Inequality, Business Cycles and Monetary-Fiscal Policy[*]

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Abstract

We study monetary and fiscal policy in a heterogeneous agents model with incomplete markets and sticky nominal prices. We develop numerical techniques that allow us to approximate Ramsey plans in economies with substantial heterogeneity. In a calibrated model that captures features of income inequality in the US, we study optimal responses of nominal interest rates and labor tax rates to productivity and cost push shocks. Optimal policy responses are an order of magnitude larger than in a representative agent economy, and for cost push shocks are of opposite signs. Taylor rules poorly approximate optimal nominal interest rates.

KEY WORDS: Sticky prices, heterogeneity, business cycles, monetary policy, fiscal policy

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1   Introduction

An empirical labor literature has documented that dispersions of labor earnings, assets, and other measures of inequality co-move with aggregate business cycle fluctuations. Meanwhile, a quantitative macroeconomics literature that studies optimal monetary and/or fiscal policy over business cycles relies almost exclusively either on a representative agent assumption or oversimplified models of heterogeneity. We want to know how those simplified treatments of heterogeneity affect quantitative prescriptions. Therefore, this paper studies optimal monetary and fiscal policies in a workhorse New Keynesian model augmented to capture rich heterogeneities across agents and empirical facts about co-movements of aggregate variables and measures of inequality.

We study a New Keynesian economy populated by a continuum of heterogeneous agents who are subject to idiosyncratic wage risks. Agents differ in permanent and transitory components of shock processes that we calibrate to emulate dynamics of the distribution of U.S. labor earnings documented by Guvenen et al. (2014). Financial markets are incomplete with agents differing in their equity holdings and their access to financial markets. We study how a Ramsey planner adjusts nominal interest rates, transfers, and proportional labor taxes in response to aggregate shocks.

In studying the optimal policies of a Ramsey planner, we confront substantial computational challenges. Existing studies of economies with heterogeneities, incomplete markets, and aggregate shocks typically approximate the ergodic distributions of competitive equilibrium prices and quantities under a given set of policies and cannot easily be extended to answer normative questions. Ramsey problems bring additional challenges. First, because they assign important roles to Lagrange multipliers on individual and aggregate forward looking constraints, they have state variables that summarize history dependencies making them larger than what are needed to characterize ordinary competitive equilibria. Second, due to market incompleteness, some elements of state spaces exhibit very slow rates of mean-reversion, implying that approximations around a mean of an invariant distribution poorly approximate an optimal policy during a transition from a given distribution.

This paper contributes a new computational technique that allows us to obtain good approximations to optimal government policies for economies with such large state spaces. Our numerical methods build on perturbation theory that uses small noise expansions with respect to a one-dimensional parameterization of uncertainty as in Fleming (1971) and Fleming and Souganidis (1986) that has been applied earlier in economics by Anderson et al. (2012). These are related to but differ from expansions in Judd and Guu (1993, 1997) and Judd (1996, 1998) that explore small noise expansions with respect to shocks and state variables

about a deterministic steady state. A key step is that at each date, we take a Taylor expansion of policy functions around the current value of state vector with respect to a parameter that scales both idiosyncratic and aggregate shocks. The current state vector can include a distribution of idiosyncratic states. We thus update the point around which local approximations are taken each period,which allows our approximations to remain accurate even in settings where transition dynamics are slow.

To manage heterogeneity, we approximate the distribution of individual state variables using a discrete grid with a sufficiently large number of points. Our contribution here is to derive explicit formulas for coefficients occurring in the Taylor expansions of individual agents' and aggregate policy functions. We show that these formulas require matrix inversions only of manageable dimensions, often equal to the number of *aggregate* variables, and that they can be efficiently computed. In this way, our procedure allows fast approximations even for a large number of agents. so we can also construct impulse responses that account for how distributions across agents evolve after an aggregate shock. In Section 3.3, we describe the steps in our algorithm and how our method compares to other approaches.

Applying our methods to a calibrated New Keynsian economy with heterogeneous agents, we find that attitudes about inequality induce the planner to use fiscal and monetary tools to redistribute resources toward agents who are especially adversely affected by recessions. We study two types of shocks: shocks to the growth rate of productivity that also change the distribution of labor earnings in ways documented by Guvenen et al. (2014) and markup shocks. We compare our results to those from a representative agent benchmark.

In response to a negative productivity shock, we find that optimal monetary policy lowers nominal rates and closely tracks real rates thereby keeping expected inflation near zero. Nevertheless, it is desirable to engineer high unanticipated inflation in recessions because that transfers resources from agents with high bond holdings toward agents with low holdings. This transfer makes up for the inability of agents to fully insure against aggregate shocks. An optimal plan induces such surprise inflation by increasing the tax rate, which raises real wages and marginal costs for firms. Furthermore, as in data, recessions in our calibrated economy are accompanied by persistent increases inequality. This generates a motive to redistribute labor income from productive agents by increasing transfers. The planner achieves this by keeping marginal labor tax rates high long after output has recovered. We find that in response to a productivity shock that lowers output growth by 3%, there is a nearly permanent increase in the labor tax rates of about 0.5 percentage points and a 0.15 percentage points jump in inflation for one period. As a point of comparison, the optimal tax rate and inflation rate in an economy without heterogeneity are an order of magnitude lower for similar shocks.

In response to a "cost push" shock that we model as a shock to the elasticity of substitution between goods that leads to an increase in the the desired mark-ups for the firms, an optimal policy calls for a significant decrease in nominal interest rates that generates an increase in inflation and output. This policy response is opposite from that found in a representative agent economy. The explanation for this difference is that in response to a cost push shock firms, want to increase their prices, and the presence of nominal rigidities makes that costly. So, in a representative agent economy, a Ramsey planner increases nominal interests rates to reduce output and marginal costs enough to offset this inflationary pressures, a policy that Galí (2015) dubs "leading against the wind". The mark-up shock also decreases the labor share and increases the profit share, which in heterogeneous agent economies redistributes resources from agents who mainly obtain income from wages to agents to with large stock holdings. The "leaning against the wind policies" exacerbate this effect. When we calibrate the distribution of equity ownership to U.S. data, we find that a 1 percentage point positive cost push shock calls for a -0.75 percentage point decrease in the nominal interest rate compared to 0.05 percentage point increase with a representative agent calibration.

We also investigate to what extent Taylor rules approximate an optimal policy. We find that in heterogeneous agent economies Taylor rules do a substantially worse job than in a representative agent counterpart. Taylor rules imply that interest rates and inflation share the same persistence and co-move positively. This behavior is sub-optimal in our economy.

We begin by describing our model and some properties of the Ramsey allocation in Section 2. The numerical method and its comparison to alternative are discussed in Section 3. We use our method to obtain quantitative results in the calibrated economy in Section 4. Section 5 studies the optimal response to "cost-push" shocks. Section 6 compare the optimal policies to those prescribed by a Taylor rule. Section 7 concludes.

2   Environment

A continuum of infinitely lived households face idiosyncratic shocks to their productivities. Individual i's preferences over stochastic processes for a final consumption good $\{c_{i,t}\}$ and labor supply $\{n_{i,t}\}$ are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{i,t}, n_{i,t}, \Theta_t\right)$$

where

$$U\left(c, n, \Theta\right) = \frac{c^{1-\nu}}{1-\nu} - \exp\left((1-\nu)\Theta\right) \frac{n^{1+\gamma}}{1+\gamma}, \tag{1}$$

$\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0,1)$ is a time discount factor. With separable preferences, scaling the disutility of labor with $\exp((1-\nu)\Theta)$ keeps n stationary when the aggregate shocks have a stochastic trend.

The economy is subject to aggregate and idiosyncratic shocks. Aggregate shocks are labor productivity Θ_t and government expenditures G_t that follow stochastic processes described by

$$\Theta_t = \bar{\Theta} + \Theta_{t-1} + \mathcal{E}_{\Theta,t},$$

$$\log G_t = \Theta_t + \bar{G} + \mathcal{E}_{G,t},$$

where $\mathcal{E}_{\Theta,t}, \mathcal{E}_{G,t}$ are mean-zero, i.i.d. random variables. Aggregate and idiosyncratic shocks relate to individual i's labor productivity $\theta_{i,t}$ by

$$\theta_{i,t} = \Theta_t + e_{i,t} + \varsigma_{i,t}, \tag{2}$$

$$e_{i,t} = \rho_e e_{i,t-1} + f(e_{i,t-1})\mathcal{E}_{\Theta,t} + \eta_{i,t}, \tag{3}$$

where $\varsigma_{i,t}$, $\eta_{i,t}$ are also mean-zero, i.i.d. random variables. This specification of idiosyncratic shocks builds closely on formulations used in labor literature, e.g. Storesletten et al. (2001), Low et al. (2010) where $\varsigma_{i,t}$ and $\eta_{i,t}$ correspond to transitory and persistent shocks to individual productivity. The function $f(e_{i,t-1})$ individuals' skills vary with aggregate shocks. It allows us to match the business cycles in cross sections that are documented by Guvenen et al. (2014). We assume that all shocks take values in a compact set.

Agent i supplies $\exp(\theta_{i,t})n_{i,t}$ units of effective labor to a competitive labor market at nominal wage $P_t W_t$, where P_t is the nominal price of the final consumption good at time t. There is a common proportional labor tax rate τ_t and a common lump transfer $T_t P_t$. Agents trade a one-period risk-free nominal bond with each other and with the government. We use $P_t b_{i,t}$, $P_t B_t$ to denote bond holdings of agent i and the debt position of the government respectively, and ι_t, π_t to denote the nominal interest rate and inflation. Finally, D_t are dividends from intermediate goods producers measured in units of the final good. We assume in our baseline specification that these dividends are distributed equally across households but drop this assumption in Section 5. We take as given an initial price level $P_{-1} < \infty$ and set $\iota_{-1} = \beta^{-1} - 1$.

Agent i's budget constraint can be written as

$$c_{i,t} + b_{i,t} = (1-\tau_t)W_t \exp(\theta_{i,t})n_{i,t} + T_t + D_t + \left(\frac{1+\iota_{t-1}}{1+\pi_t}\right)b_{i,t-1}. \tag{4}$$

The government's budget constraint at time t is

$$G_t + T_t + \left(\frac{1 + \iota_{t-1}}{1 + \pi_t}\right) B_{t-1} = \tau_t \int_i W_t \exp(\theta_{i,t}) n_{i,t} di + B_t.$$

A final good Y_t is produced by competitive firms that use a continuum of intermediate goods $\{y_t(j)\}_{j \in [0,1]}$ in a production function

$$Y_t = \left[\int_0^1 y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}.$$

The final good producer takes the final good prices P_t and intermediate goods prices $\{p_t(j)\}_j$ as given and solves

$$\max_{\{y_t(j)\}_{j \in [0,1]}} P_t \left[\int_0^1 y_t(j) dj\right]^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 p_t(j) y_t(j) dj. \tag{5}$$

Outcomes of optimization problem (5) are a demand function for intermediate goods

$$y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t,$$

and a nominal price satisfying

$$P_t = \left(\int_0^1 p_t(j)^{1-\epsilon}\right)^{\frac{1}{1-\epsilon}}.$$

Intermediate goods $y_t(j)$ are produced by monopolists having linear technologies

$$y_t(j) = \tilde{n}_t(j),$$

where $\tilde{n}_t(j)$ is the amount of effective labor hired by firm j. These monopolists face downward sloping demand curves $\left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t$ and choose prices $p_t(j)$ while bearing quadratic Rotemberg (1982) price adjustment costs $\frac{\exp(\Theta_t)\psi}{2} \left(\frac{p_t(j)}{p_{t-1}(j)} - 1\right)^2$ measured in units of the final consumption good. Firm j chooses prices $\{p_t(j)\}_t$ that solve

$$\max_{\{p_t(j)\}_t} \mathbb{E}_0 \sum_t \beta^t \left(\frac{C_t}{C_0}\right)^{-\nu} \left\{\left[\frac{p_t(j)}{P_t} - W_t\right] \left(\frac{p_t(j)}{P_t}\right)^{-\epsilon} Y_t - \frac{\exp(\Theta_t)\psi}{2} \left(\frac{p_t(j)}{p_{t-1}(j)} - 1\right)^2\right\}, \tag{6}$$

where for convenience we have imposed that each firm values profit streams with a stochastic

discount factor that is driven by aggregate consumption $C_t = \int c_{i,t} di$.[1]

In the symmetric equilibrium $p_t(j) = P_t$, $y_t(j) = Y_t$ for all j and market clearing conditions in labor, goods, and bond markets are:

$$y_t(j) = Y_t = \int \exp(\theta_{i,t}) n_{i,t} di \tag{7}$$

$$C_t + G_t = Y_t - \frac{\exp(\Theta_t) \psi}{2} \pi_t^2 \tag{8}$$

$$\int_i b_{i,t} di = B_t. \tag{9}$$

Definition 1. An *allocation* is a sequence $\{c_{i,t}, n_{i,t}\}_{i,t}$. A *bond profile* is a sequence $\{\{b_{i,t}\}_i, B_t\}_t$. A *price* system is a sequence $\{W_t, P_t\}_t$. A *monetary policy* is a sequence $\{\iota_t, T_t\}_t$. A *monetary-fiscal policy* is a sequence $\{\iota_t, T_t, \tau_t\}_t$.

Definition 2. Given an initial bond distribution $(\{b_{i,-1}, e_{i,-1}\}_i, B_{-1})$ and initial price levels $p_{-1}(j) = P_{-1}$ for all j, a *competitive equilibrium* is a monetary-fiscal policy $\{\iota_t, T_t, \tau_t\}_t$ and a sequence $\{\{c_{i,t}, n_{i,t}, b_{i,t}\}_i, B_t, W_t, P_t\}_t$ such that: (i) $\{c_{i,t}, l_{i,t}, b_{i,t}\}_{i,t}$ maximize (1) subject to (4) and natural debt limits; (ii) final goods firms choose $\{y_t(j)\}_j$ to maximize (5); (iii) intermediate goods producers' prices solve (6) and satisfy $p_t(j) = P_t$; and (iv) market clearing conditions (7), (8) and (9) are satisfied.

A utilitarian Ramsey planner orders allocations by

$$\mathbb{E}_0 \int \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{i,t}^{1-\nu}}{1-\nu} - \exp((1-\nu)\Theta_t) \frac{n_{i,t}^{1+\gamma}}{1+\gamma} \right] di. \tag{10}$$

Definition 3. Given a tax sequence $\tau_t = \bar{\tau}$ for some $\bar{\tau}$ and initial conditions $(\{b_{i,-1}, e_{i,-1}\}_i, B_{-1})$, an *optimal monetary policy* is a sequence $\{\iota_t, T_t\}_t$ that supports a competitive equilibrium allocation that maximizes (10). Given initial conditions $(\{b_{i,-1}, e_{i,-1}\}_i, B_{-1})$, an *optimal monetary-fiscal policy* is a sequence $\{\iota_t, T_t, \tau_t\}_t$ that support a competitive equilibrium allocation that maximizes (10). A maximizing monetary or monetary-fiscal policy is called a *Ramsey plan*; an associated allocation is called a *Ramsey allocation*.

The distinction between optimal monetary and monetary-fiscal policies is that the former takes tax rates as given while the latter also optimizes with respect to tax rates. A common

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[1] In economies with heterogeneous agents and incomplete markets one has to take a stand on how firms are valued. Using aggregate consumption to drive a stochastic discount factor process allows us to get a representative agent economy as a special case of our heterogeneous agent economy by appropriately setting some of our parameters. This choice aligns with Kaplan et al. (2016).

argument is that institutional constraints make it difficult to adjust tax rates in response to typical business cycle shocks, leaving nominal interest rates as the government's only tool for responding to such shocks. We will capture that argument by studying optimal monetary policy when tax rates $\{\tau_t\}_t$ are fixed at some level $\bar{\tau}$. The monetary-fiscal Ramsey plan evaluates the optimal policies when this restriction is dropped.

2.1 Ramsey Plans

As in Kydland and Prescott (1980) and Farhi (2010), we use firms' and household's optimality conditions to derive implementability constraints and express a Ramsey problem in terms of two Bellman equations, a continuation Ramsey problem for $t \geq 1$, and a $t = 0$ Ramsey problem. We relegate descriptions of these equations to Appendix A and state only the $t \geq 1$ continuation problem here.

We use hats to denote variables scaled by aggregate productivity. For example, $\hat{c}_{i,t} = \frac{c_{i,t}}{\exp(\Theta_t)}$, $\hat{b}_{i,t} = \frac{b_{i,t}}{\exp(\Theta_t)}$, $\hat{T}_t = \frac{T_t}{\exp(\Theta_t)}$ and so on for other idiosyncratic and aggregate variables. The period utility function is

$$U(c, n, \Theta) = \exp((1-\nu)\Theta)\left(\frac{\hat{c}^{1-\nu}}{1-\nu} - \frac{n^{1+\gamma}}{1+\gamma}\right) = \exp((1-\nu)\Theta) U(\hat{c}, n, 1)$$

We use $U_{\hat{c}}$ and U_n to denote derivatives of $U(\hat{c}, n, 1)$ with respect to first and second arguments, respectively. State variables for the $t \geq 1$ continuation value function are marginal utility adjusted assets, $\hat{a}_{it} \equiv \hat{c}_{i,t}^{-\nu}\hat{b}_{i,t}$, inverse marginal utility $\hat{m}_{i,t} \equiv \hat{c}_{i,t}^{\nu}$, and the persistent component of idiosyncratic shocks $e_{i,t}$. Let $z_{i,t} \equiv (\hat{a}_{i,t}, \hat{m}_{i,t}, e_{i,t})$ and let Z_t denote the distribution of individual states $z_{i,t}$, an aggregate state vector for our problem.

In an optimum, aggregate allocations in period t are functions of the previous period's aggregate state Z_{t-1} and the aggregate shocks $\mathcal{E}_t = (\mathcal{E}_{\Theta,t}, \mathcal{E}_{G,t})$. Individual variables are functions of $(Z_{t-1}, \mathcal{E}_t)$, individual state z_{t-1} and idiosyncratic shocks $\varepsilon_t = (\eta_t, \varsigma_t)$. Let Φ and ϕ denote distributions of $\mathcal{E}_t$ and ε_t. It will also be convenient to define $g(z, \varepsilon, \mathcal{E}) \equiv \exp[e + \varsigma + \eta + f(e, Z, \mathcal{E}_\Theta)\mathcal{E}_\Theta]$ as the productivity of an agent scaled by the aggregate productivity in state $(z, \varepsilon, \mathcal{E})$. Finally, let $\mathbb{E}_z$ denote a mathematical expectation of an individual variable conditional on z, Z. Let $\hat{V}(Z)$ be the value function for Ramsey monetary-fiscal problem, so

$$\hat{V}(Z) = \max_{\substack{\hat{c}, n, \hat{a}', \\ \hat{C}, W, \hat{Y}, \hat{D}, \tau, \hat{T}, \hat{Y}, \pi, \alpha}} \int e^{(1-\nu)[\bar{\Theta}+\mathcal{E}_\Theta]} \left\{ U(\hat{c}(z, \varepsilon, \mathcal{E}), n(z, \varepsilon, \mathcal{E}), 1) \, d\phi dZ + \beta \hat{V}(Z') \right\} d\Phi$$

$$(11)$$

subject to

$$\frac{\exp\left\{-\left[\bar{\Theta}+\mathcal{E}_\Theta\right]\right\}\hat{a}\left(z\right)U_{\hat{c}}(z,\varepsilon,\mathcal{E})(1+\pi(\mathcal{E}))^{-1}}{\beta\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta}+\mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(z,\varepsilon,\mathcal{E})(1+\pi(\mathcal{E}))^{-1}]}$$
$$= U_{\hat{c}}(z,\varepsilon,\mathcal{E})\left[\hat{c}(z,\varepsilon,\mathcal{E})-\hat{D}(\mathcal{E})-\hat{T}(\mathcal{E})\right]+U_n(z,\varepsilon,\mathcal{E})n(z,\varepsilon,\mathcal{E})+\hat{a}'(z,\varepsilon,\mathcal{E}) \tag{12}$$

$$\alpha = \hat{m}(z)\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta}+\mathcal{E}_\Theta\right]\right\}U_{\hat{c}}(z,\varepsilon,\mathcal{E})(1+\pi(\mathcal{E}))^{-1}] \tag{13}$$

$$U_n(z,\varepsilon,\mathcal{E}) = -(1-\tau(\mathcal{E}))W(\mathcal{E})U_{\hat{c}}(z,\varepsilon,\mathcal{E})g(z,\varepsilon,\mathcal{E}), \tag{14}$$

for all $(z,\varepsilon,\mathcal{E})$, and

$$\hat{C}(\mathcal{E}) = \int \hat{c}(z,\varepsilon,\mathcal{E})d\phi dZ \tag{15}$$

$$\hat{Y}(\mathcal{E}) = \int n(z,\varepsilon,\mathcal{E})g(z,\varepsilon,\mathcal{E})d\phi dZ \tag{16}$$

$$\hat{C}(\mathcal{E}) + \hat{G}(\mathcal{E}) = \hat{Y}(\mathcal{E}) - \frac{\psi}{2}\pi(\mathcal{E})^2 \tag{17}$$

$$\hat{D}(\mathcal{E}) = (1-W(\mathcal{E}))\hat{Y}(\mathcal{E}) - \frac{\psi}{2}\pi(\mathcal{E})^2. \tag{18}$$

for all $\mathcal{E}$. The distribution Z' is generated by $\hat{a}', \hat{m}'$ and shocks $\varepsilon, \mathcal{E}$.

The steps to obtain these equations are standard. Equation (12) is obtained by dividing the budget constraint (4) by $\exp(\Theta_t)$, then multiplying with $U_{\hat{c}_i,t}$ and substituting for the labor-leisure optimality condition in equation (14). Equations (13) and (14) use agents' intra and inter-temporal marginal conditions. Equations (15) - (18) are aggregate market clearing conditions.

The Ramsey monetary problem is written in a similar way except that τ_t is replaced with $\bar{\tau}$ in (14). Since the two problems are very similar, for concreteness we focus on the Ramsey monetary-fiscal policy in this section and the next.

Bellman equation (11) closely resembles one obtained in representative agent economies, except now the state is a distribution over individual state variables, a highly dimensional object for realistic amounts of heterogeneity. That makes it impossible to solve Bellman equation (11) directly, and motivates us instead to approach the problem by approximating the policy rules that solve the first order conditions of the planner's problem..

We use tildes to denote policy functions that attain the right side of Bellman equation (11). Let $\tilde{x}(z, Z, \varepsilon, \mathcal{E})$ and $\tilde{X}(Z, \mathcal{E})$ denote vectors of individual and aggregate policy functions, respectively. Let $\tilde{z}(z, Z, \varepsilon, \mathcal{E})$ and $\tilde{Z}(Z, \mathcal{E})$ be laws of motion for individual and aggregate states respectively. Let $\tilde{z}$ be a component of $\tilde{x}$. And as noted above, we use $\mathbb{E}_z\tilde{x}$ to denote the expectation of $\tilde{x}(z, Z, \cdot, \cdot)$ conditional on (z, Z) and $\mathbb{E}_{\tilde{z}}\tilde{\tilde{x}}$ the expectation of

$\tilde{x}\left(\tilde{z}\left(z,Z,\epsilon,\mathcal{E}\right),\tilde{Z}(\Psi,\mathcal{E}),\cdot,\cdot\right)$ conditional on $\left(\tilde{z},\tilde{Z}\right)$.

From problem (11), $\tilde{x}$ includes $\{\hat{c}, n, \hat{a}'\}$ and Lagrange multipliers on constraints (12) - (14), $\tilde{X}$, while contains $\left\{\hat{C}, W, \hat{Y}, \hat{D}, \tau, \hat{T}, \hat{Y}, \pi, \alpha\right\}$ and the Lagrange multipliers on (28) - (18). Let N_z, N_x, and N_X be numbers of elements in z, x, and X respectively. Substitute (15) into (17), (18) into (12), and follow Marcet and Marimon (2011) to use the Lagrange multiplier on (12), the co-state variable of $\hat{a}(z)$, in place of $\hat{a}(z)$ in z to end up with $N_z = 3$, $N_x = 6$, and $N_X = 4$. More details are in Appendix B.

We start with the set of individual constraints and first-order conditions with respect to x for the problem defined by (11). There exists a vector-valued function F that lets us write these equations as

$$F\left(z, \mathbb{E}_z \tilde{x}, \tilde{x}(z, Z, \varepsilon, \mathcal{E}), \mathbb{E}_{\tilde{z}}\tilde{\tilde{x}}, \tilde{X}(Z,\mathcal{E}), \varepsilon, \mathcal{E}\right) = \mathbf{0} \quad \text{for all } z, \varepsilon, \mathcal{E}. \tag{19}$$

Similarly, there exists a vector-valued function R such that first-order conditions with respect to $\tilde{X}$ along with market clearing constraints can be written compactly as

$$\int R(z, \tilde{x}(z, Z, \varepsilon, \mathcal{E}), \tilde{X}(Z, \mathcal{E}), \varepsilon, \mathcal{E})d\phi dZ = \mathbf{0} \quad \text{for all } \mathcal{E}. \tag{20}$$

Explicit functional forms of F and R are described in Appendix B. Our goal is to approximate $\tilde{x}\left(z, Z, \varepsilon, \mathcal{E}\right)$ and $\tilde{X}\left(Z, \mathcal{E}\right)$ that satisfy (19) and (20) for arbitrary Z.

3   Numerical Method

Our starting point is the perturbation theory of Fleming (1971), Fleming and Souganidis (1986) that uses small noise expansions. Consider a family of stochastic processes parameterized by a positive scalar σ that scales all shocks $(\varepsilon, \mathcal{E})$. Let $\tilde{x}\left(z, Z, \sigma \cdot \varepsilon, \sigma \cdot \mathcal{E}; \sigma\right)$ and $\tilde{X}(Z, \sigma \cdot \mathcal{E}; \sigma)$ denote policy functions when the scaling parameter equal σ. For our application, we will assume that autocorrelation of the persistent component of idiosyncratic shock is parameterized as $\rho_e = 1 - \hat{\rho}_e \sigma$.

Consider a second-order Taylor expansion with respect to σ around $\sigma = 0$ at *a given state Z*:[2]

$$\tilde{X}\left(Z, \sigma\mathcal{E}; \sigma\right) = \bar{X} + \sigma\left(X_{\mathcal{E}}\mathcal{E} + X_\sigma\right) + \frac{\sigma^2}{2}\left(\mathcal{E}^\mathsf{T} X_{\mathcal{E}\mathcal{E}}\mathcal{E} + 2X_{\mathcal{E}\sigma}\mathcal{E} + X_{\sigma\sigma}\right) + \mathcal{O}(\sigma^3) \tag{21}$$

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[2]We use $\mathcal{E}^\mathsf{T} X_{\mathcal{E}\mathcal{E}}\mathcal{E}$ to denote the vector who's ith element is given by the quadratic form $\mathcal{E}^\mathsf{T} X_{\mathcal{E}\mathcal{E}}^i \mathcal{E}$

and

$$\tilde{x}\left(z, Z, \sigma\varepsilon, \sigma\mathcal{E}; \sigma\right) = \bar{x} + \sigma\left(x_{\mathcal{E}}\mathcal{E} + x_{\varepsilon}\varepsilon + x_{\sigma}\right)$$
$$+ \frac{\sigma^2}{2}\left(\mathcal{E}^{\mathrm{T}}x_{\mathcal{E}\mathcal{E}}\mathcal{E} + \varepsilon^{\mathrm{T}}x_{\varepsilon\varepsilon}\varepsilon + 2\mathcal{E}^{\mathrm{T}}x_{\mathcal{E}\varepsilon}\varepsilon + 2x_{\mathcal{E}\sigma}\mathcal{E} + 2x_{\varepsilon\sigma}\varepsilon + x_{\sigma\sigma}\right) + \mathcal{O}(\sigma^3), \quad (22)$$

where $\bar{X} \equiv \tilde{X}\left(Z, 0; 0\right)$, $\bar{x} \equiv \tilde{x}\left(z, Z, 0, 0; 0\right)$, $X_{\mathcal{E}}, X_{\sigma}$ denote derivatives of $\tilde{X}\left(Z, \sigma\mathcal{E}; \sigma\right)$ with respect to the second and third arguments evaluated at $\sigma = 0$, and the derivatives of $\tilde{x}$ and higher order derivatives defined in similar ways.

Our main contribution is to show that the right sides of these expressions can be computed quickly even when the dimension of the underlying discretized state Z is very large. With realistic heterogeneity, it is important to have a large number of points on the grid for Z in order to capture a distribution of individual states (in our numerical application we discretize Z with $K =$10,000 elements). In particular, we show: (a) that the "no uncertainty" terms $\bar{X}$, $\bar{x}$ solve a simple system of non-linear equations corresponding to a static economy, and (b) explicit formulas for higher order terms $X_{\mathcal{E}}, x_{\varepsilon}, \ldots$ that can involve only linear algebra. We attained these formulas by overcoming two significant problems. The first problem is that the number of unknowns in the first-order terms is proportional to K^2, while the number of unknowns for the second order terms is proportional to K^3. A second problem is that computing these unknowns directly requires inverting at least $K \times K$ matrices. When K is large, inverting $K \times K$ matrices and solving for unknowns whose number grow exponentially in K become impractical. We overcome these problems in the following ways. First, we show that all unknowns can be computed as a product of either two (in the case of first-order expansions) or three (in the case of the second-order expansion) $K \times 1$ vectors with unknown coefficients, so that the number of unknowns grows linearly rather than exponentially with K. Furthermore, we can compute these unknowns by inverting matrices with dimensions of at most $N_x \times N_x$. Since the inversion of 6×6 matrices (unlike $10,000 \times 10,000$ matrices) can be done quickly, all unknowns can be computed quickly even when K is very large.

3.1 Step 1: computing points of expansion

Our next proposition describes how to simplify the 'zeroth-order' terms $\bar{X}$, $\bar{x}$.

Proposition 1. *The individual and aggregate states are stationary in the non-stochastic limit, i.e.,*
$$\tilde{z}(z, Z, 0, 0; 0) = z \text{ and } \tilde{Z}(Z, 0; 0) = Z.$$

Therefore $\left(\bar{x}, \bar{X}\right)$ solve

$$F\left(z, \bar{x}, \bar{x}, \bar{x}, \bar{X}, 0, 0\right) = \mathbf{0} \quad \textit{for all } z, \tag{23}$$

$$\int R(z, \bar{x}, \bar{X}, 0, 0) dZ = \mathbf{0}. \tag{24}$$

The explanation for these outcomes is that in the absence of shocks, markets are complete and the pair $\left(\bar{x}, \bar{X}\right)$ corresponds to a stationary economy in which all households completely smooth consumption, which implies that the aggregate state Z stays unchanged too. In our case, we can invert (23) to express $\bar{x}$ in terms of $\bar{X}$ and then use a standard numerical root finder to solve (24) as a system of equations in N_X unknowns.[3]

Once $\left(\bar{x}, \bar{X}\right)$ is known, we evaluate first-, second- and higher-order derivatives of functions F and R at an allocation. These derivatives can be found efficiently using automatic differentiation routines. Let $F_0^k, F_{x-}^k, F_x^k, F_{x+}^k, F_X^k$ be the derivatives of F with respect to its first five arguments evaluated at $\left(z_k, \bar{x}, \bar{x}, \bar{x}, \bar{X}, 0, 0\right)$ for $k = 1, .., K$. Similarly, let R_0^k, R_x^k, R_X^k be derivatives of R with respect to its first three arguments evaluated at $\left(z_k, \bar{x}, \bar{X}, 0, 0\right)$. Define higher order terms in similar ways.

3.2   Step 2: finding derivatives of policy functions

We now describe how to compute derivatives of policy functions $\tilde{x}$ and $\tilde{X}$. Given our discretization procedure, the state variable for $\tilde{X}$ is an $N_z \times K$-dimensional object, and the state variable for $\tilde{x}$ is an $N_z \times (K+1)$ dimensional object. We let X_k, x_k^l denote derivatives of $\tilde{X}\left(Z, \cdot; \cdot\right)$ and $\tilde{x}\left(z_l, Z, \cdot, \cdot; \cdot\right)$ with respect to the kth element of Z evaluated at $\sigma = 0$ and $z = z_l$; and x_0^l for $l \in \{1, ..., K\}$ denote the derivative of $\tilde{x}$ with respect to the individual state z evaluated at $\sigma = 0$ and $z = z_l$. It is convenient to define a matrix Q that selects state variables $\tilde{z}$ from the vector $\tilde{x}$, i.e., $\tilde{z} = Q\tilde{x}$ and similarly use z_k^l, z_0^l to define the derivatives of $\tilde{z}$. Higher order derivatives of $\tilde{x}$ and $\tilde{X}$ are defined in similar ways.

Although derivatives of policy functions only with respect to shocks $(\varepsilon, \mathcal{E})$ appear in the Taylor expansion (21) - (22), computing those derivatives requires also computing the derivatives of $\tilde{x}$ and $\tilde{X}$ with respect to the state variables (z, Z). This creates an obstacle. Consider the first-order terms. Total differentiation of (19) and (20) generates a linear system that determines $\left\{X_k, x_k^l, x_0^l\right\}_{k,l}$. This is a system with $KN_X N_z + K^2 N_x N_z + KN_x N_z$ unknowns that requires inverting a $KN_x + N_X \times KN_x + N_X$ dimensional Jacobian matrix. Since the number of terms grows with K^2, these inversions are not computationally feasible for large K. The next lemma significantly simplifies things.

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[3]In Section (5) we extend our analysis to the cases when zeroth-order expansions cannot be found using Proposition (1)

Lemma 1. *For $z = z_l$, let x_X^l, x_0^l be matrices of dimension $N_x \times N_X$ and $N_x \times N_z$ defined as*

$$x_X^l = - \left[F_{x-}^l + F_x^l + F_{x+}^l \right]^{-1} F_X^l.$$

Derivatives of policy rules with respect to states $\left\{ x_0^l, X_k, x_k^l \right\}_{k,l}$ are

$$x_0^l = - \left[F_{x-}^l + F_x^l + F_{x+}^l \right]^{-1} F_0^l$$

$$X_k = \left(\sum_l \left(R_x^l x_X^l + R_X^l \right) \right)^{-1} \left[R_0^k + R_x^k x_0^k \right]$$

$$x_k^l = x_X^l X_k.$$

Lemma 1 is a critical step that makes our approach computationally feasible. It allows us to decompose a $\left\{ x_k^l \right\}_{l,k}$ that has $N_z \times K^2$ elements into a product of two objects, $\left\{ x_X^l \right\}_l$ and $\left\{ X_k \right\}_k$, each of which has $N_z \times K$ elements. This means that computational complexity grows linearly in K rather than K^2. Critically, computing x_X^l and X_k requires inversions of $\max\{N_X, N_x\} \times \max\{N_X, N_x\}$ matrices, which can be done quickly when the number of aggregate variables is small.

Economic structure that yields these simplifications are described by Evans (2015). The term x_k^l captures the effect on the group of agents having state variable $z = z_l$ by a small change in the state variable of the group $z = z_k$. In the present economy, interactions between groups are entirely intermediated through aggregates like prices and taxes. The means that pairwise effects x_k^l can be divided into how group k affect the aggregates X and how a change in aggregates X affect individuals in group l. Ultimately, the number of computations scales linearly in the number of points required to approximate Z.

Lemma 1 allows us to compute coefficients of the first-order expansion of policy functions.

Proposition 2. *Coefficients in a first-order expansion of the policy function $\tilde{x}$ are given by:* $x_\sigma^l = 0$ *and*

$$x_\varepsilon^l = - \left[F_x^l + F_{x+}^l x_0^l Q \right]^{-1} F_\varepsilon^l$$

$$x_{\mathcal{E}}^l = \left(x_{\mathcal{E},1}^l + x_{\mathcal{E},3}^l \left(I - \sum_k X_k Q x_{\mathcal{E},3}^k \right)^{-1} \left(\sum_k X_k Q x_{\mathcal{E},1}^k \right) \right)$$

$$+ \left(x_{\mathcal{E},2}^l + x_{\mathcal{E},3}^l \left(I - \sum_k X_k Q x_{\mathcal{E},3}^k \right)^{-1} \left(\sum_k X_k Q x_{\mathcal{E},2}^k \right) \right) X_{\mathcal{E}}$$

$$x_\sigma^l = \left(x_{\sigma,1}^l + x_{\sigma,3}^l \left(I - \sum_k X_k Q x_{\sigma,3}^k \right)^{-1} \left(\sum_k X_k Q x_{\sigma,1}^k \right) \right)$$
$$+ \left(x_{\sigma,2}^l + x_{\sigma,3}^l \left(I - \sum_k X_k Q x_{\sigma,3}^k \right)^{-1} \left(\sum_k X_k Q x_{\sigma,2}^k \right) \right) X_\sigma.$$

where

$$x_{\mathcal{E},1}^l = - \left(F_x^l + F_{x+}^l x_0^l Q \right)^{-1} F_{\mathcal{E}}^l, \quad x_{\mathcal{E},2}^l = - \left(F_x^l + F_{x+}^l x_0^l Q \right)^{-1} F_X^l, \quad x_{\mathcal{E},3}^l = - \left(F_x^l + F_{x+}^l x_0^l Q \right)^{-1} F_{x+}^l x_X^l,$$

$$x_{\sigma,1}^l = - \left(F_{x-}^l + F_x^l + F_{x+}^l + F_{x+}^l x_0^l Q \right)^{-1} F_\sigma^l, \quad x_{\sigma,2}^l = - \left(F_{x-}^l + F_x^l + F_{x+}^l + F_{x+}^l x_0^l Q \right)^{-1} F_X^l,$$

$$x_{\sigma,3}^l = - \left(F_{x-}^l + F_x^l + F_{x+}^l + F_{x+}^l x_0^l Q \right)^{-1} F_{x+}^l x_X^l.$$

Approximate policy functions $\tilde{X}$ satisfy

$$X_{\mathcal{E}} = - \left(\sum_k \left[R_X^k + R_x^k x_{\mathcal{E},2}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l \right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},2}^l \right) \right] \right)^{-1}$$
$$\times \left(\sum_k \left[R_\varepsilon^k + R_x^k x_{\mathcal{E},1}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l \right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},1}^l \right) \right] \right)$$

and

$$X_\sigma = - \left(\sum_k \left[R_X^k + R_x^k x_{\sigma,2}^k + R_x^k x_{\sigma,3}^k \left(I - \sum_l X_l Q x_{\sigma,3}^l \right)^{-1} \left(\sum_l X_l Q x_{\sigma,2}^l \right) \right] \right)^{-1}$$
$$\times \left(\sum_k \left[R_x^k x_{\sigma,1}^k + R_x^k x_{\sigma,3}^k \left(I - \sum_l X_l Q x_{\sigma,3}^l \right)^{-1} \left(\sum_l X_l Q x_{\sigma,1}^l \right) \right] \right).$$

Another message from Proposition 2 is that derivatives necessary for the approximations can be expressed in terms of matrices that are $N_x \times N_x$ or $N_X \times N_X$ and do not scale with the number of agents K. This allows us to handle very large state spaces.

Similar reasoning allows us to compute find higher-order derivatives with respect to $(z, Z, \varepsilon, \mathcal{E})$ and also derivatives of policy functions with respect to σ. It might seem that the number of terms required for higher order derivatives grows exponentially. For example, the second-order derivatives with respect to the state variables $\{x_{jk}^l, X_{jk}, x_{0,k}^l\}_{k,l,j}$ would require solving for $K^3 N_x N_z + K^2 N_X N_z + K^2 N_x N_z$ terms, which grow at the rate K^3. But a counterpart of Lemma 1 allows computations of all the second-order derivatives to be

subdivided into simpler terms that scale at most linearly with K and that only require inversion of low dimensional matrices. This logic preserves the computational advantages of our approach for higher-order expansions. The next proposition summarizes these findings. We provide formulas for all second-order terms in Appendix C.3.

Proposition 3. *The derivatives* $\left\{ x_{jk}^l, X_{jk}, x_{0,k}^l \right\}_{k,l,j}$ *can be expressed in terms that scale at most linearly with* K. *Derivatives* $\left\{ x_{\varepsilon\varepsilon}^l, x_{\mathcal{E}\varepsilon}^l, x_{\varepsilon\mathcal{E}}^l, x_{\mathcal{E}\mathcal{E}}^l, x_{\sigma\sigma}^l, X_{\mathcal{E}\mathcal{E}}, X_{\sigma\sigma} \right\}$ *in the second-order expansion can be expressed in terms of matrices of dimensions at most* $\max\left\{ N_x, N_X \right\}$ *by* $\max\left\{ N_x, N_X \right\}$.

The inclusion of second-order terms improves accuracy of approximations by capturing interesting economic behavior. For instance, the derivative $x_{\sigma\sigma}$ describes how agents with different asset holdings save differently in anticipation of future uninsurable shocks. In our economy, self insurance motives affect individuals' savings behavior and through market clearing conditions change aggregate prices and optimal responses of government policies. Such aggregate responses are encoded in the $X_{\sigma\sigma}$ terms. Our formulas for second-order derivatives combine such partial and general equilibrium effects of future risks on individual behavior. From a computational standpoint, these terms are important in capturing the evolution of the distributional state variable Z. In our quantitative Section 4, we show that ignoring second order terms significantly affects the quantitative magnitudes of policy responses to aggregate shocks.

3.3 Comparison to Other Methods

Our method is related to perturbation techniques of Judd and Guu (1993, 1997) and Judd (1996, 1998) that were subsequently extended to heterogeneous agent economies by Campbell (1998), Reiter (2009), Mertens and Judd (2013), Ahn et al. (2017), Winberry (2016), and Legrand and Ragot (2017).

Those approaches approximate responses to aggregate shocks by using first-order expansions of policy rules around a steady state Z^{SS} obtained by shutting down aggregate shocks. Ahn et al. (2017), which represents the current frontier,[4] extends Reiter (2009) into continuous time and performs a sophisticated form of model reduction by projecting a distribution of individual states onto a lower-dimensional subspace that is designed to do a good job of approximating impulse response functions of key variables like prices. In doing so, they

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[4]Mertens and Judd (2013) approximate around a point of no heterogeneity. Winberry (2016) uses a variant in which parametric forms capture the steady state distributions rather than the histograms used by Reiter (2009). Legrand and Ragot (2017) study an optimal fiscal policy problem with idiosyncratic risk and aggregate shocks after truncating individual histories, which limits the amount of heterogeneity that they can consider.

can incorporate a larger number of individual state variables than was previously possible. Lastly, except for Legrand and Ragot (2017), the studies cited earlier in this paragraph focus on competitive equilibria under fixed policies, rather than finding optimal policies.

We differ from these contributions in two ways. First, our points of approximation, Z_{t-1}, are dynamic and history dependent. By building on Fleming (1971), Fleming and Souganidis (1986), and Anderson et al. (2012), we take Taylor expansions with respect to uncertainty at each date as aggregate shocks push the economy through time.

There are several reasons that approximating around Z^{SS} is not a good way to approximate economies like ours. First, computing Z^{SS} in a Ramsey setting is difficult. Z^{SS} is an endogenous object that depends on a key object to be computed, an optimal policy. That requires jointly solving for agents' optimal behaviors, which depend on the government's policies, and optimal optimal policies. Even for a deterministic setting, that would require using computationally challenging non-linear solution methods. We are not aware of methods to do that quickly. But even if Z^{SS} could be found, it would be unlikely to be a good point of expansion. That is because in Ramsey settings with incomplete markets, speeds of mean-reversion to Z^{SS} are typically extremely slow because state variables are driven by martingale-like dynamics that drift slowly.[5] From a computational point of view, using perturbation around the fixed point provides a poor approximation for the optimal policy in many states that are away from Z^{SS}.

Secondly, paralleling Evans (2015), in Proposition 2 and 3 we are able to characterize the derivatives used in a small noise expansion in terms of matrices of small (typically $N_x \times N_x$) dimension. This allows us to perform not only first-order but also higher-order expansions quickly. As mentioned earlier, higher order terms are required to compute transition paths and accurate responses of aggregate variables to shocks because agents' responses to the idiosyncratic risks that they face puts slow drifts into equilibrium distributions of their state variables.

Alternatives to perturbation methods the literature have also used *projection methods* like Krusell and Smith (1998), Den Haan (1997), Algan et al. (2010). Projection methods summarize the infinite dimensional state variable using a subset of moments and approximate value functions and policy functions by using functional approximations and simulations for aggregate laws of motion that describe the ergodic behavior of moments.

Like the perturbation methods cited above, projection methods that approximate around the long run ergodic distribution Z^{SS} are problematic in Ramsey settings. Projections

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[5]To give an extreme example, debt follows a random walk in a canonical incomplete market model of Barro (1979), so that the speed of the mean reversion is 0. Aiyagari et al. (2002) showed that a slow margtingale-like component is generally typically present in a Ramsey plan in an incomplete market economy. In Bhandari et al. (2017) we compute analytically the speed of mean reversion for several incomplete markets economies.

methods work well only when an economy exhibits what Krusell and Smith termed an "approximate aggregation" property in which a function of the first moment of Z predicts next period's prices accurately. In our setting, Z is distributed over $\mathbb{R}^3$, which makes it much more difficult to summarize in terms of one or two dimensional statistics. Moreover, there is little reason to believe that our economy with heterogeneous loadings on aggregate shocks (and, in later extensions, heterogeneous participation in asset markets) exhibits approximation aggregation.

4 Quantitative Application

We apply our equilibrium approximation algorithm to an economy whose initial conditions are calibrated to recent U.S. data, assess quantitatively the properties of Ramsey policies, and contrast them with those of benchmark representative agent settings.

4.1 Calibration of Baseline Economy

In our baseline economy, we study effects of productivity shocks. We set $\nu = 1$ and $\gamma = 2$ to attain an intertemporal elasticity of substitution of 1 and a Frisch elasticity of labor supply equal to 0.5. The mean and standard deviation of the growth rate in productivity Θ_t are set at 2% and 3%, respectively, to match the mean and standard deviation of growth rate in output per hour in the US. The elasticity of substitution is set at $\epsilon = 6$ to target a value-added markup of 20%.

We choose initial conditions $\{b_{i,-1}, e_{i,-1}\}_i$ using data from the 2013 wave of the Survey of Consumer Finance (SCF). We restrict the SCF sample to married households and use information on households' total labor earnings, hours worked by the primary and secondary earners, and assets. From labor earning and hours we compute average households wages, $\{e_{i,-1}\}$. We then split wages into 20 quantiles and compute average households' holdings of government debt in each quantile.[6] This gives us a joint frequency distribution of $\{b_{i,-1}, e_{i,-1}\}_i$. High wage earners hold more government debt, the correlation coefficient between $e_{i,-1}$ and $b_{i,-1}$ being 0.47.

Remaining parameters are calibrated by insisting that competitive equilibrium outcomes given policies $\{\iota_t, \tau_t\}_t$ match stylized facts about U.S. policies. In particular, we set

$$\tau_t = \bar{\tau} \tag{25}$$

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[6]We sum direct holdings plus indirect holdings through government bond mutual funds (taxable and nontaxable), saving bonds, money market accounts, and components of retirement accounts that are invested in government bonds.

with $\bar{\tau} = 24\%$ to match the federal average marginal income tax estimated by Barro and Redlick (2011). We set ι_t to follow a Taylor rule

$$\iota_t = \left(\frac{1}{\beta \mathbb{E} exp\left\{ -\bar{\Theta} \right\}} - 1 \right) + 1.5\pi_t \tag{26}$$

Our choice of constant tax rates and Taylor rule that responds more than one to one to inflation and has a constant intercept features a desirable property that it can implement the optimal allocation in the absence of heterogeneity. The reason for this is that the real interest in a flexible price economy and with i.i.d. growth rate shocks and our choice of preferences is constant. Our baseline calibration is not sensitive to alternative specifications for fiscal and monetary policy that allow the nominal interest rate and tax rate to feedback on output or the output gap.[7]

We set the discount factor β to match an interest rate of 4% per year. The mean of government expenditures divided by labor productivity $\bar{G}$ and its standard deviation $\mathcal{E}_G$ are set to match the level and the volatility of the observed ratio of government spending to GDP.[8] Following Schmitt-Grohé and Uribe (2004), we use the estimates of Sbordone (2002) to calibrate the menu cost ψ. [9]

The persistence and volatility of idiosyncratic shocks and their loading on aggregate shocks are chosen to match several stylized facts about labor earnings. The standard deviation of $\varsigma_{i,t}$ is chosen to match the standard deviation of change in log earnings. As discussed in Storesletten et al. (2004), the parameter ρ_e and the standard deviation of $\eta_{i,t}$ can be inferred from the slope and curvature of the variance of log earnings at horizon $t + h$ as a function of horizon h. The left panel of Figure I shows how the variance-age plot for log earnings simulated from the model compares with the data. For both the date and the model outcomes, we plot the variance of log earnings horizon $t + h$ minus the variance of log earnings at $h = 0$.

The loading function $f(e)$ is constructed to match the evidence in Guvenen et al. (2014) on how recessions affects households in different parts of labor earning distribution. Following

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[7]An empirical literature about Taylor rules typically estimates loadings on output that are small and statistically close to zero. See for example Bhandari et al. (2017) for a discussion of fiscal policy rules and Clarida et al. (2000) for a discussion of monetary policy rules.

[8]For the measure of government spending we use federal government current expenditures net of transfer payments from NIPA. The average ratio for the period 1960-2016 is 7% with a standard deviation of 2%.

[9]Using quarterly data inflation and measures for marginal cost mc_t, Sbordone estimates a relationship $\pi_t = \alpha_1 \mathbb{E}_t \pi_{t+1} + \alpha_0 mc_t$ with $\alpha_1 \approx 1$ and α_0^{-1} in the range of 10 - 20 depending on particular measure of marginal cost. In a linearized version of Phillps curve equation 30, $\alpha_0 = \frac{\theta}{\epsilon-1}$. Using $\epsilon = 6$ and $\alpha_0^{-1} = 15$ implies a $\psi = 75$. Since in our model a period is an year, we set $\theta = \frac{75}{4} = 18.75$. As explained in Sbordone (2002), in a Calvo type price setting friction, this estimate corresponds to firms changing prices every 9 months. In appendix D we show how our results change when we vary ψ.

Parameters	Values	Targeted moment	Values
Preferences, technology			
ν^{-1}	1	inter temporal elasticity	1
γ^{-1}	2	Frisch elasticity	2
β	.98	risk free rate	4%
ϵ	6	average markups	20%
ψ	18.75	estimates from Sbordone (2002)	see footnote 9
Aggregate shocks			
$\bar{\Theta}$	2%	mean labr productivity growth	2%
$\bar{G}$	7%	mean govt. expenditure to GDP ratio	7%
s.d of $\mathcal{E}_\Theta$	3%	s.d of labor productivity growth	3%
s.d of $\mathcal{E}_G$	2%	s.d of govt. expenditure to GDP ratio	2%
Idiosyncratic shocks			
s.d of ς	0.25	s.d. of earnings growth	0.53
ρ_e and s.d. of η	$0.998, 0.11$	log earnings variance - age profile	Figure I
f_2, f_1, f_0	$0.28, -0.52, 0.00$	earnings losses $5^{th}, 50^{th}$ & 95th percentiles	

Table I: Baseline calibration

the empirical procedure in Guvenen et al. (2014), we rank workers by their average log labor earnings 5 years prior to simulating a 3% fall in aggregate output. We then compute the percent income loss for each worker following the recession and calibrate the parameters of a quadratic function for $f(e) = f_0 + f_1 e + f_2 e^2$ to match income losses of the 5th, 50th and 95th percentiles. The right panel of Figure I compares earnings loss patterns simulated from our model with corresponding data summarized by moments in Guvenen et al. (2014).

Parameters of our baseline specification are summarized in Table I.

4.2   Results

In this section we show the optimal response to a one standard deviation negative labor productivity shock. When we consider purely a monetary policy response, we fix the tax rate at $\tau_t = \tau^*$, where τ^* is the optimal tax rate in the non-stochastic environment. Since Ramsey policies at time 0 typically differ from continuation Ramsey policies at $t \geq 1$, we report impulse responses for a shock that occurs at $t = 10$.

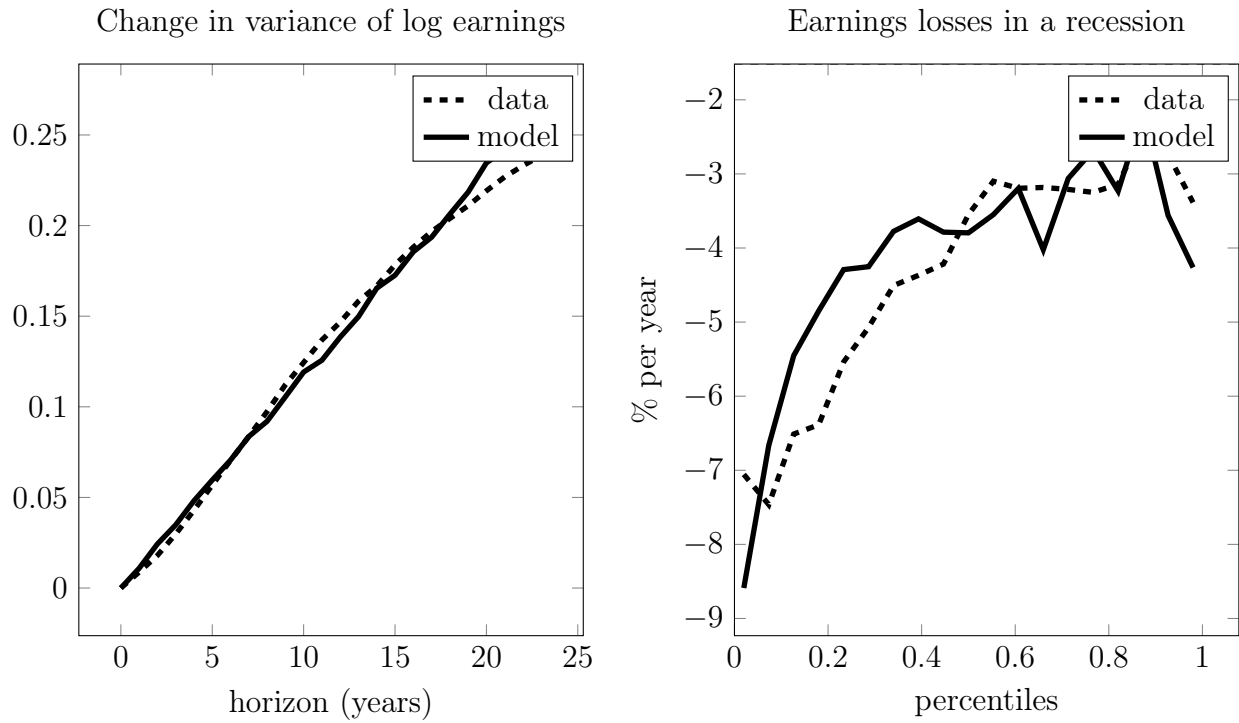Change in variance of log earnings   Earnings losses in a recession

Figure I: The left panel plots change in variance of log earnings for several horizons using simulated earnings from the model and data from Guvenen et al. (2012). The right panel plots annual earnings losses using simulated earnings from the model and data in Guvenen et al. (2014).

4.2.1 Optimal Response to Productivity Shock

We first consider an optimal monetary policy. Figure III depicts responses to a one-time, one standard deviation negative impulse to aggregate productivity $\mathcal{E}_\Theta$ occurring at $t = 10$. On impact, this shock induces a drop in growth rate of output of about 3 percentage points. The solid lines represent responses in our calibrated heterogeneous agent New Keynesian (HANK) economy, the dashed line show responses in its representative agent counterpart (RANK). Here we have set $b_{1,-1} = B_{-1}$, $\theta_{i,t} = \Theta_t$ for all i and $f = 0$.[10]

In the representative agent version, the economy's response to a productivity shock is efficient without policy adjustments. As a result, the Ramsey planner keeps nominal interest rates unchanged to keep inflation stable. Tax rates, which are unchanged by assumption, are shown to ease comparisons with later experiments.

Such a hands-off monetary policy is not optimal when agents are heterogeneous. A productivity shock affects different agents differently and because markets are incomplete, agents cannot insure those risks. A monetary policy response indirectly provides insurance, partly compensating for market incompleteness.

Productivity shocks differentially affect agents for two reasons. One arises from wealth heterogeneity. Because an adverse productivity shock permanently lowers all wages, the consumption of agents having few financial assets falls by more than consumption of agents with more financial assets. To provide insurance against that adverse aggregate shock, the planner desires to lower returns on assets. She can achieve this in two ways. First, the planner can reduce the *ex post* realized real return on debt by engineering a surprise inflation at the time of the shock. Second, the planner can tilt the path of future nominal interest rates to reduce *ex ante* returns on savings in subsequent periods. Both of these effects appear in Figure III. The planner cuts interests rates on impact of the shock, thereby generating a spike in inflation and a drop in *ex post* real asset returns, and then commits higher nominal interest rates that raises real rates.

The second motive for government intervention comes from productivity shocks having different effects on high-wage and low-wage agents. As we saw in panel B of Figure I, low-wage agents are more adversely affected by an adverse productivity shock. To provide insurance indirectly, the government would like to design a policy that effectively redistributes

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[10]To compute the impulse responses we simulate a 20 sequences of $\{\mathcal{E}_{G,t}, \mathcal{E}_{\Theta,t}\}$ of length 25 and simulate the economy twice changing only the shock at period 10. For the first path $\{\mathcal{E}_{G,t}, \mathcal{E}_{\Theta,t}\}_{t=10} = (0, -0.03)$ and for the second $\{\mathcal{E}_{G,t}, \mathcal{E}_{\Theta,t}\}_{t=10} = (0, 0)$. For each sequence we subtract the the path of endogenous variables with $\{\mathcal{E}_{G,t}, \mathcal{E}_{\Theta,t}\}_{t=10} = (0, 0)$ from the path corresponding to the sequence $\{\mathcal{E}_{G,t}, \mathcal{E}_{\Theta,t}\}_{t=10} = (0, -0.03)$ and report the mean across the 20 sequences. The distribution of the IRF is quite tight for our case and so we do not report the standard error bands around the mean path. All variables show deviations in percentage points.
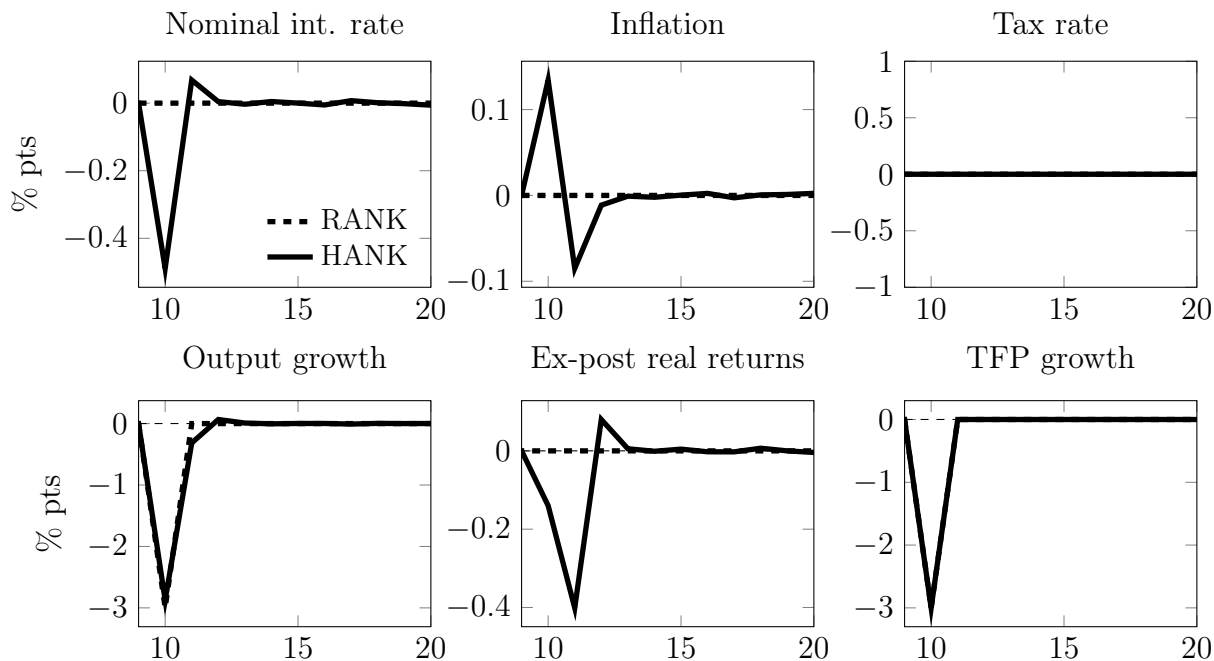
Figure II: Optimal monetary response to a productivity shock

resources from high-wage earners to low-wage earners. Monetary policy can redistribute indirectly by affecting returns on financial assets. Recall from Section 4.1 that wages and asset holdings are positively correlated, both in the data and in our model. Thus, a policy that reduces asset returns effective redistributes resources from high to low wage individuals. A desire to use this channel reinforces the direction of the optimal response that discussed above.

When a government also has access to fiscal policy, increasing progressiveness of the labor taxes also induces desirable redistributions by directly offsetting adverse differential affects of an adverse productivity shock on the distribution of wages. Figure III shows the optimal response of monetary-fiscal policy to one standard deviation negative aggregate labor productivity shock. Because an adverse TFP shock leads to a persistent increase in the dispersion of log wages, a government optimally responds by permanently increasing tax rates one period after the shock. The delayed increased in tax rates is optimal as it also helps to lower real rate at the time of the shock (for the same reason tax rates are temporarily decreased on the impact of the shock). As the result, the path of nominal rates and inflation is smoother when fiscal policy is active, which helps the planner to reduce to costs of price adjustments.
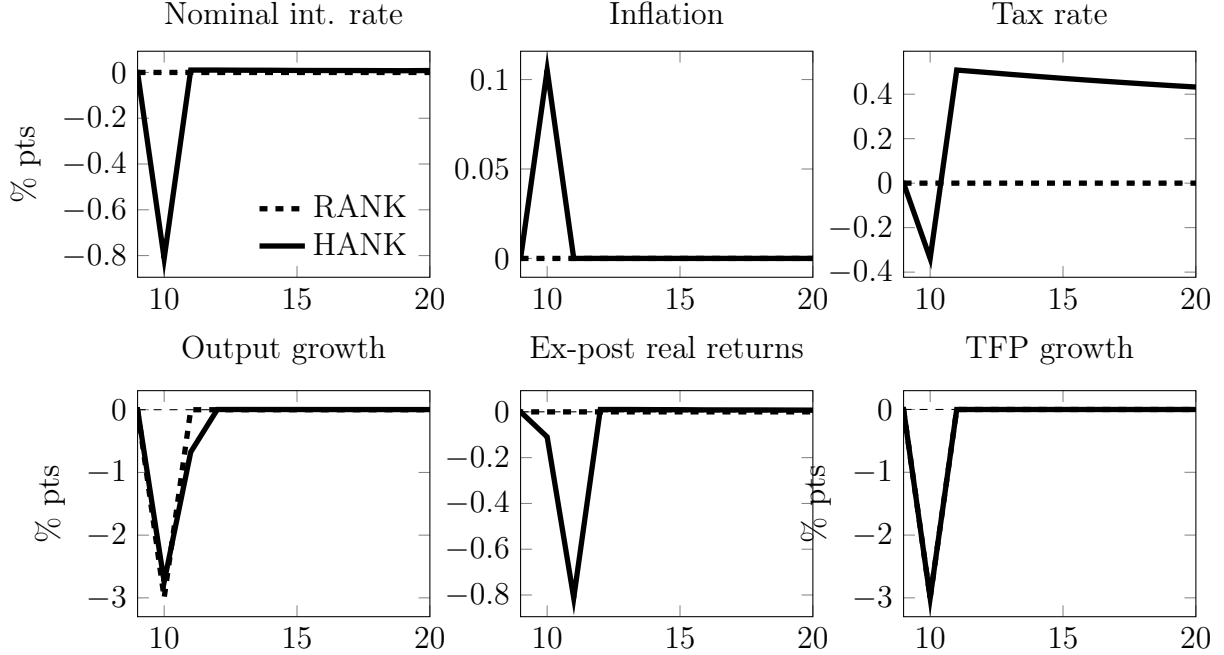
Figure III: Optimal monetary-fiscal response to a productivity shock

4.2.2 Robustness of baseline findings

We now discuss the robustness of our baseline findings and the roles of several assumptions.

We begin with the role of Pareto weights. Since agents are heterogeneous, a Ramsey planner wants to redistribute resources; how and how much depends on the Pareto weights. Therefore, Pareto weights affect the *average* levels of the interest rate, tax rate, and transfers. Policy responses to shocks are mainly driven by the planner's desire to provide insurance. The planner's desire to insure is distinct from its desire to redistribute. As a result, policy responses to aggregate shocks remain quite similar when we assign Pareto weights other than those assigned by the utilitarian in government objective function (10), or if we were to have made alternative assumptions about $\bar{\tau}$ in our analysis of the optimal monetary policy. We illustrate this assertion in figures VIII and IX of Appendix D.

The planner's preference for supplying insurance driven by the assumption that aggregate shocks affect consumption of agents differentially. A planner's preference to supply insurance arises from two features of our baseline calibration: agents differ in their holdings of the nominal bonds as well as in their exposures to the aggregate shocks through the loading function that we have f calibrated to match the evidence in Guvenen et al. (2014). In Figure X of Appendix D, we indicate optimal responses in a economy in which $f = 0$ and find that asset heterogeneity alone contributes to 30% of the policy responses.

In our experiments the planner chooses transfers T_t at each t. Authors of RANK models

including Schmitt-Grohé and Uribe (2004) and Siu (2004) typically restrict $T_t = 0$ for all t. When we re-compute optimal policies in the RANK version or our economy with $T_t = 0$ for all t, we recover outcomes like those discovered by Schmitt-Grohé and Uribe and Siu, namely, that paths of inflation and interest rates in that economy, while not longer constant, are extremely smooth. This reaffirms those authors' insight that the cost of price changes in calibrated RANK economies is sufficiently high that a Ramsey planner chooses to abstain from using inflation fluctuations to smooth distortions coming from aggregate shocks. The peak change in nominal interest rates and inflation is 0.05% and 0.03% in the RANK economy, which is an order of magnitude smaller than our baseline HANK outcomes.

We calibrated menu costs to match the the slope of the Phillips curve. Lower costs of changing prices imply that lowering ex-post returns through inflation is a cheaper tool for the planner to insure agents. We use Figure XI of Appendix D to study the role of price stickiness by computing the optimal monetary-fiscal policy when menu cost parameter $\psi = 0$. We find that an inflation response of about 7 percentage points, which is about 5 times larger than in the benchmark economy.

We evaluate the importance of second-order terms in our approximation. In Figure XII we find that ignoring second-order terms would result in underestimating the optimal responses by 50%. With only first-order terms, the model misses agents' precautionary savings and therefore the evolution of the asset distribution. The fact that impulse responses are computed at different locations in the state-space under first-order and second-order approximations causes the first-order and second-order approximations to differ.

In the baseline we imposed natural borrowing limits. A tractable way to add credit frictions in our framework is to allow for segmented markets and along the lines of Campbell and Mankiw (1989) we introduce "hand-to-mouth" agents. To discipline the mass and characteristics of such hand-to0mouth agents, we use the SCF to compute the fraction of households who report zero bond holdings in each of the 20 wage bins ordered by the wage quantiles and adjust our initial conditions. In the model these agents do not trade nominal bonds and finance their consumption using after tax labor income and profits from the firm.[11] The optimal monetary fiscal responses with hand-to-mouth agents are reported in Figure XIII. In comparison to the baseline, the optimal interest rate response is about 50% larger in magnitude and the response of the tax rate is about the same.

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[11]To maintain the comparability with the baseline and other exercises in this section, we maintain the assumption that the holdings for the firm are uniformly distributed. It is straightforward to relax this assumption as we do in Section 5.

5 Cost-Push Shocks

The New Keynesian literature has emphasized cost-push shocks in accounting for business cycle fluctuations (see, e.g., Smets and Wouters (2007)) and studied their implications for optimal monetary policy in a representative agent framework (see, e.g., Clarida et al. (2001), Galí (2015), Woodford (2003)). Here we study normative implications of such shocks in our heterogeneous agents economy. To allow for cost push shocks we make two departures from the Section 2 environment. First, we follow Galí (2015) by modeling cost-push shocks as shocks to the the elasticity of substitution parameter ϵ. In particular, we assume that the elasticity of substitution follows an AR(1) process

$$\ln(\epsilon_t) = (1 - \rho_\epsilon) \ln(\bar{\epsilon}) + \rho_\epsilon \ln(\epsilon_{t-1}) + \zeta_t.$$

Second, we relax the assumption of the previous section that all agents hold equal number of firm shares. Instead, we assume that agent i holds s_i shares of firms so that agent i's budget constraint (4) becomes

$$c_{i,t} + b_{i,t} = (1 - \tau_t) W_t \exp(\theta_{i,t}) n_{i,t} + T_t + s_i D_t + \left(\frac{1 + \iota_{t-1}}{1 + \pi_t} \right) b_{i,t-1}.$$

5.1 Calibration and Calculations

These assumptions introduce two additional aggregate state variables to a continuation planner's problem: κ_{t-1} the co-state of the Phillips curve and the elasticity of substitution among goods, $\ln(\epsilon_{t-1})$. Both state variables have deterministic dynamics in the non-stochastic limit. The section 3 computational algorithm must be adjusted to handle such state variables.

We define a new set of aggregate state variables $\mathcal{Z}_t$ that have deterministic dynamics in the non stochastic zero noise limit. We adjust the optimal individual and aggregate policy functions explicitly to depend on these additional states: $\tilde{x}(z, Z, \mathcal{Z}, \varepsilon, \mathcal{E})$ and $\tilde{X}(Z, \mathcal{Z}, \mathcal{E})$, but continue to approximate around a sequence of current distributions of idiosyncratic states date by date. To achieve this, let $\bar{\mathcal{Z}}(Z)$ be the non-stochastic steady state values of $\mathcal{Z}$ associated with the current distribution of idiosyncratic states Z and scale both the size of the shocks $\varepsilon, \mathcal{E}$ and deviations of $\mathcal{Z}$ from their steady state levels with the same scaling parameter σ. We can write policy rules explicitly as functions of σ: $\tilde{x}(z, Z, \bar{\mathcal{Z}} + \sigma \cdot (\mathcal{Z} - \bar{\mathcal{Z}}), \sigma \cdot \varepsilon, \sigma \cdot \mathcal{E}; \sigma)$ and $\tilde{X}(Z, \bar{\mathcal{Z}} + \sigma \cdot (\mathcal{Z} - \bar{\mathcal{Z}}), \sigma \cdot \mathcal{E}; \sigma)$ and take a truncated Taylor expansion with respect to σ around $\sigma = 0$ to approximate policies.[12]

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[12]An alternative approach would be to scale only the shocks: $\tilde{x}(z, Z, \mathcal{Z}, \sigma \cdot \varepsilon, \sigma \cdot \mathcal{E}; \sigma)$ and $\tilde{X}(Z, \mathcal{Z}, \sigma \cdot \mathcal{E}; \sigma)$. Such an approach would require solving and approximating around a deterministic path of $\mathcal{Z}$. We leave

In Appendix C.4, we show that numerical methods from Section 3 extend to this setting: all required derivatives can be computed by inverting at most $\max\{N_X, N_x\} \times \max\{N_X, N_x\}$ matrices and computing terms that grow linearly in K, where K is the number of elements in discrete approximation of Z.

Our calibration remains identical to the one presented in Section 4 except for the stochastic process for ϵ_t and an initial distribution of shares. To calibrate the distribution of shares, s_i, we average stock holdings of the 2013 Survey of Consumer Finances by wage quantile.[13] The correlation of wages and stock holdings is 0.52. Initial shares are defined to be average stock holding scaled to make average shares held by all individuals be 1. We normalize ζ_t so that a one standard deviation shock to ζ_t changes markups by 1% and following Smets and Wouters (2007) set the persistence of $\ln(\epsilon_t)$ to be 0.65.

5.2 Optimal Responses to Cost-Push Shocks

We report responses to a one standard deviation positive shock ζ_t. Experiments were constructed in similar ways to those conducted in Section 4.2. A key finding is that the trade-offs faced by the policy maker in a heterogeneous agents setting differ substantially from those in a representative agent economy, leading to policy prescriptions that an order of magnitude larger in the HANK economy and can also have opposite signs from those in the RANK economy. We start with differences in monetary policy when labor taxes are fixed at τ^* and then study the case with optimal monetary and fiscal policy.

Optimal monetary responses to a cost-push shock are shown in Figure IV. Although the representative agent model calls for a moderate tightening of monetary policy following a cost-push shock, the heterogeneous agents economy requires a substantial decrease in nominal interest rates and a positive spike in inflation.

To understand this result, it is useful first to analyze the optimal response in the RANK model. A positive cost-push shock increases firms' desired mark ups over marginal costs. Since price changes are costly in New Keynesian models, the planner offsets this effect by lowering marginal costs and thereby pushes output below its natural level.[14] Galí (2015) dubs this policy "leaning against the wind". The reduction in output is achieved by committing to a tight monetary policy in the future that lowers aggregate demand.

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studies of that approach to future work and scale $\mathcal{Z} - \bar{\mathcal{Z}}$ with σ partly because $\mathcal{Z} - \bar{\mathcal{Z}}$ remains small throughout our simulations.

[13]We interpret stock holdings to be the sum of direct stock and mutual funds and indirect holdings in the through retirement accounts.

[14]It is feasible to implement an allocation that sets $\pi_t = 0$ but that requires that wages adjust to offset markups and associated deviations of $(1 - \tau)W$ from one, which is costly. We return to this consideration when we discuss an optimal monetary-fiscal response to cost-push shocks.
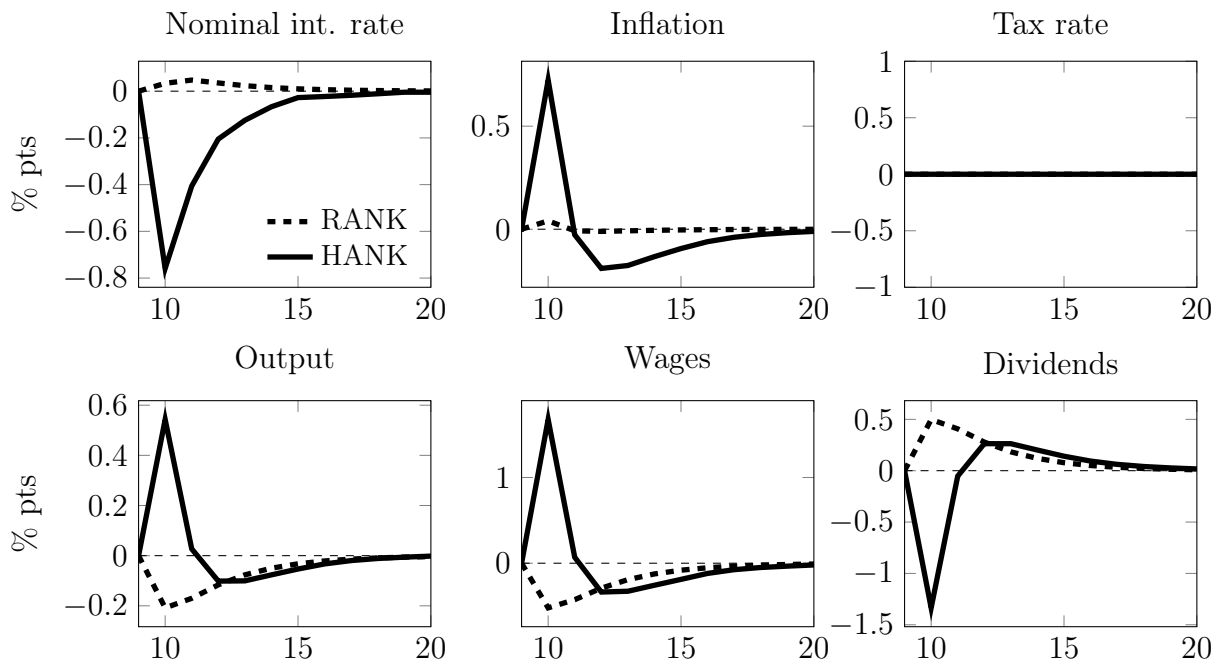
Figure IV: Optimal monetary response to a markup shock

Not only positive mark-up shocks induce inflationary pressure, they also decrease labor's share and increase profit's share (see the last row in Figure IV). In the representative agent economy, these effects on factor shares are of second-order since workers and firm owners are the same person. In the HANK economy stock ownership is heterogeneous and, thus, a mark up shock naturally redistributes resources from agents with low stock ownership to agents with high stock ownership. Since stock ownership and labor earnings are correlated, this effectively redistributes resources from low wage workers to high wage workers. Leaning against the wind policies exacerbate this effect.

When markets are incomplete, agents cannot insure against the cost push shock and the Ramsey planner sets policies indirectly to provide insurance by offsetting the distributional effects of a cost-push shock. Quantitatively, this consideration dominates the planner's desire to reduce costs of price changes. The planner induces a desired redistribution by significantly lowering interest rates immediately and committing to low interest rates in the future. That boosts aggregate demand and thereby raises wages and lowers dividends. A notable feature of the optimal policy is the increases in wages that occur in the period that the shock hits. Postponing wage increases would be detrimental because firms would respond to anticipated wage increases by raising current prices thereby generating extra inflation, which is costly, while not getting the benefits from lowering dividends.

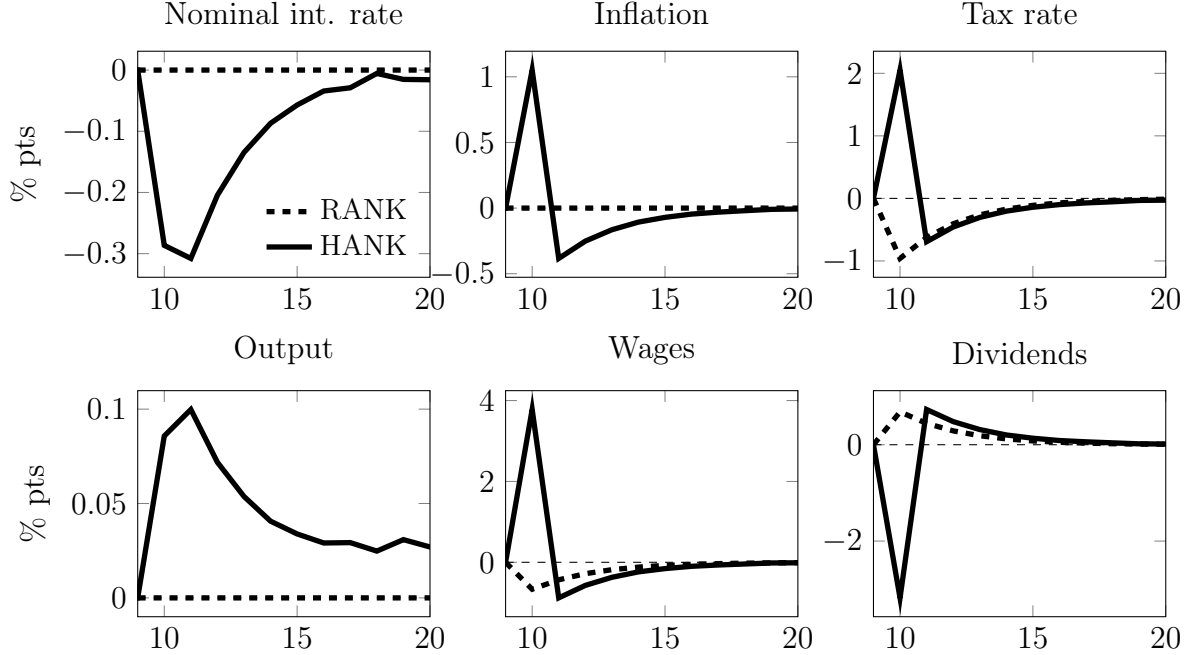With fixed tax rates, a cost-push shock sets up a tension between movements in the labor

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Figure V: Optimal fiscal monetary response to a markup shock

wedge, i.e., deviations in $(1-\tau)W$ from one, and costs of inflation. In a representative agent economy, when the planner has access to fiscal policy, a first-best allocation characterized by $\pi_t = 0$ and $(1-\tau)W = 1$ is feasible and can be implemented using a labor tax subsidy that offsets the time varying markup and nominal rates that do not respond to to the cost-push shock. This summarizes the dashed lines in Figure V.

Optimal fiscal policy in a HANK economy also stands starkly in contrast to what it is in a RANK economy. In the HANK economy, the planner *raises* taxes in response to an adverse cost-push shock. The aim of planner is still to transfer resources from high-wage owners of the firms to lower-wage agents but, with labor taxes available, the planner has a more direct way of influencing wages and firm profits. Higher tax rates contracts labor supply, raise wages, and lowers dividends. That arrests some of the adverse distributional effects of markup shocks. As before, tax changes are concentrated on impact of the shock in order to make the wage increase and the resulting inflation both be unanticipated. Following the shock, the planner implements a tax subsidy like the one used in the representative agent economy. The nominal rate closely tracks the real rate, which is also low on impact. These responses are summarized in the solid lines of Figure V.
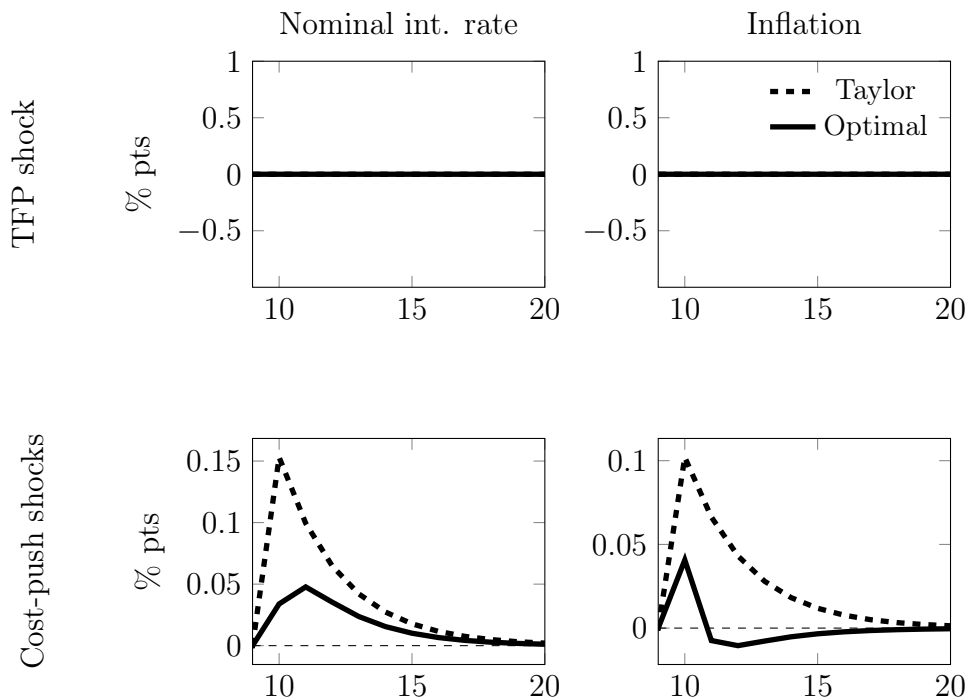
Figure VI: Comparing optimal monetary responses to Taylor rule in RANK model. The solid line is the optimal response and the dashed line is the response in a competitive equilibrium with $i_t = \bar{i} + 1.5\pi_t$.

6   Taylor Rules

In this section we study how well standard Taylor rules approximate the optimal policy in heterogeneous agents settings. We impose a Taylor rule of the form (26) and compare responses to TFP shocks and cost-push shocks with responses under an optimal policy.[15]

We begin with the RANK economy. In Figure VI, we see that the Taylor rule economy implements an optimal allocation in an economy with only productivity shocks. It also leads to outcomes similar to those for an optimal policy in response to a cost-push shock. A key feature of the optimal response is low and stable prices that can be implemented by a Taylor rule that features a sufficiently large response of nominal rates to inflation. These findings confirm conclusions of Woodford (2003) and Galí (2015) who also find that Taylor rules are to being optimal rules.

Things differ in the HANK model. In response to both types of adverse shocks – a negative aggregate productivity shock or a positive cost-push shock that raises markups – the optimal plan seeks to transfer resources from high wage earners who receive asset income to low wage earners who rely primarily on wage income. Transfers are implemented by

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[15]As in sections 4.2.1 and 5.2, we we keep tax rates constant at optimal τ^* value and focus on monetary responses for all the experiments.
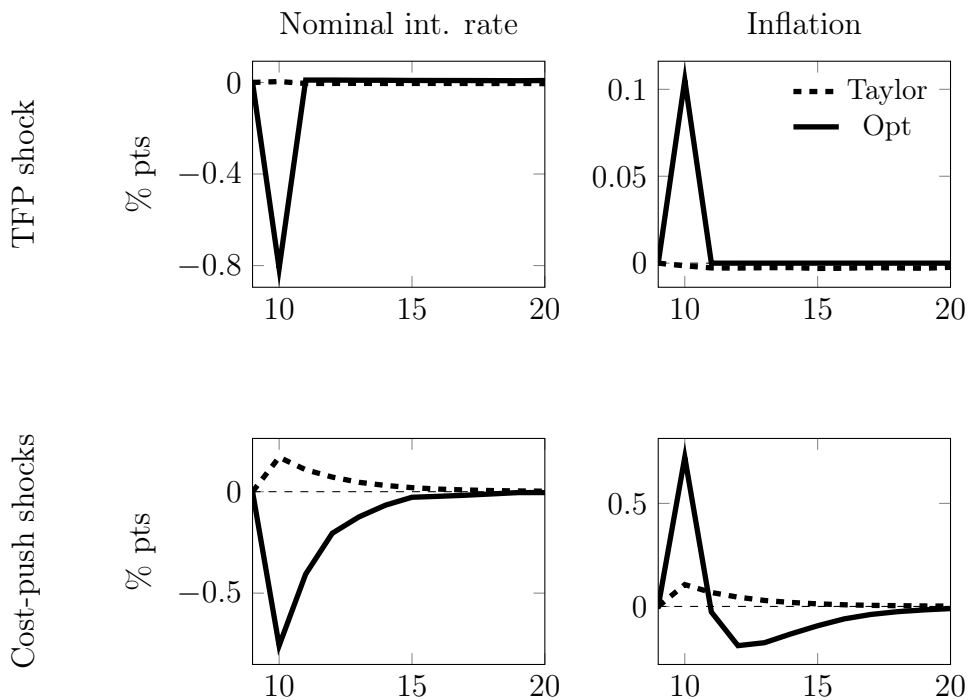
Figure VII: Comparing optimal monetary responses to Taylor rule in HANK model. The solid line is the optimal response and the dashed line is the response in a competitive equilibrium with $i_t = \bar{i} + 1.5\pi_t$.

lowering nominal rates, thereby raising aggregate demand, wages, and the price level. Under the Taylor rule, the responses of aggregates in the HANK economy are very close to those of RANK economy. The Taylor rule continues to recommend no response to productivity shocks, and contrary to an optimal response, an increase in interest rates after a cost push shock. Thus, Taylor rules do a poor job of approximating an optimal policy. Figure VII summarizes these findings.

7 Concluding Remarks

James Tobin described macroeconomics as a field that explains aggregate quantities and prices while ignoring distribution effects. Tobin's characterization also describes much work subsequent to his in the real business cycle, asset pricing, Ramsey tax and debt, and New Keynesian research traditions. In each of these lines of research, an assumptions of complete markets and/or of a representative consumer allows the analyst to compute aggregate quantities and prices without also determining distributions across agents.

This paper departs from Tobin's "aggregative economics" in two ways. First, we assume incomplete markets – which means that aggregate quantities, prices, and allocations

across agents must be determined jointly, not recursively as in complete markets models. And second, we specify technology and relative skills shocks in a way that makes contact with findings of Guvenen et al. (2014) that US cross section distributions of labor earnings have moved systematically over business cycles. A common shock affects *both* an aggregate technology shock and the cross-section distribution of skills. Cross-section dispersions in labor earnings and asset holdings shape both aggregate outcomes and choices confronting a Ramsey planner.

Finally, an incomplete markets model goes a long way toward framing an optimal policy problem when it sets the menu of assets. By specifying that the only asset traded in our model is a risk-free *nominal* bond, we activate a beneficial role for fiscal and monetary policy to make nominal interest rates fluctuate in ways that hedge inequality-increasing shocks to distributions of labor earnings.[16]

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[16]It is fruitful to compare our assumptions with those of Musto and Yilmaz (2003), who focus on how markets that allow citizens to insure outcomes of voting affect the efficacy of redistribution.

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A Scaled Bellman Equation

After history $\{\mathcal{E}_{\Theta,s}, \mathcal{E}_{G,s}, \varsigma_{i,s}, \eta_{i,s}\}_{s=0}^{t-1}$, the continuation value for agent i for an allocation $\{c_{i,t}, n_{it}\}_t$ is

$$W_{t-1}^{HH}\left(\{c_{i,t}, n_{it}\}_t\right) \equiv \mathbb{E}_{t-1}\sum_{j=0}^{\infty}\beta^{t+j}U\left(c_{i,t+j}, n_{i,t+j}, \Theta_{t+j}\right).$$

Scaling $\hat{W}_{t-1}^{HH} = e^{-(1-\nu)\Theta_{t-1}}W_{t-1}^{HH}$, we then have

$$\hat{W}_{t-1}^{HH}\left(\{c_{i,t}, n_{it}\}_t\right) = \mathbb{E}_{t-1}e^{(1-\nu)\left[\bar{\Theta}+\mathcal{E}_{\Theta,t}\right]}\left[U\left(\hat{c}_t, n_t, 1\right) + \beta\hat{W}_t^{HH}\left(\{c_{i,t}, n_{it}\}_t\right)\right]. \qquad (27)$$

Household budget constraint at date t with scaled variables is

$$\hat{c}_{i,t} + \hat{b}_{i,t} = (1-\tau_t)n_{i,t}W_t\exp(\theta_{i,t}-\Theta_t) + \hat{T}_t + \hat{D}_t + \left(\frac{1+\iota_{t-1}}{1+\pi_t}\right)\exp\left\{-\left[\bar{\Theta}+\mathcal{E}_{\Theta,t}\right]\right\}\hat{b}_{i,t-1}. \quad (28)$$

For a given monetary-fiscal policy, the household maximizes $\hat{W}_0^{HH}\left(\{c_{i,t}, n_{it}\}_t\right)$ subject to (28) for all $t \geq 0$ and natural debt limits. The optimality conditions are (??) and

$$\mathbb{E}_{t-1}\left\{\beta\left(\frac{1+\iota_{t-1}}{1+\pi_t}\right)\exp\left\{-\nu\left[\bar{\Theta}+\mathcal{E}_{\Theta,t}\right]\right\}\frac{U_{\hat{c}_{it}}}{U_{\hat{c}_{it-1}}}\right\} = 1. \qquad (29)$$

The firms' optimal pricing and symmetry after dividing by $\exp(\Theta_t)$ gives us

$$\left(\frac{\hat{Y}_t\left[1-\epsilon\left(1-W_t\right)\right]}{\psi}\right) - \pi_t(1+\pi_t) + \beta\mathbb{E}_t\exp\left((1-\nu)\left[\bar{\Theta}+\mathcal{E}_{\Theta,t+1}\right]\right)\left(\frac{\hat{C}_{t+1}}{\hat{C}_t}\right)^{-\nu}\pi_{t+1}(1+\pi_{t+1}) = 0. \qquad (30)$$

The market clearing conditions using the scaled variables are

$$\hat{G}_t + \hat{T}_t + \left(\frac{1+\iota_{t-1}}{1+\pi_t}\right)\exp\left\{-\left[\bar{\Theta}+\mathcal{E}_{\Theta,t}\right]\right\}\hat{B}_{t-1} = \tau_t\int_i n_{i,t}\exp(\theta_{i,t}-\Theta_t)W_t di + \hat{B}_t. \qquad (31)$$

$$\hat{Y}_t = \int n_{i,t}\exp(\theta_{i,t}-\Theta_t)di, \qquad (32)$$

$$\hat{C}_t + \hat{G}_t = \hat{Y}_t - \frac{\psi}{2}\pi_t^2, \qquad (33)$$

$$\int \hat{b}_{i,t}di = \hat{B}_t, \qquad (34)$$

$$\hat{D}_t = (1-W_t)\hat{Y}_t - \frac{\psi}{2}\pi_t^2. \qquad (35)$$

The planner maximizes (10) subject to (28) - (35). The next lemma shows that (30) and (35) are slack at the optimal allocation.

Lemma 2. *Constraint (30) and (35) do not bind at the optimal allocation.*

Proof. Consider an allocation, bond profile, price system, and monetary-fiscal policy

$$\{\hat{c}_{i,t}, n_{i,t}\}_{i,t}, \left\{\left\{\hat{b}_{i,t}\right\}_i, \hat{B}_t\right\}_t, \{W_t, R_t, P_t\}_t, \{\tau_t, \hat{T}_t, i_t\}$$

that satisfies constraints (28) - (35) except (30) and (35). We can construct an alternative price system and monetary-fiscal policy that does satisfy all equations (28) - (35) and attains the same value to the Planner.

First, choose a sequence $\{\check{W}_t\}_t$ that makes constraint (30) satisfied. Then choose $\left\{\check{\tau}_t, \check{T}_t, \check{D}_t\right\}$ so that

$$(1 - \check{\tau}_t)\,\check{W}_t = (1 - \tau_t)\,W_t$$

$$\check{T}_t + \check{W}\hat{Y}_t = T_t + W_t\hat{Y}_t.$$

$$\check{D}_t = (1 - \check{W}_t)\hat{Y}_t - \frac{\psi}{2}\pi_t{}^2$$

Evidently $\{\hat{c}_{i,t}, n_{i,t}\}_{i,t}, \left\{\left\{\hat{b}_{i,t}\right\}_i, \hat{B}_t\right\}_t, \{\check{W}_t, R_t, P_t\}_t, \{\check{\tau}_t, \check{T}_t, i_t\}$ satisfies (28) - (35) and is thus implementable. Furthermore, since the allocation, $\{\hat{c}_{i,t}, n_{i,t}\}_{i,t}$ is unchanged, the value that the planner assigns to the equilibrium allocation is also unchanged. $\qquad\square$

B First-Order Conditions for $t \geq 1$ Continuation Plan

For brevity we will use s to denote the joint of states and shocks $(z, \varepsilon, \mathcal{E})$. Let $\mu(s), \rho(s), \phi(s)$ be the multipliers on the individual constraints (12) - (14) and $\chi(\mathcal{E}), \xi(\mathcal{E}), \lambda(\mathcal{E})$ be the multipliers on the aggregate constraints (15) - (17).

Lemma 2 shows that (18) does not bind and hence $W(\mathcal{E})$ only appears in the form of term $(1 - \tau(\mathcal{E}))W(\mathcal{E})$ and $\hat{D}$ in the form of term $\hat{D} + \hat{T}$. Similarly the state variable $\hat{m}$ only enters in equation (13). This means that in order to solve for the optimal allocation we need to find the product $(1 - \tau(\mathcal{E}))W(\mathcal{E})$ which we denote by $\mathrm{W}(\mathcal{E})$, the sum $\hat{D} + \hat{T}$ denoted by $\hat{\mathrm{T}}$ and scale $\hat{m}'$ by a constant of proportionality such that $\hat{m}'(s)U_{\hat{c}}(s) = \aleph(\mathcal{E})$ and

$$\int \hat{m}'(s)d\phi dZ = 1.$$

Following Marcet and Marimon (2011) we also replace $\hat{a}$ in z with a transformation of its associated co-state variable $\hat{\mu} \equiv \frac{\mathbb{E}_z\left[\exp\left\{-[\bar{\Theta}+\mathcal{E}_\Theta]\right\}\hat{U}_c(1+\pi)^{-1}\mu\right]}{\beta\mathbb{E}_z\left[\exp\left\{-\nu[\bar{\Theta}+\mathcal{E}_\Theta]\right\}\hat{U}_c(1+\pi)^{-1}\right]}$.

The list of equations that comprise functions F are

$$0 = \frac{\exp\left\{-\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}\hat{a}(s)\,U_{\hat{c}}(s)(1 + \pi(\mathcal{E}))^{-1}}{\beta\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}]}$$
$$- U_{\hat{c}}(s)\left[\hat{c}(s) - \hat{\mathrm{T}}(\mathcal{E})\right] - U_n(s)n(s) + \hat{a}(s'), \tag{36a}$$

$$0 = -\alpha + \hat{m}(z)\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}], \tag{36b}$$

$$0 = -U_n(s) - \mathrm{W}(\mathcal{E})U_{\hat{c}}(s)g(s), \tag{36c}$$

$$0 = -\aleph(\mathcal{E}) + \hat{m}'(s)U_{\hat{c}}(s), \tag{36d}$$

$$0 = \frac{\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}\hat{a}(s)U_{\hat{c}\hat{c}}(s)(1 + \pi(\mathcal{E}))^{-1}}{\beta\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}]}\left(\hat{\mu}'(s) - \hat{\mu}(z)\right)$$
$$- \exp\left\{(1 - \nu)\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}\hat{\mu}'(s)\left(U_{\hat{c}\hat{c}}(s)\left[\hat{c}(s) - \hat{\mathrm{T}}(\mathcal{E})\right] + U_{\hat{c}}(s)\right)$$
$$- \rho(s)\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(s)(1 + \pi(\mathcal{E}))^{-1}$$
$$+ \mathrm{W}(\mathcal{E})U_{\hat{c}\hat{c}}(s)g(s)\phi(s) - \chi(\mathcal{E}) + e^{(1-\nu)[\bar{\Theta} + \mathcal{E}_\Theta]}\left(w(z)U_{\hat{c}} - \beta\varrho(s')\aleph(\mathcal{E})\frac{U_{\hat{c}\hat{c}}(s)}{(U_{\hat{c}}(s))^2}\right), \tag{36e}$$

$$0 = w(z)e^{(1-\nu)[\bar{\Theta} + \mathcal{E}_\Theta]}\hat{U}_n(s) - \exp\left\{(1 - \nu)\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}\hat{\mu}'(s)\left(U_{nn}(s)n(s) + U_n(s)\right)$$
$$+ \phi(s)U_{nn}(s) - g(s)\xi(\mathcal{E}), \tag{36f}$$

$$0 = -\hat{\mu}(z) + \frac{\mathbb{E}_z\left[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}\hat{\mu}'\right]}{\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}]}, \tag{36g}$$

$$0 = -\varrho(s) - \rho(s)\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_\Theta\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}], \tag{36h}$$

$$0 = -e'(s) + \rho_e e(z) + f\left(z, Z, \mathcal{E}\right)\mathcal{E}_\Theta + \eta(\varepsilon), \tag{36i}$$

$$0 = -\omega'(s) + \omega(z). \tag{36j}$$

To impose a measurability restriction that $\hat{a}$ and ρ are choice variables that do not depend on shocks $\varepsilon, \mathcal{E}$ we require

$$\hat{a}(s) = \mathbb{E}_z\hat{a}(s), \quad \rho(s) = \mathbb{E}_z\rho(s). \tag{36k}$$

and include them in F. Next, the list of equations in function R are

$$0 = -\hat{C}(\mathcal{E}) + \int \hat{c}(s)d\phi dZ, \tag{37a}$$

$$0 = -\hat{Y}(\mathcal{E}) + \int n(s)g(s)d\phi dZ, \tag{37b}$$

$$0 = -\hat{Y}(\mathcal{E}) + \frac{\psi}{2}\pi(\mathcal{E})^2 + \hat{C}(\mathcal{E}) + \hat{G}(\mathcal{E}), \tag{37c}$$

$$0 = \chi(\mathcal{E}) - \lambda(\mathcal{E}), \tag{37d}$$

$$0 = \xi(\mathcal{E}) + \lambda(\mathcal{E}), \tag{37e}$$

$$0 = \int \phi(s)U_{\hat{c}}(s)g(s)d\phi dZ, \tag{37f}$$

$$0 = \int \hat{\mu}'(s)U_{\hat{c}}(s)d\phi dZ, \tag{37g}$$

$$0 = \int \rho(s)d\phi dZ \tag{37h}$$

$$0 = -\int \frac{\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}\hat{a}(s)U_{\hat{c}}(s)(1 + \pi(\mathcal{E}))^{-2}}{\beta\mathbb{E}_z[\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(1 + \pi)^{-1}]}\left(\hat{\mu}'(s) - \hat{\mu}(z)\right)d\phi dZ,$$
$$+ \int \rho(z)m(z)\exp\left\{-\nu\left[\bar{\Theta} + \mathcal{E}_{\Theta,t}\right]\right\}U_{\hat{c}}(s)(1 + \pi(\mathcal{E}))^{-2}d\phi dZ - \psi\pi(\mathcal{E})\lambda(\mathcal{E}). \tag{37i}$$

C   Taylor Expansion

C.1   Proof of Proposition 1

Let $\tilde{\mu}'(z, Z, \sigma\varepsilon, \sigma\mathcal{E}; \sigma)$ and $\tilde{m}'(z, Z, \sigma\varepsilon, \sigma\mathcal{E}; \sigma)$ be optimal policies for the state variable $\hat{\mu}'$ and $\hat{m}'$. From equation (36g) when when $\sigma = 0$

$$\hat{\mu}(z) = \frac{\tilde{\mu}'(z, Z, 0, 0; 0)\tilde{U}_c(z, Z, 0, 0; 0)(1 + \tilde{\pi}(Z, 0; 0))^{-1}}{\tilde{U}_c(z, Z, 0, 0; 0)(1 + \tilde{\pi}(Z, 0; 0))^{-1}} = \tilde{\mu}'(z, Z, 0, 0; 0).$$

From (36b) we have, when $\sigma = 0$,

$$\tilde{\alpha}(Z, 0; 0) = \hat{m}(z)\tilde{U}_{\hat{c}}(z, Z, 0, 0; 0)(1 + \tilde{\pi}(Z, 0; 0))^{-1}.$$

and (36d) implies

$$\tilde{m}'(z, Z, 0, 0; 0)\tilde{U}_{\hat{c}}(z, Z, 0, 0; 0) = \tilde{\aleph}(Z, 0; 0), \quad \int \tilde{m}'(z, Z, 0, 0; 0)dZ = 1.$$

Thus

$$\tilde{m}'(z, Z, 0, 0; 0) = \frac{\tilde{\aleph}(Z, 0; 0)(1 + \tilde{\pi}(Z, 0; 0))^{-1}}{\tilde{\alpha}(Z, 0; 0)}m(z)$$

and integrating both sides with respect to dZ we find $\frac{\tilde{\aleph}(Z,0;0)(1+\tilde{\pi}(Z,0;0))^{-1}}{\tilde{\alpha}(Z,0;0)} = 1$ and hence

$$\tilde{m}'(z, Z, 0, 0; 0) = m(z).$$

Finally, the stochastic process (36i) for shocks implies

$$e'(z, Z, 0, 0; 0) = e(z).$$

We conclude that

$$\tilde{z}(z, Z, 0, 0; 0) = z$$

and therefore the $\sigma = 0$ allocation is stationary.

C.2   First-Order Terms

We first prove Lemma 1:

Proof. Proposition 1 implies that $\frac{\partial}{\partial z}\tilde{z}(z, Z, 0, 0; 0) = 1$ for all (z, Z) and therefore $z_0^l = 1$ for all l. Using this fact, differentiate (19) with respect to z, evaluated at z^l, and re-arrange to get

$$x_0^l = -\left[F_{x-}^l + F_x^l + F_{x+}^l\right]^{-1}F_0^l \text{ for all } l.$$

Differentiation of (19) and (20) with respect to the kth argument of Z gives

$$F_{x-}^l x_k^l + F_x^l x_k^l + F_{x+}^l x_k^l + F_X X_k = 0 \text{ for all } l, k, \tag{38}$$

$$\left(R_0^k + R_x^k x_0^k\right) + \sum_{l=1}^K \left(R_x^l x_k^l + R_X^l X_k\right) = 0 \text{ for all } k. \tag{39}$$

Then solving (38) for x_k^l we get

$$x_k^l = -\left[F_{x-}^l + F_x^l + F_{x+}^l\right]^{-1}F_X^l X_k \equiv x_X^l X_k. \tag{40}$$

Substituting x_k^l using (40) allows us to solve for Y_k as

$$X_k = \left(\sum_l \left(R_x^l x_X^l + R_X^l\right)\right)^{-1}\left[R_0^k + R_x^k x_0^k\right]. \tag{41}$$

After we have found X_k, we compute x_k^l from (40). □

We now are ready to prove Proposition 2

Proof. The total derivative of (19) and (20) with respect to σ along with $\mathbb{E}_z[\varepsilon] = \mathbb{E}_z[\mathcal{E}] = 0$ yields[17]

$$0 = F_x^l(x_\varepsilon^l \varepsilon + x_\mathcal{E}^l \mathcal{E} + x_\sigma^l) + F_X^l(X_\mathcal{E}\mathcal{E} + X_\sigma) + F_\mathcal{E}^l \mathcal{E} + F_\varepsilon^l \varepsilon \tag{42}$$

$$+ F_{x+}^l \left[x_0^l Q(x_\varepsilon^l \varepsilon + x_\mathcal{E}^l \mathcal{E} + x_\sigma^l) + \sum_k x_k^l Q(x_\mathcal{E}^k + x_\sigma^k) + x_\sigma^l \right]$$

and

$$\sum_k \left(R_x^k(x_\mathcal{E}^k \mathcal{E} + x_\sigma^k) + R_\mathcal{E}^k \mathcal{E} + R_X^k (X_\mathcal{E}\mathcal{E} + X_\sigma) \right) = 0. \tag{43}$$

As equations (42) and (43) must hold for all $\mathcal{E}$ and ε, combining the terms loading on ε yields the first result of Proposition 2

$$x_\varepsilon^l = \left[F_x^l + F_{x+}^l x_0^l Q \right]^{-1} F_\varepsilon^l.$$

The terms multiplying $\mathcal{E}$ produce the equations

$$F_x^l x_\mathcal{E}^l + F_X^l X_\mathcal{E} + F_\mathcal{E}^l + F_{x+}^l \left[x_0^l Q x_\mathcal{E}^l + \sum_k x_k^l Q x_\mathcal{E}^k \right] = 0 \tag{44}$$

and

$$\sum_k \left(R_x^k x_\mathcal{E}^k + R_\mathcal{E}^k + R_X^k X_\mathcal{E} \right) = 0. \tag{45}$$

After substituting $x_k^l = x_X^l X_k$ and solving equation (44) for $x_\mathcal{E}^l$

$$x_\mathcal{E}^l = - \left[F_x^l + F_{x+}^l x_0^l Q \right]^{-1} \left(F_\mathcal{E}^l + F_X^l X_\mathcal{E} + F_{x+}^l x_X^l \sum_k X_k Q x_\mathcal{E}^k \right)$$

$$\equiv x_{\mathcal{E},1}^l + x_{\mathcal{E},2}^l X_\mathcal{E} + x_{\mathcal{E},3}^l \sum_k X_k Q x_\mathcal{E}^k.$$

Applying $\sum_l X_l Q$ to both sides of this equation gives

$$\sum_k X_k Q x_\mathcal{E}^k = \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l \right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},1}^l + \sum_l X_l Q x_{\mathcal{E},2}^l X_\mathcal{E} \right),$$

---

[17] As ε is mean 0, to first-order it will not affect the distribution Z, thus there is no x_ε^k in the laster term of the derivative of F or R.

which can be substituted back into $x_{\mathcal{E}}^l$ to yield the next result of the proposition

$$
x_{\mathcal{E}}^l = \left(x_{\mathcal{E},1}^l + x_{\mathcal{E},3}^l \left(I - \sum_k X_k Q x_{\mathcal{E},3}^k \right)^{-1} \left(\sum_k X_k Q x_{\mathcal{E},1}^k \right) \right)
$$

$$
+ \left(x_{\mathcal{E},2}^l + x_{\mathcal{E},3}^l \left(I - \sum_k X_k Q x_{\mathcal{E},3}^k \right)^{-1} \left(\sum_k X_k Q x_{\mathcal{E},2}^k \right) \right) X_{\mathcal{E}}.
$$

The response of aggregate variables can be found by substituting for $x_{\mathcal{E}}^l$ in (45) and then solving for $X_{\mathcal{E}}$

$$
X_{\mathcal{E}} = - \left(\sum_k \left[R_X^k + R_x^k x_{\mathcal{E},2}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l \right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},2}^l \right) \right] \right)^{-1}
$$

$$
\times \left(\sum_k \left[R_{\mathcal{E}}^k + R_x^k x_{\mathcal{E},1}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l \right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},1}^l \right) \right] \right).
$$

The remaining terms of (42) and (43) give

$$
F_\sigma + F_{x-}^l x_\sigma^l + F_x^l x_\sigma^l + F_X^l X_\sigma + F_{x+}^l \left[x_0^l Q x_\sigma^l + \sum_k x_k^l Q x_\sigma^k + x_\sigma^l \right] = 0 \tag{46}
$$

and

$$
\sum_k \left(R_x^k x_\sigma^k + R_X^k X_\sigma \right) = 0. \tag{47}
$$

Solving for (46) for x_σ^l yields

$$
x_\sigma^l = - \left(F_{x-}^l + F_x^l + F_{x+}^l + F_{x+}^l x_0^l Q \right)^{-1} \left(F_\sigma + F_X^l X_\sigma + F_{x+}^l x_X^l \sum_k X_k Q x_\sigma^k \right)
$$

$$
\equiv x_{\sigma,1}^l + x_{\sigma,2}^l X_\sigma + x_{\sigma,3}^l \left(\sum_k X_k Q x_\sigma^k \right).
$$

Applying $\sum_l X_l Q$ to both sides of this equation gives

$$
\sum_k X_k Q x_\sigma^k = \left(I - \sum_l X_l Q x_{\sigma,3}^l \right)^{-1} \left(\sum_l X_l Q x_{\sigma,1}^l + \sum_l X_l Q x_{\sigma,2}^l X_\sigma \right),
$$

which can be substituted back into $x^l_{\mathcal{E}}$ to yield the next result of the proposition

$$
x^l_\sigma = \left(x^l_{\sigma,1} + x^l_{\sigma,3} \left(I - \sum_k X_k Q x^k_{\sigma,3} \right)^{-1} \left(\sum_k X_k Q x^k_{\sigma,1} \right) \right)
$$
$$
+ \left(x^l_{\sigma,2} + x^l_{\sigma,3} \left(I - \sum_k X_k Q x^k_{\sigma,3} \right)^{-1} \left(\sum_k X_k Q x^k_{\sigma,2} \right) \right) X_\sigma.
$$

The response of aggregate variables can be found by substituting for x^l_σ in (45) and then solving for X_σ

$$
X_\sigma = - \left(\sum_k \left[R^k_X + R^k_x x^k_{\sigma,2} + R^k_x x^k_{\sigma,3} \left(I - \sum_l X_l Q x^l_{\sigma,3} \right)^{-1} \left(\sum_l X_l Q x^l_{\sigma,2} \right) \right] \right)^{-1}
$$
$$
\times \left(\sum_k \left[R^k_x x^k_{\sigma,1} + R^k_x x^k_{\sigma,3} \left(I - \sum_l X_l Q x^l_{\sigma,3} \right)^{-1} \left(\sum_l X_l Q x^l_{\sigma,1} \right) \right] \right).
$$

$\square$

C.3 Proof of Proposition 3

In this section we document the properties of the second-order approximation. To express the terms compactly we will use tensor notation. In particular, suppose that A is a $n_1 \times n_2 \times n_3$ dimensional tensor, H is a $n_2 \times n_4$ dimensional tensor and L is a $n_3 \times n_5$ dimensional tensor. Let $A_{\mathtt{ijk}}$ a particular element of the tensor A and $H_{\mathtt{jl}}$ be an element of the tensor H and $L_{\mathtt{km}}$ be an element of tensor L. Define $\langle A, H, L \rangle$ as the $n_1 \times n_4 \times n_5$ tensor given by

$$
\langle A, H, L \rangle_{\mathtt{ilm}} \equiv \sum_{j,k} A_{\mathtt{ijk}} H_{\mathtt{jl}} L_{\mathtt{km}},
$$

and similarly $\langle A, \cdot, L \rangle$, $\langle A, H, \cdot \rangle$ as

$$
\langle A, \cdot, L \rangle_{\mathtt{ijl}} \equiv \sum_l A_{\mathtt{ijk}} L_{\mathtt{kl}} \quad \langle A, H, \cdot \rangle_{\mathtt{ikl}} \equiv \sum_j A_{\mathtt{ijk}} H_{\mathtt{jl}}
$$

For convenience, we also define z^l_0 to be the identity matrix, $(x-)^l_0 = x^l_0$, $(x-)^l_X = x^l_X$, $(x+)^l_0 = x^l_0$, $(x+)^l_X = x^l_X$, and X^l_X to be the identity matrix. We begin with a lemma that is the counterpart of Lemma 1 for higher order derivatives.

Lemma 3. $\{x_{jk}^l, X_{jk}\}_{k,l,j}$ *satisfy*

$$x_{00}^l = -\left(F_{x-}^l + F_x^l + F_{x+}^l\right)^{-1} \left(\sum_{\alpha \in \{z,x-,x,x+\}} \sum_{\beta \in \{z,x-,x,x+\}} \langle F_{\alpha\beta}^l, \alpha_0^l, \beta_0^l \rangle \right), \tag{48a}$$

$x_{0k}^l = \langle x_{0X}^l, \cdot, X_k \rangle$ *for $k \geq 1$ (symmetrically for x_{j0}^l) with*

$$x_{0X}^l = -\left(F_{x-}^l + F_x^l + F_{x+}^l\right)^{-1} \left(\sum_{\alpha \in \{z,x-,x,x+\}} \sum_{\beta \in \{x-,x,x+,X\}} \langle F_{\alpha\beta}^l, \alpha_0^l, \beta_X^l \rangle \right), \tag{48b}$$

and $x_{jk}^l = \langle x_{XX}^l, X_j, X_k \rangle + x_X^l X_{jk}$ with

$$x_{XX}^l = -\left(F_{x-}^l + F_x^l + F_{x+}^l\right)^{-1} \left(\sum_{\alpha \in \{x-,x,x+,X\}} \sum_{\beta \in \{x-,x,x+,X\}} \langle F_{\alpha\beta}^l, \alpha_X^l, \beta_X^l \rangle \right). \tag{48c}$$

Finally

$$X_{jk} = 1_{jk} X_{00}^j + \langle X_{jX}, \cdot, X_k \rangle + \langle X_{Xk} X_j, \cdot \rangle + \langle X_{XX}, X_j, X_k \rangle \tag{49a}$$

where 1_{jk} is 1 if $j = k$ and 0 otherwise

$$X_{00}^j = -\left[\sum_l \left(R_x^l x_X^l + R_X^i \right) \right]^{-1} \left(R_x^j x_{00}^j + \sum_{\alpha \in \{z,x\}} \sum_{\beta \in \{z,x\}} \langle R_{\alpha\beta}^j, \alpha_0^j, \beta_0^j \rangle \right) \tag{49b}$$

$$X_{jX} = -\left[\sum_l \left(R_x^l x_X^l + R_X^i \right) \right]^{-1} \left(R_x^j x_{0X}^j + \sum_{\alpha \in \{z,x\}} \sum_{\beta \in \{x,X\}} \langle R_{\alpha\beta}^j, \alpha_0^j, \beta_X^j \rangle \right) \tag{49c}$$

$$X_{XX} = -\left[\sum_l \left(R_x^l x_X^l + R_X^i \right) \right]^{-1} \left(\sum_l \left[R_x^l x_{XX}^l + \sum_{\alpha \in \{x,X\}} \sum_{\beta \in \{x,X\}} \langle R_{\alpha\beta}^l, \alpha_X^l, \beta_X^l \rangle \right] \right) \tag{49d}$$

and symmetrically for X_{Xk}.

Proof. From Proposition 1, we know several key features of the policy functions when $\sigma = 0$. For the first-order terms $z_0 = I$, $z_k = 0$ for all $k \geq 1$, and $Z_k^l = I$ if $l = k$ and 0 otherwise. For the second-order terms, $z_{jk} = 0$ and $Z_{jk}^l = 0$ for all j, k, l. With this, total differentiation of (19), evaluated at z^l, twice with respect to z yields

$$F_{x-}^l x_{00}^l + F_x^l x_{00}^l + F_{x+}^l x_{00}^l + \sum_{\alpha \in \{z,x-,x,x+\}} \sum_{\beta \in \{z,x-,x,x+\}} \langle F_{\alpha\beta}^l, \alpha_0^l, \beta_0^l \rangle = 0,$$

where latter sum captures the contribution of all the first-order terms. Solving for x_{00}^l produces equation (48a). Following similar procedures we can produce the other terms (48b) and (48c). To find X_{jk}, total differentiate equation (20) twice with respect to the jth and kth arguments of Z to obtain

$$
\begin{aligned}
0 = \sum_l &\left(R_x^l x_{jk}^l + R_X^l X_{jk} + \sum_{\alpha \in \{x,X\}} \sum_{\beta \in \{x,X\}} \langle R_{\alpha\beta}^l, \alpha_X^l X_j, \beta_X^l X_k \rangle \right) \\
&+ R_x^j \langle x_{0X}^j, \cdot, X_k \rangle + \sum_{\alpha \in \{z,x\}} \sum_{\beta \in \{x,X\}} \langle R_{\alpha\beta}^j, \alpha_0^j, \beta_X^j X_k \rangle \\
&+ R_x^k \langle x_{X0}^k, X_j, \cdot \rangle + \sum_{\alpha \in \{x,X\}} \sum_{\beta \in \{z,x\}} \langle R_{\alpha\beta}^k, \alpha_X^k X_j, \beta_0^k \rangle \\
&+ 1_{jk} R_x^j x_{00}^j + 1_{jk} \sum_{\alpha \in \{z,x\}} \sum_{\beta \in \{z,x\}} \langle R_{\alpha\beta}^j, \alpha_0^j, \beta_0^j \rangle.
\end{aligned}
$$

Substituting for $x_{jk}^l = \langle x_{XX}^l, X_j, X_k \rangle + x_X^l X_{jk}$ and then solving for X_{jk} gives the expressions in equations (49). $\qquad \square$

In addition to Lemma 3 we require expressions governing the interactions of the individual and aggregate states (z, Z) with the perturbation parameter σ. For convenience, we also define $\sigma_\sigma^l = 1, (x-)_\sigma^l = x_\sigma, X_\sigma^l = X_\sigma$ and $(x+)_\sigma^l = x_\sigma^l + x_0^l Q x_\sigma^l + \sum_k x_k^l Q x_\sigma^k$.

Lemma 4. $\{x_{k\sigma}^l, X_{k\sigma}\}_{k,l}$ *satisfy*

$$
x_{0\sigma}^l = -\left(F_{x-}^l + F_x^l + F_{x+}^l + F_{x+}^l x_0^l Q \right)^{-1} \left(\sum_{\alpha \in \{z,x-,x,x+\}} \sum_{\beta \in \{\sigma,x-,x,X,x+\}} \langle F_{\alpha\beta}^l, \alpha_0^l, \beta_\sigma^l \rangle \right)
$$

and

$$
\begin{aligned}
x_{k\sigma}^l = &\left(x_{Z\sigma,1}^l + x_{\sigma,3}^l \left(I - \sum_m X_m Q x_{\sigma,3}^m \right)^{-1} \left(\sum_m X_m Q x_{Z\sigma,1}^m \right) \right) X_k \\
&+ \left(x_{Z\sigma,2}^l + x_{\sigma,3}^l \left(I - \sum_m X_m Q x_{\sigma,3}^m \right)^{-1} \left(\sum_m X_m Q x_{Z\sigma,2}^m \right) \right) \left(\langle X_{00}^k, \cdot, Q x_\sigma^k \rangle + \langle X_{kX}, \cdot, X_\sigma' \rangle \right) \\
&+ \left(x_{\sigma,2}^l + x_{\sigma,3}^l \left(I - \sum_m X_m Q x_{\sigma,3}^m \right)^{-1} \left(\sum_m X_m Q x_{\sigma,2}^m \right) \right) X_{k\sigma}
\end{aligned}
$$

$where X'_\sigma = \sum_j X_j Q x^j_\sigma$ and

$$z^l_{Z\sigma,1} = -\left(F^l_{x-} + F^l_x + F^l_{x+} + F^l_{x+} x^l_0 Q\right)^{-1}\left(F^l_{x+}\langle x^l_{XX}, \cdot, X'_\sigma\rangle + F^l_{x+} x^l_X\langle X_{XX}, \cdot, X'_\sigma\rangle\right.$$

$$\left.+ F^l_{x+}\langle x^l_{X0}, \cdot, Q x^l_\sigma\rangle + \sum_{\alpha\in\{x-,x,X,,x+\}}\sum_{\beta\in\{\sigma,x-,x,X,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_X, \beta^l_\sigma\rangle\right)$$

$$z^l_{Z\sigma,2} = -\left(F^l_{x-} + F^l_x + F^l_{x+} + F^l_{x+} x^l_0 Q\right) F^l_{x+} x^l_X.$$

Finally

$$X_{k\sigma} = -\left(\sum_l\left[R^l_x x^l_{\sigma,2} + R^l_x x^l_{\sigma,3}\left(I - \sum_m X_m Q x^m_{\sigma,3}\right)^{-1}\left(\sum_m X_m Q x^m_{\sigma,2}\right) + R^l_X\right]\right)^{-1}$$

$$\times\left(\sum_l\left[R^l_x x^l_{Z\sigma,1} + R^l_x x^l_{\sigma,3}\left(I - \sum_m X_m Q x^m_{\sigma,3}\right)^{-1}\left(\sum_m X_m Q x^m_{Z\sigma,1}\right) + \sum_{\alpha,\beta\in\{x,X\}}\langle R^l_{\alpha\beta}, \alpha^l_X, \beta^l_\sigma\rangle\right]X_k\right.$$

$$+ \sum_l\left[x^l_{Z\sigma,2} + x^l_{\sigma,3}\left(I - \sum_m X_m Q x^m_{\sigma,3}\right)^{-1}\left(\sum_m X_m Q x^m_{Z\sigma,2}\right)\right]\left(\langle X^k_{00}, \cdot, Q x^k_\sigma\rangle + \langle X_{kX}, \cdot, X'_\sigma\rangle\right)$$

$$\left.+ R^k_x x^k_{0\sigma} + \sum_{\alpha\in\{z,x\}}\sum_{\beta\in\{x,X\}}\langle R^k_{\alpha\beta}, \alpha^k_0, \beta^l_\sigma\rangle\right).$$

and symmetrically for $\{x^l_{\sigma k}, X_{\sigma k}\}_{k,l}$.

Proof. Total differentiation of (19) with respect to z and σ gives

$$F^l_{x-} x^l_{0\sigma} + F^l_x x^l_{0\sigma} + F^l_{x+} x^l_{0\sigma} + F^l_{x+} x^l_0 Q x^l_{0\sigma} + \sum_{\alpha\in\{z,x-,x,x+\}}\sum_{\beta\in\{\sigma,x-,x,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_0, \beta^l_\sigma\rangle = 0.$$

Directly solving for $x^l_{0\sigma}$ yields the expression in the Lemma. For $x^l_{k\sigma}$ and $X^l_{k\sigma}$, total differentiation of (19) and (20) with respect to Z_k and σ obtains

$$0 = F^l_{x-} x^l_{k\sigma} + F^l_x x^l_{k\sigma} + F^l_X X_{k\sigma} + F^l_{x+} x^l_{k\sigma} + F^l_{x+} x^l_0 Q x^l_{k\sigma} + F^l_{x+}\sum_m\langle x^l_{km}, \cdot, Q x^m_\sigma\rangle \qquad (50)$$

$$+ F^l_{x+}\langle x^l_{k0}, \cdot, Q x^l_\sigma\rangle + F^l_{x+}\sum_m x^l_m Q x^m_{k\sigma} + \sum_{\alpha\in\{x-,x,X,,x+\}}\sum_{\beta\in\{\sigma,x-,x,X,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_k, \beta^l_\sigma\rangle$$

and

$$0 = \sum_l \left(R_x^l x_{k\sigma}^l + R_X^l X_{k\sigma} + \sum_{\alpha,\beta \in \{x,X\}} \langle R_{\alpha\beta}^l, \alpha_X^l X_k, \beta_\sigma^l \rangle \right) \tag{51}$$
$$+ R_x^k x_{0\sigma}^k + \sum_{\alpha \in \{z,x\}} \sum_{\beta \in \{x,X\}} \langle R_{\alpha\beta}^k, \alpha_0^k, \beta_\sigma^l \rangle.$$

Solving equation (50) for $x_{k\sigma}^l$ gives

$$x_{k\sigma}^l = x_{Z\sigma,1}^l X_k + x_{Z\sigma,2}^l \left(\langle X_{00}^k, \cdot, Q x_\sigma^k \rangle + \langle X_{kX}, \cdot, X_\sigma' \rangle \right) + x_{\sigma,2}^l X_{k\sigma} + x_{\sigma,3}^l \sum_m X_m Q x_{k\sigma}^m$$

where $x_{Z\sigma,1}^l$ and $x_{Z\sigma,2}^l$ are given in the statement of the Lemma and $x_{\sigma,2}^l$ and $x_{\sigma,3}^l$ are in the statement of Proposition 2. Applying $\sum_l X_l Q$ to both sides of this equation and solving for $\sum_l X_l Q x_{k\sigma}^l$ yields the expression for $x_{k\sigma}^l$ found in the Lemma. Substituting for $x_{k\sigma}^l$ in (51) and solving for $X_{k\sigma}$ generates the expression for $X_{k\sigma}$ in the statement of the Lemma. $\qquad\square$

Once again we are able to decompose complicated terms such as x_{jk}^l and $x_{k\sigma}^l$ which depend on multiple agents into terms that only depend on a single agent. We exploit this in the following lemma that gives the quadratic terms in the Taylor expansion of $\tilde{x}$ and $\tilde{X}$. For convenience, define $\mathcal{E}_{\mathcal{E}}^l$ as the identity matrix, $X_{\mathcal{E}}^l = X_{\mathcal{E}}$ and $(x+)_{\mathcal{E}}^l = x_{\mathcal{E}}^l + x_0^l Q x_{\mathcal{E}}^l + \sum_k x_k^l Q x_{\mathcal{E}}^k$.

Lemma 5. *The terms* $\left\{ x_{\varepsilon\varepsilon}^l, x_{\varepsilon\mathcal{E}}^l, x_{\varepsilon\sigma}^l, x_{\mathcal{E}\mathcal{E}}^l, x_{\mathcal{E}\sigma}^l, x_{\sigma\sigma}^l, X_{\mathcal{E}\mathcal{E}}, X_{\mathcal{E}\sigma} X_{\sigma\sigma} \right\}$ *in the second-order expan-*

sion for the individual policies $\tilde{x}$ are given by

$$x_{\varepsilon\varepsilon}^l = -\left[F_x^l + F_{x+}^l x_0^l Q\right]^{-1}\left(\sum_{\alpha,\beta\in\{\varepsilon,x,x+\}}\langle F_{\alpha\beta}^l, \alpha_\varepsilon^l, \beta_\varepsilon^l\rangle + F_{x+}^l\langle x_{00}^l, Qx_\varepsilon^l, Qx_\varepsilon^l\rangle\right)$$

$$x_{\varepsilon\sigma}^l = -\left[F_x^l + F_{x+}^l x_0^l Q\right]^{-1}\left(\sum_{\alpha\in\{\varepsilon,x,x+\}}\sum_{\beta\in\{\sigma,x-,x,X,x+\}}\langle F_{\alpha\beta}^l, \alpha_\varepsilon^l, \beta_\sigma^l\rangle\right.$$

$$\left.+ F_{x+}^l\langle x_{0\sigma}^l, Qx_\varepsilon^l, \cdot\rangle + F_{x+}^l\langle x_{00}^l, Qx_\varepsilon^l, Qx_\sigma^l\rangle + F_{x+}^l\sum_k\langle x_{0k}^l, Qx_\varepsilon^l, Qx_\sigma^k\rangle\right)$$

$$x_{\varepsilon\mathcal{E}}^l = -\left[F_x^l + F_{x+}^l x_0^l Q\right]^{-1}\left(\sum_{\alpha\in\{\varepsilon,x,x+\}}\sum_{\beta\in\{\mathcal{E},x,X,x+\}}\langle F_{\alpha\beta}^l, \alpha_\varepsilon^l, \beta_\mathcal{E}^l\rangle\right.$$

$$\left.+ F_{x+}^l\langle x_{00}^l, Qx_\varepsilon^l, Qx_\mathcal{E}^l\rangle + F_{x+}^l\sum_k\langle x_{0k}^l, Qx_\varepsilon^l, Qx_\mathcal{E}^k\rangle\right)$$

$$x_{\mathcal{E}\mathcal{E}}^l = \left(x_{\mathcal{E}\mathcal{E},1}^l + x_{\mathcal{E},3}^l\left(I - \sum_k X_k Qx_{\mathcal{E},3}^k\right)^{-1}\left(\sum_k X_k Qx_{\mathcal{E}\mathcal{E},1}^k\right)\right)$$

$$+ \left(x_{\mathcal{E},2}^l + x_{\mathcal{E},3}^l\left(I - \sum_k X_k Qx_{\mathcal{E},3}^k\right)^{-1}\left(\sum_k X_k Qx_{\mathcal{E},2}^k\right)\right)X_{\mathcal{E}\mathcal{E}},$$

$$x_{\sigma\mathcal{E}}^l = \left(x_{\sigma\mathcal{E},1}^l + x_{\mathcal{E},3}^l\left(I - \sum_k X_k Qx_{\mathcal{E},3}^k\right)^{-1}\left(\sum_k X_k Qx_{\sigma\mathcal{E},1}^k\right)\right)$$

$$+ \left(x_{\mathcal{E},2}^l + x_{\mathcal{E},3}^l\left(I - \sum_k X_k Qx_{\mathcal{E},3}^k\right)^{-1}\left(\sum_k X_k Qx_{\mathcal{E},2}^k\right)\right)X_{\sigma\mathcal{E}},$$

$$x_{\sigma\sigma}^l = \left(x_{\sigma\sigma,1}^l + x_{\sigma,3}^l\left(I - \sum_k X_k Qx_{\sigma,3}^k\right)^{-1}\left(\sum_k X_k Qx_{\sigma\sigma,1}^k\right)\right)$$

$$+ \left(x_{\sigma,2}^l + x_{\sigma,3}^l\left(I - \sum_k X_k Qx_{\sigma\sigma,3}^k\right)^{-1}\left(\sum_k X_k Qx_{\sigma,2}^k\right)\right)X_{\sigma\sigma}$$

and for the aggregate policy function $\tilde{X}$ are given by

$$X_{\mathcal{E}\mathcal{E}} = -\left(\sum_k \left[R_X^k + R_x^k x_{\mathcal{E},2}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l\right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},2}^l\right)\right]\right)^{-1}$$

$$\times \left(\sum_k \left[R_x^k x_{\mathcal{E}\mathcal{E},1}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l\right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E}\mathcal{E},1}^l\right)\right.$$

$$\left. + \sum_{\alpha,\beta\in\{\mathcal{E},x,X\}} \langle R_{\alpha\beta}^k, \alpha_{\mathcal{E}}^k, \beta_{\mathcal{E}}^k\rangle\right]\right)$$

$$X_{\sigma\mathcal{E}} = -\left(\sum_k \left[R_X^k + R_x^k x_{\mathcal{E},2}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l\right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E},2}^l\right)\right]\right)^{-1}$$

$$\times \left(\sum_k \left[R_x^k x_{\sigma\mathcal{E},1}^k + R_x^k x_{\mathcal{E},3}^k \left(I - \sum_l X_l Q x_{\mathcal{E},3}^l\right)^{-1} \left(\sum_l X_l Q x_{\mathcal{E}\mathcal{E},1}^l\right)\right.$$

$$\left. + \sum_{\alpha\in\{x,X\}} \sum_{\beta\in\{\mathcal{E},x,X\}} \langle R_{\alpha\beta}^k, \alpha_{\sigma}^k, \beta_{\mathcal{E}}^k\rangle\right]\right)$$

$$X_{\sigma\sigma} = -\left(\sum_k \left[R_X^k + R_x^k x_{\sigma,2}^k + R_x^k x_{\sigma,3}^k \left(I - \sum_l X_l Q x_{\sigma,3}^l\right)^{-1} \left(\sum_l X_l Q x_{\sigma,2}^l\right)\right]\right)^{-1}$$

$$\times \left(\sum_k \left[R_x^k x_{\sigma\sigma,1}^k + R_x^k x_{\sigma,3}^k \left(I - \sum_l X_l Q x_{\sigma,3}^l\right)^{-1} \left(\sum_l X_l Q x_{\sigma\sigma,1}^l\right)\right.\right.$$

$$\left.\left. \sum_{\alpha,\beta\in\{\varepsilon,x\}} \mathbb{E}_z\left[\langle R_{\alpha\beta}^k, \alpha_{\varepsilon}^k\varepsilon, \beta_{\varepsilon}^k\varepsilon\rangle\right] + R_x^k \mathbb{E}_z\left[\langle x_{00}^k, \varepsilon, \varepsilon\rangle\right]\right] + \sum_{\alpha,\beta\in\{x,X\}} \langle R_{\alpha\beta}^k, \alpha_{\sigma}^k, \beta_{\sigma}^k\rangle\right).$$

Where

$$x^l_{\mathcal{E}\mathcal{E},1} = -(F^l_x + F^l_{x+}x^l_0 Q)^{-1}\left(F^l_{x+}\sum_{j,k}\langle x^l_{jk}, Qx^j_{\mathcal{E}}, Qx^k_{\mathcal{E}}\rangle + F^l_{x+}\sum_{j}\langle x^l_{j0}, Qx^j_{\mathcal{E}}, Qx^l_{\mathcal{E}}\rangle \right.$$

$$+ F^l_{x+}\sum_{k}\langle x^l_{0k}, Qx^l_{\mathcal{E}}, Qx^k_{\mathcal{E}}\rangle + F^l_{x+}\langle x^l_{00}, Qx^l_{\mathcal{E}}, Qx^l_{\mathcal{E}}\rangle$$

$$\left. + \sum_{\alpha,\beta\in\{\mathcal{E},x,X,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_{\mathcal{E}}, \beta^l_{\mathcal{E}}\rangle \right),$$

$$x^l_{\sigma\mathcal{E},1} = -(F^l_x + F^l_{x+}x^l_0 Q)^{-1}\left(F^l_{x+}\sum_{j,k}\langle x^l_{jk}, Qx^j_{\sigma}, Qx^k_{\mathcal{E}}\rangle + F^l_{x+}\sum_{j}\langle x^l_{j0}, Qx^j_{\sigma}, Qx^l_{\mathcal{E}}\rangle \right.$$

$$+ F^l_{x+}\sum_{k}\langle x^l_{0k}, Qx^l_{\sigma}, Qx^k_{\mathcal{E}}\rangle + F^l_{x+}\langle x^l_{00}, Qx^l_{\sigma}, Qx^l_{\mathcal{E}}\rangle$$

$$+ F^l_{x+}\sum_{k}x^l_{\sigma k}Qx^k_{\mathcal{E}} + F_{x+}x^l_{\sigma 0}Qx^l_{\mathcal{E}}$$

$$\left. + \sum_{\alpha\in\{\sigma,x-,x,X,x+\}}\sum_{\beta\in\{\mathcal{E},x,X,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_{\sigma}, \beta^l_{\mathcal{E}}\rangle \right),$$

$$x^l_{\sigma\sigma,1} = -(F^l_{x-} + F^l_x + F^l_{x+} + F^l_{x+}x^l_0 Q)^{-1}\left(F^l_{x+}\sum_{j,k}\langle x^l_{jk}, Qx^j_{\sigma}, Qx^k_{\sigma}\rangle + F^l_{x+}\sum_{j}\langle x^l_{j0}, Qx^j_{\sigma}, Qx^l_{\sigma}\rangle \right.$$

$$+ F^l_{x+}\sum_{k}\langle x^l_{0k}, Qx^l_{\sigma}, Qx^k_{\sigma}\rangle + F^l_{x+}\langle x^l_{00}, Qx^l_{\sigma}, Qx^l_{\sigma}\rangle$$

$$+ F^l_{x+}\sum_{k}x^l_{\sigma k}Qx^k_{\sigma} + F^l_{x+}x^l_{\sigma 0}Qx^l_{\sigma} + F^l_{x+}x^l_{0\sigma}Qx^l_{\sigma}$$

$$+ F^l_{x+}\sum_{k}x^l_{k\sigma}Qx^k_{\sigma} + \left(F^l_{x-} + F^l_{x+}\right)\mathbb{E}_z\left[\langle x^l_{\varepsilon\varepsilon}, \varepsilon, \varepsilon\rangle + \langle x^l_{\mathcal{E}\mathcal{E}}, \mathcal{E}, \mathcal{E}\rangle\right]$$

$$+ F^l_{x+}x^l_X\sum_{k}X_k Q\mathbb{E}_z\left[\langle x^k_{\varepsilon\varepsilon}, \varepsilon, \varepsilon\rangle\right]$$

$$+ F^l_{x+}x^l_X\sum_{k}\mathbb{E}_z\left[\langle X^k_{00}, Qx^k_{\varepsilon}\varepsilon, Qx^k_{\varepsilon}\varepsilon\rangle\right]$$

$$\left. + \sum_{\alpha,\beta\in\{\sigma,x-,x,X,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_{\sigma}, \beta^l_{\sigma}\rangle \right).$$

Proof. As in Proposition 2, proceed by total differentiating equations (19) and (20) twice with respect to σ. To simplify exposition we will only report the component parts of this derivatives. Combining the terms loading on $\varepsilon\varepsilon$ yields

$$F^l_x x^l_{\varepsilon\varepsilon} + F^l_{x+}x^l_0 Qx^l_{\varepsilon\varepsilon} + \sum_{\alpha,\beta\in\{\varepsilon,x,x+\}}\langle F^l_{\alpha\beta}, \alpha^l_{\varepsilon}, \beta^l_{\varepsilon}\rangle + F^l_{x+}\langle x^l_{00}, Qx^l_{\varepsilon}, Qx^l_{\varepsilon}\rangle = 0.$$

Solving this equation for $x_{\varepsilon\varepsilon}^l$ obtains the expressions in the proposition. A similar procedure with produces the expression for $x_{\varepsilon\varepsilon}^l$ and $x_{\varepsilon\sigma}^l$. After combining the terms loading on $\mathcal{E}\mathcal{E}$, from the derivative of (19), we see

$$
\begin{aligned}
0 =& F_x^l x_{\mathcal{E}\mathcal{E}}^l + F_X^l X_{\mathcal{E}\mathcal{E}} + F_{x+}^l x_0^l Q x_{\mathcal{E}\mathcal{E}}^l + F_{x+}^l \sum_k x_k^l Q x_{\mathcal{E}\mathcal{E}}^k \\
& + F_{x+}^l \langle x_{00}^l, Q x_{\mathcal{E}}^l, Q x_{\mathcal{E}}^l \rangle + F_{x+}^l \sum_j \langle x_{0k}^l, Q x_{\mathcal{E}}^l, Q x_{\mathcal{E}}^k \rangle \\
& + F_{x+}^l \sum_k \langle x_{k0}^l, Q x_{\mathcal{E}}^k, Q x_{\mathcal{E}}^l \rangle + F_{x+}^l \sum_{j,k} \langle x_{jk}^l, Q x_{\mathcal{E}}^j, Q x_{\mathcal{E}}^k \rangle \\
& + \sum_{\alpha,\beta\in\{\mathcal{E},x,X,x+\}} \langle F_{\alpha\beta}^l, \alpha_{\mathcal{E}}^l, \beta_{\mathcal{E}}^l \rangle.
\end{aligned}
$$

After substituting $x_k^l = x_X^l X_k$ and solving for $x_{\mathcal{E}\mathcal{E}}^l$ gives

$$
x_{\mathcal{E}\mathcal{E}}^l = x_{\mathcal{E}\mathcal{E},1}^l + x_{\mathcal{E},2}^l X_{\mathcal{E}\mathcal{E}} + x_{\mathcal{E},3}^l \sum_k X_k Q x_{\mathcal{E}\mathcal{E}}^k,
$$

where $x_{\mathcal{E}\mathcal{E},1}^l$ is the expression from the Lemma and $x_{\mathcal{E},2}^l, x_{\mathcal{E},3}^l$ are the same terms from Proposition 2. Applying $\sum_l X_l Q \cdot$ to both sides and then solving for $\sum_k X_k Q x_{\mathcal{E}\mathcal{E}}^k$ yields the expression for $x_{\mathcal{E}\mathcal{E}}^l$ in the proposition. The terms loading on $\mathcal{E}\mathcal{E}$ in the derivative of (20) imply

$$
0 = \sum_k \left(R_X^k X_{\mathcal{E}\mathcal{E}} + R_x^k x_{\mathcal{E}\mathcal{E}}^k + \sum_{\alpha,\beta\in\{\mathcal{E},x,X\}} \langle R_{\alpha\beta}^k, \alpha_{\mathcal{E}}^k, \beta_{\mathcal{E}}^k \rangle \right).
$$

Substituting for $x_{\mathcal{E}\mathcal{E}}^l$ and solving for $X_{\mathcal{E}\mathcal{E}}$, produces the expression for $X_{\mathcal{E}\mathcal{E}}$ in the Lemma.

Combining the loading on $\mathcal{E}\sigma$, from the derivative of (19) gives

$$
\begin{aligned}
0 =& F_x^l x_{\sigma\mathcal{E}}^l + F_X^l X_{\sigma\mathcal{E}\sigma} + F_{x+}^l x_0^l Q x_{\sigma\mathcal{E}}^l + F_{x+}^l \sum_k x_k^l Q x_{\sigma\mathcal{E}}^k \\
& + F_{x+}^l \sum_{j,k} \langle x_{jk}^l, Q x_\sigma^j, Q x_{\mathcal{E}}^k \rangle + F_{x+}^l \sum_j \langle x_{j0}^l, Q x_\sigma^j, Q x_{\mathcal{E}}^l \rangle \\
& + F_{x+}^l \sum_k \langle x_{0k}^l, Q x_\sigma^l, Q x_{\mathcal{E}}^k \rangle + F_{x+}^l \langle x_{00}^l, Q x_\sigma^l, Q x_{\mathcal{E}}^l \rangle \\
& + F_{x+}^l \sum_k x_{\sigma k}^l Q x_{\mathcal{E}}^k + F_{x+} x_{\sigma 0}^l Q x_{\mathcal{E}}^l \\
& + \sum_{\alpha\in\{\sigma,x-,x,X,x+\}} \sum_{\beta\in\{\mathcal{E},x,X,x+\}} \langle F_{\alpha\beta}^l, \alpha_\sigma^l, \beta_{\mathcal{E}}^l \rangle.
\end{aligned}
$$

After substituting $x_k^l = x_X^l X_k$ and solving for $x_{\sigma\mathcal{E}}^l$ gives

$$x_{\sigma\mathcal{E}}^l = x_{\sigma\mathcal{E},1}^l + x_{\mathcal{E},2}^l X_{\mathcal{E}\mathcal{E}} + x_{\mathcal{E},3}^l \sum_k X_k Q x_{\mathcal{E}\mathcal{E}}^k,$$

where $x_{\mathcal{E}\mathcal{E},1}^l$ is the expression from the Lemma and $x_{\mathcal{E},2}^l, x_{\mathcal{E},3}^l$ are the same terms from Proposition 2. Applying $\sum_l X_l Q \cdot$ to both sides and then solving for $\sum_k X_k Q x_{\sigma\mathcal{E}}^k$ yields the expression for $x_{\sigma\mathcal{E}}^l$ in the proposition. The terms loading on $\sigma\mathcal{E}$ in the derivative of (20) imply

$$0 = \sum_k \left(R_X^k X_{\sigma\mathcal{E}} + R_x^k x_{\sigma\mathcal{E}}^k + \sum_{\alpha \in \{x,X\}} \sum_{\beta \in \{\mathcal{E},x,X\}} \langle R_{\alpha\beta}^k, \alpha_\sigma^k, \beta_\mathcal{E}^k \rangle \right).$$

Substituting for $x_{\sigma\mathcal{E}}^l$ and solving for $X_{\sigma\mathcal{E}}$ yields the expression for $X_{\mathcal{E}\mathcal{E}}$ in the Lemma.

The remaining terms capture the direct dependence on $\sigma\sigma$. From the derivative of (19) :

$$
\begin{aligned}
0 = & F_x^l x_{\sigma\sigma}^l + F_X^l X_{\sigma\sigma} + F_{x+}^l x_{\sigma\sigma}^l + F_{x+}^l x_0^l Q x_{\sigma\sigma}^l + F_{x+}^l \sum_k x_k^l Q x_{\sigma\sigma}^k \\
& + F_{x-}^l \left(\mathbb{E}_z \left[\langle x_{\varepsilon\varepsilon}^l, \varepsilon, \varepsilon \rangle \right] + \mathbb{E}_z \left[\langle x_{\mathcal{E}\mathcal{E}}^l, \mathcal{E}, \mathcal{E} \rangle \right] \right) + F_{x+}^l \left(\mathbb{E}_z \left[\langle x_{\varepsilon\varepsilon}^l, \varepsilon, \varepsilon \rangle \right] + \mathbb{E}_z \left[\langle x_{\mathcal{E}\mathcal{E}}^l, \mathcal{E}, \mathcal{E} \rangle \right] \right) \\
& + F_{x+}^l x_X^l \sum_k X_k Q \mathbb{E}_z \left[\langle x_{\varepsilon\varepsilon}^k, \varepsilon, \varepsilon \rangle \right] + F_{x+}^l x_X^l \sum_k \mathbb{E}_z \left[\langle X_{00}^k, Q x_\varepsilon^k \varepsilon, Q x_\varepsilon^k \varepsilon \rangle \right] \\
& + F_{x+}^l \sum_{j,k} \langle x_{jk}^l, Q x_\sigma^j, Q x_\sigma^k \rangle + F_{x+}^l \sum_j \langle x_{j0}^l, Q x_\sigma^j, Q x_\sigma^l \rangle \\
& + F_{x+}^l \sum_k \langle x_{0k}^l, Q x_\sigma^l, Q x_\sigma^k \rangle + F_{x+}^l \langle x_{00}^l, Q x_\sigma^l, Q x_\sigma^l \rangle \\
& + F_{x+}^l \sum_k x_{\sigma k}^l Q x_\sigma^k + F_{x+}^l x_{\sigma 0}^l Q x_\sigma^l + F_{x+}^l x_{0\sigma}^l Q x_\sigma^l \\
& + F_{x+}^l \sum_k x_{k\sigma}^l Q x_\sigma^k + \sum_{\alpha,\beta \in \{\sigma, x-, x, X, x+\}} \langle F_{\alpha\beta}^l, \alpha_\sigma^l, \beta_\sigma^l \rangle.
\end{aligned}
$$

The final line line captures the effect of the idiosyncratic shocks on the distribution Z and hence of future policies. Solving this equation for $x_{\sigma\sigma}^l$ gives

$$x_{\sigma\sigma}^l = x_{\sigma\sigma,1}^l + x_{\sigma\sigma,2}^l X_{\sigma\sigma} + x_{\sigma\sigma,3}^l \sum_k X_k Q x_{\sigma\sigma}^k,$$

where $x_{\sigma\sigma,1}^l, x_{\sigma\sigma,2}^l$ and $x_{\sigma\sigma,3}^l$ are the terms given in the Lemma. Applying $\sum_l X_l Q \cdot$ to both sides and then solving for $\sum_k X_k Q x_{\sigma\sigma}^k$ yields the expression for $x_{\sigma\sigma}^l$. Finally the remaining

terms from the derivative of (20) are

$$0 = \sum_k \left(R_x^k x_{\sigma\sigma}^k + R_X^k X_{\sigma\sigma} + \sum_{\alpha,\beta\in\{\varepsilon,x\}} \mathbb{E}_z \left[\langle R_{\alpha\beta}^k, \alpha_\varepsilon^k \varepsilon, \beta_\varepsilon^k \varepsilon \rangle \right] + R_x^k \mathbb{E}_z \left[\langle x_{00}^k, \varepsilon, \varepsilon \rangle \right] \right).$$

The expression for $X_{\sigma\sigma}$ in the proposition is obtained by substituting for $x_{\sigma\sigma}^l$ and solving for $X_{\sigma\sigma}$. $\qquad\square$

Finally we note that all the expressions in Lemma 5 involve expressions that inverses of matrices of order at most $\max\{N_x, N_X\}$. And sum over at most K elements. The latter can be seen as all of the subcomponents of any sum can be decomposed into terms depending on only one group of agents Z_k. For example

$$\sum_{j,k} \langle x_{jk}^l, Qx_\sigma^j, Qx_{\mathcal{E}}^k \rangle = \sum_{j,k} \left(\langle x_{XX}^l, X_j Qx_\sigma^j, X_k Qx_\sigma^k \rangle + x_X^l \langle X_{jk}, Qx_\sigma^j, Qx_{\mathcal{E}}^k \rangle \right)$$

$$= \langle x_{XX}^l, \sum_j X_j Qx_\sigma^j, \sum_k X_k Qx_\sigma^k \rangle + x_X^l \sum_{j,k} 1_{jk} \left(\langle X_{00}^j, Qx_\sigma^j, Qx_{\mathcal{E}}^k \rangle \right.$$

$$\left. + \langle X_{Xk}, X_j Qx_\sigma^j, Qx_\sigma^k \rangle + \langle X_{jX}, Qx_\sigma^j, X_k Qx_\sigma^k \rangle + \langle X_{XX}, X_j Qx_\sigma^j, X_k Qx_\sigma^k \rangle \right)$$

$$= \langle x_{XX}^l, X_\sigma', X_\sigma' \rangle + x_X^l \left(\sum_k \langle X_{00}^k, Qx_\sigma^k, Qx_{\mathcal{E}}^k \rangle + \langle X_{X\sigma}', X_\sigma', \cdot \rangle \right.$$

$$\left. + \langle X_{\sigma X}', \cdot, X_\sigma' \rangle + \langle X_{XX}, X_\sigma', X_\sigma' \rangle \right).$$

where $X_\sigma' = \sum_k X_k Qx_\sigma^k$, $X_{X\sigma}' = \sum_k \langle X_{Xk}, \cdot, Qx_\sigma^k \rangle$ and symmetrically for $X_{\sigma X}'$.

C.4   Extension to Cost-Push Shocks

[TBA]

C.5 Simulation and Choice of K

In simulating the optimal policy two tasks must be accomplished: we must track the evolution of the distribution of individual state variables and then discretize the distribution using K points. In order to track the evolution of the distribution we approximate the continuum of agents with a large number, $N = 100,000$, of agents, with each receiving his/her own idiosyncratic shock. In principle, each period it would be possible to approximate the policy

rules around this discretized group of agents, but this would be unecessarily intensive. Instead we follow a two step procedure: first we approximate the $N = 100,000$ group of agents with a smaller group of $K = 10,000$ points using a k-means algorithm. Next we approximate the policy rules around the Z constructed with the k-means algorithm and simulate 1 period of the economy with the N agents using the procedure presented in section 3. Additionally, the derivatives provided in sections C.2-C.3 with respect to the state variables allow us to partially correct the errors introduced, with the k-means approximation, by adding additional terms to the Taylor expansion. The advantage of this approach is that it can reduce the computational time by, in this case, a factor of 10. We choose our K such that increasing K does not change the impulse responses reported in section 4.

D   Robustness

In this section we show the robustness of our main results. We compare the optimal monetary-fiscal response to alternative economies changing one feature at time. Apart from Figure IX, in all the figures VIII-XII, the solid line is our baseline calibration and coincides with the "HANK" lines of Figure III. In Figure IX we compare the optimal monetary response and the solid line corresponds to that in Figure II.

We start with alternative Pareto weights. Let the initial Pareto weights be denoted by ω_i , we use the functional form $\omega_i \propto \theta_{i,-1}^h$ and choose h to make the optimal tax rate in an economy with no risk equal the observed marginal tax rate of 24%. In Figure VIII, we see taxes are slightly more volatile but overall the results are essentially unchanged.

In the main text Section 4.2, when we reported the optimal monetary responses, we fixed tax rate to their non-stochastic optimal value. Our results are driven by the need for insurance against aggregate shocks and hence robust to alternative choices of tax rates. As an example, in Figure IX we compare the optimal monetary responses when $\tau_t = 24\%$.

To isolate the role of initial asset heterogeneity and the heterogeneous exposure that comes directly from the loadings function $f(.)$ we set $f = 0$. The productivity shock now leads to a parallel shift in skills for all agents. The optimal responses to such a shock is in Figure X. As one can expect, the responses are mitigated and we conclude that asset heterogeneity by itself accounts for a third of the response with the more substantial two third portion coming from the heterogeneous exposures to the aggregate shocks.

Menu costs affect the tradeoffs for the planner to use inflation as against tax rates to lower holding period returns. In Figure XI we see that when $\psi = 0$ the tax rate no longer dips on impact but stays high for a long time. The planner need not move the tax rate in order to lower real returns but can instead raise inflation to achieve the same outcome
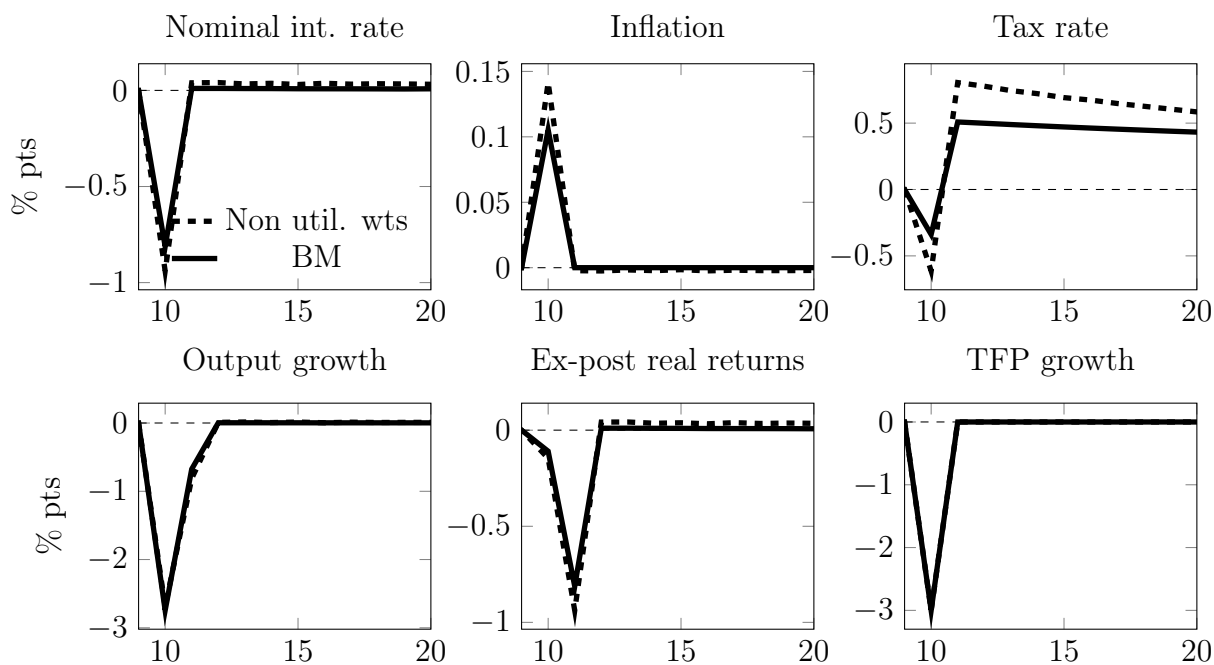
Figure VIII: Optimal monetary-fiscal response to a productivity shock with non-utilitarian Pareto weights
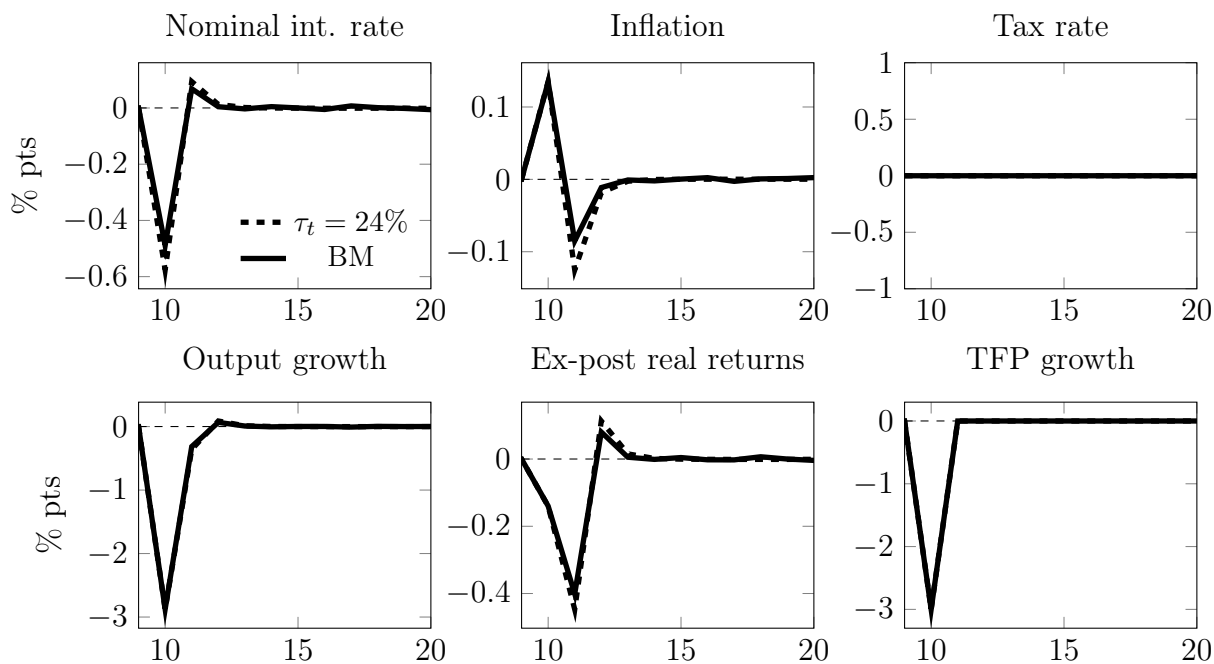

Figure IX: Optimal monetary response to a productivity shock with non optimal tax rate, $\tau_t = 24\%$
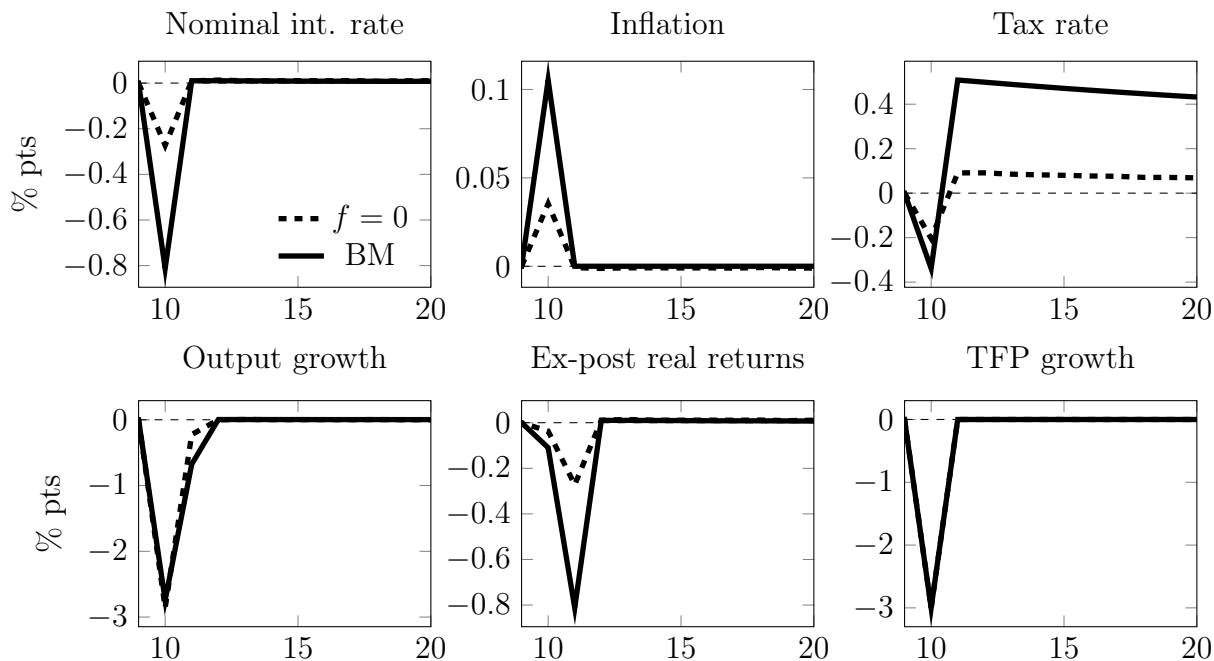
Figure X: Optimal monetary-fiscal response to a productivity shock with $f = 0$

at a lower welfare cost. The inflation rate jumps by 7 percentage points as against 0.12 percentage points in our baseline heterogeneous agent economy.

A useful contribution is that we have derived analytical expressions for derivatives that allow us to compute higher order terms terms in the Taylor expansions. In Figure XII we compare our results to the case where we only use the first-order terms. We find that the optimal responses are very different and lowered by about 50%.

Lastly, in Figure XIII we compute the optimal responses with hand-to-mouth agents.
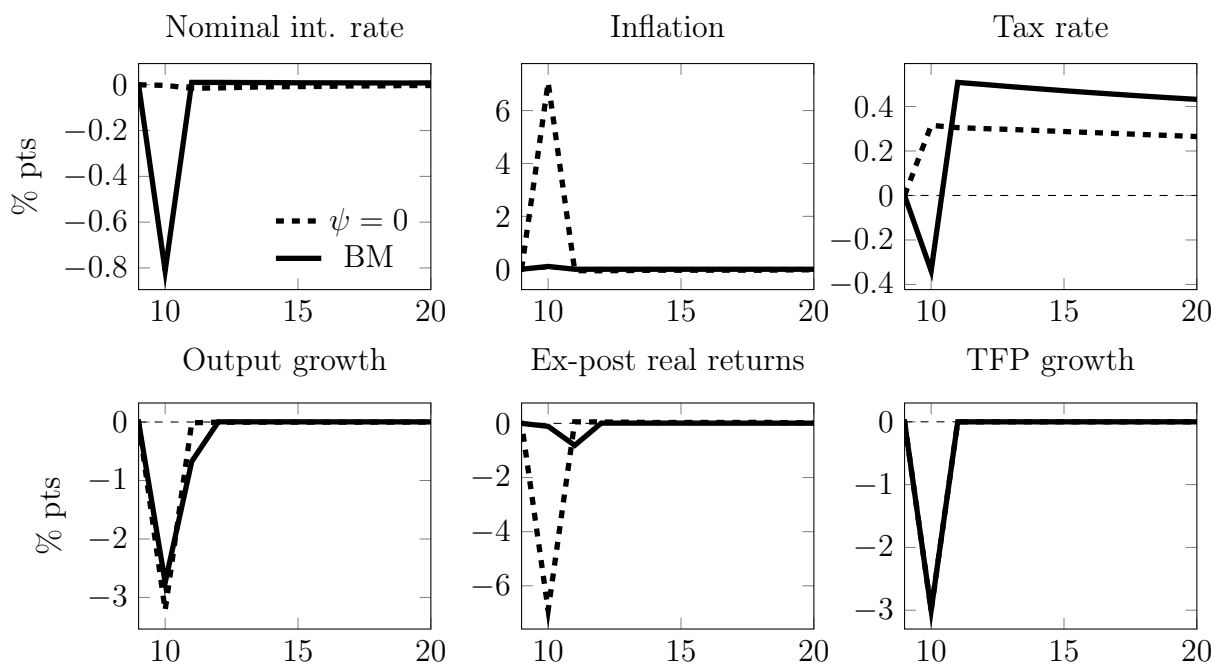
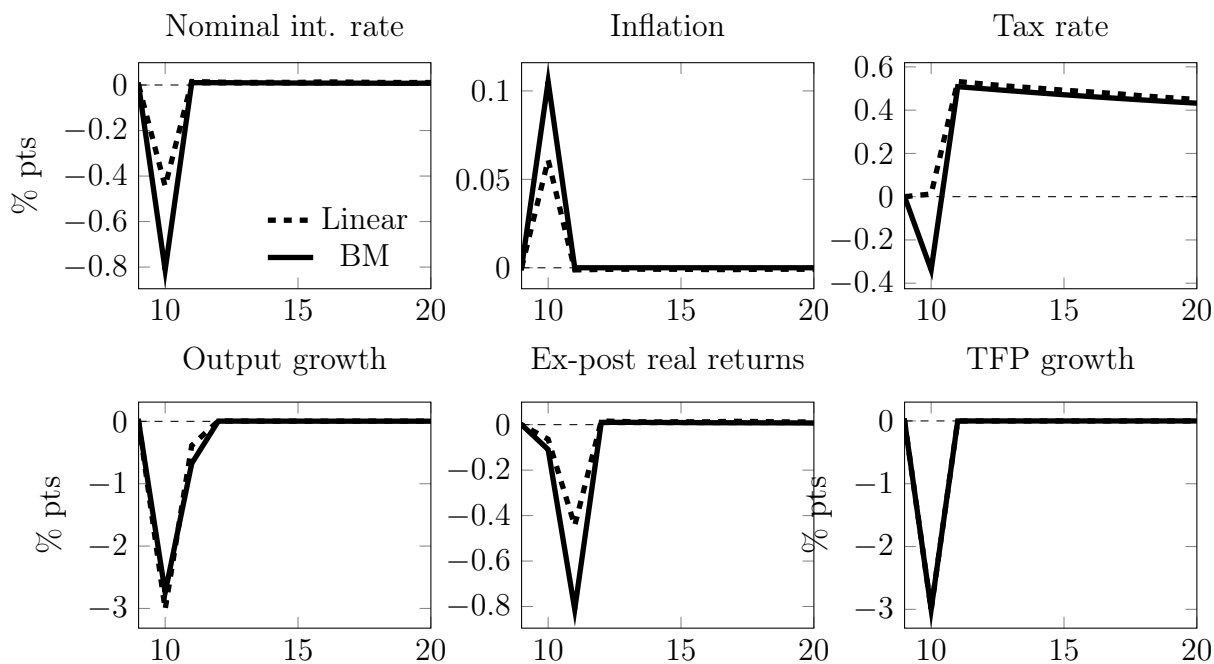Figure XI: Optimal monetary-fiscal response to a productivity shock with $\psi = 0$

Figure XII: Optimal monetary-fiscal response to a productivity shock with no second-order terms
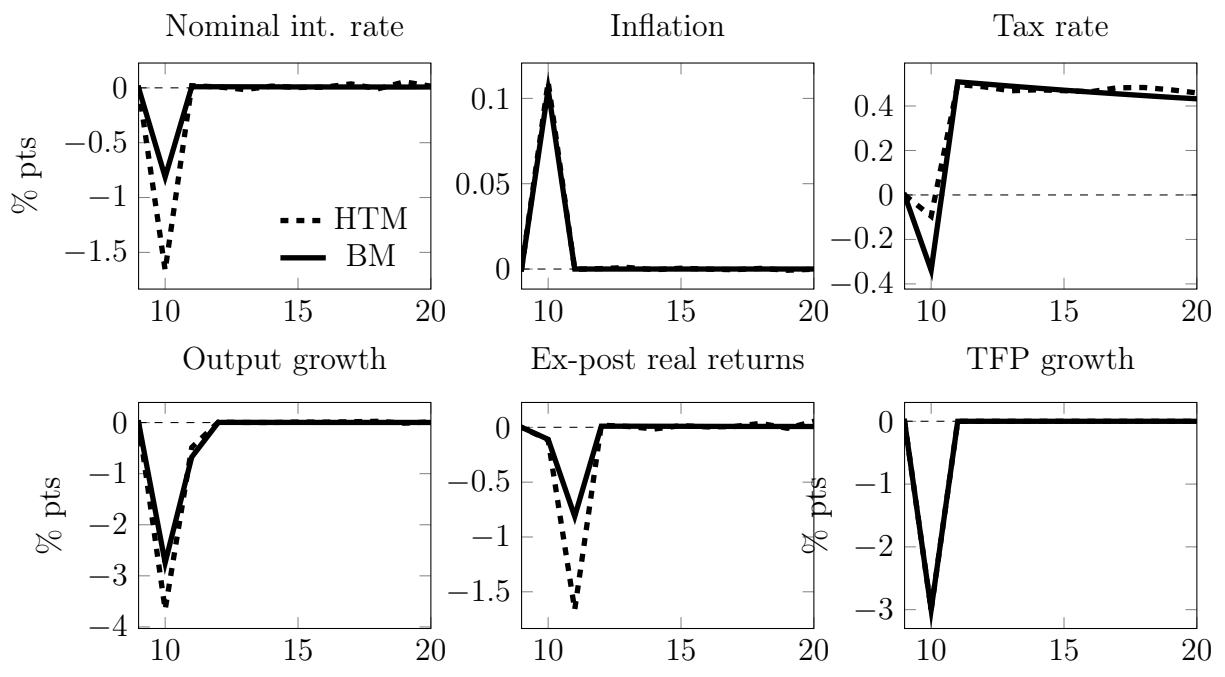
Figure XIII: Optimal monetary-fiscal response to a productivity shock with hand to mouth agents.