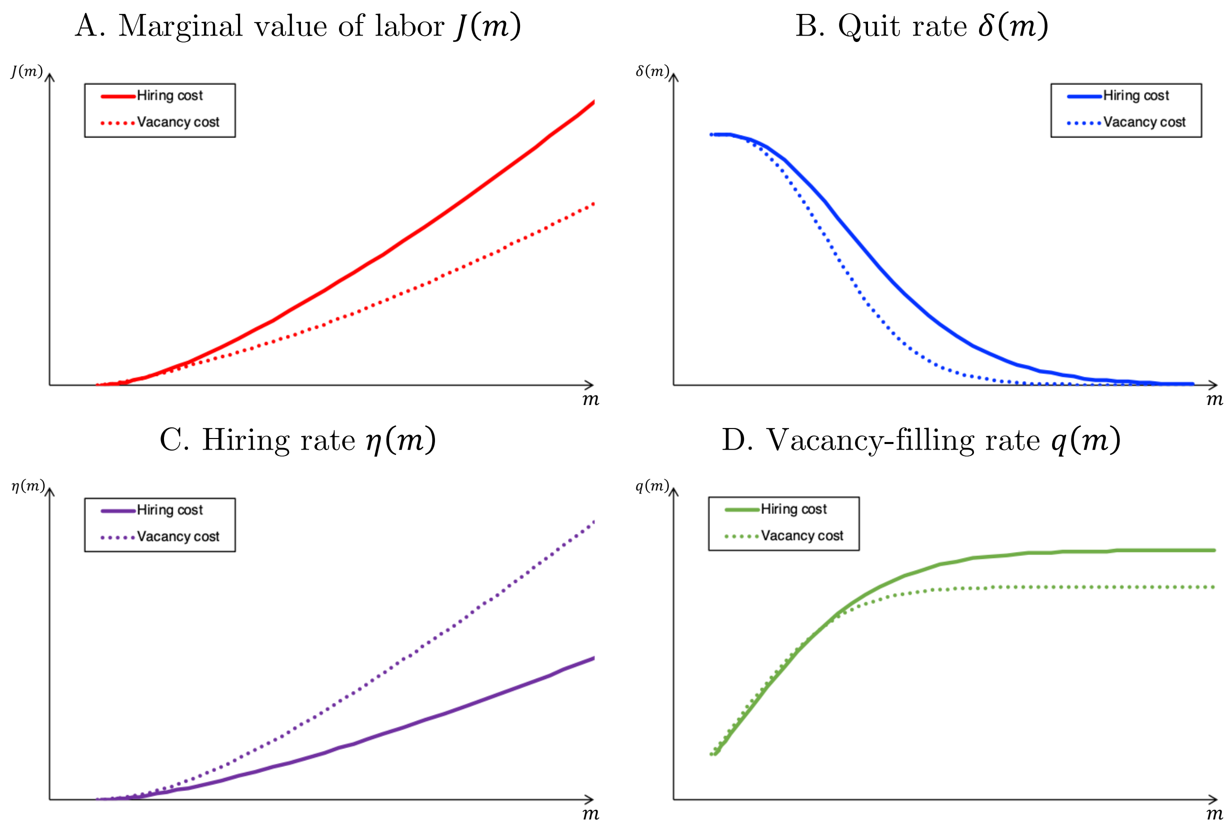
The reiteration of the aggregation identities for the hiring rate $\eta(m)$ and the vacancy-filling rate $q(m)$ in (46) underscores the usefulness of the analytical results of the preceding sections to these alternative environments. Notably, since $q(m)$ involves an integral of the quit rate $\delta(m)$, inspection of the ingredients of Lemma 6 makes clear that they can be rewritten as a *fourth-order* differential equation in the quit rate. It is in this formal sense that the interaction of optimal hiring and retention with the vacancy-filling rate gives rise to still richer firm dynamics.

The upshot is that the determination of the firm's vacancy-filling rate $q(m)$ is now intertwined with the determination of the firm's marginal value $J(m)$, its quit rate $\delta(m)$, and its hiring rate $\eta(m)$. Thus, the system of differential equations summarized by Lemma 6 now determines four endogenous functions that are mutually interdependent. Nonetheless, despite its further layers of complexity, Lemma 6 again distils an otherwise-complex dynamic problem into a simpler system of differential equations, aided by the analytical results of earlier sections.

**Illustrative simulations**. Figure 6 illustrates numerical solutions to the systems of differential equations summarized by Lemma 5 for the case of a convex hiring cost, and Lemma 6 for the case of a convex vacancy cost. For concreteness, we focus on the quadratic case with $\gamma = 1$. The remaining parameters are not chosen to match empirical moments, but rather to illustrate the qualitative form of the solutions. For this reason, scales are omitted. Thanks to the characterization of the dynamic systems in Lemmas 5 and 6, numerical solution requires only a standard ordinary differential equation solver, obviating the need for brute-force simulation of firms' decisions, and their aggregate implications. A consequence is that numerical solution is, in practice, very fast.

Figure 6 highlights several intuitive properties of the solutions. Consider first the case of a convex hiring cost, illustrated by the bold lines in Figure 6. The solution now has the property that the firm's marginal value $J(m)$ is strictly increasing for all $m > m_l$. Likewise, consistent with the first-order condition for optimal hiring in (39), the hiring rate $\eta(m)$ also is strictly increasing over this range. Intuitively, positive innovations to a firm's productivity are now resolved by an incremental increase in hiring, the marginal costs of which are offset by an incremental rise in the firm's marginal value of labor. Consistent with the hiring rate rising less aggressively in the presence of a convex cost, the quit rate $\delta(m)$ now descends less abruptly with the marginal product, and indeed asymptotes to zero.

Figure 6. Model outcomes with convex hiring and vacancy costs




Notes. Illustrative simulations of the model with convex hiring and vacancy costs, respectively. Solutions to the differential equations in Lemmas 5 and 6 are plotted for the quadratic case ($\gamma = 1$). The figure is intended to provide a qualitative sense of the solutions, and so we omit scales.

Now consider the case of a convex vacancy cost, illustrated by the dotted lines in Figure 6. Qualitatively, outcomes in this case bear a strong resemblance to those in the convex hiring cost case. The difference is the role of the vacancy-filling rate $q(m)$ in endogenously shaping firm's marginal hiring costs. Again, the results are intuitive. Because firms with higher marginal products are able to fill their vacancies more quickly, their effective marginal hiring costs are smaller. It follows that the marginal value of labor $J(m)$ rises less steeply, and the hiring rate $\eta(m)$ more steeply, with $m$. In turn, the steeper rise in hiring rates is accompanied by a faster decline in the quit rate $\delta(m)$ which, via (46), accumulates in a shallower rise in the vacancy-filling rate $q(m)$.

# Appendix

## A. Proofs of Lemmas and Propositions

**Proof of Lemma 1.** We first verify that, under an $m$-solution, the worker surplus in (9) is a function solely of $m$. Denoting the firm's hiring and separation rates by $\mathrm{d}H^*/n = \eta(m; \mathrm{d}t)$ and $\mathrm{d}S^*/n = \varsigma(m; \mathrm{d}t)$, we can rewrite (9) as

$$
\begin{aligned}
rW(m)\mathrm{d}t = \Bigg[ & \frac{\beta m}{1-\beta(1-\alpha)} + \omega_0 - b - \lambda \int \widetilde{W} d\Phi(\widetilde{W}) \\
& + s\lambda \int_{W(m)} [\widetilde{W} - W(m)]\mathrm{d}\Phi(\widetilde{W}) + [\mu + (1-\alpha)\delta(W(m))]mW'(m) \\
& + \frac{1}{2}\sigma^2 m^2 W''(m) \Bigg] \mathrm{d}t - (1-\alpha)[\eta(m;\mathrm{d}t) - \varsigma(m;\mathrm{d}t)]mW'(m) \\
& - \varsigma(m;\mathrm{d}t)W(m),
\end{aligned} \tag{48}
$$

which is a function of the single idiosyncratic state $m$, as required.

We now establish monotonicity of $W$ in $m$. Consider two firms with different marginal products $m'$ and $m$ with $m' > m$ at a given point in time $t$. Clearly, the flow worker surplus is higher in firm $m'$. In addition to this, worker surpluses in each firm incorporate capital gains from three sources: changes in idiosyncratic productivity $x$, changes in firm employment through hiring and firing, and the arrival of superior outside offers.

Fix, for both firms, a given sample path for changes in idiosyncratic productivity. Furthermore, suppose that the worker employed in firm $m'$ implements, for all future periods, the same job acceptance policy as the optimal policy for the worker employed in firm $m$. The implied worker surplus for the worker in firm $m'$ is thus weakly lower than would be implied had she implemented her optimal job acceptance policy.

Despite implementing a suboptimal job acceptance policy, it is clear that the worker in firm $m'$ receives a higher worker surplus: She receives a higher flow wage. She accepts the same outside offers. And since, under an $m$-solution, sample paths for employment are continuous, the same is true for the path of the marginal product. Thus, for any given sample path for changes in idiosyncratic productivity, the path for the marginal product of firm $m'$ must be weakly higher than that for firm $m$. Since the worker surplus is based on expectations over sample paths, monotonicity of $W$ in $m$ directly follows.

**Proof of Proposition 1**. Given that $\delta(m) = s\lambda$ and $\delta'(m) = 0$ in the natural wastage region, the recursion for the firm's marginal value (14) takes the simple form

$$(r + s\lambda)J(m) = (1-\omega_1)m - \omega_0 + [\mu + (1-\alpha)s\lambda]mJ'(m) + \frac{1}{2}\sigma^2 m^2 J''(m). \tag{49}$$

The latter resembles canonical firm dynamics problems studied by Bentolila and Bertola (1990) and Abel and Eberly (1996). It involves finding a solution to the recursion for the marginal surplus $J(m)$ in (49) subject to two pairs of boundary conditions that are implied by optimality,

$$J(m_l) = 0, \text{ and, } J(m_m) = c, \tag{50}$$

together with

$$J'(m_l) = 0, \text{ and, } J'(m_m) = 0. \tag{51}$$

It can be verified that the stated solution for $J(m)$ satisfies (49). Furthermore, the coefficients $J_1$ and $J_2$, and the boundaries $m_l$ and $m_m$, that satisfy the boundary conditions (50) and (51) can be inferred from the solution provided by Abel and Eberly (1996). Applying their result *mutatis mutandis* yields the coefficients

$$J_1 = -\frac{(1-\omega_1)\vartheta(\mathcal{G})m_l^{1-\gamma_1}}{\gamma_1\rho(1)}, \text{ and, } J_2 = -\frac{(1-\omega_1)[1-\vartheta(\mathcal{G})]m_l^{1-\gamma_2}}{\gamma_2\rho(1)}, \tag{52}$$

where

$$\mathcal{G} \equiv \frac{m_m}{m_l}, \text{ and, } \vartheta(\mathcal{G}) \equiv \frac{\mathcal{G}^{\gamma_2} - \mathcal{G}}{\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1}}. \tag{53}$$

In turn, the geometric gap between the middle and lower boundaries $\mathcal{G}$ is the solution to

$$\frac{\omega_0 + \rho(0)c}{\omega_0}\varphi(\mathcal{G}) - \mathcal{G}\varphi(\mathcal{G}^{-1}) = 0, \tag{54}$$

where

$$\varphi(\mathcal{G}) \equiv \frac{1}{\rho(1)}\left\{1 - \frac{\vartheta(\mathcal{G})}{\gamma_1} - \frac{1-\vartheta(\mathcal{G})}{\gamma_2}\right\}. \tag{55}$$

Finally, the boundaries solve

$$(1-\omega_1)m_l = \frac{\omega_0}{\rho(0)\varphi(\mathcal{G})}, \text{ and, } (1-\omega_1)m_m = \frac{\omega_0 + \rho(0)c}{\rho(0)\varphi(\mathcal{G}^{-1})}. \tag{56}$$

**Proof of Proposition 2.** Given that $J(m) = c$, and $J'(m) = J''(m) = 0$ in the hiring region, the recursion for the firm's marginal value (14) becomes a differential equation in the quit rate $\delta(m)$,

$$\{r + [\delta(m) - (1-\alpha)m\delta'(m)]\}c = (1-\omega_1)m - \omega_0. \tag{57}$$

It is straightforward to verify that the solution takes the form

$$\delta(m) = \frac{(1-\omega_1)m}{\alpha c} - \frac{\omega_0}{c} - r + \delta_1 m^{\frac{1}{1-\alpha}}, \tag{58}$$

for all $m \in (m_m, m_u)$.

It remains to infer the coefficient $\delta_1$, and the upper boundary for the marginal product in the hiring region, $m_u$. These are determined by boundary conditions for the quit rate,

$$\delta(m_m) = s\lambda, \text{ and, } \delta(m_u) = 0. \tag{59}$$

It follows from the first boundary condition that

$$\delta_1 = \left(r + s\lambda + \frac{\omega_0}{c}\right) m_m^{-\frac{1}{1-\alpha}} - \frac{1-\omega_1}{\alpha c} m_m^{1-\frac{1}{1-\alpha}}. \tag{60}$$

Inserting the latter into (58) yields the stated solution for $\delta(m)$.

Turning now to the upper boundary $m_u$, the second condition in (59) implies

$$\left[m_m - \alpha\frac{\omega_0 + (r+s\lambda)c}{1-\omega_1}\right]\left[\left(\frac{m_u}{m_m}\right)^{\frac{1}{1-\alpha}} - 1\right] = \frac{\alpha c s\lambda}{1-\omega_1} + (m_u - m_m). \tag{61}$$

Using the solution for $m_m$ in (56), we can write the leading coefficient in the latter as

$$m_m - \alpha\frac{\omega_0 + (r+s\lambda)c}{1-\omega_1} = \frac{\omega_0 + (r+s\lambda)c}{1-\omega_1}\left[\frac{1}{(r+s\lambda)\varphi(\mathcal{G}^{-1})} - \alpha\right]. \tag{62}$$

Abel and Eberly (1996) prove that $\mathcal{G} > 1$ implies that $\varphi(\mathcal{G}^{-1}) < \varphi(1) = 1/(r+s\lambda)$. It follows that

$$m_m - \alpha\frac{\omega_0 + (r+s\lambda)c}{1-\omega_1} > \frac{\omega_0 + (r+s\lambda)c}{1-\omega_1}(1-\alpha) > 0. \tag{63}$$

This implies that there exists a unique $m_u > m_m$ that satisfies (61).

Now consider the slope of $\delta(m)$. Differentiating (17), applying the solution for $m_m$ in (56), and once again noting that $\mathcal{G} > 1$ implies that $\varphi(\mathcal{G}^{-1}) < \varphi(1) = 1/(r+s\lambda)$ yields

$$\delta'(m) = \frac{1-\omega_1}{\alpha c}\left\{1 - \frac{1}{1-\alpha}[1-\alpha(r+s\lambda)\varphi(\mathcal{G}^{-1})]\left(\frac{m}{m_m}\right)^{\frac{\alpha}{1-\alpha}}\right\}$$
$$< \frac{1-\omega_1}{\alpha c}\left[1-\left(\frac{m}{m_m}\right)^{\frac{\alpha}{1-\alpha}}\right]. \tag{64}$$

It follows that $\delta'(m_m^+) < 0$ and that $\delta(m)$ is declining for all $m \in (m_m, m_u)$. Finally, differentiating (17) once more, and following the same steps,

$$\delta''(m) = -\frac{1-\omega_1}{c(1-\alpha)^2}\frac{1}{m_m}[1-\alpha(r+s\lambda)\varphi(\mathcal{G}^{-1})]\left(\frac{m}{m_m}\right)^{\frac{2\alpha-1}{1-\alpha}} < 0. \tag{65}$$

**Proof of Proposition 3.** (i) Denote the logarithm of the marginal product $\mathscr{m} \equiv \ln m$. In the natural wastage region, this evolves according to the stochastic law of motion

$$\mathrm{d}\mathscr{m} = \mathrm{d}\ln x - (1-\alpha)\mathrm{d}\ln n = \left[\mu - \frac{1}{2}\sigma^2 + (1-\alpha)s\lambda\right]\mathrm{d}t + \sigma \mathrm{d}z \equiv \mu_{\mathscr{m}}\mathrm{d}t + \sigma\mathrm{d}z. \tag{66}$$

This process can be approximated by the following discrete-time, discrete-state process (Dixit 1993):

$$\mathscr{m}_{t+\mathrm{d}t} = \begin{cases} \mathscr{m}_t + \Delta & \text{with probability} \quad \mathscr{p}, \\ \mathscr{m}_t - \Delta & \text{with probability} \quad \mathscr{q}, \end{cases} \tag{67}$$

where $\Delta = \sigma\sqrt{\mathrm{d}t}$, $\mathscr{p} = \frac{1}{2}\left(1 + \frac{\mu_{\mathscr{m}}}{\sigma}\sqrt{\mathrm{d}t}\right)$, and $\mathscr{q} = \frac{1}{2}\left(1 - \frac{\mu_{\mathscr{m}}}{\sigma}\sqrt{\mathrm{d}t}\right)$.

Consider a worker at $\mathscr{m}_l$. With probability $\mathscr{q}$, her firm crosses the lower boundary and fires a fraction $\Delta/(1-\alpha)$ of its employees such that it returns to $\mathscr{m}_l$. Denoting the stationary density of employees at $\mathscr{m}_l$ by $\mathscr{g}(\mathscr{m}_l)$, the fraction of total employment that separates into unemployment is given by

$$\varsigma \mathrm{d}t = \mathscr{q}\frac{\Delta}{1-\alpha}\cdot[\mathscr{g}(\mathscr{m}_l)\cdot\Delta] = \frac{\sigma^2/2}{1-\alpha}\mathscr{g}(\mathscr{m}_l)\mathrm{d}t + o(\mathrm{d}t). \tag{68}$$

Mapping back from logarithms to levels, $\mathscr{g}(\mathscr{m}_l) = m_l g(m_l)$, yields the stated result,

$$\varsigma = \frac{\sigma^2/2}{1-\alpha}m_l g(m_l). \tag{69}$$

It will be useful in what follows to derive the flow-balance condition for the steady-state density at the lower boundary $\mathscr{g}(\mathscr{m}_l)$. Setting outflows equal to inflows,

$$\mathscr{p}\mathscr{g}(\mathscr{m}_l) + \mathscr{q}\frac{\Delta}{1-\alpha}\mathscr{g}(\mathscr{m}_l) + s\lambda\mathrm{d}t\mathscr{g}(\mathscr{m}_l) = \mathscr{q}\mathscr{g}(\mathscr{m}_l + \Delta). \tag{70}$$

Expanding $\mathscr{g}(m_l+\Delta)$, using the definitions of $\mathscr{p}$, $\mathscr{q}$ and $\Delta$, collecting terms in orders of $\sqrt{\mathrm{d}t}$, and eliminating terms of order higher than $\mathrm{d}t$ yields

$$\left[\left(\mu_m+\frac{\sigma^2/2}{1-\alpha}\right)\mathscr{g}(m_l)-\frac{1}{2}\sigma^2\mathscr{g}'(m_l)\right]\sqrt{\mathrm{d}t} \\ =\frac{\sigma}{2}\left[\left(\frac{1}{1-\alpha}\mu_m-2s\lambda\right)\mathscr{g}(m_l)-\mu_m\mathscr{g}'(m_l)+\frac{1}{2}\sigma^2\mathscr{g}''(m_l)\right]\mathrm{d}t. \tag{71}$$

As $\mathrm{d}t\to 0$, the terms of order $\sqrt{\mathrm{d}t}$ dominate, and therefore must cancel,

$$\left(\mu_m+\frac{\sigma^2/2}{1-\alpha}\right)\mathscr{g}(m_l)-\frac{1}{2}\sigma^2\mathscr{g}'(m_l)=0. \tag{72}$$

Noting that $\mathscr{g}(m_l)=m_l g(m_l)$ and $\mathscr{g}'(m_l)=m_l g(m_l)+m_l^2 g'(m_l)$, recalling the definition of $\mu_m$, and imposing the aggregate stationarity condition $\mu+\frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}=0$ yields

$$\left[(1-\alpha)s\lambda-\frac{1}{2}\sigma^2\right]g(m_l)=\frac{1}{2}\sigma^2 m_l g'(m_l). \tag{73}$$

(ii) and (iii). To infer the stationary distribution of marginal products across employees $g(m)$, and thereby the vacancy-filling rate $q(m)=\chi[\psi+(1-\psi)G(m)]$, we first infer the stochastic law of motion for the marginal product, $\mathrm{d}m/m=(\mathrm{d}x/x)-(1-\alpha)(\mathrm{d}n/n)$, on the interval $m\in(m_l,m_u)$. The evolution of productivity $x$ is given by (1). The evolution of employment $n$ is as follows: There are outflows of employment due to quits, $\delta(m)n\mathrm{d}t$. But there are also potential inflows due to hires: The hiring rate at $m$, denoted $\eta(m)$, can be written as the total measure of hires at $m$, $f(m)Vq(m)$, divided by the total measure of employment at $m$, $g(m)N$; or, more succinctly,

$$\eta(m)=-\frac{\delta'(m)q(m)}{q'(m)}. \tag{74}$$

Thus, the stochastic law of motion for the marginal product is

$$\frac{\mathrm{d}m}{m}=\left\{\mu+(1-\alpha)\left[\frac{\delta'(m)q(m)}{q'(m)}+\delta(m)\right]\right\}\mathrm{d}t+\sigma\mathrm{d}z. \tag{75}$$

The latter describes the motion of the marginal product for an employee that remains in a given firm. However, additional flows of employees across marginal products arise due to the presence of search. Specifically, the net inflow of density into $g(m)$ from this channel is given by the measure of hires less quits,

$$[\eta(m)-\delta(m)]g(m)=-\frac{\partial}{\partial m}\frac{[\delta(m)q(m)]}{\chi(1-\psi)}. \tag{76}$$

The Fokker-Planck (Kolmogorov Forward) equation for the worker density $g(m)$ is thus given by

$$\begin{aligned}\frac{\partial g(m)}{\partial t}=-\frac{\partial}{\partial m}\frac{[\delta(m)q(m)]}{\chi(1-\psi)}-\frac{\partial}{\partial m}\left[\left\{\mu+(1-\alpha)\left[\frac{\delta'(m)q(m)}{q'(m)}+\delta(m)\right]\right\}mg(m)\right] \\ +\frac{1}{2}\sigma^2\frac{\partial^2}{\partial m^2}[m^2g(m)].\end{aligned} \tag{77}$$

Noting that $g(m)=q'(m)/\chi(1-\psi)$, and that $\partial g(m)/\partial t=0$ in steady state, we can rewrite the latter as

$$\frac{\partial}{\partial m}[\delta(m)q(m)]+\frac{\partial}{\partial m}\left\{\mu mq'(m)+(1-\alpha)m\frac{\partial}{\partial m}[\delta(m)q(m)]\right\}=\frac{1}{2}\sigma^2\frac{\partial^2}{\partial m^2}[m^2q'(m)]. \tag{78}$$

Integrating once,

$$\delta(m)q(m)+\mu mq'(m)+(1-\alpha)m\frac{\partial}{\partial m}[\delta(m)q(m)]=\frac{1}{2}\sigma^2\frac{\partial}{\partial m}[m^2q'(m)]+C_1, \tag{79}$$

where $C_1$ is a constant of integration. Evaluating at $m=m_l$, imposing the boundary condition for $g(m_l)=q'(m_l)/\chi(1-\psi)$ in (73), noting that $\delta(m_l)=s\lambda$, $\delta'(m_l)=0$, $q(m_l)=\chi\psi$, and recalling the aggregate stationarity condition, $\mu+\frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}=0$, yields

$$C_1=s\lambda\chi\psi-\frac{\sigma^2/2}{1-\alpha}m_lq'(m_l)=\chi(1-\psi)\varsigma\left(\frac{\lambda U}{\varsigma N}-1\right)=0, \tag{80}$$

where the second and third equalities follow from the solution for the separation rate into unemployment $\varsigma$ in (25), established above, the definition of $\psi=U/(U+sN)$, and the fact that inflows into unemployment $\varsigma N$ must equal outflows from unemployment $\lambda U$ in steady state.

Expanding and collecting terms in (79), we can now write

$$(1-\alpha)\frac{\partial}{\partial m}\left[m^{\frac{1}{1-\alpha}}\delta(m)q(m)\right]+(\mu-\sigma^2)m^{\frac{1}{1-\alpha}}q'(m)=\frac{1}{2}\sigma^2m^{1+\frac{1}{1-\alpha}}q''(m). \tag{81}$$

Integrating again, applying integration by parts to the right-hand side, collecting terms, and imposing the aggregate stationarity condition $\mu+\frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}=0$, yields a first-order differential equation in $q(m)$,

$$(1-\alpha)\delta(m)q(m)=\frac{1}{2}\sigma^2mq'(m)+C_2m^{-\frac{1}{1-\alpha}}, \tag{82}$$

where $C_2$ is a further constant of integration. Evaluating once again at $m = m_l$ implies

$$C_2 = (1-\alpha)\chi(1-\psi)\varsigma m_l^{\frac{1}{1-\alpha}}\left(\frac{\lambda U}{\varsigma N} - 1\right) = 0. \tag{83}$$

Thus we have

$$\delta(m)q(m) = \frac{\sigma^2/2}{1-\alpha} mq'(m), \tag{84}$$

and it is straightforward to verify that the solution takes the form stated in (27).

Finally, it follows that the hiring rate can be written as

$$\eta(m) = -\frac{\sigma^2/2}{1-\alpha}\frac{m\delta'(m)}{\delta(m)}. \tag{85}$$

**Proof of Lemma 2.** Applying the same methods as those underlying Propositions 1 and 2, the marginal value of labor to the firm $J = \Pi_n$ can be written as follows in the presence of offer matching,

$$\begin{aligned} rJ(m) = m - b - (1-\xi)[\delta(m) - (1-\alpha)m\delta'(m)]J(m) \\ + [\mu + (1-\xi)(1-\alpha)\delta(m)]mJ'(m) + \frac{1}{2}\sigma^2 m^2 J''(m). \end{aligned} \tag{86}$$

In the natural wastage region, the latter simplifies to

$$[r + (1-\xi)s\lambda]J(m) = m - b + [\mu + (1-\alpha)(1-\xi)s\lambda]mJ'(m) + \frac{1}{2}\sigma^2 m^2 J''(m). \tag{87}$$

Thus, Proposition 1 holds *mutatis mutandis* with $\omega_0$, $\omega_1$ and $s\lambda$ exchanged respectively with $b$, $0$, and $(1-\xi)s\lambda$. Likewise, in the hiring region, we can write

$$\{r + (1-\xi)[\delta(m) - (1-\alpha)m\delta'(m)]\}c = m - b. \tag{88}$$

When combined with the boundary condition $\delta(m_m) = s\lambda$, it is straightforward to verify that the solution for $\delta(m)$ takes the stated form.

**Proof of Lemma 3.** Applying the same methods to the case with a per-worker hiring cost equal to $c \cdot m$ yields a recursion in the firm's marginal valuation of labor identical to that in (14). As before, in the natural wastage region, $\delta(m) = s\lambda$, $\delta'(m) = 0$, and the marginal value satisfies

$$(r + s\lambda)J(m) = (1-\omega_1)m - \omega_0 + [\mu + (1-\alpha)s\lambda]mJ'(m) + \frac{1}{2}\sigma^2 m^2 J''(m) \tag{89}$$

for all $m \in (m_l, m_m)$. However, the boundary conditions differ in the presence of a training cost. Specifically, (50) and (51) are replaced by

$$J(m_l) = 0, \text{ and, } J(m_m) = \tau \cdot m_m, \tag{90}$$

and

$$J'(m_l) = 0, \text{ and, } J'(m_m) = \tau. \tag{91}$$

It follows that the solution for $J(m)$ in the natural wastage region has the same functional form as the solution in Proposition 1, but that the solution for the coefficients, and the boundaries will differ.

In the hiring region, $J(m) = \tau m$, $J'(m) = \tau$, $J''(m) = 0$, and

$$\{r - \mu + [\alpha\delta(m) - (1-\alpha)m\delta'(m)]\}\tau m = (1-\omega_1)m - \omega_0. \tag{92}$$

Using an integrating factor, we can write the latter as

$$\frac{\partial}{\partial m}\left[(1-\alpha)m^{-\frac{\alpha}{1-\alpha}}\delta(m)\right] = \left[(r-\mu) - \frac{1-\omega_1}{\tau}\right]m^{-\frac{1}{1-\alpha}} + \frac{\omega_0}{\tau}m^{-\frac{1}{1-\alpha}-1}. \tag{93}$$

Thus

$$\delta(m) = \frac{1}{\alpha}\left[\frac{1-\omega_1}{\tau} - (r-\mu)\right] - \frac{\omega_0}{\tau m} + \delta_1 m^{\frac{\alpha}{1-\alpha}}. \tag{94}$$

Noting that $\delta(m_m) = s\lambda$ completes the solution,

$$\delta_1 = \left\{s\lambda - \frac{1}{\alpha}\left[\frac{1-\omega_1}{\tau} - (r-\mu)\right]\right\}m_m^{-\frac{\alpha}{1-\alpha}} + \frac{\omega_0}{\tau}m_m^{-\frac{\alpha}{1-\alpha}-1}. \tag{95}$$

**Proof of Lemma 4.** (i) In the absence of idiosyncratic shocks, $\mu = \sigma = 0$, and with exogenous job destruction at rate $\varsigma$, the firm's marginal value satisfies

$$\begin{aligned} rJ(m) = (1-\omega_1)m - \omega_0 - [\varsigma + \delta(m) - (1-\alpha)m\delta'(m)]J(m) \\ + (1-\alpha)[\varsigma + \delta(m)]mJ'(m). \end{aligned} \tag{96}$$

It follows that there is a hiring region such that $J(m) = c$ and $J'(m) = 0$ on its interior, and in which the quit rate is given as in Proposition 2, with $r$ exchanged with $r + \varsigma$.

(ii) Evaluating (96) to the left and right of $m_m$ implies

$$(\varsigma + s\lambda)m_m J'(m_m^-) = m_m \delta'(m_m^+)c. \tag{97}$$

Noting that $J'(m_m^-) \geq 0$ and $\delta'(m_m^+) \leq 0$ implies that $J'(m_m) = \delta'(m_m) = 0$. This in turn implies that $m_m$ solves $(r + \varsigma + s\lambda)c = (1-\omega_1)m_m - \omega_0$, as claimed.

(iii) and (iv). Retracing the steps of the proof of Proposition 3, imposing $\mu = \sigma = 0$, and noting that the total separation rate from a firm is in this case given by $\varsigma + \delta(m)$, gives rise to the following analogue to (78),

$$\frac{\partial}{\partial m}\{[\varsigma + \delta(m)]q(m)\} + \frac{\partial}{\partial m}\left\{(1-\alpha)m\frac{\partial}{\partial m}\{[\varsigma + \delta(m)]q(m)\}\right\} = 0. \tag{98}$$

Integrating once,

$$[\varsigma + \delta(m)]q(m) + (1-\alpha)m\frac{\partial}{\partial m}\{[\varsigma + \delta(m)]q(m)\} = C_1, \tag{99}$$

where $C_1$ is a constant of integration. This has solution

$$[\varsigma + \delta(m)]q(m) = C_1 + C_2 m^{-\frac{1}{1-\alpha}}. \tag{100}$$

Evaluating at $m = m_m$, noting that $\delta(m_m) = s\lambda$, and $q(m_m) = \chi\psi$,

$$(\varsigma + s\lambda)\chi\psi = C_1 + C_2 m_m^{-\frac{1}{1-\alpha}}. \tag{101}$$

Likewise, evaluating at $m = m_u$, noting that $\delta(m_u) = 0$, and $q(m_u) = \chi$,

$$\varsigma\chi = C_1 + C_2 m_u^{-\frac{1}{1-\alpha}}. \tag{102}$$

Solving for the constants yields

$$C_1 = (\varsigma + s\lambda)\chi\psi = \chi\varsigma, \text{ and}$$
$$\left[\left(\frac{m_u}{m_m}\right)^{\frac{1}{1-\alpha}} - 1\right]C_2 = \chi[\psi s\lambda - (1-\psi)\varsigma]m_u^{\frac{1}{1-\alpha}} = \chi(1-\psi)\varsigma\left(\frac{\lambda U}{\varsigma N} - 1\right)m_u^{\frac{1}{1-\alpha}} = 0, \tag{103}$$

where the latter uses the definition of $\psi = U/(U + sN)$, and the fact that inflows into unemployment $\varsigma N$ must equal outflows from unemployment $\lambda U$ in steady state. We thus obtain the following solution for the vacancy-filling rate,

$$q(m) = \frac{\varsigma\chi}{\varsigma + \delta(m)}. \tag{104}$$

It follows that the hiring rate can be written as

$$\eta(m) = -\frac{\delta'(m)q(m)}{q'(m)} = \varsigma + \delta(m). \tag{105}$$

It follows that the drift for each $m$ is zero, firm marginal products are constant over time, and thus the natural wastage region is never entered.

**Proof of Lemma 5.** The firm's value is given by

$$r\Pi \mathrm{d}t = \max_{h\geq 0, \mathrm{d}S\geq 0} \left\{ \left[ xn^{\alpha} - wn - \frac{ch}{1+\gamma}\left(\frac{h}{n}\right)^{\gamma} + h\Pi_n - \delta n\Pi_n + \mu x\Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx} \right] \mathrm{d}t \right. \\ \left. - \Pi_n \mathrm{d}S \right\}. \tag{106}$$

The wage remains $w = \beta m/[1-\beta(1-\alpha)] + \omega_0$ . Note that, since $m$ is a state variable, and hiring is incremental, $h$ has no effect on $m$. The first-order conditions for optimal hires and fires are thus

$$\left[ -c\left(\frac{h}{n}\right)^{\gamma} + \Pi_n \right] h = 0, \quad \text{and} \quad \Pi_n \mathrm{d}S = 0. \tag{107}$$

It follows that the optimized value function is given by

$$r\Pi = xn^{\alpha} - wn + \gamma C(h^*; n) - \delta n\Pi_n + \mu x\Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx}, \tag{108}$$

where $h^*$ denotes optimal hires. Differentiating with respect to $n$ yields

$$rJ = (1-\omega_1)m - \omega_0 + \gamma\left[ C_1(h^*; n)\frac{\partial h^*}{\partial n} + C_2(h^*; n) \right] - \frac{\partial}{\partial n}(\delta nJ) + \mu xJ_x \\ + \frac{1}{2}\sigma^2 x^2 J_{xx}. \tag{109}$$

Under an $m$-solution, $h^* = \eta(m)n$ and $\delta = \delta(m)$. Using these, and the definition of $C(h; n)$ in (37), we can write the firm's marginal value as in (38).

**Proof of Lemma 6.** The firm's value is given by

$$r\Pi \mathrm{d}t = \max_{v\geq 0, \mathrm{d}S\geq 0} \left\{ \left[ xn^{\alpha} - wn - \frac{cv}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma} + qv\Pi_n - \delta n\Pi_n + \mu x\Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx} \right] \mathrm{d}t \right. \\ \left. - \Pi_n \mathrm{d}S \right\}. \tag{110}$$

The wage remains $w = \beta m/[1-\beta(1-\alpha)] + \omega_0$ . Note that, since $m$ is a state variable, and hiring is incremental, $v$ has no effect on $m$. The first-order conditions for optimal vacancies and fires are

$$\left[ -c\left(\frac{v}{n}\right)^{\gamma} + q\Pi_n \right] v = 0, \quad \text{and,} \quad \Pi_n \mathrm{d}S = 0. \tag{111}$$

It follows that the optimized value function is given by

$$r\Pi = xn^{\alpha} - wn + \gamma C(v^*; n) - \delta n \Pi_n + \mu x \Pi_x + \frac{1}{2}\sigma^2 x^2 \Pi_{xx} \tag{112}$$

where $v^*$ denotes optimal vacancies. Differentiating with respect to $n$ yields

$$\begin{aligned} rJ = (1-\omega_1)m - \omega_0 + \gamma\left[C_1(v^*; n)\frac{\partial v^*}{\partial n} + C_2(v^*; n)\right] - \frac{\partial}{\partial n}(\delta n J) + \mu x J_x \\ + \frac{1}{2}\sigma^2 x^2 J_{xx}. \end{aligned} \tag{113}$$

Under an $m$-solution, $\delta = \delta(m)$, $q = q(m)$ and

$$v^* = \frac{\eta(m)}{q(m)} n \equiv \nu(m) n, \tag{114}$$

where $\nu(m)$ is the firm's optimal vacancy rate. Using these, and the definition of $C(v; n)$ in (42), we can write the firm's marginal value as in (43).

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