A multi-horizon perspective

- Normative analysis of optimal portfolio allocation for horizons beyond a single time interval
  - is relevant if returns are not i.i.d.
  - is difficult

- Result: researchers/practitioners often resort to tools developed for myopic situations

- Changing the perspective:
  - analyze returns at different frequencies (multi-horizon returns; MHR)
  - time-varying conditional return dynamics show up in unconditional moments of MHR
  - example from prior literature: variance ratios
This paper

- Use MHR to test asset-pricing models
  - Frame models in terms of the stochastic discount factor (SDF)
  - SDF encodes risk-return trade-off across multiple assets and multiple horizons

- Develop GMM estimation / test using MHR as input

- Application: linear factor models $M = a - b^\top F$
  - $F$ is vector of excess returns. Under null, $b^\top F$ is MVE
  - Find pervasive misspecification, large long-run pricing errors
  - Adding multi-horizon returns to test assets (overidentifying restrictions) allow us to correct dynamic properties of SDF
Multi-horizon returns and the SDF

- SDF from $t - 1$ to $t$, $M_t$, satisfies
  \[ E_{t-1}(M_{t-1,t} R_{t-1,t}) = 1, \]
  where $R_t$ is a one-period gross return on an asset (LOOP)

- Consider a candidate SDF that prices a set of assets unconditionally
  \[ E(M_{t-1,t} R_{t-1,t}) = 1 \]

- What do we gain by testing model on multi-horizon returns?
  - E.g., test $E(M_{t-1,t} M_{t+1,t+1} R_{t-1,t} R_{t,t+1}) = 1$?
Multi-horizon returns: A conditional test of the SDF

- Consider two-period returns:

\[
E(M_{t-1,t} M_{t,t+1} R_{t-1,t} R_{t,t+1}) = E(M_{t-1,t} R_{t-1,t}) \cdot E(M_{t,t+1} R_{t,t+1}) + \text{Cov}(M_{t-1,t} R_{t-1,t}, M_{t,t+1} R_{t,t+1})
\]

- Additional restriction: \( \text{Cov}(M_{t-1,t} R_{t-1,t}, M_{t,t+1} R_{t,t+1}) = 0 \)
  
  - Follows from \( E_{t-1}(M_{t-1,t} R_{t-1,t}) = 1 \)
  
  - Multi-horizon returns uncover persistent pricing errors

- Moment conditions

\[
f_{t+1} = \begin{pmatrix}
M_{t,t+1} R_{t,t+1} - 1 \\
M_{t-1,t} R_{t-1,t} \cdot (M_{t,t+1} R_{t,t+1} - 1)
\end{pmatrix}
\]

  - Why? \( E(f_t f_{t+1}) = E(f_t E_t(f_{t+1})) = 0 \)
Linear Factor Models

\[ M_{t,t+1} = a - b^\top F_{t,t+1} \]

Under null, \( b^\top F_{t,t+1} \) is unconditional MVE portfolio

Standard: CAPM, CAPM+BAB, FF3, FF5

Alternative: DMRS versions of FF; volatility managed FF5; Stambaugh and Yuan

Sample: monthly, 1963 – 2017

Test assets:

- Gross risk-free rate
- Excess returns to factors at horizons 1, 2, 3, 6, 12, and 24 months
Cumulative autocorrelations and variance ratios of factors

(A) Fama-French factors

(B) DMRS factors

(C) Stambaugh-Yuan factors

(D) Variance ratios
Term structure of pricing errors, $E(M \cdot (R - R_f))$

(A) MKT + BAB

(B) Stambaugh-Yuan

(C) Fama-French

(D) DMRS
GMM tests of linear factor models

- MKT
- BAB
- FF3
- FF5
- FF5
- MOM
- FF3
- DMRS
- FF5
- DMRS
- SY
- FF5

p-values from bootstrapped MHR-based model tests
Single-horizon Max Sharpe ratio vs. multi-horizon errors
Long-run factor mispricing as monthly factor return instruments: GRS tests

Test asset indexed $i = (k, h)$ is a strategy

$$R_{i,t,t+1} - R_{f,t,t+1} \equiv z_{k,t}^{(h)} F_{k,t,t+1}, \quad z_{k,t}^{(h)} = M_{t-h,t} R_{k,t-h,t}$$

Run the time-series regression

$$R_{i,t,t+1} - R_{f,t,t+1} = \alpha_i + \beta_i^\top F_{t,t+1} + \varepsilon_{i,t+1}$$

for each test asset $i$ and report the standard joint (GRS) test that $\alpha_i = 0$ for all $i$.

- We reject all models, except $FF5_{VolMan}$
- MacKinlay (1995) unbiased IR’s average about 0.5
  - Compare to model SR’s ranging from about 0.5-1.5
A fix?

- Conditional model will work, given our test assets.
  - I.e., \( M_{t,t+1} = a_t + b_t^\top F_{t,t+1} \)
  - Next, we use the multi-horizon returns to help estimate \( b_t \)
  - \( a_t \) nailed using the conditional risk-free rate

- Applying conditional Law of One Price, we have
  \[
  b_t = \left[ R_{f,t,t+1} V_t(F_{t,t+1}) \right]^{-1} E_t(F_{t,t+1})
  \]
  - Hard to estimate \( E_t(F_{t,t+1}) \)
  - Alternative in 4 steps

1. Assume single-factor structure of \( b_t \) regardless of the model
2. Construct in-sample MVE portfolio \( \tilde{F}_{t,t+1} = b^\top F_{t,t+1} \)
3. Construct “observed” \( b_t : \tilde{b}_{t+1} = \left[ R_{f,t,t+1} V_t(\tilde{F}_{t,t+1}) \right]^{-1} \tilde{F}_{t,t+1} \)
4. Posit \( \tilde{b}_{t+1} \) to be an “AR(12)”:

\[
\begin{align*}
\tilde{b}_{t+1} &= c_0 + \sum_{j=1}^{12} c_j \tilde{b}_{t-j+1} + c_{13}[R_{f,t,t+1} V_t(\tilde{F}_{t,t+1})]^{-1} + e_{t+1}, \quad c_3 = \ldots = c_{12} \\
\quad b_t &= E_t \tilde{b}_{t+1}
\end{align*}
\]
GMM tests of linear factor models with time-varying $b$

- **MKT**
- **BAB**
- **FF3**
- **FF5**
- **MOM**
- **DMRS**
- **SY**
- **VolMan**

$p$-values from bootstrapped MHR-based model tests
MVE portfolio Sharpe ratios when $b$ is constant and time-varying

![Bar chart showing Sharpe ratios for different portfolios with constant and time-varying $b$.](chart.png)

- **SR: Constant $b_t$**
- **SR: Time-varying $b_t$**
- **IR**

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<th>Portfolio</th>
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Time-series of price of risk $\lambda_t$, averaged across models
Dynamics of time-varying prices of risk: Relation to standard instruments

\[ R^2 = 9.6\% \]
Conclusion

- Literature typically only characterizes one-horizon risk-return trade-off
  - Economically, we care also about multi-horizon risk-return trade-off

- MHR-based tests are tests of models’ conditional implications
  - Use models’ overidentifying restrictions, no auxiliary variables introduced
  - Tests are informative, reject state-of-the-art empirical factor models
  - Information Ratios of about 0.6 – 0.8

- MHR-based estimation of conditional prices of risk present a challenge to standard models
  - Substantial variation in risk prices
  - Low in recessions and when market return variance is high
  - Positively related to term spread, but most variation not accounted for