A Walrasian Theory of Sovereign Debt Auctions with Asymmetric Information

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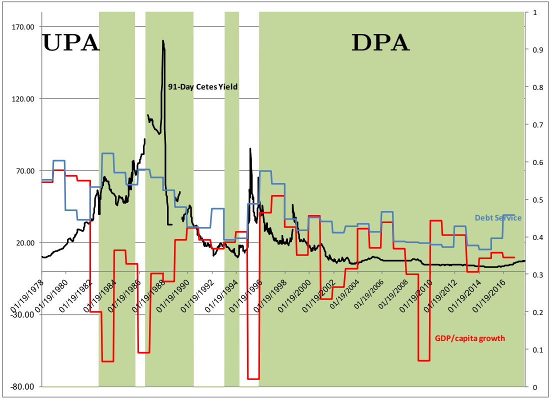
Motivation



- Gov't yields experience wild and tranquil periods.
- Not clearly correlated with publicly observed "fundamentals."
- Why? Does information environment and market structure matter?

Overview

1. Study role of asymmetric information in gov't bond auctions.
   (a) Obtain clear characterization by studying *Walrasian limit*
      - Many bidders and perfect divisibility.
      - Ex-post risk and information acquisition.

   (b) Link risk premia to participation and adverse selection.
   (c) Find that auction protocol induces equilibrium multiplicity.
   Uninformed vs. (multiple) informed equlibria, Pareto-ranked.

2. Compare discriminatory and uniform price auctions.
   (a) Strong tradeoff between protocols only if information is asymmetric.
   $\Rightarrow$ DPA: higher avg. debt burden, less exposure to demand shocks.
   (b) Information is more likely to be asymmetric in DPA.

Relationship to the Literature

- Fits into the efforts to understand sovereign bond prices.
- Fits into the classic GE discussion "where do prices and the information in them come from?"
  - Walras auctioneer, market games, etc.
  - Grossman/Stiglitz (1980).
  - Auctions are ways of micro founding prices & info. (Milgrom 1981).
- Fits into a particular corner of auction theory.
  - Theory: Focus on strategic considerations (few bidders).
  - Empirics: Hortacsu and McAdams (2010), Kastl (2011), Gupta and Lamba (2017).

    For us, many bidders + divisible good ≈ price-taking.

Model

- **Government** needs to raise D (to rollover debt) by selling bonds.
  - Promises to repay 1 per unit of bond, but pays 0 if it defaults.
  - If raises D, defaults with probability κ_θ, where $\theta \in \{b, g\}$ with $\kappa_b > \kappa_g$ and $\sum_\theta f(\theta) = 1$. Otherwise it always defaults.

- Unit mass of risk-averse potential **investors** with wealth W.
  - Access to a risk-free bond with return 1.
  - A random share η of investors do not show up to buy bonds, with $\eta \in [0, \eta_M]$, $\eta_M < 1$ and $\int_\eta g(\eta) = 1$.

- **Information** structure $s \in \mathcal{S} = \{g, b\} \times [0, \eta_M]$.
  - No investor knows demand shock η.
  - A fraction n knows θ ($i = I$). The rest do not ($i = U$).

Walrasian Auctions

1. Assume government sells debt at *Walrasian auction*:
   (i) Perfect divisibility + many bidders.
   (ii) Investors take set of marginal prices as given.

2. Investors submit bid schedules $B^I(P|\theta)$ and $B^U(P)$ for all P.
   - No short-selling: $B \geq 0$ for all P and θ.
   - May (and will) choose to bid at multiple prices.
   - Bids = *commitments* to buy if accepted.

     (Investors may infer θ from P^* ex-post, but cannot revise bids.)

3. Government executes bids in descending order of P
   - Stops when $Demand \geq D$: Marginal price, $P(s)$. Ration if needed.

Two Types of Auctions

- Uniform-Price Auction (UP):
  - If bid accepted, the bidder pays the lowest accepted bid.
- Discriminatory-Price Auction (DP):
  - If bid accepted, the bidder pays his/her bid.

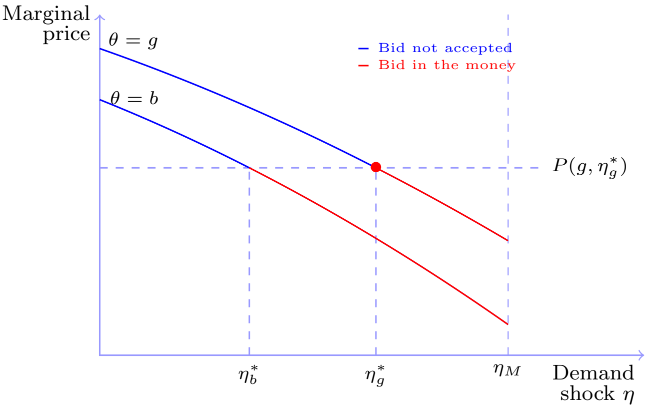
- Marginal price $P(s)$ in state s is the *highest* price such that

$$(1-\eta)\int_{P(s)}^{1}\left[nB^{I}(P|\theta(s))+(1-n)B^{U}(P)\right]\overline{P}dP \geq D$$

UP: $\overline{P} = P(s)$ DP: $\overline{P} = P$

- In equilibrium, the auction will clear with equality.

Which Bids are Accepted?



1. All bids above marginal price accepted ⇒ never bid at non-marginal price
2. Concern: **overpaying** (DP) and/or **buying too much** in bad states
3. Bids executed in different θ-states ⇒ need to infer expected default probability.

Changing the Notation to Walras

Price takers only bid at marginal prices.

Definition

For each state $s = (\theta, \eta) \in \mathcal{S}$,

- The marginal price is denoted $P(s)$ and set by $\mathcal{P}$.
- Uninformed investors choose $B^U(s)$, # of units bid at $P(s)$.
- Informed investors choose $B^I(s, \hat{\theta})$, # of units bid at $P(s)$ when the realized θ is $\hat{\theta}$.

Bids at two states s and s' where $P(s) = P(s')$ are perfect substitutes!
(The bidder buys (or not) the sum of the bids in both states.)

Auction Equilibrium: Uninformed

The expected payoff to an uninformed investor is given by

$$\sum_{\theta \in \{g,b\}} \int_{\eta} \left\{ \begin{array}{c} U(B^U_{RF}([\theta,\eta]))\kappa_\theta + \\ U\left(B^U_{RF}([\theta,\eta]) + \mathcal{B}^U_R([\theta,\eta])\right)(1-\kappa_\theta) \end{array} \right\} f(\theta)g(\eta)d\eta$$

The total risky bonds purchased $\mathcal{B}^U_R(s)$, is

$$\mathcal{B}^U_R(s) = \sum_{s':P(s')\geq P(s)} B^U(s'),$$

sum of in-the-money bids. (Notation abuse warning.)

Auction Equilibrium: Uninformed

Expenditures on risk-free bonds $B^U_{RF}([s])$ are a residual:

$$\begin{aligned} \text{UP auction} \quad &: \quad B^U_{RF}(s) = W - \left[\sum_{s': P(s') \geq P(s)} B^U(s') \right] P(s), \\ \text{DP auction} \quad &: \quad B^U_{RF}(s) = W - \left[\sum_{s': P(s') \geq P(s)} B^U(s') P(s') \right]. \end{aligned}$$

The investor cannot short-sell or borrow, so nonegativity constraint

$$B^U(s) \geq 0 \text{ and } B^U_{RF}(s) \geq 0 \qquad \forall s \in \mathcal{S}.$$

Auction Equilibrium: Informed

The expected payoff (given θ) to an informed investor is

$$\int_{\eta} \left\{ \begin{array}{c} U(B_{RF}^{I}([\theta,\eta],\theta))\kappa_{\theta} + \\ U\left(B_{RF}^{I}([\theta,\eta],\theta) + \mathcal{B}_{R}^{I}([\theta,\eta],\theta)\right)(1-\kappa_{\theta}) \end{array} \right\} g(\eta)d\eta \qquad \forall \theta \in \{g,b\},$$

where risky bond purchases are

$$\mathcal{B}_{R}^{I}(s,\theta) = \sum_{s':P(s')\geq P(s)} B^{I}(s',\theta) \qquad \forall \theta \in \{g,b\},$$

Auction Equilibrium: Informed

Total expenditures on risk-free bonds are a residual:

$$\begin{aligned}
\text{UP auction} \quad &: \quad B^I_{RF}(s,\theta) = W - \left[\sum_{s':P(s')\geq P(s)} B^I(s',\theta) \right] P(s), \\
\text{DP auction} \quad &: \quad B^I_{RF}(s,\theta) = W - \left[\sum_{s':P(s')\geq P(s)} B^I(s',\theta) P(s') \right],
\end{aligned}$$

and the nonegativity constraints are

$$B^I(s,\theta) \geq 0 \text{ and } B^I_{RF}(s,\theta) \geq 0 \qquad \forall s \in \mathcal{S} \text{ and } \forall \theta \in \{g,b\}.$$

Trivially, only bid at prices $P(\theta,\eta)$ and not at prices $P(\theta',\eta)$ $(\theta' \neq \theta)$

Linear Algebra Structure

E.g. assume 4 states $P_j > P_{j+1}$. For the uniform protocol expenditures at auction are

$$\mathbf{X^i_{UP}} = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ P_2 & P_2 & 0 & 0 \\ P_3 & P_3 & P_3 & 0 \\ P_4 & P_4 & P_4 & P_4 \end{bmatrix} * \begin{bmatrix} B^i_1 \\ B^i_2 \\ B^i_3 \\ B^i_4 \end{bmatrix} = \mathbf{P^{UP}} * \vec{B}^i$$

and for the discriminating protocol expenditures at auction are

$$\mathbf{X^i_{DP}} = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ P_1 & P_2 & 0 & 0 \\ P_1 & P_2 & P_3 & 0 \\ P_1 & P_2 & P_3 & P_4 \end{bmatrix} * \begin{bmatrix} B^i_1 \\ B^i_2 \\ B^i_3 \\ B^i_4 \end{bmatrix} = \mathbf{P^{DP}} * \vec{B}^i.$$

While the gross return $\mathbf{R} = \mathbf{1} - \mathbf{P}$ is similar with P_j replaced by $1 - P_j$ in the price matrix. Auction clearing in all states are

$$[1 - \vec{\eta}] \cdot \left(n * \mathbf{X^I} + (1-n) * \mathbf{X^U}\right) = D$$

Bid-Overhang Constraint

- Recall: Marginal price $P(s)$ = highest price s.t. $Demand \geq D$.
- **Requirement**: For all s, there *cannot* exist a state $\tilde{s}$ such that:
    1. $P(\tilde{s}) > P(s)$,
    2. *Demand* given $P(\tilde{s})$ is enough to cover supply in state s.
- This constraint may bind, but only in the UP auction.
- For DP, high price bids reduce remaining supply at low prices.
- Removes source of multiplicity relative to C.Eq.

Auction Equilibrium

Definition

An equilibrium of a Walrasian auction is defined as a price function $P : \mathcal{S} \to [0, 1]$, *and bidding functions* $B^U : \mathcal{S} \to [0, \infty)$ *and* $B^I : \mathcal{S} \times \{g, b\} \to [0, \infty)$, *such that*

1. *each type of investor's bid function solves their problem,*
2. *the auction clearing condition is satisfied for all* $s \in \mathcal{S}$, *and*
3. *the bid-overhand constraint is satisfied at each* $s \in \mathcal{S}$.

Property of Price Functions

Proposition

For both auction formats the price function $P(\theta, \eta)$ is decreasing in η.

Hence, a bid at a price $P(\theta, \hat{\eta})$ is in-the-money for all $\eta \geq \hat{\eta}$, given θ. If there are two states such that $P(\bar{\theta}, \bar{\eta}) = P(\theta, \hat{\eta})$, then the bid is also in-the-money for all $\eta \geq \bar{\eta}$ when $\bar{\theta}$.

Proposition

Since the price schedule conditional on θ is bounded and monotonic, it follows that it is both continuous and differentiable almost everywhere.

Specialize Model and Solve

We specialize the model to get simple expressions and illustrate forces.

- Preferences are log.
- We assume η is distributed uniformly on $[0, \eta_M]$.

Numerical Example:

- $\kappa_g = 0.15$, $\kappa_b = 0.35$ and $f(b) = 0.5$.
- Wealth of lenders, $W = 250$. Debt rolled over, $D = 60$.
- $\eta_M = 0.17$

Example chosen so there is perfect revelation ex-post and short sale constraints do not bind.

Uniform Price Auction

Symmetric Ignorance ($n = 0$)

- Prices cannot depend upon θ, $P(g, \eta) = P(b, \eta)$ for all $\eta \in \mathcal{H}$. Hence, write $P(\eta)$ for prices and $B(\eta)$ for bond purchases.
- As prices convey no information about θ, the ex-ante probability of default is $\tilde{\kappa}(P) = \kappa^U = f(g)\kappa_g + f(b)\kappa_b$, for all η.
- As $P(\eta)$ decline in η, $B(\eta) > 0$ for all η.

Two Polar Cases: $n = 0$ and $n = 1$

1. **Symmetric ignorance** (n=0): prices independent of θ.
   - Ex-ante default prob $= \tilde{\kappa}(P) = \kappa^U = f(g)\kappa_g + f(b)\kappa_b$, for all η.
   - Block-recursive problem from the top down. Prices in closed-form:

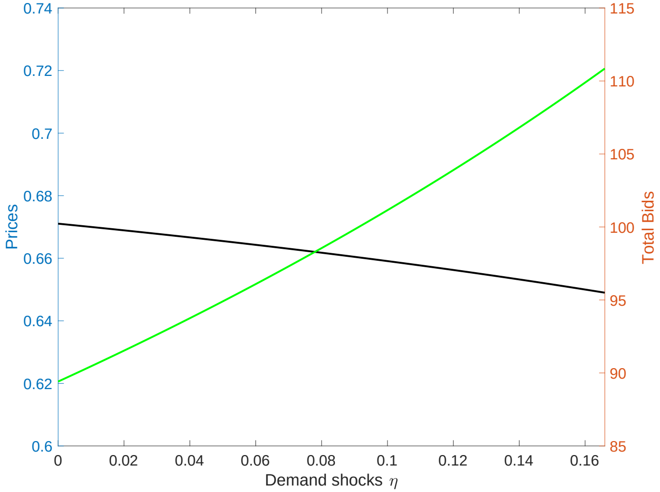
$$P(\eta) = 1 - \frac{\kappa^U}{1 - \frac{D}{W}\frac{1}{1-\eta}} \qquad \forall \eta.$$

2. **Symmetric Information** (n=1): prices contingent on θ
   - Analogous block-recursive construction. Prices in closed-form:

$$P(\theta, \eta) = 1 - \frac{\kappa_\theta}{1 - \frac{D}{W}\frac{1}{1-\eta}} \qquad \forall \eta, \theta$$

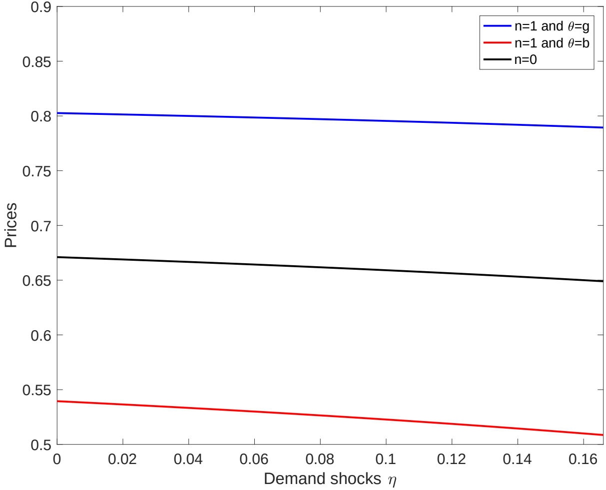
   with state-contingent default probability $\kappa(\theta)$.

Symmetric Ignorance ($n = 0$)



Gains to being informed?

Symmetric Information ($n = 1$)



Costs of being uninformed? **None** if perfect replication is possible.

Replication in UP Auction

Proposition. In UP auctions, U-investors can **perfectly replicate** the portfolio and payoffs of I-investors if and only if

1. Each marginal price is associated with a unique state in $\mathcal{S}$.
   ($\Leftrightarrow$ bid-overhang constraint does not bind).
2. The short-sale constraints do not bind for the uninformed at the informed bids. Sufficient condition: $B^I(g, \eta_M) \leq B^I(b, 0)$.

The bid-overhang constraint binds when the uninformed become too many (when $n < \eta$), which forces price pooling.

Price pooling violates condition 1 and requires belief consistency.

Uninformed Investors' Inference

The uninformed investor does not know θ, but can make an inference about the probability of default given a price, $\widetilde{\kappa}(P)$.

- Easy if P *only* corresponds to one state: $\widetilde{\kappa}(P(\theta, \eta)) = \kappa_\theta$
- More difficult if P corresponds to more than one state.
  - When $P(g, \eta_g) = P(b, \eta_b)$ use mass of η's in $[P(\theta, \cdot) - \epsilon, P(\theta, \cdot) + \epsilon]$ to determine relative likelihood of each θ.
  - This computation depends on the slope of P w.r.t. η.
    A flatter slope in a schedule means more mass of η's in a given range 2ϵ around P, and then such schedule is more likely.

Once Bid-Overhang Forces Pooling

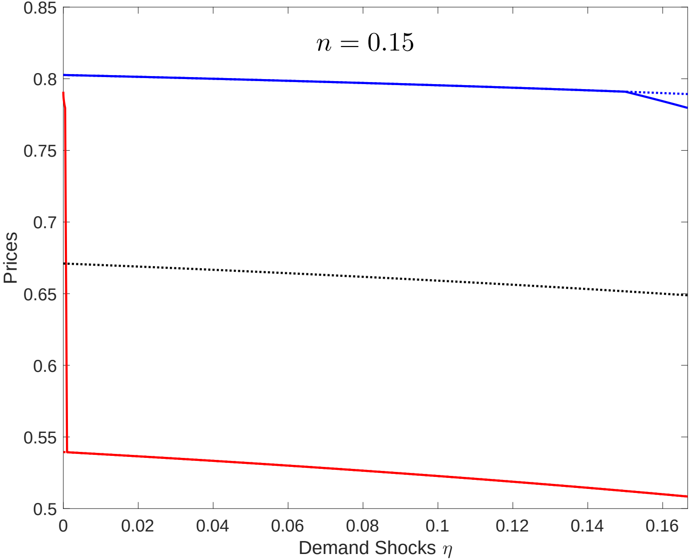
Take any two states $s = [g, \eta_g]$ and $s' = [b, \eta_b]$ with a common price:

$$n\left(\frac{1-\kappa_g-P}{1-P}\right)+(1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right)=\frac{D}{W}\frac{1}{1-\eta_g},$$

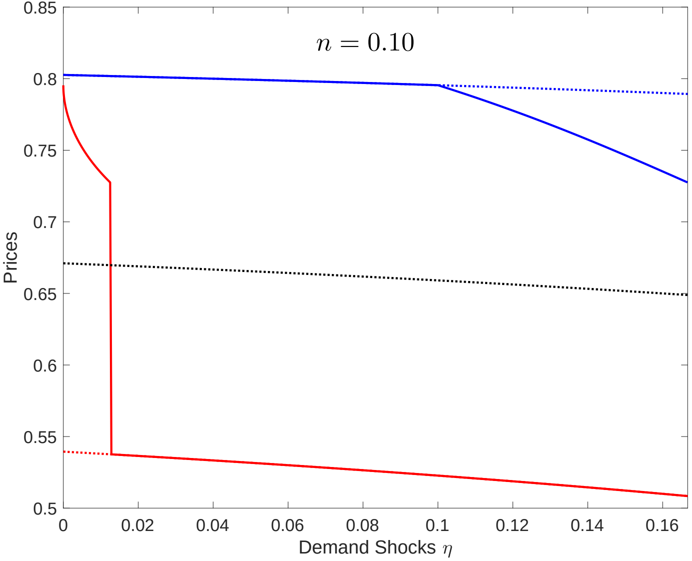
$$n\max\left[\left(\frac{1-\kappa_b-P}{1-P}\right),0\right]+(1-n)\left(\frac{1-\tilde{\kappa}-P}{1-P}\right)=\frac{D}{W}\frac{1}{1-\eta_b}. \quad (1)$$

- Short-sale constraint can bind on informed when quality is κ_b.
- Cannot bind on the uninformed (of course).

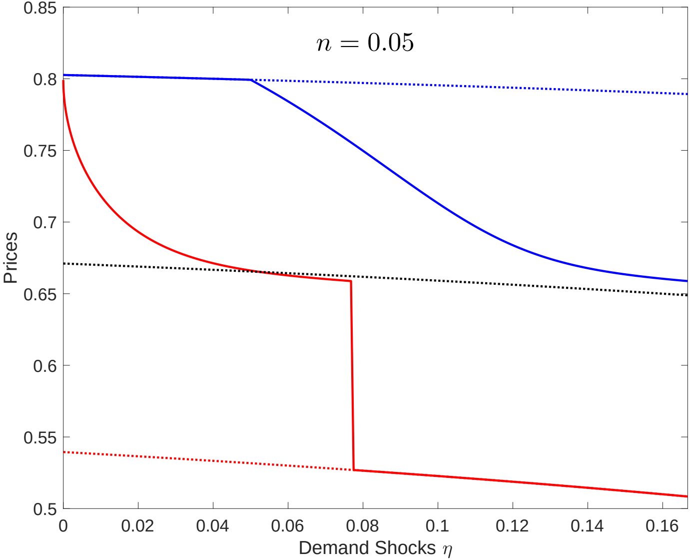
Prices when changing n



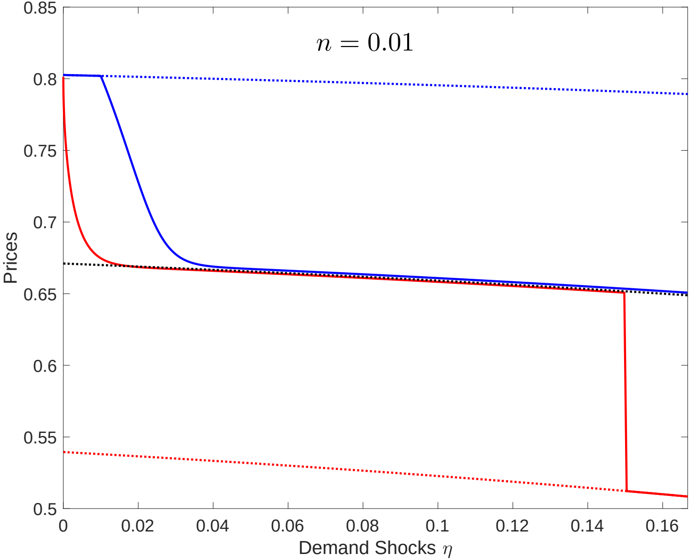
PRICES WHEN CHANGING n



Prices when changing n



Prices when changing n



Convergence at extremes

Proposition

For both UP and DP auctions, price schedules $P([g, \eta]; n)$ and $P([b, \eta]; n)$ converge to each other (for interior η) as $n \to 0$.

Proof.

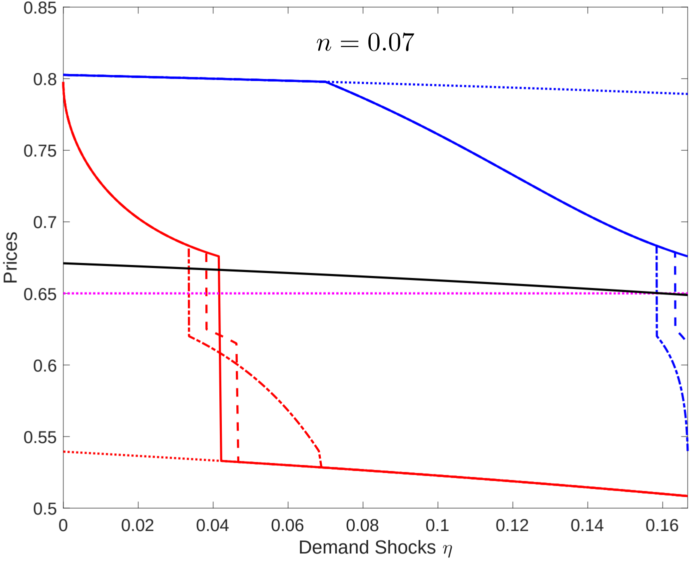
For n sufficiently close to 0, η must partially order the price schedules.

If $\kappa_\theta < \kappa_{\theta'}$ and $\eta > \eta'$ then $P([\theta, \eta]; n) < P([\theta', \eta']; n) < P([\theta, \eta']; n)$.

- Given θ prices are decreasing in η: $P([\theta, \eta]; n) < P([\theta, \eta']; n)$
- As $n \to 0$: $P([\theta', \eta']; n) \to P([\theta, \eta']; n)$

□

Multiple Equilibria



Discriminatory Price Auction

Similar...yet very different

- Now concerned about *buying too much* **and** *paying too much.*
- Bids executed at different prices $\rightarrow$ price dispersion.
- "Inference" problem replaced by "in-the-money" problem.

  (because bids are executed at bid price)

- Impossible to perfectly replicate informed portfolio.

Symmetric Ignorance ($n = 0$)

Assuming log preferences. First order conditions in vector matrix form are:

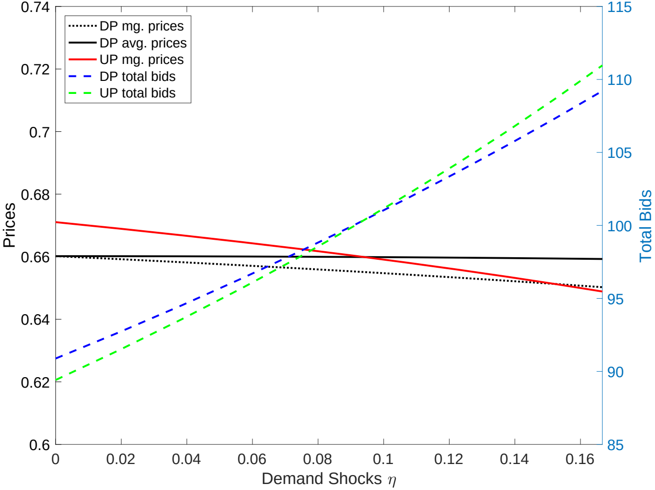
$$-\left(W - \mathbf{P^{DP}} \times \vec{B}^U\right)^{-1} \cdot \vec{P} \cdot \kappa^U + \left(W + [\mathbf{1} - \mathbf{P^{DP}}] \times \vec{B}^U\right)^{-1} \cdot \left[1 - \vec{P}\right] * [1 - \kappa^U] = 0.$$

Auction clearing is simply:

$$[1 - \vec{\eta}] \cdot \left[\mathbf{P^{DP}} \times \vec{B}^U\right] = D$$

This is **NOT** block-recursive and then all prices have to be solved simultaneouly.

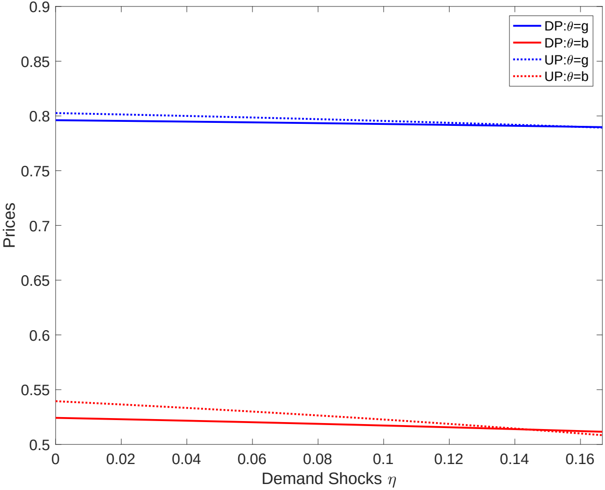
Symmetric Ignorance ($n = 0$)



DP price and bond schedule flatter than UP!

Gains to being informed? Very similar to UP auction!

Symmetric Information $(n = 1)$



Under symmetry, little difference in prices across auctions.

Costs of being uninformed? Very different from UP auction!

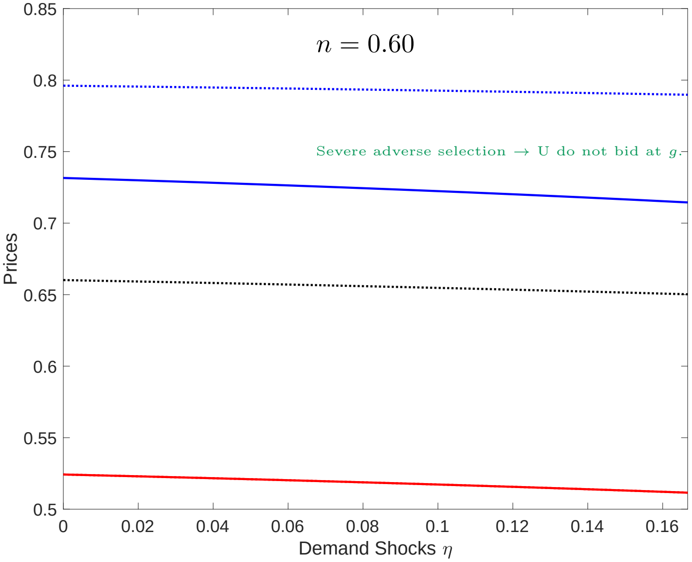
No Replication in DP auction

Proposition

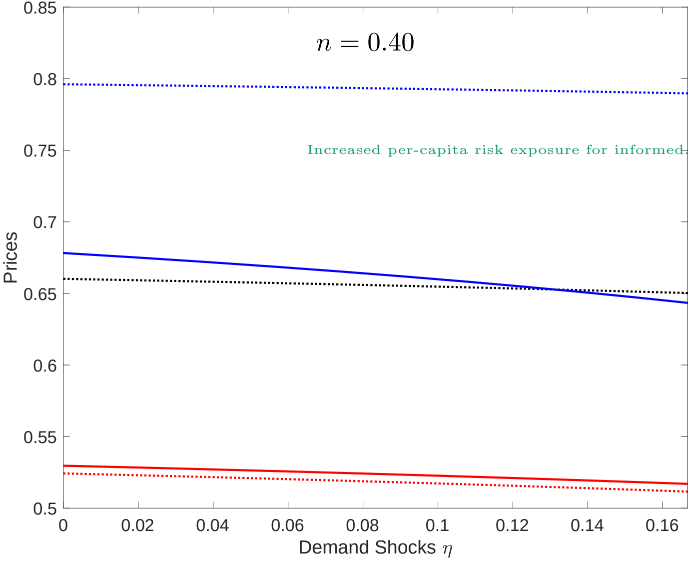
In a DP auction, the uninformed will never be able to replicate the bids of the informed, and hence their payoffs, so long as

1. $\kappa_g \neq \kappa_b$ *and* $f(g)$ *and* $f(b)$ *are both positive*
2. *Informed investors bid positive amounts for both* $\theta = g$ *and* $\theta = b$ *for some values of* η*.*

Prices when changing n



Prices when changing n



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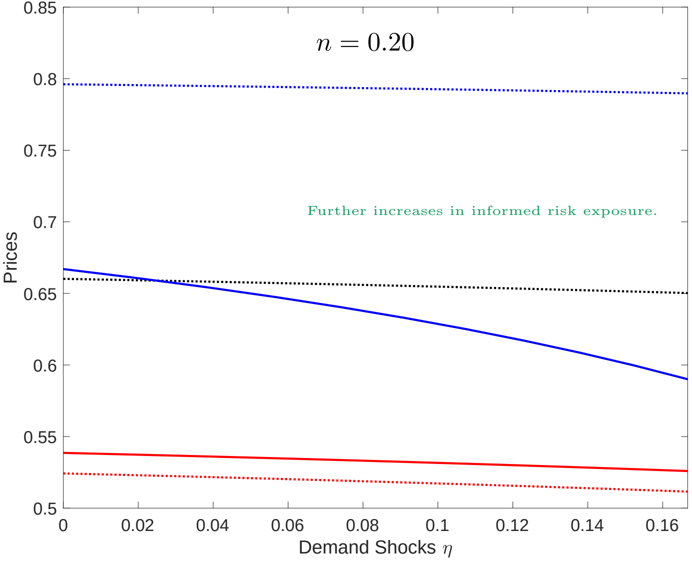

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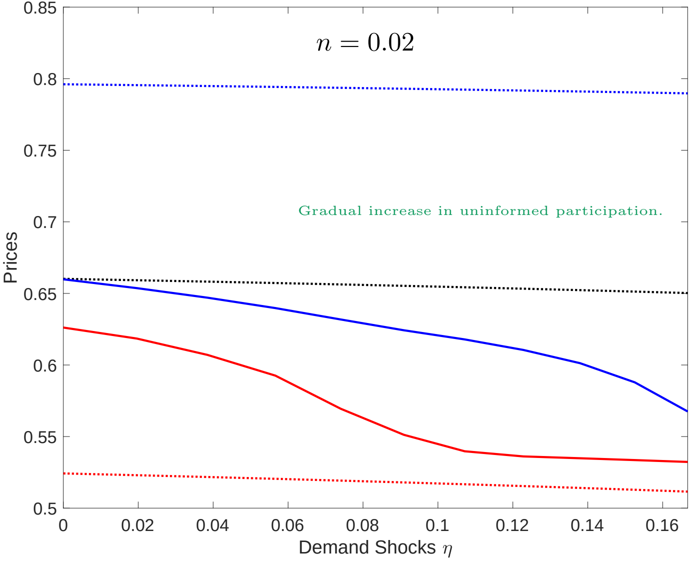


Comparison to Competitive Equilibrium

- In competitive equilibrium (CEq), there is *a single realized price* and bids at prices *other than the one realized* are not binding.

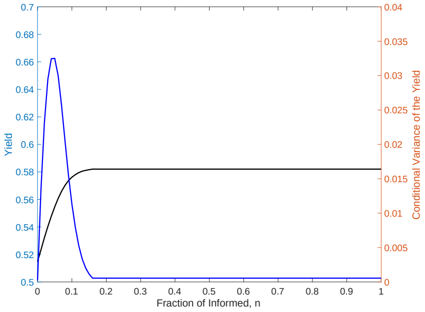
- DP auction is not a CEq (several prices are realized given a state).
- UP auction may be a CEq (single price is realized given a state)
  1. When short-sale constraints do not bind anywhere.

     In CEq short-sale constraints affect total purchases, not each bid.
  2. When the bid-overhang constraint does not bind.

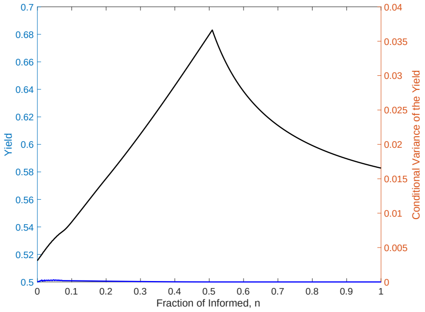
     In CEq the marginal investor is always informed.

Comparing Protocols

Yields and Conditional Variances



(A) Uniform Price Auction



(B) Discriminating Price Auction

- With symmetric information (or ignorance) yields are similar.
- With asymmetric information yields are quite different: $DP > UP$...
- ...and so is the conditional variance, but $DP < UP$ so risk trade-off.

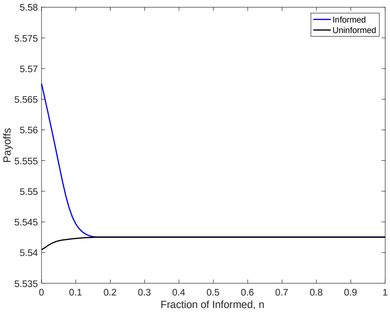
Endogenous Information Acquisition

- Allow investors to acquire information about θ at utility cost K.
- Then n is endogenous.

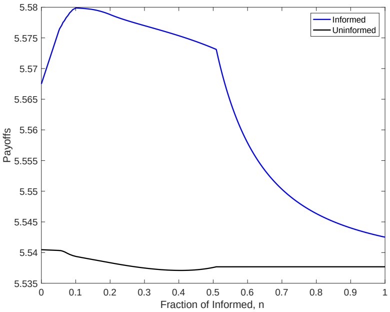
$$
\begin{aligned}
\overbrace{V^I(g)f(g) + V^I(b)f(b)}^{V^I} - V^U &\geq K \qquad \text{if } n > 0 \\
V^I(g)f(g) + V^I(b)f(b) - V^U &\leq K \qquad \text{if } n < 1.
\end{aligned}
$$

- Solve for $V^I(\theta)$ and V^U for all n. Then obtain n^*.

Payoffs to Investors

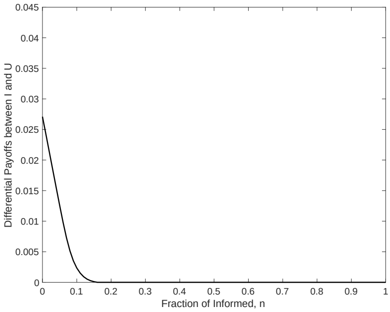


(C) Uniform Price Auction

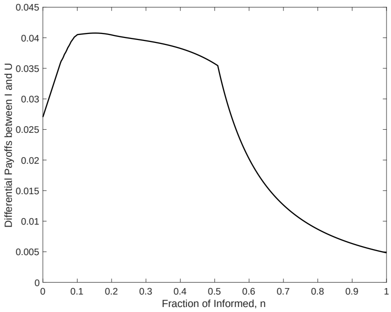


(D) Discriminating Price Auction

Equilibrium with Information Acquisition



(E) Uniform Price Auction



(F) Discriminatory Price Auction

Turbulence and Stability

- Sources of Turbulence
  - Both in UP and DP: High degree of asymmetry (high n^*)

    Price schedules are very different and sensitive to quality shocks.
  - Only in UP. Low degree of asymmetry (low n^*).

    Price schedules are very sensitive to demand shocks (both in terms of slopes and multiplicity).
  - Only in DP. Switch of informational regimes.

    Sometimes prices react to quality shocks, sometimes not.
- Stability is maximized when $n^* = 0$ (symmetric ignorance).

- **DP auctions are in average more exposed to quality shocks.**
- **DP auctions may switch their sensitivity to quality shocks.**
- **UP auctions are in average more exposed to demand shocks.**

Final Remarks

- Novel analysis of auctioning divisible goods to many buyers.
  - DP and UP similar under *symmetric information* or *ignorance*.
  - Surprisingly different under *asymmetric information* ($n \in (0,1)$).

- DP auctions may lead to multiple information regimes.
- Asymmetric information regime displays a tradeoff.
  - UP auctions: Lower debt burden and exposure to quality shocks.
  - DP auctions: Lower exposure to demand shocks.
  - In either case, lower welfare (costly information here is a waste).

- Potential application beyond auctions, such as limit-order trading.

Differences across Auction Protocols

- Starting with $n = 1$,
  - DP: Severe adverse selection $\rightarrow$ uninformed buy on $\theta = b$ schedule only.
  - UP: replication implies full participation by the uninformed.
- Shrink $n \downarrow \eta_M$,
  - DP: Informed forced to hold more risk per-capita: $P(g, \eta)$ declines.
  - UP: no change because of replication.
- Shrink $n < \eta_M$,
  - DP: I's risk exposure + U's adverse selection drives $P(g, \eta) < P^U(\eta)$.
  - UP: blending at prices close to $P(g, \eta)$ due to bid overhang.
- Shrink $n \rightarrow 0$,
  - DP: prices overlap, less adverse selection: $\widetilde{\kappa} = \kappa_u$ so $P(\theta, \eta) \rightarrow P^U(\eta)$.
  - UP: blending everywhere but extremes: $\widetilde{\kappa} \rightarrow \kappa_u$ so $P(\theta, \eta) \rightarrow P^U(\eta)$.

Expected Probability of Default

- Given $P = \phi(\theta, \eta)$, define $\eta = \phi^{-1}(P|\theta)$.
- Define the probability of a set of prices, $\mathbf{P} \subset \mathcal{P}$, as

$$h(\mathbf{P}) = \sum_\theta f(\theta) \int_{\eta : P(\theta,\eta) \in \mathbf{P}} g(\eta) d\eta = \sum_\theta f(\theta) \underbrace{\int_{\widetilde{P} : \phi^{-1}(\widetilde{P}|\theta) \in \phi^{-1}(\mathbf{P}|\theta)} \frac{\partial \phi^{-1}(P|\theta)}{\partial P} g(\eta) d\widetilde{P}}_{Pr(\mathbf{P}|\theta)}$$

- Infer probability from each θ and hence probability of default.
- Shrink $\mathbf{P} \to P$ to get expected probability of default given P:

$$\widetilde{\kappa}(P) = \frac{\sum_\theta f(\theta) Pr(P|\theta) \kappa_\theta}{h(P)}$$