

Declining Search Frictions, Unemployment and Growth*

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Abstract

The Diamond-Mortensen-Pissarides theory argues that unemployment and vacancies emerge because of search frictions in the labor market. Yet, over the last century, US unemployment and vacancy rates show no trend, even though search efficiency in the labor market must have improved thanks to the diffusion of telephones, computers and the Internet. We resolve this puzzle using a search model where firm-worker matches are inspection goods. We show that iff the distribution of idiosyncratic productivity for new matches is Pareto, then unemployment, vacancy, job-finding and job-loss rates remain constant while the efficiency of search grows over time. Improvements in search technology show up in productivity growth. A corollary of our theory is that population growth does not affect unemployment and vacancy rates even under non-constant returns to scale in the search process. We develop and implement a strategy to measure the growth rate of the search technology, the returns to scale of the search process, and their contribution to productivity growth.

JEL Codes: E24, O40, R11.

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1 Introduction

The US Beveridge curve has been rather stable ever since data on unemployment and vacancies have begun being collected in the 1920s.¹ As one can see from Figure 1, the Beveridge curve has been stable during the five decades going from 1927 to 1976, it has shifted out during the 1977-1986 decade, shifted back in during the 1987-1996 decade, and it has again shifted out and back during the period between 2007 and 2016. While much research has been devoted to the cyclical shifts of the Beveridge curve (see, e.g. Kaplan and Menzio 2016, Gavazza, Mongey and Violante 2018 and Sniekers 2018), what we find truly remarkable in Figure 1 is the lack of any secular trend in the Beveridge curve. Indeed, right now the Beveridge curve is basically where it was in the late 1940s.

There have also been no secular shifts along the Beveridge curve. As illustrated in Figure 2, the unemployment rate is quite volatile at business-cycle frequency, but it displays no secular trend.² Similarly, the vacancy rate is also very volatile at business-cycle frequency, although in the opposite direction, and does not feature any secular trend. Since unemployment and vacancy rates have no secular trend, neither does the tightness of the labor market, which is defined as the vacancy-to-unemployment ratio.

The standard theory of unemployment, vacancies, and the Beveridge curve has been developed by Diamond (1982), Mortensen (1982) and Pissarides (2000). The theory is based on the view that unemployment and vacancies coexist because searching the labor market for a trading partner is a time-consuming activity. The relationship between unemployment and vacancies is downward sloping because the vacancy-to-unemployment ratio increases the speed at which unemployed workers find jobs and, hence, lowers unemployment. Formally, the theory states that the Beveridge curve is given by

$$u_t = \frac{\delta_t}{\delta_t + A_t p(v_t/u_t)}, \quad (1.1)$$

where u_t and v_t denote the unemployment and vacancy rates, δ_t denotes the rate at which employed workers become unemployed (henceforth, the EU rate), and $A_t p(v_t/u_t)$ denotes the rate at which unemployed workers become employed (henceforth, the UE rate), which is given by the product between a parameter A_t that controls the efficiency of the search process and an increasing function $p(v_t/u_t)$ of the labor market tightness.

The secular stability of the Beveridge curve is puzzling from the perspective of search

¹Figures 1 and 2 are constructed using the time-series for unemployment and vacancies in Petrosky-Nadeau and Zhang (2013). The reader can find a description of these time-series in Appendix A.

²We do not mean to say that the unemployment rate has no low-frequency movements, but that these low-frequency movements are upwards over some periods and downwards over other periods, rather than being systematically in the same direction.

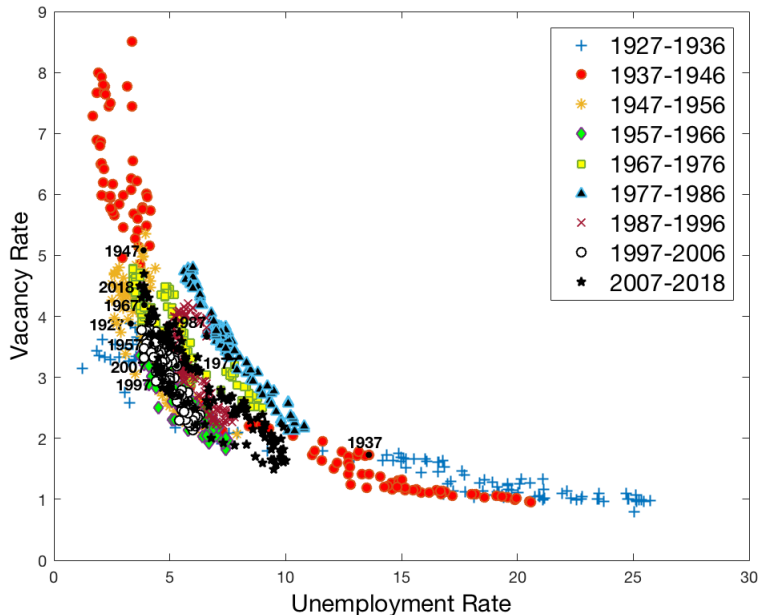


Figure 1: Beveridge Curve US: 1927-2018

theory. It suggests either that there has been no increase in the efficiency A_t of the search process from 1927 to 2018, or that every increase in A_t has been offset by an increase in the EU rate δ_t . The first possibility seems implausible, as the period from 1927 to 2018 has witnessed the development and diffusion of a great deal of communication and information technology—the telephone, the fax machine, the copying machine, the computer, the Internet—which must have had an impact on the ability of firms to announce their job openings to the market, on the ability of workers to learn about and apply to job openings and, ultimately, on the efficiency of the search process. The second possibility can be easily refuted by looking at the data. As illustrated in Figure 3, neither the UE rate nor the EU rate have a secular trend, and certainly not an upward one.³

The aim of this paper is to explain why the unemployment rate, the UE rate, the EU rate, the vacancy rate and the Beveridge curve have all been substantially stable in the face of major improvements in information technology. In a nutshell, our explanation is based on the observation that, while improvements in the information technology increase the rate at which workers learn about vacancies, they also make workers and firms more selective about the quality of the relationships that they are willing to establish. According

³The UE and EU rates are constructed as in Menzio and Shi (2011). The UE and EU rates corrected for time-aggregation as in Shimer (2005) are similar.

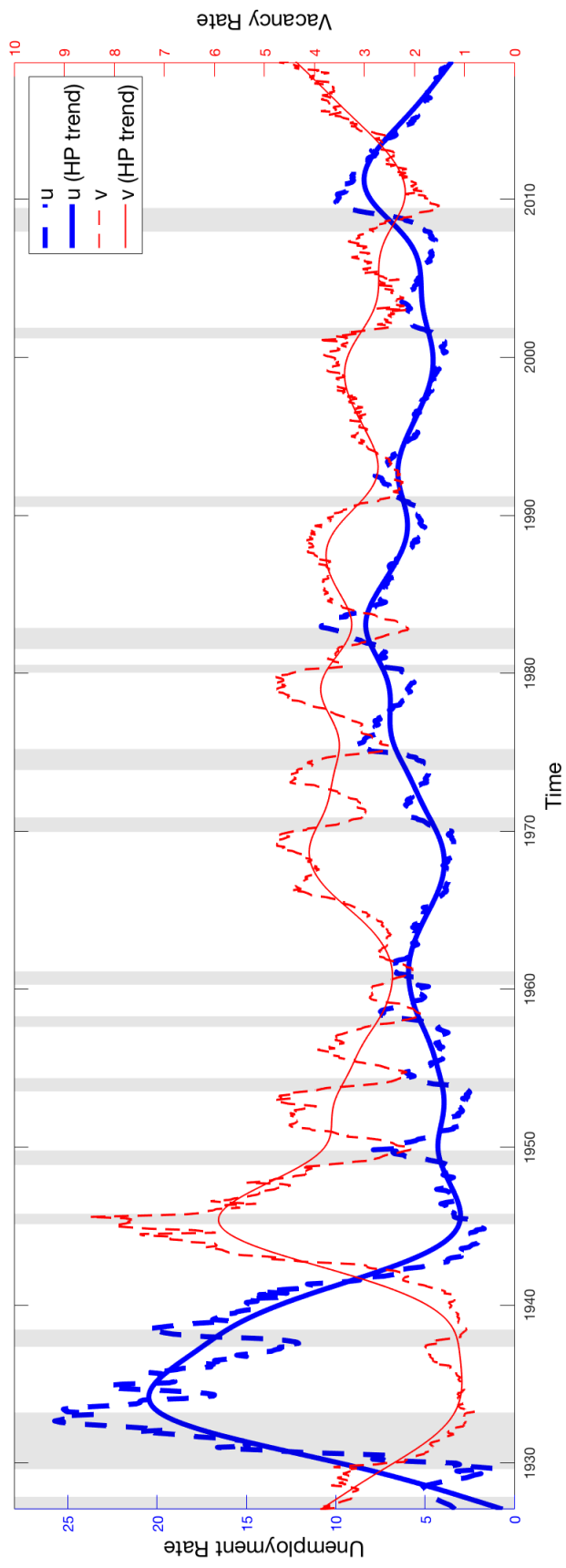


Figure 2: Unemployment and Vacancy Rates US: 1927-2018

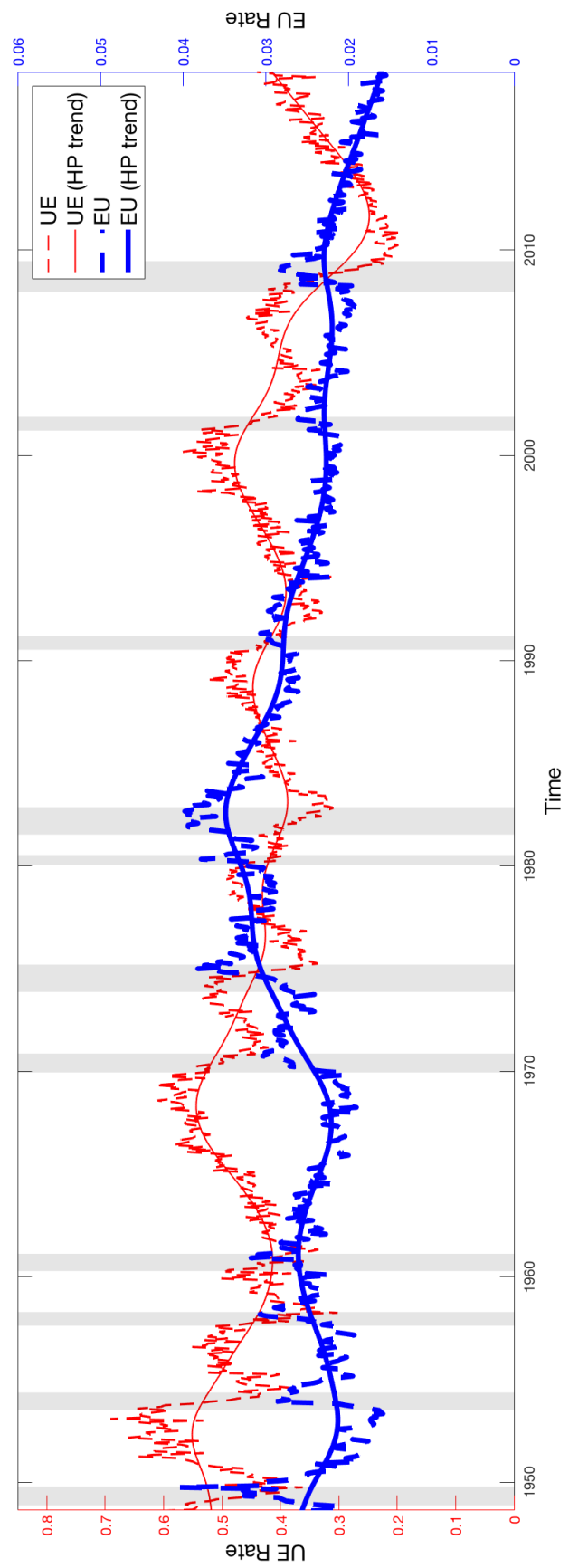


Figure 3: UE and EU Rates US: 1948-2018

to our explanation, the Beveridge curve is given by

$$u_t = \frac{\delta_t}{\delta_t + A_t p(v_t/u_t)(1 - F(R_t))}, \quad (1.2)$$

where F is the distribution of quality for a firm-worker match and R_t is the reservation quality, i.e. an endogenous object that denotes the lowest match quality such that a firm and a worker are willing to start a labor relationship. Under some conditions on the shape of F , the growth in the efficiency A_t of the search process is exactly offset by the endogenous decline in the probability $1 - F(R_t)$ that the match is viable, thus leading to stationary unemployment and vacancy rates, UE and EU rates, and labor market tightness.

In the first part of the paper, we develop our theory in the context of a growth version of Mortensen and Pissarides (1994). As in Mortensen and Pissarides (1994), unemployed workers and vacant firms look for each other in the labor market and the rate at which workers contact firms is given by $A_t p(v_t/u_t)$. In contrast to Mortensen and Pissarides (1994), matches are “inspection goods,” in the sense that—upon meeting—a firm and a worker observe the productivity of their match and, based on this information, decide whether to start an employment relationship or to keep on searching. The environment features growth in both the search technology and the production technology, as the efficiency of the search process grows at a constant rate g_A and the component of productivity that is common to all matches grows at the rate g_y .

We define a Balanced Growth Path (henceforth, BGP) as an equilibrium in which the unemployment rate, the UE rate, the EU rate, the vacancy rate and the tightness of the labor market are constant over time, while the distribution of employed workers across matches of different quality grows at some endogenous rate g_z , in the sense that every quantile of the distribution grows at the rate g_z . We find that a BGP exists if and only if the distribution F of the quality of new matches is Pareto with an arbitrary tail coefficient $\alpha > 1$, and the worker’s income from unemployment and the firm’s cost from posting a vacancy grow at the rate $g_y + g_A/\alpha$. In a BGP, unemployment, vacancies and the UE and EU rates are constant. The distribution of employed workers across matches of different qualities is a truncated Pareto that grows at the rate $g_z = g_A/\alpha$. The average productivity of labor, wages and output per capita all grow at the rate $g_y + g_A/\alpha$.

The intuition behind our findings is simple. Improvements in the search technology lead to an increase in the rate at which unemployed workers meet vacant firms. Simultaneously, improvements in the search technology make firms and workers choosier about the quality of the matches that they are willing to form, as they make it easier to experiment with alternative partners. When the distribution of match qualities is Pareto, the

two effects exactly cancel out, leading to a constant UE rate, EU rate and unemployment. The firm’s return to filling a vacancy grows at the rate $g_A + g_y/\alpha$ because of improvements in the production and search technologies. If the cost of a vacancy grows at the same rate—because, say, opening a vacancy requires the use of labor—then the tightness of the labor market remains constant as well, and so does the vacancy rate. Interestingly, while improvements in the search technology do not lead to any decline in unemployment, they contribute to the growth rate of the economy with a strength that increases with the thickness of the tail of the Pareto distribution.

In the second part of the paper, we generalize the baseline model to allow workers to search the labor market not only when they are unemployed, but also when they have a job (albeit with potentially different intensity). The generalization of the model is relevant, as we know that workers move often from one job directly to another and, thus, the workers’ opportunity cost of accepting a job out of unemployment is not to entirely give up on search. The analysis of the general model is harder, but the necessary and sufficient conditions for the existence of a BGP turn out to be the same as for the baseline model. Moreover, we show that, in any BGP of the on-the-job search model, unemployment, vacancies and the UE and EU rates remain constant over time, the distribution of employed workers across matches of different qualities is a truncated Fréchet that grows at the rate g_A/α , and the average productivity of labor, wages and output all grow at the rate $g_y + g_A/\alpha$.

In the third part of the paper, we further generalize the model to allow for growth in the size of the labor force and for non-constant returns to the scale of the labor market in the search process. We find that the conditions for the existence of a BGP are essentially the same as in the baseline model. In the BGP, unemployment, vacancies and the UE and EU rates are constant. The distribution of employed workers across matches of different qualities grows at the rate $(g_A + \beta g_N)/\alpha$, where g_N is the growth rate of the labor force and β is the parameter that controls the returns to scale in the search process—with $\beta > 0$ meaning increasing, $\beta = 0$ constant, and $\beta < 0$ decreasing returns to scale. Finally, the average productivity of labor, wages and output per capita all grow at the rate $g_y + (g_A + \beta g_N)/\alpha$. The findings are intuitive because non-constant returns to scale in the search process have the same type of effect on the rate at which workers meet firms as improvements in the search technology. The findings are interesting because they prove that—under the same conditions under which unemployment and vacancies are constant in the face of an ever improving search technology—the returns to scale in the search process cannot be detected by looking at the correlation between unemployment rates (or UE rates) and the size of the labor market.

We conclude the paper with some observations on how to measure the contribution of declining search frictions to the growth rate of the economy and with some back-of-the-envelope calculations. In a BGP, the number of applicants considered for each vacancy—which is a measure of how selective firms and workers are—grows at the rate $g_A + \beta g_N$. In a BGP, the log of the ratio of applicants per vacancy in two labor markets at the same point in time is proportional to the log of the size of the two markets, with a constant of proportionality equal to β . In a BGP, the distribution of wages for homogeneous workers is approximately Pareto with a tail coefficient α . Thus, observations on applicants-per-vacancy over time and across space, and observations on the cross-section of wages for homogeneous workers would be enough to identify the contribution g_A/α of improvements in the search technology to economic growth, the extent β of returns to scale in the search process, as well as the contribution $\beta g_N/\alpha$ of these returns to scale to economic growth.

As far as we know, there is no dataset that contains the secular evolution of applications-per-vacancy. However, the Employment Opportunity Pilot Project contains applications-per-vacancy for the US in 1980 and 1982 (see Faberman and Menzio 2018) and several job sites, such as CareerBuilder.com and SnagAJob.com, contain applications-per-vacancy for the US in the 2010s (see Marinescu and Wolthoff 2016 and Faberman and Kudlyak 2016). Using these observations, we find that applications-per-vacancy grew between 1980 and 2011 by approximately 2% per year. Using applications-per-vacancy in different commuting zones, we find that the elasticity of applications-per-vacancy with respect to the population of the commuting zone is 0.52, which implies $\beta = 0.52$. The measurement of the growth rate of applications-per-vacancy implies that declining search frictions contribute to 0.55 percentage points to the annual growth rate of the economy if $\alpha = 4$, to 0.275 percentage points if $\alpha = 8$, and to 0.14 percentage points if $\alpha = 16$. As the annual growth rate of productivity is 1.9%, these are large contributions even when the tail of the Pareto distribution of match qualities is very thin. The measurement of β —together with the fact that the US labor force has grown by 1.1% per year between 1980 and 2011—implies that 3/4 of the contribution of declining search frictions to economic growth are due to improvements in the search technology and 1/4 to increasing returns to scale. Moreover, the measurement of β implies that average labor productivity and wages in a commuting zone that is 10% larger are 1.25% higher if $\alpha = 4$, 0.6% higher if $\alpha = 8$ and 0.3% higher if $\alpha = 16$, just because of increasing returns to search.

The main contribution of the paper is to develop a theory that reconciles the search view of unemployment and vacancies with the observation that unemployment, vacancy, UE and EU rates are all substantially stable in the long-run in the face of major improvements in information technology and, presumably, in the efficiency of the search process.

Our theory implies that, while improvements in the search technology do not affect unemployment, they do contribute to the growth of the economy. Moreover, some simple back-of-the-envelope calculations suggest that, indeed, the contribution of declining search frictions to economic growth is far from negligible.

From a methodological point of view, our paper belongs to the literature that seeks conditions on fundamentals under which an economy experiencing growth in the production technology features stationarity in some of its outcome (e.g., the labor share, the capital-output ratio, the interest rate, etc...). Key contributions in this literature include King, Plosser and Rebelo (1988) and, more recently, Grossman et al. (2007). Most of this literature focuses on Walrasian models. A few exceptions, which include Aghion and Howitt (1994) and Pissarides (2000), consider search-theoretic models of the labor market. Yet, these exceptions focus on understanding the interaction between the growth rate of the production technology and unemployment, rather than the effects of declining search frictions.

From a technical point of view, our paper is closely related to recent contributions in growth theory such as Perla and Tonetti (2014), Lucas and Moll (2014), Benhabib, Perla and Tonetti (2017) and Buera and Oberfield (2017). These papers focus on the diffusion of knowledge across individuals with different human capital. In these papers, as in ours, the key economic decision involves a choice between production and search. In our paper, search is for a new partner in the labor market and growth is driven by exogenous improvements in the search technology. In these papers, search is for someone from whom to learn and growth is driven by endogenous changes in the distribution of human capital. Another paper that is technically similar to ours is Kortum (1997). This paper wants to explain why the growth rate of research output is constant in the face of an increasing fraction of labor devoted to research and development. This question is analogous to the one asked in our paper, namely why the unemployment rate is constant in the face of better and better information technology. Interestingly, the answer proposed by Kortum (1997) is conceptually similar to the one in our paper: while the rate of experimentation increases over time (where experimentation is research input in Kortum and search for a match in our paper), the probability of a successful experiments falls over time (where success is discovering a technology better than the status-quo in Kortum and finding a viable match in our paper).

From a substantive point of view, our paper is related to the fundamental idea that lower trading frictions allow agents to become more and more specialized. Kiyotaki and Wright (1993) make this point in the context of a search theoretic model of the product market. They show that the introduction of fiat money effectively reduces trading frictions

and allows agents to produce more specialized goods. Ellison and Ellison (2018) show that the reduction of trading frictions brought about by the Internet has led to better matching between products and consumers and, in doing so, to an increase in consumer surplus. These papers make the same fundamental point as ours, although they only examine a one-time rather than a continuous decline in search frictions. Moreover, these papers focus on search frictions in the product rather than in the labor market. To the best of our knowledge, ours is the first paper that analyzes the effect of declining search frictions in the labor market and tries to quantify its effect on growth.

An immediate corollary of our theory about the independence of unemployment, vacancies and UE and EU rates from the efficiency of the search technology in the time-series is that these variables will also be independent from the size of the labor market in the cross-section, even in the presence of increasing returns to scale in the search process. Thus, the same theory that explains a time-series phenomenon also explains why, in the data, unemployment and the job-finding rate are uncorrelated with the size of the labor market. Moreover, the same strategy that can be used to identify the contribution of declining search frictions to productivity in the time-series (i.e., measuring applicants-per-vacancy) can be used to identify the extent of returns to scale in the search process and its contribution to the city-size wage premium. Our preliminary implementation of this identification strategy suggests that, indeed, the search process features strong increasing returns to scale and that these returns to scale contribute to a non-negligible fraction of the city-size wage premium. The idea that an increase in how selective firms and workers are may hide the extent of increasing returns to search is not entirely novel. Petrongolo and Pissarides (2006) show—in the context of a partial equilibrium model—that changes in the workers’ reservation wage partially offset the effect of market size on the job-finding rate. Using survey data on self-reported reservation wages, they show that, in fact, reservation wages are systematically higher in larger markets. Gautier and Teulings (2009) argue that increasing returns to scale in search may be further offset by the endogenous composition of workers in larger and smaller cities. Our theory is similar in spirit, but it ties together time-series and cross-sectional facts. Moreover, our analysis is focused on finding the conditions under which the increase in selectivity exactly neutralizes the effect of the decline in search frictions on the job-finding rate, rather than simply dampening such effect.

2 Basic model

In this section, we develop the simplest version of our theory of constant unemployment, vacancies, UE and EU rates in the face of an ever-improving search technology. We consider a textbook model of the labor market in the spirit of Mortensen and Pissarides (1994), where the production technology—as captured by the component of productivity common to all firm-worker matches—and the search technology—as captured by the efficiency of the matching function—improve over time, and firm-worker matches are inspection goods—in the sense that firms and workers get to see the idiosyncratic component of productivity of their match before deciding whether to match or not. We look for conditions under which there is an equilibrium such that unemployment, vacancies, UE and EU rates are constant over time, even though the search technology gets better. We refer to this equilibrium as a Balanced Growth Path (BGP). We find that a BGP exists if and only if the distribution of idiosyncratic productivities is Pareto and the vacancy cost and unemployment benefit grow at the same rate as the economy. The growth rate of the economy depends on the growth rate of the production technology, the growth rate of the search technology, and the tail coefficient of the Pareto distribution.

2.1 Environment

The labor market is populated by a continuum of workers of measure 1. Each worker's objective is to maximize the present value of labor income discounted at the rate $r > 0$, where labor income is given by some wage w_t if the worker is employed and by an unemployment benefit b_t if the worker is unemployed. The labor market is also populated by a continuum of firms of positive measure. Each firm's objective is to maximize the present value of profits discounted at the rate r . Each firm operates a technology that turns the flow of labor from a worker into a flow of $y_t z$ units of output, where y_t is a component of labor productivity that is common to all firm-worker pairs and z is a component of productivity that is idiosyncratic to a specific firm-worker pair.

The labor market is subject to search frictions. Workers need to search the market to locate firms with vacant jobs, and firms with vacant jobs need to search the market to locate workers. Workers search the market only when they are unemployed. Firms maintain a vacant job by paying the cost k_t . The outcome of the search process is a flow $A_t M(u_t, v_t)$ of random bilateral meetings between unemployed workers and vacant jobs, where u_t is the measure of unemployed workers, v_t is the measure of vacant jobs, A_t is the efficiency of the search process, and $M(u, v)$ is a constant returns to scale function.⁴

⁴We assume that search is random. The assumption is not critical for the existence of a BGP. In fact,

The outcome of the search process implies that an unemployed worker meets a vacant job at the Poisson rate $A_t p(\theta_t)$, where θ_t denotes the tightness v_t/u_t of the labor market and $p(\theta) = M(1, \theta)$ is such that $p(0) = 0$, $p'(\cdot) > 0$, and $p(\infty) \rightarrow \infty$. Similarly, a vacant job meets an unemployed worker at the rate $A_t q(\theta_t)$, where $q(\theta) = p(\theta)/\theta$ is such that $q(0) \rightarrow \infty$, $q'(\cdot) < 0$ and $q(\infty) = 0$.

Upon meeting, a firm and a worker draw the idiosyncratic component \hat{z} of the labor productivity of their match from a cumulative distribution function F . Let \bar{F} denote the complementary cumulative distribution function $1 - F$. After drawing and observing z , the firm and the worker decide whether to match or not. If they do match, they bargain over the terms of an employment contract and start producing a flow of $y_t \hat{z}$ units of output. If they do not match, the worker remains unemployed and the firm's job remains vacant. A matched firm-worker pair keeps producing until the employment relationship is broken off.⁵

The terms of the employment contract between the worker and the firm are determined according to the generalized axiomatic Nash bargaining solution, where the threat point of the worker is the value of unemployment and the threat point of the firm is the value of a vacant job. The worker's bargaining power is denoted as $\gamma \in (0, 1)$, while the firm's bargaining power is $1 - \gamma$. The employment contract specifies, either explicitly or implicitly, a path for the wage and a break-up date. We assume that the employment contract has enough contingencies to guarantee that the break-up date maximizes the the joint value of the firm-worker match.⁶ Given this assumption, the Nash bargaining solution assigns a fraction γ of the gains from trade to the worker and a fraction $1 - \gamma$ to the firm.

The production and the search technology improve over time. Specifically, the aggregate component of labor productivity grows at the rate g_y , i.e. $y_t = y_0 \exp(g_y t)$. The assumption is meant to capture the idea that progress in the production technology allows to generate more output using the same amount of labor. The efficiency of the meeting

it is straightforward to show the conditions for the existence of a BGP would be the same in a model with directed search a la Moen (1997) or Menzio and Shi (2010, 2011). We decided to assume that search is random because random-search models are more popular. However, if we had assumed that search is directed, the equilibrium would have been block recursive and we could have established the convergence to a BGP from arbitrary starting conditions.

⁵We assume that an employment relationship is always broken off by choice, never by chance. It would be straightforward to add an exogenous break-up rate to the model.

⁶There are many employment contracts with enough contingencies to guarantee that the joint value of the match is maximized. For example, if the employment contract can specify a wage path and a break-up date, the joint value of the match is maximized. The same is true if the employment contract can only specify a wage path and the worker and firm are free to leave the match at any time. The same is true even if the employment contract can only specify a wage over the next dt interval of time and then it is re-bargained (as in Mortensen and Pissarides 1994).

function grows at the rate g_A , i.e. $A_t = A_0 \exp(g_A t)$. The assumption is meant to capture the idea that progress in communication and information technologies leads to more meetings between the same measures of unemployed workers and vacant jobs. We also assume that the cost of a vacancy grows at the constant rate g_k and the unemployment benefit grows at the rate g_b , i.e. $k_t = k_0 \exp(g_k t)$ and $b_t = b_0 \exp(g_b t)$.

The model is a version of Mortensen and Pissarides (1994) in which there is growth in the production and search technologies and in which firm-worker matches are “inspection goods”—in the sense that the idiosyncratic component of productivity of a firm-worker pair is observed before the firm and the worker decide whether to match or not. The assumption that matches are inspection goods is critical for the existence of an equilibrium in which unemployment, vacancies, and the rate at which workers move in and out of unemployment remain constant in the face of an ever improving search technology. In fact, if matches were pure “experience goods”—in the sense that the idiosyncratic component of productivity of a firm-worker pair is only observed after the firm and the worker match—the improvements in the search technology would lead to an ever increasing exit rate from unemployment, as every new meeting would look the same and, hence, lead to a match.⁷

2.2 Definition of BGP

In order to define a BGP for the economy, we need to introduce some notation. Let U_t denote the value of unemployment to a worker, let $V_t(z)$ denote the maximized joint value of a firm-worker match of quality z , and let $S_t(z)$ denote the surplus of a firm-worker match of quality z . Let θ_t denote the tightness of the labor market, let u_t denote the unemployment rate, and let $G_t(z)$ denote the distribution of employed workers across matches of different quality z .⁸

The initial state of the economy is given by the distribution of workers across employment states at date $t = 0$, i.e. u_0 and G_0 . A rational expectation equilibrium of the economy is a path for U_t , V_t , S_t , θ_t , u_t and G_t such that agents’ decisions are optimal, markets clear, and the evolution of u_t and G_t is consistent with individual decisions and the initial state of the economy u_0 and G_0 . A BGP is an initial state of the economy and an associated rational expectation equilibrium such that unemployment, labor market tightness, the rate at which workers enter and exit unemployment are all constant over time, and the distribution of employed workers G_t grows at some constant rate (in the sense that each quantile of the distribution grows at the same, constant rate). That is,

⁷We conjecture that a BGP may also exist if matches are part inspection and part experience goods, in the sense that the firm and the worker receive a noisy signal about the idiosyncratic component of productivity before deciding whether to match or not (see, e.g., Menzio and Shi 2011).

⁸Throughout the paper, we shall use \dot{x}_t to denote the time-derivative of some arbitrary variable x_t .

just like a steady-state, a BGP does not take the initial conditions of the economy as given.

We are now in the position to formally derive the conditions for a BGP. The maximized joint value $V_t(z)$ of a firm-worker match with idiosyncratic productivity z at date t is such that

$$V_t(z) = \max_{d \geq 0} \int_t^{t+d} e^{-r(\tau-t)} y_\tau z d\tau + e^{-rd} U_{t+d}. \quad (2.1)$$

The above expression is easy to understand. At date τ , the sum of the worker's labor income and the firm's profit is equal to $y_\tau z$, which is the flow of output of the match. After d units of time, the firm and the worker break up. The continuation value to the worker is U_{t+d} and the continuation value to the firm is zero. Note that the joint value of a match is well-defined only if the discount rate exceeds the growth rate of the common component of productivity, i.e. only if $r > g_y$.

The break-up date that maximizes the joint value of a match is such that

$$y_{t+d}z + \dot{U}_{t+d} \leq rU_{t+d}, \text{ and } d \geq 0, \quad (2.2)$$

where the two inequalities hold with complementary slackness. The left-hand side of (2.2) is the marginal benefit of delaying the break-up of the match, which is given by the flow of output of the match $y_{t+d}z$ plus the time-derivative of the worker's value of unemployment \dot{U}_{t+d} . The right-hand side of (2.2) is the marginal cost of delaying the break-up of the match, which is given by the annuitized values that the worker and the firm can attain separately, rU_{t+d} . Then condition (2.2) states that either $d = 0$ and the marginal cost of delaying the break-up exceeds the marginal benefit, or $d > 0$ and the marginal cost of delaying the break-up equals the marginal benefit. Condition (2.2) is not only necessary but also sufficient. In fact, in any BGP, the right-hand side of (2.2) increases over time more quickly than the left-hand side.

The reservation quality R_t is defined as

$$R_t = \frac{rU_t - \dot{U}_t}{y_t}. \quad (2.3)$$

The reservation quality R_t is the lowest idiosyncratic component of productivity such that continuing firm-worker matches are maintained at date t , and new firm-worker matches are consummated at date t . In fact, from (2.2) and (2.3), it follows that a continuing firm-worker match is maintained only if $z > R_t$. Otherwise, the firm and the worker break up. For the same reason, a new firm-worker match is consummated only if $z > R_t$. Otherwise, the firm and the worker keep on searching for other trading partners.

The surplus $S_t(z)$ of a firm-worker match with idiosyncratic productivity z at date t

is defined as

$$S_t(z) = V_t(z) - U_t. \quad (2.4)$$

That is, the surplus of a firm-worker match is defined as the difference between the maximized joint value of the match and the sum of the outside options of the worker and the firm. From (2.4) and (2.3), it follows that the surplus of a firm-worker match is strictly positive for $z > R_t$ and it is equal to zero for all $z \leq R_t$. Therefore, a continuing match is maintained if and only if the surplus is strictly positive, and a new match is consummated if and only if the surplus is strictly positive.

The value of unemployment U_t to a worker is such that

$$rU_t = b_t + A_t p(\theta) \gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \dot{U}_t. \quad (2.5)$$

The left-hand side of (2.5) is the annuitized value of unemployment. The annuitized value of unemployment is equal to the sum of the three terms on the right-hand side of (2.5). The first term is the unemployment benefit. The second term is the option value of searching. The option value of searching is the rate at which the worker meets a vacancy times the increase in the worker's value upon meeting a vacancy, which—because of the axiomatic Nash bargaining solution—is equal to a fraction γ of the expected surplus. The last term is the time-derivative of the value of unemployment. Note that U_t is well-defined only if the discount rate is greater than the growth rate of the unemployment benefit, i.e. only if $r > g_b$.

The value of a vacant job to a firm is such that

$$A_t q(\theta) (1 - \gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}) - k_t = 0. \quad (2.6)$$

The left-hand side of (2.6) is the difference between the benefit and the cost of a vacancy at date t . The cost of a vacancy is k_t . The benefit of a vacancy is the rate at which the vacancy meets an unemployed worker times the increase in the firm's value upon meeting a worker, which—because of the axiomatic Nash bargaining solution—is equal to a fraction $1 - \gamma$ of the expected surplus. The difference between the benefit and the cost of a vacancy at date t must be equal to zero.

In a BGP, unemployment, vacancies, the rate at which unemployed workers become employed (UE rate) and the rate at which employed workers become unemployed (EU rate) are constant over time. The stationarity condition for vacancies is (2.6), which requires the same vacancy-to-unemployment ratio θ to solve the zero profit conditions for vacancies at all $t \geq 0$. The stationarity conditions for UE, EU and unemployment rates

are

$$A_t p(\theta) \bar{F}(R_t) = h_{ue}, \quad (2.7)$$

$$G'_t(R_t) \dot{R}_t = h_{eu}, \quad (2.8)$$

$$u h_{ue} = (1 - u) h_{eu}. \quad (2.9)$$

Condition (2.7) states that the UE rate is equal to some constant h_{ue} at all dates $t \geq 0$. The UE rate at date t is given by the rate $A_t p(\theta)$ at which an unemployed worker meets a vacancy times the probability $\bar{F}(R_t)$ that the idiosyncratic productivity of the firm-worker pair exceeds the reservation quality R_t . Condition (2.8) states that the EU rate is equal to some constant h_{eu} at all dates $t \geq 0$. The EU rate at date t is the product between the density $G'_t(z)$ of the distribution of employed workers across idiosyncratic productivities evaluated at the reservation quality R_t and the time-derivative of R_t . Condition (2.9) states that the date $t = 0$ unemployment rate u equates the flow of workers in and out of unemployment. Since the UE and EU rates are constant, condition (2.9) implies that the unemployment rate remains equal to u at all dates $t > 0$.

In a BGP, the distribution of employed workers across idiosyncratic productivities $G_t(z)$ grows at some constant rate g_z , in the sense that every quantile of the distribution grows at the rate g_z . Formally, in a BGP, $z_t(x) = z_0(x) \exp(g_z t)$ for all $x \in [0, 1]$ and $t \geq 0$ where $z_t(x)$ denotes the x -th quantile of G_t . The condition $z_t(x) = z_0(x) \exp(g_z t)$ is satisfied if and only if

$$\begin{aligned} & (1 - u) [G_t(z_t(x)e^{g_z dt}) - G_t(z_t(x))] + u A_t p(\theta) [F(z_t(x)e^{g_z dt}) - F(R_t e^{g_z dt})] dt \\ &= (1 - u) [G_t(R_t e^{g_z dt}) - G_t(R_t)]. \end{aligned} \quad (2.10)$$

The left-hand side of (2.10) is the flow of workers into matches with idiosyncratic productivity z below the x -th quantile. The inflow is given by the sum of two terms. The first term is the measure of workers who, at date t , are employed in a match of type z above the x -th quantile and who fall below the x -th quantile in the next dt units of time. The second term is the measure of workers who, at date t , are unemployed and find a job of type z below the x -th quantile in the next dt units of time. The right-hand side of (2.10) is the flow of workers out of matches with idiosyncratic productivity z below the x -th quantile. The outflow is given by the measure of workers who, at date t , are employed in a match of type z above the reservation quality R_t and who, over the next dt units of time, move into unemployment. Dividing both sides of (2.10) by dt and taking the limit for $dt \rightarrow 0$, we obtain

$$(1 - u) G'_t(z_t(x)) z_t(x) g_z + u A_t p(\theta) [F(z_t(x)) - F(R_t)] = (1 - u) G'_t(R_t) R_t g_z. \quad (2.11)$$

The above observations lead us to the following definition of a BGP.

Definition 1: A BGP is a tuple $\{R_t, S_t, U_t, V_t, \theta, h_{ue}, h_{eu}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, S_t, U_t and V_t satisfy (2.1), (2.3), (2.4) and (2.5); (ii) θ satisfies (2.6); (iii) h_{UE}, h_{EU} and u satisfy (2.7), (2.8) and (2.9); (iv) G_t satisfies (2.11).

2.3 Necessary conditions for BGP

We now derive some restrictions on the fundamentals of the model that are necessary for the existence of a BGP. We start by deriving a necessary condition for the exogenous distribution F of idiosyncratic productivities for new matches. The stationarity condition (2.7) for the UE rate implies that

$$\dot{A}(t)[1 - F(R_t)] = A_t F'(R_t) \dot{R}_t = 0, \forall t \geq 0. \quad (2.12)$$

In words, the increase in the rate at which unemployed workers meet vacant jobs that is caused by improvements in the search technology (the left-hand side of (2.12)) must be exactly offset by a decline in the probability that the idiosyncratic productivity of a firm-worker match is above the reservation quality (the right-hand side of (2.12)).

The efficiency of the search technology A_t grows at the rate g_A . The reservation quality R_t grows at the rate g_z , since R_t is the 0-th quantile of the distribution of employed workers G_t . From these two observations, it follows that

$$\frac{F'(R_t)R_t}{1 - F(R_t)} = \frac{g_A}{g_z}, \forall R_t \geq R_0. \quad (2.13)$$

The expression in (2.13) is a differential equation for F . The solution to the differential equation (2.13) that satisfies the boundary condition $F(\infty) = 1$ is⁹

$$F(z) = 1 - \left(\frac{z_\ell}{z}\right)^\alpha, \quad (2.14)$$

where $\alpha = g_A/g_z$ and z_ℓ is an arbitrary lower bound non-greater than R_0 . Therefore, a BGP may only exist if the exogenous distribution F of idiosyncratic productivities for new matches has the shape given by (2.14), which is a Pareto with some arbitrary coefficient α . This is a restriction on a fundamental of the model. Note that $\alpha = g_A/g_z$ should not be interpreted as a restriction on the shape of the Pareto distribution because g_z is an endogenous variable. Instead, the correct interpretation of $\alpha = g_A/g_z$ is that, in any BGP, the growth rate g_z of the distribution of employed workers G_t must be equal to g_A/α for

⁹The expression in (2.14) is a differential equation for $F(z)$ for $z \geq R_0$. Since the shape of the distribution $F(z)$ for $z < R_0$ is irrelevant for the outcomes of the model, we simply assume that (2.14) holds for all z .

an arbitrary α .

There is a simple intuition behind the finding that the exogenous distribution F of idiosyncratic productivities for new matches must be Pareto. The rate at which an unemployed worker meets a vacant job grows at a constant rate. Therefore, for the UE rate to remain constant over time, the probability that the idiosyncratic productivity of the firm-worker match exceeds the reservation quality must also decline at the same, constant rate. Since the reservation quality grows at the rate g_z , the probability distribution F must be such that the measure of realizations exceeding a growing threshold falls at a constant rate. Pareto is the only probability distribution to have this property.

Next, we derive a necessary condition for the exogenous growth rate g_k of the vacancy cost and for the exogenous growth rate g_b of the unemployment benefit. Using the equilibrium conditions (2.5) and (2.6) for the value of unemployment to a worker and for the value of a vacancy to a firm, we can rewrite the equilibrium condition for the reservation quality as

$$R_t = \frac{b_t}{y_t} + \frac{\gamma\theta}{1 - \gamma} \frac{k_t}{y_t}, \forall t \geq 0. \quad (2.15)$$

In a BGP, the reservation quality R_t grows at the rate g_z . The ratio b_t/y_t between the unemployment benefit and the aggregate component of productivity grows at the rate $g_b - g_y$. The ratio k_t/y_t between the vacancy cost and the aggregate component of productivity grows at the rate $g_k - g_y$. Hence, the left-hand side of (2.15) may be equal to the right-hand side of (2.15) for all $t \geq 0$ only if $g_k, g_b = g_y + g_z$, where $g_y + g_z$ is equal to $g_y + g_A/\alpha$. In other words, a BGP may only exist if the exogenous growth rate g_k of the vacancy cost and the exogenous growth rate g_b of the unemployment benefit are both equal to $g_y + g_A/\alpha$.

The necessary conditions $g_k, g_b = g_y + g_A/\alpha$ are easy to understand. In a BGP, the cost of a vacancy must equal the benefit of a vacancy for a constant market tightness θ . The cost of a vacancy grows at the rate g_k . The benefit of a vacancy is equal to the firm's option value of searching. In turn, the firm's option value of searching is proportional to $y_t R_t - b_t$, as one can see by combining the equilibrium conditions for unemployment and the reservation quality. Since the growth rate of y_t is g_y and the growth rate of R_t is $g_z = g_A/\alpha$, the cost and the benefit of a vacancy grow at the same rate only if $g_k, g_b = g_y + g_A/\alpha$.

Taken at face value, the conditions $g_k, g_b = g_y + g_A/\alpha$ imply that the existence of a BGP is a knife-edge result that may emerge only when the exogenous growth rate of the vacancy cost and of the unemployment benefit take a particular value. If that were the case, our explanation for why unemployment has no secular trend in the face of a century of progress

in information and communication technology would not be very satisfactory. However, as we shall see in the next few pages, the growth rate of k_t and b_t that is necessary for the existence of a BGP is exactly the growth rate of wages, productivity and of output per capita. Hence, if the input to produce vacancies is labor and if unemployment benefits are proportional to average wages or average productivity, k_t and b_t will endogenously grow at the rate $g_y + g_A/\alpha$. In Appendix B, we develop such a version of the model.

We summarize our findings in the following proposition.

Proposition 1. *(Necessary conditions for BGP). Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production and the search technologies.*

- (i) *A BGP may exist only if: (a) the distribution F of idiosyncratic productivity for new firm-worker matches is Pareto with an arbitrary tail coefficient α ; (b) the growth rate of vacancy cost, g_k , and the growth rate of the unemployment benefit, g_b , are $g_y + g_A/\alpha$; (c) the discount rate r is greater than $g_y + g_A/\alpha$.*
- (ii) *In any BGP, the growth rate g_z of the distribution G_t of employed workers is g_A/α .*

2.4 Existence and uniqueness of BGP

In light of Proposition 1, we now assume that the distribution F of idiosyncratic productivities for new matches is Pareto with tail coefficient α , that the growth rate g_k of the vacancy cost and the growth rate g_b of unemployment benefits are equal to $g_y + g_A/\alpha$, and that the discount rate r is greater than $g_y + g_A/\alpha$. Given these assumptions about the fundamentals of the model, we now show that a BGP exists and it is unique.

The maximized joint value $V_t(z)$ of a firm-worker match with idiosyncratic productivity $z > R_t$ and the value U_t of unemployment to a worker are given by

$$V_t(z) = \int_t^{t+d} e^{-r(\tau-t)} y_\tau z d\tau + e^{-rd} U_{t+d}, \quad (2.16)$$

$$U_t = \int_t^{t+d} e^{-r(\tau-t)} y_\tau R_\tau d\tau + e^{-rd} U_{t+d}. \quad (2.17)$$

The expression in (2.16) is the same as (2.1) evaluated at the optimal duration d of the match, which is equal to the amount of time $\log(z/R_t)/g_z$ it takes for the reservation quality to reach z . The expression in (2.17) is obtained from (2.5) after substituting in the definition of reservation quality (2.3). From (2.16) and (2.17), it follows that the surplus $S_t(z)$ of a firm-worker match with idiosyncratic productivity $z > R_t$ is given by

$$S_t(z) = \int_t^{t+d} e^{-r(\tau-t)} (y_\tau z - y_\tau R_\tau) d\tau.$$

Since y_t grows at the rate g_y , R_t grows at the rate g_z , and d is equal to $\log(z/R_t)/g_z$, we can solve the integral on the right-hand side and obtain

$$S_t(z) = y_t \left\{ \frac{z}{r - g_y} \left[1 - (R_t/z)^{\frac{r-g_y}{g_z}} \right] - \frac{R_t}{r - g_y - g_z} \left[1 - (R_t/z)^{\frac{r-g_y-g_z}{g_z}} \right] \right\}. \quad (2.18)$$

The expected surplus of a new firm-worker match is given by

$$\bar{S}_t \equiv \int_{R_t} S_t(\hat{z}) F'(\hat{z}) d\hat{z} = \Phi y_t R_t^{-(\alpha-1)}. \quad (2.19)$$

The expression on the right-hand side of (2.19) is obtained by replacing $S_t(z)$ with (2.18), replacing $F(z)$ with (2.14) and then solving the integral. The coefficient Φ is a strictly positive constant that depends only on parameters.¹⁰ Note that the expected surplus of a new firm-worker match is well defined if and only if the tail coefficient α of the Pareto distribution F is greater than one¹¹.

The reservation quality R_t is given by

$$R_t = b_t/y_t + A_t p(\theta) \gamma \Phi R_t^{-(\alpha-1)}. \quad (2.20)$$

The expression above is obtained from (2.3), using (2.5) to substitute out $rU_t - \dot{U}_t$ and using (2.19) to substitute out the expected surplus of a new firm-worker match. Evaluated at $t = 0$, (2.20) is an equation that pins down the initial reservation quality

$$R_0 = b_0/y_0 + A_0 p(\theta) \gamma \Phi R_0^{-(\alpha-1)}. \quad (2.21)$$

Evaluated at $t \geq 0$, (2.20) is an equation that pins down the growth rate g_z of the reservation quality

$$R_0 \cdot e^{g_z t} = (b_0/y_0) \cdot e^{(g_b - g_y)t} + A_0 p(\theta) \gamma \Phi R_0^{-(\alpha-1)} \cdot e^{(g_A - (\alpha-1)g_z)t}. \quad (2.22)$$

Clearly, (2.22) holds if and only if $g_z = g_A/\alpha$. That is, (2.22) holds for all $t \geq 0$ if and only if the growth rate of the reservation quality is equal to the growth rate of the search technology divided by the tail coefficient of the Pareto distribution F .

¹⁰The coefficient Φ is defined

$$\Phi \equiv \alpha z \ell^\alpha \left[g_z \frac{(r - g_y - g_z)^{-1} - (r - g_y)^{-1}}{r + (\alpha - 1)g_z - g_y} + \frac{1}{(\alpha - 1)(r - g_y)} - \frac{1}{\alpha(r - g_y - g_z)} \right].$$

It is a matter of algebra to show that Φ is strictly positive as long as $r > g_y + g_z$, which is the case as we assumed that $r > g_y + g_A/\alpha$ and, in any BGP, $g_z = g_A/\alpha$.

¹¹Note that $\alpha > 1$ is the condition that guarantees that the mean of a Pareto distribution is finite.

The zero-profit condition for vacancies is given by

$$k_t = A_t q(\theta)(1 - \gamma)\Phi y_t R_t^{-(\alpha-1)}. \quad (2.23)$$

The expression above is obtained from (2.6), using (2.19) to substitute out the expected surplus of a new firm-worker match. Evaluated at $t = 0$, (2.23) is an equation that pins down the tightness of the labor market

$$k_0 = A_0 q(\theta)(1 - \gamma)\Phi y_0 R_0^{-(\alpha-1)}. \quad (2.24)$$

To verify that the same tightness θ that solves (2.24) also implies zero profits for vacancies at all dates $t \geq 0$, it is sufficient to notice that the left-hand side of (2.23) grows at the rate $g_k = g_y + g_z/\alpha$, the right-hand side grows at the rate $g_A + g_y - (\alpha - 1)g_z$, and that these growth rates are equal because $g_z = g_A/\alpha$.

The constant-growth condition for the distribution $G_t(z)$ of employed worker is

$$(1 - u) [G'_t(z e^{g_z t}) z e^{g_z t} g_z] = u A_t p(\theta) \bar{F}(z e^{g_z t}). \quad (2.25)$$

The expression above is obtained from (2.11), using (2.8) and (2.9) to substitute out $G'_t(R_t) R_t g_z$ with $A_t p(\theta) \bar{F}(R_t)$. Evaluated at $t = 0$, the constant growth condition (2.25) is a differential equation for the initial distribution $G_0(z)$ of employed workers. The solution of the differential equation that satisfies the boundary condition $G_0(\infty) = 1$ is

$$G_0(z) = 1 - \frac{u}{1 - u} \frac{A_0 p(\theta)}{g_A} \bar{F}(z). \quad (2.26)$$

Since $G_0(R_0) = 0$, (2.26) implies that the unemployment rate u is

$$u = \frac{g_A}{g_A + A_0 p(\theta) \bar{F}(R_0)}. \quad (2.27)$$

Replacing the unemployment rate (2.27) into (2.26), we find

$$G_0(z) = 1 - \left(\frac{R_0}{z}\right)^\alpha = \frac{F(z) - F(R_0)}{1 - F(R_0)}. \quad (2.28)$$

To verify that $G_t(z e^{g_z t}) = G_0(z)$ satisfies the constant-growth condition (2.25) for all $t \geq 0$, it is sufficient to note that the left-hand side of (2.25) is constant because $G'_t(z e^{g_z t})$ is equal to $G'_0(z) \exp(-g_z t)$ and the right-hand side of (2.25) is constant because $\bar{F}(z e^{g_z t})$ is equal to $\bar{F}(z) \exp(-\alpha g_z t)$.

The UE and EU rates are respectively given by

$$h_{ue} = A_t p(\theta) \bar{F}(R_t) = A_0 p(\theta) \bar{F}(R_0), \quad (2.29)$$

$$h_{eu} = G'_t(R_t) R_t g_z = g_A. \quad (2.30)$$

The UE rate is constant over time because A_t grows at the rate g_A and $\bar{F}(R_t)$ grows at the rate $-\alpha g_z$, which is equal to the negative of g_A . The constant UE rate is $A_0 p(\theta) \bar{F}(R_0)$. The EU rate is constant over time because R_t grows at the rate g_z and $G'_t(R_t)$ grows at the rate $-g_z$. The constant EU rate is g_A . Using (2.29) and (2.30), it is easy to verify that the unemployment rate u in (2.27) equates the flows of workers in and out of unemployment and, hence, is constant over time.

The analysis above implies that a BGP exists if and only if there is a reservation quality R_0 and a market tightness θ that solve the equations (2.21) and (2.24). The solution for R_0 of the equation (2.21) exists and is unique for all $\theta \geq 0$, and we denote it as $\psi_1(\theta)$. The solution $\psi_1(\theta)$ is such that $\psi_1(0) = b_0/y_0$, $\psi'_1(\theta) > 0$ and $\psi_1(\infty) \rightarrow \infty$. The solution for R_0 of the equation (2.24) exists and is unique for all $\theta \geq 0$, and we denote it as $\psi_2(\theta)$. The solution $\psi_2(\theta)$ is such that $\psi_2(0) \rightarrow \infty$, $\psi'_2(\theta) < 0$, $\psi_2(\infty) = 0$. From these observations, it follows that there always exists one and only one pair $(R_0, \theta) \in \mathbb{R}_+^2$ that solve (2.21) and (2.24). Therefore, a BGP exists and it is unique.

We have established the following result.

Proposition 2. *(Existence and Properties of BGP) Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production and search technologies. A BGP exists if and only if F is Pareto with tail coefficient $\alpha > 1$, $g_b, g_k = g_y + g_A/\alpha$, and $r > g_y + g_A/\alpha$. If a BGP exists, it is unique and such that:*

- (i) *the labor market tightness, unemployment, UE and EU rates are constant;*
- (ii) *the reservation quality R_t grows at the rate g_A/α ;*
- (iii) *the distribution G_t of employed workers is a Pareto with tail coefficient α truncated at R_t , and it grows at the rate g_A/α ;*
- (iv) *labor productivity and output per capita grow at the rate $g_y + g_A/\alpha$.*

Proposition 2 proves that a BGP exists if and only if the distribution of idiosyncratic quality of new firm-worker matches is Pareto, and the vacancy cost and unemployment benefit grow at the rate of the economy. In a BGP, vacancies, unemployment, UE and EU rates are all constant over time even though the search technology keeps improving.

Let us first explain why the UE rate remains constant over time. Improvements in the search technology have two countervailing effects on the UE rate. On the one

hand, improvements in the search technology increase the rate at which workers meet firms. The rate at which workers meet firms grows at the rate g_A . On the other hand, improvements in the search technology increase the pickiness of firms and workers and thus lower the probability that a meeting between a worker and a firm turns into a match. The reservation quality for a match is proportional to the option value of searching to the worker, which is the product between the rate at which a worker meets a vacancy and the expected surplus of a meeting. The growth rate of the meeting rate is g_A . The growth rate of the expected surplus is $g_y - (\alpha - 1)g_z$, where g_z is the growth rate of the reservation quality. Hence, the growth rate of the reservation quality is such that $g_z = g_A + g_y - (\alpha - 1)g_z$, which implies $g_z = g_A/\alpha$. Since the reservation quality grows at the rate g_z and the distribution of qualities is a Pareto with tail coefficient α , the probability that a meeting between a worker and a firm turns into a match falls at the rate $-\alpha g_z = -g_A$. Overall, the rate at which a worker meets a firm grows at the rate g_A and the probability with which a meeting turns into a match falls at the rate g_A . Thus, the UE rate remains constant.

Next, let us explain why the EU rate remains constant over time. The initial distribution $G_0(z)$ of employed workers is equal to the sampling distribution F truncated at the reservation quality R_0 . Over time, workers employed in matches that fall below the reservation quality move into unemployment, while unemployed workers become employed in matches that are drawn from the sampling distribution F truncated at the reservation quality. Therefore, the distribution $G_t(z)$ of employed workers remains equal to the truncated sampling distribution. Given $G_t(R_t)$ and the growth rate of R_t , it follows that the EU rate is always equal to g_A . Since the EU rate and the UE rate are constant, the unemployment rate also remains constant as long as its initial value is the stationary one.

To understand why the market tightness remains constant over time, let us look at the labor market from the perspective of firms. The cost of a vacancy grows at the rate $g_k = g_y + g_A/\alpha$. The benefit of a vacancy is proportional to the rate at which a vacancy meets a worker times the expected surplus of a meeting between a vacancy and a worker. The meeting rate grows at the rate g_A for a given θ . The expected surplus grows at the rate $g_y - (\alpha - 1)g_z$. Therefore, the benefit of a vacancy grows at the rate $g_y + g_A/\alpha$, for a given θ . Therefore, the cost and the benefit of a vacancy remain equal to each other for a constant market tightness θ .

Improvements in the search technology contribute to the overall growth of the economy. In fact, note that the average productivity of labor is given by

$$\int_{R_t} y_t z G'_t(z) dz = \frac{\alpha}{\alpha - 1} y_t R_t. \quad (2.35)$$

The above expression implies that the growth rate of the average productivity of labor is the sum of the growth rate g_y of the production technology and the ratio between the growth rate g_A of the search technology and the tail coefficient α of the distribution F of idiosyncratic productivities for new matches. Since the employment rate is constant over time, the growth rate of output per capita is also $g_y + g_A/\alpha$. Intuitively, improvements in the search technology contribute to the growth rate of the economy because they allow firms and workers to become more selective with respect to the quality of the matches they create. The impact of improvements in the search technology on the growth rate of the economy depends on the shape of the distribution F . In particular, the impact is stronger when the tail of the distribution F is thicker (i.e. when α is lower). Intuitively, the thicker is the tail of F , the higher is the rate of return to faster search.

3 Search on the job

In this section, we show that our theory of constant unemployment, vacancies, UE and EU rates in the face of progress in the search technology is robust to relaxing the assumption that workers can only search the market when unemployed. We generalize the baseline model by assuming that workers can search not only off but also on the job, albeit at a different intensity. We find that the conditions for the existence of a BGP for the generalized version of the model are exactly the same as the conditions for the existence of a BGP in the baseline model. We also find that the properties of a BGP for the generalized version of the model are very similar to the properties of a BGP for the baseline model.

The robustness of our theory to relaxing the assumption that workers can only search off the job is important for two reasons. First, search on the job is empirically relevant. The rate at which workers move directly from one employer to another is around 2% a month, which is almost the same as the rate at which workers move from employment into unemployment. Also, the number of new hires who come directly from other firms is approximately as large as the number of new hires who come from unemployment. Second, search on the job is theoretically relevant. An unemployed worker's decision of accepting or rejecting a match depends on how effectively he can search on the job. A firm's decision of how many vacancies to maintain depends on how many searching workers are unemployed and, hence, have a worse outside option and how many searching workers are employed and, hence, have a better outside option. Our theory is robust to these changes in the trade-offs faced by workers and firms.

3.1 Environment

We generalize the environment of Section 2 to allow workers to search off and on the job. In particular, unemployed workers search the market with an intensity which is normalized to 1. Employed workers search the market with an intensity of $\rho \in [0, 1]$. Firms search the labor market by maintaining vacancies at the unit cost k_t . The outcome of the search process is a flow $A_t M(s_t, v_t)$ of random bilateral meetings between workers and vacancies, where $s_t = u_t + \rho(1 - u_t)$ is the intensity-weighted measure of searching workers and v_t is the measure of vacancies. The outcome of the search process implies that an unemployed worker meets a vacancy at the rate $A_t p(\theta_t)$, where $\theta_t = v_t/s_t$ denotes the tightness of the labor market. An employed worker meets a vacancy at the rate $\rho A_t p(\theta_t)$. A vacancy meets an unemployed worker at the rate $A_t q(\theta)u/[u + \rho(1 - u)]$ and it meets an employed worker at the rate $A_t q(\theta)\rho(1 - u)/[u + \rho(1 - u)]$.

Upon meeting, a firm and a worker draw the idiosyncratic productivity \hat{z} of their match from the distribution F . After drawing and observing \hat{z} , the firm and the worker decide whether to match or not. If they match, the worker and the firm start producing a flow of $y_t \hat{z}$ units of output. If they do not match, the worker remains in his previous employment position—which may be unemployment or employment at some other firm—and the firm’s job remains vacant.

The terms of the employment contract between a firm and a worker are determined according to the generalized Nash bargaining solution. The threat point of the firm is the value of a vacancy. If the worker is unemployed, his threat point is the value of unemployment. If the worker is employed, his threat point is the joint value of the match with his employer.¹² As in Section 2, we assume that the employment contract has enough contingencies to guarantee that the Nash bargaining solution maximizes the joint value of the firm-worker match. As noted in Section 2, this assumption implies that the Nash bargaining solution assigns a fraction γ of the gains from trade to the worker and a fraction $1 - \gamma$ to the firm.

¹²The assumption that the threat point of an employed worker is the joint value of his current match implies that the worker fully internalizes the effect of moving to a new firm on his current employer. The assumption implies that the worker moves to a new firm if and only if his productivity is higher with the new firm than with his current employer. The assumption is common in the literature (see, e.g., Postel-Vinay and Robin 2002, Cahuc, Postel-Vinay and Robin 2006, or Bagger et al. 2014).

3.2 Definition of BGP

The maximized joint value of a firm-worker match with idiosyncratic productivity z is such that

$$V_t(z) = \max_{d \geq 0} \left\{ \int_t^{t+d} e^{-r(\tau-t)} \mu_\tau \left[y_\tau z + A_\tau p(\theta) \rho \gamma \int_z (V_\tau(\hat{z}) - V_\tau(z)) dF(\hat{z}) \right] d\tau \right. \\ \left. + e^{-rd} \mu_{t+d} U_{t+d} \right\}, \quad (3.1)$$

where μ_τ denotes the probability that the match is still active at date τ and is equal to

$$\mu_\tau = \exp \left[- \int_t^\tau A_x p(\theta) \rho \bar{F}(z) dx \right].$$

Let us explain the above expression. Conditional on the firm-worker match surviving to date τ , the sum of the worker's labor income and the firm's profit is equal to $y_\tau z$. Moreover, at date τ , the worker meets a poaching firm at rate $A_\tau p(\theta) \rho$. If the idiosyncratic productivity \hat{z} of the match between the worker and the poaching firm is greater than z , the worker moves to the poaching firm. In this case, the worker's value is $V_\tau(z) + \gamma(V_\tau(\hat{z}) - V_\tau(z))$ and the incumbent firm's value is zero. Hence, the joint value of the firm-worker match increases by a fraction γ of the gains from trade $V_\tau(\hat{z}) - V_\tau(z)$. If the idiosyncratic productivity \hat{z} of the match between the worker and the poaching firm is smaller than z , the worker stays with the incumbent firm and there is no change in their joint value. Conditional on the firm-worker match surviving to date $t+d$, the worker and the firm voluntarily break up. In this case, the value to the worker is U_{t+d} and the value to the firm is zero. Since the firm-worker match breaks up at the rate $A_x p(\theta) \rho \bar{F}(z)$ at date x , the probability that the match survives until τ is given by $\mu(\tau)$. The difference between (3.1) and (2.1) is that the joint value of the match includes the option value of searching on the job.

The break-up date that maximizes the joint value of a match is such that

$$y_{t+d} z + A_{t+d} p(\theta) \rho \gamma \int_z (V_{t+d}(\hat{z}) - V_{t+d}(z)) dF(\hat{z}) + \dot{U}_{t+d} \leq r U_{t+d}, \text{ and } d \geq 0, \quad (3.2)$$

where the two inequalities hold with complementary slackness. The left-hand side of (3.2) is the marginal benefit of delaying the break-up of the match, which is the sum of the flow of output of the match, the option value of searching, and the time-derivative of the worker's value of unemployment. The right-hand side of (3.2) is the marginal cost of delaying the break-up of the match, which is given by the annuitized values that the worker and the firm can attain individually. The difference between (3.2) and (2.2) is that the marginal benefit of delaying the break-up of the match includes the option value of

searching on the job for an additional instant.

The reservation quality R_t is defined as

$$y_t R_t = rU_t - \dot{U}_t - A_t p(\theta) \rho \gamma \int_{R_t} (V_t(\hat{z}) - V_t(R_t)) dF(\hat{z}). \quad (3.3)$$

The definition (3.3) implies that a firm and a worker prefer staying together rather than being alone if and only if the idiosyncratic productivity of their match is greater than R_t . Similarly, a firm and an unemployed worker prefer consummating their match rather than staying alone if and only if the idiosyncratic productivity of their match is greater than R_t . Note that, other things equal, the reservation quality in (3.3) is lower than in (2.3) because the worker does not have to entirely give up searching when he keeps an old job or when he accepts a new job. Also, note that the reservation quality R_t characterizes the choice of whether a firm and a worker should be together or alone. In contrast, the choice of whether a worker should stay with an incumbent firm or move to a poaching firm is characterized by the ranking of the idiosyncratic productivities of the two available matches.

The surplus $S_t(z)$ of a firm-worker match with idiosyncratic productivity z is defined as

$$S_t(z) = V_t(z) - U_t. \quad (3.4)$$

The definition (3.4) implies that the surplus of a firm-worker match is strictly positive for $z > R_t$ and equal to zero for all $z \leq R_t$. Hence, a firm and a worker prefer staying together rather than being alone if and only if the surplus of their match is strictly positive. A firm and an unemployed worker prefer consummating their match rather than searching for alternative partners if and only if the surplus of their match is strictly positive.

The value of unemployment to a worker and the value of a vacancy to a firm are, respectively, such that

$$rU_t = b_t + A_t p(\theta) \gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \dot{U}_t. \quad (3.5)$$

and

$$\begin{aligned} & A_t q(\theta) \frac{u}{u + \rho(1-u)} (1-\gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}) \\ & + A_t q(\theta) \frac{\rho(1-u)}{u + \rho(1-u)} (1-\gamma) \int_{R_t} \left[\int_z (V_t(\hat{z}) - V_t(z)) dF(\hat{z}) \right] dG_t(z) - k_t = 0. \end{aligned} \quad (3.6)$$

Condition (3.5) for the value of unemployment to a worker is the same as condition (2.5). Condition (3.6) for the value of a vacancy to a firm is different from condition (2.6). Intuitively, when workers search both off and on the job, the vacancy meets both

unemployed and employed workers and this is reflected in the value of a vacancy. In fact, conditional on a meeting, the vacancy meets an unemployed worker with probability $u/(u + \rho(1 - u))$. In this case, the firm captures a fraction $1 - \gamma$ of the expected gains from trade $S_t(\hat{z})$. The vacancy meets a worker employed in a job of quality z with probability $[\rho(1 - u)/(u + \rho(1 - u))]G'_t(z)$. In this case, the firm captures a fraction $1 - \gamma$ of the gains from trade $V_t(\hat{z}) - V_t(z)$.

The stationarity conditions for UE, EU and unemployment rates are

$$A_t p(\theta) \bar{F}(R_t) = h_{ue}, \quad (3.7)$$

$$G'_t(R_t) \hat{R}_t = h_{eu}, \quad (3.8)$$

$$u h_{ue} = (1 - u) h_{eu}. \quad (3.9)$$

These are the same as conditions (2.7), (2.8) and (2.9). The condition $z_t(x) = z_0(x) \exp(g_z t)$ for the constant growth of the distribution of employed workers is

$$\begin{aligned} & (1 - u) G'_t(z_t(x)) z_t(x) g_z + u A_t p(\theta) [F(z_t(x)) - F(R_t)] \\ = & (1 - u) G'_t(R_t) R_t g_z + (1 - u) G_t(z_t(x)) \rho A_t p(\theta) \bar{F}(z_t(x)). \end{aligned} \quad (3.10)$$

The left-hand side of (3.10) is the flow of workers into matches with idiosyncratic productivity z below the x -th quantile. The left-hand side is the same as in (2.11). It is given by the flow of employed workers who are in a match of type z that falls below the x -th quantile in the next instant and by the flow of unemployed workers who, in the next instant, become employed in a match of type z below the x -th quantile. The right-hand side of (3.10) is the flow of workers out of matches with idiosyncratic productivity below the x -th quantile. The right-hand side is different than in (2.11). It includes the flow of employed workers who become unemployed, as well as the flow of workers who—by searching on the job—move from a match below the x -th quantile to a match above the x -th quantile.

The above observations lead us to the following definition of a BGP.

Definition 2: A BGP is a tuple $\{R_t, S_t, U_t, V_t, \theta, h_{ue}, h_{eu}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, S_t, U_t and V_t satisfy (3.1), (3.3), (3.4) and (3.5); (ii) θ satisfies (3.6); (iii) h_{UE}, h_{EU} and u satisfy (3.7), (3.8) and (3.9); (iv) G_t satisfies (3.10).

3.3 Existence of a BGP

It is easy to generalize the proof of Proposition 1 to show that, also for a version of the model in which workers search off and on the job, a BGP may exist only if the fundamentals satisfy the following conditions: (i) The distribution F of idiosyncratic productivity for new firm-worker matches is Pareto with tail coefficient α ; (ii) The vacancy

cost and the unemployment benefit grow at the rates $g_k, g_b = g_y + g_z$; (iii) The discount rate r is greater than $g_y + g_z$. Moreover, in any BGP, the growth rate g_z of the distribution G_t of employed workers is g_A/α . Therefore, we shall assume (i), (ii) and (iii) as we solve for a BGP.

The joint value $V_t(z)$ of a firm-worker match with idiosyncratic productivity $z > R_t$ and the value U_t of unemployment to a worker can be written as

$$rV_t(z) = y_t z + A_t p(\theta) \rho \gamma \int_z (S_t(\hat{z}) - S_t(z)) dF(\hat{z}) + \dot{V}_t(z), \quad (3.11)$$

$$rU_t = y_t R_t + A_t p(\theta) \rho \gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \dot{U}_t. \quad (3.12)$$

The expression in (3.11) is obtained by taking the derivative of (3.1) with respect to t . The expression in (3.12) is obtained from (3.5) after substituting in the definition of reservation quality. From (3.11) and (3.12), it follows that the surplus $S_t(z)$ of a firm-worker match with idiosyncratic productivity $z > R_t$ is given by

$$rS_t(z) = y_t(z - R_t) - A_t p(\theta) \rho \gamma \left[\int_{R_t}^z S_t(\hat{z}) dF(\hat{z}) + S_t(z) \bar{F}(z) \right] + \dot{S}_t(z) \quad (3.13)$$

Solving the differential equation in (3.13) seems rather difficult, as the equation involves both the derivative of S with respect to t and, implicitly in the integral, the derivative of S with respect to z . We manage to solve the partial differential equation in (3.13) by guessing that S is such that, when evaluated at an idiosyncratic productivity that grows at the rate g_z , the surplus of a match grows at the rate $g_y + g_z$, i.e.

$$S_t(z e^{g_z t}) = S_0(z) \cdot e^{(g_y + g_z)t}. \quad (3.14)$$

To verify the guess in (3.14), let us evaluate the partial differential equation (3.13) at $z \exp(g_z t)$ to obtain

$$\begin{aligned} rS_t(z e^{g_z t}) = & y_t(z e^{g_z t} - R_t) - A_t p(\theta) \rho \gamma \int_{R_0 e^{g_z t}}^{z e^{g_z t}} S_t(\hat{z}) dF(\hat{z}) \\ & - A_t p(\theta) \rho \gamma S_t(z e^{g_z t}) \bar{F}(z e^{g_z t}) + \dot{S}_t(z e^{g_z t}). \end{aligned} \quad (3.15)$$

Using the guess in (3.14) and the fact that $y_t = y_0 \exp(g_A t)$, $A_t = A_0 \exp(g_A t)$, $R_t =$

$R_0 \exp(g_z t)$, $\bar{F}(ze^{g_z t}) = \bar{F}(z) \exp(-\alpha g_z t)$, $g_z = g_A/\alpha$, we can rewrite (3.15) as¹³

$$rS_0(z) \cdot e^{(g_y+g_z)t} = \left\{ y_0(z - R_0) - A_0 p(\theta) \rho \gamma \int_{R_0}^z S_0(\hat{z}) dF(\hat{z}) - A_0 p(\theta) \rho \gamma S_0(z) \bar{F}(z) + [(g_y + g_z)S_0(z) - z g_z S_0'(z)] \right\} \cdot e^{(g_y+g_z)t}. \quad (3.16)$$

The left-hand side of (3.16) is an expression that depends only on S_0 and that grows over time at the rate $g_y + g_z$. The right-hand side is an expression that depends only on S_0 and that grows over time at the rate $g_y + g_z$. Thus, the guess (3.14) satisfies the partial differential equation (3.13) for all $t \geq 0$, as long as the initial surplus S_0 satisfies (3.16) at date $t = 0$.

To solve for the initial surplus S_0 , we take (3.16) evaluated at $t = 0$ and we differentiate it with respect to z . We obtain

$$rS_0'(z) = y_0 + [g_y - \sigma \gamma \bar{F}(z)] S_0'(z) - z g_z S_0''(z), \quad (3.17)$$

where σ is shorthand for $A_0 p(\theta) \rho$. The solution for $S_0'(z)$ to the differential equation (3.17) which satisfies the smooth-pasting condition $S_0'(R_0) = 0$ is

$$S_0'(z) = \frac{y_0}{g_z} \int_{R_0}^z \frac{1}{s} \exp \left\{ -\frac{1}{g_z} \left[\frac{\sigma \gamma}{\alpha} (F(z) - F(s)) + (r - g_y) \log \left(\frac{z}{s} \right) \right] \right\} ds, \quad (3.18)$$

The solution for $S_0(z)$ to the differential equation (3.18) that satisfies the value-matching condition $S_0(R_0) = 0$ is

$$S_0(z) = \frac{y_0}{g_z} \int_{R_0}^z \left[\int_{R_0}^x \frac{1}{s} \exp \left\{ -\frac{1}{g_z} \left[\frac{\sigma \gamma}{\alpha} (F(x) - F(s)) + (r - g_y) \log \left(\frac{x}{s} \right) \right] \right\} ds \right] dx. \quad (3.19)$$

Thus, the initial surplus S_0 in (3.19) together with $S_t(z \exp(g_z t)) = S_0(z) \exp(g_y + g_z)t$ provides a solution to the partial differential equation (3.13). While other solutions may exist and may be associated with different BGPs, it is straightforward to verify that all these other balanced growth paths also satisfy the properties derived below and summarized in Proposition 3.¹⁴

Using the fact that $S_t(ze^{g_z t}) = S_0(z) \exp(g_y + g_z)t$ and that $G_t(ze^{g_z t}) = G_0(z)$, we can derive some useful properties of the expected gains from trade $\bar{S}_{u,t}$ in a meeting between a firm and an unemployed worker, the expected gains from trade $S_{e,t}(ze^{g_z t})$ in a meeting between a firm and a worker employed in a match with idiosyncratic quality $z \exp(g_z t)$,

¹³The details of the derivation of (3.16) are in Appendix C.

¹⁴To see why this is the case, consider the properties of a BGP listed in Proposition 3. Property (i) follows from the definition of a BGP. Property (ii) holds in any BGP, as it is a necessary implication of a constant UE rate. Property (iii) follows directly from the constant growth condition for the employment distribution G_t . Property (iv) follows from property (iii).

and the expected gains from trade $\bar{S}_{e,t}$ in a meeting between a firm and an employed worker who is randomly drawn from the employment distribution G_t . In Appendix D, we show that all of these expected gains from trade increase over time at the rate of $g_y - (\alpha - 1)g_z$, i.e.

$$\begin{aligned} S_{e,t}(ze^{g_z t}) &\equiv \int_{ze^{g_z t}} (S_t(\hat{z}) - S_t(ze^{g_z t}))dF(\hat{z}) = S_{e,0}(z)e^{(g_y - (\alpha - 1)g_z)t}, \\ \bar{S}_{e,t} &\equiv \int_{R_t} S_{e,t}(z)dG_t(z) = \bar{S}_{e,0}e^{(g_y - (\alpha - 1)g_z)t}, \\ \bar{S}_{u,t} &\equiv \int_{R_t} S_t(\hat{z})dF(\hat{z}) = S_{u,0}e^{(g_y - (\alpha - 1)g_z)t}. \end{aligned} \quad (3.20)$$

Note that the expected gains above are well-defined only if the tail coefficient α of the distribution F is greater than 1.

We are now in the position to construct a BGP. The reservation quality R_t is given by

$$R_t = \frac{b_t}{y_t} + \frac{A_t}{y_t}p(\theta)(1 - \rho)\gamma\bar{S}_{u,t}. \quad (3.21)$$

Evaluated at $t = 0$, (3.21) is an equation that pins down the initial reservation quality R_0 as

$$R_0 = \frac{b_0}{y_0} + \frac{A_0}{y_0}p(\theta)(1 - \rho)\gamma\bar{S}_{u,0}. \quad (3.22)$$

Evaluated at $t \geq 0$, (3.21) is an equation that pins down the growth rate g_z of the reservation quality. In fact, note that the left-hand side of (3.21) grows at the rate g_z . The first term on the right-hand side of (3.21) grows at the rate $g_b - g_y = g_z$, and the second term grows at the rate $g_A - (\alpha - 1)g_z$. Thus, the growth rate g_z of the reservation quality is g_A/α .

The zero-profit condition for vacancies is given by

$$k_t = A_t q(\theta)(1 - \gamma) \left\{ \frac{u}{u + \rho(1 - u)} \bar{S}_{u,t} + \frac{\rho(1 - u)}{u + \rho(1 - u)} \bar{S}_{e,t} \right\}. \quad (3.23)$$

Evaluated at $t = 0$, (3.23) is an equation that pins down the tightness of the labor market theta as

$$k_0 = A_0 q(\theta)(1 - \gamma) \left\{ \frac{u}{u + \rho(1 - u)} \bar{S}_{u,0} + \frac{\rho(1 - u)}{u + \rho(1 - u)} \bar{S}_{e,0} \right\}. \quad (3.24)$$

To verify that the same tightness that solves (3.24) implies zero profits for vacancies at all dates $t \geq 0$, it is sufficient to notice that the left-hand side of (3.23) grows at the rate $g_k = g_y + g_A/\alpha$, the right-hand side grows at the rate $g_A + g_y - (\alpha - 1)g_z$, and that these growth rates are equal because $g_z = g_A/\alpha$.

The constant-growth condition for the distribution G_t of employed workers is given by

$$(1 - u)G'_t(ze^{g_z t})ze^{g_z t}g_z = [u + (1 - u)\rho G_t(ze^{g_z t})] A_t p(\theta) \bar{F}(ze^{g_z t}). \quad (3.25)$$

Evaluated at $t = 0$, the constant-growth condition (3.25) is a differential equation for the initial distribution $G_0(z)$ of employed workers. The solution for $G_0(z)$ of the differential equation that satisfies the boundary conditions $G_0(\infty) = 1$ and the solution for u to the boundary condition $G_0(R_0) = 0$ are respectively given by

$$G_0(z) = \frac{\exp(-A_0 p(\theta) \rho \bar{F}(z)/g_A) - \exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}{1 - \exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}, \quad (3.26)$$

and

$$u = \frac{\rho \exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}{1 - (1 - \rho) \exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}. \quad (3.27)$$

To verify that $G_t(ze^{g_z t}) = G_0(z)$ satisfies the constant-growth condition (3.25) for all $t \geq 0$, note that the left-hand side of (3.25) is constant because $G'_t(ze^{g_z t}) = G'_0(z) \exp(-g_z t)$ and the right-hand side is constant because $\bar{F}(ze^{g_z t}) = \bar{F}(z) \exp(-\alpha g_z t)$.

The UE and EU rates are respectively given by

$$h_{ue} = A_t p(\theta) \bar{F}(R_t) = A_0 p(\theta) \bar{F}(R_0), \quad (3.28)$$

and

$$h_{eu} = G'_t(R_t) R_t g_z = A_0 p(\theta) \bar{F}(R_0) \frac{\exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}{1 - \exp(-A_0 p(\theta) \rho \bar{F}(R_0)/g_A)}. \quad (3.29)$$

The UE rate is constant over time, as A_t grows at the rate g_A and $\bar{F}(R_t)$ grows at the rate $-g_z/\alpha$, which is equal to $-g_A$. The EU rate is constant over time, as $G'_t(R_t)$ grows at the rate $-g_z$ and R_t grows at the rate g_z . Using (3.28) and (3.29), it is easy to verify that the unemployment rate in (3.27) equates the flow of workers in and out of unemployment and, hence, is constant over time as well.

The analysis above implies that a BGP exists as long as there is a reservation quality R_0 and a market tightness θ that solve the equations (3.22) and (3.24). We can show that there exists a pair R_0 and θ that solves (3.22) and (3.24). Hence, a BGP does exist. However, we are not able to show that there exists a unique pair R_0, θ that solves (3.22) and (3.24). Hence, there may be multiple BGPs.

We summarize our findings in the following proposition.

Proposition 3. (*Existence and Properties of BGP with On-the-Job Search*) Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production and search technologies. A BGP exists if and only if F is Pareto with tail coefficient $\alpha > 1$, $g_b, g_k = g_y + g_A/\alpha$, and

$r > g_y + g_A/\alpha$. Any BGP is such that:

- (i) the labor market tightness, unemployment, UE and EU rates are constant;
- (ii) the reservation quality R_t grows at the rate g_A/α ;
- (iii) the distribution G_t of employed workers is a Fréchet truncated at R_t , and it grows at the rate g_A/α ;
- (iv) labor productivity and output per capita grow at the rate $g_y + g_A/\alpha$.

Proposition 3 shows that the necessary and sufficient conditions for the existence of a BGP for a version of the model where workers search off and on the job are exactly the same as for a version of the model where workers can only search while unemployed. That is, the distribution of idiosyncratic productivity for new firm-worker matches is Pareto and the vacancy cost and the unemployment benefit grow at the same rate as the economy. Further, Proposition 3 shows that the properties of a BGP are essentially the same in the models with and without on-the-job search. That is, vacancies, unemployment, UE and EU rates are constant, the reservation quality and the employment distribution grow at the rate g_A/α , and average labor productivity and output per capita grow at the rate $g_y + g_A/\alpha$.

It is worth commenting on the finding that the conditions for the existence of a BGP are the same in a model with search on the job as in a model without it. In a model with search on the job, the reservation quality equation is (3.21). This is the same equation as in the model without search on the job, except that the expected surplus of a meeting is multiplied by the difference $1 - \rho$ between the search intensity of unemployed and employed workers. Therefore, while the option of searching on the job affects the level of the reservation quality, it does not affect its growth rate g_z which remains equal to g_A/α . In turn, as the reservation quality grows at the rate g_A/α , the probability that a meeting between an unemployed worker and a firm turns into a match declines at the rate $-g_A$ and, hence, the UE rate remains constant over time.

In the model with search on the job, workers coming out of unemployment keep searching the labor market for better matches. The rate at which employed workers meet other firms grows at the rate g_A as a result of progress in the search technology. However, workers coming out of unemployment are employed in matches with an idiosyncratic productivity that grows at the rate $g_z = g_A/\alpha$ as a result of the growth in the reservation quality. For this reason, the probability that these workers find a match that is more productive than the one they currently have declines at the rate g_A . As a result, the rate at which workers move from one employer to another (the EE rate) remains constant over time.

As workers keep moving to better matches after coming out of unemployment, the distribution of employed workers will not be equal to the sampling distribution F truncated at the reservation quality. Instead, the distribution of employed workers is a Fréchet truncated at the reservation quality, as one can see by rewriting (3.26) as

$$G_0(z) = \frac{H(z) - H(R_0)}{1 - H(R_0)}, \quad (3.30)$$

where

$$H(z) = 1 - \exp \left[\frac{z}{z_\ell (A_0 p(\theta) \rho / g_A)^{1/\alpha}} \right]^{-\alpha}.$$

Clearly, H is a Fréchet distribution. The shape parameter of the distribution H is α , the tail coefficient of the sampling distribution F . The scale parameter of H is $z_\ell (A_0 p(\theta) \rho / g_A)^{1/\alpha}$. The distribution of employed workers grows at the rate $g_z = g_A / \alpha$. And, since the distribution of employed workers grows at a constant rate, the EU rate remains constant over time.

In a model with search on the job, the benefit of a vacancy to a firm is proportional to a weighted average between the expected surplus of a meeting with an unemployed worker and the expected surplus of a meeting with an employed worker, randomly drawn from the employment distribution. Clearly, the expected surplus is smaller in a meeting with an employed worker. Hence, the option of searching on the job lowers the benefit of a vacancy to a firm. However, the expected surplus in a meeting with an employed worker grows at the same rate. Hence, the option of searching on the job has no effect on the growth rate of the benefit of a vacancy, which remains equal to $g_y + g_z$. Since this is the same growth rate of the cost of a vacancy, a constant market tightness keeps the expected profit of a vacancy equal to zero at all dates.

4 Population growth

In the baseline model, we assume that the population is fixed. If the production technology and the search process both have constant returns to scale, assuming that the population is fixed is basically without loss in generality. In this section, we generalize the baseline model by assuming that the population grows over time at some arbitrary rate, and that the search process features arbitrary returns to scale. We find that the conditions for the existence of a BGP in the generalized version of the model are exactly the same as the conditions for the existence of a BGP in the baseline model. We also find that the properties of a BGP for the generalized version of the model are similar to the properties of a BGP for the baseline model. The only difference is that the growth rate of output

per capita now depends on the growth rate of the production technology, the growth rate of the search technology, the growth rate of the population, the returns to scale to search, and the tail coefficient of the Pareto distribution.

4.1 Environment

We generalize the environment of Section 2 to allow for population growth and arbitrary returns to scale in the search process. In particular, we assume that at date t the labor market is populated by a continuum of infinitely-lived workers of measure N_t , where N_t grows at the rate $g_N \geq 0$. We assume that workers enter the labor market as unemployed. The labor market is also populated by a continuum of firms with positive measure. Preferences, endowments and technologies of workers and firms are the same as in Section 2.

Unemployed workers and vacant jobs search the labor market. At date t , the measure of unemployed workers is $N_t u_t$, where u_t is the unemployment rate. At date t , the measure of vacant jobs is $N_t v_t$, where v_t is the vacancy rate. The outcome of the search process at date t is a flow $A_t N_t^{1+\beta} M(u_t, v_t)$ of random bilateral meetings between unemployed workers and vacant jobs, where A_t is the efficiency of the search technology, β is a parameter that captures the effects of scale in the search process, and M is a constant returns to scale function. If $\beta = 0$, the flow of meetings is proportional to the scale of the market N_t and, hence, the search process features constant returns to scale as in the baseline model. If $\beta > 0$, the flow of meetings is more than proportional to N_t and, hence, the search process features increasing returns to scale. If $\beta < 0$, the flow of meetings is less than proportional to N_t and, hence, the search process features decreasing returns to scale. The outcome of the search process implies that an unemployed worker meets a vacancy at the rate $\hat{A}_t p(\theta)$, where θ is the tightness of the labor market, $p(\theta) = M(1, \theta)$ and $\hat{A}_t = A_t N_t^\beta$. Similarly, a vacant firm meets an unemployed worker at the rate $\hat{A}_t q(\theta)$, where $q(\theta) = p(\theta)/\theta$.

Upon meeting, a firm and a worker draw the idiosyncratic productivity z of their match from the distribution F . They then decide whether to match or not. If the firm and the worker match, the terms of the employment contract are set according to the axiomatic Nash bargaining solution. Again, we assume that the employment contract has enough contingencies to guarantee that the Nash bargaining solution maximizes the joint value of their match.

4.2 Definition of a BGP

The definition of BGP is almost the same as in Section 2, except that: (i) A_t needs to be replaced with \hat{A}_t in all of the BGP conditions, where $\hat{A}_t = A_t N_t^\beta$ represents the overall efficiency of the search process; (ii) the stationarity condition for unemployment and the constant-growth condition for the employment distribution need to be modified to account for the flow of workers entering the labor market.

Formally, the joint value V_t of a firm-worker match, the reservation quality R_t , and the surplus $S_t(z)$ of a firm-worker match are

$$V_t(z) = \max_{d \geq 0} \int_t^{t+d} e^{-r(\tau-t)} y_\tau z d\tau + e^{-rd} U_{t+d}, \quad (4.1)$$

$$R_t = (rU_t - \dot{U}_t)/y_t, \quad (4.2)$$

$$S_t(z) = V_t(z) - U_t. \quad (4.3)$$

The value of unemployment to a worker and the value of a vacancy to a firm are

$$rU_t = b_t + \hat{A}_t p(\theta) \gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}), \quad (4.4)$$

$$0 = A_t q(\theta) (1 - \gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}) - k_t. \quad (4.5)$$

Conditions (4.1)-(4.3) are the same as conditions (2.1), (2.3) and (2.4). Conditions (4.4)-(4.5) are the same as conditions (2.5)-(2.6) with \hat{A}_t replacing A_t .

The stationarity conditions for the UE, EU and unemployment rates are

$$\hat{A}_t p(\theta) \bar{F}(R_t) = h_{ue}, \quad (4.6)$$

$$G'_t(R_t) \dot{R}_t = h_{eu}, \quad (4.7)$$

$$N_t u h_{ue} = N_t (1 - u) (h_{eu} + g_N). \quad (4.8)$$

The stationarity conditions (4.6)-(4.7) for the UE and EU rates are the same as (2.7)-(2.8) with \hat{A} replacing A_t . The stationarity condition (4.8) for unemployment is different from (2.9). The unemployment rate is stationary when the flow of workers out of unemployment, $N_t u h_{ue}$, is equal to the flow of workers entering unemployment from employment, $N_t (1 - u) h_{eu}$, plus the flow of workers entering unemployment from outside the labor market, $N_t g_N$, multiplied by the difference $1 - u$ between the unemployment rate of entering and existing workers.

The condition guaranteeing that the employment distribution G_t grows at the constant rate g_z —in the sense that $z_t(x) = z_0(x) \exp(g_z t)$ where $z_t(x)$ denotes the x -th quantile of

G_t is

$$\begin{aligned} & N_t(1-u)G'_t(z_t(x))z_t(x)g_z + N_tu\hat{A}_t p(\theta)[F(z_t(x)) - F(R_t)] \\ = & N_t(1-u)G'_t(R_t(x))R_t(x)g_z + N_tg_N(1-u)G_t(z_t(x)). \end{aligned} \quad (4.9)$$

The constant-growth condition (4.9) for the employment distribution is different from (2.10) because of the second term on the right-hand side of (4.9). The left-hand side of (4.9) is the flow of workers into matches of type z below the x -th quantile. The first term is the flow of workers who are employed in a match of type z which falls below the x -th quantile in the next instant. The second term is the flow of unemployed workers who, in the next instant, become employed in a match of type z below the x -th quantile. The right-hand side of (4.9) is the flow of workers out of matches of type z below the x -th quantile. The first term is the flow of workers who are employed and, in the next instant, move into unemployment. The second term is the flow of workers entering the labor market times the difference between the fraction of existing workers who are employed in matches below the x -th quantile (which is $(1-u)G_t(z_t(x))$) and the fraction of new workers who are employed in matches below the x -th quantile (which is zero).

The above observations lead us to the following definition of a BGP.

Definition 3: A BGP is a tuple $\{R_t, S_t, U_t, V_t, \theta, h_{ue}, h_{eu}, u, G_t\}$ such that for all $t \geq 0$: (i) R_t, S_t, U_t and V_t satisfy (4.1), (4.2), (4.3) and (4.4); (ii) θ satisfies (4.5); (iii) h_{UE}, h_{EU} and u satisfy (4.6), (4.7) and (4.8); (iv) G_t satisfies (4.9).

4.3 Existence and properties of BGP

The only difference between the conditions for the value and policy functions $R_t, S_t, U_t, V_t, \theta$ in the models with and without population growth is that \hat{A}_t replaces A_t . The only difference between the stationarity conditions and the constant growth conditions for u, h_{ue}, h_{eu} and G_t in the models with and without population growth is that g_N affects the flow of workers into unemployment.

In light of the above observations, the following proposition should not come as a surprise to our readers.

Proposition 4. (*Existence and Properties of BGP with Population Growth*) Consider arbitrary growth rates $g_y > 0, g_A$ and $g_N > 0$ for the production technology, search technology and population, with $g_A + \beta g_N > 0$. A BGP exists if and only if F is Pareto with tail coefficient $\alpha > 1$ and $g_b, g_k = g_y + (g_A + \beta g_N)/\alpha$, and $r > g_y + (g_A + \beta g_N)/\alpha$. If a BGP exists, it is unique and such that:

- (i) the labor market tightness, unemployment, UE and EU rates are constant;
- (ii) the reservation quality R_t grows at the rate $(g_A + \beta g_N)/\alpha$;

- (iii) the distribution G_t of employed workers is a Pareto truncated at R_t , and it grows at the rate $(g_A + \beta g_N)/\alpha$;
- (iv) labor productivity and output per capita grow at the rate $g_y + (g_A + \beta g_N)/\alpha$.

Proposition 4 shows that the necessary and sufficient conditions for the existence of a BGP with population growth are essentially the same as the conditions for the existence of a BGP without population growth. That is, the distribution of idiosyncratic productivity for new firm-worker matches is Pareto and the vacancy cost and the unemployment benefit grow at the same rate as the economy. The restriction on the growth rates $g_A + \beta g_N > 0$ in Proposition 4 is needed to make sure that the overall efficiency of the search process \hat{A}_t grows over time and it replaces the restriction $g_A > 0$ in Proposition 2. This finding is not surprising in light of the observation that population growth impacts value and policy functions only by substituting \hat{A}_t with A_t .

Proposition 4 also shows that the properties of a BGP with population growth are slightly different from the properties of a BGP without population growth. First, the growth rate of the reservation quality and of the distribution of employed workers g_z is $(g_A + \beta g_N)/\alpha$ rather than g_A/α . Second, the growth rate of average labor productivity and output per capita is $g_y + (g_A + \beta g_N)/\alpha$ rather than $g_y + g_A/\alpha$. These findings are easy to understand in light of the fact that the overall efficiency of the search process \hat{A}_t grows at the rate $g_A + \beta g_N$, i.e. the sum of the growth rate of the search technology plus the growth rate of the size of the market multiplied by the returns to scale coefficient in the search process.

Finally, note that it would be equally easy to introduce population growth also in the version of the model in which workers search off and on the job. We decided to add population growth to the basic model only to keep this section short and to the point.

5 Identification and back-of-the-envelope calculations

We conclude by discussing some empirical implications of our theory. If we take it as given that improvements in communication and information technology over the last 100 years have generated improvements in the efficiency with which workers and firms can contact each other, then the conditions for a BGP must be satisfied. That is, the distribution of idiosyncratic productivity for new firm-worker matches must be approximately Pareto, and the vacancy cost and the flow value of unemployment must grow at approximately the same rate as output per capita.

Since the conditions for a BGP are satisfied, u , v , h_{ue} and h_{eu} remain constant over time, irrespective of how quickly the search technology might be improving. Hence, one

cannot measure the growth rate of the search technology by looking at the time-trend of u , v , h_{ue} and h_{eu} . Moreover, when the conditions for a BGP are satisfied, u , v , h_{ue} and h_{eu} remain constant as the size of the market grows, irrespective of what the returns to scale in the search process might be. Hence, one cannot measure the returns to scale in the search process by looking at the correlation between size of the market and u , v , h_{ue} and h_{eu} .¹⁵

In other words, the fact that u has not fallen over the last 100 years does not mean that the search technology has not improved. And the fact that u has not been trending either up or down while the workforce has dramatically grown does not mean that the search process has constant returns to scale. Similarly, the fact that the u is not systematically different in large and small markets does not mean that the search process has constant returns to scale. Is there a way, then, to measure the growth rate in the search technology, the returns to scale in the search process, and the contribution of declining search frictions to the growth of the economy?

We answer the questions above using the model without on-the-job search. We do this for the sake of simplicity, but the same answers also apply to the model with on-the-job search. To recover the growth rate of the search technology and the returns to scale of the matching process, one can examine the difference in the selectivity of firms over time and across markets of different size. The average number of workers that a firm considers before filling its vacancy is given by

$$A_t N_t^\beta q(\theta) \cdot \frac{1}{A_t N_t^\beta q(\theta) F(\bar{R}_t)} \quad (5.1)$$

The first term in (5.1) is the number of workers contacted by a firm per unit of time, and the second term is the duration of a vacancy (i.e. the inverse of the rate at which the firm fills the vacancy). We refer to (5.1) as the number of applicants per vacancy. Note

¹⁵Consider an economy in which $A_t = A > 0$ and $N_t = N_0 \exp(g_N t)$. Let M_t , U_t , V_t , N_t be the number of matches, unemployed workers, vacant jobs, and total population. Suppose that $M_t = AU_t^{c_u} V_t^{c_v} \varepsilon_t = Au_t^{c_u} v_t^{c_v} N_t^{c_u + c_v} \varepsilon_t$, where $u_t = U_t/N_t$, $v_t = V_t/N_t$ and ε_t is a residual, unobservable variable.

Then, the standard estimation regression is

$$\tilde{m}_t = \eta_0 + \eta_u \tilde{u}_t + \eta_v \tilde{v}_t + \eta_n \tilde{n}_t + \tilde{\varepsilon}_t,$$

where $\tilde{x} = \ln(x)$ and $\eta_n \equiv \eta_u + \eta_v$. The hypothesis of constant returns to scale of the matching function is equivalent to $\eta_n = 1$.

Through the lens of our model, we can re-write $\tilde{\varepsilon}_t = \ln(1 - F(R_t)) + \phi_t$, for some variable ϕ_t that is uncorrelated with $\tilde{n}_t, \tilde{v}_t, \tilde{u}_t$. Since F is a Pareto distribution with tail coefficient α , and $R_t \propto \exp(g_N t / \alpha)$, $\tilde{\varepsilon}_t = \kappa - (\eta_n - 1)\tilde{n}_t$, for some constant κ . Hence, we can re-write the regression as

$$\tilde{m}_t = \hat{\eta}_0 + \eta_u \tilde{u}_t + \eta_v \tilde{v}_t + 1 * \tilde{n}_t + \tilde{\phi}_t,$$

which implies that the OLS estimate of η_n , $\hat{\eta}_n \rightarrow_p 1$. In other words, the regression would always confirm the hypothesis of constant returns to scale, irrespective of the actual value of η_n .

that the number of applicants per vacancy grows over time at the rate $g_A + \beta g_N$. In fact, the number of workers contacted by a firm per unit of time grows over time at the rate $g_A + \beta g_N$, while the duration of a vacancy is constant over time.

Next, consider two labor markets (market 1 and market 2) at the same calendar date t . The two markets have sizes $N_{1,t}$ and $N_{2,t}$. The two markets are identical in all the fundamentals, except that vacancy costs and unemployment benefits are proportional to the output per capita in the respective market. The ratio between the number of applicants per vacancy in market 1 and market 2 is given by

$$\frac{A_t N_{1,t}^\beta q(\theta_1)}{A_t N_{2,t}^\beta q(\theta_2)} \cdot \frac{A_t N_{2,t}^\beta q(\theta_2) F(\bar{R}_{2,t})}{A_t N_{1,t}^\beta q(\theta_1) F(\bar{R}_{1,t})}. \quad (5.2)$$

It is easy to verify that the tightness is the same in the two markets. Similarly, the duration of a vacancy in the two markets is the same. Therefore, the ratio between applicants per vacancy in the two markets is equal to $(N_{1,t}/N_{2,t})^\beta$.

To recover the contribution of declining search frictions to economic growth, we need to measure the tail coefficient α of the Pareto distribution F . To this aim, one can look at wages. If employment contracts are continuously renegotiated, as assumed in Mortensen and Pissarides (1994), the wage of a worker in a match of quality z at date t is given by

$$w_t(z) = \gamma y_t z + (1 - \gamma) y_t R_t. \quad (5.3)$$

The cross-sectional distribution of wages L_t is given by

$$L_t(w) = G_t \left(\frac{1}{\gamma} \frac{w}{y_t} - \frac{1 - \gamma}{\gamma} R_t \right) = 1 - \left(\frac{\gamma y_t R_t}{w - (1 - \gamma) y_t R_t} \right)^\alpha. \quad (5.4)$$

The cross-sectional wage distribution of wages is not Pareto. Yet, the right tail of the distribution is well approximated by a Pareto with tail coefficient α since

$$\lim_{w \rightarrow \infty} \frac{d \log[1 - L_t(w)]}{d \log w} = -\alpha. \quad (5.5)$$

Taken together, the above observations provide a complete identification strategy.

Proposition 5. (*Identification of Declining Search Frictions*).

- (i) The growth rate $g_A + \beta g_N$ in the overall efficiency of the search process is the growth rate of applicants per vacancy;
- (ii) The return to scale β in the search process is the coefficient of a regression of the log of applicants per vacancy on the size of the market;
- (iii) The tail coefficient α of the distribution F is the tail coefficient of the wage distrib-

ution for identical workers;

- (iv) *Given $g_A + \beta g_N$, β , α and g_N , one can measure the contribution $(g_A + \beta g_N)/\alpha$ of declining search frictions, the contribution g_A/α of progress in the search technology, and the contribution $\beta g_N/\alpha$ of population growth to the growth of labor productivity.*

We now attempt to implement the identification strategy outlined in Proposition 5. The strategy requires time-series and cross-sectional data on applications per vacancy, and cross-sectional data on the wages of identical workers. As far as we know, there does not exist a time-series for the number of applications per vacancy spanning more than just a few years. However, Faberman and Menzio (2018) report a measure of applications per vacancy for the US in the early 1980s, while Marinescu and Wolthoff (2016) and Faberman and Kudlyak (2016) report a measure of applications per vacancy for the US in the early 2010s. The data used by Marinescu and Wolthoff (2016) also contains a detailed break-down of vacancies and their applications across different commuting zones. Measuring the wage distribution for identical workers is a difficult task that we do not attempt here. Instead, we consider a wide range of values for the coefficient α . Overall, our implementation of the identification strategy in Proposition 5 should be seen as a back-of-the-envelope calculation.

Faberman and Menzio (2018) study data from the Employment Occupation Pilot Project (EOPP), which is a survey of US firms that was carried out in 1980 and 1982 and contains information about the characteristics of job openings (e.g., occupation, industry, location, etc. . .) and about the recruitment outcomes (e.g., number of applications per vacancy, number of interviews per vacancy, vacancy duration, wage paid to the hired worker, etc. . .). Faberman and Menzio (2018) find that the average number of applications per vacancy is 10.4 per week and that the average duration of a vacancy is 16.2 days. These figures imply an average number of applications per vacancy of 24.

Marinescu and Wolthoff (2016) study data from CareerBulider.com, which is the largest online job site in the US, contains over 1 million jobs at any time, and is visited by approximately 11 million unique job seekers each month. The data includes detailed information about the characteristics of job openings (e.g., job title, occupation, name of the firm, industry, etc. . .) and about outcomes (including applications per vacancy). Marinescu and Wolthoff (2016) restrict attention to vacancies posted in the Chicago and Washington DC Designated Market Areas between January and March 2011. They find that the average number of applications per vacancy is 59. In a related study, Faberman and Kudlyak (2016) use data from SnagAJob.com, an online search engine that mainly focuses on hourly-paid jobs. They find that the average number of applications per vacancy is 31.

The data reported in Faberman and Menzio (2018), Marinescu and Wolthoff (2016) and Faberman and Kudlyak (2016) suggest that, between 1982 and 2011, the average number of applications per vacancy grew from 24 to somewhere between 31 and 59. Taking the average between 31 and 59, we conclude that, between 1982 and 2011, the average growth rate of application per vacancy was approximately 2.2% per year. From (5.1), it follows that 2.2% is an estimate of $g_A + \beta g_N$, i.e. the sum of the growth rate of the efficiency of the search technology and the growth rate of the size of the market multiplied by the returns to scale in the search process.

The data from CareerBuilder.com has also information on the average number of applications per vacancy across different markets in the US and, hence, can be used to measure the returns to scale in the search process. Ioana Marinescu kindly agreed to run for us a regression of the log of applications per vacancy on the log of the population in the commuting zone of the vacancy. She estimates a regression coefficient on the log of population size is 0.52. She finds a similar coefficient also after controlling for occupation. From (5.2), it follows that 0.52 is an estimate of the return to scale coefficient β in the search process. Hence, the search process appears to have increasing returns to scale.

Between 1982 and 2011, the US labor force grew from 108 to 152 million people. These figures imply an average growth rate g_N in the labor force of 1.1% per year. Since $\beta = 0.52$ and $g_N = 1.1\%$, it follows that the growth rate of the size of the market is responsible (via increasing returns to scale) for an increase in the number of applications per vacancy of 0.6% per year, while the growth rate of the efficiency of the search technology is responsible for an increase in the number of applications per vacancy of 1.6% (i.e. $2.2\% - 0.6\%$). In other words, approximately 1/4 of the growth in applications per vacancy is due to increasing returns to scale, and 3/4 is due to improvements in the search technology.

Average labor productivity—measured, following Shimer (2005), as output per worker in the non-farm business sector—grew by 1.9% per year between 1982 and 2011. The contribution to the growth of labor productivity of declining search frictions (caused by either progress in the search technology or by increasing returns to scale) is given by the growth rate of applications per vacancy divided by the tail coefficient α of the Pareto distribution F . As mentioned above, measuring α is not easy, as it requires measuring the tail coefficient of the wage distribution for inherently identical workers. However, α is unlikely to be very large. In fact, Postel-Vinay and Robin (2002) document—using a model that is similar to ours and allows for workers’ heterogeneity that is unobserved by the econometrician—that search frictions account for almost half of the wage inequality among workers in the same occupation. For the sake of the argument, let us suppose that α is 4 (which implies a 90-to-50 percentile ratio of 1.49). Then, declining search frictions

account for 0.55 percentage points of the 1.9% annual growth in labor productivity, or about 1/4. If α is 8 (which implies a 90-to-50 percentile ratio of 1.22), declining search frictions account for 0.28 percentage points of the annual growth in labor productivity. Even if α is as high as 16 (which implies a 90-to-50 percentile ratio of 1.1), declining search frictions still account for a non-negligible 0.14 percentage points of annual growth in labor productivity. In every case, improvements in the search technology account for three fourths of the contribution of declining search frictions to labor productivity growth, while increasing returns account for the remaining fourth.

The estimate of the degree of increasing returns to scale in the search process has some interesting implications for understanding geographic differences in wages and labor productivity. The estimate of β implies that, in a commuting zone that is 10% larger, the average productivity of labor and, hence, wages are 1.25% higher if $\alpha = 4$, 0.62% higher if $\alpha = 8$, and 0.31% higher if $\alpha = 16$. Similarly, in a commuting zone of 10 million people, relative to a commuting zone of 1 million people, average labor productivity and wages are 25% higher if $\alpha = 4$, 12.5% higher if $\alpha = 8$ and 6% higher if $\alpha = 16$. Hence, increasing returns to scale in the search process can account for a sizeable fraction of the productivity and wage differential across small and large cities. Yet, increasing returns to scale in the search process do not generate any differences in unemployment, UE and EU rates across small and large cities and, hence, cannot be detected by regressing the unemployment rate on city size.

It is indeed well-documented that wages are systematically higher (see, e.g., Glaeser and Maré 2001) while the unemployment rate is not systematically different in larger than in smaller cities (see, e.g., Petrongolo and Pissarides 2006). The standard explanation for these two facts is that the search process features constant returns to scale—explaining why unemployment is independent of city size—and that, for some reason, workers are more productive in large cities than in small cities—explaining why wages are higher in larger cities. However, the standard explanation is at odds with the findings that applications per vacancy are systematically higher in larger cities. Our explanation is a corollary of the theory of constant unemployment, vacancies, UE and EU rates in the face of progress in the search technology. That is, under the same conditions under which improvements in the search technology do not generate a time-trend in unemployment, vacancies, UE and EU rates, the size of a city does not affect the level of these variables but it does affect the level of wages and labor productivity. Moreover, our explanation is consistent with (and, in fact, it is disciplined by) the observation that applications per vacancy are systematically higher in larger cities.

References

- [1] Aghion, P., and P. Howitt. 1994. "Growth and Unemployment." *Review of Economic Studies*, 61: 477-494.
- [2] Bagger, J., F. Fontaine, F. Postel-Vinay, and J. Robin. 2014. "Tenure, Experience, Human Capital, and Wages: A Tractable Equilibrium Search Model of Wage Dynamics." *American Economic Review*, 104: 1551-1596.
- [3] Barnichon, R. 2010. "Building a Composite Help-Wanted Index." *Economic Letters*, 109: 175-178.
- [4] Benhabib, J., J. Perla, and C. Tonetti. 2017. "Reconciling Models of Diffusion and Innovation: A Theory of the Productivity Distribution and Technology Frontier." Manuscript, New York University.
- [5] Buera, F., and E. Oberfield. 2018. "The Global Diffusion of Ideas." Manuscript, Princeton University.
- [6] Cahuc, P. F. Postel-Vinay and J. Robin. 2006. "Wage Bargaining with On-the-Job Search: Theory and Evidence." *Econometrica*, 74: 323-364.
- [7] Diamond, P. 1982. "Wage Determination and Efficiency in Search Equilibrium," *Review of Economic Studies*, 49: 217-227.
- [8] Ellison, G., and S. Ellison. 2018. "Match Quality, Search, and the Internet Market for Used Books." NBER Working Paper 24197.
- [9] Faberman, J., and M. Kudlyak. 2016. "The Intensity of Job Search and Search Duration." FRB San Francisco Working Paper 2016-13.
- [10] Faberman, J., and G. Menzio. 2018. "Evidence on the Relationship Between Recruiting and Starting Wage." *Labour Economics*, 50: 67-79.
- [11] Gautier, P., and C. Teulings. 2009. "Search and the City." *Regional Science and Urban Economics*, 39: 251-265.
- [12] Gavazza, A., S. Mongey, and G. Violante. 2018. "Aggregate Recruiting Intensity." *American Economic Review*, Forthcoming.
- [13] Glaeser, E., and X. Maré. 2001. "Cities and Skills." *Journal of Labor Economics*, 19: 316-342.

- [14] Grossman, G., E. Helpman, E. Oberfield, and T. Sampson. 2017. “Balanced Growth Despite Uzawa.” *American Economic Review*, 107: 1293-1312.
- [15] Kaplan, G., and G. Menzio. 2016. “Shopping Externalities and Self Fulfilling Unemployment Fluctuations.” *Journal of Political Economy*, 124: 771-825.
- [16] King, R., C. Plosser, and S. Rebelo. 1988. “Production, Growth and Business Cycles I. The Basic Neoclassical Model.” *Journal of Monetary Economics*, 21: 195-232.
- [17] Kiyotaki, N., and R. Wright. 1993. “A Search-Theoretic Approach to Monetary Economics.” *American Economic Review*, 83: 63-77
- [18] Kortum, S. 1997. “Research, Patenting, and Technological Change.” *Econometrica*, 65: 1389-1419.
- [19] Lucas, R., and B. Moll. 2014. “Knowledge Growth and the Allocation of Time.” *Journal of Political Economy*, 122: 1-51.
- [20] Marinescu, I., and R. Wolthoff. 2016. “Opening the Black Box of the Matching Function: The Power of Words.” Manuscript, University of Toronto.
- [21] Menzio, G., and S. Shi. 2010. “Block Recursive Equilibria for Stochastic Models of Search on the Job.” *Journal of Economic Theory*, 145: 1453-1494.
- [22] Menzio, G., and S. Shi. 2011. “Efficient Search on the Job and the Business Cycle.” *Journal of Political Economy*, 119: 468-510.
- [23] Moen, E. 1997. “Competitive Search Equilibrium.” *Journal of Political Economy*, 105: 694-723.
- [24] Mortensen, D. 1982. “Property Rights and Efficiency in Mating, Racing and Related Games.” *American Economic Review*, 72: 968-979.
- [25] Mortensen, D. and C. Pissarides. 1994. “Job Creation and Job Destruction in the Theory of Unemployment.” *Review of Economic Studies*, 61: 397-415.
- [26] Perla, J., and C. Tonetti. 2014. “Equilibrium Imitation and Growth.” *Journal of Political Economy*, 122: 52-76.
- [27] Petrongolo, B., and C. Pissarides. 2006. “Scale Effects in Markets with Search.” *Economic Journal*, 116: 21-44.
- [28] Petrosky-Nadeau, N., and L. Zhang. 2013. “Unemployment Crises.” Manucript, Federal Reserve Bank of San Francisco.

- [29] Pissarides, C. 2000. *Equilibrium Unemployment Theory*. MIT Press, Cambridge, MA.
- [30] Postel-Vinay, F., and J. Robin. 2002. "Equilibrium Wage Dispersion with Worker and Employer Heterogeneity." *Econometrica*, 70: 2295-2350.
- [31] Shimer, R. 2005 "The Cyclical Behavior of Equilibrium Unemployment and Vacancies." *American Economic Review*, 95: 25-49.
- [32] Sniekers, F. 2018. "Persistence and Volatility of Beveridge Cycles." *International Economic Review*.
- [33] Zagorsky, J. 1998. "Job Vacancies in the United States: 1923 to 1994." *Review of Economics and Statistics*, 80: 338-345.

Appendix

A Data

A.1 Unemployment rate

The unemployment rate is constructed using the NBER macro-history files from 1927 to 1947, and the Federal Reserve Economic Data (FRED) from 1948 to 2018. Data before 1948 is obtained by concatenating 3 series of seasonally-adjusted monthly unemployment rates: NBER data series m08292a (January 1929-February 1940), NBER data series m08292b (March 1940-December 1946), NBER data series m08292c (January 1947-December 1947). Starting from 1948, the time series corresponds to the seasonally-adjusted civilian unemployment rate from the Bureau of Labor Statistics (FRED series id: UNRATE).

A.2 Vacancy rate

The vacancy rate series is the concatenation of 4 different series: the MetLife help-wanted advertising index, NBER data series m08082a (January 1927-December 1959), the help-wanted advertising index from the Conference Board (January 1960-December 1994), the composite print and online help-wanted index from Barnichon (2010) (January 1995-December 2000), the job openings series from the Job Openings and Labor Turnover Survey, JOLTS, (January 2001-October 2018).

The MetLife index includes help-wanted ads published in 45 US cities on 100 newspapers (1927 to the early 1940s) or on 60 newspapers (thereafter). The construction of the Conference Board index tightly follows the MetLife index. The three main aspects in which the Conference Board differs from MetLife are the use of 51 newspapers in 51 different cities, the adjustment of the index to account for the different number of Sundays in each month (help-wanted ads were usually published on Sundays), and the weighting of the index computed in each city by the city's employment share (see Zagorsky 1998 for additional details). The two series coexisted between January 1951 and August 1960. Since neither of them displays any significant trend during the overlapping years, and in light of the similarity in their construction procedure, the two series are connected to each other by rescaling the Conference Board index so that it has the same value as the MetLife index in January 1960.

As online advertising became widespread after the mid 1990s, the Conference Board index has increasingly lost its ability to represent the actual dynamics of job vacancies. To

address this issue, Barnichon (2010) combines data on print and online help-wanted ads. He weights their relative importance by assuming that the diffusion of online postings followed a similar pattern as the diffusion of internet use among US households. This assumption allows him to create a composite print-online index.

Starting from 2001, vacancy rates are computed using data from the JOLTS, which is a survey of 16 thousands establishments. The series built by Barnichon covers the period until December 2016, which makes the comparison to the JOLTS data possible. The two series track each other very closely ¹⁶.

Once the time series for the help-wanted index covering the entire sample period is created, the index is divided by the labor force series in order to create a vacancy rate. Finally, the correct level of the vacancy rate is achieved by rescaling the series so that the vacancy rate is equal to 2.05% in 1965, as documented by Zagorsky (1998).

B Endogenous vacancy cost and unemployment benefit

In this Appendix, we analyze a version of the baseline model in which the cost of a vacancy and the benefit of unemployment are endogenous. We show that, in this version of the model, the vacancy cost and the unemployment benefit grow endogenously at the same rate as the economy. Hence, in this version of the model, the only substantive condition for a BGP is that the distribution of productivity for new firm-worker matches is Pareto.

There are two types of firms, production firms and recruitment firms. Production firms are the firms described in Section 2, which operate a constant returns to scale technology that turns one worker into $y_t z$ units of output, where y_t is the common component of productivity and z is the component of productivity that is idiosyncratic to a firm-worker match. Recruitment firms are firms that create the hiring services required by production firms to maintain their vacancies. In particular, production firms need to purchase 1 unit of hiring services to maintain a vacancy. Recruitment firms create hiring services according to a constant return to scale production function which turns 1 unit of labor into $A_h > 0$ units of hiring services. Recruitment firms hire labor in a frictionless and competitive market and sell hiring services in a frictionless and competitive market. We assume that recruitment firms hire labor in a frictionless market to guarantee that, even when every worker is unemployed, the economy does not shut down. Finally, the un-

¹⁶One might question the precision of Barnichon's composite index in capturing the speed of transition towards online advertisements. Reassuringly, the Conference Board index at the end of 1994 is at the same level as the JOLTS in early 2000s. It seems unlikely that the vacancy rate might have experienced any dramatic - but temporary - shift in the late 1990s.

employment benefit is determined by the government as a fraction $\eta > 0$ of the average output of workers employed by production firms. We assume that the unemployment benefit is proportional to average output so as to make it independent of any particular wage determination rule.

Let $w_{h,t}$ denote the wage paid by recruitment firms to their employees. Let $p_{h,t}$ denote the price at which recruitment firms sell hiring services to production firms. Let $e_{h,t}$ denote the measure of workers who are employed by recruitment firms. The endogenous variables $w_{h,t}$, $p_{h,t}$ and $e_{h,t}$ are such that

$$w_{h,t} = rU_t - \dot{U}_t, \quad (\text{B.1})$$

$$p_{h,t} = w_{h,t}/A_h, \quad (\text{B.2})$$

$$e_{h,t} = u\theta/A_h. \quad (\text{B.3})$$

Intuitively, the wage $w_{h,t}$ makes an unemployed worker indifferent between taking a job at a recruitment firm and searching for a job at a production firm. The price $p_{h,t}$ makes the profit of a recruitment firm equal to zero. The employment $e_{h,t}$ is such that the aggregate supply of hiring services is equal to the aggregate demand of hiring services.

The joint-value V_t , the reservation quality R_t , and the surplus S_t for a match between a production firm and a worker are such that

$$V_t(z) = \max_{d \geq 0} \int_t^{t+d} e^{-r(\tau-t)} y_\tau z d\tau + e^{-rd} U_{t+d}, \quad (\text{B.4})$$

$$R_t = (rU_t - \dot{U}_t)/y_t, \quad (\text{B.5})$$

$$S_t(z) = V_t(z) - U_t. \quad (\text{B.6})$$

The value of unemployment to a worker and the value of a vacancy to a production firm are such that

$$rU_t = \eta \int_{R_t} y_t \hat{z} dG_t(\hat{z}) + \hat{A}_t p(\theta) \gamma \int_{R_t} S_t(\hat{z}) dF(\hat{z}) + \dot{U}_t, \quad (\text{B.7})$$

$$0 = A_t q(\theta) (1 - \gamma) \int_{R_t} S_t(\hat{z}) dF(\hat{z}) - y_t R_t / A_h. \quad (\text{B.8})$$

The conditions (B.4), (B.5) and (B.6) are the same as the conditions (2.1), (2.3) and (2.4). The difference between (B.7) and (2.5) is that here the unemployment benefit is a fraction η of the average productivity of labor rather than the exogenous b_t . The difference between (B.8) and (2.6) is that here the cost of a vacancy is the price of a unit of hiring services rather than the exogenous k_t . Note that the price $p_{h,t}$ of a unit of hiring services is equal to $y_t R_t / A_h$ because $w_{h,t} = y_t R_t$ and $p_{h,t} = w_{h,t} / A_h$.

The stationarity conditions for the UE, EU and unemployment rates are

$$A_t p(\theta) \bar{F}(R_t) = h_{ue}, \quad (\text{B.9})$$

$$G'_t(R_t) \hat{R}_t = h_{eu}, \quad (\text{B.10})$$

$$u h_{ue} = (1 - u - u\theta/A_h) h_{eu}. \quad (\text{B.11})$$

The stationarity conditions for the UE and EU rates are the same as (2.7) and (2.8). The difference between (B.11) and (2.9) is that, here, the flow into unemployment is given by the product between the measure of workers employed in the production sector (rather than the total measure of employed workers) and the EU rate.

The constant-growth condition for the distribution of workers employed in the production sector is such that

$$\begin{aligned} & (1 - u - u\theta/A_h) G'_t(z_t(x)) z_t(x) g_z + u A_t p(\theta) [F(z_t(x)) - F(R_t)] \\ = & (1 - u - u\theta/A_h) G'_t(R_t(x)) R_t(x) g_z. \end{aligned} \quad (\text{B.12})$$

The difference between (B.12) and (2.10) is that, here, the first term on the left-hand side is the measure of workers employed in the production sector (rather than the total measure of employed worker) times the rate at which these workers fall below the x -th quantile of the distribution. Similarly, the term on the right-hand side is the measure of workers employed in the production sector times the rate at which these workers become unemployed.

It is easy to show that a BGP may exist only if the distribution F of idiosyncratic productivity for new firm-worker matches is a Pareto with tail coefficient $\alpha > 1$ and the discount rate r is greater than $g_y + g_A/\alpha$. Given these restrictions on the fundamentals, it easy to show that a BGP exists and is unique as long as $\eta < (\alpha - 1)/\alpha$.¹⁷ In the BGP, the reservation quality R_t grows at the rate $g_z = g_A/\alpha$ and R_0 is equal to

$$R_0 = \left(\frac{A_0 p(\theta) \gamma \Phi}{1 - \eta \alpha / (\alpha - 1)} \right)^{1/\alpha}. \quad (\text{B.13})$$

The labor market tightness θ is such that

$$\theta = q^{-1} \left(\frac{R_0^\alpha}{A_h A_0 (1 - \gamma) \Phi} \right). \quad (\text{B.14})$$

¹⁷This condition is necessary and sufficient for the unemployment benefit to be lower than the reservation quality.

The UE, EU and unemployment rates are

$$h_{ue} = A_0 p(\theta) \bar{F}(R_0), \quad (\text{B.15})$$

$$h_{eu} = g_A, \quad (\text{B.16})$$

$$u = \frac{g_A}{A_0 p(\theta) \bar{F}(R_0) + g_A} \quad (\text{B.17})$$

The distribution of workers employed by production firms grows at the rate $g_z = g_A/\alpha$ and G_0 is equal to

$$G_0(z) = \frac{F(z) - F(R_0)}{1 - F(R_0)}. \quad (\text{B.18})$$

The wage $w_{h,t}$ paid by recruitment firms is equal to $y_t R_t$ and, hence, grows at the rate $g_y + g_z/\alpha$ with $w_{h,0} = y_0 R_0$. The price $p_{h,t}$ of hiring services is equal to $w_{h,t}/A_h$ and, hence grows at the rate $g_y + g_z/\alpha$ with $p_{h,0} = y_0 R_0/A_h$. Employment $e_{h,t}$ at recruitment firms is constant and equal to $u\theta/A_h$.

We have thus established the following proposition.

Proposition 6. (*Existence and Properties of BGP*) Consider arbitrary growth rates $g_y > 0$ and $g_A > 0$ for the production and search technologies. A BGP exists if and only if F is Pareto with tail coefficient $\alpha > 1$, $r > g_y + g_A/\alpha$, and $\eta < (\alpha - 1)/\alpha$. If the BGP exists, it is unique and such that:

- (i) the labor market tightness, unemployment, UE and EU rates are constant;
- (ii) the reservation quality R_t grows at the rate g_A/α ;
- (iii) the distribution G_t of employed workers is a Pareto with tail coefficient α truncated at R_t , and it grows at the rate g_A/α ;
- (iv) labor productivity and output per capita grow at the rate $g_y + g_A/\alpha$;
- (v) vacancy cost and unemployment benefit grow at the rate $g_y + g_A/\alpha$.

At face value, Proposition 2 implies that the existence of a BGP is a knife-edge result that holds only if the exogenous growth rate of the vacancy cost and the exogenous growth rate of the unemployment benefit both happen to be equal to $g_y + g_A/\alpha$. Proposition 6 shows that the existence of a BGP is not a knife-edge result because, once the vacancy cost and the unemployment benefit are endogenized, they naturally end up growing at the rate $g_y + g_A/\alpha$. The vacancy cost grows at the rate $g_y + g_A/\alpha$ because hiring services use a constant amount of labor and the price of labor grows at the rate $g_y + g_A/\alpha$. The unemployment benefit grows at the rate $g_y + g_A/\alpha$ because it is proportional to average productivity.

C Verifying the guess for the surplus

The partial differential equation for $S_t(z)$ evaluated at $z \exp(g_z t)$ is

$$\begin{aligned} rS_t(ze^{g_z t}) &= y_t(ze^{g_z t} - R_t) - A_t p(\theta) \rho \gamma \int_{R_0 e^{g_z t}}^{ze^{g_z t}} S_t(\hat{z}) dF(\hat{z}) \\ &\quad - A_t p(\theta) \rho \gamma S_t(ze^{g_z t}) \bar{F}(ze^{g_z t}) + \dot{S}_t(ze^{g_z t}). \end{aligned} \quad (\text{C.1})$$

First, notice that (3.14) implies

$$\begin{aligned} \dot{S}_t(ze^{g_z t}) &= \lim_{dt \rightarrow 0} \frac{1}{dt} [S_{t+dt}(ze^{g_z t}) - S_t(ze^{g_z t})] \\ &= \lim_{dt \rightarrow 0} \frac{1}{dt} [S_0(z) e^{(g_y + g_z)(t+dt)} - S_0(z) e^{(g_y + g_z)t}] \\ &\quad - \lim_{dt \rightarrow 0} \frac{1}{dt} [S_0(z) e^{(g_y + g_z)(t+dt)} - S_0(ze^{-g_z dt}) e^{(g_y + g_z)(t+dt)}] \\ &= (g_y + g_z) S_0(z) e^{(g_y + g_z)t} - z g_z S'_0(z) e^{(g_y + g_z)t}. \end{aligned} \quad (\text{C.2})$$

Second, notice that (3.14) and $R_t = R_0 \exp(g_z t)$ imply

$$\begin{aligned} A_t \int_{R_t}^{ze^{g_z t}} S_t(\hat{z}) dF(\hat{z}) &= A_t \int_{R_t}^{ze^{g_z t}} S_t(\hat{z}) \left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{\alpha}{\hat{z}} d\hat{z} \\ &= A_t \int_{R_0}^z S_0(\hat{x}) e^{(g_y + g_z)t} \left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{\alpha}{\hat{x}} e^{-\alpha g_z t} d\hat{x} \\ &= A_0 \left[\int_{R_0}^z S_0(\hat{x}) dF(\hat{x}) \right] e^{(g_A + g_y - (\alpha - 1)g_z)t} \\ &= A_0 \left[\int_{R_0}^z S_0(\hat{x}) dF(\hat{x}) \right] e^{(g_y + g_z)t}, \end{aligned} \quad (\text{C.3})$$

where the second line makes use of the change of variable $\hat{x} = \hat{z} \exp(-g_z t)$ and the last line makes use of $g_z = g_A/\alpha$.

Third, note that (3.14), $A_t = A_0 \exp(g_A t)$ and $\bar{F}(ze^{g_z t}) = \bar{F}(z) \exp(-\alpha g_z t)$ imply

$$\begin{aligned} &A_t p(\theta) \rho \gamma S_t(ze^{g_z t}) \bar{F}(ze^{g_z t}) \\ &= A_0 p(\theta) \rho \gamma S_0(z) \bar{F}(z) e^{(g_A + g_y + g_z - \alpha g_z)t} \\ &= A_0 p(\theta) \rho \gamma S_0(z) \bar{F}(z) e^{(g_y + g_z)t}, \end{aligned} \quad (\text{C.4})$$

where the last line makes use of the fact that $g_z = g_A/\alpha$.

Substituting (C.2), (C.3) and (C.4) into (C.1), we obtain (3.16).

D Expected gains from trade

The expected gains from trade $\bar{S}_{u,t}$ in a meeting between a firm and an unemployed worker are

$$\begin{aligned}
\bar{S}_{u,t} &\equiv \int_{R_t} S_t(\hat{z}) dF(\hat{z}) \\
&= \int_{R_t} S_t(\hat{z}) \left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{\alpha}{\hat{z}} d\hat{z} \\
&= e^{(g_y - (\alpha-1)g_z)t} \int_{R_0} S_0(\hat{x}) \left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{\alpha}{\hat{x}} d\hat{x} \\
&= e^{(g_y - (\alpha-1)g_z)t} \bar{S}_{u,0},
\end{aligned} \tag{D.1}$$

where the third line is obtained by using (3.14), $R_t = R_0 \exp(g_z t)$, and then by changing the variable of integration from \hat{z} to $\hat{x} = \hat{z} \exp(-g_z t)$.

Similarly, the expected gains from trade $S_{e,t}(ze^{g_z t})$ in a meeting between a firm and a worker employed in a job of quality $z \exp(g_z t)$ are

$$\begin{aligned}
S_{e,t}(ze^{g_z t}) &\equiv \int_{ze^{g_z t}} (S_t(\hat{z}) - S_t(ze^{g_z t})) dF(\hat{z}) \\
&= \int_{ze^{g_z t}} (S_t(\hat{z}) - S_t(ze^{g_z t})) \left(\frac{z_\ell}{\hat{z}}\right)^\alpha \frac{\alpha}{\hat{z}} d\hat{z} \\
&= e^{(g_y - (\alpha-1)g_z)t} \int_z (S_0(\hat{x}) - S_0(z)) \left(\frac{z_\ell}{\hat{x}}\right)^\alpha \frac{\alpha}{\hat{x}} d\hat{x} \\
&= e^{(g_y - (\alpha-1)g_z)t} S_{e,0}(z).
\end{aligned} \tag{D.2}$$

Finally, the expected gains from trade $\bar{S}_{e,t}$ between a firm and an employed worker are

$$\begin{aligned}
\bar{S}_{e,t} &\equiv \int_{R_t} S_{e,t}(z) dG_t(z) \\
&= \int_{R_t} S_{e,t}^e(z) G_t'(z) dz \\
&= e^{(g_y - (\alpha-1)g_z)t} \int_{R_0} S_{e,0}^e(x) G_0'(x) dx \\
&= e^{(g_y - (\alpha-1)g_z)t} \bar{S}_{e,0},
\end{aligned} \tag{D.3}$$

where we made use of (3.14) and $G_t'(ze^{g_z t}) = G_0'(z) \exp(-g_z t)$. It is easy but tedious to verify that the integrals in (D.1)-(D.3) are finite if and only if $\alpha > 1$.