Sentiment and speculation in a market with heterogeneous beliefs

Ian Martin   Dimitris Papadimitriou

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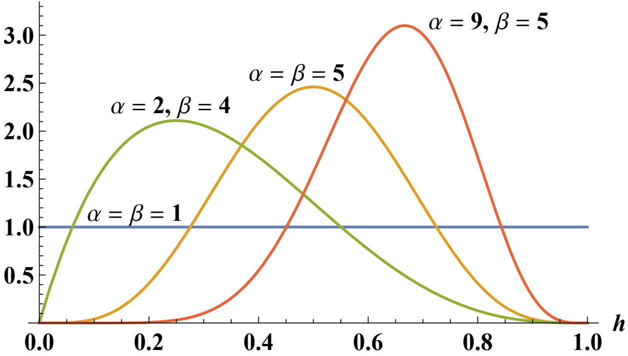
Introduction

- Agents disagree about the probabilities of good/bad news
- Optimists go long; pessimists go short
- If the market rallies, optimists get rich; if the market sells off, pessimists get rich
- In either case, ex post winners' beliefs become overrepresented in prices
- This *sentiment* effect boosts volatility, and hence risk premia
- Sentiment induces *speculation*: agents trade at prices that they think are not warranted by fundamentals, in anticipation of adjusting their positions in future

Setup

- Agents indexed by $h \in (0, 1)$ are initially endowed with one unit of a risky asset
- The asset evolves on a binomial tree with exogenous terminal payoffs
- The interest rate is normalized to zero
- Agent h thinks the probability of an up-move is h
- Agents have log utility over terminal wealth

Setup



- The mass of agents with belief h follows a beta distribution, pdf

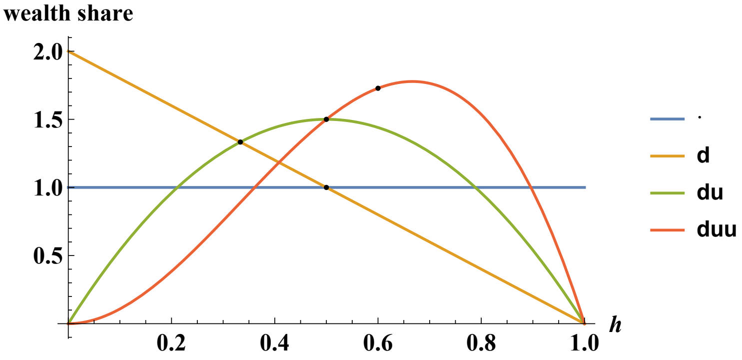
$$f(h) \propto h^{\alpha-1}(1-h)^{\beta-1} \quad \text{where } \alpha, \beta > 0$$

Equilibrium (1): individual optimization

- With log utility, agents behave myopically
- Solve backwards: the price of the risky asset is p_d or p_u next period
- Agent h chooses number of units of risky asset held:

$$x_h = w_h \left(\frac{h}{p - p_d} - \frac{1-h}{p_u - p} \right)$$

- If $p_d < p_u$ then pessimistic agents, $h \approx 0$, are short, and optimists, $h \approx 1$, are long



- After m up and n down steps, agent h's share of aggregate wealth is

$$\frac{w_h}{p} = \frac{B(\alpha, \beta)}{B(\alpha + m, \beta + n)} h^m (1 - h)^n$$

- The richest agent is $h = m/(m + n)$; this agent looks right in hindsight

Equilibrium (2): market clearing

- Price p clears the market. At time $t = m + n$,

$$p = \frac{p_u p_d}{H_{m,t} p_d + (1 - H_{m,t})\, p_u}$$

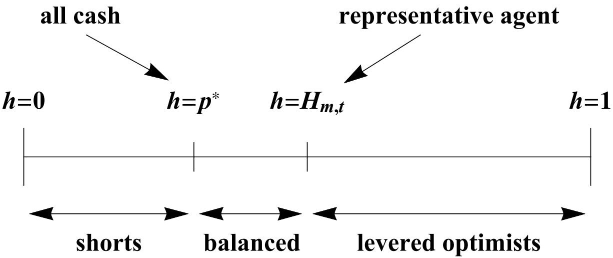
where

$$H_{m,t} = \frac{m + \alpha}{t + \alpha + \beta} = \int_0^1 h \frac{w_h f(h)}{p}\, dh$$

is wealth-weighted average belief

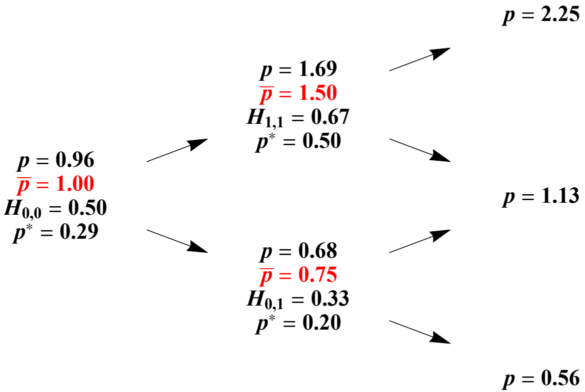
- Write p^* for risk-neutral probability of an up-move, defined via $p = p^* p_u + (1 - p^*) p_d$

Two special investors

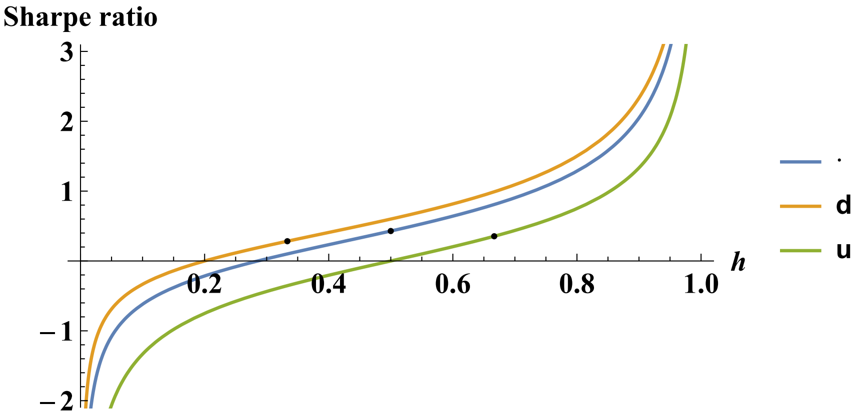


- Share of wealth agent h invests in the risky asset is $\dfrac{h-p^*}{H_{m,t}-p^*}$
- Representative agent—“Mr. Market”—with $h = H_{m,t}$ invests fully in the risky asset
- The agent with $h = p^*$ invests fully in the bond

Example 1: Geometric payoffs, uniform belief distribution



p: price. $\overline{p}$: price in homog. economy. $H_{m,t}$: identity of rep agent. p^*: risk-neutral prob.



- Mr. Market perceives a higher Sharpe ratio in the up state than the down state
- This is the opposite of what any individual thinks

Example 2: A risky bond

- $T = 50$ periods to go. Uniform beliefs
- Bond defaults (recover 30) in the bottom state. Else pays 100
- In order of increasing pessimism:
  - $h = 0.50$ thinks default prob is less than 10^{-15}
  - $h = 0.25$ thinks default prob is less than 10^{-6}
  - $h = 0.10$ thinks default prob is less than 0.6%
  - $h = 0.05$ thinks default prob is less than 8%
  - $h = 0.01$ thinks default prob is just over 60%
- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at?

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- Who would go short, at this price? everyone below $h = 0.48$!

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- Who will *stay* short?

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- What price does the bond trade at? at \$95.63
- Who would go short, at this price? everyone below $h = 0.48$!
- Who will *stay* short? marginal agent p^* at time 0, 1, 2, . . . is $h = 0.48, 0.31, 0.22, \ldots$

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- Initially, $h = 0.50$ is the representative agent
- What price does the bond trade at? at \$95.63
- Who would go short, at this price? everyone below $h = 0.48$!
- Who will *stay* short? marginal agent p^* at time 0, 1, 2, ... is $h = 0.48, 0.31, 0.22, \ldots$; only $h < 0.006$ stay short to the bitter end

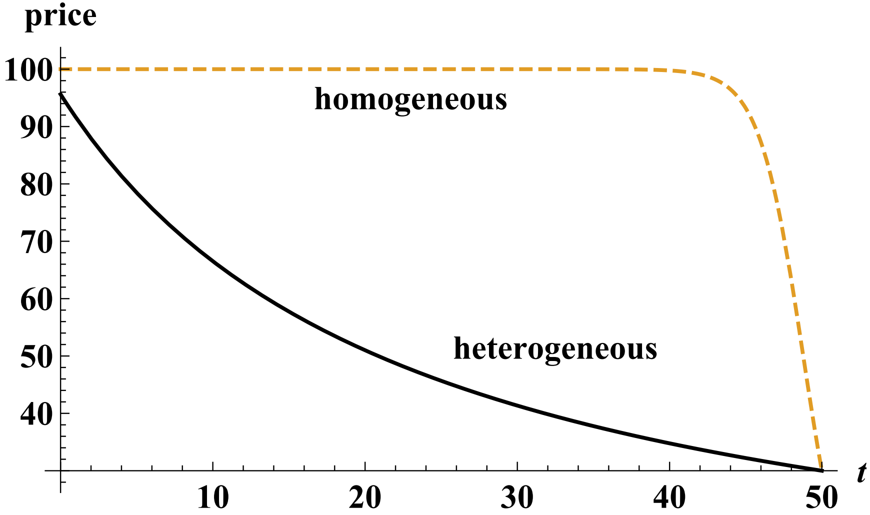

Figure: The risky bond's price over time following consistently bad news.

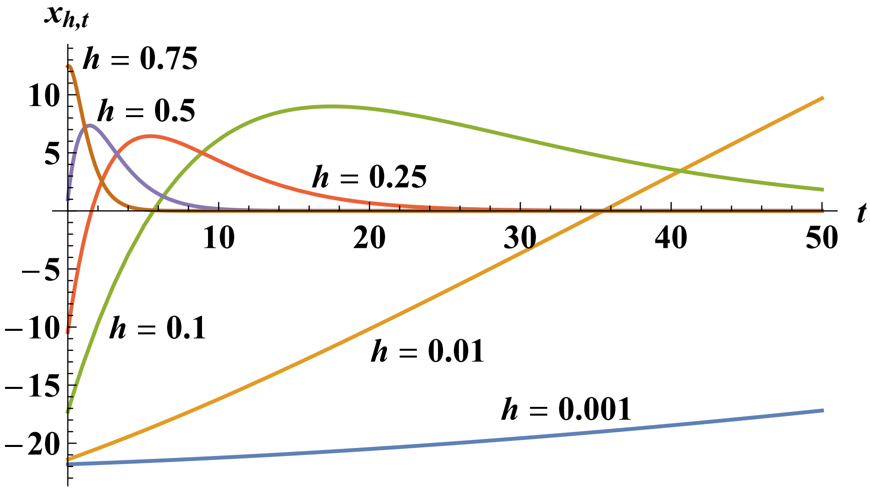

Figure: The number of units of the risky bond held by different agents, $x_{h,t}$, plotted against time.

Example 3: A diffusion limit

- Slice the period from 0 to T into $2N$ short periods
- Geometrically increasing terminal payoffs with volatility σ (Cox–Ross–Rubinstein)
- Tune down per-period disagreement by parametrizing $\alpha = \beta = \theta N$
- Low θ: lots of disagreement. $\theta \to \infty$: homogeneous economy
- The belief distribution is very spiky, so we write $h = \widetilde{\mathbb{E}}[h] + z\sqrt{\widetilde{\text{var}}[h]}$
- Now let $N \to \infty$. All agents perceive returns as lognormal and agree on ($\mathbb{P}$) volatility

$$\text{annualized return vol}_{0\to t} = \left(\frac{\theta + 1}{\theta + \frac{t}{T}}\right)\sigma$$

- But they disagree on risk premia...

Result (Subjective expectations)

Agent z's annualized expected return is

$$\frac{1}{t}\log \mathbb{E}^{(z)} R_{0\to t} = \frac{\theta+1}{\theta+\frac{t}{T}}\left[\frac{z\sigma}{\sqrt{\theta T}} + \frac{\theta+1}{\theta}\frac{\theta+\frac{t}{2T}}{\theta+\frac{t}{T}}\sigma^2\right]$$

In particular, the cross-sectional average expected return is

$$\widetilde{\mathbb{E}}\frac{1}{t}\log \mathbb{E}^{(z)} R_{0\to t} = \frac{(\theta+1)^2\left(\theta+\frac{t}{2T}\right)}{\theta\left(\theta+\frac{t}{T}\right)^2}\sigma^2$$

Disagreement is the cross-sectional standard deviation of expected returns:

$$disagreement = \frac{\theta+1}{\theta+\frac{t}{T}}\frac{\sigma}{\sqrt{\theta T}}$$

Result (Option pricing and the volatility term structure)

The time-0 price of a call option with maturity t and strike price K is

$$p_0 \Phi(d_1) - K\Phi(d_1 - \widetilde{\sigma}_t \sqrt{t})$$

where

$$d_1 = \frac{\log(p_0/K) + \frac{1}{2}\widetilde{\sigma}_t^2 t}{\widetilde{\sigma}_t \sqrt{t}} \quad \textit{and} \quad \widetilde{\sigma}_t = \frac{\theta + 1}{\sqrt{\theta(\theta + \frac{t}{T})}}\sigma$$

In particular, short-dated options have $\widetilde{\sigma}_0 = \frac{\theta+1}{\theta}\sigma$ *and long-dated options have* $\widetilde{\sigma}_T = \sqrt{\frac{\theta+1}{\theta}}\sigma$

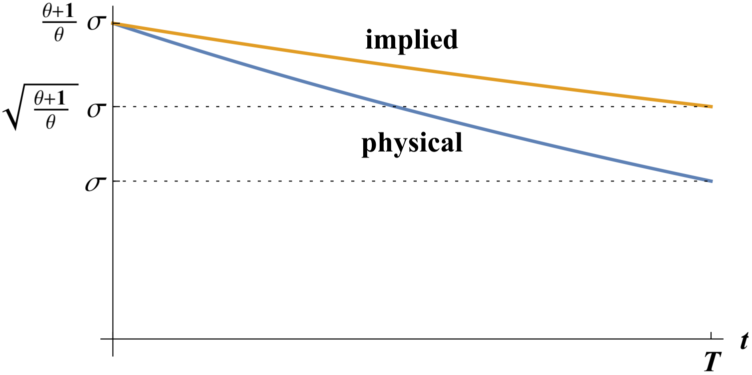

Figure: The term structures of implied and physical volatility.

- Variance risk premium $\frac{1}{T}\left(\text{var}^* \log R_{0\to T} - \text{var} \log R_{0\to T}\right) = \frac{\sigma^2}{\theta}$

An illustrative calibration

	Model	Data
1mo implied vol	18.6%	18.6%
1yr implied vol	18.2%	18.1%
2yr implied vol	17.7%	17.9%
1yr disagreement	4.4%	4.8%
10yr disagreement	2.9%	2.9%
1yr mean risk premium	3.3%	3.8%
10yr mean risk premium	1.9%	3.6%

- We set $T = 10$ and $\sigma = 12\%$
- We set belief heterogeneity parameter θ to 1.8

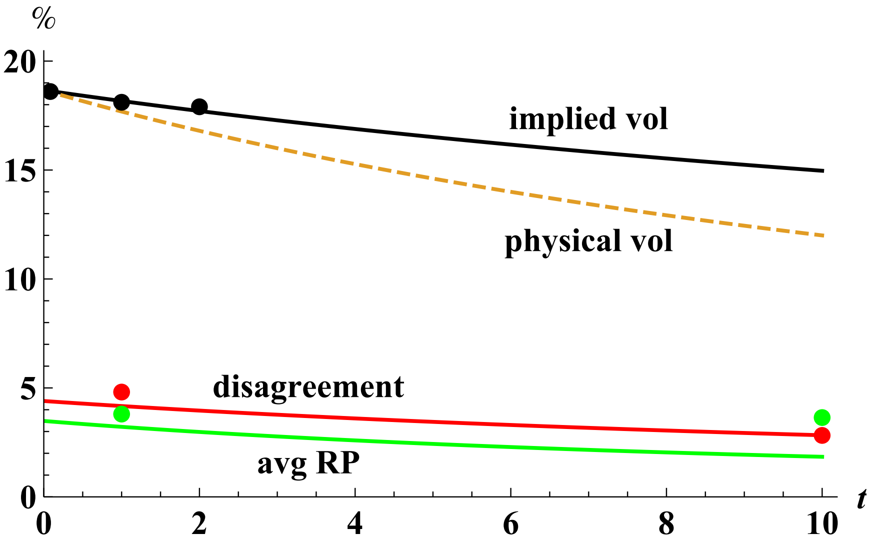

Figure: Volatility term structures in the baseline calibration with $\theta = 1.8$.

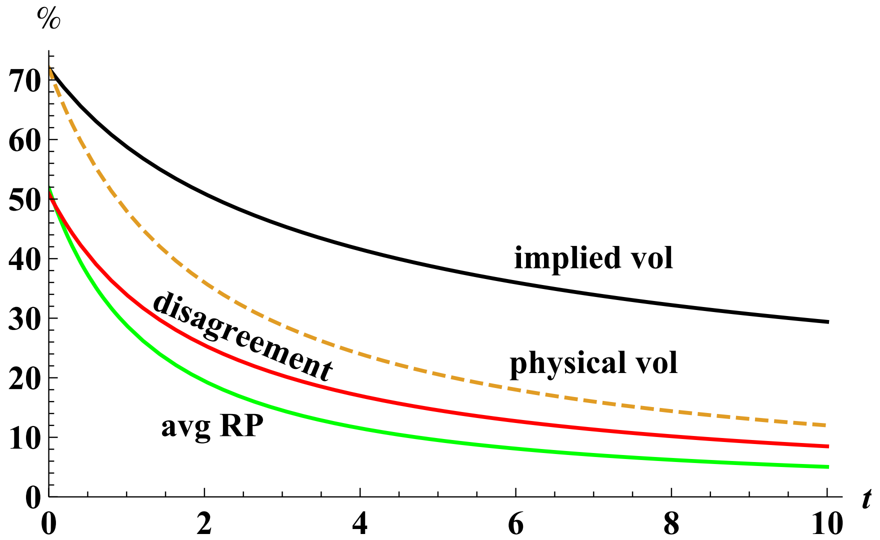

Figure: Volatility term structures in a "crisis" calibration with $\theta = 0.2$.

Speculation and Sharpe ratios

- As agents have different beliefs but agree on market prices, they have different SDFs
- Results so far supply the static Sharpe ratio of the risky asset
- But the max *dynamic* Sharpe ratio attains the Hansen–Jagannathan (1991) bound

Result

The maximum Sharpe ratio (as perceived by investor z) is finite for $\theta > 1$ and is equal to

$$MSR^{(z)}_{0\to T} = \sqrt{\frac{\theta}{\sqrt{\theta^2-1}} \exp\left\{ \frac{\left[z\sqrt{\theta} + (\theta+1)\,\sigma\sqrt{T} \right]^2}{\theta\,(\theta-1)} \right\} - 1}$$

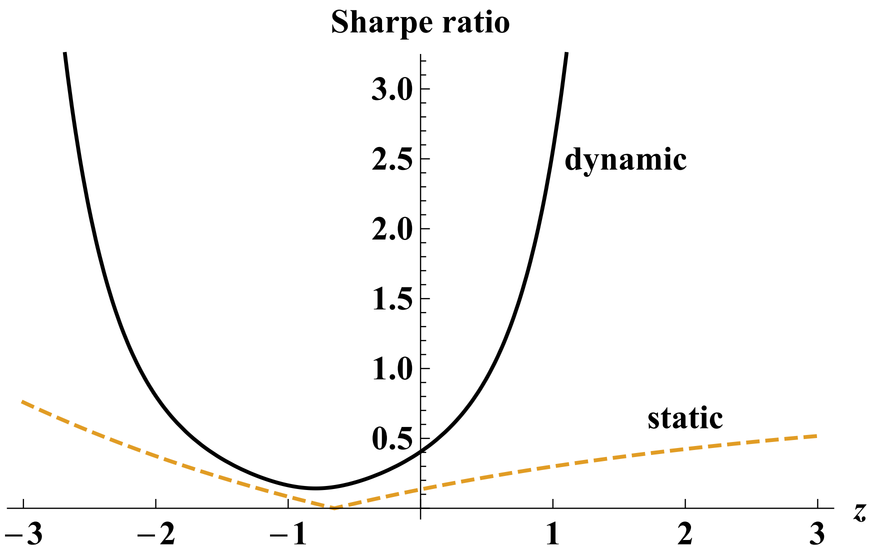

Figure: Max Sharpe ratio (annualized) as perceived by investor z. Baseline calibration.

The gloomy investor

- The gloomy investor who perceives the smallest MSR has $z = z_g$,

$$z_g = -\frac{\theta + 1}{\sqrt{\theta}} \sigma \sqrt{T}$$

- This investor perceives zero instantaneous Sharpe ratio, but a positive MSR associated with a contrarian strategy: buy if the market sells off, sell if the market rallies

$$\text{MSR}_{0 \to T}^{(z_g)} = \sqrt{\frac{\theta}{\sqrt{\theta^2 - 1}} - 1}$$

Agents have target prices

- Terminal wealth of agent z is

$$W^{(z)}(p_T) = p_0\sqrt{\frac{\theta+1}{\theta}}\exp\left\{\frac{1}{2}(z-z_g)^2 - \frac{1}{2(1+\theta)\sigma^2 T}\left[\log\left(p_T/K^{(z)}\right)\right]^2\right\}$$

- Target price $K^{(z)}$ for investor z—their ideal outcome—satisfies

$$\log K^{(z)} = \mathbb{E}^{(z)}\log p_T + (z-z_g)\sigma\sqrt{\theta T}$$

- Gloomy investor $z = z_g$ wants to be proved right: ideal outcome equals expected outcome (in logs)
- But extremists are happiest if the market moves even more than they expect

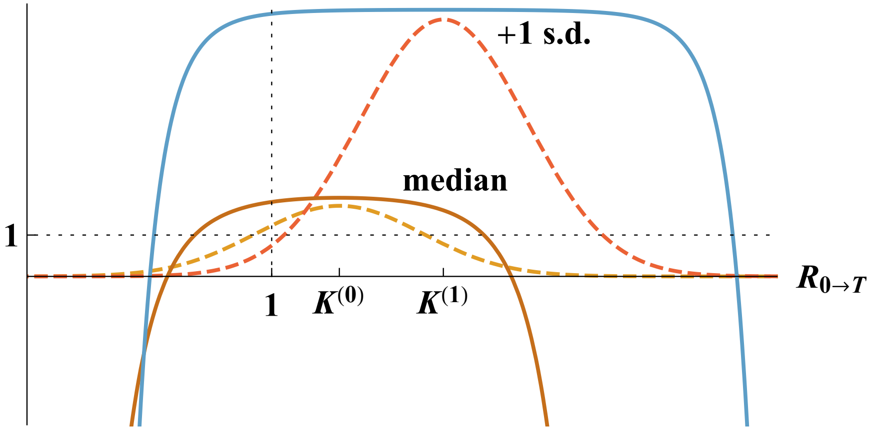

Figure: MSR strategies (solid) are far from optimal strategies (dashed). Log scale on x-axis.

Conclusions

- Sentiment generates volatility, speculation, and volume
- Sentiment can push prices either up or down
- Extreme outcomes are far more important than in a homogeneous economy
- Downward-sloping vol term structure in a diffusion limit, and a variance risk premium
- Agents have target prices
- Moderate investors are contrarian, “short vol”, supply liquidity to extremists