APPENDICES:

Appendix A: Estimation of the state-space system

Data

All data are of quarterly frequency, spanning 1971Q1:2017Q4. All series used in this exercise are for a sample of OECD economies and are produced by the OECD and can be downloaded from the OECD website. For more details on the OECD data, see the OECD Economic Outlook data inventory.

To construct the observed advanced economy interest rate, we use long-term interest rates (10-year government bond yields; database here). As explained in the main text, we calculate the arithmetic average of these interest rates across the unbalanced panel of 36 OECD countries. Our results are unchanged if a median interest rate is used.

The inflation series is the non-food, non-energy consumer price index for OECD-total sample (database metadata available here). We construct the measure of inflation expectations as a moving average of observed core inflation rates over the past four quarters. This, of course, is not an ideal measure, but it follows past efforts in this literature and allows for estimation using the data from a large bloc of countries (alternative measures of inflation expectations for a large sample of countries going this far back are not available).

Finally, the GDP data cover the OECD-total sample. These are seasonally adjusted real (constant prices) measures, calculated using fixed 2010 PPPs, and expressed in 2010 US dollars. The series we use are calculated using the expenditure approach (OECD series code VPVOBARSA).

Estimation

The estimation is a recursive process that starts with a guess for the unconditional mean and variance of the unobservable state variables (such as the equilibrium real rate r^* or the level of potential output y^*) in the initial period. These are used to produce forecasts for the observables (such as inflation π or actual output y) next period. For every $t > 1$, the procedure consists of two steps. First, the update step changes the best guess for the unobservable states based on the forecast error on the set of observables. The direction of the revision is determined by the covariance of the observable and unobservable states, and the size of the update is determined by the variance of the observable (higher variance means there is more noise relative to signal on average, which reduces the size of the update). In the second step, this latest information is used to produce a new forecast for next period.

Following the classic notation of Hamilton (1994), we can write equations 5-3 in the state-space form as follows:

$$\mathbf{y}_t = \mathbf{A}'\mathbf{x}_t + \mathbf{H}'\xi_t + \mathbf{v}_t \tag{28}$$

$$\xi_t = \mathbf{F}\xi_{t-1} + \epsilon_t. \tag{29}$$

In this system, $\mathbf{x}_t$ is the vector of exogenous or lagged state variables; ξ_t is the vector of endogenous states; and $\mathbf{v}_t$ and ϵ_t are the vector of uncorrelated Gaussian disturbances. Specifically in this model, after substituting equation (1) into (5), (28) contains two observation equations (equations (5) and (6)) and (29) are the three state equations: (2), (3) and (4).

Following Holston et al. (2017b) and as explained in detail in Holston et al. (2017a), the estimation is carried out in three stages, building up to the full model. This is done to avoid the downward bias in the estimates of the standard deviations of the shocks to z and the trend growth rate. Instead of estimating these parameters directly, they are constructed from the first and second stages and imposed in the final estimation stage. For more details, see the referenced papers.

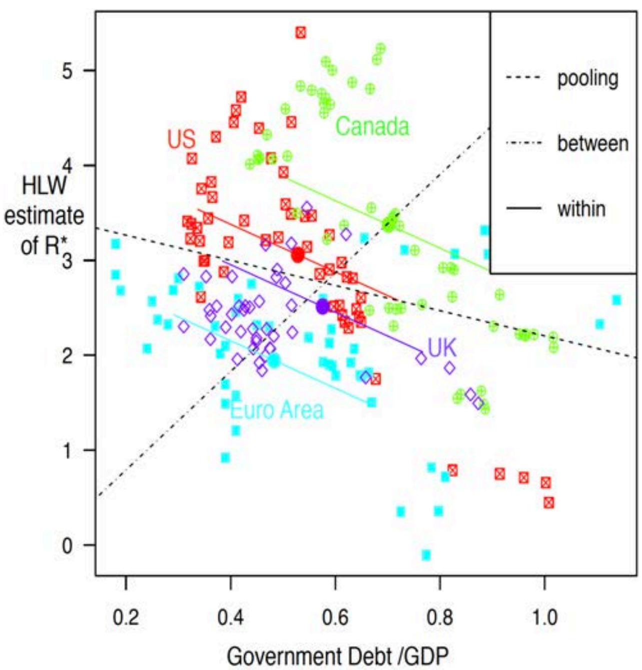
Appendix B: Illustration of the difficulties estimating the link between government debt and R*

In this Appendix we showcase the difficulties of estimating the causal link between government debt and R* described in the main text. For the purpose of this illustration, we construct a panel of equilibrium interest rates and government debt / GDP ratios for four large developed economies: the US, Canada, Euro Area and the UK. We use the estimates of R* estimated by Holston et al. (2017b), and study how these estimated equilibrium interest rates vary with the headline measure of government debt in these four economies. We run three regression specifications: the pooled regression, equivalent to OLS on the entire dataset, which effectively ignores the panel-structure of the data; the fixed effects or within regression, which controls for constant unobserved heterogeneity at the economy-level; and a 'between' regression, based on economy-level averages.[31]

Figures 12 and 13 contain the results. The difference between the two figures is the dependent variable: Figure 12 uses the estimate of R* from Holston et al. (2017b) as a dependent variable, while Figure 13 uses their estimate of the unobserved component z (which excludes the effects of declining trend growth). The latter specification is motivated by the idea that it may be easier to deduce the relationship between R* and debt after accounting for trends in productivity. In

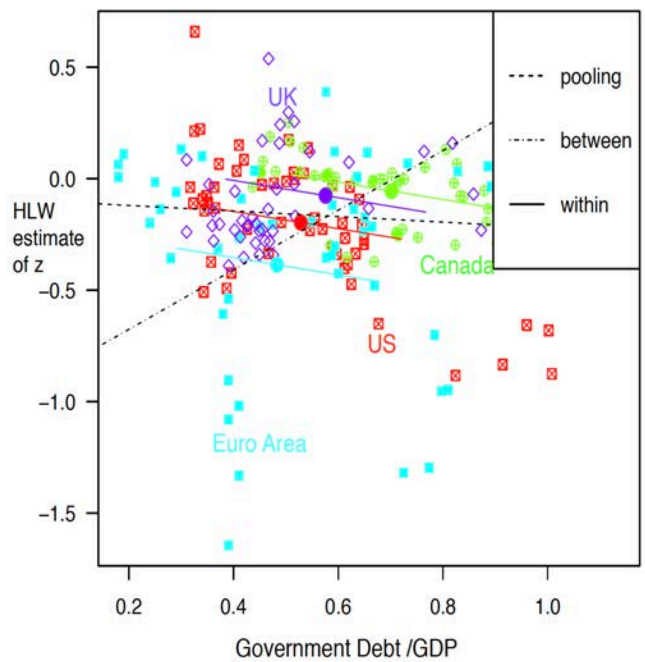
[31] Our panel comprises of annual observations spanning 1961-2013 for the US, Canada and the UK, and 1972-2013 for the Euro Area. We proxy for the government debt in the EA using data for Germany.

Figure 12: Panel regressions of R* and government debt



Source: Holston et al. (2017b) and Jorda-Schularick-Taylor database Jordà et al. (2016).

Figure 13: Panel regressions of the z component of R* and government debt



Source: Holston et al. (2017b) and Jorda-Schularick-Taylor database Jordà et al. (2016).

each figure, the different-colored squares are the data points for the four economies, and the large blurbs denote the economy-level averages. The three different kinds of lines show the estimated relationships: the broken line is the pooled regression, the solid colored lines show the fixed effects model, and the upward sloping dash/dot line shows the 'between' model. The notable result is that both in the pooled and fixed effects models the regression lines are downward sloping: the secular trend in interest rates, which coincided with increasing government debt, dominates these econometric estimates. Only the 'between' model detects a positive relationship between debt and R*, but the inference is extremely limited as the 'between' regression uses only four data points, each corresponding to one economy.[32]

Our simple exercise brings to the fore the difficult challenge that the empirical literature needs to overcome to uncover the link between debt and interest rates, namely that the secular fall in interest rates coincided with a rapid increase in advanced nations' public debt, making identification using measured macro data problematic. In the main text we discuss the papers that overcome the identification difficulties by including detailed measures of the output gap,

[32] Taken with an appropriately sized pinch of salt, the 'between' estimate suggests that a 1pp increase in government debt / GDP ratio raises the equilibrium rate of interest by about 5bps.

inflation expectations and portfolio shifts in their regressions or use fiscal forecasts rather than the realized debt/GDP ratios to alleviate the concern that cyclical variation drives the results.

Appendix C: The life-cycle model derivations

Retirees

The first order conditions of the retiree's problem yield the Euler Equation:

$$C_{t+1}^{rjk} = (R_{t+1}\beta)^{\sigma} C_t^{rjk} \tag{30}$$

Denoting by $\epsilon_t \pi_t$ the retiree's marginal propensity to consume out of wealth[33], we can write down retiree's consumption function as:

$$C_t^{rjk} = \epsilon_t \pi_t (R_t/\gamma) A_t^{rjk} \tag{31}$$

Plugging this expression into the Euler Equation yields the expression for the evolution of the retiree's MPC:

$$\epsilon_t \pi_t = 1 - (R_{t+1}^{\sigma-1} \beta^{\sigma} \gamma) \frac{\epsilon_t \pi_t}{\epsilon_{t+1} \pi_{t+1}}. \tag{32}$$

Workers

The Euler Equation from the worker's problem is:

$$\omega C_{t+1}^{rjk} + (1-\omega)\Lambda_{t+1} C_{t+1}^{rj(t+1)} = (R_{t+1}\Omega_{t+1}\beta)^{\sigma} C_t^{wj} \tag{33}$$

where Λ is the marginal rate of substitution across consumption while being a worker and a retiree, and Ω is a weighing factor which captures the fact that workers discount future more: $\Omega_{t+1} = \omega + (1-\omega)\epsilon_{t+1}^{\frac{1}{1-\sigma}}$.[34]

Denoting the MPC of the worker by π, and conjecturing that the consumption function takes the form:

$$C_t^{wj} = \pi_t (R_t A_t^{wj} + H_t^j + S_t^j) \tag{34}$$

(where H stands for human wealth and S is Social Security wealth, given respectively by $H_t^j =$

[33] Reasons for this notation will become clear momentarily.

[34] For the complete derivation of worker's Euler Equation, see the Appendix of Gertler (1999).

$\sum_{\nu=0}^{\infty} \frac{W_{t+\upsilon}-T_{t+\nu}}{\prod_{z=1}^{\upsilon} R_{t+z}\Omega_{t+z}/\omega}$ and $S_t^j = \sum_{\nu=0}^{\infty} \frac{E_{t+\nu}^j}{\prod_{z=1}^{\upsilon} R_{t+z}\Omega_{t+z}/\omega}$), we obtain the time path of worker's MPC :

$$\pi_t = 1 - (R_{t+1}\Omega_{t+1})^{\sigma-1}\beta^{\sigma}\frac{\pi_t}{\pi_{t+1}}. \tag{35}$$

Aggregation

Marginal propensities to consume are the same across all retirees, so we can just add up their consumptions to get the aggregate consumption function. With slight abuse of notation, denoting now by C_t^r, A_t^r and S_t the aggregate variables, we have that:

$$C_t^r = \epsilon_t\pi_t(R_tA_t^r + S_t) \tag{36}$$

where Social Security wealth is given by the discounted sum of Social Security payments:

$$S_t = \sum_{\nu=0}^{\infty} \frac{E_{t+\nu}}{\prod_{z=1}^{\upsilon}(1+n)R_{t+z}/\gamma}. \tag{37}$$

The evolution of $\epsilon_t\pi_t$ is governed by the Euler Equation of the retiree, given by equation (32).

Marginal propensities to consume are the same across all workers, so we can add individual worker consumption across individuals to obtain their aggregate consumption function. This will depend not only on the aggregate asset wealth but also on the aggregate human wealth and aggregate Social Security wealth:

$$C_t^w = \pi_t(R_tA_t^w + H_t + S_t^w). \tag{38}$$

The aggregate human wealth is given by

$$H_t = \sum_{\nu=0}^{\infty} \frac{N_{t+\nu}W_{t+\nu} - T_{t+\nu}}{\prod_{z=1}^{\upsilon}(1+n)R_{t+z}\Omega_{t+z}/\omega}. \tag{39}$$

Human wealth is a discounted sum of the economy-wide net-of-tax wage bill. The discount rate that is applied to the aggregate wage bill is the product of the gross population growth rate and the rate at which individual workers discount their labor income. The importance of the generation currently alive declines over time, however – they get replaced by newly born generations. So from the point of view of the current generation, the human wealth is discounted more heavily than in the infinite horizon case – the gross population growth rate $(1+n)$ enters the discount factor. In total, therefore, there are three distinct factors in the life-cycle setting that raise the discount rate on future labor income (relative to the infinite horizon case). They are: (1) finite expected time spent working (reflected by the presence of ω in the discount rate);

(2) greater discounting of the future owing to expected finiteness of life (reflected by the presence of Ω); and (3) growth of the labor force (reflected by the presence of $(1+n)$).

Social Security wealth of the workers is:

$$S_t^w = \sum_{\nu=0}^{\infty} \frac{(1-\omega)\omega^\nu N_t \left(\frac{\epsilon_{t+\nu+1} \frac{S_{t+\nu+1}}{\psi N_{t+\nu+1}}}{R_{t+\nu}\Omega_{t+\nu}} \right)}{\prod_{z=1}^{\nu} R_{t+z}\Omega_{t+z}}. \tag{40}$$

The numerator of the sum on the right hand side is a time-$t+\nu$ capitalized value of the Social Security payments to all the individuals who were in the workforce at t and retire at $t+\nu+1$. The total Social Security wealth is just the infinite sum of the discounted value of these capitalized payments.

Denoting by λ the share of assets held by retirees, we can add the two aggregate consumptions above to get aggregate consumption in the main text:

$$C_t = C_t^w + C_t^r = \pi_t \left\{ (1-\lambda_t) R_t A_t + H_t + S_t^w + \epsilon_t (\lambda_t R_t A_t + S_t^r) \right\} \tag{41}$$

The novel feature is the presence of λ. Because the MPC of retirees is higher than MPC of workers ($\epsilon > 1$), higher λ raises aggregate consumption. So transferring resources across the demographic groups changes overall demand.

The evolution of total wealth of retirees is the sum of return on their wealth from last period plus what the newly retired bring in:

$$\lambda_{t+1} A_{t+1} = \lambda_t R_t A_t - C_t^r + (1-\omega)[(1-\lambda_t) R_t A_t + W_t - C_t^w] \tag{42}$$

From this we get the explicit expression for the evolution of the retiree share:

$$\lambda_{t+1} = \omega(1-\epsilon_t \pi_t)\lambda_t R_t \frac{A_t}{A_{t+1}} + (1-\omega). \tag{43}$$

Appendix D: The model of precautionary savings: derivations and equilibrium

Equilibrium in the asset market

Equilibrium in the asset market requires that asset demand (households' desired asset holdings) equals asset supply (firms' capital plus government bonds):

$$A_t = K_t + B_t. \tag{44}$$

Because of exogenous technological progress, the equilibrium in this economy will be characterized by a balanced growth path along which the aggregate variables – K_t, w_t and Y_t – grow at rate η. Below we show how to rewrite the model with variables normalized by GDP, thus making it stationary.

Transformation into a stationary model

Growth is exogenous, driven by increases in labor augmenting technology: $x_t = e^{\eta t}x_0$. In the balanced growth equilibrium w_t, Y_t and K_t will be growing at rate η whereas the interest rate will be constant.

Let $k_t = K_t/Y_t$, $\tilde{w}_t = w_t/Y_t$, $\tilde{c}_t = c_t/Y_t$, $\tilde{a}_t = a_t/Y_t$, $\tau_t = T_t/Y_t$, $b_t = B_t/Y_t$, $\bar{a}_t = A_t/Y_t$, $tran_t = \frac{TR_t}{Y_t}$ denote the normalized variables.

Households. We begin by rewriting the consumer problem. First, note that $c_t = Y_t\tilde{c}_t$ and $Y_t = e^{\eta t}Y_0$. We can rewrite the integral as:

$$\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}dt = \int_0^\infty e^{-\rho t}\frac{\left(e^{\eta t}Y_0\tilde{c}_t\right)^{1-\gamma}}{1-\gamma}dt = \frac{Y_0^{1-\gamma}}{1-\gamma}\int_0^1 e^{-(\rho-(1-\gamma)\eta)t}\tilde{c}_t^{1-\gamma}dt. \tag{45}$$

The original budget constraint is:

$$\dot{a}_t = (1-\tau)w_t e_t + (1-\tau)r_t a_t - c_t. \tag{46}$$

Dividing through by Y_t:

$$\frac{\dot{a}_t}{Y_t} = \frac{(1-\tau)w_t e_t}{Y_t} + \frac{(1-\tau)r_t a_t}{Y_t} - \frac{c_t}{Y_t}. \tag{47}$$

Note that

$$\dot{a}_t = \frac{\partial a_t}{\partial t} = \frac{\partial \tilde{a}_t}{\partial t}Y_t + \tilde{a}_t\frac{\partial Y_t}{\partial t} = \dot{\tilde{a}}_t Y_t + \tilde{a}_t Y_0 \eta e^{\eta t}. \tag{48}$$

It follows that

$$\frac{\dot{a}_t}{Y_t} = \dot{\tilde{a}}_t + \tilde{a}_t\eta. \tag{49}$$

Thus the budget constraint in transformed variables is:

$$\dot{\tilde{a}}_t = (1-\tau)\tilde{w}_t e_t + ((1-\tau)r-\eta)\tilde{a}_t - \tilde{c}_t \tag{50}$$

And the transformed problem of the household is:

$$\max_{\{\tilde{c}_t\}} \mathbb{E}\,\frac{Y_0^{1-\gamma}}{1-\gamma}\int_0^\infty e^{-(\rho-(1-\gamma)\eta)t}\tilde{c}_t^{1-\gamma}dt \tag{51}$$

subject to

$$\begin{aligned}\dot{\tilde{a}}_t &= (1-\tau)\tilde{w}_t e_t + ((1-\tau)r-\eta)\tilde{a}_t - \tilde{c}_t \\ \tilde{c}_t &\geq 0 \\ \tilde{a}_t &\geq 0.\end{aligned}$$

This is a standard optimal control problem. Because the individual problem is recursive, its stationary version can be summarized with a Hamilton-Jacobi-Bellman (HJB) equation:[35]

$$(\rho-(1-\gamma)\eta)v_j(\tilde{a}) = \underset{\tilde{c}}{max}\left\{\frac{\tilde{c}^{1-\gamma}}{1-\gamma} + v_j'(\tilde{a})((1-\tau)\tilde{w}e_j + ((1-\tau)r-\eta)\tilde{a} - \tilde{c}) + \sum_{i\neq j} P_{j,i}v_i(\tilde{a}) - P_{j,j}v_j(\tilde{a})\right\} \tag{52}$$

where the variables with a tilde are normalized by GDP, and $P_{j,i}$ is the Poisson intensity of a change from state $e = z_j$ to state $e = z_i$. This equation has a natural economic interpretation, related to the intuition from the asset pricing literature: the required return to an asset equals the dividend plus the change in value. The left hand side of the equation is the instantaneous required return to holding assets $\tilde{a}$ in state j: it is the effective discount rate (i.e. the return $\rho-(1-\gamma)\eta)$ times the value function. The first term on the right is the 'dividend': the stream of consumption utility sustained by the given level of asset holdings. The remaining terms denote the instantaneous changes in value, due to asset accumulation and a possibility of a Poisson event that changes the state from j to i.

We solve the problem summarized in equation (52) by deriving the first order and the boundary conditions. The first order condition is obtained by differentiating (52) with respect to $\tilde{c}$:

$$\tilde{c_j}^{-\gamma} = v_j'(a). \tag{53}$$

One of the advantages of the continuous time formulation is that equation (53) holds at the constraint. We can use this to derive the boundary condition:[36]

$$v_j'(\underline{a}) \geq ((1-\tau)\tilde{w}e_j + ((1-\tau)r-\eta)\tilde{a})^{-\gamma}. \tag{54}$$

Using these two conditions we solve the HJB equation (52) utilizing the methods described in

[35] We focus our attention on stationary equilibria, so that the value function or any other variable in the HJB equation do not depend on time.

[36] To see this, note that equation (53) holding at the constraint implies that $\tilde{c_j}(\underline{a})^{-\gamma} = v_j'(\underline{a})$ where the notation $c_j(\underline{a})$ stands for the policy function in state e_j and assets $\underline{a}$. At the borrowing constraint saving cannot be negative (otherwise the constraint would be breached), so that we must have $\dot{\underline{a}} = (1-\tau)\tilde{w}e_j + ((1-\tau)r-\eta)\underline{\tilde{a}} - \underline{\tilde{c}} \geq 0$, which implies $(1-\tau)\tilde{w}e_j + ((1-\tau)r-\eta)\underline{\tilde{a}} \geq \tilde{c}(\underline{\tilde{a}})$. This, together with the concavity of utility function and equation (53), imply the boundary condition (54).

detail in Achdou et al. (2017).

Production. All markets are competitive. The aggregate production function is

$$Y_t = AK_t^{\alpha}(x_t L_t)^{1-\alpha} \tag{55}$$

where $x_t = e^{\eta t}x_0$ is the process for exogenous labor augmenting technical progress, which grows at rate η.[37] Because we normalized population to be of measure one, we have $L_t = 1$. Capital and labor demands are pinned down by the usual first order conditions:

$$\alpha A K_t^{\alpha-1} x_t^{1-\alpha} = r + \delta \tag{56}$$

$$(1-\alpha)AK_t^{\alpha} x_t^{1-\alpha} = w_t. \tag{57}$$

Capital demand is

$$K_t = \left(\frac{\alpha A}{r+\delta}\right)^{\frac{1}{1-\alpha}} x_t. \tag{58}$$

Dividing through by Y_t we get

$$k_t = \left(\frac{\alpha A}{r+\delta}\right)^{\frac{1}{1-\alpha}} \frac{x_t}{Y_t}. \tag{59}$$

Because x and Y both grow at rate η, this is the same as:

$$k_t = \left(\frac{\alpha A}{r+\delta}\right)^{\frac{1}{1-\alpha}} \frac{x_0}{Y_0}. \tag{60}$$

Similarly, labor demand is given by:

$$(1-\alpha)Ak_t^{\alpha}\left(\frac{x_0}{Y_0}\right)^{1-\alpha} = \tilde{w}_t. \tag{61}$$

When solving the model, we normalize the starting values x_0 and Y_0 to unity.

Government. Using the homogeneity of the production function, we can rewrite the government budget constraint (25) as follows. By Euler's Theorem we have:

$$w_t = Y_t - (r+\delta)K_t. \tag{62}$$

This and the asset market clearing $K + B = A$ together imply that (25) becomes:

$$\dot{B}_t = G_t + TR_t + rB_t - \tau(Y_t - (r+\delta)K_t + rK_t + rB_t). \tag{63}$$

Simplifying:

[37] To see this, note that $\frac{\dot{x}_t}{x_t} = \frac{\partial x_t}{\partial t}\frac{1}{x_t} = \eta$.

$$\dot{B}_t = G_t + TR_t + (1-\tau)r_t B_t - \tau(Y_t - \delta K_t). \tag{64}$$

Dividing through by Y_t and rearranging we get the transformed government budget constraint:[38]

$$g_t + tran_t + ((1-\tau)r_t - \eta)b_t - \dot{b}_t = \tau(1-\delta k_t) \tag{65}$$

In steady state, government debt / GDP ratio is constant, so that for given values of r, b and g, the government budget constraint pins down the tax rate:

$$\tau = \frac{g + tran + b(r-\eta)}{1 - \delta k + rb}. \tag{66}$$

In the main text we set $TR_t = 0\ \forall t$.

Stationary Equilibrium

The following set of equations fully characterizes the stationary equilibrium:

- The HJB equation summarizing the household's problem:

$$(\rho - (1-\gamma)\eta)v_j(\tilde{a}) = \max_{\tilde{c}} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + v_j'(\tilde{a})((1-\tau)\tilde{w}e_j + ((1-\tau)r - \eta)\tilde{a} - \tilde{c}) + \sum_{i \neq j} P_{j,i} v_i(\tilde{a}) - P_{j,j} v_j(\tilde{a}) \right\} \tag{67}$$

- The Kolmogorov Forward Equations which characterize the distributions of workers in the four income states. In stationary equilibrium these take the following form:

$$0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \sum_{i \neq j} \lambda_i g_i(a) \tag{68}$$

where $s_j(a)$ is the saving policy function from the HJB equation and $g_j(a)$ denotes the distribution (density) of type-j worker, so that

$$\int_{\underline{a}}^{\infty} (g_1(a) + g_2(a) + g_3(a) + g_4(a))\, da = 1, \qquad g_j \geq 0\ \forall j. \tag{69}$$

- The asset market clearing condition (expressed using the transformed variables):

$$\bar{a} = \sum_j \int_{\underline{a}}^{\infty} a g_j(a) da = k + b. \tag{70}$$

[38] Note that $\frac{\dot{B}_t}{Y_t} = \frac{\frac{\partial \dot{B}_t}{\partial t}}{Y_t} = \frac{\dot{b}_t Y_t + b_t \dot{Y}_t}{Y_t} = \dot{b}_t + b_t \eta$.

Appendix E: Sensitivity of the model-based results to alternative parametrization

The key parameter that determines the overall sensitivity of the interest rate to long-term fiscal stance as well as other secular trends, such as technological and demographic change, is the intertemporal elasticity of substitution, $\frac{1}{\gamma}$. In general, the higher this elasticity, the smaller the impact of a change in the macroeconomic environment on the interest rate. The intuition is that, if consumers are very willing to substitute consumption across time, smaller changes in the interest rate will be sufficient to induce them to do so. So less of a change in the interest rate will be required to restore equilibrium.

Our simulations reported in the main text are based on the parametrization in which this elasticity is set equal to $\frac{1}{2}$, which is a standard value used in many macroeconomic models. It is also the average value of the estimated elasticity across a large number of studies described in a comprehensive review by Havranek et al. (2015). Still, some models assume a different elasticity. For example, the calibration of the Smets and Wouters (2007) model assumes that $IES = \frac{2}{3}$, and calibrations with lower IES are also common. To illustrate the sensitivity of our results, we now present the results of a robustness exercise in which we simulate the models under these alternative parametrizations.

Figure 14 illustrates how much of the decline in real rates the models can account for depending on the calibration of the IES. As we highlighted in the main text, under the central calibration, the models under-explain the overall decline. However, the sensitivity analysis shows that relatively small changes to the value of the parameters can result in large changes in the contributions of the different explanations. The conclusion that parameter uncertainty plays an important role is in line with the results of the detailed study of these sensitivities presented in Ho (2018).

Figure 14: Factors driving the decline in the equilibrium real interest rate in the life-cycle model between 1970-2018, for different values of the IES.