

Getting Signal from Interval Data: Theory and Applications to Mortality and Intergenerational Mobility*

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Abstract

We propose a method for the partial identification of conditional expectation functions with interval censored conditioning variables. We focus on two out of many potential applications. First, we resolve a known problem in the estimation of mortality as a function of education: because individuals with high school or less are a smaller and thus more negatively selected group over time, estimates of their mortality change are likely to be biased. Our method makes it possible to estimate mortality at constant education percentiles, revealing that current estimates of rising mortality for less educated women are biased upward in some cases by a factor of three. The method is also applicable to the estimation of education gradients in patterns of fertility, marriage and disability, among others, where similar compositional problems arise. Second, we apply the method to the estimation of intergenerational mobility, where researchers frequently use coarsely measured education data in the many contexts where matched parent-child income data are unavailable. We show that conventional measures like the rank-rank correlation may be uninformative once interval censoring is taken into account. We propose a new CEF-based measure of mobility that can be bounded tightly. Earlier methods in both mortality and mobility produced results that were either biased or had bounds too wide to be meaningful. Our bounds are applicable in a wide range of contexts, rely on minimal structural assumptions, and generate parameters that are useful for policy.

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1 Introduction

Interval data often pose a barrier to empirical research. One example that has drawn substantial recent attention is in the analysis of U.S. mortality change. Many researchers have noted that mortality is rising in subgroups defined by low levels of education, such as those with a high school degree or less (LEHS) (Meara et al., 2008; Cutler and Lleras-Muney, 2010b; Cutler et al., 2011; Olshansky et al., 2012; Case and Deaton, 2015; Case and Deaton, 2017). But because of rising education, this group is smaller and thus more negatively selected over time.¹ LEHS individuals are likely to have lower relative socioeconomic status today than in the past; if low socioeconomic status is a driver of mortality, then negative selection could explain all of the mortality increases among the less educated (Dowd and Hamoudi, 2014; Bound et al., 2015; Currie, 2018). There is considerable debate in the mortality literature over whether rising mortality among the less educated is a mechanical or a substantive result. Studying the time series of mortality in education percentiles would solve the selection problem. Because education is measured coarsely with bin boundaries that change over time, this is an interval data problem. But current interval data methods in the literature produce bounds on mortality change that are too wide to be meaningful.

A similar challenge arises in the study of intergenerational educational mobility. Recent approaches to intergenerational mobility focus on the expected rank of a child born to a parent at the 25th percentile of the socioeconomic rank distribution, called absolute upward mobility or p_{25} (Chetty et al., 2014b). How does one identify this value when the 25th percentile parent is observed only in a wide bin? In older generations in India, for instance, 60% of parents report education in a bottom-coded bin. Conventional approaches to the measurement of p_{25} (as well as other mobility measures, like the rank-rank gradient) that assume away the censoring problem are implicitly imposing strong and often implausible

¹64% of 50–54 year old women had less than or equal to a high school education (LEHS) in 1992, compared with 39% in 2015. Figure 1 shows mortality for 50–54 year old U.S. women as a function of the median education rank in each of three educational categories, illustrating the simultaneous changes in mortality and in the distribution of education.

assumptions. Measuring changes in educational mobility over time is even more challenging, as the bin boundaries in the parent education distribution change over time.

These are specific examples of a broader class of problems, that of estimating $E(y|x)$ when x is interval censored. In this paper, we propose an empirical approach to this problem that results in tight bounds on the parameters of interest, generating new facts on both mortality and mobility. The method is also applicable to the relationship between education and birth outcomes, marriage patterns and disability, all areas where the changing distribution of education may be an important barrier to estimation (Cutler and Lleras-Muney, 2010a; Aizer and Currie, 2014; Bertrand et al., 2016). Most broadly, it is relevant in any setting with interval censored conditioning variables, which appear in contexts as diverse as bond ratings, top-coded incomes, or studies using Likert scales.

We propose three innovations that substantially tighten known bounds on $E(y|x)$, sometimes by multiple orders of magnitude. First, we derive analytical bounds for $E(y|x)$ when x is interval censored and from a known distribution. Recognizing the distribution (of ranks, for example) can permit a substantial reduction in bound width compared with the more general bounds on $E(y|x)$ derived by Manski and Tamer (2002).² Second, we derive analytical bounds for the expectation of the CEF in some interval $[a, b]$ and show that $E(y|x \in [a, b])$ can often be tightly bounded even when $E(y|x = i)$ cannot for any value of i . For example, $E(\text{Mortality}|\text{Ed Rank} \in [0, 50])$ can be tightly bounded, but bounds are too wide to be meaningful for either $E(\text{Mortality}|\text{Ed Rank} = 25)$ or $E(\text{Mortality}|\text{Ed Rank} = 50)$. Third, we present a numerical method that further narrows the bounds by imposing a curvature constraint on the CEF. The limiting case of this approach is a maximal curvature constraint which corresponds to a linear estimation, which is often the default choice of researchers faced with interval data like these.

We use these innovations to calculate bounds on the mortality change from 1992–2015 of

²For some conditioning variables, no additional assumptions are required; for example, ranks are uniform by construction. For others (such as income), distributional assumptions on the variable of interest are common and reasonable, and results under alternative assumptions can be tested.

less educated 50–54 year olds, a group that has been the attention of considerable recent mortality research (Case and Deaton, 2015; Case and Deaton, 2017). The mortality change for women with a high school degree or less (the bottom 64% in 1992 and the bottom 39% in 2015) is 129 additional deaths per 100,000, or a 28% increase in the mortality rate. In contrast, bounds on mortality change among the bottom 64% of the education distribution (a similar group of individuals, but one that is fixed in size and relative position in the distribution) suggest a mortality increase between 29 and 38 additional deaths per 100,000, a 6–8% increase. The constant rank measure provides an estimate of mortality change in a quantile of the education distribution, comparable to estimates of mortality change in quantiles of the income distribution.³ This is particularly useful as vital statistics data on U.S. mortality do not contain a measure of income. The mortality change in the constant percentile group is considerably smaller than the unadjusted estimate, though still concerning given improvements in survival in other high income countries. For some age-gender groups, the unadjusted estimates are substantially biased or even have the incorrect sign; for others, the bias is small.

Next, we show that our approach considerably improves the analysis of interval censored data in an entirely different setting: intergenerational mobility. Because the common mobility measures p_{25} and the rank-rank gradient can be described as functions of the CEF, we can bound them using the method described above. We show that if the interval censoring in the Indian data is taken seriously, both of these measures have bounds that are too wide to be informative. In contrast, when interval censoring in the education data is ignored, estimates of p_{25} and the rank-rank gradient are biased and misleadingly precise.

We propose a new measure of upward mobility, *upward interval mobility*, which describes the mean value of the child CEF in the bottom half of the parent rank distribution, *i.e.* $E(y|x \in [0, 50])$.⁴ This measure is of similar policy relevance to absolute upward mobility,

³In a context where education is strictly considered as an input to the production function (such as in estimating the returns to education), unadjusted estimates may be preferred. We take no stand on the health production function or whether the mortality-education relationship should be treated causally.

⁴Compare this to absolute upward mobility, which measures the expected outcome of the child born to the median parent in the bottom half of the rank distribution.

but upward interval mobility can be tightly bounded given coarse education data while absolute upward mobility cannot. Further, this measure is to our knowledge the first measure of intergenerational educational mobility that is suitable for cross-group comparisons.⁵

The interval problem for educational mobility is most severe in developing countries, but is important in other contexts as well. In wealthier countries, it is common for a large share of the population to be in a top-coded education bin.⁶ Internationally comparable censuses also frequently report education in as few as four categories, making our method particularly relevant for cross-country comparison.

This paper makes three contributions. First and most generally, we derive sharp bounds on $E(y|x)$ and related measures when x is only observed in intervals and is from a known distribution. We also provide a tractable numerical framework for calculating nonparametric bounds under more complex constraints, including for contexts without monotonicity, which was previously a challenge for interval data methods. While our analysis is limited to conditional expectation functions with a single parameter, this setup nevertheless describes a broad class of interval censoring problems beyond the education-related examples described here; extensions to more complex models are a subject for future research.⁷ Our contribution here is largely practical; it comes as no surprise that bounds are tighter under known than under unknown distributions. But the combination of our three innovations generates tight bounds in multiple contexts where researchers previously faced substantial barriers to estimation. The method is easily generalized to use additional structural assumptions, or to measure

⁵As noted by Hertz (2005) and Aaronson and Mazumder (2008), the widely used rank-rank gradient is not meaningful for population subgroups, because it describes rank gains relative to the subgroup rather than to the population. Cross-group comparisons of absolute upward mobility are meaningful, but of limited value when absolute upward mobility cannot be tightly bounded.

⁶In one mobility study from Sweden, for example, 40% of adoptive parents were top-coded with 15 or more years of education (Björklund et al., 2006). Studies on the persistence of occupation across generations also frequently use a small number of categories and face a similar challenge when the occupational structure changes significantly over time, as it has with farm work in the United States. See, for example, Long and Ferrie (2013), Xie and Killewald (2013) and Guest et al. (1989).

⁷For additional background on partial identification, see Manski (2003), Tamer (2010), and Ho and Rosen (2015). Similar problems to ours are addressed by Cross and Manski (2002), Magnac and Maurin (2008), Bontemps et al. (2012) and Chandrasekhar et al. (2012) (all discussed in Section 2), but none are directly applicable to our context.

other conditional parameters, like a median or other percentile of the outcome distribution.

Second, we provide a new method for analysis of mortality change at the bottom of the education distribution. Earlier work on this topic had limited policy relevance because of the unknown compositional bias from changing education levels. Our work is related to several other papers in this literature. Goldring et al. (2016) derive a one-tailed test for changes in the mortality-education gradient, but they do not calculate the bias in existing mortality estimates, nor do they estimate mortality in constant rank bins. Bound et al. (2015) proposes measurement of mortality in constant education quantiles, but achieves point estimation by implicitly assuming that mortality is constant across education ranks within observed intervals, an assumption that is unlikely to hold. Other approaches to this problem have been proposed by Case and Deaton (2015) and Cutler et al. (2011). Our method is the first that can generate mortality estimates in constant percentile bins that takes seriously the information loss from interval censoring.

Finally, we provide a tractable framework for the measurement of intergenerational educational mobility in contexts where education is coarsely observed. Our new measure of upward mobility is easy to calculate and generates precise estimates in many contexts where conventional measures provide information that is too coarse to be useful. To our knowledge, upward interval mobility is also the first measure that permits meaningful comparisons of intergenerational educational mobility across groups that are robust to changes in the education distribution.

In the next section, we describe the setup, prove the new bounds and present the numerical solution framework. In Section 3, we validate the bounds and demonstrate some of their properties in a simulation. Sections 4 and 5 present the applications to the measurement of mortality and of intergenerational mobility. Section 6 concludes. Stata and Matlab code to implement all methods in the paper is posted online.⁸

⁸See <https://github.com/paulnov/anr-bounds>.

2 Bounds on CEFs with a Known Conditioning Distribution

In this section, we derive analytical and numerical bounds on a CEF where the conditioning variable is interval censored but has a known distribution. The bounds are sharp and depend either on the assumption of a weakly monotonic CEF or on the assumption that the CEF has limited curvature. The method can further bound any statistic that can be derived from the CEF (e.g. the best linear approximator to the CEF), or other conditional measures, like the conditional median.

We work through an example motivated by Figure 1, which plots total mortality against mean education rank, where education is only observed in one of three education bins: (i) less than or equal to high school (LEHS); (ii) some college; or (iii) bachelor's degree or higher.⁹ We focus on women aged 50–54, because this age group has been highlighted in other recent research, and this group has experienced substantial education gains over the sample period. We wish to estimate mortality in 1992 and 2015 for a group occupying the same education rank or set of ranks over time. This is challenging because the rank bin boundaries change between 1992 and 2015. In 1992, 64% of women had LEHS education, while in 2015, this number was 39%.

We work through the example of calculating bounds on mortality among the least educated 64% of women in all years, the set of education ranks occupied by LEHS women in 1992. This value can be point estimated from the data in 1992, but can at best be bounded in other years where there is no bin boundary that falls right at this percentile.

We require the concept of a continuous, latent education rank. Such a concept arises naturally out of a standard human capital accumulation model, where years of schooling have a convex cost and individual-specific differences in costs and benefits of schooling are constant shifters (Card, 1999). Individuals within any given schooling bin can be ranked according to the amount that the cost of schooling would have to change for them to prefer a higher level of schooling. The individuals with highest latent ranks within a bin are the ones

⁹Points are plotted at the midpoint of the education rank bins.

who would attain a higher level of education given only a marginal shift in cost. In a context where years of education have a strictly causal relationship with some outcome variable, the latent education rank would not be a useful concept, because all individuals within a narrowly-defined bin could have the same level of education. However, education is used as a proxy for socioeconomic status in both the mortality and the mobility literatures; we can expect individuals with higher latent education ranks (holding constant observed education bin) to have lower mortality and better off children. In both of our contexts, education is also measured in bins much more coarse than the number of years of education, so the latent rank variable is meaningful even if we treat education as strictly causing the outcomes of interest.

We proceed in a two step process. First, we derive bounds on the CEF of mortality at each latent education rank. Second, we extend these bounds to generate estimates of average mortality in a bin with arbitrary rank boundaries. This section of the paper focuses on the problem of identifying a CEF given interval data, and Section 4 explores the findings on mortality in more detail.

Figure 2 depicts the setup for 2015. The points show mortality at the midpoints of three education bins and the vertical lines show the rank bin boundaries. The lines plot two (of many) possible nonparametric CEFs, each of which fit the sample means with zero error. These two functions have the same mean in each bin, even if they do not cross the mean at the bin midpoint.¹⁰ These are the functions we aim to bound. We begin with the assumption that mortality is weakly decreasing in latent rank, and we then show how a curvature constraint can supplement or substitute for this assumption.

Our work is related to several other papers. Manski and Tamer (2002) prove sharp bounds on $E(y|x)$ when x is interval censored and from an unknown distribution. These bounds, which we take as our starting point, are too wide to be informative about mortality among the less educated, and have thus not been considered for this problem. We improve on these bounds with structural assumptions that are inherent in the mortality problem. Cross and

¹⁰A naive polynomial fit to the midpoints in the graph would be a biased fit to the data because of Jensen's Inequality.

Manski (2002) examine bounds on $E(y|x)$ when the probability distributions of x and y are known but the joint distribution is unknown. In our case, the distribution of y is unknown, but the observation of the bin means (which are unknown in Cross and Manski (2002)) is considerably more powerful in tightening the bounds. Magnac and Maurin (2008) focus on cases with binary dependent variables, showing that bounds are tighter under known distributions. Bontemps et al. (2012) present partial identification results for cases where the y variable is interval-censored. Chandrasekhar et al. (2012) study inference on the best linear approximator to the conditional expectation function; they require a sharp set of conditional expectation *functions*, which cannot be characterized in our context because we can only characterize the set of expectations evaluated at a *point*.

2.1 Nonparametric Estimation with Interval Data

Define the outcome as y and the conditioning variable as x ; the conditional expectation function is $Y(x) = E(y|x)$. Let the function $Y(x)$ be defined on $x \in [0, 100]$, and assume $Y(x)$ is integrable. We also assume throughout that $\underline{Y} \leq Y(x) \leq \bar{Y}$, that is, the function is bounded absolutely.¹¹

With interval data, we do not observe x directly, but only that it lies in one of K bins. Let $f_k(x)$ be the probability density function of x in bin k . Define the expected outcome in the k^{th} bin as

$$r_k = E(y|x \in [x_k, x_{k+1}]) = \int_{x_k}^{x_{k+1}} Y(x)f_k(x)dx,$$

where x_k and x_{k+1} define the bin boundaries of bin k . This expression holds due to the law of iterated expectations. The limits of the conditioning variable are assumed to be known, and are denoted by x_1 and x_{K+1} . Further define the expected outcomes in the intervals directly above and below the intervals of interest as $r_{k+1} = E(y|x \in [x_{k+1}, x_{k+2}])$ and $r_{k-1} = E(y|x \in [x_{k-1}, x_k])$, if they exist. Define $r_0 = \underline{Y}$ and $r_{K+1} = \bar{Y}$. The sample analog

¹¹In most applications, parameters of interest are likely to have upper and lower bounds either in theory or in practice. Loosening the absolute upper and lower bound restriction would result in wider bounds for the CEF in the bottom or top intervals, but informative estimates are possible even in these outer bins. In the case of mortality, we will impose that the upper bound is a mortality rate of 100%.

to r_k is the observed mean outcome in bin k , which we denote \bar{r}_k .

Sharp bounds on $E(y|x)$ given interval measurement of x are derived by Manski and Tamer (2002), when the distribution of x is unknown. The essential structural assumption that constrains the CEF is Monotonicity (M):

$$E(y|x) \text{ must be weakly increasing in } x. \quad (\text{Assumption M})$$

Note that we apply this assumption to the survival rate, which is one minus the mortality rate; however, our graphs show the mortality rate which is the parameter of interest. The CEF in the monotonic graphs is thus monotonically decreasing.¹² Manski and Tamer (2002) also introduce the following Interval (I) and Mean Independence (MI) assumptions. For x which appears in the data as lying in bin k ,

$$x \text{ in bin } k \implies P(x \in [x_k, x_{k+1}]) = 1. \quad (\text{Assumption I})$$

$$E(y|x, x \text{ lies in bin } k) = E(y|x). \quad (\text{Assumption MI})$$

Assumption *I* states that the rank of all people who report education ranks in category k are actually in bin k . Assumption *MI* states that censored observations are not different from uncensored observations. These always hold in our context because all of the data are interval censored.

If all observations of x are interval censored, the Manski and Tamer (2002) bounds are:

$$r_{k-1} \leq E(y|x) \leq r_{k+1} \quad (\text{Manski-Tamer bounds})$$

In words, the value of the CEF in each bin is bounded by the means in the previous and next bins.

We can improve upon these bounds if the distribution of x is known. In some cases, as with

¹²Mortality is decreasing in educational attainment for every group and time period in the CDC data; it is also a monotonically decreasing function of income (Chetty et al., 2016).

ranks, the distribution is given by the definition of the variable. In other cases, conventional distributions are frequently assumed (such as lognormal or Pareto for income data). Alternatively, data could be transformed into a known distribution, for example, by transforming the conditioning variable into ranks. We first show bounds under the assumption that x has a uniform distribution because the analytical results are particularly parsimonious, but we derive all of our results under a general known distribution. We therefore consider the following assumption (U):

$$x \sim U(x_0, x_{K+1}) \quad (\text{Assumption U})$$

where U is the uniform distribution.

If x is uniformly distributed, we know that:

$$E(x|x \in [x_k, x_{k+1}]) = \frac{1}{2}(x_{k+1} - x_k). \quad (2.1)$$

We derive the following proposition.

Proposition 1. *Let x be in bin k . Under assumptions M, I, MI and U, and without additional information, the following bounds on $E(y|x)$ are sharp:*

$$\begin{cases} r_{k-1} \leq E(y|x) \leq \frac{1}{x_{k+1}-x} ((x_{k+1} - x_k) r_k - (x - x_k) r_{k-1}), & x < x_k^* \\ \frac{1}{x-x_k} ((x_{k+1} - x_k) r_k - (x_{k+1} - x) r_{k+1}) \leq E(y|x) \leq r_{k+1}, & x \geq x_k^* \end{cases}$$

where

$$x_k^* = \frac{x_{k+1}r_{k+1} - (x_{k+1} - x_k)r_k - x_k r_{k-1}}{r_{k+1} - r_{k-1}}.$$

The proposition is obtained from the insight that the value of $E(y|x = i)$ at a point i in bin k (below the midpoint) will only be minimized if all points in bin k to the left of i have the same value. Since all points to the right of i are constrained by the outcome value in the subsequent bin $k + 1$, $E(y|x = i)$ will need to rise above the Manski and Tamer (2002) lower bound as i increases, in order to meet the bin mean. Intuitively, consider the point $E(y|x =$

$x_{k+1} - \varepsilon$). In order for this point to take on a value below the bin mean r_k , it needs to be the case that virtually all of the density in bin k lies between $x_{k+1} - \varepsilon$ and x_{k+1} . This is ruled out by the uniform distribution, and indeed by most distributions; for many distributions, therefore, the Manski and Tamer (2002) bounds are excessively conservative, especially near bin boundaries. We prove the proposition and provide additional intuition in Appendix B.

We generalize the proposition to obtain the following result for an arbitrary known distribution of x :

Proposition 2. *Let x be in bin k . Let $f_k(x)$ be the probability density function of x in bin k . Under assumptions M , I , MI , and without additional information, the following bounds on $E(y|x)$ are sharp:*

$$\begin{cases} r_{k-1} \leq E(y|x) \leq \frac{r_k - r_{k-1} \int_{x_k}^x f_k(s) ds}{\int_{x_k}^x f_k(s) ds}, & x < x_k^* \\ \frac{r_k - r_{k+1} \int_x^{x_{k+1}} f_k(s) ds}{\int_x^{x_{k+1}} f_k(s) ds} \leq E(y|x) \leq r_{k+1}, & x \geq x_k^* \end{cases}$$

where x_k^* satisfies:

$$r_k = r_{k-1} \int_{x_k}^{x_k^*} f_k(s) ds + r_{k+1} \int_{x_k^*}^{x_{k+1}} f_k(s) ds.$$

A proof of the proposition is in Appendix B.

Figure 3 compares Manski and Tamer (2002) bounds to those obtained under the additional assumption of uniformity, using the mortality data. The new bounds are a significant improvement, especially where the data are particularly coarse and near the bin boundaries. For example, without using information on the distribution type, one could not reject that the mortality rate for people in the first bin is 100,000 per 100,000 until just before the first bin boundary. The improvements in the other bins are less extreme but still substantial.

In addition to bounding the value of $Y = E(y|x)$ at any given point, we can also bound many functions of the CEF, which we represent in the form $M(Y)$. One function of interest is the slope of the best linear approximation to the CEF; this is difficult to bound analytically, but we bound this numerically in Section 2.4.

Here, we highlight a function that describes the average value of the CEF over an arbitrary interval of the conditioning space, or $\mu_a^b = E(y|x \in [a, b])$. This function has several desirable properties. First, it can be bounded analytically. Second, it is frequently bounded more tightly than $E(y|x)$. Third, it has a similar interpretation to $E(y|x)$ and is thus likely to be policy-relevant. We show in Sections 4 and 5 that for our applications, μ_a^b can be bounded considerably more tightly than $E(y|x)$.

Let $f(x)$ represent the probability density function of x . Define μ_a^b as

$$\mu_a^b = \frac{\int_a^b E(y|x)f(x)dx}{\int_a^b f(x)dx}. \quad (2.2)$$

We now state analytical bounds on μ_a^b given uniformity. Let Y_x^{max} be the analytical upper bound on $E(y|x)$, given by Proposition 1. Let Y_x^{min} be the analytical lower bound on $E(y|x)$. The following proposition defines sharp bounds on μ_a^b under the assumption that x is uniformly distributed:

Proposition 3. *Let $a \in [x_h, x_{h+1}]$ and $b \in [x_k, x_{k+1}]$, with $a < b$. Let assumptions M , I , MI and U hold. Then, if no additional information is available, the following bounds are sharp:*

$$\left\{ \begin{array}{ll} Y_b^{min} \leq \mu_a^b \leq Y_a^{max} & h = k \\ \frac{r_h(x_k - a) + Y_b^{min}(b - x_k)}{b - a} \leq \mu_a^b \leq \frac{Y_a^{max}(x_k - a) + r_k(b - x_k)}{b - a} & h + 1 = k \\ \frac{r_h(x_{h+1} - a) + \sum_{\lambda=h+1}^{k-1} r_\lambda(x_{\lambda+1} - x_\lambda) + Y_b^{min}(b - x_k)}{b - a} \leq \mu_a^b \leq \frac{Y_a^{max}(x_{h+1} - a) + \sum_{\lambda=h+1}^{k-1} r_\lambda(x_{\lambda+1} - x_\lambda) + r_k(b - x_k)}{b - a} & h + 1 < k. \end{array} \right.$$

We prove this proposition under uniformity and under an arbitrary known conditioning distribution in Appendix B.

We note two special cases. First, if $a = b$, then $\mu_a^b = E(y|x = a)$. Second, if a and b correspond exactly to bin boundaries, then the bounds on μ_a^b collapse to a point: in this case, μ_a^b is just a weighted average of the bin means between a and b .

In fact, μ_a^b can be very tightly bounded whenever a and b are close to bin boundaries. For

intuition, consider the following examples. If $\delta \in [a, b]$, μ_a^b can be written as a weighted mean of the two subintervals $\frac{\delta-a}{b-a}\mu_a^\delta + \frac{b-\delta}{b-a}\mu_\delta^b$.¹³ If μ_a^δ is known (because there are bin boundaries at a and δ), then any uncertainty about the value of the CEF in the range $[a, \delta]$ is not consequential for the bounds on μ_a^b . If b is close to δ , the weight on the unknown value μ_δ^b is very small, and μ_a^b can be tightly bounded. Similarly, if instead μ_a^b is known, and b is again close to δ , then μ_a^δ can be tightly estimated even if μ_δ^b has wide bounds.

2.2 Bounds on More Complex Functions of the CEF

Bounds on other functions of the CEF may be difficult to calculate analytically, but can be defined as the set of solutions to a pair of minimization and maximization problems that take the following structure. We write the conditional expectation function in the form $Y(x) = s(x, \gamma)$, where γ is a finite-dimensional vector that lies in parameter space G and serves to parameterize the CEF through the function s . For example, we could estimate the parameters of a linear approximation to the CEF by defining $s(x, \gamma) = \gamma_0 + \gamma_1 * x$. We can approximate an arbitrary nonparametric CEF by defining γ as a vector of discrete values that give the value of the CEF in each of N partitions; we take this approach in our numerical optimizations, setting N to 100.¹⁴ Any statistic m that is a single-valued function of the CEF, such as the average value of the CEF in an interval (μ_a^b), or the slope of the best fit line to the CEF, can be defined as $m(\gamma) = M(s(x, \gamma))$.

Let $f(x)$ again represent the probability distribution of x . Define Γ as the set of parameterizations of the CEF that obey monotonicity and minimize mean squared error with

¹³The weights on each subcomponent here assume that x is uniformly distributed. A different distribution would use different weights.

¹⁴For example, $s(x, \gamma_{50})$ would represent $E(y|x \in [49, 50])$.

respect to the observed interval data:

$$\Gamma = \operatorname{argmin}_{g \in G} \sum_{k=1}^K \left\{ \int_{x_k}^{x_{k+1}} f(x) dx \left(\left(\frac{1}{\int_{x_k}^{x_{k+1}} f(x) dx} \int_{x_k}^{x_{k+1}} s(x, g) f(x) dx \right) - \bar{r}_k \right)^2 \right\} \quad (2.3)$$

such that

$E(y|x)$ is weakly increasing in x .

(Monotonicity)

Decomposing this expression, $\frac{1}{\int_{x_k}^{x_{k+1}} f(x) dx} \int_{x_k}^{x_{k+1}} s(x, g) f(x) dx$ is the mean value of $s(x, g)$ in bin k , and $\int_{x_k}^{x_{k+1}} f(x) dx$ is the width of bin k . The minimand is thus a bin-weighted MSE.¹⁵

Recall that for the rank distribution, $x_1 = 0$ and $x_{K+1} = 100$.

The bounds on $m(\gamma)$ are therefore:

$$m^{\min} = \inf\{m(\gamma) \mid \gamma \in \Gamma\} \quad (2.4)$$

$$m^{\max} = \sup\{m(\gamma) \mid \gamma \in \Gamma\}.$$

For example, bounds on the best linear approximation to the CEF can be defined by the following process. First, consider the set of all CEFs that satisfy monotonicity and minimize mean-squared error with respect to the observed bin means.¹⁶ Next, compute the slope of the best linear approximation to each CEF. The largest and smallest slope constitute m^{\min} and m^{\max} . Note that this definition of the best linear approximator to the CEF corresponds to the *least squares set* defined by Ponomareva and Tamer (2011).

Stata code to generate bounds on the CEF and on μ_a^b , and Matlab code to run these numerical optimizations for more complex functions (as well as with the curvature constraints described below) are posted on the corresponding author's web site.

¹⁵While we choose to use a weighted mean squared error penalty, in principle Γ could use other penalties.

¹⁶In many cases, and in all of our applications, there will exist many such CEFs that exactly match the observed data and the minimum mean-squared error will be zero.

2.3 CEF Bounds Under Constrained Curvature

The candidate CEFs that underlie the bounds in Proposition 1 are step functions with substantial discontinuities. If such functions are implausible descriptions of the data, then the researcher may wish to impose an additional constraint on the curvature of the CEF, which will generate tighter bounds. For example, examination of the mortality-income relationship (which can be estimated at each of 100 income ranks, displayed in Figure A1) suggests no such discontinuities.¹⁷ Alternately, in a context where continuity has a strong theoretical underpinning but monotonicity does not, a curvature constraint can substitute for a monotonicity constraint and in many cases deliver useful bounds.

We consider a curvature restriction with the following structure:

$$s(x, \gamma) \text{ is twice-differentiable and } |s''(x, \gamma)| \leq \bar{C}. \quad (\text{Curvature Constraint})$$

This is analogous to imposing that the first derivative is Lipschitz.¹⁸ Depending on the value of \bar{C} , this constraint may or may not bind.

The most restrictive curvature constraint, $\bar{C} = 0$, is analogous to the assumption that the CEF is linear. Note that the default practice in many studies of mortality is to estimate the best linear approximation to the CEF of mortality given education (e.g., Cutler et al. (2011) and Goldring et al. (2016)). In the study of intergenerational mobility (Section 5), the best linear approximation to $E(\text{rank}_{child}|\text{rank}_{parent})$ is a canonical estimator. A moderate curvature constraint is therefore a *less* restrictive assumption than the approach in many studies. We discuss the choice of curvature restriction below.

¹⁷More complex structural restrictions can also be imposed. For example, the CEF might be continuous within education bins, but there could be large discontinuities due to sheepskin effects at the education bin boundaries (Hungerford and Solon, 1987).

¹⁸Let X, Y be metric spaces with metrics d_X, d_Y respectively. The function $f : X \rightarrow Y$ is *Lipschitz continuous* if there exists $K \geq 0$ such that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

In the rest of this section, we show results under a range of curvature restrictions to shed light on how these additional assumptions affect bounds in an empirical application. In our applications in Sections 4 and 5, we show all results under the most conservative approach of $\bar{C} = \infty$.

2.4 Numerical Calculation of CEF Bounds

This section describes a method to numerically solve the constrained optimization problem suggested by Equations 2.3 and 2.4. We take a nonparametric approach for generality: explicitly parameterizing an unknown CEF with limited data is unsatisfying and could yield inaccurate results if the interval censoring conceals a non-linear within-bin CEF. In the context of mortality (and mobility, Section 5), many CEFs of interest do not appear to obey a familiar parametric form (see Figures A1 and A3).

To make the problem numerically tractable, we solve the discrete problem of identifying the feasible mean value taken by $E(y|x)$ in each of N discrete partitions of x . We thus assume $E(y|x) = s(x, \gamma)$, where γ is a vector that defines the mean value of the CEF in each of the N partitions. We use $N = 100$ in our analysis, corresponding to integer rank bins, but other values may be useful depending on the application. Given continuity in the latent function, the discretized CEF will be a very close approximation of the continuous CEF; in our applications, increasing the value of N increases computation time but does not change any of our results.

We solve the problem through a two-step process. Define a N -valued vector $\hat{\gamma}$ as a candidate CEF. First, we calculate the minimum MSE from the constrained optimization problem given by Equation 2.3. We then run a second pair of constrained optimization problems that respectively minimize and maximize the value of $m(\hat{\gamma})$, with the additional constraint that the MSE is equal to the value obtained in the first step, denoted MSE. Equation 2.5 shows the second stage setup to calculate the lower bound on $m(\hat{\gamma})$. Note that this particular setup is

specific to the uniform rank distribution, but setups with other distributions would be similar.

$$m^{min} = \min_{\hat{\gamma} \in [0,100]^N} m(\hat{\gamma}) \quad (2.5)$$

such that

$$s(x, \hat{\gamma}) \text{ is weakly increasing in } x \quad (\text{Monotonicity})$$

$$|s''(x, \hat{\gamma})| \leq \bar{C}, \quad (\text{Curvature})$$

$$\sum_{k=1}^K \left[\frac{\|X_k\|}{100} \left(\left(\frac{1}{\|X_k\|} \sum_{x \in X_k} s(x, \hat{\gamma}) \right) - \bar{r}_k \right)^2 \right] = \underline{\text{MSE}} \quad (\text{MSE Minimization})$$

X_k is the set of discrete values of x between x_k and x_{k+1} and $\|X_k\|$ is the width of bin k .

The complementary maximization problem obtains the upper bound on $m(\hat{\gamma})$.

Note that setting $m(\gamma) = \gamma_x$ (the x^{th} element of γ) obtains bounds on the value of the CEF at point x . Calculating this for all ranks x from 1 to 100 generates analogous bounds to those derived in proposition 1, but satisfying the additional curvature constraint. Similarly $m(\gamma) = \frac{1}{b-a} \sum_{x=a}^b \gamma_x$ obtains bounds on μ_a^b .

2.5 Example with Sample Data

In this section, we demonstrate the bounding method using data on the mortality of 50–54 year-old U.S. women in 2015. We focus here on the properties of the bounds under different assumptions. We explore mortality change in more detail in Section 4.

Panel A of Figure 4 graphs the analytical upper and lower bounds on $E(y|x)$ at each value of x under just the assumption of monotonicity. These bounds do not reflect statistical uncertainty but uncertainty about the CEF in the unobserved parts of the latent rank distribution.¹⁹

We next consider a curvature-constrained CEF.²⁰ The mortality-education data are not

¹⁹We do not present standard errors because we are working with the universe of deaths in a large country and statistical imprecision is very small in this context. We discuss and present bootstrap confidence sets in Section 5 where statistical imprecision is more important.

²⁰With neither the monotonicity nor the curvature constraint, the CEF cannot be bounded except by the maximum possible value of the variable of interest.

in themselves informative regarding which curvature restriction to choose. To identify a conservative curvature constraint, we examine the curvature of a closely related conditional expectation function that is not interval censored: the CEF of mortality given income rank. We show this CEF in Figure A1, using data from Chetty et al. (2016). Using a spline approximation to income rank data for 52-year-old women in 2015, we calculate a maximum \bar{C} of 1.6; we use a constraint approximately twice as high as a conservative starting point. Panel B of Figure 4 shows the bounds obtained under curvature constraints of 2, 3 and 5, but *without* the assumption of monotonicity. Relative to those under monotonicity, the curvature-constrained bounds are less informative at the tails of the distribution, and more informative close to the bin midpoints.

In Panel C, we impose the monotonicity and curvature constraints simultaneously. Panel D shows the limit case with $\bar{C} = 0$; the CEF in this figure is identical to the predicted values from a regression of mortality on median education rank. Note that while stricter curvature restrictions can tighten the bounds, this may come at the expense of ruling out a plausible CEF, even if the MSE remains zero. In Figure 4, only Panel D has a non-zero MSE.

Table 1 presents estimates of $p_x = E(y|x)$ and $\mu_a^b = E(y|x \in [a, b])$ for women ages 50–54 in 2015, for various values of x , a and b , under different constraints. In 1992, 64% of women had high school education or less, and thus occupied the bottom rank bin in the education distribution. We can describe mortality change for this group with the measures p_{32} and μ_0^{64} , which respectively describe the median and mean mortality of the bottom 64% in all periods. These measures describe mortality at constant education ranks, even though the distribution of education levels is changing over time. We also show measures for the bottom 20%, 50% and 39% (the share of women with LEHS in 2015).

We draw attention to two features of the table. First, the interval mean estimates (μ_a^b) are in most cases considerably more tightly bounded than estimates of the CEF value at the midpoint of the interval (p_x). μ_0^{64} is nearly point identified in 1992 because 0 and 64 are

very close to bin boundaries in 1992.²¹ μ_0^{64} is tightly bounded in 2015 as well, regardless of the constraint set used. μ_0^{64} and p_{32} are both useful summaries of mortality among the less educated, but μ_0^{64} is estimated with at least 22 times more precision than p_{32} . The advantage of μ_a^b over p_x (where $x = \frac{a+b}{2}$) is greatest when a and b are close to boundaries in the data.

Second, μ_0^{39} and μ_0^{64} are very robust to different bounding assumptions. Estimation is more difficult for a measure like μ_0^{20} with boundaries far from any in the data, and the width of the bounds depends strongly on the assumptions being made. μ_0^{20} (mortality in the least educated 20%) thus cannot be tightly bounded even if it is of policy interest.

Finally, Column 5 shows the predicted values for each measure from the best linear approximation to the mortality-education CEF, an object of interest in the literature. These measures are point estimated, but their precision is misleading, as they implicitly assume away large increases in mortality at the bottom of the distribution, increases that are consistent with the data and in fact suggested by Figure A1. This highlights the strength of our method, which generates consistent bounds on mortality across the education distribution under considerably less restrictive assumptions.

3 Simulation: Bounds on the U.S. Mortality-Income CEF

In this section, we validate our method in a simulation by taking data from the fully supported U.S. mortality-income CEF, interval censoring that data based on bin boundaries in the education data, and then recovering bounds on the true CEF from the interval censored data. The exercise illustrates that studying partially identified bounds permits the recovery of important features of the CEF that might be missed by a parametric fit to the bin means. We use data on mortality by income percentile, gender, age and year (Chetty et al., 2016). We focus on women aged 52 in 2014, the group most comparable what we have examined so far.

First, we estimate the true CEF from the mortality-income data by fitting a cubic spline

²¹Similarly, μ_0^{39} is nearly point estimated in 2015, where 39% of women had attained high school or less. We have used integer approximations to these parameters for convenience; if we used the average mortality for the precise proportion of women with less than or equal to a high school degree ($\mu_0^{63.658}$), then the parameter would be precisely point identified.

with four knots to the data, the same spline used to obtain an estimate of \bar{C} . We plot this in Appendix Figure A1.²²

Next, we simulate interval censoring by obtaining the mean of the true CEF within income rank bins based on the 2015 education rank boundaries.²³ After interval censoring, we have a dataset with average mortality in each of three bins, comparable to the data from Section 2. We compute bounds on the CEF using only the binned data.

Panels A–D of Figure 5 present CEF bounds generated from the binned data, under monotonicity and curvature limits that vary from ∞ (unconstrained) to 1. The dashed lines show the underlying data. The solid circles show the constructed bin means of the censored data; these are the only data that we use for the optimization. The solid lines show the upper and lower envelopes that we calculate for the nonparametric CEF.

The suggested curvature constraint ($\bar{C} = 3$) yields bounds that contain the true CEF at every point; but when we impose $\bar{C} = 1$, the constraint is excessive and the bounds do not contain the true CEF. The true CEF is not always centered within the bounds; from ranks 25 to 40, the true CEF is near the bottom bound, and from ranks 90 to 100, it is nearer the upper bound.

The exercise further illustrates that assuming a parametric form for the underlying CEF can yield misleading results. A quadratic or linear fit to the data would fail to identify the convexity at the bottom of the distribution. The strength of our method is that it makes transparent how the structural assumptions affect the CEF bounds.

Table 2 shows bounds on a range of statistics of interest under different curvatures, as well as the true estimate. We highlight three results. First, the interval mean measures (μ_a^b) generate tighter bounds than the CEF values p_x , with no greater propensity for error. Second, \bar{C} is consequential for p_x , but considerably less important for μ_a^b ; we obtain tight

²²We use a spline approximation rather than the raw data because the variation across neighboring rank bins is most likely idiosyncratic given the small number of deaths in an age bin defined by a single year. By using information from neighboring points, the spline is a better estimate of mortality risk than the individual rank bin means.

²³We round to the nearest integer, since we only observe integer percentiles in the data from Chetty et al. (2016).

bounds even without the curvature constraint. Third, the linear estimates generated with $\bar{C} = 0$ are biased by as much as 25% relative to the true estimates, and sometimes produce estimates that are outside the bounds even of CEFs with unconstrained curvature.

4 Application: U.S. Mortality Change in Constant Education Rank Bins

In this section, we apply our method to study changes in U.S. mortality for individuals at constant ranks in the education distribution. We explore how the bounded estimates and the bound widths change under different structural assumptions, and how this range of estimates compares to naive estimation of changing mortality at *levels* of education rather than ranks.

The importance of compositional bias in estimates of mortality among the less educated has received substantial attention in the literature, but the problem is largely unsolved. Note first that if the causal effect of education on mortality is the only mechanism relating these two variables, then the comparison over time of mortality at constant levels of education would be directly meaningful. However, if a third variable, like socioeconomic status, affects both mortality and education, then secular increases in education for the entire population may mechanically cause higher measured mortality at all levels of education even if mortality at each education percentile is unchanged. Estimating mortality among the bottom X% of the education distribution is analogous to estimating mortality in the bottom X% of the income distribution, which is generally agreed to be a policy-relevant measure even though levels of income at each percentile are changing over time.²⁴

Past researchers have taken various approaches to address this issue. Cutler et al. (2011) adjust for compositional shifts by predicting propensity to attend college using region, marital status and income, and then using this propensity as a conditioning variable. They argue that compositional shifts are not important for mortality changes from the 1970s to the 1990s. This approach is limited by the extent to which changes in mortality are correlated with the predicting variables; in many cases (e.g. with vital statistics data), these additional variables are unavailable. Case and Deaton (2015) and Case and Deaton (2017) argue that changes in

²⁴See, for example, Chetty et al. (2016).

the proportion of middle-aged whites with LEHS from the 1990s to the present are too small to influence mortality rates; this approach is useful in their context but substantially limits the scope of analysis. Dowd and Hamoudi (2014) and Bound et al. (2015) perform analytical exercises that suggest that compositional shifts can explain most or all of recent mortality changes. Bound et al. (2015) estimate mortality for the bottom quartile of the education distribution, implicitly assuming that mortality is constant within each interval-censored mortality rank bin. As noted in Section 2, this assumption is unlikely to hold in a standard human capital accumulation framework, especially when education is measured at coarse schooling levels rather than in years of education.²⁵ Goldring et al. (2016) propose a one-tailed test that describes whether the mortality-education gradient is changing; our framework can generate a similar test, but can also bound mortality estimates in arbitrary bins of the latent rank space. In contrast with the existing work, our method requires no additional covariates, produces estimates of interest under minimal and reasonable structural assumptions, and is transparent regarding the uncertainty that arises from coarse measurement of education.

Mortality by education records come from the U.S. Center for Disease Control’s WONDER database and total population by age, gender and education come from the Current Population Survey, as in Case and Deaton (2017). Additional details on data construction are available in Appendix D.1.

As above, we assume that the observed mortality data describe a monotonic relationship between mortality and the latent education rank. Results are virtually identical if we constrain curvature using the parameter suggested in Section 2 and forgo the monotonicity constraint. We focus in this section on women aged 50–54, because this is a group whose education composition has shifted substantially over time.²⁶

Panel A of Figure 6 plots mean total mortality for women age 50-54 in each education group in 1992 and 2015, along with analytical bounds on CEFs with unconstrained curvature.

²⁵The implicit Bound et al. (2015) mortality CEF is contained within our bounds, but under our weaker assumptions, it is only one of many possible CEFs.

²⁶We use 5-year bins for ages rather than larger bins to ensure that the average age in the bin does not change over time (Gelman and Auerbach, 2016).

The bounds at each latent education rank i are the bounds on the measure $p_i = E(Y|X = i)$ described above. The bounds are largely overlapping across the entire education distribution, and too wide to infer very much about changes in mortality. In Panel B, we restrict curvature to approximately twice the maximum curvature from the income-mortality data, as discussed in Section 2. Panel B identifies a clear decline in mortality at the top of the education distribution, but the bounds remain minimally informative at the bottom.

The interval mean measures (μ_a^b) are more informative. We focus on mean mortality in the bottom 64% (μ_0^{64}), as this describes the set of women with high school or less in 1992.²⁷ In 2015, the bottom 64% includes all women with high school or less, and some women with some college education, but none with bachelor’s degrees or higher. For men, we focus on the bottom 54%, the LEHS population share with in 1992; by 2015, 44% of men have LEHS, so the bottom 54% again includes some men with two-year college degrees. As in Chetty et al. (2016), we rank men and women against members of their own gender, estimating mortality for a given percentile group of men or women; that is, the least educated group can be interpreted as the “the 64% of least educated women,” rather than “women in the bottom 64% of the population education distribution.” We chose own-gender reference points because women’s and men’s labor market opportunities and choices are often different and because women and men often share households and incomes, making population ranks misleading. However, alternate choices could be considered and estimated with the same method.

Panel A of Figure 7 shows bounds on total mortality for women aged 50-54 in the bottom 64%. Mortality in 1992 can be nearly point estimated, because the 0-64 rank bin interval is almost exactly observed in the data. As education levels diverge from those in 1992, the bounds progressively widen. The “x” markers in the figure plot the unadjusted estimates of mortality among women with less than or equal to high school education; these mortality estimates describe a group occupying a shrinking and more negatively selected share of the population over time. The unadjusted estimates, which are the object of study in most ear-

²⁷Specifically, we calculate $\mu_0^{63.658}$ for analytical monotonic bounds (because 63.658% of women have LEHS education) and μ_0^{64} for numerical curvature constrained bounds.

lier work on the mortality-education relationship, significantly overstate mortality increases relative to the constant rank group. The upper bound on mortality gain for the bottom 64% is 8.5%, compared to the unadjusted estimate of 28%. Panel B shows the same figure for men. The unadjusted estimates are closer to the bounds here because men have gained less education than women over this period. We can bound the mortality change for men in the interval $[-7.1\%, +0.3\%]$, compared with the unadjusted estimate of $+1.2\%$.²⁸ Panels C and D present analogous results for combined deaths from suicide, poisoning and liver disease, described by Case and Deaton (2017) as “deaths of despair.” The unadjusted mortality estimates continue to overstate the constant rank mortality changes, but the difference here is small because (i) deaths of despair have increased substantially among all education levels; and (ii) the education gradient in deaths of despair was small in 1992. Appendix Figure A2 shows the same plots, but removes the monotonicity assumption and instead imposes the curvature restriction of $\bar{C} = 3$ suggested above. The plots are highly similar; as discussed in Section 2, when a and b are close to bin boundaries, the bounds on μ_a^b are very robust to alternate bounding assumptions.

The extent of bias in naive estimates is increasing in the magnitude of the mortality-education gradient, and in the magnitude of the shift in bin boundaries. Blanket assumptions about the existence or lack of bias in unadjusted mortality differences are therefore unlikely to be useful. This can be seen in Table 3, which shows unadjusted and constant-rank estimates of women’s mortality changes from 1992-2015 for age groups from 20 to 69, for all education categories. Education rank bins are fixed based on the 1992 rank bin divisions; results are very similar if we fix estimates at the 2015 boundaries. The unadjusted estimates systematically overstate mortality increases for all groups, because the mean rank in each group has declined over this period. Unadjusted estimates of men’s mortality changes from

²⁸This result is not directly comparable to Case and Deaton (2015, 2017), who focus on *white* men and women, whose unadjusted mortality is rising more substantially among the less educated. Estimating separate bounds for different racial groups requires additional assumptions about the relative positions of these groups in the unobserved part of the latent education distribution, and we leave this exercise for future work. The increases in mortality for less educated women that we identify here are lower than naive estimates, but remain striking compared with other rich countries.

1992-2015 are close to the constant rank bounds, as are unadjusted estimates of deaths of despair for both men and women. For women’s total mortality, however, the naive estimates overstate mortality increases in many cases by a factor of three or more, and in some cases they have the wrong sign.

5 Application: Bounds on Intergenerational Educational Mobility

The study of intergenerational mobility is another context where the conditional expectation function of interest in many cases has an interval-censored conditioning variable.²⁹ Studies of intergenerational mobility typically rely upon some measure of rank in the social hierarchy which can be observed for both parents and children (Chetty et al., 2014a; Chetty et al., 2017b). In many contexts, the only measure of social rank available for parents is their level of education. In richer countries, this arises for studies of mobility in eras that predate the availability of administrative income data.³⁰ In developing countries, matched parent-child data are considerably more rare, and educational mobility is often the only feasible object of study.³¹ Interval-censored parent education data is ubiquitous in studies of intergenerational educational mobility. Table A1 reports the number of parent education bins used in a set of recent studies of intergenerational mobility from several rich and poor countries. Several of the studies observe education in fewer than ten bins, the population share in the bottom bin is often above 20%, and sometimes it is above 50%.³²

Studies on educational mobility have typically focused on linear estimators of the parent-child outcome relationship, such as the slope of the best linear approximator to the CEF of child education rank given parent education rank, *i.e.*, the rank-rank gradient. This measure has two important limitations. First, it is not useful for cross-group comparison. The within-group rank-rank gradient measures children’s outcomes against better off members of their

²⁹For a review of intergenerational mobility, see Solon (1999), Hertz (2008), Corak (2013), Black and Devereux (2011), and Roemer (2016).

³⁰See, for example, Black et al. (2005) and Güell et al. (2013).

³¹See, for example, Wantchekon and Stanig (2015), Hnatkowska et al. (2013) or Emran and Shilpi (2015).

³²We specifically selected a set of studies where coarse data is likely to be an important factor. Note that internationally comparable censuses often report education in only four or five categories.

own group; a subgroup can therefore have a lower gradient (suggesting more mobility) in spite of having worse outcomes than other groups at every point in the parent distribution.³³ Second, the rank-rank gradient aggregates information about mobility at the top and at the bottom of the parent distribution; it is not directly informative about upward mobility in the bottom half of the distribution.

Because of these limitations, recent studies have focused on measures based on the value of the parent-child CEF at a point in the parent distribution, termed absolute mobility at percentile x by Chetty et al. (2014a) and denoted p_x . For example, Chetty et al. (2014a) focus on p_{25} , which describes the expected outcome of a child born to the median family in the bottom half of the rank distribution. These CEF-based measures are informative about child outcomes at arbitrary points in the parent rank distribution and can be meaningfully compared across population subgroups. These measures are central to current research on mobility, but there is no established method for calculating such measures with education data, where any given percentile in the parent distribution lies within some larger bin. The problem is most stark when the bins are very large, so we focus our application on measuring intergenerational mobility in India, where over 50% of older generation parents are in a bottom-coded education bin (< 2 years of education). Appendix Table A2 shows the education transition matrices for decadal birth cohorts from 1950 to 1989.

We combine data from two sources, including administrative data on the education of every person in India in 2012, to obtain a representative sample of every father-son pair in India.³⁴ The details of data construction are described in Appendix D.2.

We observe education for both fathers and sons in seven categories.³⁵ Because sons' education levels are also reported categorically, we do not directly observe the expected child

³³An extreme example makes this clear. Suppose children in some population subgroup A all end up at the 10th percentile of the outcome distribution with certainty. The rank-rank gradient for this group would be zero (assuming some variation in parent outcomes), implying perfect mobility. But in fact the group would have virtually no upward mobility.

³⁴We are restricted to the study of fathers and sons because the data do not match daughters to parents or children to mothers when they do not live in the same household.

³⁵The categories are (i) less than two years of education; (ii) at least two years but no primary; (iii) primary; (iv) middle school; (v) secondary; (vi) senior secondary; and (vii) post-secondary or higher.

outcome in each parent education bin. In this section, we instead assign to children the midpoint of their rank bin. We show in Appendix C that data on son wages (for which the rank distribution is uncensored) suggests that the midpoint is a very close approximation to the true expected rank, because the residual correlation of father education and son wages is very small once son’s education is controlled for. Note that such an exercise is impossible for interval censoring of parent data, because no additional data on parents is available, as is typical in studies of intergenerational mobility.³⁶ Appendix C also provides a method that generates bounds under joint censoring, which can be used in contexts where additional data on sons is not available. An alternate approach would be to estimate child rank directly using a socioeconomic measure for sons that can be observed continuously.

5.1 Bounding the CEF of Child Rank Given Parent Rank

Panel A of Figure 8 shows the raw data for cohorts born in the 1950s and in the 1980s. Each point plots the midpoint of a father education rank bin against the expected child rank in that bin. The vertical lines plot the boundary for the lowest education bin for each cohort, which corresponds to fathers with less than two years of education. In the 1950s birth cohort (solid line), this group represents 60% of the population; it represents 38% for the 1980s cohort (dashed line). The points in the figure suggest that the rank-rank CEF has not changed in the bottom half of the parent distribution over this period: the bottom point in the 1950s lies almost directly between the bottom two points in the 1980s. However, when we estimate the rank-rank gradient directly on the bin means, we find small but unambiguous mobility gains over this 30-year period: the gradient declines from 0.59 to 0.54). The graph makes clear that the decrease in the gradient is driven by changes in mobility in the top half of the distribution. Alternately, if we treat the data as uncensored, such that the expected child outcome is the same at all latent ranks within each parent bin, we would conclude that absolute upward mobility (p_{25} , or the expected child outcome at the 25th parent percentile) has unambiguously

³⁶Because parental education is often obtained by asking children, it is common to have data on many child outcomes, but only the education level of parents, as we do here.

fallen from the 1950s to the 1980s. But this result is clearly an artifact of the changing bin boundaries. Neither of these conclusions represents the true change in mobility.³⁷

We take the following approach. We assume that the latent parent-child rank CEF can be described by an increasing monotonic function; this relationship is monotonic in virtually every country (Dardanoni et al., 2012), as well across every rank bin in every year of our data on India. We then use the method from Section 2 to obtain bounds on the CEF of child rank given parent rank, and various functions of that CEF.³⁸

Panel B of Figure 8 shows bounds on the parent-child CEFs for these birth cohorts, i.e. $E(Y|X = i, \forall i)$; we select a curvature constraint of 0.1, which is approximately 1.5 times the maximum curvature observed in uncensored parent-child *income* data from the United States, Denmark, Sweden and Norway.³⁹ The bounds on the CEF are widest at the bottom of the distribution where interval censoring is most severe, and are worse for the older generation with the larger bottom rank bin. The bounds in the bottom half of the distribution are consistent with both large positive and large negative changes in mobility, and thus uninformative. Absolute upward mobility can evidently not be bounded informatively.⁴⁰

We can considerably tighten the bounds by focusing on an interval-based measure such as $\mu_a^b = E(y|x \in [a, b])$. We call this measure *interval mobility*, and focus in particular on $\mu_0^{50} = E(y|x \in [0, 50])$, which we call *upward interval mobility*. This statistic is closely related to absolute upward mobility (p_{25}). Absolute upward mobility describes the expected outcome of a child born to the median parent in the bottom half of the parent distribution,

³⁷Note also that the CEF is evidently non-linear, so a naive nonlinear parametric fit to the bin midpoints would be biased due to Jensen’s Inequality. It also assumes away concavity at the bottom of the distribution, which is observed in many other countries (see Appendix Figure A3).

³⁸Our method is loosely related to Chetty et al. (2017a), who use a numerical procedure with similar constraints to bound absolute mobility at the 25th percentile, given just the marginal distributions of children’s and parents’ incomes and no information on the joint distribution. However, the substantive problem they solve is very different from ours.

³⁹We selected these countries because we were able to obtain precise uncensored parent-child income rank data for them from Chetty et al. (2014b), Boserup et al. (2014) and Bratberg et al. (2015). Graphs for the spline estimations used to calculate the curvature constraints are displayed in Appendix Figure A3. Results are similar under different curvature constraints.

⁴⁰Note that a more restrictive curvature constraint would narrow the bounds, but at the expense of imposing excessive structure that would rule out plausible CEFs, especially given the evident nonlinearity in the data.

whereas upward interval mobility describes the *mean* child outcome in the bottom half of the parent distribution. These measures are of similar economic importance, but we show here that upward interval mobility can be bounded tightly in contexts with severe interval censoring, while absolute upward mobility cannot.

Figure 9 shows bounds on the three mobility statistics discussed for each decadal cohort: the rank-rank gradient, absolute upward mobility (p_{25}), and upward interval mobility (μ_0^{50}).⁴¹ For reference, we plot recent estimates of similar mobility measures from USA and Denmark.^{42,43} When we recognize the interval censoring and allow expected child outcomes to vary within the bottom parent education bin, both the rank-rank gradient and absolute upward mobility have wide and minimally informative bounds. In contrast, upward interval mobility is estimated with tight bounds in all periods. We can now see that upward mobility has changed very little over the four decades studied, as suggested by the population moments; there is a small gain from the 1950s to the 1960s, followed by a small decline from the 1960s to the 1980s. On average, Indian mobility is as far below that in the United States as mobility in the United States is below Denmark. Table 4 reports the bounds for each measure and cohort with bootstrap confidence sets under a range of curvature restrictions. Moderate curvature restrictions generate substantial improvements on the estimation of the value of the CEF (e.g., p_{25}), but are considerably less important for the interval measures (e.g., μ_0^{50}), which are tightly bounded even with unconstrained curvature.

In conclusion, the most widely used mobility estimator, the rank-rank gradient, presents an incomplete and potentially biased picture of intergenerational educational mobility. Upward interval mobility, in contrast, yields informative estimates even without a curvature constraint. Our method thus makes it possible to study upward mobility in the lower-ranked

⁴¹The bounds on the rank-rank gradient describe the slopes of the set of best linear approximators to feasible CEFs.

⁴²Rank-rank correlations of education are from Hertz (2008), which are equal to the slope of the rank-rank regression coefficient if estimated on uncensored rank data. For absolute mobility, we calculate p_{25} for the U.S. and Denmark from the income distributions shown in Figure A3, with data from Chetty et al. (2014a).

⁴³We calculate and show bootstrap confidence sets using 1,000 bootstrap samples from the underlying datasets, following methods described in Imbens and Manski (2004) and Tamer (2010).

parts of the distribution even in a context with extreme interval censoring. The advantage of upward interval mobility is likely to be replicated in mobility studies in the many other countries where older generations are clustered in less educated bins.

6 Conclusion

Interval data are ubiquitous in economic research. We have shown in this paper that the shortcuts taken by applied researchers to transform data intervals to points often implicitly make strong assumptions. We have further shown that many parameters of interest, but by no means all, can be tightly bounded even when interval data are severe. Accurate estimation requires careful consideration of which parameters can and cannot be bounded meaningfully.

Our proposed methods are particularly relevant to studies describing characteristics of subpopulations defined by low levels of education, which cover areas as broad as employment, health, marriage and fertility. All of these studies must contend with the fact that a population defined by a level of education represents a different subset of the education rank distribution at different point in time. This problem is typically dealt with on an ad hoc basis. Our paper provides a clear and tractable framework for the analysis and resolution of this problem, and shows its utility in the two substantively different domains of mortality and intergenerational mobility.

A useful thought experiment when working with interval data is to explore how estimates are affected as intervals become more or less granular. The bounds presented in this paper become wider when the data become more coarse, as should be expected given that information has been removed. In contrast, with conventional point estimation approaches, the use of coarser intervals can lead to different point estimates and even increases in precision, thus obscuring the loss of information to the researcher. Like many partial identification methods, the key advantage of our approach is that it is transparent about what is known and what is not known, and how results depend upon the assumptions about the CEF in question.

The use of structural assumptions to tighten bounds is a staple of partial identification analysis. We have shown in this paper that a very limited set of assumptions can generate

substantial new information, solving known problems in the study of mortality and of intergenerational mobility. While our applications have focused on estimating outcomes that condition on education, the broader contribution to the analysis of interval data is widely applicable. Interval data are found in the study of incomes, Likert scales, bond ratings and a wide range of other domains. The innovations described here have the potential to generate findings in some of these other areas even when the degree of interval censoring is substantial.

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Figure 1
Women's Total Mortality by Education Group,
Age 50-54, 1992-2015

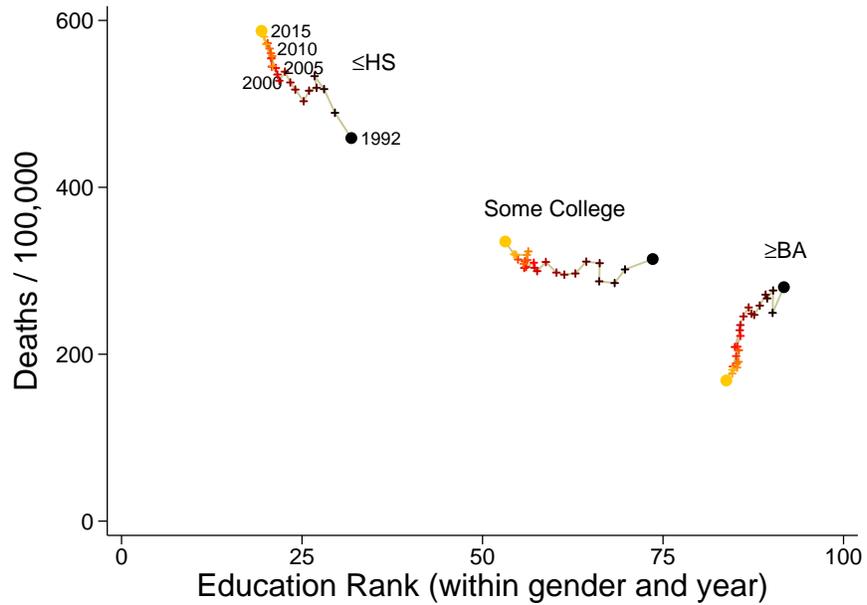


Figure 1 plots mortality rates vs. mean education rank for three groups: women with less than or equal to a high school degree, women with some college education, and women with a BA or more. Each point represents a mortality rate (in deaths per 100,000) within a year and education group. The lighter colored points correspond to later years. Ranks are calculated within gender and year.

Figure 2
Candidate Functions for Conditional Expectation of Mortality
given Education Rank

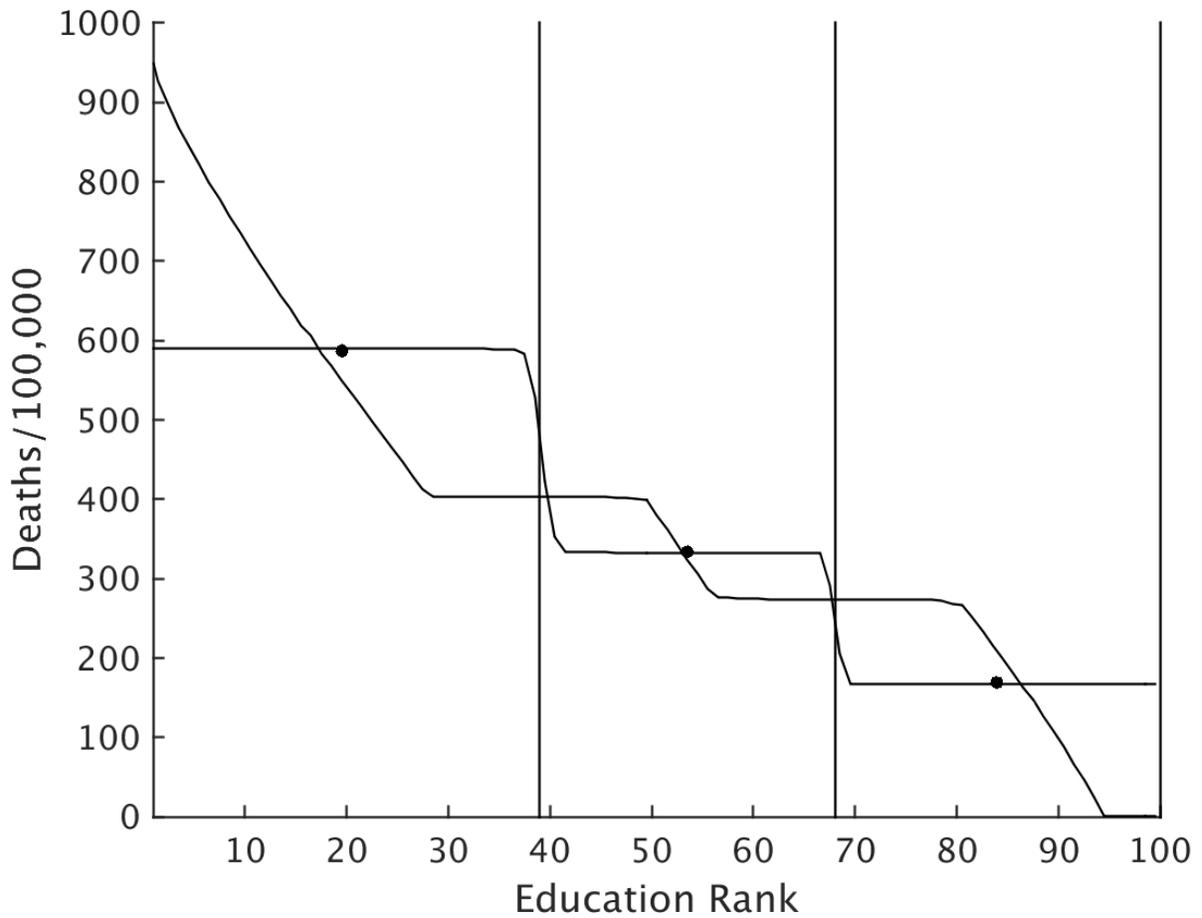


Figure 2 shows two candidate conditional expectation functions of mortality given education rank for women aged 50–54 in the United States in 2015. The vertical lines show the bin boundaries and the points show mean mortality and mean child rank in each bin.

Figure 3

Analytical Bounds on the CEF of Mortality given Education Rank

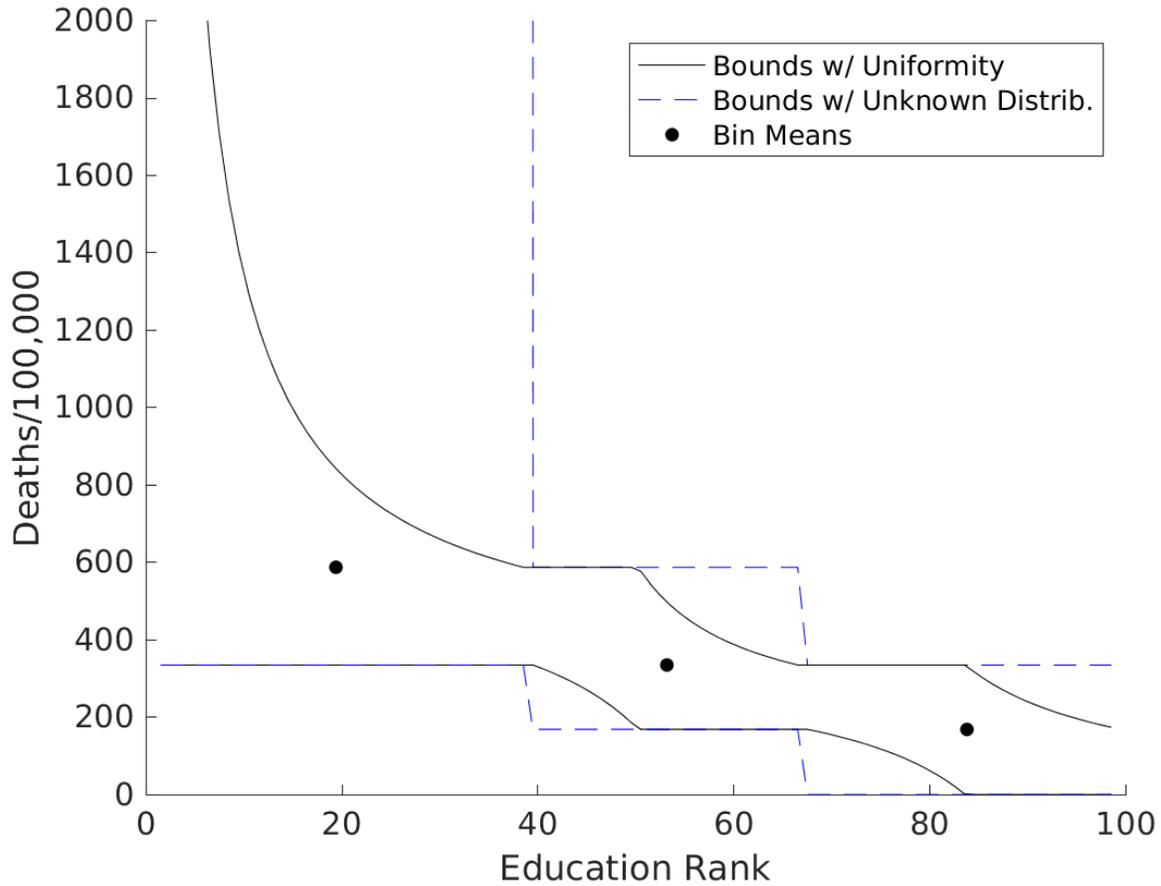


Figure 3 shows bounds on the conditional expectation of mortality given education rank for women aged 50–54 in the United States in 2015. The vertical lines show the bin boundaries and the points show the mean total mortality and child rank in each bin. The dashed lines show analytical bounds when the distribution of the x variable is unknown (Manski and Tamer, 2002). The solid line shows analytical bounds when the distribution of the x variable is uniform.

Figure 4
 CEF of Mortality given Education Rank:
 Bounds Under Different Constraint Assumptions

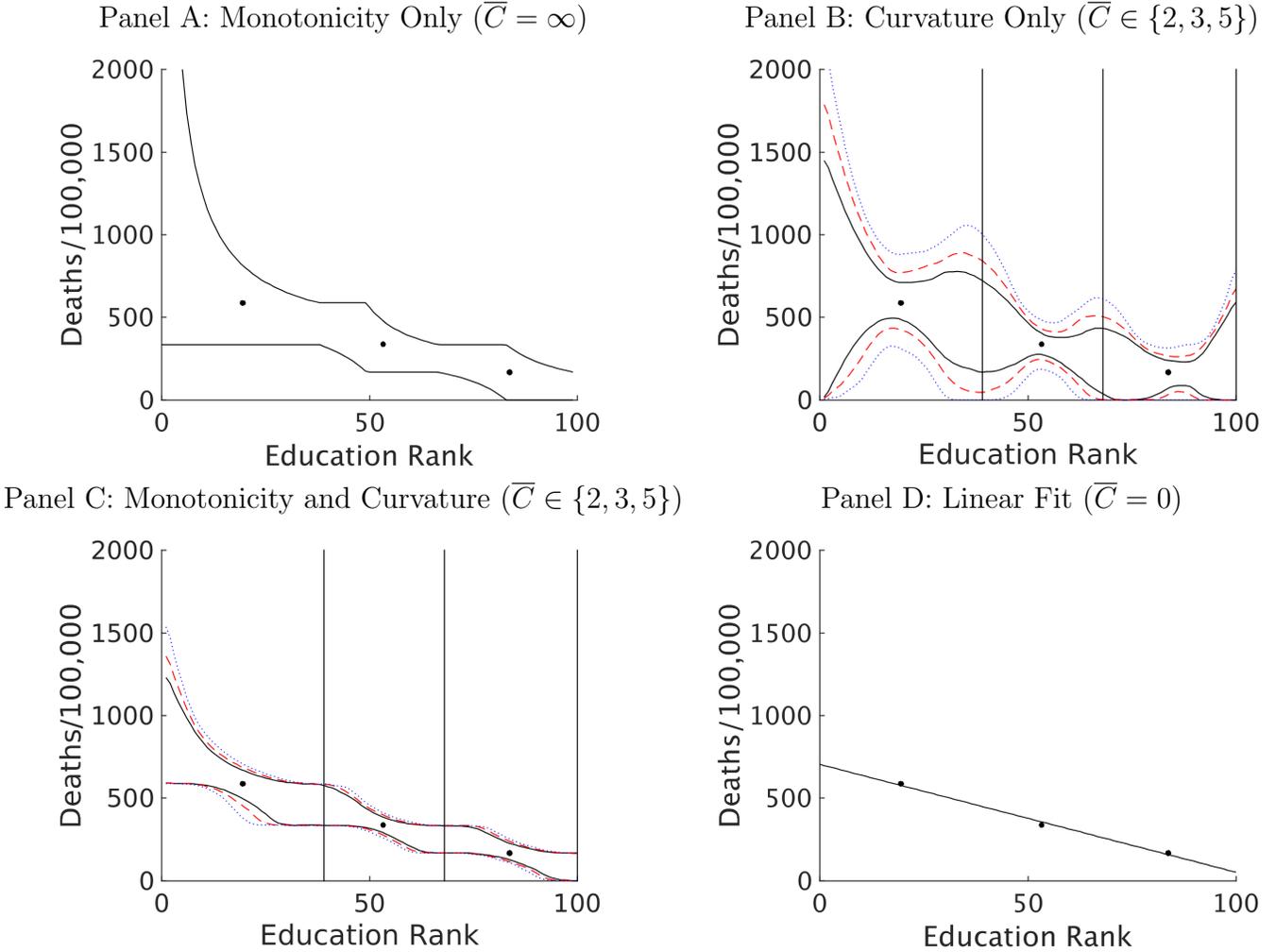


Figure 4 presents bounds on conditional expectation functions of mortality given education ranks for women aged 50–54 in 2015 under different assumptions sets. Education rank is measured relative to the set of all women aged 50–54. The lines in each panel represent the upper and lower bounds on the CEF at each rank, obtained under different monotonicity or curvature restrictions. Panel A imposes monotonicity only. Panel B imposes curvature constraints only, with the solid, dashed and dotted lines respectively showing bounds with $\bar{C} = 2$, $\bar{C} = 3$ and $\bar{C} = 5$. Panel C imposes monotonicity and curvature constraints, with the same limits as Panel B. Panel D imposes linearity, by setting \bar{C} to zero. The points show the mean mortality and education rank of women in each bin in the education distribution.

Figure 5

Simulated Interval Censoring and Bounds using U.S. Mortality-Income Data

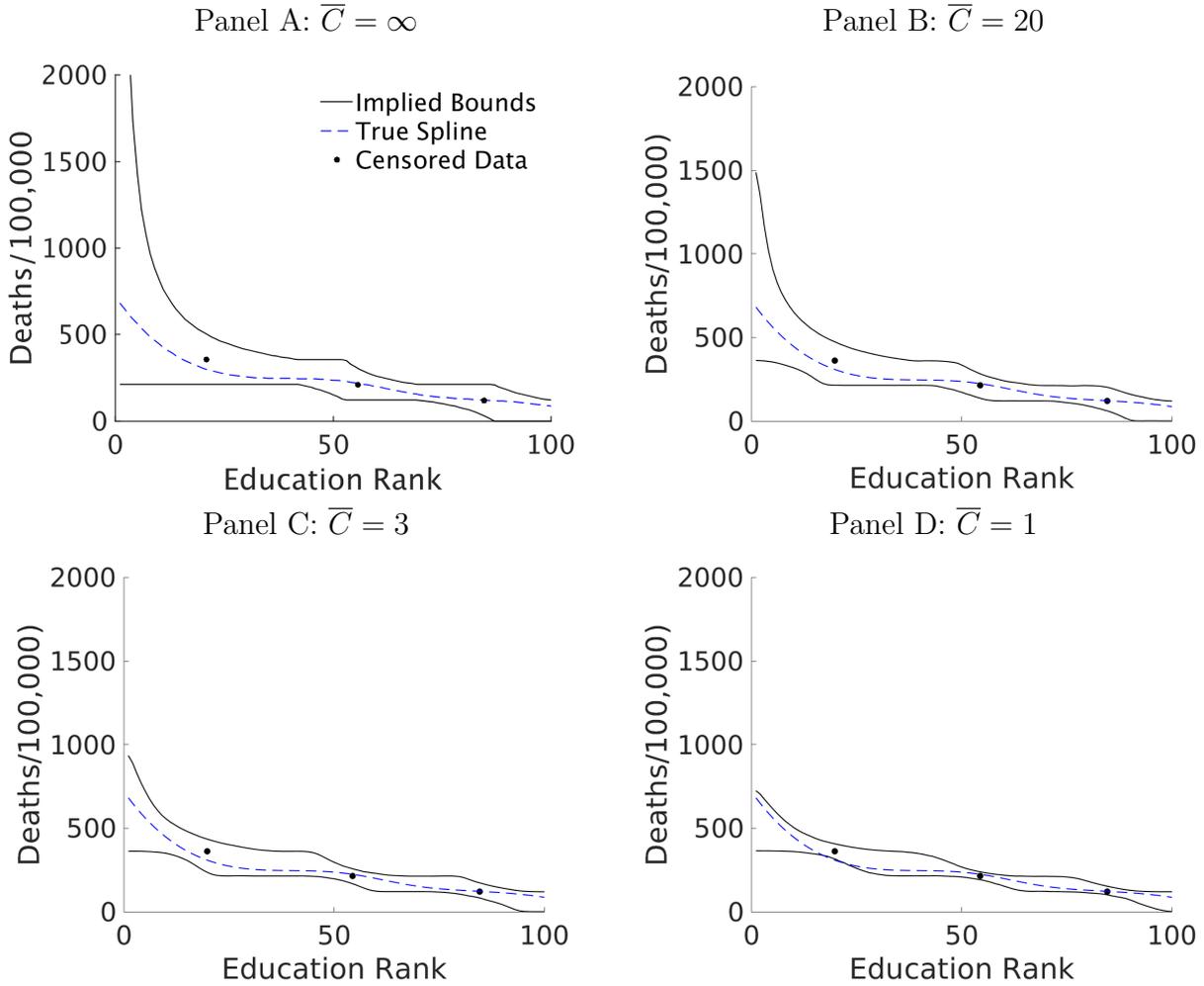
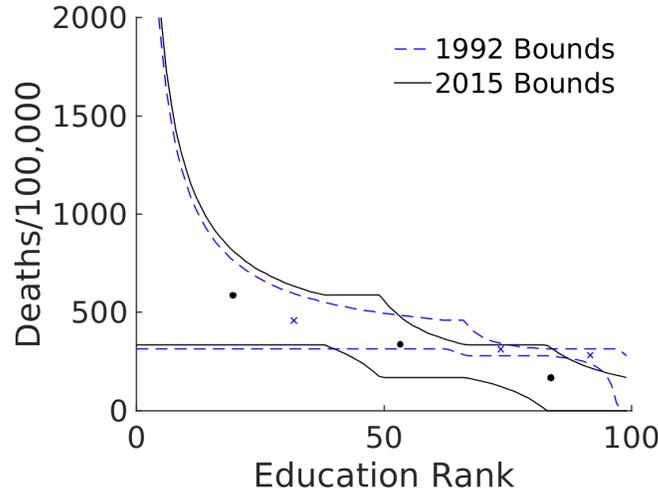


Figure 5 shows results from a simulation using matched mortality-income rank data from Chetty et al. (2016) in 2014 for women aged 52. We simulated interval censoring along the bin boundaries from the 2014 education-mortality data, so that the only observable data were the points in the graphs, which show mean mortality and education rank in each education bin. We then calculated bounds under four different curvature constraints, indicated in the graph titles. The solid lines show the upper and lower bound of the CEF at each point in the parent distribution, and the dashed line shows the spline fit to the underlying data (described in Figure A1).

Figure 6
 Change in Total Mortality of U.S. Women, Age 50-54
 Bounds on Conditional Expectation Functions

Panel A: Monotonicity Only



Panel B: Monotonicity and $\bar{C} = 3$

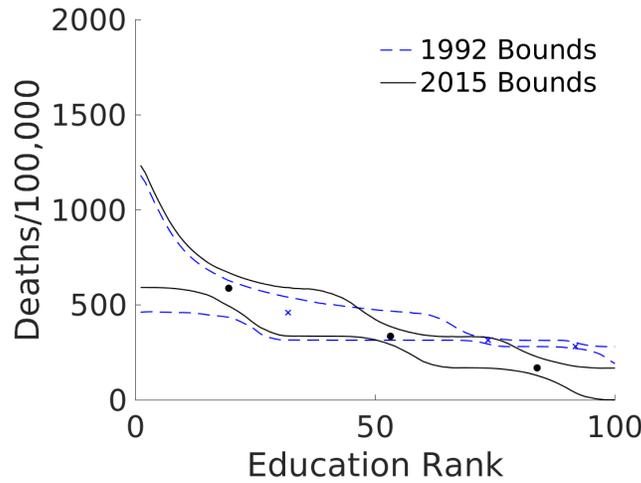


Figure 6 shows bounds on the conditional expectation function of mortality as a function of latent educational rank. The sample consists of U.S. women aged 50-54; mortality is measured in deaths per 100,000 women. The points in the graph show the mean education rank and mortality in each year for individuals with (i) less than or equal to high school education; (ii) some college; and (iii) a B.A. or higher. The curves show the bounds on the expected mortality at each latent parent rank ($p_i = E(Y|X = i)$ in the text). Panel A shows analytical bounds with no curvature constraint. Panel B uses the curvature constraint suggested in Section 2. Education rank is measured relative to the set of all women aged 50-54.

Figure 7
 Changes in U.S. Mortality, Age 50-54, 1992-2015:
 Constant Rank Interval Estimates (High School or Less in 1992)

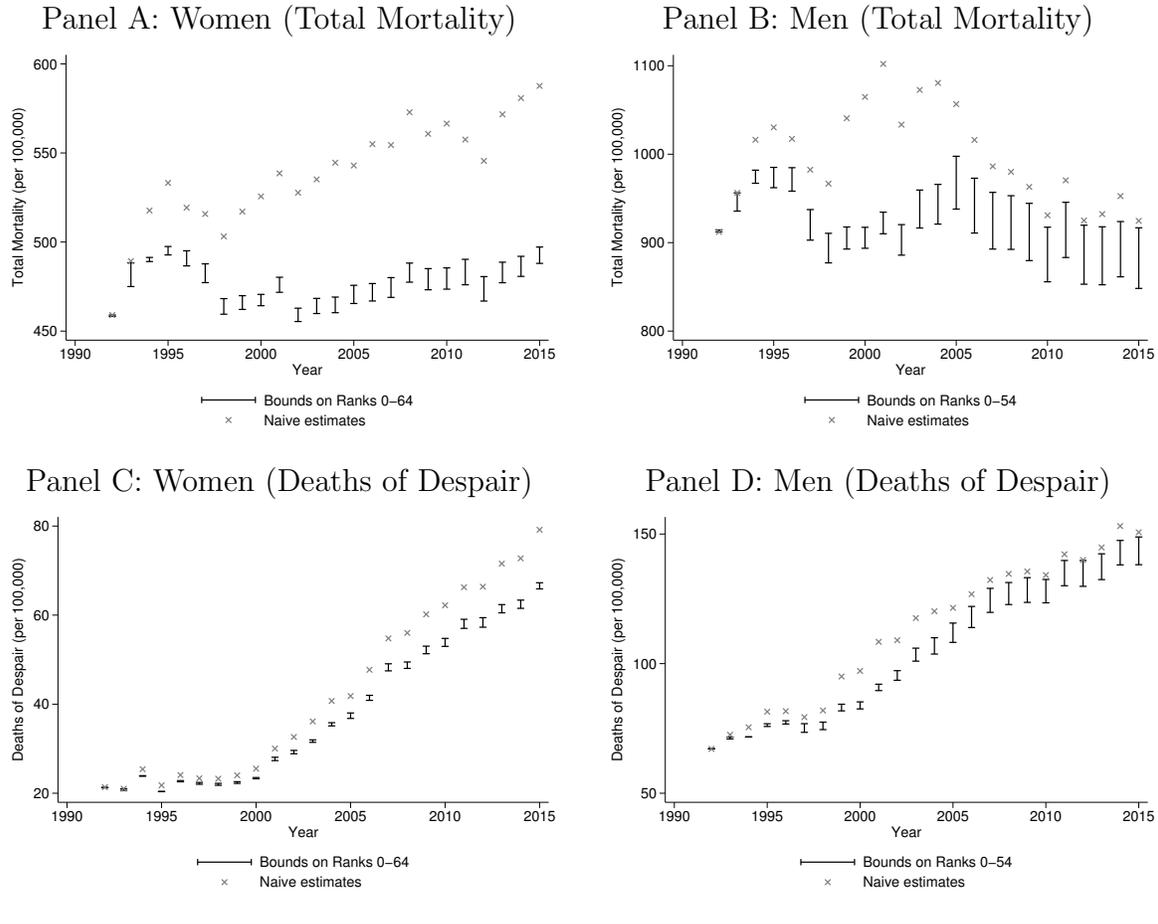
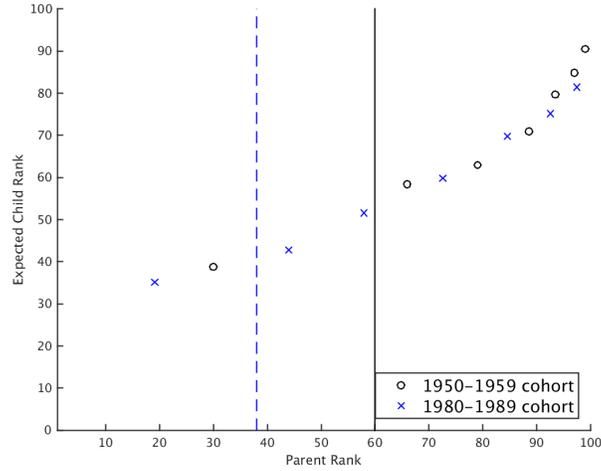


Figure 7 shows bounds on mortality change for less educated men and women aged 50–54 over time, as well as unadjusted estimates. The points show the total mortality of men and women with high school education or less (LEHS), from 1992–2015. The vertical lines show the bounds on mortality for the group of men or women who occupy a constant set of ranks corresponding to the ranks of men and women with LEHS in 1992. For women, these are ranks 0-64, and for men they are ranks 0-54. Panel A shows estimates for women age 50-54, and Panel B for men. Panels C and D show analogous plots for mortality deaths of despair for both groups, defined as deaths from suicide, poisoning or chronic liver disease. All bounds are calculated analytically under the assumptions of monotonicity and unconstrained curvature.

Figure 8

Changes in Intergenerational Educational Mobility in India from 1950s to 1980s Birth Cohorts

Panel A: Rank Bin Midpoints



Panel B: CEF Bounds

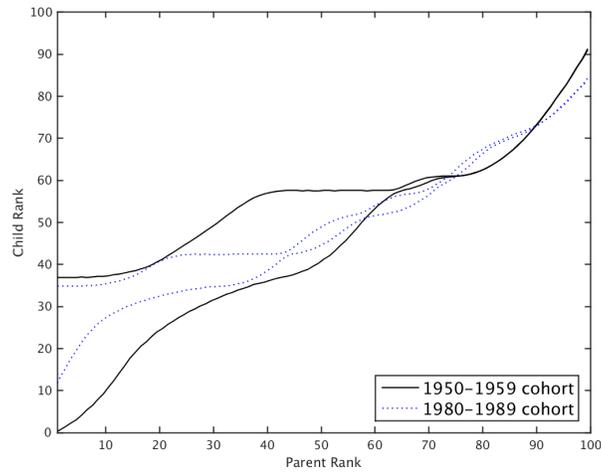
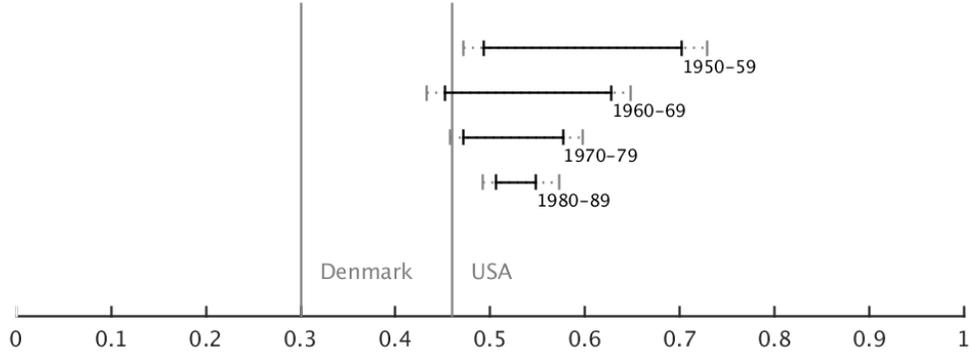


Figure 8 presents the change over time in the rank-rank relationship between Indian fathers and sons born in the 1950s and the 1980s. Panel A presents the raw bin means in the data. The vertical lines indicate the size of the lowest parent education rank bin, representing fathers with less than two years of education; the solid line shows this value for the 1950s cohort, and the dashed line for the 1980s cohort. Panel B presents the bounds on the CEF of child rank at each parent rank, under the curvature constraint $\bar{C} = 0.10$.

Figure 9
Mobility Bounds for 1950s to 1980s Birth Cohorts

Panel A: Rank-Rank Gradient



Panel B: Absolute and Interval Mobility: p_{25} and μ_0^{50}

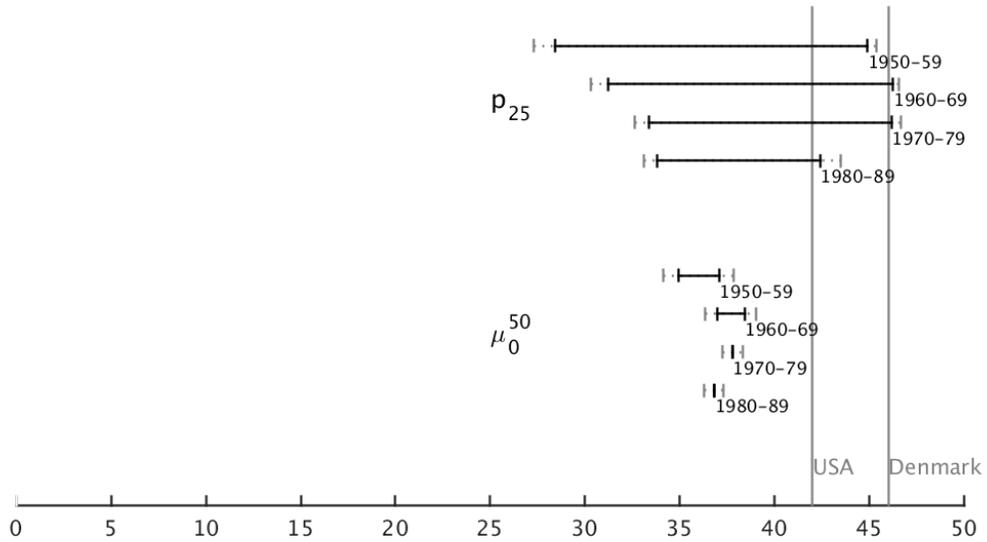


Figure 9 shows bounds on three mobility statistics, estimated on four decades of matched Indian father-son pairs. The solid lines show the estimated bounds on each statistic and the gray dashed lines show the 95% bootstrap confidence sets, based on 1000 bootstrap samples. Each of these statistics was calculated using monotonicity and the curvature constraint $\bar{C} = 0.10$. For reference, we display the rank-rank education gradient for USA and Denmark (from Hertz (2008)), and p_{25} for USA and Denmark (from Chetty et al. (2014a)). The rank-rank gradient is the slope coefficient from a regression of son education rank on father education rank. p_{25} is absolute upward mobility, which is the expected rank of a son born to a family at the 25th percentile. μ_0^{50} is upward interval mobility, which is the expected rank of a son born below to a family below the 50th percentile.

Table 1

Sample Statistics for Mortality of Less Educated Women Ages 50–54

Panel A: 1992

Statistic	Monotonicity Only ($\bar{C} = \infty$)	Curvature Only ($\bar{C} = 3$)	Monotonicity and Curvature $\bar{C} = 3$	Linear Fit ($\bar{C} = 0$)
p_{10} : First Quintile Median	[314.0, 1236.1]	[223.8, 1008.0]	[453.9, 813.9]	526.4
p_{25} : Bottom Half Median	[314.0, 683.1]	[226.1, 738.7]	[384.9, 585.5]	479.3
p_{32} : Median \leq High School (1992)	[314.0, 602.4]	[176.9, 694.9]	[346.1, 536.3]	457.3
p_{19} : Median \leq High School (2015)	[314.0, 799.6]	[253.4, 750.3]	[421.2, 642.8]	498.1
μ_0^{20} : First Quintile Mean	[459.0, 775.5]	[11.2, 1287.5]	[467.6, 749.0]	526.4
μ_0^{50} : Bottom Half Mean	[459.0, 498.6]	[423.1, 565.1]	[460.7, 496.2]	479.3
μ_0^{64} : Mean \leq High School (1992)	[458.2, 459.0]	[459.0, 459.0]	[459.0, 459.0]	457.3
μ_0^{39} : Mean \leq High School (2015)	[459.0, 550.7]	[316.5, 608.4]	[462.1, 544.9]	496.5

Panel B: 2015

Statistic	Monotonicity Only ($\bar{C} = \infty$)	Curvature Only ($\bar{C} = 3$)	Monotonicity and Curvature $\bar{C} = 3$	Linear Fit ($\bar{C} = 0$)
p_{10} : First Quintile Median	[335.1, 1313.5]	[473.4, 899.1]	[577.2, 846.8]	640.8
p_{25} : Bottom Half Median	[335.1, 726.7]	[345.3, 718.2]	[376.9, 628.8]	542.6
p_{32} : Median \leq High School (1992)	[335.1, 641.1]	[260.5, 721.5]	[345.3, 593.8]	496.7
p_{19} : Median \leq High School (2015)	[335.1, 850.3]	[434.0, 746.4]	[464.5, 672.8]	581.9
μ_0^{20} : First Quintile Mean	[587.7, 824.8]	[326.5, 956.9]	[590.0, 806.9]	640.8
μ_0^{50} : Bottom Half Mean	[531.0, 587.7]	[523.1, 576.9]	[534.1, 567.5]	542.6
μ_0^{64} : Mean \leq High School (1992)	[488.2, 497.3]	[488.4, 498.4]	[490.2, 498.0]	496.7
μ_0^{39} : Mean \leq High School (2015)	[586.3, 587.7]	[587.5, 587.5]	[587.5, 587.5]	578.6

Table 1 presents bounds on various mortality statistics under different constraints. The last column in each panel presents point estimates obtained from the best linear approximation to the mean mortality observed in each bin. p_x is the value of the CEF at x ; μ_a^b is the average value of the CEF between points a and b . Panel A presents statistics for women in 1992, and Panel B for 2015.

Table 2
Simulation Results:
Mortality at Different Income Ranks

Statistic	Value from			
	Linear Fit	True Value	$\bar{C} = \infty$	$\bar{C} = 3$
p_{10} : First Quintile Median	393.4	444.9	[213.7, 797.3]	[352.6, 519.5]
p_{25} : Bottom Half Median	337.1	272.3	[213.7, 447.1]	[285.6, 393.7]
p_{32} : Median \leq High School (1992)	310.8	251.0	[213.7, 396.1]	[236.3, 367.3]
p_{19} : Median \leq High School (2015)	359.6	315.4	[213.7, 520.9]	[323.8, 429.5]
μ_0^{20} : First Quintile Mean	393.4	456.7	[361.4, 505.5]	[362.2, 500.1]
μ_0^{50} : Bottom Half Mean	337.1	335.3	[330.4, 361.4]	[332.2, 353.7]
μ_0^{64} : Mean \leq High School (1992)	310.8	307.6	[304.9, 312.5]	[306.4, 311.7]
μ_0^{39} : Mean \leq High School (2015)	357.7	361.4	[361.4, 363.3]	[361.9, 364.9]

Table 2 presents bounds on mortality statistics computed in a simulation exercise. We begin with mortality-income rank data on women aged 52 in 2014 from Chetty et al. (2016) and compute the best-fit spline to the data to obtain a close estimate of the true CEF for this distribution. We then simulate interval censoring according to the education bins for women aged 50–54 in 2014 used elsewhere in the paper. We then compute bounds on mortality statistics obtained data with simulated censoring. p_x is the value of the CEF at x ; μ_a^b is the average value of the CEF between points a and b .

Table 3
 Unadjusted and Constant Rank Estimates of Mortality Changes for
 Women 50-54, 1992 to 2015

Age	<u>≤ High School</u>		<u>Some College</u>		<u>B.A. or Higher</u>	
	Unadjusted Estimate	Constant Rank Bounds	Unadjusted Estimate	Constant Rank Bounds	Unadjusted Estimate	Constant Rank Bounds
25-29	42.8	[16.1, 24.1]	9.6	[-28.0, 6.3]	-8.8	[-28.6, -8.8]
30-34	46.7	[17.6, 27.4]	22.2	[-22.6, 21.3]	-9.0	[-42.5, -9.0]
35-39	33.1	[8.9, 26.3]	24.4	[-39.0, 24.4]	-17.0	[-53.9, -17.0]
40-44	47.6	[16.4, 44.2]	35.8	[-55.7, 35.8]	-34.5	[-82.9, -34.5]
45-49	74.9	[22.3, 46.6]	33.7	[-80.4, 33.7]	-56.9	[-127.5, -56.9]
50-54	128.6	[30.0, 40.1]	21.1	[-145.5, 21.1]	-111.8	[-272.8, -111.8]
55-59	84.6	[-47.4, -37.2]	39.3	[-168.6, 39.3]	-158.4	[-398.9, -158.4]
60-64	5.0	[-177.8, -168.5]	-89.7	[-372.7, -89.7]	-242.1	[-625.3, -242.1]
65-69	-82.7	[-266.8, -293.7]	-135.9	[-475.9, -328.0]	-627.1	[-1093.1, -627.1]

Table 3 compares unadjusted estimates to bounds on total mortality changes for less educated women age 50-54. The unadjusted estimate is the change in total mortality of women with high school education or less (LEHS), from 1992–2015. The bounds describe the mortality change for the group of women who occupy ranks 0-64 in the education distribution, which are the ranks occupied by LEHS women in 1992. The lower bound on mortality increase is the lower bound in 2015 minus the upper bound in 1992, and vice versa for the upper bound on mortality increase. Bounds are computed analytically under the assumptions of monotonicity and unconstrained curvature.

Table 4

Bounds on Intergenerational Educational Mobility in India

Cohort	Gradient	$\bar{C} = \infty$		$\bar{C} = 0.20$		
		p_{25}	μ_0^{50}	Gradient	p_{25}	μ_0^{50}
1950-59	[0.457, 0.742]	[13.0, 58.3]	[34.8, 38.7]	[0.474, 0.722]	[25.5, 48.1]	[34.8, 37.9]
	(0.447, 0.763)	(10.2, 59.8)	(34.0, 39.0)	(0.464, 0.745)	(24.2, 48.3)	(34.0, 38.4)
1960-69	[0.436, 0.655]	[22.2, 54.5]	[37.0, 39.1]	[0.444, 0.639]	[29.3, 49.3]	[37.0, 38.8]
	(0.421, 0.677)	(19.7, 54.8)	(36.3, 39.5)	(0.429, 0.661)	(28.0, 49.5)	(36.3, 39.3)
1970-79	[0.463, 0.595]	[29.0, 48.6]	[37.8, 37.8]	[0.468, 0.584]	[32.2, 48.3]	[37.8, 37.8]
	(0.455, 0.616)	(26.8, 49.7)	(37.3, 38.0)	(0.461, 0.603)	(31.9, 49.1)	(37.3, 38.0)
1980-89	[0.500, 0.565]	[32.3, 42.3]	[36.8, 36.8]	[0.505, 0.556]	[33.3, 42.8]	[36.8, 36.8]
	(0.488, 0.591)	(30.2, 43.6)	(36.4, 37.3)	(0.492, 0.582)	(32.8, 43.6)	(36.4, 37.3)

Cohort	Gradient	$\bar{C} = 0.10$		$\bar{C} = 0$		
		p_{25}	μ_0^{50}	Gradient	p_{25}	μ_0^{50}
1950-59	[0.492, 0.702]	[28.4, 44.9]	[34.9, 37.1]	0.587	35.6	35.6
	(0.480, 0.727)	(27.2, 45.3)	(34.0, 37.8)	(0.577, 0.595)	(35.4, 35.9)	(35.4, 35.9)
1960-69	[0.452, 0.629]	[31.2, 46.3]	[37.0, 38.5]	0.538	36.8	36.8
	(0.436, 0.629)	(31.1, 46.5)	(36.9, 38.9)	(0.530, 0.553)	(36.5, 37.0)	(36.5, 37.0)
1970-79	[0.472, 0.577]	[33.4, 46.2]	[37.8, 37.8]	0.534	36.9	36.9
	(0.465, 0.597)	(32.7, 46.5)	(37.3, 38.0)	(0.524, 0.549)	(36.6, 37.2)	(36.6, 37.2)
1980-89	[0.506, 0.548]	[33.8, 42.4]	[36.8, 36.8]	0.537	36.8	36.8
	(0.494, 0.575)	(33.3, 43.1)	(36.3, 37.3)	(0.523, 0.551)	(36.5, 37.2)	(36.5, 37.2)

The table shows estimates of bounds on three scalar mobility statistics, for different decadal cohorts and under different restrictions \bar{C} on the curvature of the child rank conditional expectation function given parent rank. The rank-rank gradient is the slope coefficient from a regression of son education rank on father education rank. p_{25} is absolute upward mobility, which is the expected rank of a son born to a family at the 25th percentile. μ_0^{50} is upward interval mobility, which is the expected rank of a son born below to a family below the 50th percentile. When $\bar{C} = 0$, the bounds shrink to point estimates. Bootstrap 95% confidence sets are displayed in parentheses below each estimate based on 1000 bootstrap samples.

A Appendix A: Additional Tables and Figures

Figure A1
Spline Approximations to the Empirical Mortality-Income CEF
52-Year-Old Women in 2014

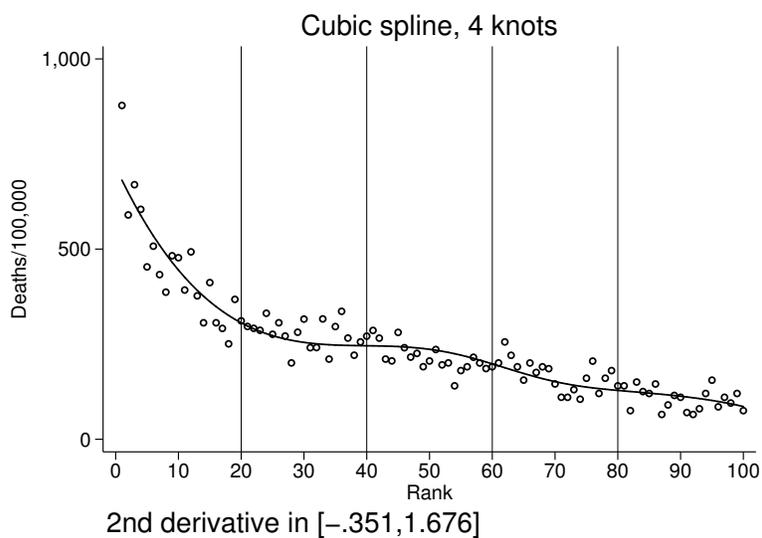


Figure A1 presents estimates of the conditional expectation function of U.S. mortality given income rank, using data from Chetty et al. (2016). The CEF is fitted using a four-knot cubic spline. The function plots the best cubic spline fit to the data series, and the circles plot the underlying data. The text under the graph shows the range of the second derivative across the support of the function.

Figure A2
 Changes in U.S. Mortality, Age 50-54, 1992-2015:
 Constant Rank Interval Estimates (High School or Less in 1992)
 Curvature Constraint Only

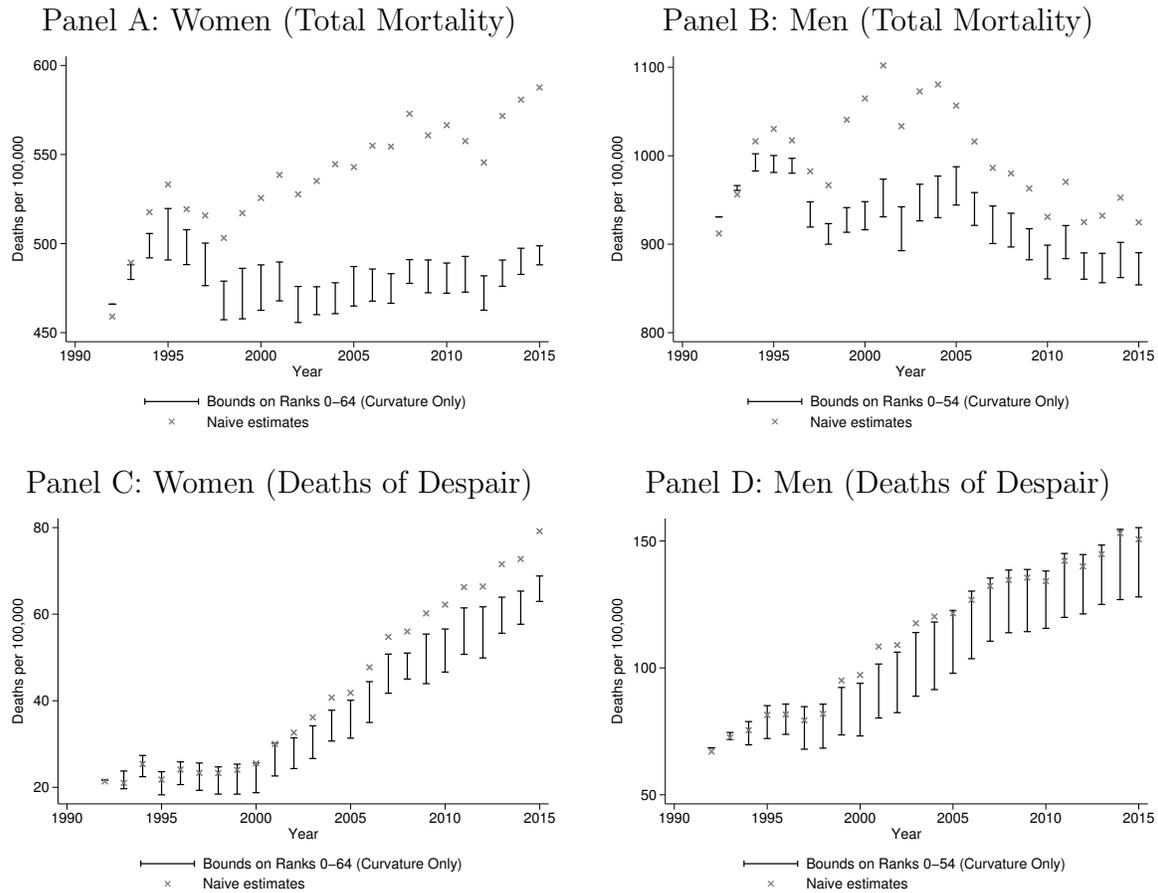


Figure A2 shows bounds on estimates of mortality for men and women aged 50–54 over time. The figure is similar to Figure 7, but the bounds here are generated under the assumption of a curvature constraint ($\bar{C} = 3$) but without the requirement of monotonicity. In contrast, Figure 7 calculates bounds under the assumption of a monotonic CEF with no curvature constraint. The sample is defined by the set of latent education ranks corresponding to a high school education or less in 1992, or ranks 0-64 for women and 0-54 for men. Panel A shows total mortality for women age 50–54, and Panel B shows total mortality for men age 50–54. Panels C and D show mortality from deaths of despair for both groups.

Figure A3

Spline Approximations to Empirical Parent-Child Rank Distributions

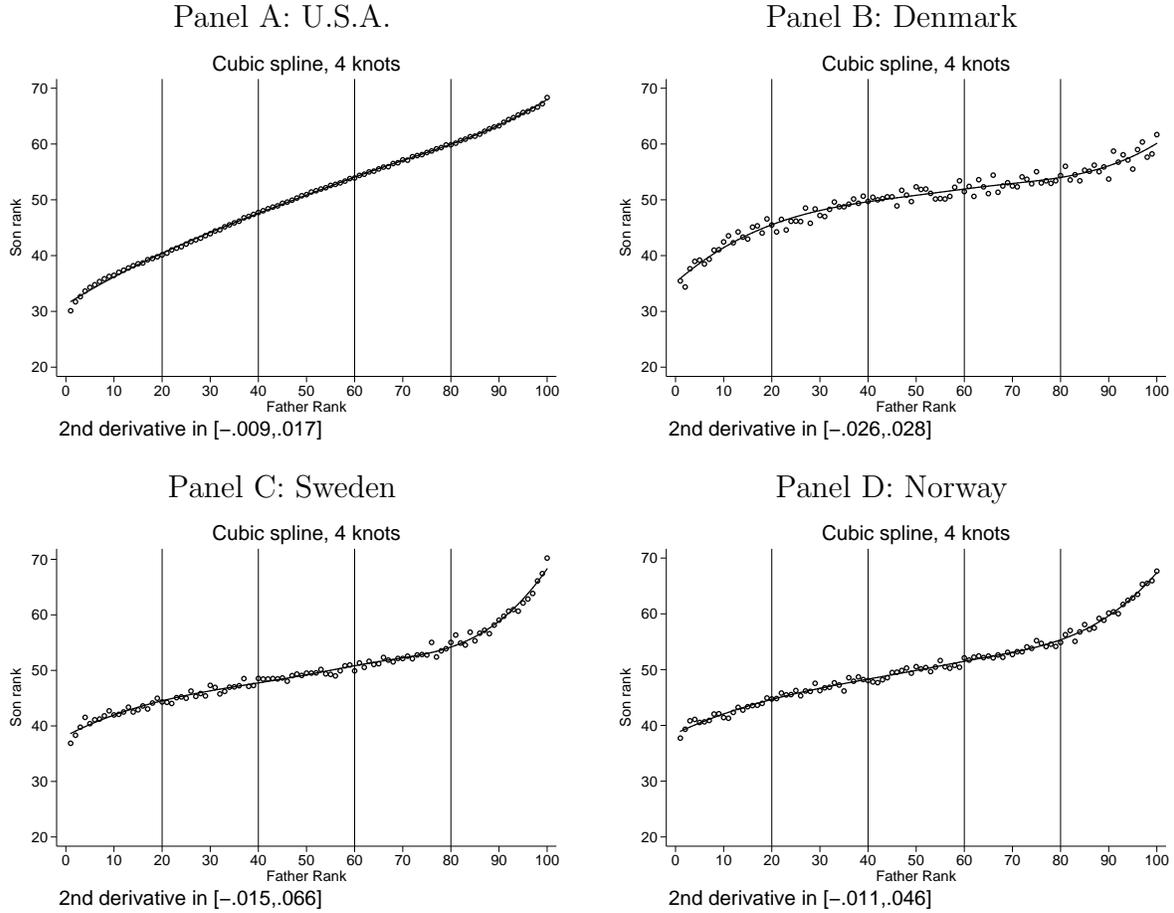


Figure A3 presents estimates of the conditional expectation functions obtained from fully supported parent-child rank-rank income distribution in several developed countries. The data for U.S.A. and Denmark come from Chetty et al. (2014a), who obtained the Denmark data from Boserup et al. (2014). The data for Sweden and Norway come from Bratberg et al. (2015). The CEFs were fitted using cubic splines, with knots at 20, 40, 60, and 80 (as indicated by the vertical lines). The functions plot the best cubic spline fit to each series, and the circles plot the underlying data. The text under each graph shows the range of the second derivative across the support of the function.

Table A1
Bin Sizes in Studies of Intergenerational Mobility

Study	Country	Birth Cohort of Son	Number of Parent Outcome Bins	Population Share in Largest Bin
Aydemir and Yazici (2016)	Turkey	1990 ⁴⁴	15	39%
	Turkey	1960 ⁴⁵	15	78%
Dunn (2007)	Brazil	1972–1981	> 18	20% ⁴⁶
Emran and Shilpi (2011)	Nepal, Vietnam	1992–1995	2	83%
Güell et al. (2013)	Spain	~ 2001	9	27% ⁴⁷
Guest et al. (1989)	USA	~ 1880	7	53.2%
Hnatkovska et al. (2013)	India	1918–1988	5	Not reported
Knight et al. (2011)	China	1930–1984	5	29% ⁴⁸
Lindahl et al. (2012)	Sweden	1865–2005	8	34.5%
Long and Ferrie (2013)	Britain	~ 1850	4	57.6%
	Britain	~ 1949–55	4	54.2%
	USA	~ 1850–51	4	50.9%
	USA	~ 1949–55	4	48.3%
Piraino (2015)	South Africa	1964–1994	6	36%

Table A1 presents a review of papers analyzing educational and occupational mobility. The sample is not representative: we focus on papers where interval censoring may be a concern. The column indicating number of parent outcome bins refers to the number of categories for the parent outcome used in the main specification. The outcome is education in all studies with the exception of Long and Ferrie (2013) and Guest et al. (1989), where the outcome is occupation.

⁴⁵Includes all people born after about 1990.

⁴⁶Includes all people born after about 1960.

⁴⁷This is the proportion of sons in 1976 who had not completed one year of education — an estimate of the proportion of fathers in 2002 with no education, which is not reported.

⁴⁸Estimate is from the full population rather than just fathers.

⁴⁹This reported estimate does not incorporate sampling weights; estimates with weights are not reported.

Table A2
Transition Matrices for Father and Son Education in India

A: Sons Born 1950-59

Father ed attained	Son highest education attained						
	< 2 yrs. (31%)	2-4 yrs. (11%)	Primary (17%)	Middle (13%)	Sec. (13%)	Sr. sec. (6%)	Any higher (8%)
<2 yrs. (60%)	0.47	0.12	0.17	0.11	0.09	0.03	0.03
2-4 yrs. (12%)	0.10	0.18	0.22	0.19	0.16	0.09	0.06
Primary (13%)	0.07	0.08	0.31	0.16	0.19	0.08	0.10
Middle (6%)	0.06	0.05	0.09	0.30	0.17	0.14	0.18
Secondary (5%)	0.03	0.02	0.04	0.12	0.37	0.11	0.30
Sr. secondary (2%)	0.02	0.00	0.03	0.11	0.11	0.35	0.38
Any higher ed (2%)	0.01	0.01	0.01	0.03	0.08	0.13	0.72

B: Sons Born 1960-69

Father ed attained	Son highest education attained						
	< 2 yrs. (27%)	2-4 yrs. (10%)	Primary (16%)	Middle (16%)	Sec. (14%)	Sr. sec. (7%)	Any higher (10%)
<2 yrs. (57%)	0.41	0.12	0.16	0.14	0.09	0.04	0.04
2-4 yrs. (13%)	0.12	0.17	0.18	0.22	0.15	0.08	0.08
Primary (14%)	0.09	0.05	0.26	0.18	0.20	0.09	0.13
Middle (6%)	0.06	0.04	0.09	0.29	0.21	0.13	0.19
Secondary (6%)	0.03	0.02	0.08	0.12	0.35	0.16	0.25
Sr. secondary (2%)	0.02	0.02	0.03	0.07	0.19	0.25	0.41
Any higher ed (2%)	0.01	0.01	0.02	0.03	0.09	0.11	0.73

C: Sons Born 1970-79

Father ed attained	Son highest education attained						
	< 2 yrs. (20%)	2-4 yrs. (8%)	Primary (17%)	Middle (18%)	Sec. (16%)	Sr. sec. (10%)	Any higher (12%)
<2 yrs. (50%)	0.33	0.10	0.19	0.17	0.12	0.05	0.04
2-4 yrs. (11%)	0.11	0.16	0.20	0.22	0.15	0.08	0.08
Primary (15%)	0.08	0.06	0.24	0.23	0.18	0.11	0.11
Middle (8%)	0.05	0.03	0.09	0.29	0.21	0.17	0.16
Secondary (9%)	0.03	0.02	0.06	0.12	0.31	0.19	0.27
Sr. secondary (3%)	0.01	0.01	0.02	0.08	0.17	0.29	0.42
Any higher ed (4%)	0.00	0.00	0.02	0.05	0.10	0.17	0.66

D: Sons Born 1980-89

Father ed attained	Son highest education attained						
	< 2 yrs. (12%)	2-4 yrs. (7%)	Primary (16%)	Middle (20%)	Sec. (16%)	Sr. sec. (12%)	Any higher (17%)
<2 yrs. (38%)	0.26	0.10	0.21	0.20	0.12	0.06	0.05
2-4 yrs. (11%)	0.08	0.17	0.19	0.24	0.15	0.09	0.08
Primary (17%)	0.05	0.04	0.22	0.23	0.20	0.13	0.13
Middle (12%)	0.03	0.02	0.10	0.28	0.20	0.17	0.20
Secondary (11%)	0.02	0.01	0.05	0.13	0.23	0.24	0.32
Sr. secondary (5%)	0.02	0.01	0.04	0.09	0.15	0.24	0.46
Any higher ed (5%)	0.01	0.01	0.02	0.05	0.10	0.16	0.65

Table A2 shows transition matrices by decadal birth cohort for Indian fathers and sons in the study.

B Appendix B: Proofs

Proof of Proposition 1.

Let the function $Y(x) = E(y|x)$ be defined on a known interval; without loss of generality, define this interval as $x \in [0, 100]$. Assume $Y(x)$ is integrable. We want to bound $E(y|x)$ when x is known to lie in the interval $[x_k, x_{k+1}]$; there are K such intervals. Define the expected value of y in bin k as

$$r_k = \int_{x_k}^{x_{k+1}} Y(x) f_k(x) dx.$$

Note that

$$r_k = E(y|x \in [x_k, x_{k+1}])$$

via the law of iterated expectations. Define $r_0 = 0$ and $r_{K+1} = 100$.

Restate the following assumptions from Manski and Tamer (2002):

$$\begin{aligned} P(x \in [x_k, x_{k+1}]) &= 1. && \text{(Assumption I)} \\ E(y|x) \text{ must be weakly increasing in } x. &&& \text{(Assumption M)} \\ E(y|x \text{ is interval censored}) &= E(y|x). && \text{(Assumption MI)} \end{aligned}$$

From Manski and Tamer (2002), we have:

$$r_{k-1} \leq E(y|x) \leq r_{k+1} \quad \text{(Manski-Tamer bounds)}$$

Suppose also that

$$x \sim U(0, 100). \quad \text{(Assumption U)}$$

In that case,

$$r_k = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} Y(x) dx,$$

substituting the probability distribution function for the uniform distribution within bin k . Then we derive the following proposition.

Proposition 1. *Let x be in bin k . Under assumptions M, I, MI (Manski and Tamer, 2002) and U, and without additional information, the following bounds on $E(y|x)$ are sharp:*

$$\begin{cases} r_{k-1} \leq E(y|x) \leq \frac{1}{x_{k+1}-x} ((x_{k+1}-x)r_k - (x-x_k)r_{k-1}), & x < x_k^* \\ \frac{1}{x-x_k} ((x_{k+1}-x)r_k - (x_{k+1}-x)r_{k+1}) \leq E(y|x) \leq r_{k+1}, & x \geq x_k^* \end{cases}$$

where

$$x_k^* = \frac{x_{k+1}r_{k+1} - (x_{k+1}-x_k)r_k - x_k r_{k-1}}{r_{k+1} - r_{k-1}}.$$

The intuition behind the proof is as follows. First, find the function z which meets the bin mean and is defined as r_{k-1} up to some point j . Because z is a valid CEF, the lower bound on $E(y|x)$ is no larger than z up to j ; we then show that j is precisely x_k^* from the statement. For points $x > x_k^*$, we show that the CEF which minimizes the value at point x must be a horizontal line up to x and a horizontal line at r_{k+1} for points larger than x . But there is only one such CEF, given that the CEF must also meet the bin mean, and we can solve analytically for the minimum value the CEF can attain at point x . We focus on lower bounds for brevity, but the proof for upper bounds follows a symmetric structure.

Part 1: Find x_k^ .* First define \mathcal{V}_k as the set of weakly increasing CEFs which meet the bin mean. Put otherwise, let \mathcal{V}_k be the set of $v : [x_k, x_{k+1}] \rightarrow \mathbb{R}$ satisfying

$$r_k = \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} v(x) dx.$$

Now choose $z \in \mathcal{V}_k$ such that

$$z(x) = \begin{cases} r_{k-1}, & x_k \leq x < j \\ r_{k+1}, & j \leq x \leq x_{k+1}. \end{cases}$$

Note that z and j both exist and are unique (it suffices to show that just j exists and is unique, as then z must be also). We can solve for j by noting that z lies in \mathcal{V}_k , so it must meet the bin mean. Hence, by evaluating the integrals, j must satisfy:

$$\begin{aligned} r_k &= \frac{1}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} z(x) dx \\ &= \frac{1}{x_{k+1} - x_k} \left(\int_{x_k}^j r_{k-1} dx + \int_j^{x_{k+1}} r_{k+1} dx \right) \\ &= \frac{1}{x_{k+1} - x_k} ((j - x_k) r_{k-1} + (x_{k+1} - j) r_{k+1}). \end{aligned}$$

Note that these expressions invoke assumption U, as the integration of $z(x)$ does not require any adjustment for the density on the x axis. For a more general proof with an arbitrary distribution of x , see section B.

With some algebraic manipulations, we obtain that $j = x_k^*$.

Part 2: Prove the bounds. In the next step, we show that x_k^* is the smallest point at which no $v \in \mathcal{V}_k$ can be r_{k-1} , which means that there must be some larger lower bound on $E(y|x)$ for $x \geq x_k^*$. In other words, we prove that

$$x_k^* = \sup \left\{ x \mid \text{there exists } v \in \mathcal{V}_k \text{ such that } v(x) = r_{k-1} \right\}.$$

We must show that x_k^* is an upper bound and that it is the least upper bound.

First, x_k^* is an upper bound. Suppose that there exists $j' > x_k^*$ such that for some $w \in \mathcal{V}_k$, $w(j') = r_{k-1}$. Observe that by monotonicity and the bounds from Manski and Tamer (2002), $w(x) = r_{k+1}$ for $x \leq j'$; in other words, if $w(j')$ is the mean of the mean of the prior bin, it can be no lower or higher than the mean of the prior bin up to point j' . But since $j' > j$, this means that

$$\int_{x_k}^{j'} w(x) dx < \int_{x_k}^{j'} z(x) dx,$$

since $z(x) > w(x)$ for all $h \in (j, j')$. But recall that both z and w lie in \mathcal{V}_k and must therefore meet the bin mean; i.e.,

$$\int_{x_k}^{x_{k+1}} w(x) dx = \int_{x_k}^{x_{k+1}} z(x) dx.$$

But then

$$\int_{j'}^{x_{k+1}} w(x) dx > \int_{j'}^{x_{k+1}} z(x) dx.$$

That is impossible by the bounds from Manski and Tamer (2002), since $w(x)$ cannot exceed r_{k+1} , which is precisely the value of $z(x)$ for $x \geq j$.

Second, j is the least upper bound. Fix $j' < j$. From the definition of z , we have shown that for some $h \in (j', j)$, $z(h) = r_{k-1}$ (and $z \in \mathcal{V}_k$). So any point j' less than j would not be a lower bound on the set — there is a point h larger than j' such that $z(h) = r_{k-1}$.

Hence, for all $x < x_k^*$, there exists a function $v \in \mathcal{V}_k$ such that $v(x) = r_{k-1}$; the lower bound on $E(y|x)$ for $x < x_k^*$ is no greater than r_{k-1} . By choosing z' with

$$z'(x) = \begin{cases} r_{k-1}, & x_k \leq x \leq j \\ r_{k+1}, & j < x \leq x_{k+1}, \end{cases}$$

it is also clear that at x_k^* , the lower bound is no larger than r_{k-1} (and this holds in the proposition itself, substituting in x_k^* into the lower bound in the second equation).

Now, fix $x' \in (x_k^*, x_{k+1}]$. Since x_k^* is the supremum, there is no function $v \in \mathcal{V}_k$ such that $v(x') = r_{k-1}$.

Thus for $x' > x_k^*$, we seek a sharp lower bound larger than r_{k-1} . Write this lower bound as

$$Y_{x'}^{min} = \min \left\{ v(x') \text{ for all } v \in \mathcal{V}_k \right\},$$

where $Y_{x'}^{min}$ is the smallest value attained by any function $v \in \mathcal{V}_k$ at the point x' .

We find this $Y_{x'}^{min}$ by choosing the function which maximizes every point after x' , by attaining the value of the subsequent bin. The function which minimizes $v(x')$ must be a horizontal line up to this point.

Pick $\tilde{z} \in \mathcal{V}_k$ such that

$$\tilde{z}(x) = \begin{cases} \underline{Y}, & x_k \leq x' \\ r_{k+1}, & x' < x_{k+1} \leq x_{k+1} \end{cases}.$$

By integrating $\tilde{z}(x)$, we claim that \underline{Y} satisfies the following:

$$\frac{1}{x_{k+1} - x_k} ((x' - x_k) \underline{Y} + (x_{k+1} - x') r_{k+1}) = r_k.$$

As a result, \underline{Y} from this expression exists and is unique, because we can solve the equation. Note that this integration step also requires that the distribution of x be uniform, and we generalize this argument in B.

By similar reasoning as above, there is no $Y' < \underline{Y}$ such that there exists $w \in \mathcal{V}_k$ with $w(x') = Y'$. Otherwise there must be some point $x > x'$ such that $w(x') > r_{k+1}$ in order that w matches the bin means and lies in \mathcal{V}_k ; the expression for \underline{Y} above maximizes every point after x' , leaving no additional room to further depress \underline{Y} .

Formally, suppose there exists $w \in \mathcal{V}_k$ such that $w(x') = Y' < \underline{Y}$. Then $w(x') < \tilde{z}(x')$ for all $x < x'$, since w is monotonic. As a result,

$$\int_{x_k}^{x'} \tilde{z}(x) dx > \int_{x_k}^{x'} w(x) dx.$$

But recall that

$$\int_{x_k}^{x_{k+1}} w(x) dx = \int_{x_k}^{x_{k+1}} \tilde{z}(x) dx,$$

so

$$\int_{x'}^{x_{k+1}} w(x) dx > \int_{x'}^{x_{k+1}} \tilde{z}(x) dx.$$

This is impossible, since $\tilde{z}(x) = r_{k+1}$ for all $x > x'$, and by Manski and Tamer (2002), $w(x) \leq r_{k+1}$ for all $w \in \mathcal{V}_k$. Hence there is no such $w \in \mathcal{V}_k$, and therefore \underline{Y} is smallest possible value at x' , i.e. $\underline{Y} = Y_{x'}^{min}$.

By algebraic manipulations, the expression for $\underline{Y} = Y_x^{min}$ reduces to

$$Y_x^{min} = \frac{(x_{k+1} - x_k) r_k - (x_{k+1} - x) r_{k+1}}{x - x_k}, \quad x \geq x_k^*.$$

The proof for the upper bounds uses the same structure as the proof of the lower bounds.

Finally, the body of this proof gives sharpness of the bounds. For we have introduced a CEF $v \in \mathcal{V}_k$ that obtains the value of the upper and lower bound for any point $x \in [x_k, x_{k+1}]$. For any value y within the bounds, one can generate a CEF $v \in \mathcal{V}_k$ such that $v(x) = y$. \square

Proof of Proposition 2. Suppose we relax assumption U and merely characterize x by some known probability density function. Then we can derive the following bounds.

Proposition 2. *Let x be in bin k . Let $f_k(x)$ be the probability density function of x in bin k . Under assumptions M, I, MI (Manski and Tamer, 2002), and without additional information, the following bounds*

on $E(y|x)$ are sharp:

$$\begin{cases} r_{k-1} \leq E(y|x) \leq \frac{r_k - r_{k-1} \int_{x_k}^x f_k(s) ds}{\int_{x_k}^x f_k(s) ds}, & x < x_k^* \\ \frac{r_k - r_{k+1} \int_{x_k}^{x_{k+1}} f_k(s) ds}{\int_{x_k}^{x_{k+1}} f_k(s) ds} \leq E(y|x) \leq r_{k+1}, & x \geq x_k^* \end{cases}$$

where x_k^* satisfies:

$$r_k = r_{k-1} \int_{x_k}^{x_k^*} f_k(s) ds + r_{k+1} \int_{x_k^*}^{x_{k+1}} f_k(s) ds.$$

The proof follows the same argument as in proposition 1. With an arbitrary distribution, \mathcal{V}_k now constitutes the functions $v : [x_k, x_{k+1}] \rightarrow \mathbb{R}$ which satisfy:

$$\int_{x_k}^{x_{k+1}} v(s) f_k(s) ds = r_k.$$

As before, choose $z \in \mathcal{V}_k$ such that

$$z(x) = \begin{cases} r_{k-1}, & x_k \leq x < j \\ r_{k+1}, & j \leq x \leq x_{k+1}. \end{cases}$$

Because the distribution of x is no longer uniform, j must now satisfy

$$\begin{aligned} r_k &= \int_{x_k}^{x_{k+1}} z(s) f_k(s) ds \\ &= r_{k-1} \int_{x_k}^j f_k(s) ds + r_{k+1} \int_j^{x_{k+1}} f_k(s) ds. \end{aligned}$$

This implies that $j = x_k^*$, precisely.

The rest of the arguments follow identically, except we now claim that for $x > x_k^*$, $\underline{Y} = Y_x^{min}$ satisfies the following:

$$r_k = \int_{x_k}^x Y_x^{min} f_k(s) ds + \int_x^{x_{k+1}} r_{k+1} f_k(s) ds.$$

By algebraic manipulations, we obtain:

$$Y_x^{min} = \frac{r_k - r_{k+1} \int_x^{x_{k+1}} f_k(s) ds}{\int_{x_k}^x f_k(s) ds}$$

and the proof of the lower bounds is complete. As before, the proof for upper bounds follows from identical logic. \square

Proof of Proposition 3. Define

$$\mu_a^b = \frac{1}{b-a} \int_a^b E(y|x) di.$$

Let Y_x^{min} and Y_x^{max} be the lower and upper bounds respectively on $E(y|x)$ given by Proposition 1. We seek to bound μ_a^b when x is observed only in discrete intervals.

Proposition 3. *Let $b \in [x_k, x_{k+1}]$ and $a \in [x_h, x_{h+1}]$ with $a < b$. Let assumptions M, I, MI (Manski and Tamer, 2002) and U hold. Then, if there is no additional information available, the following bounds are sharp:*

$$\begin{cases} Y_b^{min} \leq \mu_a^b \leq Y_a^{max}, & h = k \\ \frac{r_h(x_k - a) + Y_b^{min}(b - x_k)}{b - a} \leq \mu_a^b \leq \frac{Y_a^{max}(x_k - a) + r_k(b - x_k)}{b - a}, & h + 1 = k \\ \frac{r_h(x_{h+1} - a) + \sum_{\lambda=h+1}^{k-1} r_\lambda(x_{\lambda+1} - x_\lambda) + Y_b^{min}(b - x_k)}{b - a} \leq \mu_a^b \leq \frac{Y_a^{max}(x_{h+1} - a) + \sum_{\lambda=h+1}^{k-1} r_\lambda(x_{\lambda+1} - x_\lambda) + r_k(b - x_k)}{b - a}, & h + 1 < k. \end{cases}$$

The order of the proof is as follows. If a and b lie in the same bin, then μ_a^b is maximized only if the CEF is minimized prior to a . As in the proof of proposition 1, that occurs when the CEF is a horizontal line at Y_x^{min} up to a , and a horizontal line Y_x^{max} at and after a . If a and b lie in separate bins, the value of the integral in bins that are contained between a and b is determined by the observed bin means. The portions of the integral that are not determined are maximized by a similar logic, since they both lie within bins. We prove the bounds for maximizing μ_a^b , but the proof is symmetric for minimizing μ_a^b .

Part 1: Prove the bounds if a and b lie in the same bin. We seek to maximize μ_a^b when $a, b \in [x_k, x_{k+1}]$. This requires finding a candidate CEF $v \in \mathcal{V}_k$ which maximizes $\int_a^b v(x)dx$. Observe that the function $v(x)$ defined as

$$v(x) = \begin{cases} Y_a^{min}, & x_k \leq x < a \\ Y_a^{max}, & a \leq x \leq x_{k+1} \end{cases}$$

has the property that $v \in \mathcal{V}_k$. For if $a \geq x_k^*$, $v = \tilde{z}$ from the second part of the proof of proposition 1. If $a < x_k^*$, the CEF in \mathcal{V}_k which yields Y_a^{max} is precisely v (by a similar argument which delivers the upper bounds in proposition 1).

This CEF maximizes μ_a^b , because there is no $w \in \mathcal{V}_k$ such that

$$\frac{1}{b-a} \int_a^b w(x)dx > \frac{1}{b-a} \int_a^b v(x)dx.$$

Note that for any $w \in \mathcal{V}_k$, $\frac{1}{x_{k+1}-x_k} \int_{x_k}^{x_{k+1}} w(x)dx = \frac{1}{x_{k+1}-x_k} \int_{x_k}^{x_{k+1}} v(x)dx = r_k$. Hence in order that $\int_a^b w(x)dx > \int_a^b v(x)dx$, there are two options. The first option is that

$$\int_{x_k}^a w(x)dx < \int_{x_k}^a v(x)dx.$$

That is impossible, since there is no room to depress w given the value of v after a . If $a < x_k^*$, then it is clear that there is no w giving a larger μ_a^b , since $r_{k-1} \leq w(x)$ for $x_{k-1} \leq x \leq a$, so w is bounded below by v . If $a \geq x_k^*$, then $v(x) = r_{k+1}$ for all $a \leq x \leq x_{k+1}$. That would leave no room to depress w further; if $\int_{x_k}^a w(x)dx < \int_{x_k}^a v(x)dx$, then $\int_a^{x_{k+1}} w(x)dx > \int_a^{x_{k+1}} v(x)dx$, which cannot be the case if $v = r_{k+1}$, by the bounds given in Manski and Tamer (2002).

The second option is that

$$\int_b^{x_k} w(x)dx < \int_b^{x_k} v(x)dx.$$

This is impossible due to monotonicity. For if $\int_a^b w(x)dx > \int_a^b v(x)dx$, then there must be some point $x' \in [a, b]$ such that $w(x') > v(x')$. By monotonicity, $w(x) > v(x)$ for all $x \in [x', x_{k+1}]$ since $v(x) = Y_a^{max}$ in that interval. As a result,

$$\int_b^{x_k} w(x)dx > \int_b^{x_k} v(x)dx,$$

since $b \in (x', x_{k+1})$. (If $b = x_{k+1}$, then only the first option would allow w to maximize the desired μ_a^b .)

Therefore, there is no such w , and v indeed maximizes the desired integral. Integrating v from a to b , we obtain that the upper bound on μ_a^b is $\frac{1}{b-a} \int_a^b Y_a^{max} dx = Y_a^{max}$. Note that there may be many functions which maximize the integral; we only needed to show that v is one of them.

To prove the lower bound, use an analogous argument.

Part 2: Prove the bounds if a and b do not lie in the same bin. We now generalize the set up and permit $a, b \in [0, 100]$. Let \mathcal{V} be the set of weakly increasing functions such that $\frac{1}{x_{k+1}-x_k} \int_{x_k}^{x_{k+1}} v(x)dx = r_k$ for all $k \leq K$. In other words, \mathcal{V} is the set of functions which match the means of every bin. Now observe that for

all $v \in \mathcal{V}$,

$$\begin{aligned}\mu_a^b &= \frac{1}{b-a} \int_a^b v(x) dx \\ &= \frac{1}{b-a} \left(\int_a^{x_{h+1}} v(x) dx + \int_{x_{h+1}}^{x_k} v(x) dx + \int_{x_k}^b v(x) dx \right),\end{aligned}$$

by a simple expansion of the integral.

But for all $v \in \mathcal{V}$,

$$\int_{x_{h+1}}^{x_k} v(x) dx = \sum_{\lambda=h+1}^{k-1} r_\lambda (x_{\lambda+1} - x_\lambda)$$

if $h+1 < k$ and

$$\int_{x_{h+1}}^{x_k} v(x) dx = 0$$

if $h+1 = k$. For in bins completely contained inside $[a, b]$, there is no room for any function in \mathcal{V} to vary; they all must meet the bin means.

We proceed to prove the upper bound. We split this into two portions: we wish to maximize $\int_a^{x_{h+1}} v(x) dx$ and we also wish to maximize $\int_{x_k}^b v(x) dx$. The values of these objects are not codependent. But observe that the CEFs $v \in \mathcal{V}_k$ which yield upper bounds on these integrals are the very same functions which yield upper bounds on $\mu_a^{x_{h+1}}$ and $\mu_{x_k}^b$, since $\mu_s^t = \frac{1}{t-s} \int_s^t v(x) dx$ for any s and t . Also notice that a and x_{h+1} both lie in bin h , while b and x_k both lie in bin k , so we can make use of the first portion of this proof.

In part 1, we showed that the function $v \in \mathcal{V}$, $v : [x_h, x_{h+1}] \rightarrow \mathbb{R}$, which maximizes $\mu_a^{x_{h+1}}$ is

$$v(x) = \begin{cases} Y_a^{min}, & x_h \leq x < a \\ Y_a^{max}, & a \leq x \leq x_{h+1}. \end{cases}$$

As a result

$$\max_{v \in \mathcal{V}} \left\{ \int_a^{x_{h+1}} v(x) dx \right\} = \int_a^{x_{h+1}} Y_a^{max} dx = Y_a^{max} (x_{h+1} - a).$$

Similarly, observe that x_k and b lie in the same bin, so the function $v : [x_k, x_{k+1}] \rightarrow \mathbb{R}$, with $v \in \mathcal{V}$ which maximizes $\int_{x_k}^b v(x) dx$ must be of the form

$$v(x) = \begin{cases} Y_{x_k}^{min}, & x_k \leq x < a \\ Y_{x_k}^{max}, & b \leq x \leq x_{k+1}. \end{cases}$$

With identical logic,

$$\max_{v \in \mathcal{V}} \left\{ \int_{x_k}^b v(x) dx \right\} = \int_{x_k}^b Y_{x_k}^{max} dx = Y_{x_k}^{max} (b - x_k).$$

And by proposition 1, $x_k \leq x_k^*$ so $Y_{x_k}^{max} = r_k$. (Note that if $x_k = x_k^*$, substituting x_k^* into the second expression of proposition 1 still yields that $Y_{x_k}^{max} = r_k$.)

Now we put all these portions together. First let $h+1 = k$. Then $\int_{x_{h+1}}^{x_k} v(x) dx = 0$, so we maximize μ_a^b by

$$\frac{1}{b-a} (Y_a^{max} (x_{h+1} - a) + r_k (b - x_k)).$$

Similarly, if $h+1 < k$ and there are entire bins completely contained in $[a, b]$, then we maximize μ_a^b by

$$\frac{1}{b-a} \left(Y_a^{max} (x_{h+1} - a) + \sum_{\lambda=h+1}^{k-1} r_\lambda (x_{\lambda+1} - x_\lambda) + r_k (b - x_k) \right).$$

The lower bound is proved analogously. Sharpness is immediate, since we have shown that the CEF which delivers the endpoints of the bounds lies in \mathcal{V} . As a result, there is a function delivering any intermediate value for the bounds. \square

Extension of Proposition 3 to an arbitrary known distribution

Proposition 4. *Let $a \in [x_h, x_{h+1}]$ and $b \in [x_k, x_{k+1}]$. Let assumptions M, I, MI hold. Let Y_x^{min} and Y_x^{max} be the lower and upper bounds respectively on $E(y|x)$ given by proposition 2. Let the probability distribution of x be $f(x)$. Then, if no additional information is available, the following bounds are sharp:*

$$\left\{ \begin{array}{l} Y_b^{min} \leq \mu_a^b \leq Y_a^{max} \\ \frac{r_h \int_a^{x_k} f(x)dx + Y_b^{min} \int_{x_k}^b f(x)dx}{\int_a^b f(x)dx} \leq \mu_a^b \leq \frac{Y_a^{max} \int_a^{x_k} f(x)dx + r_k \int_{x_k}^b f(x)dx}{\int_a^b f(x)dx} \\ \frac{r_h \int_a^{x_{h+1}} f(x)dx + \sum_{\lambda=h+1}^{k-1} r_\lambda \int_{x_\lambda}^{x_{\lambda+1}} f(x)dx + Y_b^{min} \int_{x_k}^b f(x)dx}{\int_a^b f(x)dx} \leq \mu_a^b \\ \leq \frac{Y_a^{max} \int_a^{x_{h+1}} f(x)dx + \sum_{\lambda=h+1}^{k-1} r_\lambda \int_{x_\lambda}^{x_{\lambda+1}} f(x)dx + r_k \int_{x_k}^b f(x)dx}{\int_a^b f(x)dx} \end{array} \right. \begin{array}{l} h = k \\ h + 1 = k \\ h + 1 < k. \end{array}$$

Proposition 4 generalizes proposition 3 to an arbitrary distribution, but its proof is identical. The only difference is in the weight given to components of μ_a^b that lie in different bins; these weights are given by integrating the maximizing function $v \in \mathcal{V}$, while accounting for the probability distribution $f(x)$.

We consider only maximizing μ_a^b . To prove the first part of proposition 4, we obtain $v \in \mathcal{V}$ defined as

$$v(x) = \begin{cases} Y_a^{min}, & x_k \leq x < a \\ Y_a^{max}, & a \leq x \leq x_{k+1}. \end{cases}$$

As before, μ_a^b given by $\frac{\int_a^b v(x)f(x)dx}{\int_a^b f(x)dx}$ will maximize μ_a^b .

If the first part of the proposition holds, then the rest follows. For $h \neq k$, then

$$\begin{aligned} \mu_a^b &= \frac{1}{\int_a^b f(x)dx} \int_a^b v(x)f(x)dx \\ &= \frac{1}{\int_a^b f(x)dx} \left(\int_a^{x_{h+1}} v(x)f(x)dx + \int_{x_{h+1}}^{x_k} v(x)f(x)dx + \int_{x_k}^b v(x)f(x)dx \right), \end{aligned}$$

As before, $\int_{x_{h+1}}^{x_k} v(x)f(x)dx = 0$ if $h + 1 = k$. If $h + 1 < k$, then

$$\int_{x_{h+1}}^{x_k} v(x)f(x)dx = \sum_{\lambda=h+1}^{k-1} r_\lambda \int_{x_\lambda}^{x_{\lambda+1}} f(x)dx.$$

We maximize the objects $\int_a^{x_{h+1}} v(x)f(x)dx$ and $\int_{x_k}^b v(x)f(x)dx$ by using the expression from the first part of this proof. We therefore have that the maximum of $\int_a^{x_{h+1}} v(x)f(x)dx$ over $v \in \mathcal{V}$ is obtained by $Y_a^{max} \int_a^{x_{h+1}} f(x)dx$. By the same argument, we maximize $\int_{x_k}^b v(x)f(x)dx$ with $r_k \int_{x_k}^b f(x)dx$. Putting these expressions together, the proof is complete. \square

C Appendix C: CEF Bounds When x and y are Interval Censored

In the main part of the paper, we focus on bounding a function $Y(x) = E(y|x)$ when y is observed without error, but x is observed with interval censoring. In this section, we modify the setup to consider simultaneous interval censoring in the conditioning variable x and in observed outcomes y . This arises, for example, in the study of educational mobility, where latent education ranks of both parents and children are observed with interval censoring.

We first present a setup that takes a similar approach to the bounding method presented in Section 2. We can define bounds on the CEF $E(y|x)$ when both y and x are interval-censored as a solution to a constrained optimization problem. The number of parameters is an order of magnitude higher than the problem in Section 2, and proved too computationally intensive to solve in the Indian test case (where interval censoring is severe). We therefore present a sequential approach that yields theoretical bounds on the double-censored CEF for the case of intergenerational mobility.

Specifically, we define the theoretical best- and worst-case latent distributions of y variables for a given intergenerational mobility statistic. The best- and worst-case assumptions each generate a bound on the feasible value of y for each x bin. We then use the method in Section 2 to calculate bounds on the mobility statistic under each case. The union of these bounds is a conservative bound on the mobility statistic given censoring in both the y and x variables.

Finally, we can shed light on the distribution of the true value of y in each x bin if other data is available. In the context of intergenerational mobility, and in our specific empirical context, it is frequently the case that more information is available about children than about their parents. We use data on child wages to predict whether the true latent child rank distribution (y) is better represented by the best- or worst-case mobility scenario. The joint wage distribution suggests that the true latent distribution of y in each bin is very close to the best case distribution, which we used in Section 5, because there is little effect of parent education on child wages after conditioning on child education.

C.1 Solution Definition for CEF Bounds with Double Censoring

We are interested in bounding a function $E(y|x)$, where y is known only to lie in one of H bins defined by intervals of the form $[y_h, y_{h+1}]$, and x is known only to lie in one of K bins defined by intervals of the form $[x_k, x_{k+1}]$. For simplicity, we focus on the case where both y and x are uniformly distributed on the interval $[0, 100]$.⁵⁰

Where Section 2 focused on bounding the *cumulative expectation function* (CEF) of y given x , we focus

⁵⁰Taking a different known distribution into account would require imposing different weights on the mean-squared error function and budget constraint below, but would otherwise not be substantively different.

here on bounding a separate *conditional distribution function* (CDF) for y , given each value of x . Each value of x implies a different CDF for y , as follows:

$$F(r, x) = P(y \leq r | x = X) \tag{C.1}$$

This CDF is related to the CEF $E(y|x)$ as follows:

$$E(y|x) = \int_0^{100} r f(x, r) dr \tag{C.2}$$

where $f(x, r)$ is the probability density function corresponding to the CDF in Equation C.1, when the conditioning variable takes the value x . Note that r in this case represents a child rank. This expression simply denotes that $E(y|x)$ is the average value from 0 to 100 on the y -axis, holding x fixed.

We do not observe the sample analog of $F(x, r)$ directly. Rather, we observe the sample analog of the following expression for each of $H * K$ bin combinations:

$$P(y \leq y_{h+1} \mid x \in [x_k, x_{k+1}]) = \frac{1}{x_{k+1} - x_k} \int_{q=x_k}^{x_{k+1}} F(q, y_{h+1}) dq \tag{C.3}$$

We denote this sample analog $\hat{P}(y \leq y_{h+1} \mid x \in [x_k, x_{k+1}])$ as $\hat{R}(k, h)$. Equation C.3 states that the probability that y is less than y_{h+1} is the average value of the CDF in that bin. Since x is uniform, we can write its probability distribution function within the bin as $\frac{1}{x_{k+1} - x_k}$.

We parameterize each CDF as $F(x, r) = S(x, r, \gamma_x)$, where r is the outcome variable, x is the conditioning variable, and γ_x is a parameter vector in some parameter space G_x . Similarly let $f(x, r) = s(x, r, \gamma_x)$. In our numerical calculation, we define G_x as $[0, 1]^{100}$, a vector which gives the value of the cumulative distribution function at each of 100 conditioning variable percentiles on the y -axis, for a given value of x . Put otherwise, holding x fixed, we seek the 100-valued column vector γ_x which contains the value of the CDF at each of the 100 possible y values: $y = 1, y = 2, \dots, y = 100$. As a result, γ_x must lie within $[0, 1]^{100}$. Note that there are as many vectors γ_x as there are possible values for the conditioning variable x . If we discretize also x as $1, 2, \dots, 100$, then we define the matrix of 100 CDFs, indexed by x , as $\gamma^{100} = [\gamma_1 \ \gamma_2 \ \dots \ \gamma_{100}]$. To be explicit, γ^{100} is a 100×100 matrix constructed by setting its x^{th} column as γ_x . We write that $\gamma^{100} \in G^{100}$.

We also introduce a new monotonicity condition for this context. In this set up, monotonicity implies that the outcome distribution for any value of x first-order stochastically dominates the outcome distribution at any lower value of x . Put otherwise,

$$s(x, r, g_x) \text{ is weakly decreasing in } x \tag{Monotonicity}$$

In the mobility context, this statement implies that the child rank distribution of a higher-ranked parent stochastically dominates the child rank distribution of a lower-ranked parent.^{51,52}

The following minimization problem defines the set of feasible values of γ_x for each value x :

$$\Gamma = \operatorname{argmin}_{\mathbf{g} \in \mathbf{G}^{100}} \left\{ \sum_{k=1}^K \sum_{h=1}^H \left(\int_{q=x_k}^{x_{k+1}} S(q, h, g_q) dq - \hat{R}(k, h) \right)^2 \right\} \quad (\text{C.4})$$

such that

$$s(x, r, g_x) \text{ is weakly decreasing in } x \quad (\text{Monotonicity})$$

$$\frac{1}{100} \sum_{x=1}^{100} S(x, r, g_x) = r \quad (\text{Budget Constraint})$$

$$S(x, 0, g_x) = 0 \quad (\text{End Points})$$

$$S(x, 100, g_x) = 1.$$

In the above minimization problem, \mathbf{g} is a candidate vector satisfying the conditions; each g_x describes the candidate CDF holding x fixed. A valid set of cumulative distribution functions is one that minimizes error with respect to all of the observed data points and obeys the monotonicity condition. The budget constraint requires that the weighted sums of CDFs across all conditioning groups must add up to the population CDF. For example, $J\%$ of children must on average attain less than or equal to the J^{th} percentile. The constraints on the end points of the CDF are redundant given the other constraints, but are included to highlight how the end points constrain the set of possible outcomes. For simplicity, we have not included a curvature constraint, but such a constraint would be a sensible further restriction on the feasible parameter space in many contexts.

Once a set of candidate CDFs have been identified, they have a one-to-one correspondence with the CEF given an interval censored conditioning variable, (described by Equation C.2), and thus with any function of the CEF. These statistics can be numerically bounded as in Section 2.

This problem is computationally more challenging than the problem of censoring only in the conditioning variable dealt with in Section 2. In the case of the rank distribution, if we discretize both outcome and conditioning variables into 100 separate percentile bins, then the problem has 10,000 parameters and 10,000

⁵¹Dardanoni et al. (2012) find that a similar conditional monotonicity holds in almost all mobility tables in 35 countries.

⁵²A stronger monotonicity assumption would require that the hazard function is decreasing in x . This is equivalent to stating that the CDF must be weakly decreasing in x conditional upon x being above some value. In the mobility case, for example, the stronger assumption would imply that conditional on being in high school, a child of a better off parent must have a higher latent rank than the child of a worse off parent.

constraints, and an additional 9800 curvature constraint inequalities if desired. This problem proved computationally too difficult to resolve. Restricting the set of discrete bins (e.g. to deciles) is unsatisfying because it requires significant rounding of the raw data which could substantively affect results. We proceed instead by taking advantage of characteristics that are specific to the problem of intergenerational mobility.

C.2 Best and Worst Case Mobility Distributions

Our goal is to bound the parent-child rank CEF given interval censored data on both parent and child ranks. In this section, we take a sequential approach to the double-censoring problem. We use additional information about the structure of the mobility problem to obtain worst- and best-case parent CDFs for intergenerational mobility. From these cumulative distribution functions, we can obtain worst- and best-case CEFs using Equation C.2. First, we calculate bounds on the average value of the child rank in each child rank * parent rank cell. We then apply the methods from Section 2 on the best and worst case bounds; the union of resulting bounds describes the bounds on the mobility statistic of interest. We focus on the rank-rank gradient and on μ_0^{50} .

Given data where child rank is known only to lie in one of h bins, there are two hypothetical scenarios that describe the best and worst cases of intergenerational mobility. Mobility will be lowest if child outcomes are sorted perfectly according to parent outcomes *within* each child bin, and highest if there is no additional sorting within bins.⁵³

Consider a simple 2x2 case. In the 1960s birth cohort in India, 27% of boys attained less than two years of education, the lowest recorded category. 55% of these had fathers with less than two years of education, and 45% had fathers with two or more years of education. We do not observe how the children of each parent group are distributed *within* the bottom 27%. For this case, mobility will be lowest if children of the least educated parents occupy the bottom ranks of this bottom bin, or ranks 0 through 15, and children of more educated parents occupy ranks 16 through 27. Mobility will be highest if parental education has no relationship with rank, conditional on the child rank bin. We do not consider the case of perfectly reversed sorting, where the children of the least educated parents occupy the highest ranks within each child rank bin, as it would violate the stochastic dominance condition (and is implausible).

Appendix Figure C1 shows two set of CDFs that correspond to these two scenarios for the 1960–69 birth cohort. In Panel A, children’s ranks are perfectly sorted according to parent education within bins. Each line shows the CDF of child rank, given some father education. The points on the graph correspond to the observations in the data—the value of each CDF is known at each of this points. Children below the 27th

⁵³Specifically, these scenarios respectively minimize and maximize both the rank-rank gradient and μ_0^x for any value of x . To minimize and maximize p_x , a different within-bin arrangement is required for every x . We leave this out for the sake of brevity, and because bounds on p_x are minimally informative even with uncensored y .

percentile are in the lowest observed education bin. Within this bin, the CDF for children with the least educated parents is concave, and the CDF for children with the most educated parents is convex—indicating that children from the best off families have the highest ranks within this bin. This pattern is repeated within each child bin. Panel B presents the high mobility scenario, where children’s outcomes are uniformly distributed within child education bins, and are independent of parent education within child bin.

According to Equation C.2, each of these CDFs corresponds to a single mean child outcome in a given parent bin, or $Y = E(y|x \in [x_k, x_{k+1}])$. From these expected values, we can then calculate bounds on any mobility statistic, as in Sections 2 and 5. Table C1 shows the expected child rank by parent education for the high and low mobility scenario, as well as bounds on the rank-rank gradient and on μ_0^{50} . Taking censoring in the child distribution into account widens the bounds on all parameters. The effect is proportionally the greatest on the interval mean measure, because it was so precisely estimated before—the bounds on μ_0^{50} approximately double in width when censoring of son data is taken into account.

These bounds are very conservative, as the worst case scenario is unlikely to reflect the true uncensored joint parent-child rank distribution, due to the number and sharpness of kinks in the CDFs in Panel A of Figure C1. A curvature constraint on the CDF would move the set of feasible solutions closer to the high mobility scenario. We next draw on additional data on children, which suggests that the best case mobility scenario is close to the true joint distribution.

C.3 Estimating the Child Distribution Within Censored Bins

Because we have additional data on children, we can estimate the shape of the child CDF within parent-child education bins using rank data from other outcome variables that are not censored. Under the assumption that latent education rank is correlated with other measures of socioeconomic rank, this exercise sheds light on whether Panel A or Panel B in Figure C1 better describes the true latent distribution.

Figure C2 shows the result of this exercise using wage data from men in the 1960s birth cohort. To generate this figure, we calculate children’s ranks first according to education, and then according to wage ranks within each education bin.⁵⁴ The solid lines depict this uncensored rank distribution for each father education; the dashed gray lines overlay the estimates from the high mobility scenario in Panel B of Figure C1.

If parent education strongly predicted child wages *within* each child education bin, we would see a graph like Panel A of Figure C1. The data clearly reject this hypothesis. There is some additional curvature in the expected direction in some bins, particularly among the small set of college-educated children, but the distri-

⁵⁴We limit the sample to the 50% of men who report wages. Results are similar if we use household income, which is available for all men. Household income has few missing observations, but in the many households where fathers are coresident with their sons, it is impossible to isolate the son’s contribution to household income from the father’s, which biases mobility estimates downward.

bution of child cumulative distribution functions is strikingly close to the high mobility scenario, where father education has little predictive power over child outcomes after child education is taken into account. The last row of Table C1 shows mobility estimates using the within-bin parent-child distributions that are predicted by child wages; the mobility estimates are nearly identical to the high mobility scenario. This result supports the assumption made in Section 5 that latent child rank within a child rank bin is uncorrelated with parent rank.

Note that there is no comparable exercise that we can conduct to improve upon the situation when parent ranks are interval censored, because we have no information on parents other than their education, as is common in mobility studies. If we had additional information on parents, we could conduct a similar exercise. The closest we can come to this is by observing the parent-child rank distribution in countries with more granular parent ranks, as we did in Section 2. The results in that section suggest that interval censoring of parent ranks does indeed mask important features of the mobility distribution.

An additional factor that makes censoring in the child distribution a smaller concern is the fact that children are more educated than parents in every cohort, and thus the size of the lowest education bin is smaller for children than for parents. This result is likely to be true in many other countries where education is rising. Of course, in other contexts, we may lack additional information about the distribution of the x and y variable within bins, and researchers may prefer to work with conservative bounds as described in C.2.

Figure C1
 Best- and Worst-Case Son CDFs
 by Father Education (1960-69 Birth Cohort)

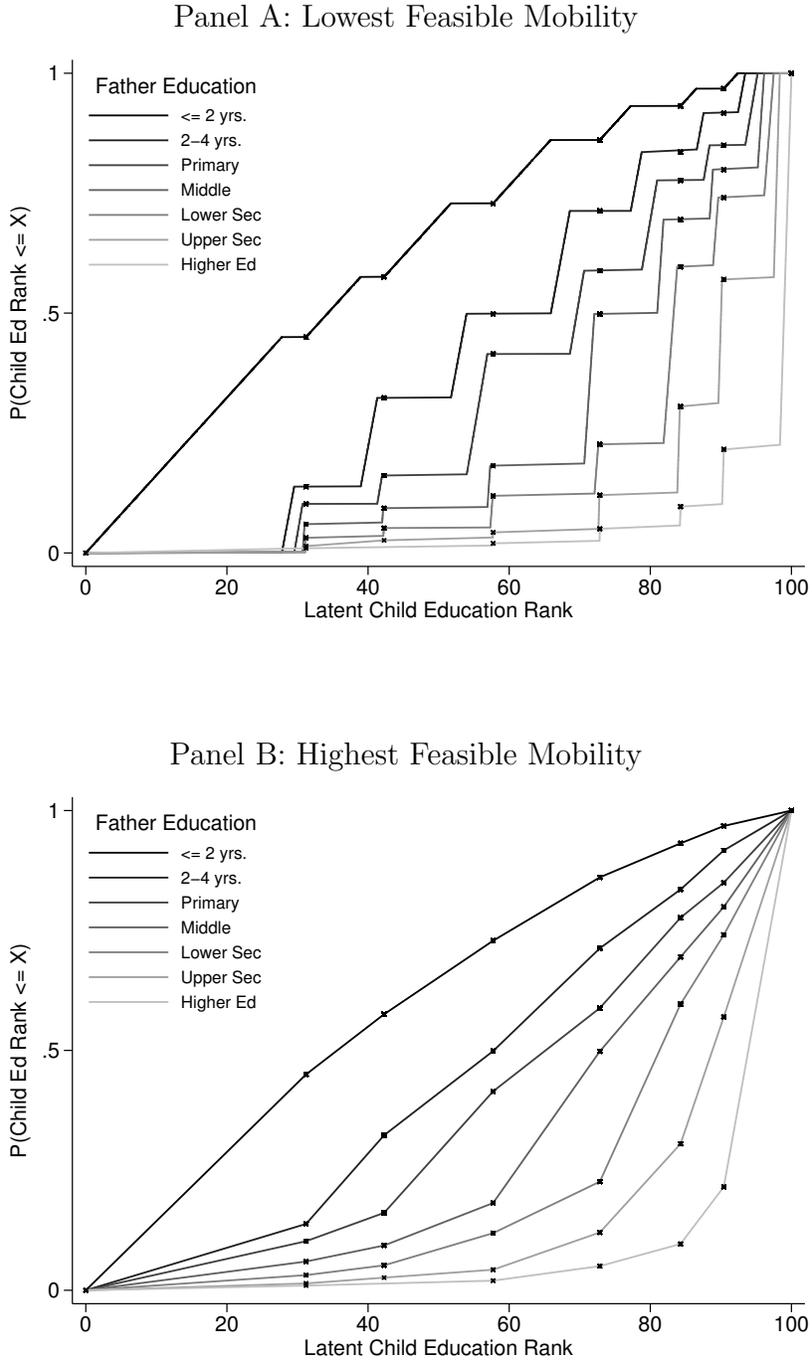


Figure C1 shows bounds on the CDF of child education rank, separately for each father education group. The lines index father types. Each point on a line shows the probability that a child of a given father type obtains an education rank less than or equal to the value on the X axis in the national education distribution. The large markers show the points observed in the data.

Figure C2
Son Outcome Rank CDF
 by Father Education (1960-69 Birth Cohort)
 Joint Education/Wage Estimates

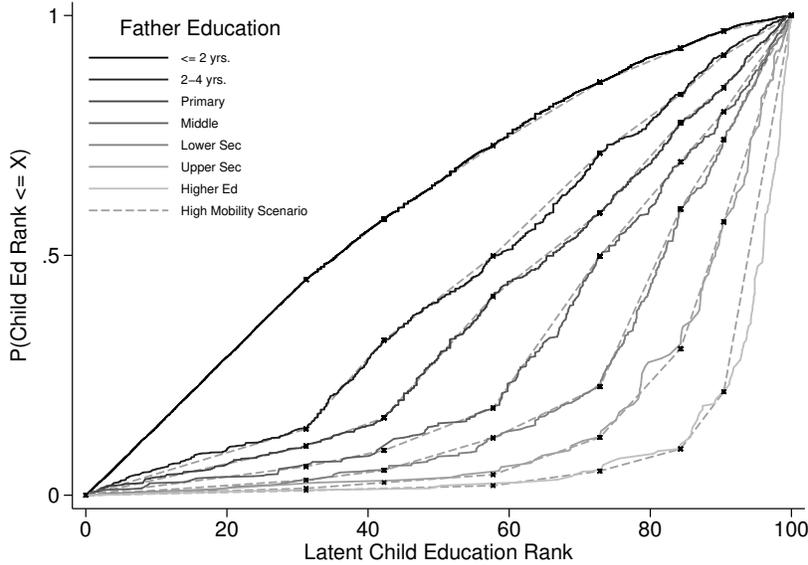


Figure C2 plots separate son rank CDFs separately for each father education group, for sons born in the 1960s in India. Sons are ranked first in terms of education, and then in terms of wages. Sons not reporting wages are dropped. For each father type, the graph shows a child’s probability of attaining less than or equal to the rank given on the X axis.

Table C1
 Mobility Estimates under Double-Censored CEF

	Upward Interval	Rank-Rank
	Mobility (μ_0^{50})	Gradient (β)
Low mobility scenario	[32.33, 35.90]	[0.55, 0.80]
High mobility scenario	[35.86, 38.80]	[0.45, 0.67]
Wage imputation scenario	[35.79, 38.70]	[0.46, 0.67]

Table C1 presents bounds on μ_0^{50} and the rank-rank gradient β under three different sets of assumptions about child rank distribution within child rank bins. The low mobility scenario assumes children are ranked by parent education within child bins. The high mobility scenario assumes parent rank does not affect child rank after conditioning on child education bin. The wage imputation predicts the within-bin child rank distribution using child wage ranks and parent education.

D Appendix D: Data Sources

D.1 Data on Mortality in the United States

For comparability, we follow the data construction procedure used in Case and Deaton (2017). We are grateful that these authors shared software for data construction on their paper’s website to simplify this process.

Death records come from the CDC WONDER database. We have deaths counts by race, gender, and education from 1992–2015, as well as information on cause of death. To obtain mortality rates by year, we obtain the number of people in each age-race-gender-education cell from the Current Population Survey.

The death records contain the universe of deaths in the U.S. The CPS only interviews people who are not institutionalized — e.g., not in a prison or health institution. As a result, the denominator used by Case and Deaton (2017) is slightly smaller than the true denominator. To account for people who are institutionalized, we obtain the number of institutionalized people missing from the CPS in the U.S. Census for 1990 and 2000, and the American Communities Survey for 2005–2015. For non-Census years prior to 2005, we linearly impute the number of institutionalized people in each age-race-gender-education cell; e.g., for 1995, we take the midpoint of the observed number of institutionalized people in 1990 and 2000. For instance, among women ages 50–54 in 1992, just under 0.4% with a high school degree or less are institutionalized. Among that group, mortality falls from 460.8 to 459.0 once we include institutionalized people in the denominator.

The mortality records are characterized by some data with missing education. We follow standard practice in assuming that the education data are missing at random; we assign the missings the educations of the observed educations in the age-race-gender cell whose deaths we observe in that year. Case and Deaton (2017) drop several states that inconsistently report education. After 2005, state identifiers are available only in restricted access data, so we do not yet take this step, and we apply the imputation procedure described above

for all people. We have applied for the restricted data from the National Center for Health Statistics, and the revision of this paper will use only the states with constant data. To predict whether these exclusions are likely to affect our results, we calculated bounds on mortality change from 1992-2004 for all states, and for the subset of states with consistent reporting. Estimates of mortality change differed by at most 0.2%, suggesting that exclusion of these states in all periods will also minimally affect results.

D.2 Intergenerational Mobility: Matched Parent-Child Data from India

To estimate intergenerational educational mobility in India, we draw on two databases that report matched parent-child educational attainment. The first is an administrative census dataset describing the education level of all parents and their coresident children. Because coresidence-based intergenerational mobility estimates may be biased, we supplement this with a representative sample of non-coresident father-son pairs. We focus on fathers and sons because we do not have data on non-coresident mothers and/or daughters. This section describes the two datasets.

The Socioeconomic and Caste Census (SECC) was conducted in 2012, to collect demographic and socioeconomic information determining eligibility for various government programs.⁵⁵ The data was posted on the internet by the government, with each village and urban neighborhood represented by hundreds of pages in PDF format. Over a period of two years, we scraped over two million files, parsed the embedded data into text, and translated the text from twelve different Indian languages into English.⁵⁶ The individual-level data that we use describe age, gender, and relationship with household head. Assets and income are reported at the household rather than the individual level, and thus cannot be used to estimate mobility. The SECC provides the education level of every parent and child residing in the same household. Sons who can be matched to fathers through coresidence

⁵⁵It is often referred to as the 2011 SECC, as the initial plan was for the survey to be conducted between June and December 2011. However, various delays meant that the majority of surveying was conducted in 2012. We therefore use 2012 as the relevant year for the SECC.

⁵⁶Additional details of the SECC and the scraping process are described in Asher and Novosad (2018).

represent about 85% of 20-year-olds and 7% of 50-year-olds. Education is reported in seven categories.⁵⁷ To ease the computational burden of the analysis, we work with a 1% sample of the SECC, stratified across India’s 640 districts.

We supplement the SECC with data from the 2011-2012 round of the India Human Development Survey (IHDS). The IHDS is a nationally representative survey of 41,554 households in 1,503 villages and 971 urban neighborhoods across India. Crucially, the IHDS solicits information on the education of fathers of household heads, even if the fathers are not resident, allowing us to fill the gaps in the SECC data. Since the SECC contains data on all coresident fathers and sons, our main mobility estimates use the IHDS strictly for non-coresident fathers and sons. IHDS contains household weights to make the data nationally representative; we assign constant weights to SECC, given our use of a 1% sample. By appending the two datasets, we can obtain an unbiased and nationally representative estimate of the joint parent-child education distribution.⁵⁸ IHDS reports neither the education of non-coresident mothers nor of women’s fathers, which is why our estimates are restricted to fathers and sons.

IHDS records completed years of education. To make the two data sources consistent, we recode the SECC into years of education, based on prevailing schooling boundaries, and we downcode the IHDS so that it reflects the highest level of schooling completed, *i.e.*, if someone reports thirteen years of schooling in the IHDS, we recode this as twelve years, which is the level of senior secondary completion.⁵⁹ The loss in precision by downcoding the IHDS is minimal, because most students exit school at the end of a completed schooling level.

We estimate changes in mobility over time by examining the joint distribution of fathers’ and sons’ educational attainment for sons in different birth cohorts. All outcomes are measured in 2012, but because education levels only rarely change in adulthood, these measures

⁵⁷The categories are (i) illiterate; (ii) literate without primary (iii) primary; (iv) middle; (v) secondary (vi) higher secondary; and (vii) post-secondary.

⁵⁸We verified that IHDS and SECC produce similar point estimates for the coresident father-son pairs that are observed in both datasets. Point estimates from the IHDS alone (including coresident and non-coresident pairs) match our point estimates, albeit with larger standard errors.

⁵⁹We code the SECC category “literate without primary” as two years of education, as this is the number of years that corresponds most closely to this category in the IHDS data, where we observe both literacy and years of education. Results are not substantively affected by this choice.

capture educational investments made decades earlier. We use decadal cohorts reflecting individuals' ages at the time of surveying. To allay concerns that differential mortality across more or less educated fathers and sons might bias our estimates, we replicated our analysis on the *same* birth cohorts using the IHDS 2005. By estimating mobility on the same cohort at two separate time periods, we identified a small survivorship bias for the 1950-59 birth cohort (reflecting attrition of high mobility dynasties), but zero bias for the cohorts from the 1960s forward. The fact that mobility in the 1950s is biased slightly downward only strengthens our conclusions about zero mobility change (see Figure 9).