Is Index Trading Benign?

Shmuel Baruch

Xiaodi Zhang*

March 2018

Abstract

We develop a conditional capital asset pricing model (CAPM) that maintains the rationale for index investment: In equilibrium, it is optimal for nonindex investors to index. The model demonstrates that index investment is not benign. As more nonindexers become indexers, the proportion of an asset's idiosyncratic risk to total risk increases; correlation in asset prices increases; correlation in returns decreases; and for any portfolio other than the market portfolio, the Sharpe ratio decreases and the conditional variance of payoff increases. As a limiting case of our model, we present an equilibrium in which all investors are indexers.

JEL: G11, D82

Keywords: CAPM, index investment, Sharpe ratio, partially revealing equilibrium

*Baruch: David Eccles School of Business, University of Utah, Salt Lake City, UT 84112; shmuel.baruch@business.utah.edu. Zhang: Department of Finance, University of Central Florida, Orlando, FL 32816; Xiaodi.Zhang@ucf.edu. We thank Kerry Back, Dan Bernhardt, Joel Hasbrouck, Yud Izhakian, Davar Khoshnevisan, Mark Loewenstein, Matt Ringgenberg, Yajun Wang, and seminar participants at the SEC and Utah for their helpful comments.

It has now been over forty years of the first edition of A Random Walk Down Wall Street. The message of the original edition was a very simple one: Investor would be far better off buying and holding an index fund than attempting to buy and sell individual securities...Now, over forty years later, I believe even more strongly in the original thesis.

Burton Malkiel

1 Introduction

The separation theorem provides the intellectual underpinning of index investment, and the capital asset pricing model (CAPM) is the most important pricing implication of the separation theorem.[1] But the CAPM is silent about the impact of index investment on pricing. In this paper, we examine the implications of indexing in an extension of the mean–variance equilibrium analysis in which (i) the separation result holds and (ii) a conditional CAPM holds. Thus, the equilibrium we present in this paper maintains the rationale for index investment, and the results are framed in the standard CAPM terminology.

To build our model, we depart from the standard CAPM in two ways. First, we adopt a rational expectation setup (Grossman, 1976) in which investors combine their private information with the information contained in equilibrium prices. In a multiasset extension of his model, Grossman (1978) shows that a conditional CAPM emerges. But Grossman's conditional CAPM is silent about the impact of indexing in exactly the same way the standard CAPM is. We therefore deviate further from the standard model by explicitly dividing the investors into two groups, indexers and nonindexers. Index investors confine themselves to

[1]In August 1976, when Vanguard started offering its index fund, Samuelson (1976) articulated the separation theorem in the popular press: "What each prudent investor must do is to decide what fraction of savings he can afford, in this age of inflation, to keep in equities and in other things. An unmanaged, low-turnover, low-fee index fund is merely an efficient way of holding that part deemed appropriate for equities." Lo (2016) writes that it was the academic research, specifically the CAPM and the efficient market hypothesis, that "provided the seeds from which the index fund business grew."

combinations of the risk-free asset and the market portfolio. Nonindex investors, by contrast, solve an unconstrained investment problem. The presence of index investors in our model is the second and last departure from the standard CAPM world. In a small economy, we do not justify why some investors index.[2] In a large economy, modeled as the limit of a sequence of finite economies (Hellwig, 1980), we show that it is rational for index investors to index.

We prove the existence of a partially revealing rational expectation equilibrium in which Tobin's separation result holds: nonindexers' investment positions are located, in the volatility–return plane, on the capital market line. Similar to the fully revealing equilibrium described by Grossman (1978), a conditional CAPM relation holds.[3] However, asset prices and betas depend on the specific partition of investors into indexers and nonindexers. We show that this dependency is also present in the large economy. Moreover, in the large economy it is optimal for index investors to index. I.e. the constraint to stay on the capital market line is not binding.

By means of comparative statics, we find that as more nonindex investors become index investors, the proportion of idiosyncratic risk to total risk increases.[4] This increase manifests in different ways. The statistical fit (measured by R^2) of the CAPM regression decreases. Correlation in returns decreases in the sense that, provided an asset is positively correlated with the portfolio of remaining assets, this correlation decreases. Correlation in asset prices increases in the sense that, provided an asset price is positively correlated with the price of the portfolio of remaining assets, this correlation increases.[5] For any portfolio other than the

[2] We note that some investors may not have a choice (e.g. investors that only invest through their retirement accounts may not be able to invest in individual stocks).

[3] A fully revealing equilibrium is an equilibrium in which, given the aggregate information in the economy, prices are jointly sufficient statistics for the payoff ("future prices") of the assets. Our model also possesses a fully revealing equilibrium.

[4] Idiosyncratic risk is the "unexplained variance" of the stock's return. The proportion of idiosyncratic risk to total risk is the fraction of unexplained variance, and it equals $1 - R^2$, where R^2 is the coefficient of determination of the CAPM regression.

[5] Like in other conditional CAPM models, prices and betas are realizations of random variables. It is therefore meaningful to study their statistical properties.

market portfolio, the portfolio's Sharpe ratio decreases, and the variance of the portfolio's payoff increases.[6] By contrast, both the market portfolio's Sharpe ratio and the variance of the market portfolio's payoff are unaffected. Finally, using numerical computations, we find that the distributions of betas become less dispersed.

Whereas the results discussed thus far highlight the impact of index investment, we also identify some market outcomes that do not depend on the specific partition of investors into indexers and nonindexers. Considering the complete set of signals (of both indexers and nonindexers) as the "data" and the payoff of the market portfolio as the "parameter," we show that the forward price of the market portfolio is a minimal sufficient statistic for the payoff of the market portfolio.[7]

Our model is relevant to a market with many layman investors who costlessly observe extremely noisy signals.[8] Even though the market is large, the law of large numbers is not applicable; even though each signal is very noisy, equilibrium prices are informative. The signals are a mathematical representation of information readily available in the investors' environments. Sources of the signals can be, for example, having experience with the customer services of firms, being enthusiastic about products of certain brands, or even noting how full retailers' parking lots are. Each individual signal can hardly be called informative; however, when all the signals in the economy are aggregated, valuable information emerges (Treynor 1987). This is our interpretation of the model.

Our model adds to the literature on partially revealing equilibria. To avoid the fully revealing outcome, this literature relies on noise trading (Kyle 1985), supply uncertainty (Hellwig 1980 and Admati 1985), extrinsic noise (DeMarzo and Skiadas 1998), or preference uncertainty

[6] In this paper, a portfolio with a return r has a Sharpe ratio $(Er - r_f)/\,\mathrm{sd}(r)$.

[7] We do not have a derivative securities market in our model, but nevertheless we can compute synthetic forward prices. Forward prices are prices divided by the price of the risk-free bond. In other words, these are the prices at which the bond acts as the numeraire.

[8] Evidence that the trade of layman investors conveys information is provided by Kaniel, Liu, Saar, and Titman (2012), Kelley and Tetlock (2013), and Boehmer, Jones, and Zhang (2016).

(see Ausubel 1990 and the dynamic model of Detemple 2002). Our model relies on none of these. Instead, the equilibrium we compute is partially revealing because index investors solve a constrained optimization problem.

In addition, our model is related to those of Levy (1978) and Malkiel and Xu (2002). These authors also study markets in which some investors—much like the index investors in our model—do not solve a complete portfolio optimization problem. These two papers demonstrate how idiosyncratic risk can become relevant. By contrast, in our model, idiosyncratic risk is irrelevant because no investor (indexer or nonindexer) is exposed to it.

Interestingly, Campbell, Lettau, Malkiel, and Xu (2001) find that from 1962 to 1997, market variance was stable while firms' variances more than doubled. They find that comovement in returns decreased, and the coefficient of determination also decreased. The authors provide several possible explanations for their findings, such as the breaking up of conglomerates. This paper provides a new explanation, namely the rise of index investment (Heath, Macciocchi, Michaely, and Ringgenberg 2018).

Our model also adds to the literature that examines comovements and exchange-traded funds. In Barberis and Shleifer (2003), rational traders take advantage of the extrapolative expectations of switchers who move their holding from one set of assets to another. Barberis, Shleifer, and Wurgler (2005) review additional theories of comovement that stem from market frictions or noise traders' sentiment. Comovement also shows up in market structure–type models. According to Bhattacharya and O'Hara (2016), the source of comovement is the inability to precisely tease out information relevant to individual assets from the exchange-traded funds. Also, Cong and Xu (2016) and Glosten, Nallareddy, and Zou (2016) show that the presence of a composite security, created to cater to factor investors, enhances comovement.

The remainder of this paper is organized as follows. In Section 2, we describe the model.

In Section 3, we expand Grossman's (1978) notion of artificial economies to a world with index investors. In Section 4, we compute a partially revealing equilibrium. In Section 5, we present comparative statics. In Section 6, we present, as a limiting case of our model, an equilibrium in which all investors are indexers. In Section 7, we study model index investment in a large economy. In Section 8, we conclude.

2 The Model

We consider a two-period, single-good exchange economy with one financial (i.e., zero-net-supply) risk-free asset (a bond) and n risky real assets (firms). The prices of the assets are denominated in units of the time-zero consumption good, and the assets' payoffs are denominated in units of the time-one consumption good. The consumption good is perishable, so the only way to transfer consumption between periods is through the capital market.

The payoff of the risk-free asset is one. The random payoff of the risky assets is denoted by $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}'$.[9] We assume $\mathbf{v} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{v}}, \boldsymbol{\Sigma}_{\mathbf{vv}})$, where $\boldsymbol{\Sigma}_{\mathbf{vv}}$ is a symmetric, positive definite matrix.[10] The price vector of the risky assets is denoted by $\mathbf{p} = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix}'$. The price of the bond is denoted by p_f. We define the risk-free interest rate as follows: $r_f = 1/p_f - 1$.

> **Comment 1:** Why do we deviate from the literature that assumes an exogenous price for the risk-free asset? The covariance matrix, $\boldsymbol{\Sigma}_{\mathbf{vv}}$, is positive definite and

[9] Notation: All vectors are column vectors. The transpose operation is denoted by a single quotation mark. Bold lowercase (Greek or upright Roman) letters are used for vectors. Bold uppercase (Greek or upright Roman) letters are used for matrices. We have no special notation to distinguish random variables from their realizations. The context should make our intention clear.

[10] Given two random vectors, $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}'$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_m \end{bmatrix}'$, we interchangeably use the notations $\text{cov}(\mathbf{z}, \mathbf{y})$ and $\Sigma_{\mathbf{zy}}$ to denote the $n \times m$ covariance matrix $\begin{bmatrix} \text{cov}(z_i, y_j) \end{bmatrix}_{n \times m}$. Consequently, using submatrix notation, we have

$$\text{cov}\left(\mathbf{z}, \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}\right) = \begin{bmatrix} \text{cov}(\mathbf{x}, \mathbf{y}_1) & \text{cov}(\mathbf{x}, \mathbf{y}_2) \end{bmatrix}$$

hence it has a full rank. This implies that the risk-free asset is a nonredundant asset. The equilibrium we study is not fully revealing. Thus, the risk-free asset may serve as a channel for information transmission, which is exactly what happens in the paper by Detemple (2002). Moreover, the risk-free rate may depend on the level of index investment in the economy. Ignoring this possibility may distort our comparative statics analysis. For these two reasons, we choose to deviate from the literature.

For each risky asset, we normalize the number of outstanding shares to one. A portfolio (of risky assets) is a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}'$ with the interpretation that x_i is the number of shares of the ith risky asset.[11] Let $\mathbf{1} \in R^n$ denote the vector of all-ones, so $\mathbf{1}$ is the *market portfolio*. We say that an investor *holds the market* if the investor's portfolio is such that $x_1 > 0$ and for every i, $x_i/x_1 = 1$. This is the same as requiring that the portfolio is a strictly positive scalar multiplication of $\mathbf{1}$. The cost of the portfolio $\mathbf{x}$ is $\mathbf{x}'\mathbf{p}$; the random payoff of the portfolio is $\mathbf{x}'\mathbf{v}$; the mean of the payoff is $\mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}}$; and the variance of the payoff is $\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{x}$.

Every portfolio with a nonzero cost has a return $\mathbf{x}'\mathbf{v}/(\mathbf{x}'\mathbf{p})-1$. Two portfolios have the same return if one is a strictly positive scalar multiplication of the other. This is an equivalence relation that is invariant under change of prices.[12] In the portfolio analysis literature, an equivalent class is identified with a vector of market-value weights, termed portfolio weights. (Often, the word "weights" is omitted.) But, under a different set of prices, the same weights represent a different equivalence class. (For example, the weights that represent the market portfolio change as we change prices.) We therefore avoid market-value weights altogether.

There are m investors, labeled $k = 1, \dots, m$. Investors observe the realization of private

[11] In this paper, the word "portfolio" is a shorthand for portfolio of risky assets.

[12] In other words, if two portfolios have the same return under one set of prices, and we then change prices, then either both portfolios will not have a return or both portfolios will have the same return.

signals centered around $\mathbf{v}$. The signals are

$$\forall k = 1 \dots m, \quad \mathbf{s}_k = \begin{bmatrix} s_{1k} \\ \vdots \\ s_{nk} \end{bmatrix} = \mathbf{v} + \boldsymbol{\epsilon}_k$$

where

$$\boldsymbol{\epsilon}_k \sim N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} \right)$$

$\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}$ is a positive definite matrix, and the random vectors $\{\mathbf{v}, \boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_m\}$ are jointly normally distributed and mutually independent.

Let $\bar{c}_k, \bar{b}_k$, and $\bar{\mathbf{x}}_k$ denote the kth investor's endowment. The budget set of the kth investor is

$$\mathcal{B}_k(p_f, \mathbf{p}) = \left\{ c \in R, b \in R, \mathbf{x} \in R^n : \bar{c}_k - c + (\bar{b}_k - b)p_f + (\bar{\mathbf{x}}_k - \mathbf{x})'\mathbf{p} = 0 \right\} \tag{1}$$

Subject to the budget constraint, the kth investor chooses the number of time-zero consumption units, c; the number of bonds, b; and a portfolio of risky assets, $\mathbf{x}$, to maximize the expected value of the utility function:

$$U_k(c, b, \mathbf{x}'\mathbf{v}) \equiv -e^{-\rho_k c} - e^{-\rho_k (b + \mathbf{x}'\mathbf{v})}$$

We let

$$\bar{\rho} = \left(\frac{1}{m} \sum_{k=1}^{m} \rho_k^{-1} \right)^{-1} \tag{2}$$

denote the harmonic mean of the coefficients of risk aversion.

Investors belong to one of two groups, indexers or nonindexers. Index investors, for exogenous reasons, confine themselves to combining the risk-free asset with the market portfolio. Nonindex investors solve a complete portfolio selection problem. We denote the set of indices of index investors by $\mathcal{I}$ and the set of indices of the nonindex investors by $\mathcal{NI}$. We then have $|\mathcal{I}| + |\mathcal{NI}| = m$. For expositional reasons, we assume both types of investors are present. In other words, $0 < |\mathcal{I}| < m$. We remove this assumption in Section 6.

We can write the investors' problems as follows:[13]

$$\forall k \in \mathcal{NI}, \quad \max_{c,b,\mathbf{x}} E\left[U_k(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{s}_k,\mathbf{p},p_f\right]$$
$$\text{s.t. } (c,b,\mathbf{x}) \in \mathcal{B}_k(p_f,\mathbf{p})$$
$$\forall k \in \mathcal{I}, \quad \max_{c,b,q} E\left[U_k(c,b,q\mathbf{1}'\mathbf{v})\,|\mathbf{s}_k,\mathbf{p},p_f\right]$$
$$\text{s.t. } (c,b,q\mathbf{1}) \in \mathcal{B}_k(p_f,\mathbf{p})$$

Let c_k, b_k, and $\mathbf{x}_k$ denote the optimal solution of the above maximization problems, where $\mathbf{x}_k \equiv q_k\mathbf{1}$, whenever $k \in \mathcal{I}$. The solutions of the maximization problems of the investors are quantities that depend on the realization of the prices and signals.

Denote by $\mathbf{s}$ the concatenation of all signals. A *rational expectation equilibrium* is a random pair $(\mathbf{p}, p_f)$ such that for each joint realization of $\mathbf{s}$ and $(\mathbf{p}, p_f)$, the market for the consumption good, the market for debt, and the market for risky assets clear:

$$\sum_{k=1}^{m} c_k = \sum_{k=1}^{m} \bar{c}_k, \qquad \sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \bar{b}_k = 0, \qquad \sum_{k\in\mathcal{NI}} \mathbf{x}_k + \sum_{k\in\mathcal{I}} q_k\mathbf{1} = \sum_{k=1}^{m} \bar{\mathbf{x}}_k = \mathbf{1}$$

3 Artificial Economies

Grossman (1978) devises a heuristic for finding a rational expectation equilibrium. He considers an artificial economy in which each investor has access to all private information in the economy. He proves that if the equilibrium price vector in this artificial economy is a sufficient statistic for the mean of the investors' signals (which is the payoff vector, $\mathbf{v}$), then this price vector is also a rational expectation equilibrium in the actual economy.

[13]Indexers can infer information from the prices of individual assets. Some readers may object to this modeling choice, preferring a model where an indexer can observe only the price of the market portfolio. However, unless her endowment is the market portfolio, to know what her budget is, an indexer has to know the prices of individual assets. If an indexer is endowed with the market portfolio, then, in the equilibrium presented in this paper, the objective of the indexer can be replaced with $E\left[U_k(c,b,q\mathbf{1}'\mathbf{v})\,|\mathbf{s}_k,\mathbf{1}'\mathbf{p},p_f\right]$ instead of $E\left[U_k(c,b,q\mathbf{1}'\mathbf{v})\,|\mathbf{s}_k,\mathbf{p},p_f\right]$, thus removing the objection some readers may have. We choose not to impose assumptions on initial endowments. Hence, our modeling choice is to allow the indexer to observe the equilibrium vector of prices.

The heuristic can be successfully applied to our model because in Grossman's *fully revealing equilibrium* everyone holds the market portfolio; therefore, in the fully revealing equilibrium, the additional constraint on index investors is nonbinding. This means that the fully revealing equilibrium is silent about the implications of index investment in exactly the same way that the standard CAPM is. We are searching for a different rational expectation equilibrium.

For the purpose of finding a new rational expectation equilibrium, we note that statistical sufficiency of prices is a requirement stronger than needed. Indeed, when the price vector is a sufficient statistic, *any decision maker* is indifferent between knowing all the information and knowing only the prices. But we should heed only what the decision makers in our model prefer (in particular, those index investors who solve a constraint problem), not what every hypothetical decision maker prefers.

That said, as in Grossman's fully revealing equilibrium, we are searching for a rational expectation equilibrium in which the equilibrium prices reveal "information to each trader which is of 'higher quality' than his own information" (Grossman 1976):

$$\forall k \in \mathcal{NI}, \forall \mathbf{x} \in R^n \;\; E\left[U_k(c, b, \mathbf{x}'\mathbf{v})\,|\mathbf{s}_k, \mathbf{p}, p_f\right] = E\left[U_k(c, b, \mathbf{x}'\mathbf{v})\,|\mathbf{p}, p_f\right]$$

$$\forall k \in \mathcal{I}, \forall q \in R \;\; E\left[U_k(c, b, q\mathbf{1}'\mathbf{v})\,|\mathbf{s}_k, \mathbf{p}, p_f\right] = E\left[U_k(c, b, q\mathbf{1}'\mathbf{v})\,|\mathbf{p}, p_f\right]$$

3.1 The Artificial Economy $\mathcal{E}_{\mathbf{y}}$

We fix an arbitrary, nondegenerate, multivariate normal random vector $\mathbf{y}$, such that $\mathbf{y}$ and $\mathbf{v}$ are jointly normal. We do not specify the dimension of the vector $\mathbf{y}$.[14]

Let $\boldsymbol{\mu}_{\mathbf{y}}$ be the expected value of $\mathbf{y}$. Because $\mathbf{v}$ and $\mathbf{y}$ are jointly normally distributed, a standard result in probability theory is that $\mathbf{v}$ conditional on the realization of $\mathbf{y}$ is multivariate

[14] A multivariate normal random vector is nondegenerate if its covariance matrix is invertible.

normal with a conditional mean and a conditional (deterministic) covariance matrix:

$$\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \tag{3}$$

$$\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{vv}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} \tag{4}$$

We define the artificial economy $\mathcal{E}_{\mathbf{y}}$ as follows. Investors have the same initial endowment as in the actual economy. All investors are nonindexers, and investors do not observe realizations of private signals. Instead, they observe the realization of the random vector $\mathbf{y}$. An equilibrium in $\mathcal{E}_{\mathbf{y}}$ is a pair $(p_f, \mathbf{p})$ such that for each realization of $\mathbf{y}$, and for each $k = 1, \dots, m$, $(c_k, b_k, \mathbf{x}_k)$ solves

$$\begin{aligned} &\max_{c,b,\mathbf{x}} E\left[U_k(c, b, \mathbf{x}'\mathbf{v})\,|\mathbf{y}\,\right] \\ &\text{s.t. } (c, b, \mathbf{x}) \in \mathcal{B}_k(p_f, \mathbf{p}) \end{aligned}$$

and the three markets clear:

$$\sum_{k=1}^m c_k = \sum_{k=1}^m \bar{c}_k, \qquad \sum_{k=1}^m b_k = \sum_{k=1}^m \bar{b}_k = 0, \qquad \sum_{k=1}^m \mathbf{x}_k = \sum_{k=1}^m \bar{\mathbf{x}}_k = \mathbf{1}$$

The following two results are standard.

Theorem 3.1. *The artificial economy $\mathcal{E}_{\mathbf{y}}$ has a unique equilibrium. The equilibrium asset prices are defined implicitly as follows. Let*

$$\mathbf{f} := \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{5}$$

The equilibrium price of the bond, p_f, is given by

$$\log(p_f) = -\frac{\bar{\rho}}{m}\left(\mathbf{1}'\mathbf{f} + \frac{\bar{\rho}}{2m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right) + \frac{\bar{\rho}}{m}\sum_{k=1}^m \bar{c}_k \tag{6}$$

and the equilibrium price of the risky assets is

$$\mathbf{p} = p_f \mathbf{f} \tag{7}$$

In equilibrium, the portfolio holding of risky assets is:

$$\mathbf{x}_k = \frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1} \tag{8}$$

The proof of Theorem 3.1 is in Appendix A.

Note that in the artificial economy, the separation result holds: each investor holds the market. From Equation 7, we see that $\mathbf{f}$ is the vector of synthetic *forward prices*. In other words, the price of the risky assets denominated in units of the bond is $\mathbf{f}$. Next, we show that the CAPM risk–return relation holds in the equilibrium in the artificial economy.

Let $r_{\text{mkt}} = \mathbf{1}'\mathbf{v}/(\mathbf{1}'\mathbf{p}) - 1$, $r_i = v_i/p_i - 1$, and

$$\beta_i = \frac{\text{cov}(r_{\text{mkt}}, r_i|\mathbf{y})}{\text{var}(r_{\text{mkt}}|\mathbf{y})} \tag{9}$$

Theorem 3.2. *In the artificial economy $\mathcal{E}_{\mathbf{y}}$, the CAPM holds:*

$$E[r_i|\mathbf{y}] = r_f + \beta_i(E[r_{\text{mkt}}|\mathbf{y}] - r_f) \tag{10}$$

The proof of Theorem 3.2 is in Appendix A. So far, our choice of $\mathbf{y}$ has been arbitrary. We now introduce a strong assumption:

(GR) The random vector $\mathbf{y}$ is such that

$$\begin{cases} \forall k \in \mathcal{NI} & E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} & \text{var}(\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \\ \forall k \in \mathcal{I} & E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} & \text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \end{cases}$$

A random vector that trivially satisfies (GR) is $\mathbf{s}$, the concatenation of all signals. Therefore, (GR) is not vacuous.

Our next goal is to show that when $\mathbf{y}$ satisfies (GR), the equilibrium prices in the artificial economy $\mathcal{E}_{\mathbf{y}}$ are the equilibrium prices in the actual economy. To that end, we first need to establish that the prices in the artificial economy carry the same information as $\mathbf{y}$, so that the conditioning in (GR) on $\mathbf{y}$ can be replaced with conditioning on prices.

Because equilibrium prices are not normally distributed, we find it easier to replace the conditioning on random vectors by conditioning on the σ–algebras generated by the random vectors. We have

Lemma 3.3. *Assume* $\mathbf{y}$ *satisfies* (GR)*, and let* p_f *and* $\mathbf{p}$ *be the equilibrium prices in the artificial economy* $\mathcal{E}_{\mathbf{y}}$*. Then,* $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f}) = \sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) \subseteq \sigma(\mathbf{y})$.

The proof of Lemma 3.3 is in Appendix A. We also need the following simple result.

Lemma 3.4. *Let* $\mathcal{G}$ *and* $\mathcal{F}$ *be two sigma algebras such that* $\mathcal{G} \subseteq \mathcal{F}$ *(*$\mathcal{F}$ *contains more information than* $\mathcal{G}$*).*

1. *If* $E[\mathbf{v}|\mathcal{F}] \in \mathcal{G}$*, then* $E[\mathbf{v}|\mathcal{G}] = E[\mathbf{v}|\mathcal{F}]$.
2. *If* $E[\mathbf{v}|\mathcal{F}] \in \mathcal{G}$ *and* $\mathrm{var}(\mathbf{v}|\mathcal{F}) \in \mathcal{G}$*, then* $\mathrm{var}(\mathbf{v}|\mathcal{G}) = \mathrm{var}(\mathbf{v}|\mathcal{F})$.

The proof of Lemma 3.4 is in Appendix A.

Corollary 3.5. *Assume* $\mathbf{y}$ *satisfies* (GR)*. Let* p_f *and* $\mathbf{p}$ *be the equilibrium prices in the artificial economy* $\mathcal{E}_{\mathbf{y}}$*. We have*

1. *The distribution of* $\mathbf{v}$ *conditional on the realization of* p_f *and* $\mathbf{p}$ *is multivariate normal with the same mean and variance as the distribution of* $\mathbf{v}$ *conditional on* $\mathbf{y}$*:*

$$E\left[\mathbf{v}\middle|\mathbf{p}, p_f\right] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \mathrm{var}\left(\mathbf{v}\middle|\mathbf{p}, p_f\right) = \boldsymbol{\Sigma}_{\mathbf{v}\mathbf{v}|\mathbf{y}}$$

2. *For all* $k \in \mathcal{NI}$*, the distribution of* $\mathbf{v}$ *conditional on the realization of* p_f*,* $\mathbf{p}$*, and* $\mathbf{s}_k$ *is multivariate normal with the same mean and variance as the distribution of* $\mathbf{v}$ *conditional on* $\mathbf{y}$*:*

$$E\left[\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \mathrm{var}\left(\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \tag{11}$$

3. *For all* $k \in \mathcal{I}$*, the distribution of* $\mathbf{1}'\mathbf{v}$ *conditional on the realization of* p_f*,* $\mathbf{p}$*, and* $\mathbf{s}_k$ *is normal with the same mean and variance as the distribution of* $\mathbf{1}'\mathbf{v}$ *conditional on* $\mathbf{y}$*. In other words,*

$$E\left[\mathbf{1}'\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \mathrm{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{12}$$

The proof of Corollary 3.5 is in Appendix A.

Corollary 3.6. *Assume that* $\mathbf{y}$ *satisfies (GR). The equilibrium prices in the artificial economy* $\mathcal{E}_{\mathbf{y}}$ *form a rational expectation equilibrium in the actual economy. The allocations are identical in both equilibria.*

Proof of Corollary 3.6. Assume that investors in the actual economy face prices that are the equilibrium prices in the artificial economy.

Let $k \in \mathcal{NI}$. The investor's objective is to maximize $E[U_k(c, b, \mathbf{x}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f]$. We have

$$E[U_k(c, b, \mathbf{x}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] \underset{(11)}{=} -e^{-\rho_k c} - e^{-\rho_k \left(b + \mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_k}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)} = E[U_k(c, b, \mathbf{x}'\mathbf{v})|\mathbf{y}]$$

Because the budget constraints are the same in both the actual economy and the artificial economy, we conclude that the equilibrium allocation of a nonindex investor in the artificial economy is also optimal in the actual economy.

Let $k \in \mathcal{I}$. The investor's objective is to maximize $E[U_k(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f]$. We have

$$
\begin{aligned}
& \max_{c,b,q} E[U_k(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
\underset{(12)}{=} & \max_{c,b,q} -e^{-\rho_k c} - e^{-\rho_k \left(b + q\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_k}{2} q^2 \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{v}\mathbf{v}|\mathbf{y}}\mathbf{1}\right)} && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
= & \max_{c,b,q} E[U_k(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{y}] && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
= & \max_{c,b,\mathbf{x}} E[U_k(c, b, \mathbf{x}'\mathbf{v})|\mathbf{y}] && \text{s.t. } (c, b, \mathbf{x}) \in \mathcal{B}_k(p_f, \mathbf{p})
\end{aligned}
$$

where the last equality arises because in the equilibrium in the artificial economy it is optimal for investors to hold the market (see Theorem 3.1).

We have shown that the equilibrium allocations in the artificial economy are optimal in the actual economy. We are left to show that the three markets clear. Those allocations clear the three markets in the artificial economy; therefore, they also clear the markets in the actual economy.

■

4 A Partially Revealing Equilibrium

Guided by hindsight, we define the vector $\mathbf{g} \in R^n$ as follows:

$$\mathbf{g} := (m\boldsymbol{\Sigma}_{\mathbf{v}\mathbf{v}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}})^{-1} \boldsymbol{\Sigma}_{\mathbf{v}\mathbf{v}}\mathbf{1} \tag{13}$$

and for the remainder of this paper, we set $\mathbf{y} \in R^{n+1}$ to

$$\mathbf{y} := \begin{bmatrix} \frac{1}{|\mathcal{NI}|} \sum_{k \in \mathcal{NI}} \mathbf{s}_k \\ \mathbf{g}' \sum_{k=1}^{m} \mathbf{s}_k \end{bmatrix}_{(n+1)\times 1} \tag{14}$$

Theorem 4.1. *The vector* $\mathbf{y}$ *satisfies* (GR).

The proof of Theorem 4.1 is in Appendix B.

According to Corollary 3.6 and Theorem 4.1, there is a rational expectation equilibrium with prices and allocations identical to those in the artificial economy $\mathcal{E}_{\mathbf{y}}$. In particular, every investor, whether indexer or nonindexer, holds the market portfolio, and a conditional CAPM holds. Importantly, because the first n coordinates of $\mathbf{y}$ depend on the set $\mathcal{NI}$, the equilibrium prices depend on the specific partition of investors into indexers and nonindexers. The next corollary points out that the equilibrium prices are informationally equivalent to $\mathbf{y}$.

Corollary 4.2. *We have* $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f}) = \sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) = \sigma(\mathbf{y})$.

The proof of Corollary 4.2 is in Appendix C. For notational consistency and brevity, we continue to condition on $\mathbf{y}$. Corollary 4.2 implies that doing so is equivalent to conditioning on equilibrium prices.

We have constructed a rational expectation equilibrium. But what is the information content of $\mathbf{y}$ (or, equivalently, the equilibrium prices $(\mathbf{p}, p_f)$)? The first n coordinates of y are the average of the nonindexers' signals. It is a well-known result in statistics that, in the case of normal distribution with a known variance, the sample mean is a sufficient statistic for the mean. Here, the mean is the asset payoff vector, $\mathbf{v}$, and the variance is known. Thus, the first n coordinates of $\mathbf{y}$ contain all the information about $\mathbf{v}$ that there is in the entire pool of nonindexers' private information.

The next theorem articulates the informational content of the $(n+1)$th coordinate of $\mathbf{y}$, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m} \mathbf{s}_k$.

Theorem 4.3. *We have*

1. $E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|y_{n+1}]$, *and* $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|y_{n+1})$.

2. *Given* $\mathbf{s}$, y_{n+1} *is a minimal sufficient statistic for* $\mathbf{1}'\mathbf{v}$.

In part 2 of the Theorem, $\mathbf{s}$ plays the role of the data, y_{n+1} the role of the statistic, and $\mathbf{1}'\mathbf{v}$ the role of the parameter.

The proof of Theorem 4.3 is in Appendix C.

The next theorem shows that the forward price of the market portfolio, $\mathbf{1}'\mathbf{f}$, contains *all* the information in the economy about the future payoff of the market portfolio. The proof is based on the observation that $\mathbf{1}'\mathbf{f}$ is informationally equivalent to y_{n+1}.

Theorem 4.4 (Sufficient Statistics). *We have*

1. $E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|\mathbf{1}'\mathbf{f}]$, *and* $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|\mathbf{1}'\mathbf{f})$.

2. *Given* $\mathbf{s}$, $\mathbf{1}'\mathbf{f}$ *is a minimal sufficient statistic for* $\mathbf{1}'\mathbf{v}$.

The proof of Theorem 4.4 is in Appendix C.

5 Comparative Statics

So far, the sets $\mathcal{NI}$ and $\mathcal{I}$ have been fixed. In this section, we study how the equilibrium outcomes depend on the level of index investment. The next result is based on the observation that the $(n+1)$th element of $\mathbf{y}$, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m} \mathbf{s}_k$, is independent of the the partition of the group of investors into indexers and nonindexers.

Theorem 5.1. *Fix a joint realization of* $\mathbf{v}$ *and* $\mathbf{s}$. *Then,* p_f, r_f, *the price of the market portfolio, the return on the market portfolio, and the capital market line do not depend on the specific partition of investors into indexers and nonindexers.*

The proof of Theorem 5.1 is in Appendix C.

Figure 1 shows an example in which the realization of the signals is fixed. The figure depicts, in the volatility–return plane, the capital market line and the efficient frontiers for two different partitions of the set of investors. As stated in Theorem 5.1, the capital market line is the same in both examples. The figure shows the most common situation we find in the many simulations we tried: The efficient frontier that corresponds to a partition of the investors with a large set of indexers is nested in the efficient frontier that corresponds to a partition with a smaller set of indexers.

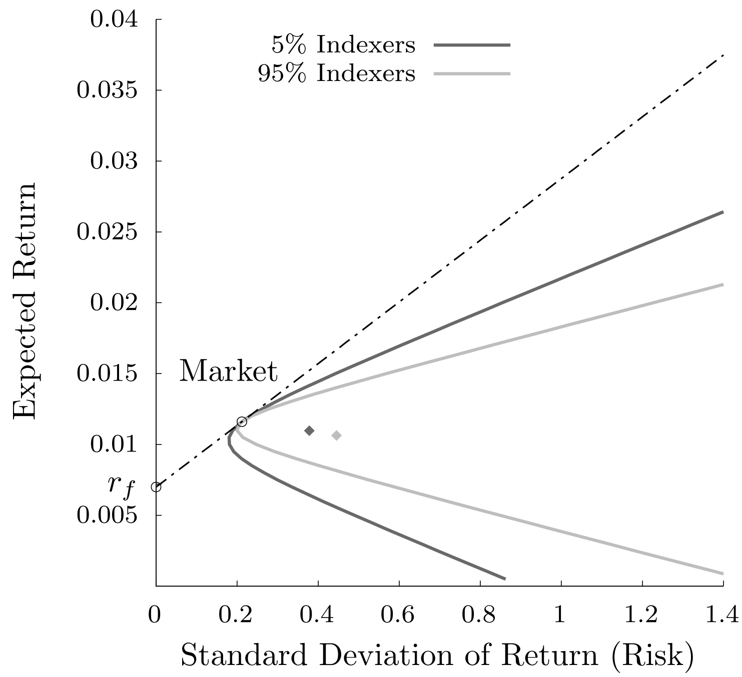


Figure 1: The efficient frontiers and the capital market line in the volatility–return plane. These are the realized efficient frontiers in an example with six risky assets and 10,000 investors. The capital market line and the position of the market portfolio (*tangency point*) are the same in both cases. The two points in the center of the frontiers stand for the same risky asset. The conditional Sharpe ratios are the slopes of the straight lines joining these points and the risk-free asset (*denoted by* r_f). These lines are not drawn, but it is apparent the Sharpe ratio is lower (the slope of the invisible line is gradual) when 95% of the investors are indexers.

In Theorem 5.1, the realizations of the signals and payoffs were fixed. In contrast, in the remainder of this section, we study outcomes that depend solely on conditional covariance matrices, which are matrices of scalars. Even the conditional Sharpe ratio (Theorem 5.3), after algebraic simplifications, can be expressed in terms of scalars taken from covariance matrices. Thus, in the rest of this section, we need not assume that the realization of $\mathbf{v}$ or the realization of $\mathbf{s}$ is fixed. In particular, when we compare two partitions of the set of investors, one with $\mathcal{I}_1$ and the other with $\mathcal{I}_2$ such that $|\mathcal{I}_1| < |\mathcal{I}_2|$, we do not assume that $\mathcal{I}_1 \subset \mathcal{I}_2$ (the two sets may even be disjoint).

Theorem 5.2 (Conditional Variance of the Portfolio's Payoff). *Let* $\mathbf{x} \in R^n$ *be a portfolio, and consider changes as we increase* $|\mathcal{I}|$. *If* $\mathbf{x}$ *is a scalar multiplication of the market portfolio, then the conditional variance* $\mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ *does not change. For all other portfolios,* $\mathrm{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ *strictly increases.*

The proof of Theorem 5.2 is in Appendix C.

For any portfolio with a return (i.e., a portfolio with a nonzero cost), we define the conditional Sharpe ratio to be

$$\frac{E\left[\dfrac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \,\middle|\, \mathbf{y}\right] - r_f}{\sqrt{\mathrm{var}\left(\dfrac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \,\middle|\, \mathbf{y}\right)}}$$

Theorem 5.3 (Conditional Sharpe Ratio). *Let* $\mathbf{x} \in R^n$ *be a portfolio with a positive Sharpe ratio, and consider changes as we increase* $|\mathcal{I}|$. *If* $\mathbf{x}$ *is a scalar multiplication of the market portfolio, then the Sharpe ratio does not change. For all other portfolios, the Sharpe ratio strictly decreases.*

The proof of Theorem 5.3 is in Appendix C.

We can always write

$$r_i = r_f + \beta_i(r_{\mathrm{mkt}} - r_f) + \epsilon \tag{15}$$

where

$$\epsilon \equiv r_i - r_f - \beta_i(r_{\text{mkt}} - r_f)$$

Because the conditional CAPM holds, we know that $E[\epsilon|\mathbf{y}] = 0$. From the definition of β_i (see Equation 9), it is immediately apparent that $E\left[(r_{\text{mkt}} - r_f)\epsilon \,|\, \mathbf{y}\right] = 0$. Thus, the theory of linear regression is applicable, and it is meaningful to use R^2 as a measure of the strength of the CAPM regression (Equation 15). In practice, one computes the sample R^2. We compute the population R^2. We now ask how R^2 depends on $|\mathcal{I}|$.

Theorem 5.4 (Conditional $\boldsymbol{R}^2$). *For every asset i, R^2 of the CAPM relation decreases with $|\mathcal{I}|$.*

The proof of Theorem 5.4 is in Appendix C.

Fix i. Let $\mathbf{1} - \mathbf{e}_i$ be the portfolio that includes all assets except for asset i. Denote the payoff and return by v_{-i} and r_{-i}, respectively.

Theorem 5.5 (Conditional Correlation in Returns). *Assume* $\text{corr}(r_i, r_{-i}|\mathbf{y}) > 0$*. Then, this correlation decreases with $|\mathcal{I}|$.*

The proof of Theorem 5.5 is in Appendix C.

Theorem 5.6 (Unconditional Variance of Portfolio's Price). *Let $\mathbf{x} \in R^n$ be a portfolio, and consider changes as we increase $|\mathcal{I}|$. If $\mathbf{x}$ is a scalar multiplication of the market portfolio, then* $\text{var}(\mathbf{x}'\mathbf{p})$ *does not change. For all other portfolios,* $\text{var}(\mathbf{x}'\mathbf{p})$ *strictly decreases.*

The proof of Theorem 5.6 is in Appendix C.

Theorem 5.7 (Unconditional Correlation in Asset Prices). *For every i, assume* $\text{corr}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1} - \mathbf{e}_i)'\mathbf{p}) > 0$*. Then, this correlation strictly increases with $|\mathcal{I}|$.*

Proof of Theorem 5.7. We have

$$\text{corr}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p}) = \frac{\text{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p})}{\sqrt{\text{var}(\mathbf{e}_i'\mathbf{p})}\sqrt{\text{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p})}}$$

To see that the denominator decreases and the numerator increases, we write

$$\text{var}(p_{\text{mkt}}) = \text{var}(\mathbf{e}_i'\mathbf{p}) + \text{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p}) + 2\,\text{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p}))$$

From Theorem 5.6, $\text{var}(p_{\text{mkt}})$ does not change, whereas both $\text{var}(\mathbf{e}_i'\mathbf{p})$ and $\text{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p})$ decrease. Hence, $\text{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p})$ increases.

■

Like in other conditional CAPM models, betas are realizations of random variables. In our model, the distributions of betas are ratios of Gaussian random variables, and therefore their moments do not exist. However, we can draw their densities. Figure 2 depicts a typical example. We see that (i) the dispersion of a beta decreases with $\mathcal{I}$, while (ii) the abscissa of the global maximum of its density is unaffected.

6 The Limiting Case

We have already pointed out that, regardless of what the set of index investors is, the concatenation of all signals, $\mathbf{s}$, satisfies (GR). Therefore, regardless of what the partition of investors into indexers and nonindexers is, there is always a fully revealing equilibrium. We think that in the presence of indexers, the fully revealing equilibrium is unappealing.

That said, the fully revealing equilibrium is the equilibrium that corresponds to the limiting case of the partially revealing equilibrium, in which all investors are nonindexers; that is, $|\mathcal{NI}| = m$. When all investors are nonindexers, the $(n+1)$th coordinate of $\mathbf{y}$ is a linear function of the first n coordinates. Therefore, $\mathbf{y}$ is degenerate, and its covariance matrix is noninvertible. We can still use the artificial economy apparatus. We simply remove the

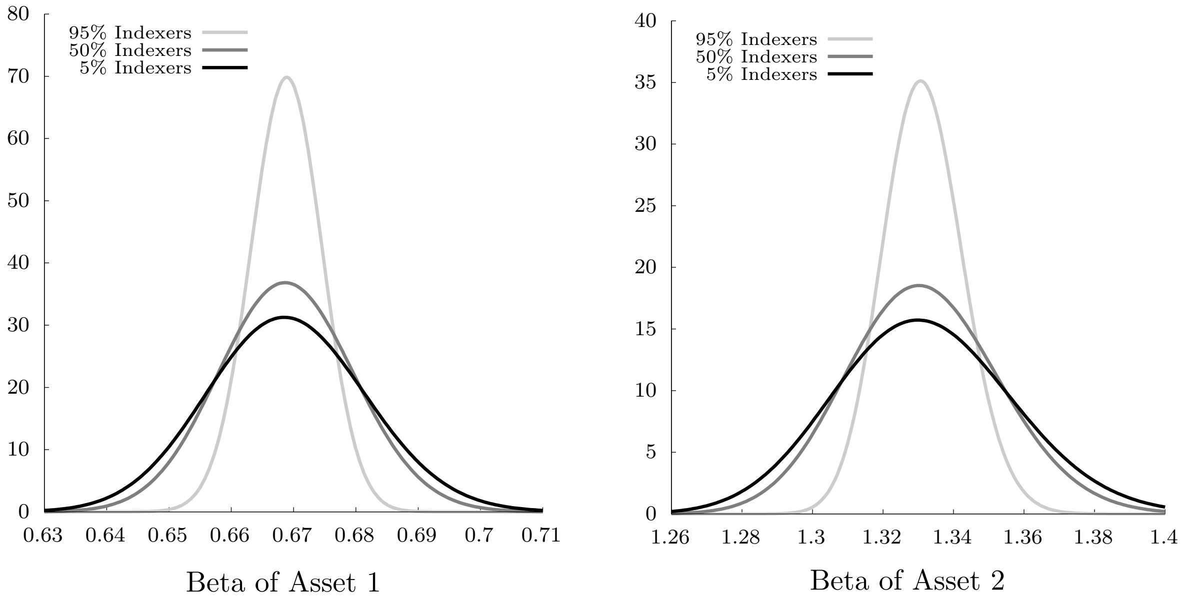


Figure 2: Densities of betas. These are the densities of betas in an example with two risky assets. The densities are shown for three different levels of index investment. The parameters in this example are $n = 2$, $m = 10,000$, , $\bar{\rho} = 2$, $\boldsymbol{\mu}_{\mathbf{v}} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$, $\boldsymbol{\Sigma}_{\mathbf{vv}} = \begin{pmatrix} 1000 & 10 \\ 10 & 2000 \end{pmatrix}$, and $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} = 1,000 \times \boldsymbol{\Sigma}_{\mathbf{vv}}$.

redundant $(n+1)$th coordinate, and we are left with the nondegenerate n-dimensional vector, $\frac{1}{m}\sum_{k=1}^{m} \mathbf{s}_k$. This vector satisfies (GR) because a sample mean of a multivariate normal random vector with a known covariance matrix is a sufficient statistic for its mean. Moreover, this is a fully revealing equilibrium.[15] Therefore, the limiting case in which all investors are nonindexers is a fully revealing equilibrium.

When all investors are indexers, it is not clear how individual assets are priced. It is conceivable that there are infinitely many equilibria, all of which agree on the price of the market portfolio but disagree on the prices of individual assets. Our model can be used to pick one of these equilibria, the equilibrium that corresponds to the limiting case of our model in which all investors are indexers.

When all investors are indexers, the random vector $\mathbf{y}$ is not defined (because of the division by

[15]The artificial economy in which everyone observes the sufficient statistic has the same outcomes as the artificial economy in which everyone observes $\mathbf{s}$ (Grossman, 1978).

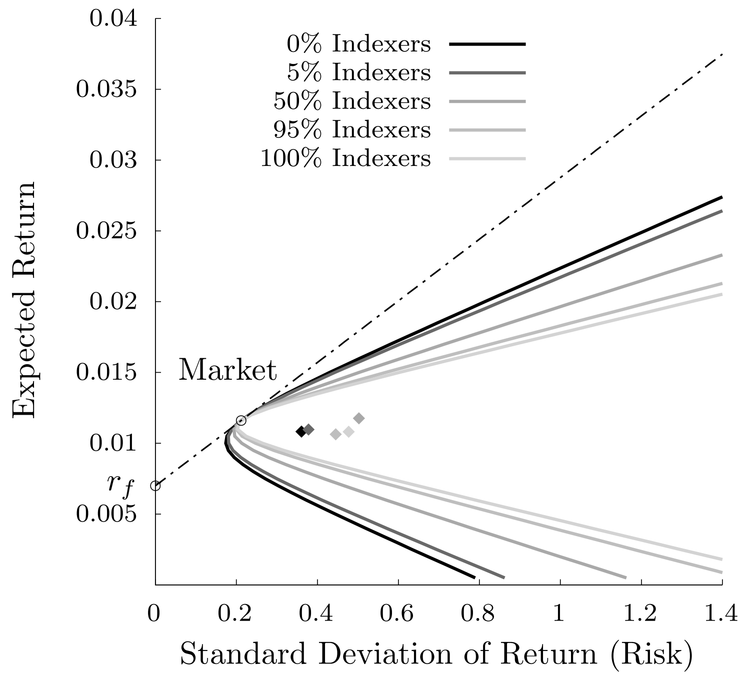


Figure 3: The efficient frontiers and the capital market line. This is a continuation of the example shown in Figure 1. These are the realized efficient frontiers in an example with six risky assets and 10,000 investors. The capital market line and the position of the market portfolio (*tangency point*) are the same in all cases. The five points in the center of the frontiers stand for the same risky asset. The conditional Sharpe ratios are the slopes of the straight lines joining these points and the risk-free asset (*denoted by* r_f). These lines are not drawn, but it is apparent the Sharpe ratio is lower (the slope of the invisible line is gradual) when there are more indexers.

zero of the first n coordinates). It is natural to remove the first n coordinates altogether, and use only the $(n+1)$th coordinate. Indeed, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m}$ satisfies (GR) because, according to Theorem 4.3, this random variable is a sufficient statistic for $\mathbf{1}'\mathbf{v}$. Hence, the conditions in (GR) with regard to the set of indexers are satisfied. Moreover, the conditions in (GR) with regard to the set of nonindexers are trivially satisfied because the set of nonindexers is empty. Hence, the equilibrium in the artificial economy in which everyone observes only y_{n+1} is also a rational expectation equilibrium when all investors are indexers.

Figure 3 repeats the same example shown in Figure 1, except that we add the middle case (50% indexers) and the two extreme cases. It is apparent from the figures that even when

100% of the investors are indexers, the efficient frontier is not degenerated.

7 Large Economy and Optimal Index Investment

In this section, we follow Hellwig (1980) and study a large economy by means of taking the limit of a sequence of properly scaled competitive economies. Here, the size of the economy grows to infinity, while the accuracies of the private signals shrink to zero. When the economy is large, the price taking assumption is appealing. When the signals are pure noise, the assumption that signals are costless is appealing. We show that as we pass to the limit (i) the indexers' constraint to remain on the capital market line is no longer binding, and (ii) the qualitative comparative statics results of the paper continue to hold.

> **Comment 2:** The results in this section depend on what we call "properly scaled sequence." Typically, in models of large economies, the aggregation of all signals reveals the value of the asset (to avoid the fully revealing outcome, noise is separately introduced). In those models, the scaling scheme meets the conditions for the law of large numbers, or the models assume from the onset the presence of a continuum of investors and posit a law of large numbers for a continuum of random variables.
>
> In this paper, we capture the idea that layman investors costlessly observe extremely noisy signals that are readily available in their environment. The aggregation of these signals is valuable but not sufficient to reveal the value of the asset. To sum up, the results in this section are not an artifact of some arbitrary scaling scheme. The scaling scheme captures the essence of our model.

We now describe the growing sequence of economies whose limit we want to study. When we need to emphasize ordinality, we use superscript m. We let $\mathcal{I}$ and $\mathcal{NI}$ denote a partition of the set of natural numbers. Though the number of investors is countably infinite, in the mth economy only m of them are included. We let $\mathcal{I}^m = \mathcal{I} \cap \{1, \ldots, m\}$ and $\mathcal{NI}^m = \mathcal{NI} \cap \{1, \ldots, m\}$ denote the partition of investors into indexers and nonindexers in the mth economy. We let $\pi^m = |\mathcal{I}^m|/m$ denote fraction of indexers in the mth economy, and we assume that $\lim_m \pi^m$ exists and it is strictly between zero and one.

The kth investor has a coefficient of risk aversion ρ_k, and the investor either belongs to the set of indexers or the set of nonindexers. Note that if the kth investor is part of the mth economy (i.e. $k \leq m$), then the investor is also included in every larger economy. Let $\bar{\rho}^m$ be the harmonic mean of the first m ρ_k's. We assume that $\lim_m \bar{\rho}^m$ exists and it is strictly positive.

We let $\bar{\mathbf{v}} \sim \mathcal{N}(\bar{\boldsymbol{\mu}}_{\mathbf{v}}, \bar{\boldsymbol{\Sigma}}_{\mathbf{vv}})$ and $\bar{\boldsymbol{\epsilon}}_k \sim N\left(\mathbf{0}, \bar{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}\right)$ be non-degenerate multivariate normal random vectors that are jointly normal and mutually independent. In the mth economy, the payoff of the risky assets is $\mathbf{v}^m = m\bar{\mathbf{v}}$. Dividing the payoff by m, we see the payoff per investor does not depend on m.[16] The noise term in the kth investor's signal is $\boldsymbol{\epsilon}_k^m = m^{3/2}\bar{\boldsymbol{\epsilon}}_k$. Dividing the kth signal by m, we see that the signal, $\mathbf{s}_k^m = \mathbf{v}^m + \boldsymbol{\epsilon}_k^m$, is informationally equivalent to $\bar{\mathbf{v}} + m^{1/2}\bar{\boldsymbol{\epsilon}}_k$. It is apparent that larger m is associated with greater noise.

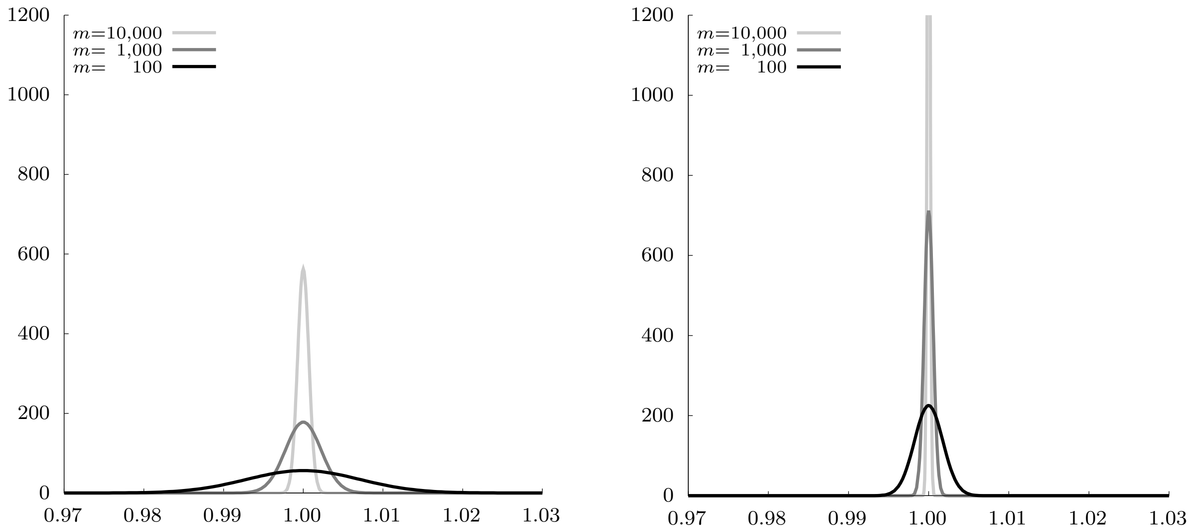
Each of the investors is endowed with units of time zero consumption good, bonds, and a portfolio of risky assets. These endowments can change from economy to economy, but we require that in the mth economy

$$\frac{1}{m}\sum_{k=1}^{m} \bar{c}_k^m = \bar{c}^m, \qquad \frac{1}{m}\sum_{k=1}^{m} \bar{b}_k^m = 0, \qquad \frac{1}{m}\sum_{k=1}^{m} \bar{\mathbf{x}}_k^m = \frac{1}{m}\mathbf{1}$$

where $\bar{c}^m$ is some converging sequence.

This completes the description of the sequence of economies. The theory we have developed

[16]The setup is equivalent to the assumption that in the mth economy, each risky asset has m shares outstanding, and each share has the same payoff distribution regardless of what m is; i.e. each share has the payoff $\bar{\mathbf{v}}$. Since we have hardcoded the assumption that the number of shares outstanding is one, the scaling scheme is a workaround.



Density of x_2^m/x_1^m when $\bar{\rho}^m = 1$ Density of x_2^m/x_1^m when $\bar{\rho}^m = 4$

Figure 4: Optimality of Index Investment in the large economy. $\mathbf{x}^m = [x_1^m \ \dots \ x_n^m]'$ is the optimal unconstrained portfolio of an indexer. The figure shows the densities of x_2^m/x_1^m, demonstrating $\lim_m x_2^m/x_1^m = 1$. The parameters in this example are $n = 2$, at every m, $\pi^m = 1/2$, and the investor's coefficient of risk aversion and $\bar{\rho}^m$ are either 1 (left figure) or 4 (right figure). $\bar{\boldsymbol{\Sigma}}_{\mathbf{vv}} = \left(\begin{smallmatrix} 10 & 1 \\ 1 & 20 \end{smallmatrix}\right)$, and $\bar{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}} = 10 \times \bar{\boldsymbol{\Sigma}}_{\mathbf{vv}}$. The values of $\bar{\boldsymbol{\mu}}_{\mathbf{v}}$ and $\bar{c}^m$ are irrelevant for the purpose of computing these densities.

so far applies to each of the finite economies in the sequence. In particular, the equilibrium portfolio holding of the kth investor, regardless of whether $k \in \mathcal{I}$ or $k \in \mathcal{NI}$, is (see 8)

$$\frac{1}{m}\frac{\bar{\rho}^m}{\rho_k}\mathbf{1}$$

Theorem 7.1 (The Optimality of Index Investment in the Large Economy). *Fix $k \in \mathcal{I}$. For any $m \geq k$, let $\mathbf{x}_k^m = \begin{bmatrix} x_{k1}^m & \dots & x_{kn}^m \end{bmatrix}'$ denote the optimal portfolio of this index investor when the investor takes the equilibrium prices in the mth economy as given, but ignores the constraint to stay on the capital market line. Then*

$$\mathbf{x}_k^m = \frac{1}{m}\frac{\bar{\rho}^m}{\rho_k}\mathbf{1} + \frac{1}{\rho_k}\mathbf{e}_k^m$$

where $\frac{1}{\rho_k}\mathbf{e}_k^m$ is a zero mean random portfolio (i.e. $\mathbf{e}_k^m$ depends on the realizations of signals)

Moreover,

$$\lim_m m\mathbf{e}_k^m = \mathbf{0}_{n\times 1} \qquad \text{with probability one}$$

and in particular,

$$\forall i \in \{1, \ldots, n\}, \quad \lim_m \frac{x_{ki}^m}{x_{k1}^m} = 1 \qquad \text{with probability one}$$

The proof of Theorem 7.1 is in Appendix C.

In a large economy, an index investor is indifferent between constraining himself to the capital market line or not. Hence, any partition of investors into indexers and nonindexers is rational. Our final task is to illustrate that the impact of index investment on asset prices passes to the limit.

The qualitative comparative statics analysis presented in Section 5 is mostly the study of the sign of certain derivatives. It is possible that a sequence of those derivatives be strictly positive, but its limit is nevertheless zero. Our goal is to prove that the limit of those derivatives is bounded away from zero, and hence the qualitative results pass to the limit.

Consider the conditional Sharpe ratio for the return on a portfolio $\mathbf{x}$. In Theorem 5.3, we proved that the Sharpe ratio strictly decreases as we increase the numbers of index investors, provided the portfolio is not a scalar multiplication of the market portfolio. Clearly, Theorem 5.3 can be expressed in terms of the proportion of index investors, rather that the number of index investors.

Lemma 7.2. *Consider the sequence of economies parametrized by m. Let $\bar{\mathbf{x}}$ be a specific portfolio. There exists a function f such in the mth economy, the conditional Sharpe ratio equals*

$$\bar{\rho}^m f(\bar{\mathbf{x}}, \overline{\mathbf{\Sigma}}_{\mathbf{vv}}, \overline{\mathbf{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \pi^m)$$

In addition, if $\mathbf{x}$ is not a positive scalar multiplication of the market portfolio, then $\dfrac{\partial f}{\partial \pi}(\bar{\mathbf{x}}, \overline{\mathbf{\Sigma}}_{\mathbf{vv}}, \overline{\mathbf{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \pi^m) < 0$.

The proof of Lemma 7.2 is in Appendix C.

The function f in the lemma is homogenous of degree zero because Sharpe ratio is a property of the return. The function f in the lemma is strictly decreasing with respect to π because Theorem 5.3 can be applied to each of the economies in the sequence. The term $\bar{\rho}^m$ shows up because it is inherited from the equilibrium prices $(p_f^m, \mathbf{p}^m)$. In other words, for the purpose of computing the Sharpe ratio, heterogeneity in risk aversion is relevant. Importantly, $\mathbf{y}^m$, m, and how endowments are distributed do not show up. In particular, if $m_1 \neq m_2$, but $\bar{\rho}^{m_1} = \bar{\rho}^{m_2}$ and $\pi^{m_1} = \pi^{m_2}$, then Sharpe ratio is the same in both economies.

We now consider two sequences of finite economies that are identical in every aspect, except for the decomposition of investors into indexers and nonindexers. We assume that $\lim_m \pi_1^m < \lim_m \pi_2^m$. We fix a portfolio $\bar{\mathbf{x}}$. We want to show that the Sharpe ratio is smaller in the large economy with larger fractions of indexers:

$$\lim_m f(\bar{\mathbf{x}}, \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}, \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \pi_1^m) \overset{?}{>} \lim_m f(\bar{\mathbf{x}}, \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}, \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \pi_2^m)$$

According to Lemma 7.2, f is differentiable w.r.t. π. Hence it is also continuous, and the inequality above can be written as

$$f(\bar{\mathbf{x}}, \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}, \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \lim_m \pi_1^m) \overset{?}{>} f(\bar{\mathbf{x}}, \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}, \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}, \lim_m \pi_2^m)$$

According to Lemma 7.2, f is decreasing in π, and this confirms the inequality. In a similar manner, we can show the other qualitative results we proved in Section 5 pass to the limit.

8 Concluding Remarks

Markowitz (1952) studies mean-variance portfolio selection and discovers the efficient frontier. Tobin (1958) adds the risk free asset, discovers the separation theorem (Tobin 1958, page 84), and explains that "Markowitz's main interest is prescription of rules of rational behaviour for investors; [while] the main concern of [my] paper is the implications for economic theory, mainly comparative statics, that can be derived from assuming that investors do in fact follow [Markowitz's] rules" (Tobin 1958, page 85). We add costless private signals, and the main concern of our paper is the implications, mainly comparative statics, that can be derived from assuming that some investors follow Tobin's rule. In the same manner that Markowitz's rules are compatible with Tobin's extension, Tobin's rule is compatible with our extension: In our model, it is optimal for nonindex investors to index.

The model shows that index investment is not benign. As more non-index investors become index investors, the proportion of idiosyncratic risk to total risk increases, the R^2 of the CAPM regression decreases, comovement in returns decreases, comovement in asset prices increases. For any portfolio other than the market portfolio, the portfolio's Sharpe ratio decreases, and the variance of the portfolio's payoff increases. The following examples illustrate some of the implications of our model.

There is a known link between corporate underinvestment in real projects and total risk. Panousi and Papanikolaou (2012) find that the link is stronger when managers hold a large equity stake in the firm, and Deng, Chen, and Kong (2014) go even further and document that the link tends to be insignificant when managerial ownership is very low. Those empirical papers support the notion that when managers are exposed to firm's total risk, they are reluctant to invest in risky real projects. Our model shows that the larger the set of index investors is, the greater is the uncertainty about future value of individual assets. Thus, our model suggests a possible link between index investment and corporate underinvestment.

Similarly, the cost of financial hedging depends on the volatility of future payoff of the asset. In particular, our model suggests the corporate practice of awarding options to management is costlier in the presence of index investment.

Finally, a firm's ability to raise financing (bank loans, or bonds) is likely to depend on its stock price. Our model shows that the larger the set of index investors is, the greater is the comovement in pricing. It is therefore conceivable that in the presence of index investors, a large negative shock to one firm may make it difficult for another firm to raise the capital needed to invest in real projects or repay old debt. The former implies corporate underinvestment; the latter implies financial contagion.

Appendices

A Artificial Economies: Proofs

Proof of Theorem 3.1. As stated in the theorem, $\mathbf{f}$ is given in Equation 5; the price of the bond, p_f, is given in Equation 6; and $\mathbf{p}$ is given in Equation 7. Our goal is to show not only that these prices clear the markets but also that they are the only prices that can clear the markets.

In the artificial economy, the problem of the kth investor is

$$\begin{aligned}
&\max_{c,b,\mathbf{x}} E\left[U_k(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{y}\right], \text{ subject to } (c,b,\mathbf{x}) \in \mathcal{B}_k(p_f,\mathbf{p}) \\
&= \max_{c\in R,\mathbf{x}\in R^n} E\left[U_k\left(c, (\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + (\bar{\mathbf{x}}_k - \mathbf{x})'\frac{1}{p_f}\mathbf{p}, \mathbf{x}'\mathbf{v}\right)\Bigg|\,\mathbf{y}\right] \\
&= \max_{c\in R,\mathbf{x}\in R^n} -e^{-\rho_k c} - e^{-\rho_k\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + (\bar{\mathbf{x}}_k - \mathbf{x})'\frac{1}{p_f}\mathbf{p} + \mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_k}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)} \\
&= \max_{c\in R} -e^{-\rho_k c} + e^{-\rho_k\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p}\right)} \times \max_{\mathbf{x}\in R^n} -e^{-\rho_k\left(\mathbf{x}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\rho_k}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)}
\end{aligned} \tag{A.1}$$

We solve the maximization problem "backward." The first-order condition with respect to $\mathbf{x}$ is

$$-\frac{1}{p_f}\mathbf{p} + \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_k}{2}\left(\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} + \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}'\right)\mathbf{x} = 0$$

We replace $\mathbf{x}$ with $\mathbf{x}_k$ to emphasize that this is the optimal portfolio of the kth investor. Using the symmetry of the covariance matrix, we rearrange and obtain

$$\mathbf{x}_k = \frac{1}{\rho_k}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}^{-1}\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) \tag{A.2}$$

When summing all portfolios, market clearing implies that in equilibrium they add up to the market portfolio. When summing all reciprocals of coefficient of risk aversions, the definition of the harmonic mean, (2), implies they add up to $m/\bar{\rho}$. Thus, summing the first order conditions, (A.2), we get

$$\begin{array}{ccc}\sum_{k=1}^{m}\mathbf{x}_k & = & \sum_{k=1}^{m}\frac{1}{\rho_k}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}^{-1}\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{1}{p_f}\mathbf{p}\right)\\ \| & & \| \\ \mathbf{1} & & \frac{m}{\bar{\rho}}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}^{-1}\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{1}{p_f}\mathbf{p}\right)\end{array}$$

We multiply the above parity by $(\bar{\rho}/m)\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$, and conclude that whatever $\mathbf{p}$ and p_f are, their ratio in any equilibrium is uniquely defined:

$$\frac{1}{p_f}\mathbf{p} = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{A.3}$$

In other words, we have shown that the market for risky assets clears if and only if the ratio of prices is given by (A.3).

Plugging the equilibrium ratio of prices, (A.3), back into the first order condition, (A.2), we conclude that

$$\mathbf{x}_k = \frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1} \tag{A.4}$$

Inserting (A.4) back into the investors' problem (Equation A.1), we can write the problem of the kth investor as

$$\max_{c\in R} \; -e^{-\rho_k c} - e^{-\rho_k\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)}$$

The first-order condition with respect to c is

$$\rho_k e^{-\rho_k c} = \frac{\rho_k}{p_f}e^{-\rho_k\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)}$$

We take log on both sides, simplify, and replace c with c_k to emphasize that this is optimal consumption of the kth investor:

$$c_k = \frac{1}{\rho_k}\log(p_f) + (\bar{c}_k - c_k)\frac{1}{p_f} + \bar{b}_k$$
$$+ \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\bar{\rho}}{2m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

Adding up the m first-order conditions of all investors yields

$$\sum_{k=1}^{m} c_k = \left(\sum_{k=1}^{m}\frac{1}{\rho_k}\right)\log(p_f) + \left(\sum_{k=1}^{m}(\bar{c}_k - c_k)\right)\frac{1}{p_f} + \sum_{k=1}^{m}\bar{b}_k$$
$$+ \left(\sum_{k=1}^{m}\bar{\mathbf{x}}_k'\right)\frac{1}{p_f}\mathbf{p} + \left(\sum_{k=1}^{m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\right)\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\bar{\rho}}{2m}\left(\sum_{k=1}^{m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\right)\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

The market clearing conditions imply $\sum_{k=1}^{m} c_k = \sum_{k=1}^{m}\bar{c}_k$, $\sum_{k=1}^{m}\bar{b}_k = 0$, and $\sum_{k=1}^{m}\bar{\mathbf{x}}_k = \mathbf{1}$. In addition, the definition of the harmonic mean, (2), implies

$$\left(\sum_{k=1}^{m}\frac{1}{m}\frac{\bar{\rho}}{\rho_k}\right) = 1$$

Thus, the sum of the m first order conditions and the market clearing conditions imply that in every equilibrium we must have

$$\sum_{k=1}^{m}\bar{c}_k = \frac{m}{\bar{\rho}}\log(p_f) + \mathbf{1}'\frac{1}{p_f}\mathbf{p} + \mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\bar{\rho}}{2m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

which simplifies to

$$\sum_{k=1}^{m}\bar{c}_k = \frac{m}{\bar{\rho}}\log(p_f) + \mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{2m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right) \tag{A.5}$$

We conclude that in every equilibrium, the logarithm of the bond price is

$$\log(p_f) = -\frac{\bar{\rho}}{m}\left(\mathbf{1}'\mathbf{f} + \frac{\bar{\rho}}{2m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right) + \frac{\bar{\rho}}{m}\sum_{k=1}^{m}\bar{c}_k \tag{A.6}$$

We already proved that in every equilibrium, the ratio of prices must satisfy (A.3), and since we have identified that in all equilibria the bond price is given by (A.6), we conclude that also the vector of asset prices is uniquely defined and given by

$$\mathbf{p} = p_f \left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{m} \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \mathbf{1} \right) \tag{A.7}$$

To conclude the proof, we note that the equilibrium prices stated in the theorem and the equilibrium portfolio of risk assets are the same at the equilibrium identities we computed in the proof. That is, Equations (5) and (7) are (A.3) and (A.7), equation (6) is (A.6), and the equilibrium portfolio (8) is (A.4).

■

Proof of Theorem 3.2. Fix an investor k.

Let c_k^*, b_k^*, and x_k^* be the investor's decisions in equilibrium. From Theorem 3.1, we know that the investor holds the market. In other words, there is a scalar q_k^* such that $x_k^* = q_k^* \mathbf{1}$.[17]

Define $w_k^* = b_k^* p_f + q_k^* \mathbf{1}'\mathbf{p}$, where p_f and $\mathbf{p}$ are the equilibrium prices in the artificial economy. We invoke a calculus of variation-type argument. Instead of looking at the full-blown problem, we restrict our attention to a subclass of feasible allocations that includes the optimal one. Specifically, let us say that the investor contemplates consuming the optimal c_k^* and investing a fraction α of w_k^* in the market portfolio, a fraction κ of w_k^* in asset i, and the remaining $(1-\alpha-\kappa)w_k^*$ in bonds. In other words, the allocation the investor contemplates is to buy $w_k^*(1-\alpha-\kappa)/p_f$ bonds, a fraction $\frac{w_k^*\alpha}{\mathbf{1}'\mathbf{p}}$ of the market portfolio $\mathbf{1}$, and an additional $\frac{w_k^*\kappa}{p_i}$ shares of asset i. The optimal fraction invested in asset i must satisfy $\kappa = 0$.

[17] In Theorem 3.1, we have shown that the scalar is $\bar{\rho}/(m\rho_k)$. But for the proof, we only need to know that the investor holds the market.

We can write the investor's problem as follows:

$$\begin{aligned}
&\max_{c,b,\mathbf{x}} E\left[U_k(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{y}\right], \text{ subject to } (c,b,\mathbf{x}) \in \mathcal{B}_k(p_f,\mathbf{p}) \\
&= \max_{b,\mathbf{x}\in R^n} E\left[U_k\left(c_k^*, b, \mathbf{x}'\mathbf{v}\right)\middle|\,\mathbf{y}\right], \text{ subject to } bp_f + \mathbf{x}'\mathbf{p} = w_k^* \\
&= -\,e^{-\rho_k c^*} + \max_{b,\mathbf{x}\in R^n} -E\left[e^{-\rho_k\,(b+\mathbf{x}'\mathbf{v})}\middle|\,\mathbf{y}\right], \text{ subject to } bp_f + \mathbf{x}'\mathbf{p} = w_k^* \\
&= -\,e^{-\rho_k c^*} + \max_{\alpha,\kappa} -E\left[\exp\left(-\rho_k w_k^*\left(\frac{1-\alpha-\kappa}{p_f} + \frac{\alpha}{\mathbf{1}'\mathbf{p}}\mathbf{1}'\mathbf{v} + \frac{\kappa}{p_i}v_i\right)\right)\middle|\,\mathbf{y}\right]
\end{aligned}$$

Thus, the maximization problem is equivalent to

$$\begin{aligned}
\max_{\alpha,\kappa}(1-\alpha-\kappa)(1+r_f) + \alpha(1+E[r_{\text{mkt}}|\mathbf{y}]) + \kappa(1+E[r_i|\mathbf{y}]) \\
-\frac{\rho_k w_k^*}{2}\left(\alpha^2\operatorname{var}\left(r_{\text{mkt}}|\,\mathbf{y}\right) + 2\alpha\kappa\operatorname{cov}\left(r_{\text{mkt}}, r_i|\,\mathbf{y}\right) + \kappa^2\operatorname{var}\left(r_i|\,\mathbf{y}\right)\right)
\end{aligned}$$

Taking the first-order condition with respect to α, and evaluating at $\kappa = 0$, yields

$$E[r_{\text{mkt}}|\mathbf{y}] - r_f - \alpha\rho_k w_k^*\operatorname{var}\left(r_{\text{mkt}}|\,\mathbf{y}\right) = 0 \quad\longrightarrow\quad \alpha\rho_k w_k^* = \frac{E[r_{\text{mkt}}|\mathbf{y}] - r_f}{\operatorname{var}\left(r_{\text{mkt}}|\,\mathbf{y}\right)}$$

Taking the first-order condition with respect to κ, and evaluating at $\kappa = 0$, yields

$$E[r_i|\mathbf{y}] - r_f - \alpha\rho_k w_k^*\operatorname{cov}\left(r_{\text{mkt}}, r_i|\,\mathbf{y}\right) = 0$$

Combining both conditions, we obtain

$$E[r_i|\mathbf{y}] - r_f - \beta_i(E[r_{\text{mkt}}|\mathbf{y}] - r_f) = 0$$

■

Proof of Lemma 3.3. In this proof, we repeatedly use the Doob–Dynkin lemma. The pair $(\mathbf{p}, p_f)$ is defined as a measurable mapping of $\mathbf{f}$. Indeed, p_f, given in Equation 6, is a measurable function of $\mathbf{f}$. So $\mathbf{p}$, given in Equation 7, is also a function of $\mathbf{f}$. Therefore, $\sigma(\mathbf{p}, p_f) \subseteq \sigma(\mathbf{f})$. The reverse is also true: Given the pair $(\mathbf{p}, p_f)$, we have $\mathbf{f} = \frac{1}{p_f}\mathbf{p}$. Therefore, $\sigma(\mathbf{p}, p_f) \supseteq \sigma(\mathbf{f})$, and we conclude that $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f})$. Next, Equation 5 implies $\sigma(\mathbf{f}) =$

$\sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}})$. Finally, from the definition of conditional expectation, we know that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ is a measurable function of $\mathbf{y}$, and hence $\sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) \subseteq \sigma(\mathbf{y})$.

$\blacksquare$

Proof of Lemma 3.4. For part 1, we have

$$E[\mathbf{v}|\mathcal{G}] \underset{\mathcal{G}\subseteq\mathcal{F}}{=} E[E[\mathbf{v}|\mathcal{F}]|\mathcal{G}] \underset{E[\mathbf{v}|\mathcal{F}]\in\mathcal{G}}{=} E[\mathbf{v}|\mathcal{F}]$$

For part 2, we use the result from part 1:

$$\begin{aligned}
\mathrm{var}(\mathbf{v}|\mathcal{G}) &\underset{\text{def.}}{=} E\big[\left(\mathbf{v} - E[\mathbf{v}|\mathcal{G}]\right)\left(\mathbf{v} - E[\mathbf{v}|\mathcal{G}]\right)' \big|\mathcal{G}\big] \\
&\underset{\text{part 1}}{=} E\big[\left(\mathbf{v} - E[\mathbf{v}|\mathcal{F}]\right)\left(\mathbf{v} - E[\mathbf{v}|\mathcal{F}]\right)' \big|\mathcal{G}\big] \\
&\underset{\mathcal{G}\subseteq\mathcal{F}}{=} E\Big[E\big[\left(\mathbf{v} - E[\mathbf{v}|\mathcal{F}]\right)\left(\mathbf{v} - E[\mathbf{v}|\mathcal{F}]\right)' \big|\mathcal{F}\big]\Big|\mathcal{G}\Big] \\
&\underset{\text{def.}}{=} E\big[\mathrm{var}(\mathbf{v}|\mathcal{F})\big|\mathcal{G}\big] \underset{\mathrm{var}(\mathbf{v}|\mathcal{F})\in\mathcal{G}}{=} \mathrm{var}(\mathbf{v}|\mathcal{F})
\end{aligned}$$

$\blacksquare$

Proof of Corollary 3.5. We study conditional distributions, where the conditioning is on prices (and, in parts 2 and 3, also on signals). Lemma 3.3 implies that whenever we condition on prices, we can condition instead on $\mathbf{f}$, which is Gaussian. Thus, all the conditional distributions stated in the corollary are indeed Gaussian. We now verify that the means and the covariance matrices are as stated in each part of the corollary.

For part 1, we have $\sigma(p_f, \mathbf{p}) \subseteq \sigma(\mathbf{y})$ and $E[\mathbf{v}|\mathbf{y}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \in \sigma(p_f, \mathbf{p})$. Thus, we can apply part 1 of Lemma 3.4 to conclude that $E[\mathbf{v}|p_f, \mathbf{p}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$. Next, we have $\mathrm{var}(\mathbf{v}|\mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\mathrm{var}(\mathbf{v}|p_f, \mathbf{p}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$. This concludes the proof of part 1.

For part 2, let $k \in \mathcal{NI}$. We have $\sigma(\mathbf{s}_k, p_f, \mathbf{p}) \subseteq \sigma(\mathbf{s}_k, \mathbf{y})$. According to (GR), $E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] =$

$\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$, which is measurable with respect to $\sigma(\mathbf{s}_k, p_f, \mathbf{p})$. Thus, we can apply part 1 of Lemma 3.4 to conclude that $E[\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$.

Next, according to (GR), $\text{var}(\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\text{var}(\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$. This concludes the proof of Equation 11.

For part 3, let $k \in \mathcal{I}$. We have $\sigma(\mathbf{s}_k, p_f, \mathbf{p}) \subseteq \sigma(\mathbf{s}_k, \mathbf{y})$. According to (GR), $E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$, which is measurable with respect to $\sigma(\mathbf{s}_k, p_f, \mathbf{p})$. Thus, we can apply part 1 of Lemma 3.4 to conclude that $E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$.

Finally, according to (GR), $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$. This concludes the proof of Equation 12.

■

B Proof of Theorem 4.1

Before we prove the theorem, we need some preliminary results.

Define the auxiliary matrix as follows:

$$\mathbf{M} := \begin{bmatrix} \mathbf{I}_{n\times n} & |\mathcal{N}\mathcal{I}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}^{-1}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) \\ \mathbf{0}_{n\times n} & -|\mathcal{N}\mathcal{I}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}^{-1}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) + m\mathbf{g} \end{bmatrix}_{2n\times(n+1)} \tag{B.1}$$

We can now write

$$\boldsymbol{\Sigma}_{\mathbf{vy}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & m\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g} \end{bmatrix} \tag{B.2}$$

$$= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix} \mathbf{M} \tag{B.3}$$

Also,

$$\forall k \in \mathcal{NI}, \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_k\right) = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \dfrac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}_{(n+1)\times n} \tag{B.4}$$

$$\forall k \in \mathcal{I} \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_k\right) = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}_{(n+1)\times n} \tag{B.5}$$

So, for any $nind \in \mathcal{NI}$ and $ind \in \mathcal{I}$, we have

$$\mathbf{1}'\begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{1\times n} & 1_{1\times 1} \end{bmatrix}\begin{bmatrix} \text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right) \end{bmatrix} \tag{B.6}$$

and

$$\begin{bmatrix} \text{cov}(\mathbf{y}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{y}, \mathbf{s}_{ind}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \dfrac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} & \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}_{(n+1)\times 2n}$$

The latter implies that

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \dfrac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} & m\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g} \end{bmatrix} = \begin{bmatrix} \text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right) \end{bmatrix}\mathbf{M} \tag{B.7}$$

Lemma B.1. *Let $\mathbf{x} \in R^n$ be an arbitrary portfolio. By means of matching terms, define $\mathbf{q} \in R^n$ and the scalar q to be*

$$\begin{bmatrix} \mathbf{q}' & q \end{bmatrix}_{1\times(n+1)} := \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1} \tag{B.8}$$

Then, $\mathbf{q} = \mathbf{0}_{n\times 1}$ if and only if $\mathbf{x}$ is a scalar multiplier of $\mathbf{1}$ with $\mathbf{x} = q\mathbf{1}$.

Proof of Lemma B.1. Let $\mathbf{x}$ be arbitrary, and multiply both sides of Equation B.8 by $\boldsymbol{\Sigma}_{\mathbf{yy}}$ on the right:

$$\begin{bmatrix} \mathbf{q}' & q \end{bmatrix}\boldsymbol{\Sigma}_{\mathbf{yy}} = \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vy}}$$

We use Equations B.2 and B.7 to obtain

$$\left[\mathbf{q}'\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right) + q\mathbf{1}'\mathbf{\Sigma_{vv}} \quad \mathbf{q}'|\mathbf{\Sigma_{vv}}\mathbf{1} + mq\mathbf{1}'\mathbf{\Sigma_{vv}}\mathbf{g}\right]_{1\times(n+1)} = \left[\mathbf{x}'\mathbf{\Sigma_{vv}} \quad m\mathbf{x}'\mathbf{\Sigma_{vv}}\mathbf{g}\right]_{1\times(n+1)}$$

Matching terms makes it clear that $\mathbf{q} = \mathbf{0}$ implies $\mathbf{x} = q\mathbf{1}$, and hence $\mathbf{x}$ is a scalar multiplier of $\mathbf{1}$. We need to show that the opposite is also true. Let us say that $\mathbf{x} = q\mathbf{1}$, and assume by means of contradiction that $\mathbf{q} \neq \mathbf{0}$. Matching terms must yield

$$\mathbf{q}'\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right) = \mathbf{0}_{1\times n}$$

Multiplying on the right by $\mathbf{q}$ and noticing that $\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right)$ is a positive definite matrix, we obtain a contradiction:

$$0 < \mathbf{q}'\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right)\mathbf{q} = \mathbf{0}_{1\times n}\mathbf{q} = 0$$

■

Applying Lemma B.1 to $\mathbf{x} = \mathbf{1}$, we obtain

$$\left[\mathbf{0}_{1\times n} \quad 1\right] = \mathbf{1}'\mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}} \tag{B.9}$$

We define the n-dimensional random vector as follows:

$$\mathbf{z} := \mathbf{v} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{y} \tag{B.10}$$

Lemma B.2. *We have*

$$E\mathbf{z} = \boldsymbol{\mu}_{\mathbf{v}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\boldsymbol{\mu}_{\mathbf{y}} \tag{B.11}$$

$$\text{var}(\mathbf{z}) = \mathbf{\Sigma_{vv|y}} \tag{B.12}$$

Proof of Lemma B.2. Equation B.11 follows from the definition of $\mathbf{z}$. As for Equation B.12, we have

$$\begin{aligned}\text{var}(\mathbf{z}) &= \text{var}(\mathbf{v} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}) \\ &= \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yy}}\Sigma_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} \\ &= \boldsymbol{\Sigma}_{\mathbf{vv}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} \\ &\underset{(4)}{=} \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\end{aligned}$$

■

Lemma B.3. *We have*

1. $\text{cov}(\mathbf{z}, \mathbf{y}) = \mathbf{0}_{n\times(n+1)}$.
2. $\forall k \in \mathcal{NI}$, $\text{cov}(\mathbf{z}, \mathbf{s}_k) = \mathbf{0}_{n\times n}$.
3. $\forall k \in \mathcal{I}$, $\text{cov}(\mathbf{1}'\mathbf{z}, \mathbf{s}_k) = \mathbf{0}_{1\times n}$.

Proof of Lemma B.3. For part 1 of the lemma,

$$\text{cov}(\mathbf{z}, \mathbf{y}) = \text{cov}(\mathbf{v} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}, \mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vy}} - \boldsymbol{\Sigma}_{\mathbf{vy}} = 0$$

For part 2 of the lemma, it is convenient to compute an $n \times 2n$ covariance matrix. For any $nind \in \mathcal{NI}$ and any $ind \in \mathcal{I}$, we have

$$\begin{aligned}&\begin{bmatrix}\text{cov}(\mathbf{z}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{z}, \mathbf{s}_{ind})\end{bmatrix} = \begin{bmatrix}\text{cov}(\mathbf{v}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{v}, \mathbf{s}_{ind})\end{bmatrix} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix}\text{cov}(\mathbf{y}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{y}, \mathbf{s}_{ind})\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix}\text{cov}(\mathbf{y}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{y}, \mathbf{s}_{ind})\end{bmatrix}\end{aligned}$$

Multiply both sides of the equations on the right by $\mathbf{M}$:

$$\begin{aligned}\begin{bmatrix}\text{cov}(\mathbf{z}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{z}, \mathbf{s}_{ind})\end{bmatrix}\mathbf{M} &= \underbrace{\begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix}\mathbf{M}}_{\underset{(B.3)}{=}\boldsymbol{\Sigma}_{\mathbf{vy}}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\underbrace{\begin{bmatrix}\text{cov}(\mathbf{y}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{y}, \mathbf{s}_{ind})\end{bmatrix})\mathbf{M}}_{\underset{(B.7)}{=}\boldsymbol{\Sigma}_{\mathbf{yy}}} \\ &= \boldsymbol{\Sigma}_{\mathbf{vy}} - \boldsymbol{\Sigma}_{\mathbf{vy}} \\ &= \mathbf{0}_{n\times(n+1)}\end{aligned}$$

In submatrix notation, we can write the above, using the definition of $\mathbf{M}$ from Equation B.1, as

$$\begin{bmatrix}\text{cov}(\mathbf{z},\mathbf{s}_{nind}) & \mathbf{t}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1}\end{bmatrix}$$

where $\mathbf{t}$ is some vector.[18] This proves that $\text{cov}(\mathbf{z},\mathbf{s}_{nind}) = \mathbf{0}$.[19]

Next, for part 3 of the lemma, we need to prove that $\text{cov}(\mathbf{1}'\mathbf{z},\mathbf{s}_{ind}) = \mathbf{0}_{1\times n}$. Again, to facilitate linear algebra manipulations, it is convenient to compute a vector of dimension $2n$:

$$\begin{aligned}
&\begin{bmatrix}\text{cov}\left(\mathbf{1}'\mathbf{z},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{1}'\mathbf{z},\mathbf{s}_{ind}\right)\end{bmatrix} = \mathbf{1}'\begin{bmatrix}\text{cov}\left(\mathbf{z},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{z},\mathbf{s}_{ind}\right)\end{bmatrix}\\
&= \mathbf{1}'\begin{bmatrix}\text{cov}\left(\mathbf{v},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{v},\mathbf{s}_{ind}\right)\end{bmatrix} - \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix}\text{cov}\left(\mathbf{y},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y},\mathbf{s}_{ind}\right)\end{bmatrix}\\
&= \mathbf{1}'\begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix} - \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix}\text{cov}\left(\mathbf{y},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y},\mathbf{s}_{ind}\right)\end{bmatrix}\\
&\underset{(B.9)}{=} \mathbf{1}'\begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix} - \begin{bmatrix}\mathbf{0}_{1\times n} & 1_{1\times 1}\end{bmatrix}\begin{bmatrix}\text{cov}\left(\mathbf{y},\mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y},\mathbf{s}_{ind}\right)\end{bmatrix}\\
&\underset{(B.6)}{=} \mathbf{0}_{1\times 2n}
\end{aligned}$$

■

Lemma B.4. *For all $k \in \mathcal{NI}$, we have*

$$E\left[\mathbf{v}\,|\mathbf{y},\mathbf{s}_k\right] = E[\mathbf{v}|\mathbf{y}]$$
$$\text{var}\left(\mathbf{v}\middle|\mathbf{y},\mathbf{s}_k\right) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$$

Proof of Lemma B.4. Let $k \in \mathcal{NI}$. Part 1 of Lemma B.3 states that $\text{cov}(\mathbf{z},\mathbf{y}) = \mathbf{0}$, and part 2 states that $\text{cov}(\mathbf{z},\mathbf{s}_k) = \mathbf{0}$. Therefore, $\text{cov}\left(\mathbf{z},\begin{bmatrix}\mathbf{y}\\ \mathbf{s}_k\end{bmatrix}\right) = \mathbf{0}_{n\times(2n+1)}$.

Thus,

$$E\left[\mathbf{z}\,|\mathbf{y},\mathbf{s}_k\right] = E\mathbf{z} \tag{B.13}$$
$$\text{var}\left(\mathbf{z}\middle|\mathbf{y},\mathbf{s}_k\right) = \text{var}(\mathbf{z}) \tag{B.14}$$

[18] Specifically,

$$\mathbf{t} = \left(\text{cov}(\mathbf{z},\mathbf{s}_{nind}) - \text{cov}(\mathbf{z},\mathbf{s}_{ind})\right)|\mathcal{NI}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}^{-1}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) + m\,\text{cov}(\mathbf{z},\mathbf{s}_{ind})\mathbf{g}$$

[19] Although it does not prove, and it is not true, that $\text{cov}(\mathbf{z},\mathbf{s}_{ind}) = \mathbf{0}$.

We have

$$E\left[\mathbf{v}\,|\mathbf{y},\mathbf{s}_k\right] \underset{(B.10)}{=} E\left[\mathbf{z}+\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\,|\mathbf{y},\mathbf{s}_k\right] \underset{(B.13)}{=} E\mathbf{z}+\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \underset{(B.11)}{=} \boldsymbol{\mu}_{\mathbf{v}}+\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{y}}) \underset{(3)}{=} E[\mathbf{v}|\mathbf{y}] \tag{B.15}$$

Next,

$$\operatorname{var}\left(\mathbf{v}\,|\mathbf{y},\mathbf{s}_k\right) \underset{(B.10)}{=} \operatorname{var}\left(\mathbf{z}+\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\,|\mathbf{y},\mathbf{s}_k\right) = \operatorname{var}\left(\mathbf{z}\,|\mathbf{y},\mathbf{s}_k\right) \underset{(B.14)}{=} \operatorname{var}(\mathbf{z}) \underset{(B.12)}{=} \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$$

■

Lemma B.5. *For all $k \in \mathcal{I}$, we have*

$$E\left[\mathbf{1}'\mathbf{v}\,|\mathbf{y},\mathbf{s}_k\right] = \mathbf{1}'E[\mathbf{v}|\mathbf{y}]$$
$$\operatorname{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{y},\mathbf{s}_k\right) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

Proof of Lemma B.5. Let $k \in \mathcal{I}$. Part 1 of Lemma B.3 implies that $\operatorname{cov}(\mathbf{1}'\mathbf{z},\mathbf{y}) = \mathbf{0}$, and part 3 states that $\operatorname{cov}(\mathbf{1}'\mathbf{z},\mathbf{s}_k) = \mathbf{0}$. Therefore, $\operatorname{cov}\left(\mathbf{1}'\mathbf{z}, \begin{bmatrix}\mathbf{y}\\ \mathbf{s}_k\end{bmatrix}\right) = \mathbf{0}_{1\times(2n+1)}$.

Thus,

$$E\left[\mathbf{1}'\mathbf{z}\,|\mathbf{y},\mathbf{s}_k\right] = E\mathbf{1}'\mathbf{z} \tag{B.16}$$

$$\operatorname{var}\left(\mathbf{1}'\mathbf{z}\middle|\mathbf{y},\mathbf{s}_k\right) = \operatorname{var}(\mathbf{1}'\mathbf{z}) \tag{B.17}$$

Now,

$$\begin{aligned} E\left[\mathbf{1}'\mathbf{v}\,|\mathbf{y},\mathbf{s}_k\right] &\underset{(B.10)}{=} E\left[\mathbf{1}'\mathbf{z}\,|\mathbf{y},\mathbf{s}_k\right]+\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \underset{(B.16)}{=} E\mathbf{1}'\mathbf{z}+\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \\ &= \mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}}+\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{y}})\right) \underset{(3)}{=} \mathbf{1}'E[\mathbf{v}|\mathbf{y}] \end{aligned} \tag{B.18}$$

Next,

$$\operatorname{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{y},\mathbf{s}_k\right) \underset{(B.10)}{=} \operatorname{var}\left(\mathbf{1}'\mathbf{z}+\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\middle|\mathbf{y},\mathbf{s}_k\right) = \operatorname{var}\left(\mathbf{1}'\mathbf{z}\middle|\mathbf{y},\mathbf{s}_k\right) \underset{(B.17)}{=} \operatorname{var}(\mathbf{1}'\mathbf{z}) \underset{(B.12)}{=} \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

■

Proof of Theorem 4.1. The proof follows from the fact that Lemmas B.5 and B.4 show that the four conditions in (GR) are satisfied.

■

C Additional Lemmas and Proofs

Lemma C.1. *We have*

$$\mathbf{\Sigma_{vv|y}}\mathbf{1} = \mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g}) \tag{C.1}$$

$$\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v|y}} = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) \tag{C.2}$$

$$\mathbf{1}'\mathbf{f} = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) - \frac{\bar{\rho}}{m}\mathbf{1}'\mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g}) \tag{C.3}$$

Proof of Lemma C.1. To prove Equation C.1, we have

$$\begin{aligned}
\mathbf{\Sigma_{vv|y}}\mathbf{1} &\underset{(4)}{=} \mathbf{\Sigma_{vv}}\mathbf{1} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}}\mathbf{1} \\
&\underset{(B.9)}{=} \mathbf{\Sigma_{vv}}\mathbf{1} - \mathbf{\Sigma_{vy}}\begin{bmatrix}\mathbf{0}_{n\times 1} \\ 1_{1\times 1}\end{bmatrix} \\
&\underset{(B.2)}{=} \mathbf{\Sigma_{vv}}\mathbf{1} - m\mathbf{\Sigma_{vv}}\mathbf{g} \\
&= \mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g})
\end{aligned}$$

This proves Equation C.1. To prove Equation C.2, we have

$$\begin{aligned}
\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v|y}} &\underset{(3)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{1}'\mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \\
&\underset{(B.9)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \begin{bmatrix}\mathbf{0}_{1\times n} & 1_{1\times 1}\end{bmatrix}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \\
&\underset{(14)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}})
\end{aligned}$$

This proves Equation C.2.

Now, we use Equation C.1 to re-express Equation 5, which yields the following equation.

$$\mathbf{f} = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})$$

We multiply the above by $\mathbf{1}'$ from the left, and we use Equation C.2 to obtain Equation C.3.

■

Proof of Corollary 4.2. This corollary adds only one additional property to Lemma 3.3: The last set inequality in Lemma 3.3 is an equality. Therefore, to prove the corollary, it is sufficient to show that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals $\mathbf{y}$.

Equation C.2 implies that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals the $(n+1)$th coordinate of $\mathbf{y}$. Indeed, rearrenging Equation C.2, we have $y_{n+1} = \mathbf{1}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \boldsymbol{\mu}_{\mathbf{v}}) + m\mathbf{g}'\boldsymbol{\mu}_{\mathbf{v}}$.

Taking advantage of the fact that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals the $(n+1)$th coordinate of $\mathbf{y}$, we rewrite Equation 3:

$$\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix} y_1 - Ey_1 \\ \vdots \\ y_n - Ey_n \\ y_{n+1} - Ey_{n+1} \end{bmatrix} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix} y_1 - Ev_1 \\ \vdots \\ y_n - Ev_n \\ \mathbf{1}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \boldsymbol{\mu}_{\mathbf{v}}) \end{bmatrix}$$

To obtain the first n coordinates of $\mathbf{y}$, we solve the above system of n linear equations; the unknowns are the first n coordinates of $\mathbf{y}$. Thus, $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals all $n+1$ coordinates of $\mathbf{y}$.

■

Proof of Theorem 4.3. Because $\mathbf{s}$ and $\mathbf{1}'\mathbf{v}$ are jointly normally distributed, the conditional distribution of $\mathbf{1}'\mathbf{v}$, given $\mathbf{s}$, is normal with mean $E[\mathbf{1}'\mathbf{v}|\mathbf{s}]$ and variance $\mathrm{var}(\mathbf{1}'\mathbf{v}|\mathbf{s})$.

We have

$$\boldsymbol{\Sigma}_{\mathbf{ss}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} \end{bmatrix}_{nm \times nm}$$

Thus,

$$\begin{bmatrix}\mathbf{I}_{n\times n} & \cdots & \mathbf{I}_{n\times n}\end{bmatrix}_{n\times nm} = \begin{bmatrix}(m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1} & \cdots & (m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1}\end{bmatrix}_{n\times nm} \boldsymbol{\Sigma}_{\mathbf{ss}} \tag{C.4}$$

We also have

$$\boldsymbol{\Sigma}_{\mathbf{vs}} = \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix}$$

Therefore, by multiplying each side of Equation C.4 on the right by $\boldsymbol{\Sigma}_{\mathbf{ss}}^{-1}$ and on the left by $\boldsymbol{\Sigma}_{\mathbf{vv}}$, we obtain

$$\boldsymbol{\Sigma}_{\mathbf{vs}}\boldsymbol{\Sigma}_{\mathbf{ss}}^{-1} = \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}}\,(m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}}\,(m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1}\end{bmatrix}_{n\times nm}$$

Multiplying both sides of the above equation on the left by $\mathbf{1}'$, and recalling the definition of $\mathbf{g}$ (see Equation 13), we conclude that

$$\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vs}}\boldsymbol{\Sigma}_{\mathbf{ss}}^{-1} = \begin{bmatrix}\mathbf{g}' & \cdots & \mathbf{g}'\end{bmatrix}_{1\times nm}$$

Thus,

$$\begin{aligned} E[\mathbf{1}'\mathbf{v}|\mathbf{s}] &= \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vs}}\boldsymbol{\Sigma}_{\mathbf{ss}}^{-1}(\mathbf{s} - E\mathbf{s}) = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) \\ &= (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} + y_{n+1} \end{aligned} \tag{C.5}$$

depends on $\mathbf{s}$ only through y_{n+1}. Therefore, applying part 1 of Lemma 3.4, we conclude that

$$E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|y_{n+1}]$$

Next, we note that

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s})$$

is merely a scalar;[20] therefore, we can apply part 2 of Lemma 3.4 to conclude that

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|y_{n+1})$$

[20]Specifically,

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} - \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vs}}\boldsymbol{\Sigma}_{\mathbf{ss}}^{-1}\boldsymbol{\Sigma}_{\mathbf{sv}}\mathbf{1} = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} - m\mathbf{g}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1}$$

To prove that y_{n+1} is a sufficient statistic, we note that the density of a normal random variable depends only on the mean and variance. Therefore, from part 1 of the lemma, we conclude that the conditional density satisfies $f(\mathbf{1}'\mathbf{v}|\mathbf{s}) = f(\mathbf{1}'\mathbf{v}|y_{n+1})$. In our model, the "data," $\mathbf{s}$; the "statistic," y_{n+1}; and the "parameter," $\mathbf{1}'\mathbf{v}$, have a known joint distribution. Thus, we use Bayes's rule (see Theorem 2.2.1 of Geweke 2005) to conclude that y_{n+1} is a sufficient statistic for $\mathbf{1}'\mathbf{v}$.

Now, let $t(\mathbf{s})$ be another arbitrary sufficient statistic. To show that y_{n+1} is minimal, we need to show that y_{n+1} is a measurable function of $t(\mathbf{s})$.[21] From Bayes's rule (again, see Theorem 2.2.1 of Geweke 2005), it follows that $f(\mathbf{1}'\mathbf{v}|\mathbf{s}) = f(\mathbf{1}'\mathbf{v}|t(\mathbf{s}))$. Thus,

$$E[\mathbf{1}'\mathbf{v}|t(\mathbf{s})] = E[\mathbf{1}'\mathbf{v}|\mathbf{s}] \underset{\text{(C.5)}}{=} (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} + y_{n+1}$$

In other words,

$$y_{n+1} = E[\mathbf{1}'\mathbf{v}|t(\mathbf{s})] - (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}}$$

which shows that y_{n+1} is a measurable function of $t(\mathbf{s})$, and hence minimal.

■

Proof of Theorem 4.4. From Equation C.3, we have

$$\mathbf{1}'\mathbf{f} = y_{n+1} + (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} - \frac{\bar{\rho}}{m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})$$

Thus, there is a one-to-one mapping between $\mathbf{1}'\mathbf{f}$ and y_{n+1}. As a result, $\sigma(\mathbf{1}'\mathbf{f}) = \sigma(y_{n+1})$. The proof follows from Theorem 4.3.

■

Proof of Theorem 5.1. The price of the bond, $p_f = 1/(1 + r_f)$, is given in Equation 6. Two terms in Equation 6 can cause p_f to depend on how we divide the investors into index and

[21] The intuition that underlies the formal definition of the minimality of a sufficient statistic is as follows. If y_{n+1} is a measurable function of $t(\mathbf{s})$, then $y_{n+1} \in \sigma(t(\mathbf{s}))$. Therefore, $\sigma(y_{n+1}) \subseteq \sigma(t(\mathbf{s}))$, so y_{n+1} contains less (not necessarily in a strict sense) information than $t(\mathbf{s})$.

nonindex investors. The first term is $\mathbf{\Sigma_{vv|y}1}$. Equation C.1 shows that this term depends on $\mathbf{\Sigma_{vv}}$ and $\mathbf{g}$, which do not depend on the specific partition. The second term is $\mathbf{1'f}$. According to Equation C.3, $\mathbf{1'f}$ does not depend on the specific partition, so we conclude that p_f and r_f also do not depend on the specific partition.

The price of the market portfolio is $\mathbf{1'p} \underset{(7)}{=} \frac{\mathbf{1'f}}{1+r_f}$. Therefore, the price of the market portfolio is also independent of the specific partition. Because payoffs are exogenous, and the price is independent, we conclude that the return on the market portfolio is independent of the specific partition. In particular, the conditional expected return on the market portfolio and the conditional variance of return on the market portfolio must be independent of the specific partition, so the capital market line remains unchanged as we change $\mathcal{I}$.

■

Proof of Theorem 5.2. We use the following standard notation. If $\mathbf{A} = \left[a_{ij}\right]$ is an arbitrary matrix with elements that depend on a parameter π, then $\frac{\partial \mathbf{A}}{\partial \pi}$ is the matrix $\left[\frac{\partial a_{ij}}{\partial \pi}\right]$. If $\mathbf{B}$ is an arbitrary matrix with elements that do not depend on the parameter π, then it is well known that

$$\frac{\partial \mathbf{B}\mathbf{A}^{-1}\mathbf{B}'}{\partial \pi} = -\mathbf{B}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \pi}\mathbf{A}^{-1}\mathbf{B}' \tag{C.6}$$

For a qualitative comparative statics, any monotonic transformation of $|\mathcal{I}|$ is a measure of the level of index investment. It is convenient to use as a measure of index investment

$$\pi \equiv \frac{1}{m-|\mathcal{I}|} = \frac{1}{|\mathcal{NI}|}$$

Although π takes values only on a discrete set of numbers, we can nevertheless differentiate $\mathbf{x}'\mathbf{\Sigma_{vv|y}}\mathbf{x}$ with respect to π.

From Equation 4, we have

$$\mathbf{\Sigma_{vv|y}} = \mathbf{\Sigma_{vv}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}}$$

where $\mathbf{\Sigma_{vv}}$ and $\mathbf{\Sigma_{vy}} = \begin{bmatrix}\mathbf{\Sigma_{vv}} & m\mathbf{\Sigma_{vv}g}\end{bmatrix}$ are independent of π. We use Equation C.6 with $A = \mathbf{\Sigma_{yy}}$ and $B = \mathbf{\Sigma_{vy}}$.

Equation B.7 shows that

$$\frac{\partial \mathbf{\Sigma_{yy}}}{\partial \pi} = \begin{bmatrix}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0_{1\times 1}\end{bmatrix} \tag{C.7}$$

Finally, by matching terms, we define the vector $\mathbf{q}$ and the scalar q such that $\begin{bmatrix}\mathbf{q}', q\end{bmatrix} = \mathbf{x}'\mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}$.

We have

$$\frac{\partial \operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}{\partial \pi} = \frac{\partial \mathbf{x}'\mathbf{\Sigma_{vv|y}}\mathbf{x}}{\partial \pi} = \mathbf{x}'\mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\frac{\partial \mathbf{\Sigma_{yy}}}{\partial \pi}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}}\mathbf{x} = \mathbf{q}'\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\mathbf{q} \geq 0$$

where the inequality arises because $\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}$ is positive definite. The inequality is strict whenever $\mathbf{q} \neq \mathbf{0}$. According to Lemma B.1, $\mathbf{q} = \mathbf{0}$ if and only if $\mathbf{x}$ is a scalar multiplication of $\mathbf{1}$.

■

Proof of Theorem 5.3. The numerator of the Sharpe ratio is

$$E\left[\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \middle| \mathbf{y}\right] - r_f = \frac{1}{\mathbf{x}'\mathbf{p}}\left(\mathbf{x}'\boldsymbol{\mu}_{\mathbf{v|y}} - \mathbf{x}'\mathbf{p}(1+r_f)\right) \underset{(7)}{=} \frac{1}{\mathbf{x}'\mathbf{p}}\frac{\bar{\rho}}{m}\mathbf{x}'\mathbf{\Sigma_{vv|y}}\mathbf{1} \underset{(C.1)}{=} \frac{1}{\mathbf{x}'\mathbf{p}}\frac{\bar{\rho}}{m}\mathbf{x}'\mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g})$$

The denominator of the Sharpe ratio is

$$\sqrt{\operatorname{var}\left(\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \middle| \mathbf{y}\right)} = \frac{1}{\mathbf{x}'\mathbf{p}}\sqrt{\operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}$$

Therefore, the Sharpe ratio equals

$$\frac{\frac{\bar{\rho}}{m}\mathbf{x}'\mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g})}{\sqrt{\operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}}$$

and its numerator is a scalar that is independent of the sets $\mathcal{NI}$ and $\mathcal{I}$. The numerator must be positive because we assume the Sharpe ratio is positive. Theorem 5.2 states that as long as $\mathbf{x}$ is not a scalar multiplication of the market portfolio, the denominator increases as $|\mathcal{I}|$ increases.

■

Proof of Theorem 5.4. We have

$$
\begin{aligned}
R^2 &= \operatorname{corr}^2(r_i, r_f + \beta_i(r_{\text{mkt}} - r_f)|\mathbf{y}) = \frac{\operatorname{cov}^2(r_i, \beta_i r_{\text{mkt}}|\mathbf{y})}{\operatorname{var}(r_i|\mathbf{y}) \times \operatorname{var}(\beta_i r_{\text{mkt}}|\mathbf{y})} \\
&= \frac{\operatorname{cov}^2(v_i, \mathbf{1}'\mathbf{v}|\mathbf{y})}{\operatorname{var}(v_i|\mathbf{y}) \times \operatorname{var}(\mathbf{1}'\mathbf{v}|\mathbf{y})} = \frac{\left(\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)^2}{\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y}) \times \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}} \\
&\underset{(C.1)}{=} \frac{\left(\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})\right)^2}{\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y}) \times \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})}
\end{aligned}
$$

where $\mathbf{e}_i$ is the vector with one in the ith coordinate and zero elsewhere.

Because R^2 is positive and the only term that depends on the level of index investment is $\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})$, whether or not R^2 decreases depends on whether or not $\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})$ increases. We apply Theorem 5.2 with $\mathbf{x} = \mathbf{e}_i$ to conclude that this conditional variance increases with $|\mathcal{I}|$. Therefore, R^2 of the CAPM regression decreases with $|\mathcal{I}|$.

■

Proof of Theorem 5.5. We have

$$
\operatorname{corr}(r_i, r_{-i}|\mathbf{y}) = \frac{\operatorname{cov}(r_i, r_{-i}|\mathbf{y})}{\sqrt{\operatorname{var}(r_i|\mathbf{y})}\sqrt{\operatorname{var}(r_{-i}|\mathbf{y})}} = \frac{\operatorname{cov}(v_i, v_{-i}|\mathbf{y})}{\sqrt{\operatorname{var}(v_i|\mathbf{y})}\sqrt{\operatorname{var}(v_{-i}|\mathbf{y})}}
$$

$$
= \frac{\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} - \operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})}{\sqrt{\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})}\sqrt{\operatorname{var}((\mathbf{1} - \mathbf{e}_i)'\mathbf{v}|\mathbf{y})}}
$$

According to Equation C.1, the first term in the numerator is independent of the level of index investment. From Theorem 5.2, applied to the portfolio $\mathbf{e}_i$, we conclude that the numerator decreases as we increase $|\mathcal{I}|$. We also know that the denominator is positive, and Theorem 5.2 (applied to $\mathbf{e}_i$, then separately applied to $\mathbf{1}-\mathbf{e}_i$) implies that the denominator is increasing. In summary, we can formally write the correlation as

$$\frac{N(\mathcal{I})}{D(\mathcal{I})}$$

where we have shown that $N(\mathcal{I})$ decreases with $|\mathcal{I}|$ and that $D(\mathcal{I})$ is positive and increases with $|\mathcal{I}|$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be such that $|\mathcal{I}_1| < |\mathcal{I}_2|$. We have

$$\frac{N(\mathcal{I}_1)}{D(\mathcal{I}_1)} \underset{\substack{\uparrow \\ 0<D(\mathcal{I}_1)<D(\mathcal{I}_2) \\ 0<N(\mathcal{I}_1)}}{>} \frac{N(\mathcal{I}_1)}{D(\mathcal{I}_2)} \underset{\substack{\uparrow \\ N(\mathcal{I}_1)>N(\mathcal{I}_2)}}{>} \frac{N(\mathcal{I}_2)}{D(\mathcal{I}_2)}$$

which shows that the correlation decreases.

■

Proof of Theorem 5.6. Let $\mathbf{x} \in R^n$. Our goal is to examine $\operatorname{var}(\mathbf{x}'\mathbf{p})$.

We have

$$\begin{aligned}\operatorname{var}(\mathbf{x}'\mathbf{p}) &\underset{(7)}{=} \operatorname{var}(p_f\mathbf{x}'\mathbf{f}) = \operatorname{var}(p_f\mathbf{x}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})) \\ &= \operatorname{var}\left(E[p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})|\mathbf{y}]\right)\end{aligned} \tag{C.8}$$

The law of total variance, applied to the random variable $p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})$, states that

$$\begin{aligned}\operatorname{var}\left(p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\right) &= E\operatorname{var}\left(p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\Big|\,\mathbf{y}\right) + \operatorname{var}\left(E[p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})|\mathbf{y}]\right) \\ &\underset{(C.8)}{=} E\operatorname{var}\left(p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\Big|\,\mathbf{y}\right) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\ &= Ep_f^2\operatorname{var}\left(\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\Big|\,\mathbf{y}\right) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\ &= Ep_f^2\operatorname{var}(\mathbf{x}'\mathbf{v}|\,\mathbf{y}) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\ &= \operatorname{var}(\mathbf{x}'\mathbf{v}|\,\mathbf{y})\,Ep_f^2 + \operatorname{var}(\mathbf{x}'\mathbf{p})\end{aligned}$$

where the last equality arises because $\text{var}\left(\mathbf{x}'\mathbf{v}|\,\mathbf{y}\right)$ is a scalar.

Now, the left-hand side, $\text{var}(p_f\mathbf{x}'(\mathbf{v} - \frac{\bar{\rho}}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}))$, is independent of the specific partition of investors into indexers and nonindexers because each element inside the variance operator is independent. Indeed, p_f is independent (see Theorem 5.1), $\mathbf{x}$ is an arbitrary vector of scalars, $\mathbf{v}$ is exogenous and hence independent, and finally $\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$ is independent because the right-hand side of Equation C.1 is independent of the specific partition.

Also, Ep_f^2 is positive and independent of the level of index investment. Therefore, $\text{var}(\mathbf{x}'\mathbf{p})$ decreases as we increase $|\mathcal{I}|$ if and only if $\text{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ increases as we increase $|\mathcal{I}|$. Thus, the proof follows from Theorem 5.2.

■

D Proofs of Theorems on Large Economy

In this appendix, we study the limit of the finite sequence of economies described in Section 7.

Proof of Theorem 7.1. To prove the theorem, we need to compute the unconstrained optimal portfolio of an indexer, given the equilibrium prices. For the rest of the proof we fix $k_0 \in \mathcal{I}$. Let m be such that $m \geq k_0$ so that the investor is part of the mth economy.

Let

$$\bar{\mathbf{g}} = \left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} + \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}\right)^{-1} \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\mathbf{1}$$

In the mth economy (see (14))

$$\mathbf{y}^m = \begin{bmatrix} \frac{1}{|\mathcal{NI}^m|} \sum_{k \in \mathcal{NI}^m} \mathbf{s}_k^m \\ \\ \frac{1}{m} \bar{\mathbf{g}}' \sum_{k=1}^{m} \mathbf{s}_k^m \end{bmatrix}_{(n+1)\times 1} = \begin{bmatrix} \frac{m}{|\mathcal{NI}^m|} \sum_{k \in \mathcal{NI}^m} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \\ \\ \bar{\mathbf{g}}' \sum_{k=1}^{m} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\pi^m} \sum_{k \in \mathcal{NI}^m} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \\ \\ \bar{\mathbf{g}}' \sum_{k=1}^{m} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \end{bmatrix} \tag{D.1}$$

Because the equilibrium prices $(p_f^m, \mathbf{p}^m)$ are informationally equivalent to $\mathbf{y}^m$ (Corollary 4.2), the information available to the kth investor is the pair $(\mathbf{s}_{k_0}^m, \mathbf{y}^m)$.

The optimal unconstrained portfolio is

$$\mathbf{x}_{k_0} = \frac{1}{\rho_{k_0}} \operatorname{var}^{-1}(\mathbf{v}^m | \mathbf{y}^m, \mathbf{s}_{k_0}^m) \left(E[\mathbf{v}^m | \mathbf{y}^m, \mathbf{s}_{k_0}^m] - \frac{1}{p_f^m} \mathbf{p}^m \right)$$

We plug in the above equilibrium prices (Theorem 3.1) and rearrange:

$$\mathbf{x}_{k_0} = \frac{1}{m} \frac{\bar{\rho}^m}{\rho_{k_0}} \operatorname{var}^{-1}(\mathbf{v}^m | \mathbf{y}^m, \mathbf{s}_{k_0}^m) \operatorname{var}(\mathbf{v}^m | \mathbf{y}^m) \mathbf{1} + \frac{1}{\rho_{k_0}} \operatorname{var}^{-1}(\mathbf{v}^m | \mathbf{y}^m, \mathbf{s}_{k_0}^m) \left(E[\mathbf{v}^m | \mathbf{y}^m, \mathbf{s}_{k_0}^m] - E[\mathbf{v}^m | \mathbf{y}^m] \right)$$

We define $\mathbf{y}_{k_0}^m$ to be as (D.1), only that k_0 is "treated" as a nonindexer.

$$\mathbf{y}_{k_0}^m := \begin{bmatrix} \frac{1}{|\mathcal{NI}^m| + 1} \sum_{k \in \mathcal{NI}^m \bigcup \{k_0\}} \mathbf{s}_k^m \\ \\ \frac{1}{m} \bar{\mathbf{g}}' \sum_{k=1}^{m} \mathbf{s}_k^m \end{bmatrix}_{(n+1)\times 1} = \begin{bmatrix} \frac{1}{1-\pi^m + 1/m} \sum_{k \in \mathcal{NI}^m \bigcup \{k_0\}} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \\ \\ \bar{\mathbf{g}}' \sum_{k=1}^{m} \bar{\mathbf{v}} + m^{1/2} \bar{\boldsymbol{\epsilon}}_k \end{bmatrix}_{(n+1)\times 1} \tag{D.2}$$

In the technical lemma presented at the start of Appenix C, we proved that $\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \mathbf{1}$ does not depend on the partition of investors (see C.1). Hence, $\operatorname{var}(\mathbf{v}^m | \mathbf{y}_{k_0}^m) \mathbf{1} = \operatorname{var}(\mathbf{v}^m | \mathbf{y}^m) \mathbf{1}$, which implies

$$\text{var}^{-1}(\mathbf{v}^m|\mathbf{y}^m_{k_0})\,\text{var}(\mathbf{v}^m|\mathbf{y}^m)\mathbf{1} = \mathbf{1} \tag{D.3}$$

It is straightforward to check that $E[\mathbf{v}^m|\mathbf{y}^m, \mathbf{s}^m_{k_0}] = E[\mathbf{v}^m|\mathbf{y}^m_{k_0}]$, and $\text{var}(\mathbf{v}^m|\mathbf{y}^m, \mathbf{s}^m_{k_0}) = \text{var}(\mathbf{v}^m|\mathbf{y}^m_{k_0})$, so the optimal unconstrained portfolio is

$$\begin{aligned}
\mathbf{x}_{k_0} &= \frac{1}{m}\frac{\bar{\rho}^m}{\rho_{k_0}}\,\text{var}^{-1}(\mathbf{v}^m|\mathbf{y}^m_{k_0})\,\text{var}(\mathbf{v}^m|\mathbf{y}^m)\mathbf{1} + \frac{1}{\rho_{k_0}}\,\text{var}^{-1}(\mathbf{v}^m|\mathbf{y}^m_{k_0})\left(E[\mathbf{v}^m|\mathbf{y}^m_{k_0}] - E[\mathbf{v}^m|\mathbf{y}^m]\right) \\
&\underset{\text{(D.3)}}{=} \underbrace{\frac{1}{m}\frac{\bar{\rho}^m}{\rho_{k_0}}\mathbf{1}}_{\text{market portfolio}} + \underbrace{\frac{1}{\rho_{k_0}}\,\text{var}^{-1}(\mathbf{v}^m|\mathbf{y}^m_{k_0})\left(E[\mathbf{v}^m|\mathbf{y}^m_{k_0}] - E[\mathbf{v}^m|\mathbf{y}^m]\right)}_{\text{zero-mean random portfolio}}
\end{aligned}$$

This proves the first part of the Theorem, with

$$\mathbf{e}^m_{k_0} = \text{var}^{-1}(\mathbf{v}^m|\mathbf{y}^m_{k_0})\left(E[\mathbf{v}^m|\mathbf{y}^m_{k_0}] - E[\mathbf{v}^m|\mathbf{y}^m]\right)$$

Our next goal is to show that $\lim m\mathbf{e}^m_{k_0} = \mathbf{0}_{n\times 1}$. To proves this we need some preliminary computations. In particular, we need to express the conditional expectations and conditional variances in terms of the primitives of the sequence of economies; i.e. $\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}$ and $\overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}}$.

Define

$$\overline{\boldsymbol{\Sigma}}_1 = \left[\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} \quad \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} + \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}})^{-1}\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\mathbf{1}\right]$$

For a scalar π and h, define

$$\overline{\boldsymbol{\Sigma}}_2(\pi, h) = \begin{bmatrix} \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} + \dfrac{1}{1-\pi+h}\overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}} & \overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\mathbf{1} \\ \mathbf{1}'\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} & \mathbf{1}'\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} + \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}})^{-1}\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\mathbf{1} \end{bmatrix}$$

We have

$$\text{cov}(\mathbf{v}^m, \mathbf{y}^m) = m^2\overline{\boldsymbol{\Sigma}}_1, \quad \text{var}(\mathbf{y}^m) = m^2\overline{\boldsymbol{\Sigma}}_2(\pi^m, 0), \qquad \text{var}(\mathbf{v}^m|\mathbf{y}^m) = m^2\left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} - \overline{\boldsymbol{\Sigma}}_1\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\overline{\boldsymbol{\Sigma}}_1'\right)$$

where in the above we evaluated $\overline{\boldsymbol{\Sigma}}_2(\pi, h)$ at $(\pi, h) = (\pi^m, 0)$. We also have

$$\text{cov}(\mathbf{v}^m, \mathbf{y}^m_{k_0}) = m^2\overline{\boldsymbol{\Sigma}}_1, \quad \text{var}(\mathbf{y}^m_{k_0}) = m^2\overline{\boldsymbol{\Sigma}}_2(\pi^m, 1/m), \quad \text{var}(\mathbf{v}^m|\mathbf{y}^m_{k_0}) = m^2\left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} - \overline{\boldsymbol{\Sigma}}_1\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\overline{\boldsymbol{\Sigma}}_1'\right)$$

where this time, we evaluated $\overline{\boldsymbol{\Sigma}}_2(\pi, h)$ at $(\pi, h) = (\pi^m, 1/m)$.

We use those expressions to rewrite $m\mathbf{e}_{k_0}^m$:

$$m\mathbf{e}_{k_0}^m = \left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} - \overline{\boldsymbol{\Sigma}}_1 \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\overline{\boldsymbol{\Sigma}}_1'\right)^{-1} \overline{\boldsymbol{\Sigma}}_1 \\ \times \frac{1}{m}\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\left(\mathbf{y}_{k_0}^m - E\mathbf{y}_{k_0}^m\right) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\left(\mathbf{y}^m - E\mathbf{y}^m\right)\right)$$

This is a product of two terms. The first term is non random, has dimension $n \times (n+1)$, and has the finite limit :

$$\left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} - \overline{\boldsymbol{\Sigma}}_1 \overline{\boldsymbol{\Sigma}}_2^{-1}(\bar{\pi}, 0)\overline{\boldsymbol{\Sigma}}_1'\right)^{-1} \overline{\boldsymbol{\Sigma}}_1$$

where $\bar{\pi} \equiv \lim \pi^m$.

Thus, to prove that $\lim m\mathbf{e}_{k_0}^m = \mathbf{0}_{n\times 1}$ with probability one, it is sufficient to prove that with probability one

$$\lim \frac{1}{m}\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\left(\mathbf{y}_{k_0}^m - E\mathbf{y}_{k_0}^m\right) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\left(\mathbf{y}^m - E\mathbf{y}^m\right)\right) = \mathbf{0}_{(n+1)\times 1} \tag{D.4}$$

For a scalar π and h, define the matrices $A(\pi, h)$ and $B(\pi, h)$

$$\mathbf{A}(\pi, h) := \begin{bmatrix} \frac{1-\pi}{1-\pi+h} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1-\pi}{1-\pi+h} & \cdots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & \frac{1-\pi}{1-\pi+h} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\mathbf{B}(\pi, h) := \begin{bmatrix} \frac{h}{1-\pi+h} & 0 & \cdots & 0 \\ 0 & \frac{h}{1-\pi+h} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \frac{h}{1-\pi+h} \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times n}$$

Then, from the definition of $\mathbf{y}^m$ (see D.1) and the definition of $\mathbf{y}^m_{k_0}$ (see D.2), we can write

$$\mathbf{y}^m_{k_0} = \mathbf{A}(\pi^m, 1/m)\mathbf{y}^m + \mathbf{B}(\pi^m, 1/m)(\bar{\mathbf{v}} + m^{1/2}\bar{\boldsymbol{\epsilon}}_{k_0})$$

where in the above, we evaluated $A(\pi, h)$ and $B(\pi, h)$ at $(\pi, h) = (\pi^m, 1/m)$. In particular, $E\mathbf{y}^m_{k_0} = E\mathbf{y}^m$, and $\mathbf{A}(\pi, 0)$ is the identity matrix.

We can now use this relation to expand the term in (D.4):

$$\begin{aligned}
&\frac{1}{m}\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\left(\mathbf{y}^m_{k_0} - E\mathbf{y}^m_{k_0}\right) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\left(\mathbf{y}^m - E\mathbf{y}^m\right)\right) \\
&= \frac{1}{m}\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{A}(\pi^m, 1/m) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\mathbf{A}(\pi^m, 0)\right)\left(\mathbf{y}^m - E\mathbf{y}^m\right) \qquad \text{(I)} \\
&+ \frac{1}{m}\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{B}(\pi^m, 1/m)\left(\bar{\mathbf{v}} - \bar{\boldsymbol{\mu}}_{\mathbf{v}} + m^{1/2}\bar{\boldsymbol{\epsilon}}_{k_0}\right) \qquad \text{(II)}
\end{aligned}$$

Consider the term denoted (II). Because $\lim_m \pi^m$ exists, for $\bar{\boldsymbol{\mu}}_{\mathbf{v}}$ and every realization of $\bar{\mathbf{v}}$ and $\boldsymbol{\epsilon}^m_{k_0}$, the limit, as m goes to infinity, is zero. Thus, to complete the proof that $\lim_m m\mathbf{e}^m_{k_0} = \mathbf{0}$ with probability one, we only need to show that the limit of the term denoted denoted (I) is, with probability one, zero.

We multiply the denominator of (I) by $1 = m \times \frac{1}{m}$, so (I) can be written as

$$\begin{aligned}
&\frac{1}{m}\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{A}(\pi^m, 1/m) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\mathbf{A}(\pi^m, 0)\right)\left(\mathbf{y}^m - E\mathbf{y}^m\right) \\
&= \frac{1}{m^2}\frac{\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{A}(\pi^m, 1/m) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\mathbf{A}(\pi^m, 0)\right)}{1/m}\left(\mathbf{y}^m - E\mathbf{y}^m\right) \\
&= \frac{1}{m^{1/2}} \times \frac{\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{A}(\pi^m, 1/m) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\mathbf{A}(\pi^m, 0)\right)}{1/m} \times \frac{1}{m^{3/2}}\left(\mathbf{y}^m - E\mathbf{y}^m\right)
\end{aligned}$$

We have decomposed the term into a product of three terms. The limit of the first term, $\frac{1}{m^{1/2}}$, is zero. The middle term, $\frac{\left(\overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 1/m)\mathbf{A}(\pi^m, 1/m) - \overline{\boldsymbol{\Sigma}}_2^{-1}(\pi^m, 0)\mathbf{A}(\pi^m, 0)\right)}{1/m}$, has a finite limit equals

$$\frac{\partial}{\partial h}\overline{\boldsymbol{\Sigma}}_2^{-1}(\bar{\pi}, 0)A(\bar{\pi}, 0)$$

where $\bar{\pi} = \lim \pi^m$.[22] The limit of the third term,

$$\frac{1}{m^{3/2}}\Big(\mathbf{y}^m - E\mathbf{y}^m\Big) \underset{(D.1)}{=} \begin{bmatrix} \frac{1}{m^{1/2}}\left(\bar{\mathbf{v}} - \bar{\boldsymbol{\mu}}_{\mathbf{v}}\right) + \frac{1}{|\mathcal{NI}^m|}\sum_{k\in\mathcal{NI}^m}\bar{\boldsymbol{\epsilon}}_k \\ \\ \frac{1}{m^{1/2}}\bar{\mathbf{g}}'\left(\bar{\mathbf{v}} - \bar{\boldsymbol{\mu}}_{\mathbf{v}}\right) + \bar{\mathbf{g}}'\frac{1}{m}\sum_{k=1}^m \bar{\boldsymbol{\epsilon}}_k \end{bmatrix}$$

is zero, with probability one, thanks to the strong law of large numbers.

This completes our proof that for every $k_0 \in \mathcal{I}$, $\lim m\mathbf{e}^m_{k_0} = \mathbf{0}_{(n+1)\times 1}$ with probability one.

Let now $k \in \mathcal{I}$ and let $i \in \{1, \ldots, n\}$. The third and last statement of the theorem is that with probability one

$$\lim \frac{x^m_{ki}}{x^m_{k1}} = 1$$

We fix realization of the random vectors $(\bar{\mathbf{v}}, \bar{\boldsymbol{\epsilon}}_1, \bar{\boldsymbol{\epsilon}}_2, \ldots)$. From the first two statements of the theorem, we know that with probability one: $\lim m x^m_{ki} = \dfrac{\lim_m \bar{\rho}^m}{\rho_k}$.

This concludes our proof.

■

Proof of Theorem 7.2. To prove the lemma one has to compute the conditional Sharpe ratio. Since the derivation is straightforward, we simply report what $f(\mathbf{x}, \pi)$ is:

$$f(\mathbf{x}, \pi) = \frac{\mathbf{x}'\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\left(\mathbf{1} - (\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} + \overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}})^{-1}\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}}\mathbf{1}\right)}{\sqrt{\mathbf{x}'\left(\overline{\boldsymbol{\Sigma}}_{\mathbf{vv}} - \overline{\boldsymbol{\Sigma}}_1\overline{\boldsymbol{\Sigma}}_2^{-1}\overline{\boldsymbol{\Sigma}}_1'\right)\mathbf{x}}}$$

Having shown what f is, it is clear that f is homogenous of degree zero with respect to $\mathbf{x}$. That f is stricly decreasing with respect to π follows from the proof of Theorem 5.3 in which

[22] Using rules for differentiation for matrices, we can compute this partial derivative. It is equal to:

$$\frac{1}{(1-\bar{\pi})^2}\overline{\boldsymbol{\Sigma}}_2^{-1}(\bar{\pi}, 0)\begin{bmatrix}\overline{\boldsymbol{\Sigma}}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0\end{bmatrix}\overline{\boldsymbol{\Sigma}}_2^{-1}(\bar{\pi}, 0) - \frac{1}{1-\bar{\pi}}\overline{\boldsymbol{\Sigma}}_2^{-1}(\bar{\pi}, 0)\begin{bmatrix}\mathbf{I}_{n\times n} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0\end{bmatrix}$$

we proved the Shrape ratio is decreasing with respect to π in general. In particular, also in the sequence of economies we have considered here.

■

References

Admati, A. R., 1985. A noisy rational expectations equilibrium for multi-asset securities markets. Econometrica: Journal of the Econometric Society, 629–657.

Ausubel, L. M., 1990. Partially-revealing rational expectations equilibrium in a competitive economy. Journal of Economic Theory 50 (1), 93–126.

Barberis, N., Shleifer, A., 2003. Style investing. Journal of financial Economics 68 (2), 161–199.

Barberis, N., Shleifer, A., Wurgler, J., 2005. Comovement. Journal of Financial Economics 75 (2), 283–317.

Berk, J. B., DeMarzo, P. M., 2014. Corporate finance, 3rd Edition. Pearson Education.

Bhattacharya, A., O'Hara, M., 2016. Can ETFs increase market fragility? Effect of information linkages in ETF markets. Working Paper.

Boehmer, E., Jones, C. M., Zhang, X., 2016. Tracking retail investor activity. Working paper.

Campbell, J. Y., Lettau, M., Malkiel, B. G., Xu, Y., 2001. Have individual stocks become more volatile? an empirical exploration of idiosyncratic risk. The Journal of Finance 56 (1), 1–43.

Cong, L. W., Xu, D. X., 2016. Rise of factor investing. Working Paper.

DeMarzo, P., Skiadas, C., 1998. Aggregation, determinacy, and informational efficiency for a class of economies with asymmetric information. journal of economic theory 80 (1), 123–152.

Deng, K., Chen, H., Kong, D., 2014. The effect of idiosyncratic risk on firm decisions: underinvestment or diversification? China Finance Review International 4 (3), 271–288.

Detemple, J. B., 2002. Asset pricing in an intertemporal partially-revealing rational expectations equilibrium. Journal of Mathematical Economics 38 (1), 219–248.

Geweke, J., 2005. Contemporary Bayesian econometrics and statistics. Wiley Series in Probability and Statistics.

Glosten, L. R., Nallareddy, S., Zou, Y., 2016. ETF trading and informational efficiency of underlying securities. Working Paper.

Grossman, S., 1976. On the efficiency of competitive stock markets where trades have diverse information. The Journal of finance 31 (2), 573–585.

Grossman, S., 1978. Further results on the informational efficiency of competitive stock markets. Journal of Economic Theory 18 (1), 81–101.

Heath, D., Macciocchi, D., Michaely, R., Ringgenberg, M. C., 2018. Passive investors are passive monitors. Working Paper.

Hellwig, M. F., 1980. On the aggregation of information in competitive markets. Journal of economic theory 22 (3), 477–498.

Kaniel, R., Liu, S., Saar, G., Titman, S., 2012. Individual investor trading and return patterns around earnings announcements. The Journal of Finance 67 (2), 639–680.

Kelley, E. K., Tetlock, P. C., 2013. How wise are crowds? insights from retail orders and stock returns. The Journal of Finance 68 (3), 1229–1265.

Kyle, A., 1985. Continuous auctions and insider trading. Econometrica 53 (6), 1315–1335.

Levy, H., 1978. Equilibrium in an imperfect market: A constraint on the number of securities in the portfolio. The American Economic Review 68 (4), 643–658.

Lo, A. W., 2016. What is an index? The Journal of Portfolio Management 42 (2), 21–36.

Malkiel, B. G., 2016. A Random Walk Down Wall Street, 11th Edition. WW Norton & Company.

Malkiel, B. G., Xu, Y., 2002. Idiosyncratic risk and security returns. University of Texas at Dallas (November 2002).

Markowitz, H., 1952. Portfolio selection. The journal of finance 7 (1), 77–91.

Panousi, V., Papanikolaou, D., 2012. Investment, idiosyncratic risk, and ownership. The Journal of Finance 67 (3), 1113–1148.

Samuelson, P., August 1976. Index-fund investing. Newsweek.

Stanyer, P., 2014. The Economist Guide to Investment Strategy: How to Understand Markets, Risk, Rewards, and Behaviour, 3rd Edition. The Economist.

Tobin, J., 1958. Liquidity preference as behavior towards risk. The review of economic studies 25 (2), 65–86.

Treynor, J. L., 1987. Market efficiency and the bean jar experiment. Financial Analysts Journal 43 (3), 50–53.