

# Central Bank Balance Sheet Policies without Rational Expectations\*

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## Abstract

We study the effects of central bank balance sheet policies—quantitative easing and foreign exchange interventions—in a model without rational expectations. We use the “level- $k$  thinking” belief formation and the associated reflective equilibrium that rationalize the idea that it is difficult to change expectations about economic outcomes even if it is easy to shift expectations about the policy. We emphasize two main results. First, under a broad set of conditions, central bank interventions are neutral in the rational expectations equilibrium, while they are effective in the reflective equilibrium. Second, we derive testable predictions that can be used to differentiate this “bounded rationality channel” from the alternative “portfolio balance” and “signaling” channels of balance sheet policies.

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# 1 Introduction

Central bank's balance sheets are among the most important stabilization policy tools (Bernanke, 2012; Draghi, 2015; Yellen, 2016). A recent example is the policy of quantitative easing (QE)—a purchase of long-term public and risky private assets financed with central bank liabilities. Several central banks in developed countries have recently used this policy to stimulate their economies when the conventional nominal interest rate tool reached its effective zero lower bound.<sup>1</sup> Yet balance sheet policies are not confined to liquidity traps. Another example of such policies, which is arguably more popular across countries and over time, is foreign exchange (FX) interventions—a purchase of sovereign bonds denominated in foreign currency, usually financed by selling holdings of domestic sovereign bonds. Advanced economies had to routinely rely on this policy during periods of fixed exchange rate arrangements (e.g., the Gold Standard, the Bretton Woods, the European Exchange Rate Mechanism). What is more, during the recent financial crisis, some economies have again resorted to such interventions to tame speculative capital flows (e.g., in Switzerland and Israel) and to stimulate domestic production (e.g., in the Czech Republic). Finally, emerging economies have also been using FX interventions to limit exchange rate fluctuations and to accumulate buffers against future sudden stops.

Despite their popularity, central bank's balance-sheet policies are among the least understood. First, from an empirical perspective, it is challenging to identify a causal effect: Balance-sheet policies are usually implemented in response to shocks that hit the economy creating an endogeneity problem. There is nonetheless some empirical evidence for QE and FX interventions effects on assets markets and the real economy.<sup>2</sup> Second, from a theoretical perspective, a wide class of standard macroeconomic models predicts that balance-sheet policies have no effect on the economy. More precisely, Wallace (1981) showed that these policies are irrelevant because investors completely undo central bank

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<sup>1</sup>The Bank of Japan implemented its first round of quantitative easing as early as 2001 in an attempt to fight deflation. The Federal Reserve and the European Central Bank increased their balance sheet fourfold in response to the global financial crisis.

<sup>2</sup>Using high frequency financial data, Gagnon *et al.* (2011), Krishnamurthy and Vissing-Jorgensen (2011), Hancock and Passmore (2011) found that large-scale mortgage-backed securities (MBS) purchases by the Fed have affected mortgage market yields and have spread to other assets markets. At the same time, Stroebel and Taylor (2012) find no effects of MBS purchases in the first round of QE by the Fed. Di Maggio *et al.* (2016) and Chakraborty *et al.* (2016) found evidence of the effects of Fed's MBS purchases on mortgage lending. Fieldhouse *et al.* (2017) show that purchases of MBS by the government sponsored enterprises in the US, which can be interpreted as QE, affected not only mortgage rates and lending but also residential investments.

Dominguez and Frankel (1990, 1993) estimate significant effects of foreign exchange interventions. Sarno and Taylor (2001) summarize the earlier literature presenting a more balanced view. Kearns and Rigobon (2005) study a "natural experiment" in which Japan and Australia "exogenously" changed their FX policies which resulted in significant change in their exchange rate. Blanchard *et al.* (2014) present evidence of the significant effects of currency interventions using more recent data.

interventions. There are two main steps in Wallace’s argument, which is essentially an application of the Ricardian equivalence to the consolidated government. First, when a central bank purchases, for example, private risky assets and issues safe liabilities (as in the case of QE), investors understand that gains or losses incurred on the central bank’s portfolio will be directly transferred to the fiscal authority and, through taxes, they will indirectly return to investors. In turn, to hedge against this tax risk, investors reduce their demand for risky assets. Second, in rational expectations equilibrium, investors do not make systematic mistakes and perceive the magnitude of the tax risk accurately. As a result, they reduce the demand for risky assets exactly by the amount of the central bank risky assets purchases, which completely undoes the direct effect of the policy intervention. The same logic applies to FX interventions.<sup>3</sup>

In this paper, we study the effects central bank’s balance-sheet policies when investors fail to correctly foresee the future consequences of policy interventions. More specifically, we build a dynamic equilibrium model based on [Wallace \(1981\)](#) and extend it to an international setting along the lines of [Jeanne and Rose \(2002\)](#) and [Bacchetta and Van Wincoop \(2006\)](#) to be able to discuss FX interventions. In the model, agents do not hold rational expectations, but forecast future endogenous variables following the “level- $k$  thinking” deductive procedure ([Nagel, 1995](#); [Stahl and Wilson, 1995](#)). This form of expectations formation was first used in macro by [Evans and Ramey \(1992, 1998\)](#) to study conventional monetary policy, and it was recently applied to resolve the “forward-guidance puzzle” by [García-Schmidt and Woodford \(2015\)](#) and [Farhi and Werning \(2016\)](#).<sup>4</sup>

In the model we propose, the monetary authority commits to the policy interventions path, which is fully believed by the public. The fiscal authority then adjusts taxes as a function of the realized returns on the central bank’s portfolio. Equilibrium variables, such as taxes and asset prices, are not announced, thus, agents have to derive the equilibrium implications of policy interventions, but they never achieve rational expectations. The latter are often justified as the limit of some learning process, therefore, a deviation from rationality is likely to be more severe when agents are confronted with policies, such as QE and FX intervention, that are new or whose effects on the economy are not precisely estimated or even understood. In these cases, the level- $k$  thinking cognitive process of belief formation provides a plausible alternative to learning as it does not require the knowledge of past policy effects.

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<sup>3</sup>We discuss some mechanisms through which balance sheet policies can have affect on the economy in the literature review.

<sup>4</sup>In the recent influential paper, [Gabaix \(2016\)](#) assumes that current consumption is less sensitive to changes in future income which helps resolve the “forward guidance puzzle.” With “level- $k$  thinking,” current aggregate consumption is also less sensitive to future aggregate income but this arises “endogenously” due to non-rational expectations about future income. [Angeletos and Lian \(2016\)](#) show how alternative deviation from rational expectations, lack of common knowledge, helps to solve the forward guidance puzzle.

Level- $k$  thinking and the equilibrium notions associated with it work as follows. Agents are perfectly aware of current balance-sheet policies (assets purchased by the government) as well as their own income and asset positions. However, they have to form expectations about the effects of the policy on future taxes and asset prices. They form expectations according to the following iterative procedure. “Level-1 thinking” assumes that agents keep expectations identical to those before the change in the policy. As a result, “level-1 thinkers” completely ignore the Wallace irrelevance result.<sup>5</sup> Given their expectations, these agents choose consumption and portfolio holdings. Markets clear on period-by-period basis which may result in outcomes that are inconsistent with level-1 thinkers naive beliefs. By assumption, these agents cannot incorporate the consequences of future market-clearing into their expectations.

“Level-2 thinking” assumes that an agent understands equilibrium outcomes of the balance sheet policy conditional on believing that economy is populated by level-1 thinkers only. As a result, this agent will expect future to depend on the policy, however, these expectations may not coincide with rational expectations. Conditional on these updated expectations, agents make consumption and portfolio decisions and markets clear on a period-by-period basis.

These deduction rounds can be generalized to “level- $k$  thinking” and carried over to infinity. Following [García-Schmidt and Woodford \(2015\)](#), we assume that the economy is populated by agents with all levels of thinking, with the mass of agents of each level of thinking given by an exogenous distribution. This results in a notion of reflective equilibrium which we describe in Section 2.

Our first main result shows that the central bank’s balance-sheet policies affect asset prices in the reflective equilibrium while they are neutral in the rational expectations equilibrium. Intuitively, because agents do not hold rational expectations about future endogenous variables, and, in fact, they always underestimate the reaction of these variables after policy interventions, agents do not change their assets demand enough to undo the intervention. This leads to a non-neutrality of balance sheet policies.

Reflective equilibrium outcomes coincide with the rational expectations equilibrium in the limit as the average level of sophistication in the economy increases. However, we show that this convergence may be non-monotonic. Specifically, an increase in agents’ sophistication has two opposing effects when the policy intervention is expected to persist over time. On the one hand, more sophisticated agents can better foresee the effects of policies on future endogenous variables, bringing the policy closer to full neutrality. On the other hand, however, more sophisticated agents become endogenously more forward looking, thus, making the policies that persist into the future more effective.

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<sup>5</sup>In other words, these agents do not hold “Ricardian expectations” ([Woodford, 2013](#)).

We present a number of comparative statics and numerical exercises that help make sense of the magnitudes of the effects. The asset price effect of quantitative easing over the price effect of similar size increase in foreign demand for same assets equals the inverse of the average sophistication of agents in the economy. For example, if one believes that the average level of sophistication of agents in the economy is  $\bar{k} = 2$  (i.e., an average agent thinks that all other agents do not change their expectations following a policy change) then QE is only twice weaker than foreign purchases of assets. Although we do not try to calibrate our very stylized model, we nevertheless numerically evaluate a policy experiment that resembles the Fed's mortgage-backed securities (MBS) purchases in 2009-10. When the home government buys 10% of the overall supply of a risky asset with excess return of 2% then price of these assets change by 0.1% if policy is short-lived and by 0.2% in case of the permanent policy.

Our second main result characterizes the behavior of forecast errors of assets prices following the policy interventions. We show that individual and average across agents forecast errors are related to policy interventions. This result helps differentiate the mechanism proposed in this paper from the models that assume limited market participation but that retain rational expectations. Moreover, we argue that in models with limited information the predictability of forecast errors arises from misinformation about policy interventions. Once the effect of misinformation is corrected for, the forecast errors should not be predictable. In contrast, with level- $k$  thinking systematic forecast error arise even when agents are perfectly aware of the policy interventions.

**Related literature.** Our paper is related to several strands of literature. First and foremost, we contribute to the theoretical literature that proposes mechanisms of the effects of the balance sheet policies. An important starting point is the [Wallace \(1981\)](#) irrelevance result, which we mentioned earlier. [Backus and Kehoe \(1989\)](#) reach an even stronger conclusion than the Wallace irrelevance result. They show that, following an intervention in foreign exchange markets, the portfolio of the government has exactly the same state-contingent payoffs as the original one, therefore, future taxes are not affected by the policy. To deviate from the irrelevance result, the literature augmented the Wallace's model with various frictions. The two main frictions are incomplete information and market segmentation. The former friction generates the so-called "signaling" channel and the latter generates the "portfolio balance" channel of the balance sheet policies.

According to the signaling channel view, changes in the composition of central bank's balance sheet does not have a direct effect on the economy but rather serve as a signal of the central bank private information about its objectives and economic fundamentals. [Mussa \(1981\)](#), [Bhattacharya and Weller \(1997\)](#), [Popper and Montgomery \(2001\)](#), [Vitale](#)

(1999, 2003) applied this idea to FX interventions. Some authors considered a situation when the central bank cannot commit to a desired future monetary policy and use the costly balance sheet policy as a signal about future intentions. See [Jeanne and Svensson \(2007\)](#) and [Bhattarai \*et al.\* \(2015\)](#) for FX and QE interventions respectively.

The portfolio-balance channel posits that changes in the supplies of different assets available to private sector affect their prices perhaps due to imperfect assets substitutability related to differences in risk or market segmentation due to fixed costs of entry. [Kouri \(1976\)](#) and more recently [Gabaix and Maggiori \(2015\)](#) apply this idea to FX interventions, while [Vayanos and Vila \(2009\)](#), [Curdia and Woodford \(2011\)](#), [Chen \*et al.\* \(2012\)](#), [Hamilton and Wu \(2012\)](#) apply this idea to quantitative easing, and [Krishnamurthy and Vissing-Jorgensen \(2011\)](#) summarizes the recent literature. [Cavallino \(2017\)](#) and [Fanelli and Straub \(2016\)](#) solve for optimal exchange rate policies when markets are segmented, and [Amador \*et al.\* \(2017\)](#) consider the restrictions on these policies imposed by the zero lower bound. [Reis \(2017\)](#) proposes that quantitative easing is a powerful stabilization tool in times of fiscal crisis.

Our paper is also related to the literature that studies the consequences of deviations from equilibrium belief formations. [Woodford \(2013\)](#); [García-Schmidt and Woodford \(2015\)](#); [Farhi and Werning \(2016\)](#); [Gabaix \(2016\)](#) are recent examples of the applications of non-rational expectations formation to macro models.

The rest of the paper is organized as follows. Section 2 presents the model and introduces equilibrium concepts. Section 3 analyzes the effects of balance sheet policies. Section 4 discusses the testable implications. Section 5 concludes.

## 2 A Model of the World Economy

We now present our baseline model, which we use to investigate the effects of domestic (e.g., quantitative easing) and international (e.g., foreign exchange interventions) balance-sheet policies. The model features a nominal friction in the form of demand for money and an international dimension so that we can analyze both closed- and international-economy interventions involving nominal variables in a single environment.

We build on [Wallace \(1981\)](#) by adding an international dimension following the open-economy models of [Jeanne and Rose \(2002\)](#) and [Bacchetta and Van Wincoop \(2006\)](#).

### 2.1 Agents, Assets, and Expectations

**Countries.** There are two countries: home and foreign. Foreign country variables will bear an asterisk. Both countries produce the same good, which is traded freely across

borders. As a result, the law of one price applies and we have  $P_t = E_t P_t^*$ , where  $P_t$  and  $P_t^*$  are the nominal price levels in the home and foreign countries, respectively, and  $E_t$  is the nominal exchange rate. The exchange rate is defined as the quantity of home currency bought by one unit of foreign currency. Consequently, an increase in  $E_t$  corresponds to depreciation of home currency. For convenience, we let  $e_t \equiv \log E_t$  and  $p_t \equiv \log P_t$ .

**Time.** Time is discrete, infinite, and indexed by  $t = 0, 1, 2, \dots$

**Assets.** There are several assets in the world. Households in the home country can hold money issued by their own country,<sup>6</sup> nominal bonds issued by both countries—which pay, respectively, interest rates  $i_t$  and  $i_t^*$ —a riskless technology asset, available in perfectly elastic supply, that pays off a real return  $r \equiv \log R$ , and a risky asset that pays off a return  $r_{t+1}^x$  in units of consumption good in the following period. The total supply of the risky asset is  $\bar{X}$  and they trade at price  $q_t$ . Similarly, households in the foreign country can hold money issued by their own country, nominal bonds issued by both countries, and the riskless technology asset. The assumption that foreign-country households cannot invest in this risky technology is made to simplify the analysis, but has no substantial consequence for the results.<sup>7</sup>

**Risk.** There are three sources of risk in the economy. First, returns on the risky asset satisfy  $r_t^x = r^x + \epsilon_t^x$ . Second, both home and foreign-country money supplies follow stochastic processes given by  $\log M_{t+1} = \log \bar{M} + \epsilon_t^h$  and  $\log M_{t+1}^* = \log \bar{M} + \epsilon_t^f$ . The disturbances  $\epsilon_t^x$ ,  $\epsilon_t^h$ , and  $\epsilon_t^f$  are assumed to be independent from each other, iid over time, and Normally distributed with mean 0 and standard deviation  $\sigma_x$ ,  $\sigma_h$ , and  $\sigma_f$ , respectively.

**Households.** We explicitly consider home-country households, foreign-country households are symmetric. Households live for two periods. In the first period of their life, they receive real endowment  $w$  and can buy four types of assets: home currency, home and foreign nominal bonds, and the riskless asset. In the second period of their lives, they get a return on their portfolio, pay taxes, and consume. Each period there is a mass of  $\omega$  of “young” households and a mass of  $\omega$  of “old” households (the size of the foreign country is  $1 - \omega$ ).

Households need to form expectations about future variables, both exogenous and endogenous ones. We describe expectations in detail below, for now we use a tilde on the

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<sup>6</sup>We follow the international economics literature and make the simplifying and realistic assumption that households in a country can only hold the money of the country they live in.

<sup>7</sup>Note that this assumption does not explicitly prohibit the government of the foreign country to invest in risky asset.



expectation operator to emphasize the fact that, when making their choices, households may use a probability distribution over future variables that is not necessarily consistent with equilibrium outcomes. Specifically, given beliefs and prices, households choose consumption, investment in the safe and risky technology, real money holdings, and saving in home and foreign bonds, so as to solve the following problem in period  $t$ :

$$\max_{\substack{b_{t+1}, x_{t+1}, b_{H,t+1}, b_{F,t+1}, \\ m_{H,t+1}, c_{t+1}}} -\frac{1}{\gamma} \tilde{\mathbb{E}}_t \exp \left[ -\gamma \left( c_{t+1} - \frac{m_{t+1} [\log (m_{t+1} / \bar{m}) - 1]}{v} \right) \right]$$

subject to the current budget constraint

$$P_t (s_{t+1} + q_t x_{t+1} + b_{H,t+1} + m_{t+1}) + E_t P_t^* b_{F,t+1} \leq P_t w, \quad (1)$$

and the future budget constraint

$$\begin{aligned} P_{t+1} (c_{t+1} + T_{t+1}) \leq & P_{t+1} [e^r s_{t+1} + (r_{t+1}^x + q_{t+1}) x_{t+1}] , \\ & + P_t \left( e^{it} b_{H,t+1} + m_{t+1} \right) + E_{t+1} P_t^* e^{it} b_{F,t+1} \end{aligned} \quad (2)$$

where  $s_{t+1}$  is investment in the riskless technology,  $x_{t+1}$  is investment in the risky technology,  $b_{H,t+1}$  and  $b_{F,t+1}$  are purchases of home and foreign bonds, respectively, and  $m_{t+1}$  denotes a choice of period- $t$  home real money balances expressed in units of period  $t$  consumption. All variables are expressed in units of domestic consumption good and converted into units of domestic money.

Preferences are assumed to depend also on real home money balances. Money in the utility function is a standard approach to introduce demand for money in macroeconomics and international finance. In addition, the particular functional form assumed here simplifies the analysis by making money demand independent of the consumption choice. Note that utility is increasing in  $m_{t+1}$  for  $m_{t+1} \leq \bar{m}$  and decreasing for  $m_{t+1} > \bar{m}$ . We thus assume that  $m_{t+1} \leq \bar{m}$ .

It is worth explaining our modelling choice of the overlapping generations framework. In such an environment, unborn generations are excluded from participating in current asset markets, thus, government asset purchases can affect the economy even under rational expectations. The overlapping generations model is therefore an example of models with limited participation where the Wallace irrelevance result does not apply. In this sense, bounded rationality—the focus on this paper—is not necessary to make asset purchases effective.

There are two main reasons that lead us to choose this particular environment. First, maximization of the CARA preferences with Gaussian shocks is equivalent to maximiza-



tion of the mean-variance preferences. This property makes the analysis extremely tractable when incorporating the effects of uncertainty. In fact, such an environment has been a workhorse model in finance literature starting from seminal contribution by [De Long et al. \(1990\)](#). Second, we can obtain the Wallace irrelevance result even in this environment. In fact, [Wallace \(1981\)](#) also uses a two-period overlapping generations model to derive his irrelevance result. To derive the irrelevance result, we assume that agents are only taxed when they are old, as can be seen from budget constraints (1) and (2). In addition, the government imposes its gains or losses from the portfolio choice in period  $t$  on the households in period  $t + 1$  (see the formal assumption on the government behavior below). Because of these two assumptions, the agents who actively participate on the risky assets market are those agents who will be exposed to future taxation risk. With similar assumptions on fiscal variables, limited participation models and, in particular, OLG models reproduce the irrelevance result. Once we guarantee that in the baseline model asset purchases are irrelevant when expectations are rational, we can focus on the effects of deviating from such expectations while, at the same time, keep the tractability of the OLG framework.

**Expectations.** We introduce bounded rationality by letting households hold beliefs that may in principle differ from what equilibrium reasoning implies. More precisely, consistently with the idea that households understand policy announcements but may be unable to solve for the equilibrium of the economy, we assume that household beliefs of future *exogenous* variables coincide with their true distributions—there are three such variables:  $M_{t+1}$ ,  $M_{t+1}^*$ , and  $r_{t+1}^x$ . On the contrary, we allow household beliefs of *endogenous* variables at time  $t \geq 1$  to differ from the distributions implied in equilibrium. More precisely, to solve their problem, households need to forecast the future realizations of the vector of endogenous variables  $(q_t, p_t, i_t, T_t, p_t^*, i_t^*, T_t^*)$ , which we denote with  $Z_t$ . We assume that, conditional on information at time  $t$ , households believe that each component of  $Z_{t+s}$  is a linear function of the exogenous shocks at time  $t + s$ , i.e.,  $(\epsilon_{t+s}^x, \epsilon_{t+s}^h, \epsilon_{t+s}^f)$ . Formally, we assume that household beliefs are described by a Gaussian distribution  $\tilde{F}$ , which we parametrize by assuming that each component of  $Z_t$  satisfies

$$\tilde{Z}_{i,t} = \alpha_{i,t} + \beta_{i,t}^x \epsilon_t^x + \beta_{i,t}^h \epsilon_t^h + \beta_{i,t}^f \epsilon_t^f, \quad (3)$$

for some scalars  $\alpha_{i,t}$ ,  $\beta_{i,t}^x$ ,  $\beta_{i,t}^h$ , and  $\beta_{i,t}^f$ .<sup>8</sup> Here, the  $\alpha_{i,t}$  represents the time- $t$  expected average of  $Z_{i,t}$ , while the  $\beta$ s capture the expected sensitivity of  $Z_{i,t}$  to the aggregate shocks.

The assumption about  $\tilde{F}$  may seem restrictive and, in fact, we could in principle solve

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<sup>8</sup>See Section III-A in [Evans and Ramey \(1992\)](#) for a similar treatment of expectations formation.

for the temporary equilibrium (see the definition below) of the economy with any  $\tilde{F}$ . With our preferences, however, it is very convenient to have beliefs that are normally distributed. What is more, for all other equilibrium concepts we will be considering—rational expectations, level- $k$ , and reflective—it turns out that beliefs (3) are rich enough, that is, even if we started from a general  $\tilde{F}$  we would obtain the same equilibrium. For example, expectations in the rational expectations equilibrium benchmark take the form of (3). In fact, depending on the equilibrium concept considered, many of the coefficients in (3) will be zero.

An important assumption behind (3) is that expectations of endogenous variables do not depend on the time at which they are made. This assumption would be violated if instead agents were allowed to update their beliefs over time, for example, through a learning process. To isolate the effects of bounded rationality from other mechanisms, we begin by abstracting from any process of belief updating.

Finally, we define inflation expectations at time  $t$  as follows:

$$\tilde{\pi}_{t,t+s} = \begin{cases} \tilde{p}_{t+s} - \tilde{p}_{t+s-1}, & \text{for } s \geq 2, \\ \tilde{p}_{t+1} - p_t, & \text{for } s = 1. \end{cases}$$

Note that inflation expectations are conditional on the time at which they are made. We finally impose that the law of one price holds in expectations as well, i.e.,

$$\tilde{\pi}_{t,t+s} = \tilde{\pi}_{t,t+s}^* + \tilde{e}_{t+s} - \tilde{e}_{t+s-1}.$$

This relation is necessary for an equilibrium to exist: If it was violated, there would be arbitrage opportunities between holding money in the two countries.

**Government.** We explicitly specify the behavior of government in both countries. We use capital letters to denote government choices. The government of the home country controls real taxes  $\{T_t\}$ , nominal money supply  $\{M_t\}$ , the real amount of private risky assets  $\{X_t\}$ , the *real* amount of home currency nominal bonds  $\{B_t^h\}$ , and the *real* amount of foreign-bond purchases  $\{B_t^f\}$ . We let  $\Pi_t = \{T_t, M_t, X_t, B_t^h, B_t^f\}$ . At time 0 the government announces the process for money supply and balance-sheet policies  $\{M_t, X_t, B_t^h, B_t^f\}$ , which becomes common knowledge in both countries. The consolidated government budget constraint is

$$\begin{aligned} P_t q_t X_{t+1} + P_t B_{t+1}^h + E_t P_t^* B_{t+1}^f + M_t \\ = P_t (r_t^x + q_t) X_t + e^{it-1} P_{t-1} B_t^h + E_t e^{it-1} P_{t-1}^* B_t^f + \omega P_t T_t + M_{t+1}. \end{aligned} \quad (4)$$

The left-hand side represents government's nominal outlays, consisting of purchases of home-country risky assets  $P_t q_t X_{t+1}$ , purchases of home-country nominal bonds  $P_t B_{t+1}^h$ , purchases of foreign-country nominal bonds  $E_t P_t^* B_{t+1}^f$ , and repayment of money liabilities. The right-hand side is government income.

We assume that only the home-country government conducts balance-sheet policies. More specifically, for simplicity we assume that the foreign government sets money supply and taxes so as to keep a constant level of real bonds. Formally, the foreign government chooses  $\Pi_t^* = \{T_t^*, M_t^*, B^*\}$ , where  $B^*$  is the constant level of real foreign-country bonds, that satisfy the budget constraint

$$e^{i_{t-1}^*} P_{t-1}^* B^* + M_t^* = P_t^* B^* + \nu P_t^* T_t^* + M_{t+1}^*. \quad (5)$$

We call QE a policy of risky-asset purchases financed with the issuance of home-country bonds. Similarly, a policy of foreign-bond purchases financed with the issuance of home-country bonds will be referred to as FX intervention.

## 2.2 Equilibrium Concepts

**Temporary Equilibrium.** Our goal is to investigate the equilibrium implications of letting agents' expectations deviate from standard rational expectations. In standard rational expectations equilibrium (REE), expectations are required to be consistent with equilibrium objects. This requirement is obviously too restrictive for our purpose, therefore, we adopt a more general notion of equilibrium known as temporary equilibrium (TE). Intuitively, such concept generalizes the standard REE insofar as it does not restrict beliefs about endogenous variables, which are free to deviate from equilibrium outcomes. More specifically, a TE takes as given household beliefs about future endogenous variables and requires only that (i) households optimize given these beliefs and that (ii) markets clear in every period. Importantly, a TE does not require beliefs to be *consistent* with equilibrium predictions.

**Definition** (Temporary Equilibrium). Given beliefs that satisfy (3), a temporary equilibrium is a collection of household choices  $\{c_t, x_{t+1}, b_{H,t+1}, b_{F,t+1}\}$  and  $\{c_t^*, b_{H,t+1}^*, b_{F,t+1}^*\}$ , government policies  $\{\Pi_t, \Pi_t^*\}$ , and prices  $\{q_t, i_t, p_t, i_t^*, p_t^*, e_t\}$  such that

1. Given beliefs and prices, households choose consumption and portfolio optimally;
2. Money and bond markets clear;
3. Government budget constraints (4) and (5) are satisfied for all  $t$ .

**Rational Expectations Equilibrium.** A special case of TE is the REE, which adds the extra requirement that expectations must be consistent with equilibrium variables.

**Definition** (Rational Expectations Equilibrium). A REE is a TE which satisfies

$$\tilde{Z}_t = Z_t, \text{ for all } t.$$

**Level- $k$  Equilibrium.** We now consider an alternative process of belief formation. This process will capture the idea that, when confronted with a new policy in a complex environment, households might not have access to all the necessary information or they might be incapable to assess the implications of such policy. What is more, even assuming that some households might be able to fully understand the effects of new policies, such households might nonetheless believe that other agents are incapable of doing so.

More specifically, the behavior of agents will likely depend on whether they have to predict exogenous or endogenous variables. The latter, in fact, result from the interaction of many different agents and, thus, require complex equilibrium considerations. Rational agents need not only to understand how markets aggregate individual actions, but also how other agents behave. What is more, even if all agents understood how the economy works, to upset the predictions of a REE, it would be enough that a fraction of them *believes* that other agents will not behave rationally. Clearly, in a REE, consistency of beliefs forbids agents to hold incorrect beliefs about the behavior of other agents. In our model, instead, agents are allowed to hold incorrect beliefs and, as a consequence, they will make mistakes that could have been predicted and avoided under rational expectations.

The alternative beliefs introduced in this section are known in the literature as level- $k$  beliefs, where  $k$  denotes the level of sophistication of an agent (which we define formally below). Intuitively, agents are assumed to have the correct model of the economy, but they hold incorrect beliefs about the sophistication of other agents. Suppose, for example, a new policy is announced. Agents will correctly understand the policy, but they will incorrectly assume that other agents will fail to respond to the new policy as prescribed by the REE. Since endogenous variables are the sum of individual actions, level- $k$  agents will incorrectly predict such variables.

By definition, we can think of a TE as a mapping from sequences of beliefs and policies into equilibrium outcomes, which we can conveniently represent as

$$\{Z_t\} = \Psi(\{\tilde{Z}_t\}, \{X_{t+s}, B_{t+s}^h, B_{t+s}^f\}), \quad (6)$$

for all  $t$ . Moreover, by definition REE sequences  $\{Z_t^{REE}\}$  are a fixed point of (6).

We start with  $k = 1$ , the lowest level of sophistication. Level-1 agents are assumed to

believe that all other agents do not respond to policy announcements. Following a policy announcement, therefore, they will hold incorrect beliefs that asset prices and future taxes will coincide with those in the REE *before* government intervention. Formally, we assume households have beliefs  $\{\tilde{Z}_t^1\} = \{Z_t^{REE}\}$ .<sup>9</sup> We define a level-1 equilibrium as a TE where agents' beliefs are given by  $\{\tilde{Z}_t^1\}$ .

Starting from level-1 agents, we define level- $k$  agents and level- $k$  equilibria recursively. Let  $\{\tilde{Z}_t^k\}$  be the beliefs of a level- $k$  agent,  $k \geq 1$ , and define a level- $k$  equilibrium accordingly. From (6), we obtain equilibrium asset prices and taxes in a level- $k$  equilibrium. We assume now that these equilibrium outcomes coincide with the beliefs of level- $k + 1$  agents. We can therefore summarize the entire process of belief formation with the following recursion:

$$\{\tilde{Z}_t^{k+1}\} = \Psi(\{\tilde{Z}_t^k\}, \{X_{t+s}, B_{t+s}^h, B_{t+s}^f\}), \quad (7)$$

for all  $k$  and  $t$ .

**Definition (Level- $k$  Equilibrium).** Given REE sequences  $\{Z_t^{REE}\}$  and sequences of government purchases  $\{X_{t+s}, B_{t+s}^h, B_{t+s}^f\}$ , a level- $k$  equilibrium is a TE where beliefs are obtained recursively from (7) with initial condition  $\{\tilde{Z}_t^1\} = \{Z_t^{REE}\}$ .

**Reflective Equilibrium.** All the equilibrium concepts so far assumed that agents in the economy were homogenous, in particular, they shared the same beliefs. We now consider an economy populated by households who are heterogeneous in their beliefs. In particular, suppose we can partition the population in different groups depending on household beliefs. In particular, each group contains households with the same level  $k$  and has mass given by the probability density function  $f(k)$ .

**Definition (Reflective Equilibrium).** Given beliefs  $\{Z_t^k\}_{t,k}$ , with  $\{Z_t^1\} = \{Z_t^{REE}\}$ , a Reflective Equilibrium (RE) is a collection of household choices  $\{c_t^k, x_{t+1}^k, b_{H,t+1}^k, b_{F,t+1}^k\}_{t,k}$  and  $\{c_t^{*,k}, b_{H,t+1}^{*,k}, b_{F,t+1}^{*,k}\}$ , government policies  $\{\Pi_t, \Pi_t^*\}$ , and prices  $\{q_t, i_t, p_t, i_t^*, p_t^*, e_t\}$  such that

1. Given beliefs and prices, households choose consumption and portfolio optimally;
2. Markets for money holdings, home-country bond, and foreign-country bond clear;
3. Government budget constraints (4) and (5) are satisfied for all  $t$ ;
4. Beliefs are generated recursively by (7).

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<sup>9</sup>Note that we introduced an additional superscript to denote "level-1" beliefs.

### 3 Equilibrium Effects of Balance Sheet Policies

**Household behavior.** We begin with the household problem and then derive the TE for general sequences of balance-sheet policies.

Let  $\mathcal{R}_{t+1} = (r_{t+1}^x + q_{t+1} - Rq_t, i_t - \pi_{t+1} - r, i_t^* - \pi_{t+1}^* - r)$  be the vector of realized excess returns for the risky asset and the two bonds. Let  $\tilde{\Sigma}_t$  denote the variance matrix of  $\mathcal{R}_{t+1}$  under the distribution  $\tilde{F}$  parametrized by (3) and *conditional* on time- $t$  information. Similarly, we will use a tilde on a moment computed using the distribution  $\tilde{F}$ . We make two simplifying assumptions. First, we combine the budget constraints (1) and (2) into a single intertemporal budget constraint, we take a first-order approximation, and treat the resulting budget constraint as exact:<sup>10</sup>

$$c_{t+1} = (1 + r)w + \mathcal{R}_{t+1} \cdot (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - (r + \pi_{t+1})m_{t+1} - T_{t+1},$$

where the prime next denotes the transpose.

Secondly, we focus on the “cashless” limit as money holdings in the economy vanish. The cashless limit is a standard assumption employed in, for example, New Keynesian Open Economy literature to eliminate the real effects of nominal money supply above and beyond its effect on inflation and nominal interest rate. We can thus abstract from money holdings when computing the TE.

**Definition** (Cashless limit). A cashless limit of the economy described above is a limit of TE such that  $\bar{m} \rightarrow 0$ ,  $\bar{M} \rightarrow 0$ , and  $\bar{m}/\bar{M} \rightarrow 1$ .

When preferences are CARA and disturbances are normally distributed, the household problem becomes a standard mean-variance portfolio problem, which has a simple closed-form solution.

**Lemma 1.** *Given beliefs (3), in the cashless limit, optimality implies that household asset choices satisfy*

$$\begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \end{pmatrix} = \frac{1}{\gamma} \Sigma_t^{-1} \tilde{\mathbb{E}}_t [\mathcal{R}'_{t+1}] + \Sigma_t^{-1} \tilde{Cov}_t (\mathcal{R}'_{t+1}, T_{t+1}), \quad (8)$$

with analogous equations for  $(b_{H,t+1}^*, b_{F,t+1}^*)$ .

Equation (8) is the standard asset demand equation in standard mean-variance portfolio analysis. The first term captures the excess return of each asset divided by the coef-

<sup>10</sup>Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006) make the same simplifying assumption. Alternatively, we could assume that households hold the mean-variance preferences instead of exponential. This would allow us to avoid linearization because one can express mean and variance of log-normally distributed random variables in closed forms.

efficient of absolute risk aversion  $\gamma$  and by the variance matrix of returns. The second term captures a hedging motive coming from the fact that, when the government conducts balance-sheet policies, households expect future taxes to correlate with asset returns. For example, suppose agents expect future taxes to be positively related to the return on the risky asset, that is,  $\widetilde{Cov}_t(r_{t+1}^x + q_{t+1} - (1+r)q_t, T_{t+1}) > 0$ . The risky asset is then a good hedge against future tax risk, which contributes towards a higher demand for such asset. The same mechanism works for the other assets.

A convenient feature of the equations in Lemma 1 is that we can impose market-clearing and, thus, solve for the asset price, interest rates, and price levels, without solving for the optimal choice of consumption and investment in the safe technology. Asset demands (8) suggest that, by affecting beliefs of future taxes, balance-sheet policies have the potential to influence investor demands. In addition, when asset demands change, prices must respond to clear markets, thus, there is a further effect on household choices.

### 3.1 Neutrality under Rational Expectations

The next proposition, however, shows that there is an important benchmark where balance-sheet policies are completely irrelevant.

**Proposition 1.** *For any sequences of balance-sheet policies  $\{X_t, B_t^h, B_t^f\}$ , there is a unique REE. Furthermore, in the REE the asset price is constant:*

$$q_t = q^{REE} \equiv \frac{1}{R-1} \left( r^x - \frac{1}{\omega} \gamma \sigma_x^2 \bar{X} \right);$$

*the home-country nominal interest rate and price level are linear functions of the home-country monetary shock:*

$$\begin{aligned} i_t^{REE} &= r - \frac{1}{1+v} \epsilon_t^h, \\ p_t^{REE} &= vr + \frac{1}{1+v} \epsilon_t^h; \end{aligned}$$

*with analogous expressions for the foreign country. Finally, the exchange rate satisfies*

$$e_t^{REE} = \frac{1}{1+v} (\epsilon_t^h - \epsilon_t^f).$$

*In particular, balance-sheet policies are irrelevant.*

Proposition 1 states that, when agents anticipate future taxes correctly, government intervention is irrelevant. This is the celebrated result that in an economy where “Ricardian



equivalence” holds asset purchases by the government—or, equivalently, by the central bank—are irrelevant. The reason is that, while assets are apparently removed from households’ budgets, they are still present indirectly through future taxes. In a REE, households correctly anticipate that future taxes will depend on government purchases and react by adjusting their demand for risky assets. In the end, equilibrium allocations and prices are unaffected. Crucially, this reasoning requires households to hold correct expectations about future endogenous variables such as taxes.

Interestingly, all the variables in Proposition 1 take a particularly simple form. First, the risky-asset price equals the average return minus a term capturing the risk premium required by the risk-averse investors. Secondly, interest rates are given by the risk-free real interest rate plus a term that reflects expected inflation. To see this, suppose  $\epsilon_t$  is positive, that is, money supply in period  $t$  is unexpectedly high. From Proposition 1, the current price level increases while the future price level does not change on average. Agents, therefore, expect lower inflation, which reduces the nominal interest rate.

### 3.2 Non-neutrality under Level- $k$ Thinking

We now depart from rational expectations and assume that, following an announcement of intervention, households in both countries form expectations following the level- $k$  process described in Section 2.2. As a starting point, we further assume that, before the announcement, the world economy is in its REE, thus, households in both countries correctly forecast the behavior of future taxes in each country. In particular, households correctly understand that the home-country government has zero holdings of government bonds, thus, future taxes are expected to be zero. In addition, households understand that the foreign-country government has a real value  $B^*$  of outstanding bonds, thus, future taxes in the foreign country are expected to be a function of the future foreign money supply. One can justify such an assumption, for example, through learning: If governments kept their outstanding supply of debt at constant (real) levels for a long enough time, households would have time to learn the stochastic process of taxes. We refer to the REE *before* the foreign government intervention as the “status quo” and we use the superscript “*REE*” to denote status-quo variables. By assumption, the status quo is the REE in the absence of asset purchases, that is, status-quo beliefs  $\tilde{Z}_t^{REE}$  are a fixed point of (6) with  $X_t = B_t^h = B_t^f = 0$ , for all  $t$ .

We begin with the TE under the assumption that households are level-1, that is, we assume that households do not change their status-quo beliefs after new balance-sheet policies are announced. From (6), we then obtain the equilibrium variables  $Z_t$  that are compatible with level-1 beliefs and the new policy. In turn, these values coincide with the

beliefs of level-2 agents. In general, using (7), we can obtain the beliefs and the equilibrium outcomes in any level- $k$  equilibrium recursively.

As an example, it is instructive to consider the recursion for the risky-asset price in a level- $k$  equilibrium:

$$q_t^k = \frac{R-1}{R}q^{REE} + \frac{1}{R}q_{t+1}^{k-1}. \quad (9)$$

Equation (9) can be iterated forward until we reach level-1 equilibrium:

$$q_t^1 = q^{REE} + \frac{1}{R} \frac{\gamma \sigma_x^2 X_{t+1}}{\omega};$$

The iterative procedure implied by (9) is depicted in Figure 1, where the horizontal axis represents time and the vertical axis plots the level  $k$ . Every bold dot is the equilibrium price  $q_t^k$  of a particular level- $k$  equilibrium. This figure visually shows that if one wants to compute, for example, the asset price at time 0 of a level-5 equilibrium by iterating equation (9) forward, one has to move diagonally and compute the asset price at time-1 in a level-4 equilibrium,  $q_1^4$ , the asset price at time-2 in a level-3 equilibrium,  $q_2^3$ , and so on.

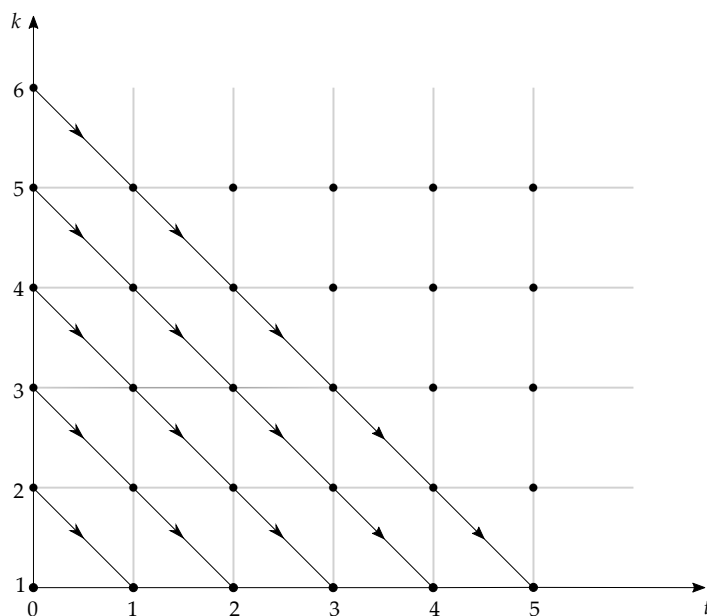


Figure 1: Level- $k$  price equilibrium solution.

Importantly, these iterations always stop when level-1 equilibrium is reached because level-1 agents do not change their beliefs when the new policy is announced. The reason is that, households in a level-1 equilibrium do not understand the connection between future taxes and government purchases, thus, they do not react by varying their demand for the risky asset, as it was the case in the REE. In turn, since level-1 households do not

change their beliefs, government purchases of risky assets affect the asset price in a level-1 equilibrium and, through (9), in any  $k$ -level equilibrium with  $k > 1$ . What is more, if we start from a higher  $k$ , equation (9) shows that the weight on  $q^{REE}$  increases, thus, asset purchases become less effective when sophistication increases. In the example of a level-5 equilibrium, it is as if agents were fully discounting any changes happening after period 4.

Having characterized beliefs for any level of sophistication, we turn to our last equilibrium concept, the RE, which allows heterogeneous agents to coexist in the economy. We specify the following pdf for the distribution of level- $k$  agents in the economy:

$$f(k) = \begin{cases} 1 - \lambda_1, & k = 1, \\ \lambda_1 (1 - \lambda) \lambda^{k-2}, & k \geq 2, \end{cases}$$

where  $\lambda, \lambda_1 \in [0, 1]$ . This distribution is exponential starting from  $k = 2$ , but the mass of level-1 agents is allowed to be an arbitrary fraction  $1 - \lambda_1$ . When  $\lambda_1 = \lambda$ , the overall distribution becomes exponential. In this case, when  $\lambda$  tends to zero, the distribution puts more weight on lower levels of  $k$ , that is, the economy becomes less “sophisticated.” On the contrary, when  $\lambda$  approaches 1,  $f(k)$  puts more weight on higher levels of  $k$  and the economy becomes increasingly more “sophisticated.” We assume this flexible specification to disentangle the effect of level-1 beliefs in the belief formation process from the presence of actual level-1 agents in the economy. For example, if  $\lambda_1 = 1$  there are no level-1 agents in the economy, yet the beliefs of level- $k$  households, with  $k \geq 2$ , are formed using level-1 beliefs as a starting point.

The following proposition contains the main result of this section.

**Proposition 2.** *For sequences of balance-sheet policies  $\{X_t, B_t^h, B_t^f\}$ , in the RE the asset price is*

$$q_t = q^{REE} + \frac{\gamma \sigma_x^2}{\omega} \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k};$$

*the home-country nominal interest rate and price level are*

$$i_t = i_t^{REE} + \gamma \sigma_h^2 \frac{1}{v} \left( \frac{1}{1+v} \right)^2 \sum_{k=1}^{\infty} f(k) \left( \frac{v}{1+v} \right)^k \left( -B_{t+k}^h \right),$$

$$p_t = p_t^{REE} + \gamma \sigma_h^2 \left( \frac{1}{1+v} \right)^2 \sum_{k=1}^{\infty} f(k) \left( \frac{v}{1+v} \right)^k \left( -B_{t+k}^h \right);$$

with analogous expressions for the foreign country. Finally, the exchange rate satisfies

$$e_t = e_t^{REE} + \gamma \left( \frac{1}{1+v} \right)^2 \sum_{k=1}^{\infty} f(k) \left( \frac{v}{1+v} \right)^k \left( \sigma_f^2 B_{t+k}^f - \sigma_h^2 B_{t+k}^h \right).$$

In particular, equilibrium variables depend on balance-sheet policies.

Proposition 2 shows that, in a RE, balance sheet-policies are effective tools in controlling asset prices. In particular, by purchasing risky assets, the government—or, equivalently, the central bank—can increase the price of this type of asset. Moreover, since the risk-free real return is fixed by the storage technology, we can relate the inverse of the risk-asset price to the risk-premium required by investors. Asset purchases, therefore, affect asset prices by lowering the equilibrium risk-premium. Similarly, Proposition 2 states that the nominal interest rate and the price level—and, therefore, the exchange rate—are now functions of the entire path of nominal-bond purchases or issuances (remember that an issuance is a negative value for  $B_t^h$ ). Take for example, the nominal bond in the home country. Suppose, for simplicity, that the home-country government issues some of these bonds at time  $t$ , i.e.,  $B_{t+1}^h < 0$ . In a RE, households fail to anticipate that, since bonds promise a risk-free *nominal* payment, future taxes will now depend on the fluctuations of the future price level. They will thus fail to hedge the tax risk by increasing their demand for nominal bonds. To induce households to hold the additional bonds and clear the market, therefore, the interest rate and the current price level have to increase.

We now consider several special cases to highlight important properties of government interventions.

**Absence of level-1 households.** First, we assume that there are no level-1 agents in the economy. Formally, this assumption corresponds to  $f(1) = 0$  or, equivalently,  $\lambda_1 = 1$ . In this case, the infinite sums in Proposition 2 start from  $k = 2$ , implying that the current (i.e., period  $t$ ) government purchases do not affect the equilibrium prices. The other future purchases have the same effect as before. Note that it is nevertheless crucial that level- $k$  households, with  $k \geq 2$ , form their beliefs starting from level-1 agents, despite the fact that there are no level-1 households in the economy.

**Opposing effects of bounded rationality on effects of future interventions.** There are two effects of level- $k$  thinking on the strength of, for example, quantitative easing. First, in the current model, the further away the agents are from rational expectations (i.e., the lower is the average level- $k$  in the economy), the stronger is the direct effect of QE. Second, the further away the agents are from rational expectations the stronger they discount the

effects of future QE. Recall that level- $k$  households “discount” the future completely after  $k$  periods. They thus do not take into account any policy of QE occurring more than  $k$  periods ahead.

To illustrate this point formally, take the risky-asset price and suppose the government announces that it will purchase assets only in the next period. For concreteness assume that  $f(k)$  is exponential, i.e.,  $\lambda_1 = \lambda$ . The risky asset price today is

$$q_t = q^{REE} + \frac{\gamma\sigma_x^2}{\omega} \cdot \frac{(1-\lambda)\lambda}{R^2} X_{t+2}.$$

The nonlinear effect of  $\lambda$  on the price is clear from this formula. If  $\lambda < 1/2$ , then a small increase in the average sophistication of agents in the economy increases the effect of this policy. The strength of QE peaks at  $\lambda = 1/2$  and then declines to zero as  $\lambda$  approaches 1.

**Persistence.** Suppose that the path of government asset purchases follows an exponentially decaying process  $X_{t+1+s} = X_{t+1}\mu^s$ , for  $s \geq 0$ , and  $\mu \in [0, 1]$ . A one-time purchase in the current period is a special case of this process with  $\mu = 0$ . A permanent increase in asset purchases corresponds to  $\mu = 1$ . When the distribution of households  $f(k)$  is exponential, we have

$$q_t = q^{REE} + \frac{\gamma\sigma_x^2}{\omega} \cdot \frac{1-\lambda}{R-\lambda\mu} X_{t+1}.$$

There are two important observations. First, the higher the persistence of QE (higher  $\mu$ ), the higher the effect on the price. Second, in line with our discussion about sophistication and discounting of the future, there are still two effects of bounded rationality on the effectiveness of QE. However, in this example, the discounting effect is weaker and, as a result, a higher level of sophistication leads to lower prices, i.e.,  $dq_t/d\lambda < 0$ .

### 3.3 Quantitative Illustration

In this section, we present the results of several numerical exercises. Our aim is to illustrate the sensitivity of results to changes in various parameters. We stress that our numerical exercises is not a serious numerically calibrated policy evaluation.

**Quantitative easing.** Consider quantitative easing in home country. We posit that the government purchases of risky assets and issuance of riskless nominal bonds start from some initial values and decay exponentially. Specifically,  $\{X_t, B_t^h\}$  follows  $X_{t+1} = X_1\mu^t$  and  $B_{t+1}^h = B_1^h\mu^t$  for all  $t \geq 0$ , positive initial values  $X_1$  and  $B_1^h$ , and  $\mu \in [0, 1]$ . Note that it is not necessary that  $B_1^h$  and  $X_1$ , and their subsequent values, are related to each other

because the government can close any gap in financing through taxes. For example, if in period 0 the amount of issued government bonds is not enough to cover the purchases of private risky assets, the government taxes young households and closes the financial gap.

Using the risky asset price derived in Proposition 2, we can express the deviation of the price from its rational expectations equilibrium value as

$$\frac{q_t - q^{REE}}{q^{REE}} = \mathbb{E}_t(\mathcal{R}_{t+1}^x) \cdot \frac{X_{t+1}}{\bar{X}} \cdot \frac{1 - \lambda}{R - \lambda\mu}, \quad (10)$$

where  $\mathbb{E}_t(\mathcal{R}_{t+1}^x)$  is the expected excess return on risky asset in rational expectations equilibrium, which is given by

$$\mathbb{E}_t(\mathcal{R}_{t+1}^x) = \frac{r^x + q^{REE}}{q^{REE}} - R = \frac{\gamma\sigma_x^2\bar{X}}{\omega q^{REE}}.$$

We can quantify equation (10) as follows. We set the excess return on risky asset to 2%, which roughly corresponds to excess return on the Bloomberg Barclays U.S. MBS Index in the period of 2005-15. The government initial purchases of risky assets are 10% of the overall supply of the risky assets, i.e.,  $X_0/\bar{X} = 0.1$ . This amount roughly corresponds to \$1 trillion of mortgage-backed securities purchased by the Fed from January 2009 and June 2010 relative to overall value of the MBS market of about \$10 trillion in that period. Finally, we set the safe rate of return is  $R = 1.01$ .

The risky assets price in equation (10) is positively related to the persistence of the quantitative easing policy. We now present two polar cases—permanent and one-period long intervention—to assess the magnitude of the intervention. If the government purchases are permanent, i.e.,  $\mu = 1$ , the risky asset price is

$$\frac{q_t - q^{REE}}{q^{REE}} = 0.02 \cdot 0.1 \cdot \frac{1 - \lambda}{1.01 - \lambda} \approx 0.002 = 0.2\%.$$

Note that the magnitude is not sensitive to changes in the distribution of sophistication in the economy, governed by parameter  $\lambda$ , when  $\lambda$  is not very close to one. As a result, the magnitude of 0.2% obtains under the wide range of the average levels of sophistication of agents in the economy.

If the government intervention continues for only one period, i.e.,  $\mu = 0$ , then, according to price equation (10), the price effect is inversely proportional to the average effect of households sophistication  $\bar{k} \equiv 1/(1 - \lambda)$ . If, for example,  $\bar{k} = 2$ , i.e., a typical household thinks that other households do not change their expectations after the policy interven-

tion, the increase in the risky asset price is 0.1%. This result suggests that increase the duration of quantitative easing from one period to very long horizon increases the effect of the policy on current price by only a factor two. This is a small effect relative to, for example, an infinite sum of 0.1% increase in every period discounted by  $R$  that would yield an increase of 10%. The reason why the strength of QE does not explode with the persistence of the policy has to do with endogenous discounting under level- $k$  thinking that we discussed in Section 3.2.

**Decline in the supply of risky assets.** The overall effect of quantitative easing on the price of risky assets consists of two main forces: (i) limited understanding of the effect on future taxes, and (ii) limited understanding of the effect on future prices. The first effect makes the police relevant. The second effect limits the strength of the policy. In the analysis so far, both of the effects were present together. Next, we separate the tax and price effects. To do this, we consider a change in the supply of risky assets similar to that under QE, i.e.,  $X_t = X_0\mu^t$ , however, without any consequences for current and future taxes. One interpretation of this experiment is the purchase of risky assets by the government of foreign country.<sup>11</sup>

It is straightforward to show that the change in the price of risky assets must equal

$$\frac{q_t - q^{REE}}{q^{REE}} = \frac{\gamma\sigma_x^2}{\omega q^{REE}} \cdot \frac{1}{R - \lambda\mu} \cdot \frac{X_{t+1}}{\bar{X}}. \quad (11)$$

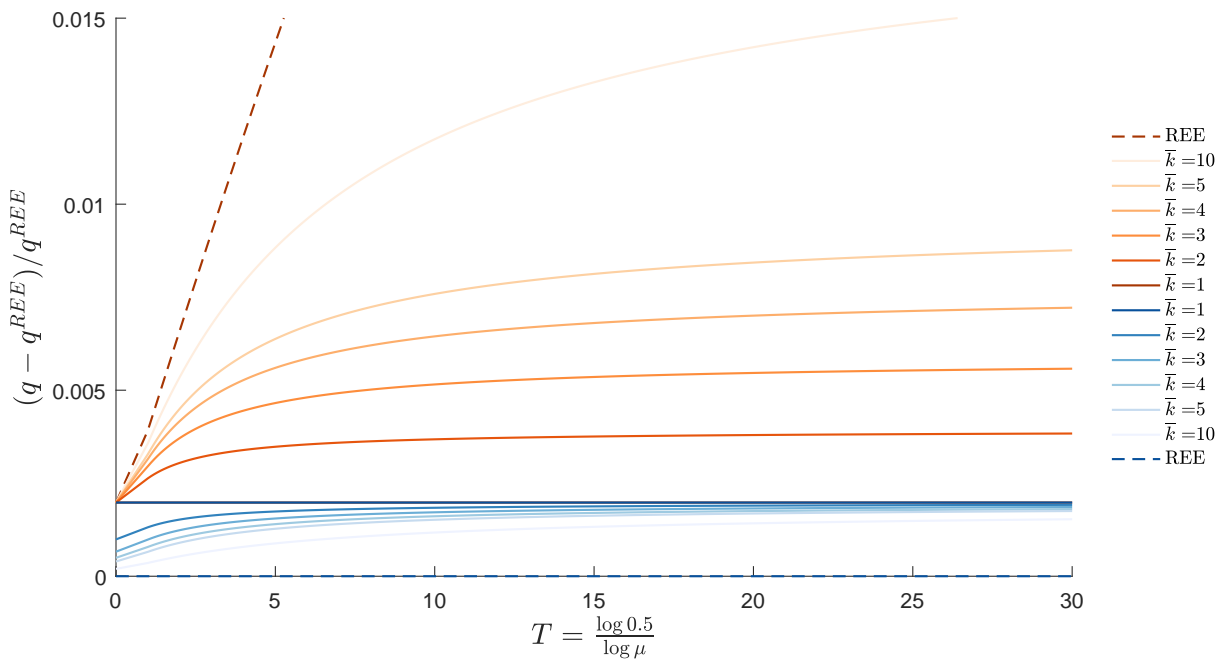
The comparison of (10) and (11) reveals that the two formulas only differ by the term  $(1 - \lambda)$ . As a result, the ratio of the price effect of quantitative easing over the price effect of a change in the supply of risky assets is constant and equals  $(1 - \lambda)$ .

We represent the price effects of the two policies in Figure 2. This figure plots the change in price of risky asset after a decline in net supply of risky assets and central bank assets purchases of same magnitude and duration as a function of the persistence of this purchase (measured in terms of half life). There are several things to note in this figure. First, when the average level of sophistication of agents in the economy equals 1 (all agents are level-1 thinkers), the effects of two policies coincide. This is represented by the horizontal line on the figure. Intuitively, when agents are level-1 thinkers they keep their expectations of future prices and taxes fixed after the changes in the economy. As a result, it does not matter if in reality the change in the net supply of risky assets is accompanied by future tax changes or not.

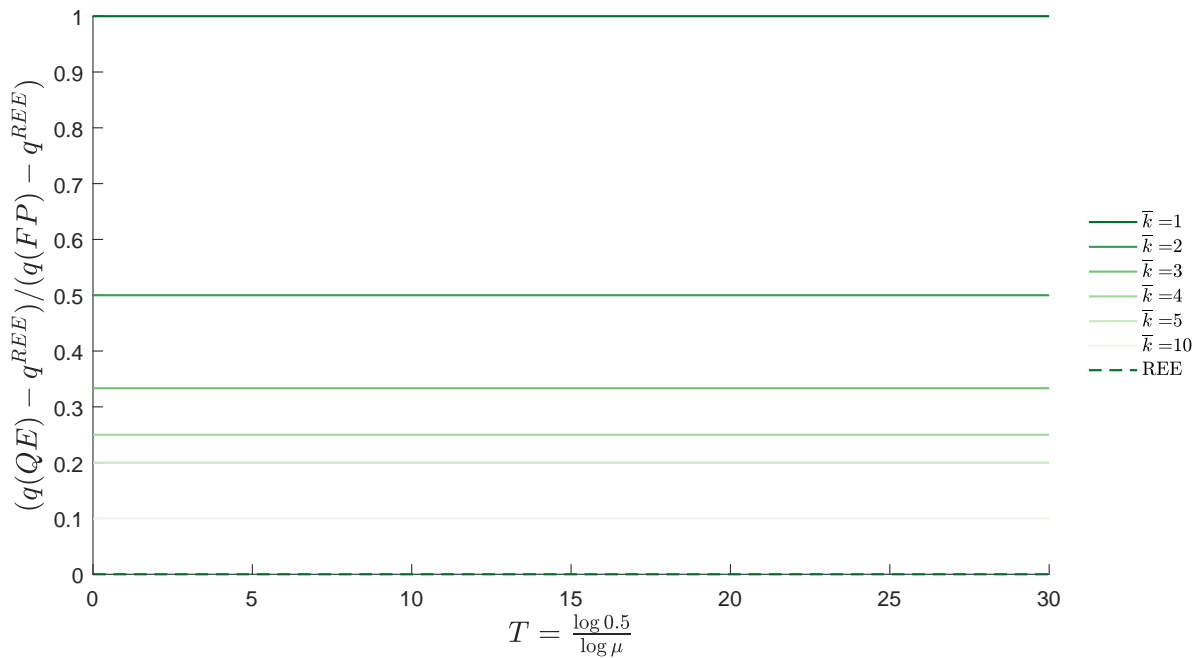
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<sup>11</sup>Note that in the model we assumed that households abroad do not have access to home risky assets market. In this interpretation of the experiment, one can assume that the government of foreign country does not face the same information and trading costs as foreign households.





**Figure 2:** Price effects of risky assets purchases. Percentage change in the price of risky assets is on the vertical axis, the half life of the persistence of the policy is on horizontal axis. The orange lines represent the effects of an increase in foreign demand, while the blue lines show the effect of central bank purchases of risky assets. Various lines correspond to different average level of sophistication of agents in the economy  $\bar{k} = 1 / (1 - \lambda)$ .



**Figure 3:** Relative Price effects of risky assets purchases by foreigners and by the central bank.

Second, as the level of sophistication of the agents in the economy increases, quantitative easing becomes less powerful: the set of blue curves move down from dark blue lines to light blue lines and, eventually, to dark blue dashed line, which represents the absence of quantitative easing effect on the price in rational expectations equilibrium. The more persistent the policy is the larger is the effect of QE policy.

Third, with more sophisticated agents, the effects of a decline in net supply of risky assets (e.g., an increase in foreign demand for these assets) becomes stronger because agents realize the future equilibrium effects on prices more. This is represented by a set of orange curves in the plot that move up from horizontal line to lighter orange lines, and, eventually to dashed orange line. The difference between the orange and blue lines corresponding to the same level of average sophistication of households reflects the negative tax effect on the strength of QE. Interestingly, the ratio of the QE policy price effect over foreign purchases price effect does not depend on the persistence of the two policies and equals the inverse of the average level of sophistication in the economy, as depicted in Figure 3. For example, in the case of  $\bar{k} = 2$ , the quantitative easing price effect is a half of the effect of foreign purchases.

**Other prices.** Quantitative easing not only affect the price of risky assets but also the nominal price level, nominal interest and exchange rates.

## 4 Forecast Errors and Balance Sheet Policies

A special feature of the model with level- $k$  thinking is its implications about forecasts of endogenous variables. Specifically, because agents do not form expectations rationally, they make systematic errors. We can use the model to compute the errors that agents make following central bank balance sheet interventions. These predictions can help differentiate the mechanism in this paper from the mechanism relying on rational expectations and market imperfections in the form of limited participation (i.e., the portfolio balance channel) or the mechanism relying on the asymmetric information between the government and private agents (i.e., the signaling channel).

Moreover, level- $k$  thinkers in the model form expectations by taking into account the beliefs of less sophisticated thinkers: they forecast the forecasts of others. One can use model's predictions about the behavior of these forecasts to distinguish the mechanism in this paper from the limited information mechanism (e.g., [Mankiw and Reis, 2002](#)) which emphasizes that some agents do not know that the policy intervention took place and, hence, do not adjust their behavior accordingly.

We next derive the predictions about the forecast errors that agents make in reflective equilibrium. We focus on the endogenous price of risky assets. The implications for other price can be obtained analogously. A level- $k$  thinker who forms expectations about  $q_t$  uses price  $q_t^{k-1}$  which is computed under the assumption that the economy is populated by level- $(k-1)$  thinkers only. We denote forecast error as  $u_t^k \equiv q_t - q_t^{k-1}$ . Using the expression for price  $q_t$  from [Proposition 2](#) and the price  $q_t^{k-1}$  obtained from recursive equation [\(9\)](#) by iterating it  $k-1$  times forward, we express the average forecast error as

$$\bar{u}_t = \sum_{k=1}^{\infty} f(k) u_t^k = \frac{\gamma \sigma_x^2}{\omega} \left( \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k} - \sum_{k=2}^{\infty} f(k) \frac{X_{t+k-1}}{R^{k-1}} \right) = (1 - \lambda) \frac{\gamma \sigma_x^2}{\omega} \cdot \frac{1 - \lambda}{R - \lambda \mu} X_{t+1},$$

where the second equality presents the result of the special case when the types of thinking distributed according to exponential distribution across the households and government asset purchases decay exponentially at rate  $\mu$ . The last expression underscores that the average forecast error of individual agents are related to the size of government intervention. Specifically, level- $k$  thinkers necessarily make forecast errors that are positively related to the size of QE: agents always underestimate the power of the government intervention. This prediction of the model contrasts it from the predictions in the models with rational expectations and symmetric information across agents. In these models, people do not make systematic forecast errors.

One can object, however, that a class of models in which agents form expectations rationally but perhaps differ in their information sets can generate non-neutrality of bal-

ance sheet policies and predictability of forecast errors (both average across agents and individual). For example, if some people do not pay attention to government announcements, they can be unaware of an ongoing policy intervention. As a result, they do not change their behavior and their forecasts of future endogenous variables following the policy change. This may lead to real effects and forecast errors predictability after the intervention.

A possible way to differentiate the predictions of the model with level- $k$  thinking from the predictions of the models with asymmetric information is to use survey data on joint behavior of forecasts of future endogenous variables and beliefs about government interventions. For example, in the limited information model in which some agents are completely unaware of policy interventions, the agents make forecasts mistakes because their beliefs about policy intervention deviate from actual policy interventions. After controlling for the discrepancy in beliefs about policy and actual policy, forecast errors should not be predictable in models with limited information. At the same time, level- $k$  thinking belief formation process implies that agents make predictable forecast errors even if they hold identical beliefs about a change in policy.

## 5 Conclusion

We study the effects of the central bank's balance sheet policies in a model without rational expectations. The same forces that make forward guidance weaker in [García-Schmidt and Woodford \(2015\)](#) and [Farhi and Werning \(2016\)](#) make the effects of the balance sheet policies stronger.

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# A Appendix

## A.1 Proof of Lemma 1

We focus on the household problem in the home country, the problem in the foreign country is analogous. If we combine the budget constraints (1) and (2) and approximate the resulting intertemporal budget constraint around  $i_t = \pi_{t+1} = r = 0$ , we obtain

$$c_{t+1} = R w + \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - (r + \pi_{t+1}) m_{t+1} - T_{t+1}. \quad (\text{A.1})$$

Equation (A.1) is a linear transformation of jointly normal variables, thus, standard properties of CARA preferences imply that the household maximization problem can be equivalently rewritten as

$$\max_{x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}, c_{t+1}} \tilde{\mathbb{E}}_t c_{t+1} - \frac{\gamma}{2} \tilde{\mathbb{V}}_t c_{t+1} + \gamma \frac{m_{t+1} [\log(m_{t+1}/\bar{m}) - 1]}{v},$$

subject to (A.1), where the tilde emphasizes that the household uses the distribution  $\tilde{F}$  to predict future endogenous variables. In particular, we can use (A.1) to rewrite the first term explicitly:

$$\tilde{\mathbb{E}}_t c_{t+1} = R w + \tilde{\mathbb{E}}_t \left[ \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' \right] - \left( r + \tilde{\mathbb{E}}_t \pi_{t+1} \right) m_{t+1} - \tilde{\mathbb{E}}_t T_{t+1}.$$

The variance of consumption is

$$\begin{aligned} \tilde{\mathbb{V}}_t c_{t+1} &= \tilde{\mathbb{V}}_t \left[ \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' \right] + \tilde{\mathbb{V}}_t (\pi_{t+1}) m_{t+1}^2 + \tilde{\mathbb{V}}_t (T_{t+1}) \\ &\quad - 2\widetilde{\text{Cov}}_t (\mathcal{R}_{t+1}, T_{t+1}) (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - 2\widetilde{\text{Cov}}_t \left( \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - T_{t+1}, \pi_{t+1} \right) m_{t+1}. \end{aligned}$$

We can then use the expressions for beliefs (3) to rewrite the first four terms as follows:

$$\tilde{\mathbb{V}}_t \left[ \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' \right] = (x_{t+1}, b_{H,t+1}, b_{F,t+1}) \tilde{\Sigma}_t (x_{t+1}, b_{H,t+1}, b_{F,t+1})',$$

$$\tilde{\mathbb{V}}_t (\pi_{t+1}) = \left( \beta_{p,t+1}^x \right)^2 \sigma_x^2 + \left( \beta_{p,t+1}^h \right)^2 \sigma_h^2 + \left( \beta_{p,t+1}^f \right)^2 \sigma_f^2,$$

$$\tilde{\mathbb{V}}_t (T_{t+1}) = \left( \beta_{T,t+1}^x \right)^2 \sigma_x^2 + \left( \beta_{T,t+1}^h \right)^2 \sigma_h^2 + \left( \beta_{T,t+1}^f \right)^2 \sigma_f^2,$$

$$\begin{aligned} \widetilde{\text{Cov}}_t (\mathcal{R}_{t+1}, T_{t+1}) &= \widetilde{\text{Cov}}_t \left( (r_{t+1}^x + q_{t+1} - R q_t, i_t - \pi_{t+1} - r, i_t^* - \pi_{t+1}^* - r), T_{t+1} \right) \\ &= \left( \begin{array}{l} \left( 1 + \beta_{q,t+1}^x \right) \beta_{T,t+1}^x \sigma_x^2 + \beta_{q,t+1}^h \beta_{T,t+1}^h \sigma_h^2 + \beta_{q,t+1}^f \beta_{T,t+1}^f \sigma_f^2 \\ - \beta_{p,t+1}^x \beta_{T,t+1}^x \sigma_x^2 - \beta_{p,t+1}^h \beta_{T,t+1}^h \sigma_h^2 - \beta_{p,t+1}^f \beta_{T,t+1}^f \sigma_f^2 \\ - \beta_{p^*,t+1}^x \beta_{T,t+1}^x \sigma_x^2 - \beta_{p^*,t+1}^h \beta_{T,t+1}^h \sigma_h^2 - \beta_{p^*,t+1}^f \beta_{T,t+1}^f \sigma_f^2 \end{array} \right)'. \end{aligned}$$

Finally, linearity implies

$$\widetilde{\text{Cov}}_t \left( \mathcal{R}_{t+1} (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - T_{t+1}, \pi_{t+1} \right) = \widetilde{\text{Cov}}_t (\mathcal{R}_{t+1}, \pi_{t+1}) (x_{t+1}, b_{H,t+1}, b_{F,t+1})' - \widetilde{\text{Cov}}_t (T_{t+1}, \pi_{t+1}),$$

with

$$\widetilde{\text{Cov}}_t(\mathcal{R}_{t+1}, \pi_{t+1}) = \begin{pmatrix} \left(1 + \beta_{q,t+1}^x\right) \beta_{p,t+1}^x \sigma_x^2 + \beta_{q,t+1}^h \beta_{p,t+1}^h \sigma_h^2 + \beta_{q,t+1}^f \beta_{p,t+1}^f \sigma_f^2 \\ - \left(\beta_{p,t+1}^x\right)^2 \sigma_x^2 - \left(\beta_{p,t+1}^h\right)^2 \sigma_h^2 - \left(\beta_{p,t+1}^f\right)^2 \sigma_f^2 \\ - \beta_{p^*,t+1}^x \beta_{p,t+1}^x \sigma_x^2 - \beta_{p^*,t+1}^h \beta_{p,t+1}^h \sigma_h^2 - \beta_{p^*,t+1}^f \beta_{p,t+1}^f \sigma_f^2 \end{pmatrix}'$$

and

$$\widetilde{\text{Cov}}_t(T_{t+1}, \pi_{t+1}) = -\beta_{p,t+1}^x \beta_{T,t+1}^x \sigma_x^2 - \beta_{p,t+1}^h \beta_{T,t+1}^h \sigma_h^2 - \beta_{p,t+1}^f \beta_{T,t+1}^f \sigma_f^2.$$

Taking the first-order condition w.r.t.  $x_{t+1}$  and rearranging we obtain

$$\begin{aligned} \left(\tilde{\Sigma}_{t,11}, \tilde{\Sigma}_{t,12}, \tilde{\Sigma}_{t,13}\right) \begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \end{pmatrix} &= \frac{1}{\gamma} \tilde{\mathbb{E}}_t [r_{t+1}^x + q_{t+1} - Rq_t] + \widetilde{\text{Cov}}_t(r_{t+1}^x + q_{t+1} - Rq_t, T_{t+1}) \\ &+ \widetilde{\text{Cov}}_t(r_{t+1}^x + q_{t+1} - Rq_t, \pi_{t+1}) m_{t+1} - \tilde{\Sigma}_{t,12} m_{t+1}. \end{aligned} \quad (\text{A.2})$$

where  $\tilde{\Sigma}_{t,ij}$  denotes the  $ij$ -th element of  $\tilde{\Sigma}_t$ . Similarly, the first-order conditions w.r.t.  $b_{H,t+1}$  and  $b_{F,t+1}$  are, respectively,

$$\begin{aligned} \left(\tilde{\Sigma}_{t,21}, \tilde{\Sigma}_{t,22}, \tilde{\Sigma}_{t,23}\right) \begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \end{pmatrix} &= \frac{1}{\gamma} \tilde{\mathbb{E}}_t [i_t - \pi_{t+1} - r] + \widetilde{\text{Cov}}_t(i_t - \pi_{t+1} - r, T_{t+1}) \\ &+ \widetilde{\text{Cov}}_t(i_t - \pi_{t+1} - r, \pi_{t+1}) m_{t+1} - \tilde{\Sigma}_{t,22} m_{t+1}. \end{aligned} \quad (\text{A.3})$$

and

$$\begin{aligned} \left(\tilde{\Sigma}_{t,31}, \tilde{\Sigma}_{t,32}, \tilde{\Sigma}_{t,33}\right) \begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \end{pmatrix} &= \frac{1}{\gamma} \tilde{\mathbb{E}}_t [i_t^* - \pi_{t+1}^* - r] + \widetilde{\text{Cov}}_t(i_t^* - \pi_{t+1}^* - r, T_{t+1}) \\ &+ \widetilde{\text{Cov}}_t(i_t^* - \pi_{t+1}^* - r, \pi_{t+1}) m_{t+1} - \tilde{\Sigma}_{t,32} m_{t+1}. \end{aligned} \quad (\text{A.4})$$

Finally, if we combine the first-order condition for  $m_{t+1}$  with (A.3) we obtain

$$\log\left(\frac{m_{t+1}}{\bar{m}}\right) = -vi_t. \quad (\text{A.5})$$

In the cashless limit, money holdings vanish,  $m_{t+1} \rightarrow 0$ , and we obtain the asset demand equation (8). Furthermore, (A.5) becomes

$$p_t = vi_t + \epsilon_t^h. \quad (\text{A.6})$$

## A.2 Proof of Proposition 1

In a REE beliefs are consistent with equilibrium variables, therefore, taxes can be computed from the government budgets (4) and (5). In particular, the latter impose the following restrictions on the sensitivity of

taxes to the aggregate shocks:

$$\begin{aligned}\omega\beta_{T,t+1}^{x,REE} &= \beta_{q,t+1}^{x,REE} X_{t+2} - \left(1 + \beta_{q,t+1}^{x,REE}\right) X_{t+1} + \beta_{p,t+1}^{x,REE} B_{t+1}^h + \beta_{p^*,t+1}^{x,REE} B_{t+1}^f, \\ \omega\beta_{T,t+1}^{h,REE} &= \beta_{q,t+1}^{h,REE} X_{t+2} - \beta_{q,t+1}^{h,REE} X_{t+1} + \beta_{p,t+1}^{h,REE} B_{t+1}^h + \beta_{p^*,t+1}^{h,REE} B_{t+1}^f, \\ \omega\beta_{T,t+1}^{f,REE} &= \beta_{q,t+1}^{f,REE} X_{t+2} - \beta_{q,t+1}^{f,REE} X_{t+1} + \beta_{p,t+1}^{f,REE} B_{t+1}^h + \beta_{p^*,t+1}^{f,REE} B_{t+1}^f,\end{aligned}$$

for the home country, and

$$\begin{aligned}(1 - \omega)\beta_{T^*,t+1}^{x,REE} &= -B^* \beta_{p^*,t+1}^{x,REE}, \\ (1 - \omega)\beta_{T^*,t+1}^{h,REE} &= -B^* \beta_{p^*,t+1}^{h,REE}, \\ (1 - \omega)\beta_{T^*,t+1}^{f,REE} &= -B^* \beta_{p^*,t+1}^{f,REE},\end{aligned}$$

for the foreign one. In equilibrium, markets have to clear, that is,

$$\omega x_{t+1} = \bar{X} - X_{t+1}, \quad (\text{A.7})$$

$$\omega b_{H,t+1} + (1 - \omega)b_{F,t+1}^* = -B_{t+1}^h, \quad (\text{A.8})$$

$$\omega b_{F,t+1} + (1 - \omega)b_{H,t+1}^* = B^* - B_{t+1}^f, \quad (\text{A.9})$$

where optimal choices  $(x_{t+1}, b_{H,t+1}, b_{F,t+1})$  are obtained from (8) together with the restrictions on expectations of taxes above. From Lemma 1 and equation (A.6), home-country asset demands satisfy

$$\begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \end{pmatrix} = \frac{1}{\gamma} \Sigma_t^{-1} \begin{pmatrix} r^x + \mathbb{E}_t q_{t+1}^{REE} - Rq_t^{REE} \\ \frac{1}{v} (p_t^{REE} - \epsilon_t^h) - \mathbb{E}_t p_{t+1}^{REE} + p_t^{REE} - r \\ \frac{1}{v} (p_t^{*,REE} - \epsilon_t^f) - \mathbb{E}_t p_{t+1}^{*,REE} + p_t^{*,REE} - r \end{pmatrix} + \Sigma_t^{-1} \text{Cov}_t \left( \begin{pmatrix} q_{t+1}^{REE} \\ -p_{t+1}^{REE} \\ -p_{t+1}^{*,REE} \end{pmatrix}, T_{t+1} \right), \quad (\text{A.10})$$

where we dropped the tilde to emphasize that expectations are rational. A similar expressions holds for the foreign country.

We conjecture and later verify that  $(q_t^{REE}, p_t^{REE}, p_t^{*,REE})$  are linear functions of the underlying shocks. Standard properties of Normal distributions then imply that the conditional first and second moments are functions of time only. In addition, since balance-sheet policies are assumed to be only functions of time, equations (A.7)-(A.9) imply that  $q_t^{REE}$  will be a deterministic function of time, while  $p_t^{REE}$  and  $p_t^{*,REE}$  will depend only on time and on the monetary shocks  $\epsilon_t^h$  and  $\epsilon_t^f$ , respectively. Formally,  $\beta_{q,t+1}^{x,REE} = \beta_{p,t+1}^{x,REE} = \beta_{p^*,t+1}^{x,REE} = 0$ ,  $\beta_{q,t+1}^{h,REE} = \beta_{p^*,t+1}^{h,REE} = 0$ , and  $\beta_{q,t+1}^{f,REE} = \beta_{p,t+1}^{f,REE} = 0$  and, in particular,

$$\Sigma_t = \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & (\beta_{p,t+1}^{h,REE})^2 \sigma_h^2 & 0 \\ 0 & 0 & (\beta_{p^*,t+1}^{f,REE})^2 \sigma_f^2 \end{pmatrix}. \quad (\text{A.11})$$

We focus on the risky-asset market and derive  $q_t^{REE}$ , analogous arguments lead to  $p_t^{REE}$  and  $p_t^{*,REE}$ .

Combining the first row of (A.10) with (A.11) and imposing market-clearing (A.7), we obtain

$$\bar{X} - X_{t+1} = \omega \frac{1}{\gamma \sigma_x^2} \left( r^x + q_{t+1}^{REE} - Rq_t^{REE} \right) + \omega \beta_{T,t+1}^{x,REE}.$$

Also, combining  $\beta_{q,t+1}^{x,REE} = \beta_{p,t+1}^{x,REE} = \beta_{p^*,t+1}^{x,REE} = 0$  with the restrictions on expectations of future taxes in a REE, gives

$$\begin{aligned}\omega\beta_{T,t+1}^{x,REE} &= \beta_{q,t+1}^{x,REE} X_{t+2} - \left(1 + \beta_{q,t+1}^{x,REE}\right) X_{t+1} + \beta_{p,t+1}^{x,REE} B_{t+1}^h + \beta_{p^*,t+1}^{x,REE} B_{t+1}^f \\ &= -X_{t+1}\end{aligned}$$

and, therefore,

$$\bar{X} = \omega \frac{1}{\gamma\sigma_x^2} \left( r^x + q_{t+1}^{REE} - Rq_t^{REE} \right). \quad (\text{A.12})$$

Iterating (A.12) forward and looking for the non-explosive path satisfying  $R^{-t}q_t^{REE} \rightarrow 0$ , gives

$$q^{REE} = \frac{1}{R-1} \left( r^x - \frac{\gamma\sigma_x^2}{\omega} \bar{X} \right),$$

where, since the right-hand side is independent of time, we omitted the subscript  $t$  from the notation.

Finally, from the price levels and the law of one price we obtain the equilibrium exchange rate:

$$e_t^{REE} = p_t^{REE} - p_t^{*,REE}.$$

### A.3 Proofs for Level- $k$ Thinking Equilibrium

We start with level-1 equilibrium. By assumption, after the announcement of the new government policy, level-1 household expectations coincide with REE variables. In turn, if expectations do not change, asset demands in Lemma 1 coincide with their REE counterparts (A.10). Specifically, from Proposition 1, expectations of the risky-asset price are pinned down by

$$\begin{pmatrix} \alpha_{q,t+1}^1 \\ \beta_{q,t+1}^{x,1} \\ \beta_{q,t+1}^{h,1} \\ \beta_{q,t+1}^{f,1} \end{pmatrix} = \begin{pmatrix} q^{REE} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{A.13})$$

expectations of the home-country price level are pinned down by

$$\begin{pmatrix} \alpha_{p,t+1}^1 \\ \beta_{p,t+1}^{x,1} \\ \beta_{p,t+1}^{h,1} \\ \beta_{p,t+1}^{f,1} \end{pmatrix} = \begin{pmatrix} vr \\ 0 \\ \frac{1}{1+v} \\ 0 \end{pmatrix}, \quad (\text{A.14})$$

and expectations of the foreign-country price level are pinned down by

$$\begin{pmatrix} \alpha_{p^*,t+1}^1 \\ \beta_{p^*,t+1}^{x,1} \\ \beta_{p^*,t+1}^{h,1} \\ \beta_{p^*,t+1}^{f,1} \end{pmatrix} = \begin{pmatrix} vr \\ 0 \\ 0 \\ \frac{1}{1+v} \end{pmatrix}. \quad (\text{A.15})$$

Finally, since we assume that the home-country government does not conduct any intervention nor it has outstanding liabilities in the status quo, taxes are expected to be 0, i.e.

$$\begin{pmatrix} \alpha_{T,t+1}^1 \\ \beta_{T,t+1}^{x,1} \\ \beta_{T,t+1}^{h,1} \\ \beta_{T,t+1}^{f,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{A.16})$$

On the contrary, in the status quo the foreign government has a real value  $B^*$  of outstanding nominal bonds, therefore, from (5),

$$\begin{pmatrix} \alpha_{T^*,t+1}^1 \\ \beta_{T^*,t+1}^{x,1} \\ \beta_{T^*,t+1}^{h,1} \\ \beta_{T^*,t+1}^{f,1} \end{pmatrix} = \begin{pmatrix} rB^* \\ 0 \\ 0 \\ -\frac{1}{1+v}B^* \end{pmatrix}. \quad (\text{A.17})$$

In equilibrium, market-clearing equations (A.7)-(A.9) must be satisfied. Importantly, as it was the case in the REE, standard properties of Normal distributions imply that the market-clearing condition (A.7) is satisfied by some price  $q_t^1$  which is only a function of time. Similarly, as it was the case in the REE, equations (A.8), (A.9), and (A.6) imply that  $p_t^1$  and  $p_t^{*,1}$  depend only on time and on the time- $t$  monetary shock in their respective country.

We now solve for equilibrium prices explicitly. Consider first market-clearing condition (A.7) and combine it with (A.11) and with the risky-asset demand given by the first row of (A.10):

$$\begin{aligned} \bar{X} - X_{t+1} &= \omega \frac{1}{\gamma\sigma_x^2} \left( r^x + \tilde{\mathbb{E}}_t [q_{t+1}] - Rq_t^1 \right) + \omega \frac{1}{\sigma_x^2} \widetilde{\text{Cov}}_t (r_{t+1}^x + q_{t+1}, T_{t+1}) \\ &= \omega \frac{1}{\gamma\sigma_x^2} \left( r^x + q^{REE} - Rq_t^1 \right), \end{aligned}$$

where we used (A.13) and (A.16).<sup>12</sup> In particular, note that agents expect future prices and taxes to be those in the status-quo REE, however, due to unexpected government purchases  $X_{t+1}$ , the equilibrium price  $q_t^1$  will be different from the status-quo price. Solving for the equilibrium price yields

$$q_t^1 = q^{REE} + \frac{\gamma\sigma_x^2}{\omega R} X_{t+1}. \quad (\text{A.18})$$

Let's now compute  $p_t^1$ . From (A.16) and (A.17),  $\widetilde{\text{Cov}}_t (p_{t+1}^1, T_{t+1}) = \widetilde{\text{Cov}}_t (p_{t+1}^1, T_{t+1}^*) = 0$  and the market-clearing condition (A.8), together with (A.6), gives

$$\frac{\frac{1}{v} (p_t^1 - \epsilon_t^h) - (\tilde{\mathbb{E}}_t [p_{t+1}] - p_t^1) - r}{\gamma\sigma_h^2 (\beta_{p,t+1}^{h,1})^2} = -B_{t+1}^h$$

or, using (A.14) and solving for the equilibrium price,

$$p_t^1 = vr - \frac{v}{1+v} \gamma\sigma_h^2 \left( \frac{1}{1+v} \right)^2 B_{t+1}^h + \frac{1}{1+v} \epsilon_t^h. \quad (\text{A.19})$$

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<sup>12</sup>We use a tilde to emphasize that expectations are no longer rational.

Finally, analogous steps together with (A.15) and (A.17) yield the foreign-country price level in the level-1 equilibrium:

$$p_t^{*,1} = vr - \frac{v}{1+v} \gamma \sigma_f^2 \left( \frac{1}{1+v} \right)^2 B_{t+1}^f + \frac{1}{1+v} \epsilon_t^f. \quad (\text{A.20})$$

The level-1 equilibrium determines the expectations of level-2 households. In particular, from (A.18),

$$\begin{pmatrix} \alpha_{q,t+1}^2 \\ \beta_{q,t+1}^{x,2} \\ \beta_{q,t+1}^{h,2} \\ \beta_{q,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} q^{REE} + \frac{\gamma \sigma_f^2}{\omega R} X_{t+1} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

from (A.19),

$$\begin{pmatrix} \alpha_{p,t+1}^2 \\ \beta_{p,t+1}^{x,2} \\ \beta_{p,t+1}^{h,2} \\ \beta_{p,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} vr - \frac{v}{1+v} \gamma \sigma_h^2 \left( \frac{1}{1+v} \right)^2 B_{t+1}^h \\ 0 \\ \frac{1}{1+v} \\ 0 \end{pmatrix},$$

and, from (A.20),

$$\begin{pmatrix} \alpha_{p^*,t+1}^2 \\ \beta_{p^*,t+1}^{x,2} \\ \beta_{p^*,t+1}^{h,2} \\ \beta_{p^*,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} vr - \frac{v}{1+v} \gamma \sigma_f^2 \left( \frac{1}{1+v} \right)^2 B_{t+1}^f \\ 0 \\ 0 \\ \frac{1}{1+v} \end{pmatrix}.$$

Finally, letting  $\Gamma_{t+1}^h \equiv \gamma \sigma_h^2 \left( \frac{1}{1+v} \right)^2 \left( \frac{v}{1+v} B_{t+2}^f - B_{t+1}^f \right)$  and  $\Gamma_{t+1}^f \equiv \gamma \sigma_f^2 \left( \frac{1}{1+v} \right)^2 \left( \frac{v}{1+v} B_{t+2}^f - B_{t+1}^f \right)$  and combining the home and foreign-government budget constraints (4) and (5) with the equilibrium asset prices, we can pin down the beliefs of future taxes:

$$\omega \begin{pmatrix} \alpha_{T,t+1}^2 \\ \beta_{T,t+1}^{x,2} \\ \beta_{T,t+1}^{h,2} \\ \beta_{T,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} -r^x X_{t+2} + q_{t+1}^1 (X_{t+2} - X_{t+1}) + B_{t+2}^h - (R + \Gamma_{t+1}^h) B_{t+1}^h + B_{t+2}^f - (R + \Gamma_{t+1}^f) B_{t+1}^f \\ -X_{t+1} \\ \frac{1}{1+v} B_{t+1}^h \\ \frac{1}{1+v} B_{t+1}^f \end{pmatrix},$$

where ,for the home country, and

$$(1 - \omega) \begin{pmatrix} \alpha_{T^*,t+1}^2 \\ \beta_{T^*,t+1}^{x,2} \\ \beta_{T^*,t+1}^{h,2} \\ \beta_{T^*,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} R - 1 + \Gamma_{t+1}^f B^* \\ 0 \\ 0 \\ -\frac{1}{1+v} B^* \end{pmatrix},$$

for the foreign one.

Starting from level-2 beliefs, we can solve for level-3 equilibrium variables. In general, we can procede

recursively and obtain  $q_t^k$ ,  $p_t^k$ , and  $p_t^{*,k}$ , for any level  $k$ . In particular, we have

$$q_t^k = q^{REE} + \frac{\gamma\sigma_x^2 X_{t+k}}{\omega R^k}, \quad (\text{A.21})$$

$$p_t^k = vr - \left(\frac{v}{1+v}\right)^k \gamma\sigma_h^2 \left(\frac{1}{1+v}\right)^2 B_{t+k}^h + \frac{1}{1+v} \epsilon_t^h, \quad (\text{A.22})$$

$$p_t^{*,k} = vr - \left(\frac{v}{1+v}\right)^k \gamma\sigma_f^2 \left(\frac{1}{1+v}\right)^2 B_{t+k}^f + \frac{1}{1+v} \epsilon_t^f. \quad (\text{A.23})$$

## A.4 Proof of Proposition 2

In a RE, there is a mass  $f(k)$  of level- $k$  households for each  $k$ , where level- $k+1$ -household beliefs are generated from level- $k$  equilibrium variables (A.21)-(A.23).

Let's start with the risky-asset market. Using asset demand in Lemma 8 with market clearing (A.7) we have that the equilibrium price  $q_t$  must satisfy

$$\omega \left( \sum_{k=1}^{\infty} f(k) \frac{r^x + \alpha_{q,t+1}^k - Rq_t}{\gamma\sigma_x^2} + \beta_{T,t+1}^{x,k} \right) = \bar{X} - X_{t+1},$$

which yields

$$q_t = q^{REE} + \sum_{k=1}^{\infty} f(k) \frac{\gamma\sigma_x^2 X_{t+k}}{\omega R^k}.$$

Similarly, the equilibrium in the home-bond market is now

$$\sum_{k=1}^{\infty} f(k) \left( \frac{i_t - (\alpha_{p,t+1}^k - p_t) - r}{\gamma\sigma_h^2 (\beta_{p,t+1}^{h,k})^2} - \omega \frac{\beta_{T,t+1}^{h,k}}{\beta_{p,t+1}^{h,k}} - (1-\omega) \frac{\beta_{T,t+1}^{f,k}}{\beta_{p,t+1}^{h,k}} \right) = -B_{t+1}^h$$

and, therefore,

$$p_t = vr - \gamma\sigma_h^2 \left(\frac{1}{1+v}\right)^2 \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{t+k}^h + \frac{1}{1+v} \epsilon_t^h.$$

Analogously,

$$p_t^* = vr - \gamma\sigma_f^2 \left(\frac{1}{1+v}\right)^2 \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{t+k}^f + \frac{1}{1+v} \epsilon_t^f.$$

Finally, as usual, the exchange rate is given by the law of one price:

$$e_t = p_t - p_t^*.$$