Risk Price Variation: The Missing Half of Empirical Asset Pricing

NBER Forecasting & Empirical Methods

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- **This paper is about cross-sectional variation in risk prices (λ)**

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- We contribute an approach to **estimate** and **test** for variation in λ across assets based on methods in machine learning

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$\Longrightarrow$ Differences in λ are pervasive and important!

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  - Panel Heterogeneity: Hahn & Moon (2010); Lin & Ng (2012); Saradis & Weber (2015); Bonhomme & Manresa (2015)

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- Implied by Errunza & Losq (1985) and Gromb & Vayanos (2018), among others

Group Assignment and Parameter Estimation

- Consider a candidate set of G groups of assets. We want to solve

$$\left(\hat{\Gamma}, \hat{\Lambda}\right) = \arg\min_{\Gamma,\Lambda} \sum_{i,t} \left(r_{it} - \alpha^{(\gamma_i)} - \sum_{k} \beta_{ik} \lambda_{kt}^{(\gamma_i)} \right)^2$$

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- And we don't have differentiability for γ_i, which complicates most standard solution methods

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  - See the paper for discussion of multi-start and genetic algorithm methods used to achieve global optima (▸ Local and Global Optima)

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  2. Existing work that accounts for this estimation step, e.g. Bonhomme and Manresa (2015), requires clusters to be "well-separated," which is not true under the null of unified prices

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- If dependence between $\mathcal{R}$ and $\mathcal{P}$ samples is limited, this split eliminates the overfitting problem arising from estimated clusters

Null Hypotheses

- We consider two tests. The null in both is of **no segmentation / equal risk prices / the Law of One Price**:

$$H_0: \ \bar{\lambda}_k^{(1)} = \bar{\lambda}_k^{(2)} = \ldots = \bar{\lambda}_k^{(G)} \ \forall k$$
$$vs. \ \ H_1: \ \bar{\lambda}_k^{(g)} \neq \bar{\lambda}_k^{(g')} \text{ for some } k, g, g'.$$

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- The first test generalizes FMB-style t tests to look at differences in **expected returns** across clusters
- The second tests enriches the first by adding information from the **dynamics** of cross-sectional slopes
- Note: these tests do not consider $\bar{\alpha}^{(g)}$ or $\alpha_t^{(g)}$ because our focus is on risk price heterogeneity, not on zero-beta rates

Test Statistics and Inference

- Our two test statistics are:

$$F^{Avg} = \frac{1}{(G-1)K} \sum_{g=1}^{G-1} \Delta\bar{\lambda}^{(g,g+1)\prime} \left(\hat{\Sigma}_{\bar{\lambda}^{(g)}} + \hat{\Sigma}_{\bar{\lambda}^{(g+1)}} \right)^{-1} \Delta\bar{\lambda}^{(g,g+1)}$$

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- We confirm in simulation studies that the tests have approximately correct size ▸ Simulation Study

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Segmentation Everywhere: Domestic Equity Portfolios

Dynamic test rejects everywhere, Avg test rejects less for P1

		Equal Avg Risk Prices		Equal Dyn Risk Prices	
	Model	1963–2016	1999–2016	1963–2016	1999–2016
P1	CAPM	0.312	0.561	0.026	0.006
	FF3F	0.057	0.011	0.000	0.000
	Carhart	0.050	0.102	0.000	0.000
	FF5F	0.078	0.375	0.000	0.000
	HKM	0.236	0.057	0.037	0.001
	HXZQ	0.000	0.671	0.000	0.000
	Carhart+3	0.005	1.000	0.000	0.000
P3	CAPM	0.052	0.006	0.000	0.000
	FF3F	0.000	0.000	0.000	0.000
	Carhart	0.000	0.000	0.000	0.000
	FF5F	0.000	0.000	0.000	0.000
	HKM	0.543	0.007	0.000	0.000
	HXZQ	0.000	0.000	0.000	0.000
	Carhart+3	0.000	0.004	0.000	0.000

Segmentation Everywhere: Placebo Portfolios

Neither test rejects more than expected by chance for placebo portfolios

		Equal Avg Risk Prices		Equal Dyn Risk Prices	
	Model	1963–2016	1999–2016	1963–2016	1999–2016
P4	CAPM	0.082	0.618	0.086	1.000
	FF3F	0.579	0.101	1.000	1.000
	Carhart	0.594	0.822	0.602	0.262
	FF5F	0.153	0.883	1.000	1.000
	HKM	1.000	0.711	0.015	0.466
	HXZQ	0.246	0.197	1.000	0.198
	Carhart+3	0.599	1.000	0.118	0.041

We find no segmentation when risk prices are the same!

(formalized in our simulation study)

Segmentation Everywhere: Int'l Equity Portfolios

Average and Dynamic tests reject everywhere

		Equal Avg Risk Prices		**Equal Dyn Risk Prices**	
	Model	1991–2016	2004–2016	1991–2016	2004–2016
P5	CAPM	0.000	0.000	0.000	0.000
	FF3F	0.000	0.000	0.000	0.000
	Carhart	0.000	0.000	0.000	0.000
	FF5F	0.000	0.000	0.000	0.000
	HKM	0.000	0.000	0.000	0.000
	HXZQ	–	–	–	–
	Carhart+3	0.002	0.001	0.000	0.000
P6	CAPM	0.000	0.000	0.000	0.000
	FF3F	0.000	0.000	0.000	0.000
	Carhart	0.000	0.000	0.000	0.000
	FF5F	0.000	0.000	0.000	0.000
	HKM	0.000	0.000	0.000	0.000
	HXZQ	–	–	–	–
	Carhart+3	0.000	0.000	0.000	0.000

Segmentation Everywhere: Multi-Asset Class Portfolios

Average and Dynamic tests reject everywhere

		Equal Avg Risk Prices		Equal Dyn Risk Prices	
	Model	1986–2010	1998–2010	1986–2010	1998–2010
P7	CAPM	0.000	0.000	0.000	0.000
	FF3F	0.000	0.000	0.000	0.000
	Carhart	0.000	0.000	0.000	0.000
	FF5F	0.000	0.000	0.000	0.000
	HKM	0.000	0.000	0.000	0.000
	HXZQ	0.000	0.000	0.000	0.000
	Carhart+3	0.000	0.000	0.000	0.000
P8	CAPM	0.000	0.000	0.000	0.000
	FF3F	0.000	0.000	0.000	0.000
	Carhart	0.000	0.000	0.000	0.000
	FF5F	0.000	0.000	0.000	0.000
	HKM	0.000	0.000	0.000	0.000
	HXZQ	0.000	0.000	0.000	0.000
	Carhart+3	0.000	0.000	0.000	0.000

Segmentation Everywhere: Summary

- **Statistical evidence** of segmented markets is **ubiquitous**. For the tests of equal factor dynamics:
  1. Domestic equities: 80/81 tests reject the null of a single cluster
  2. International equities: all 36 tests reject with p-val=0.000
  3. Multi-asset class portfolios: all 42 tests reject with p-val=0.000
- Differences in **average** risk prices are also strongly significant, reject null for 57/81, 35/36 and 41/42 cases
- But are violations of unified risk pricing also **economically meaningful?**

Economic vs. Statistical Significance of Segmentation

- We measure **economic significance** in two ways:
  1. **Increased explanatory power** for cross-section of expected returns:

$$\frac{\sigma^2_{G*}(\bar{r})}{\sigma^2_1(\bar{r})} \equiv \frac{var_i\left(\frac{1}{T}\sum_{t=1}^{T}\left(\hat{\alpha}_t^{(\hat{\gamma}_i)} + \hat{\beta}_i\hat{\lambda}_t^{(\hat{\gamma}_i)}\right)\right)}{var_i\left(\frac{1}{T}\sum_{t=1}^{T}\left(\tilde{\alpha}_t + \hat{\beta}_i\tilde{\lambda}_t\right)\right)}$$

  2. Improvements of maximal, in-sample **Sharpe ratio**:

$$\Delta SR_{G*} \equiv \sqrt{\mu'_\Lambda \Sigma_\Lambda^{-1}\mu_\Lambda} - \sqrt{\mu'_\lambda \Sigma_\lambda^{-1}\mu_\lambda}$$

Economic Importance: Domestic Portfolios

Gains in explanatory power of around 15–70%, increases in SR of around 0.15–0.80

	Model	$\sigma^2(\bar{r}_{G*})$ / $\sigma^2(\bar{r}_1)$		$SR_{G*} - SR_1$	
		63–16	**99–16**	**63–16**	**99–16**
P1	CAPM	3.77	161.88	0.26	0.15
	FF3F	1.81	1.80	0.74	-0.09
	Carhart	1.03	1.38	0.10	0.16
	FF5F	*	1.10	*	0.15
	HKM	8.25	5.30	0.33	0.38
	HXZQ	1.16	2.67	0.67	0.29
	Carhart+3	*	*	*	*
P3	CAPM	2.75	6.16	0.17	0.48
	FF3F	1.67	1.75	0.82	0.47
	Carhart	1.41	1.55	0.86	0.85
	FF5F	1.49	1.22	0.54	0.14
	HKM	6.25	10.42	0.08	0.72
	HXZQ	1.51	2.39	0.69	0.69
	Carhart+3	1.20	1.23	0.51	0.32

Economic Importance: International Portfolios

Gains in explanatory power of 100-300%, increases in SR of around 0.4-0.8

		$\sigma^2(\bar{r}_{G*})$ / $\sigma^2(\bar{r}_1)$		$SR_{G*} - SR_1$	
	Model	**91–16**	**04–16**	**91–16**	**04–16**
P5	CAPM	7.20	1.34	0.55	0.06
	FF3F	5.00	1.25	0.51	0.13
	Carhart	5.61	1.07	1.31	0.25
	FF5F	1.33	1.11	0.54	0.92
	HKM	4.15	1.30	0.40	0.48
	HXZQ	–	–	–	–
	Carhart+3	2.22	1.12	0.64	0.77
P6	CAPM	3.98	1.21	0.89	0.71
	FF3F	3.06	1.46	1.13	0.93
	Carhart	4.07	1.37	1.62	0.75
	FF5F	2.10	1.07	1.27	0.70
	HKM	3.62	1.23	1.16	0.79
	HXZQ	–	–	–	–
	Carhart+3	2.34	1.08	1.61	0.99

Economic Importance: Multi-Asset Class Portfolios

Gains in explanatory power of 5-30%, increases in SR of around 0.4-0.9

		$\sigma^2(\bar{r}_{G*})$ / $\sigma^2(\bar{r}_1)$		$SR_{G*} - SR_1$	
	Model	**86-10**	**98-10**	**86-10**	**98-10**
P7	CAPM	11.97	69.22	1.03	1.70
	FF3F	1.33	1.84	0.55	0.87
	Carhart	0.81	1.13	0.61	0.68
	FF5F	1.15	2.15	0.42	0.93
	HKM	7.48	19.69	0.79	1.40
	HXZQ	1.05	2.69	0.44	1.06
	Carhart+3	0.90	0.97	0.56	1.12
P8	CAPM	4.93	5.95	1.11	0.68
	FF3F	1.27	1.34	0.80	1.26
	Carhart	1.08	2.48	0.87	1.80
	FF5F	1.23	2.80	0.95	1.93
	HKM	4.47	2.40	0.87	1.02
	HXZQ	1.21	1.56	0.90	1.41
	Carhart+3	1.18	1.08	1.47	1.37

Detailed Example: Domestic Equity Portfolios

Domestic Equity Portfolios (P3): Domestic Carhart, 1963–2016

Table: Determining the number of clusters

	# Clusters (G)					
	1	**2**	**3**	**4**	**5**	**2–5**
Avg test *p*-val	–	**0.000**	0.487	0.490	0.000	0.000
Dyn test *p*-val	–	**0.000**	0.000	0.000	0.000	0.000
LL ($\times 10^{-6}$)	6.44	**6.51**	6.53	6.54	6.55	
AIC ($\times 10^{-6}$)	-12.81	-**12.89**	-12.86	-12.82	-12.79	

Detailed Example: Domestic Equity Portfolios

Domestic Equity Portfolios (P3): Domestic Carhart, 1963–2016

Table: Parameter estimates of 1- and G^*- cluster models

	G=1	G=2		
	All	**Grp 1**	**Grp 2**	$p_F\left(\bar{\lambda}=\right)$
$\bar{\lambda}_{MKT}$	-1.13	-0.52	2.33	0.20
t-stat	(-0.44)	(-0.16)	(1.02)	
$\bar{\lambda}_{HML}$	3.79	2.12	8.12	0.00
t-stat	(2.26)	(1.35)	(3.66)	
$\bar{\lambda}_{SMB}$	1.60	2.21	-2.54	0.03
t-stat	(0.98)	(1.21)	(-0.86)	
$\bar{\lambda}_{UMD}$	7.11	5.57	10.43	0.00
t-stat	(3.46)	(2.91)	(4.01)	
R^2_G	0.91	0.90	0.94	
$R^2_{Combined}$	0.91	0.92		

Detailed Example: Domestic Equity Portfolios

Domestic Equity Portfolios (P3): Domestic Carhart, 1963–2016

Table: Estimated group memberships

	G=1	G=2	
	All	**Grp 1**	**Grp 2**
ME 1-3	81	0	81
ME 4-5	54	54	0
Industry	49	44	5
Other	50	50	0
N_G	234	148	86
T	13469	13469	13469
	Conjectured labels:	Large cap.	Small cap.

Interpretation: Market capitalization is the **single most important** determinant of risk-price heterogeneity in domestic equity portfolios

Detailed Example: International Equity Portfolios

International Equity Portfolios (P6): Global Carhart, 1991–2016

Table: Determining the number of clusters

	# Clusters (G)					
	1	**2**	**3**	**4**	**5**	**2–5**
Avg test *p*-val	–	0.000	0.000	**0.000**	0.000	0.000
Dyn test *p*-val	–	0.000	0.000	**0.000**	0.000	0.000
LL ($\times 10^{-4}$)	5.498	6.003	6.195	**6.318**	6.328	
AIC ($\times 10^{-4}$)	-10.852	-11.718	-11.958	**-12.060**	-11.935	

Detailed Example: International Equity Portfolios

International Equity Portfolios (P6): Global Carhart, 1991–2016

Table: Parameter estimates of 1- and G^*- cluster models

	G=1	G=4				
	All	**Grp 1**	**Grp 2**	**Grp 3**	**Grp 4**	$p_F\left(\bar{\lambda} =\right)$
$\bar{\lambda}_{MKT}$	4.24	-1.57	-12.06	-0.55	2.03	0.12
t-stat	(0.73)	(-0.33)	(-2.38)	(-0.14)	(0.26)	
$\bar{\lambda}_{HML}$	1.07	2.86	9.09	4.38	6.48	0.11
t-stat	(0.42)	(1.22)	(2.36)	(1.35)	(2.22)	
$\bar{\lambda}_{SMB}$	0.05	2.82	-0.55	0.73	2.93	0.61
t-stat	(0.02)	(1.64)	(-0.16)	(0.37)	(1.02)	
$\bar{\lambda}_{UMD}$	8.07	5.79	18.69	12.16	4.00	0.00
t-stat	(2.79)	(1.96)	(3.59)	(3.70)	(0.93)	
R^2_G	0.76	0.94	0.89	0.95	0.94	
$R^2_{Combined}$	0.76		0.93			

Detailed Example: International Equity Portfolios

International Equity Portfolios (P6): Global Carhart, 1991–2016

Table: Estimated group memberships

	G=1	G=4			
	All	Grp 1	Grp 2	Grp 3	Grp 4
NA	50	50	0	0	0
AP	50	0	50	0	0
EU	50	0	0	50	0
JP	50	0	0	0	50
N_G	200	50	50	50	50
T	6783	312	312	312	312
Conjectured labels:		NA	AP	EU	JP

Interpretation: Regional stock markets are internally integrated and (perfectly) **externally segmented**

Detailed Example: Multi-Asset Class Portfolios

Cross-Asset Class Portfolios (P8): He, Kelly, and Manela (2017) Factors, 1986–2010

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	1	**2**	**3**	**4**	**5**	**2–5**
Avg test *p*-val	–	0.000	0.000	0.000	**0.000**	0.000
Dyn test *p*-val	–	0.000	0.000	0.000	**0.000**	0.000
LL ($\times 10^{-6}$)	5.19	5.46	5.55	5.61	**5.66**	
AIC ($\times 10^{-6}$)	-10.30	-10.75	-10.83	-10.88	**-10.89**	

Detailed Example: Multi-Asset Class Portfolios

Cross-Asset Class Portfolios (P8): He, Kelly, and Manela (2017) Factors, 1986–2010

Table: Parameter estimates of 1- and G^*- cluster models

	G=1	G=5					
	All	**G1**	**G2**	**G3**	**G4**	**G5**	$p_F\left(\bar{\lambda} =\right)$
$\bar{\alpha}$	0.62	-31.05	2.67	-0.13	11.42	0.54	0.00
t-stat	(6.41)	(-4.53)	(4.02)	(-0.05)	(2.76)	(6.81)	
$\bar{\lambda}_{MKT}$	7.14	45.85	10.18	10.57	-2.39	7.33	0.00
t-stat	(2.22)	(4.77)	(1.30)	(2.53)	(-0.48)	(2.10)	
$\bar{\lambda}_{HKM}$	9.30	-48.38	22.84	14.43	-8.47	9.91	0.06
t-stat	(1.18)	(-1.34)	(1.75)	(1.15)	(-0.87)	(1.14)	
R^2_G	0.74	0.98	0.58	0.85	0.91	0.84	
$R^2_{Combined}$	0.74			0.88			

Detailed Example: Multi-Asset Class Portfolios

Cross-Asset Class Portfolios (P8): He, Kelly, and Manela (2017) Factors, 1986–2010

Table: Estimated group memberships

	G=1	G=5				
	All	**G1**	**G2**	**G3**	**G4**	**G5**
Options	18	18	0	0	0	0
Commod.	23	0	14	5	0	4
US Bonds	20	0	16	0	0	4
FX	12	0	0	11	0	1
Stocks	75	2	0	16	46	11
N_G	148	20	30	32	46	20
T	300	300	300	300	300	300
Conjectured labels:		Options	Commod. / Bonds	FX+	Stocks	Other

Interpretation: Options, commodities and bonds, FX and some stock portfolios, and other domestic stock portfolios have **very different risk prices**, even when confronted by a unifying intermediary-asset pricing model

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$$\text{Model 1:} \quad r_{it} = \alpha_t^{(1)} \mathbf{1}(i \in G_1) + \alpha_t^{(2)} \mathbf{1}(i \in G_2) + \epsilon_{it},$$

$$\text{Model 2:} \quad r_{it} = \tilde{\alpha}_t + \beta_i \eta_t + \tilde{\epsilon}_{it}.$$

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- If the factor model (Model 2) is true, estimating the two-cluster model (Model 1) gives

$$\widehat{\Delta\alpha_t} = \eta_t \left(E\left[\beta_i | i \in G_1\right] - E\left[\beta_i | i \in G_2\right]\right).$$

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$\Longrightarrow$ We obtain separation in cross-sectional slopes so long as the average loadings β_i differ between "clusters" and the factor is priced

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  - The reverse also occurs: clusters can manifest as new "factors"

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  3. $K_3^* =$ AIC-optimal, up to a maximum of $(G^* - 1)(K + 1) - 1$

Extra Clusters or Factors: Domestic Equity Portfolios

Omitted factors are comparably important for P1 and large factor models

		1963–2016			1999–2016		
	Model	K_1^*	K_2^*	K_3^*	K_1^*	K_2^*	K_3^*
P1	CAPM	−−	0	+++	−−	0	−−
	FF3F	−−−	0	0	−−	0	−−
	Carhart	+++	+++	+++	++	+++	+++
	FF5F	*	*	*	−	+++	0
	HKM	−−−	0	0	++	+++	+++
	HXZQ	+++	+++	+++	*	*	*
	Carhart+3	*	*	*	*	*	*
P3	CAPM	+++	+++	+++	0	−−	0
	FF3F	+++	+++	+++	0	+++	−−−
	Carhart	+++	+++	+++	+++	+++	−−
	FF5F	+++	+++	0	0	+++	−−
	HKM	0	+++	+++	+++	+++	−−−
	HXZQ	+++	+++	+++	+++	+++	0
	Carhart+3	+++	+++	+++	+++	+++	+++

Extra Clusters or Factors: International Portfolios

Multiple risk prices are generally favored, and universally so for more variegated portfolio sets

		1991–2016			2004–2016		
	Model	K_1^*	K_2^*	K_3^*	K_1^*	K_2^*	K_3^*
	CAPM	+++	+++	+++	0	+++	0
	FF3F	+++	+++	+++	−−−	+++	−−−
	Carhart	−−−	+++	−−−	−−−	+++	−−−
P5	FF5F	+++	+++	+++	+++	+++	+++
	HKM	+++	+++	+++	+++	+++	+++
	HXZQ						
	Carhart+3	+++	+++	+++	+++	+++	+++
	CAPM	+++	+++	+++	+++	+++	+++
	FF3F	+++	+++	0	+++	+++	+++
	Carhart	+++	+++	+++	+++	+++	+++
P6	FF5F	0	+++	0	+++	+++	+++
	HKM	+++	+++	+++	+++	+++	+++
	HXZQ						
	Carhart+3	+++	+++	+++	+++	+++	0

Extra Clusters or Factors: Multi-Asset Class Portfolios

Multiple risk prices are almost always strongly preferred

		1986–2010			**1998–2010**		
	Model	K_1^*	K_2^*	K_3^*	K_1^*	K_2^*	K_3^*
	CAPM	+++	+++	+++	+++	+++	+++
	FF3F	+++	+++	+++	+++	+++	+++
	Carhart	+++	+++	+++	+++	+++	+++
P7	FF5F	+++	+++	+++	+++	+++	++
	HKM	+++	+++	+++	+++	+++	+++
	HXZQ	+++	+++	+++	+++	+++	+++
	Carhart+3	+++	+++	+++	+++	+++	+++
	CAPM	+++	+++	++	+++	+++	0
	FF3F	+++	+++	0	+++	+++	+++
	Carhart	+++	+++	+++	+++	+++	+++
P8	FF5F	+++	+++	+++	+++	+++	0
	HKM	+++	+++	+++	+++	+++	+++
	HXZQ	+++	+++	+++	+++	+++	+++
	Carhart+3	+++	+++	+++	+++	+++	+++

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  - The zoo of **"expected return factors"** may be a side effect of **heterogeneous risk prices**

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  - Comovements among securities influence portfolio dynamics and cluster assignments

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- $M = 500$ permutations, $S = 500$ replications of each design.

Simulation Study ◂ Back

- We use US domestic equity portfolios (P3 in next section) to calibrate parameters of (null) DGP

$$r_{it} = \beta_i f_t + \epsilon_{it}$$

- 16 different specifications:
  1. "Daily" data or "Monthly" data ($T = 10000$ or $T = 300$)
  2. Small or large cross-section ($N = 75$ or $N = 225$)
  3. CAPM or Carhart factor model ($K = 1$ or $K = 4$)
  4. iid or GARCH in volatility [will only show GARCH results below]
- $M = 500$ permutations, $S = 500$ replications of each design.
- Tables below show rejection frequencies of 5% level tests.

Simulation Study

Back

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- $M = 500$ permutations, $S = 500$ replications of each design.
- Tables below show rejection frequencies of 5% level tests.
- Computing time for this simulation study is $\approx 80,000$ CPU hours

Simulation Results: T=10,000 Days

Back

Rejection frequencies are all close to 0.05

N	*K*	*Test*	**G=2**	**=3**	**=4**	**=5**	$\in$**2–5**
75	**1**	**Avg**	0.06	0.08	0.07	0.05	0.07
75	**4**	**Avg**	0.05	0.05	0.04	0.05	0.07
225	**1**	**Avg**	0.04	0.06	0.07	0.06	0.05
225	**4**	**Avg**	0.06	0.06	0.06	0.06	0.08
75	**1**	**Dyn**	0.09	0.04	0.01	0.08	0.07
75	**4**	**Dyn**	0.04	0.01	0.03	0.07	0.04
225	**1**	**Dyn**	0.07	0.08	0.09	0.08	0.08
225	**4**	**Dyn**	0.07	0.08	0.07	0.04	0.06

Simulation Results: T=300 Months

◂ Back

Rejection frequencies are all close to 0.05, except when N,K are large

N	*K*	*Test*	**G=2**	**=3**	**=4**	**=5**	$\in$**2–5**
75	**1**	**Avg**	0.04	0.05	0.07	0.07	0.07
75	**4**	**Avg**	0.06	0.06	0.06	0.04	0.07
225	**1**	**Avg**	0.04	0.05	0.05	0.03	0.05
225	**4**	**Avg**	0.07	0.09	0.09	0.08	0.11
75	**1**	**Dyn**	0.05	0.05	0.05	0.13	0.08
75	**4**	**Dyn**	0.06	0.03	0.02	0.05	0.04
225	**1**	**Dyn**	0.06	0.06	0.05	0.06	0.06
225	**4**	**Dyn**	0.13	0.13	0.12	0.07	0.17

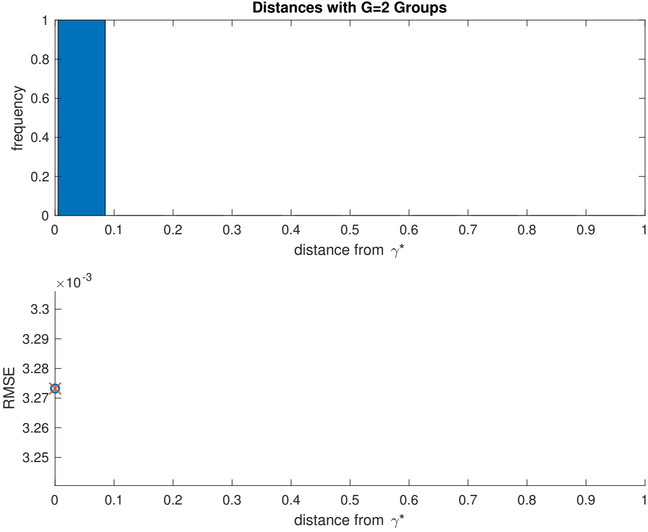
Local and Global Optima

Back

- We use two procedures to find global optima:
  1. Multi-start with $2N$ starting group assignments selected using a generalized version of k-means++
     - We use k-means++ initialization because our EM procedure can be recast as an extension of k-means
  2. Genetic algorithm solutions to optimal group assignment as a mixed-integer programming problem (MATLAB's implementation)
- Appendix A.1 of the paper provides further details
- Appendix A.2 demonstrates that global optimization is sometimes—but not always—important for obtaining the global-best group assignments

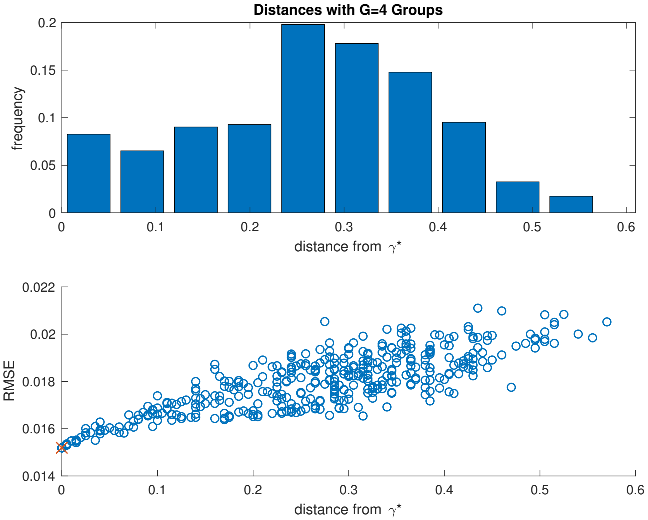
Example 1: Local and Global Optima Coincide ◂ Back

Domestic Equity Portfolios: Domestic Carhart, 1963-2016



Example 2: Local and Global Optima are "Close" Back

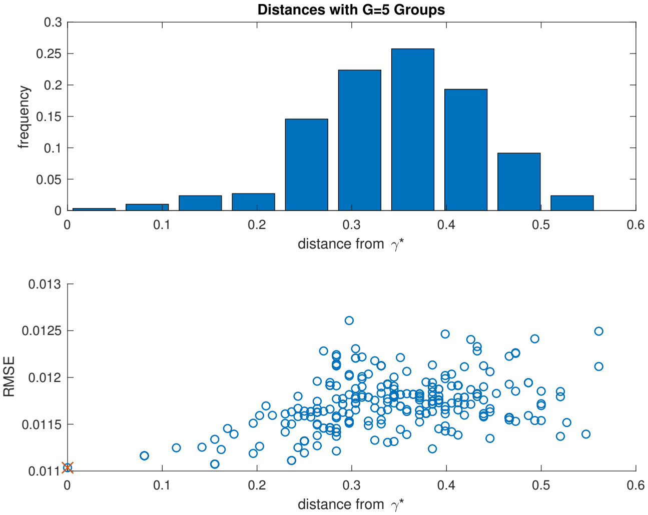
International Equity Portfolios: Global Carhart, 1991-2016



Example 3: Global Optimum is Isolated

Back

Cross-Asset Class Portfolios: He, Kelly, and Manela (2016) Factors, 1986-2010



Group Stability: Domestic Equity Portfolios ◂ Back

Domestic equity portfolio assignments are stable over time

	Model	**P1**	**P2**	**P3**
Period	CAPM	0.89	0.64	0.61
2	FF3F	0.96	0.51	0.63
1981–1998	Carhart	0.92	0.82	**0.68**
Period	FF5F	0.85	0.96	0.90
3	HKM	0.81	0.77	0.69
1999–2016	HXZQ	0.85	0.60	0.51
	Carhart+3	1.00	0.82	0.58

Table reports maximal proportion of group labels in common over all permutations of group labels

Group Stability: International Equity Portfolios

Back

International equity portfolio assignments are highly stable over time

	Model	P5	P6
Period	CAPM	0.87	0.68
1	FF3F	0.61	0.84
1991–2003	Carhart	0.74	**0.78**
Period	FF5F	0.99	0.70
2	HKM	0.55	0.91
2004–2016	HXZQ	–	–
	Carhart+3	0.59	0.65

Table reports maximal proportion of group labels in common over all permutations of group labels

Group Stability: Multi-Asset Class Portfolios Back

Dimensions of heterogeneity among asset classes change over time

	Model	P7	P8
Period	CAPM	0.70	0.70
1	FF3F	0.56	0.57
1986–1997	Carhart	0.57	0.55
Period	FF5F	0.73	0.76
2	HKM	0.62	**0.55**
1998–2010	HXZQ	0.56	0.57
	Carhart+3	0.88	0.53

Table reports maximal proportion of group labels in common over all permutations of group labels