Appendix

Appendix A analyzes long difference estimators. Appendix B reports descriptive statistics for the empirical setting. Appendix C estimates the effects of climate change by industry. Appendix D reports robustness checks for the empirical analysis. Appendix E reports estimates for other weather variables. Appendix F contains proofs.

A Long Difference Estimators

Recognizing the difficulty of accounting for adaptation, some empirical literature averages outcomes over long timesteps, a procedure known as "long differences" (e.g., Dell et al., 2012; Burke and Emerick, 2016).[42] In order to obtain sharper results, assume in this appendix that specialized forecasts are available only one period in advance and that Σ is diagonal. I use Σ_{ij} to indicate element (i, j) of Σ. Define

$$\breve{\pi}_s \triangleq \frac{1}{\delta} \sum_{t=s}^{s+\delta-1} \pi_t$$

as average payoffs over δ timesteps beginning with $t = s$. Define $\breve{w}_s$ and $\breve{f}_{1,s}$ analogously. Consider the following regression:

$$\Delta\breve{\pi}_{js} = \breve{\alpha}_j + \breve{\Lambda}\Delta\breve{w}_{js} + \breve{\lambda}\Delta\breve{f}_{j1,s} + \breve{\eta}_{js},$$

with observations only every δ timesteps (i.e., no overlap in averaging intervals). The next proposition shows that estimating this regression does not generally get us closer to the effect of climate than did estimating regression (17) with δ lags:

Proposition A-1 (Long Differences)**.** *Let Σ be diagonal and $\Sigma_{33} = 0$. Then:*

$$\text{plim}\left(\hat{\breve{\Lambda}} + \hat{\breve{\lambda}}\right) = \text{plim} \sum_{i=0}^{\delta-1} \frac{\delta - i}{\delta} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] + \text{plim} \frac{1}{\delta} \sum_{i=0}^{\delta-1} \frac{\Sigma_{22}}{\Sigma_{11} + \Sigma_{22}} \hat{\lambda}_{i+1}. \tag{A-1}$$

If $\Psi > 0$, then

$$\text{plim}\left(\hat{\breve{\Lambda}} + \hat{\breve{\lambda}}\right) = \bar{\pi}_w + \breve{\omega}\left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1 - g}\right] \left(\frac{\partial \bar{A}(K, C)}{\partial C} + \Omega\right),$$

where $\breve{\omega} \in (0, \omega_\delta)$, with $\omega_\delta \in (0, \omega)$ from Corollary 3 and ω and Ω from Proposition 2.

Proof. See Appendix F.15. □

[42] The subsequent analysis does not depend on whether the operation is summing or averaging.

Even though equation (A-1) does not explicitly include lags on the left-hand side, the estimated coefficients $\mathring{\hat{\Lambda}}$ and $\mathring{\hat{\lambda}}$ do incorporate effects of lagged weather and lagged forecasts owing to correlations between payoffs and lagged weather and forecasts within a timestep (see also Ghanem and Smith, 2021). As a result, $\mathring{\hat{\Lambda}} + \mathring{\hat{\lambda}}$ bears some resemblance to summing δ lags from regression (17). However, only observations at the very end of each long timestep have a full δ lags within the same timestep. All other observations have fewer than δ lags implicitly estimated. Equation (A-1) shows that the long difference coefficient is analogous to a summation over downweighted versions of the lag coefficients from regression (17).

As in Corollary 3, the bias introduced by $\breve{\omega}$ is particularly easy to sign when $\Psi > 0$. In this case, Corollary 3 showed that summing the first δ lags amplified the bias from historical restraints relative to summing infinite lags. We now see that implicitly summing these lags through long timesteps further amplifies that bias because nearly all observations within the long timestep have fewer than δ lags. Long differences are not generally superior to simply estimating a standard panel model with δ lags and summing the coefficients.[43]

Researchers sometimes compare long difference estimates to standard panel estimates in order to learn whether whether short-run adaptation differs from long-run adaptation. The hope is that long difference estimators are identified by spatially heterogeneous rates of climate change that manifest over decades. However, long difference estimators are in fact identified in the foregoing analysis even though there is, by construction, no climate change in the present setting (C is here constant over agents and over time). In fact, they are identified by random differences in sequences of the same transient weather shocks that identify panel estimators such as (17). This source of identification is unavoidable in applications, whether or not there is also variation in C. At best, long difference estimators conflate the identifying variation of transient weather shocks with differential rates of climate change, but at worst, they capture nothing but the familiar identifying variation of transient weather shocks. We should judge the latter case to be especially likely when long difference and panel estimators produce similar results, as has been reported in previous work (summarized in Hsiang, 2016).

B Descriptive Statistics

Table A-1 reports descriptive statistics for the empirical application, broken down by Census Region. The two panels correspond to the time periods covered by the regressions with output per capita and income per capita.

[43] Comparing long difference estimates to panel estimates with few lags does tell us something about the importance of historical restraints ($\breve{\omega}$ vs $\omega_{I'}$), which relates to the difference between long-run and short-run adaptation, but so too would simply changing the number of lags used, per Corollary 3.

Table A-1: Descriptive Statistics

	Midwest		Northeast		South		West	
	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev	Mean	Std Dev
				Years: 2002–2019				
Output ($2012,Billion)	3.16	13.62	15.28	40.26	3.91	14.28	9.13	37.90
Output p.c. ($2012,Thous)	40.78	19.83	45.51	25.39	59.90	698.59	49.07	57.27
Change in log output p.c.	0.02	0.10	0.01	0.04	0.01	0.09	0.01	0.10
Days Below 20 degF	33.55	24.38	26.10	19.21	1.98	3.84	18.20	20.70
Days In 20–50 degF	142.05	17.41	157.56	13.02	92.13	37.14	170.38	47.20
Days In 70–80 degF	61.86	21.90	46.88	22.48	91.17	21.33	35.63	30.54
Days Above 80 degF	11.54	13.54	3.29	6.36	50.59	36.94	6.50	19.25
Change in Days Below 20 degF	0.55	14.85	0.47	14.09	0.00	3.12	0.28	9.96
Change in Days In 20–50 degF	0.50	21.00	-0.25	16.98	0.06	13.86	0.24	15.49
Change in Days In 70–80 degF	0.47	16.05	0.30	14.85	-0.43	21.54	-0.22	11.39
Change in Days Above 80 degF	-0.31	9.11	0.00	5.29	1.37	18.21	-0.02	5.17
				Years: 1970–2019				
Income p.c. ($2012,Thous)	28.96	9.68	32.95	13.07	25.79	9.64	29.39	11.72
Change in log income p.c.	0.02	0.08	0.02	0.03	0.02	0.06	0.02	0.06
Days Below 20 degF	34.53	24.30	27.97	19.44	2.54	4.55	19.12	21.71
Days In 20–50 degF	141.52	17.56	159.55	13.58	92.83	36.95	172.71	46.39
Days In 70–80 degF	59.71	21.00	42.48	21.58	90.47	22.07	32.49	30.35
Days Above 80 degF	11.62	13.25	2.71	5.58	45.91	35.27	5.43	17.80
Change in Days Below 20 degF	0.05	15.40	-0.05	12.99	-0.01	3.48	0.02	11.73
Change in Days In 20–50 degF	0.01	19.69	-0.09	17.27	-0.33	13.61	0.02	17.99
Change in Days In 70–80 degF	0.12	16.10	0.08	14.43	0.01	21.48	0.06	11.61
Change in Days Above 80 degF	-0.02	10.21	0.04	4.82	0.41	18.92	0.02	5.02

C Industry-Level Results

This appendix extends the primary empirical specification from the main text to assess effects on log industry output per capita. Data are again from the Bureau of Economic Analysis, except now for each industry's output by county. I group industries using definitions from Table A2 in Colacito et al. (2019). Each industry is run in a separate regression. The only changes to the estimating equations are that they now estimate a single coefficient across Census regions and that the county weights are now the industry's initial log output instead of log population.

Table A-2 reports results from the ILS and OLS estimators for the effects of an extra day with average temperature above 80°F. For most industries, results are fairly similar across the two estimators. However, the ILS estimator predicts negative impacts that are much larger—and significantly different from—the OLS estimator's predictions for "Agriculture, forestry, fishing". And the ILS estimator predicts positive impacts that are significantly different from the OLS estimator's predictions for "Retail". Moreover, in each of these cases the estimated ratio of lagged coefficients is significantly positive (and, as implied by the theory, overlaps with values strictly less than 1). The indirect least squares estimator then implies that either actions are intertemporal complements (as with adjustment costs in capital stocks) or actions are weak intertemporal substitutes with a stock that is highly persistent. In either case, the theoretical analysis shows that ex-post adaptation to climate change will be greater than ex-post adaptation to the weather shocks observed in the data.

D Robustness

Tables A-3 and A-4 contain robustness checks. The main takeaways are fairly robust: extreme heat reduces both output per capita and income per capita in the Midwest, ILS estimates in the Northeast are imprecise, estimated effects in the South are small, extreme heat reduces income per capita in the West, and the OLS estimator underestimates the steady-state harm from additional days with extreme heat.

The first two checks address time trends. The first check drops the state-specific quadratic time trends. This change makes the ILS estimate for output per capita in the South significantly positive (albeit still small). It also substantially increases the precision of the ILS estimate for income per capita in the Northeast while reducing precision for the ILS estimate for income per capita in the West. The second check drops the years around the Great Recession (2007–2009). Estimates of effects on log output per capita change a bit due to losing more than 15% of the sample. Effects on the Midwest are dampened a bit, and effects on the South become a bit stronger. Of most interest, the estimates for the Northeast become much more precise in this case. The central estimate suggests about twice as great a reduction in log output per capita from extreme heat as in the Midwest, and the 95% confidence interval now excludes zero. Dropping these years also flips the sign of the ILS estimate for effects

Table A-2: Indirect least squares (ILS) estimates of total effects for the effects of days above 80 degF, from Proposition 7 and regression (21), versus ordinary least squares (OLS) estimates, from Proposition 1 and regression (22).

	ILS	OLS	$\hat{\phi}_{2r4}/\hat{\phi}_{1r4}$
Agriculture, forestry, fishing	-0.0229	-0.0048	1.2923
	(-20.0193,-0.0078)	(-0.0062,-0.0035)	(0.7756,2.4207)
Communication/Information	-0.0006	-0.0004	-0.9317
	(-0.0015,-0.0002)	(-0.0006,-0.0002)	(-165.8393,2.2006)
Construction	0.0001	0.0003	0.5016
	(-0.0060,0.0012)	(0.0001,0.0006)	(-10.3796,6.5682)
Finance, insurance, real estate	0.0001	0.0002	-5.1294
	(-0.0000,0.0003)	(0.0001,0.0003)	(-186.2123,9.4091)
Government	-0.0001	-0.0002	-0.5031
	(-0.0002,0.0001)	(-0.0002,-0.0001)	(-14.2248,2.3279)
Manufacturing	0.0009	0.0005	0.0494
	(0.0002,0.0023)	(0.0003,0.0007)	(-4.6717,1.7291)
Mining	-0.0005	-0.0005	12.6152
	(-0.1115,0.0001)	(-0.0010,0.0000)	(5.1391,1861.6223)
Retail	0.0049	-0.0001	1.1893
	(0.0024,0.2667)	(-0.0001,0.0000)	(0.6079,3.4696)
Services	-0.0002	0.0001	-0.6739
	(-0.0013,0.0003)	(-0.0003,0.0004)	(-74.7743,1.7257)
Transportation	0.0002	0.0005	-1.0831
	(-0.0007,0.0007)	(0.0003,0.0007)	(-230.4232,0.9609)
Utilities	0.0004	0.0003	4.0635
	(-0.0006,0.1850)	(-0.0002,0.0008)	(1.5630,4977.6216)
Wholesale	0.0003	0.0000	-0.7007
	(-0.0001,0.0009)	(-0.0001,0.0002)	(-42.0460,0.5987)

95% confidence intervals (in parentheses) bootstrapped from 1000 samples with resampling at the county level.
Regressions weighted by each county's log output for that industry in the first year of the sample.
Years: 2002–2019.

on income per capita in the Northeast, suggesting that extreme heat is beneficial. Projections for the Northeast are, in both cases, sensitive to the Great Recession.

The next two checks assess the population weights. Either changing log population weights to log output weights (based on the first year of the sample) or dropping weights altogether does not strongly affect estimated effects on output per capita. Either dropping weights or using log income weights (based on the first year of the sample) does, however, increase the precision of the ILS estimate for effects on income per capita in the Northeast. ILS estimates for income per capita in the West are sensitive to the weighting scheme.

The next checks assess robustness to including longer lags in regression (21). For both output per capita and income per capita, the deleterious effects of extreme heat in the Midwest grow larger and even more distinct from the OLS estimate. The point estimates for effects on output per capita the South switch sign, now suggesting harmful (albeit still small) effects of extreme heat. The point estimates for effects on income per capita in the Northeast jump around a bit but are significantly negative in each case. The point estimates for effects on income per capita in the West lose their significance.

The next checks assess alternate approaches to clustering standard errors (equivalently, changing the level of resampling for the bootstrap). A first check replaces county clustering with state-year clustering (i.e., draws samples at the state-year level for the bootstrap) that accounts for unobserved correlation within a state's counties at a point in time. The confidence intervals and standard errors tend to grow. The effect of extreme heat on output per capita in the Midwest is no longer significant in this case, but the estimated harmful effect of extreme heat on income per capita in the Midwest, Northeast, and West remain significant. The estimated benefits of extreme heat on income per capita in the South are no longer significant. A second check drops clustering (i.e., draws samples at the county-year level for the bootstrap), so that standard errors are robust to arbitrary heteroskedasticity but not to correlation among the residuals. Standard errors and confidence intervals are broadly similar to the base specification.

The final checks reduce or increase the discount rate by 50% from its base rate of 15% per year. Either change alters the ILS estimate mechanically (see Proposition 7) and has no effect on the OLS estimator (which does not use the discount rate). In either case, the effects are quantitatively notable only in the Northeast, because the ratio that makes the estimates imprecise also determines sensitivity to the discount rate. Using a smaller (larger) discount rate makes the estimated effect on output per capita in the Northeast less (more) precise, and the smaller discount rate actually flips the sign of the estimated effect on income per capita in the Northeast.

Table A-3: Indirect least squares estimates (ILS) of total effects for the effects of days above 80 degF on log output pc, from Proposition 7 and regression (21), versus ordinary least squares (OLS) estimates, from Proposition 1 and regression (22).

	Midwest		Northeast		South		West	
	ILS	OLS	ILS	OLS	ILS	OLS	ILS	OLS
Base	-0.0019	-0.0008	-0.0014	-0.0003	0.0003	-0.0002	0.0007	0.0002
	(-0.0030,-0.0012)	(0.0001)	(-0.0328,0.0018)	(0.0001)	(-0.0000,0.0008)	(0.0001)	(-0.0001,0.0019)	(0.0002)
No Trends	-0.0019	-0.0008	-0.0017	-0.0003	0.0006	-0.0001	0.0008	0.0001
	(-0.0030,-0.0012)	(0.0001)	(-0.0537,0.0015)	(0.0001)	(0.0002,0.0012)	(0.0001)	(-0.0003,0.0021)	(0.0002)
Without 07–09	-0.0014	-0.0006	-0.0029	-0.0006	0.0008	-0.0001	0.0006	0.0001
	(-0.0023,-0.0008)	(0.0002)	(-0.0080,-0.0005)	(0.0001)	(0.0002,0.0018)	(0.0001)	(-0.0023,0.0020)	(0.0002)
LogOutput-Weighted	-0.0020	-0.0008	-0.0013	-0.0003	0.0003	-0.0002	0.0008	0.0002
	(-0.0031,-0.0013)	(0.0001)	(-0.0406,0.0018)	(0.0001)	(-0.0000,0.0009)	(0.0001)	(-0.0001,0.0019)	(0.0002)
Unweighted	-0.0021	-0.0009	-0.0015	-0.0003	0.0003	-0.0002	0.0007	0.0003
	(-0.0032,-0.0012)	(0.0002)	(-0.0319,0.0020)	(0.0002)	(-0.0001,0.0009)	(0.0001)	(-0.0022,0.0019)	(0.0002)
Five Lags	-0.0022	-0.0008	-0.0027	-0.0003	-0.0002	-0.0002	0.0007	0.0002
	(-0.0033,-0.0014)	(0.0001)	(-0.0442,0.0063)	(0.0001)	(-0.0004,0.0001)	(0.0001)	(-0.0002,0.0019)	(0.0002)
Six Lags	-0.0025	-0.0008	-0.0023	-0.0003	-0.0001	-0.0002	0.0006	0.0002
	(-0.0041,-0.0014)	(0.0001)	(-0.0368,0.0051)	(0.0001)	(-0.0003,0.0005)	(0.0001)	(-0.0000,0.0015)	(0.0002)
State-Year Clustering	-0.0019	-0.0008	-0.0014	-0.0003	0.0003	-0.0002	0.0007	0.0002
	(-0.0056,0.0006)	(0.0004)	(-0.0254,0.0031)	(0.0003)	(-0.0002,0.0048)	(0.0002)	(-0.0004,0.0058)	(0.0002)
No Clustering	-0.0019	-0.0008	-0.0014	-0.0003	0.0003	-0.0002	0.0007	0.0002
	(-0.0030,-0.0012)	(0.0001)	(-0.0657,0.0017)	(0.0001)	(-0.0001,0.0009)	(0.0001)	(0.0000,0.0021)	(0.0002)
Small Disc	-0.0019	-0.0008	-0.0014	-0.0003	0.0003	-0.0002	0.0007	0.0002
	(-0.0030,-0.0012)	(0.0001)	(-0.1512,0.0013)	(0.0001)	(-0.0000,0.0008)	(0.0001)	(-0.0001,0.0019)	(0.0002)
Large Disc	-0.0019	-0.0008	-0.0013	-0.0003	0.0003	-0.0002	0.0007	0.0002
	(-0.0030,-0.0012)	(0.0001)	(-0.0155,0.0013)	(0.0001)	(-0.0000,0.0008)	(0.0001)	(-0.0001,0.0018)	(0.0002)

In parentheses: 95% confidence intervals (for ILS) and standard errors (for OLS) bootstrapped from 1000 samples with resampling at the county level, unless otherwise noted.
Regressions weighted by each county's log population in 2002, unless otherwise noted.
Years: 2002–2019, unless otherwise noted.

Table A-4: Indirect least squares estimates (ILS) of total effects for the effects of days above 80 degF on log income pc, from Proposition 7 and regression (21), versus ordinary least squares (OLS) estimates, from Proposition 1 and regression (22).

	Midwest		Northeast		South		West	
	ILS	OLS	ILS	OLS	ILS	OLS	ILS	OLS
Base	-0.0014	-0.0009	-0.0089	-0.0001	0.0003	-0.0001	-0.0020	-0.0000
	(-0.0017,-0.0011)	(0.0001)	(-1.5524,-0.0049)	(0.0001)	(0.0001,0.0006)	(0.0000)	(-0.2369,-0.0008)	(0.0001)
No Trends	-0.0013	-0.0008	-0.0048	-0.0001	0.0002	-0.0001	-0.0012	-0.0000
	(-0.0016,-0.0011)	(0.0001)	(-0.4166,-0.0016)	(0.0001)	(0.0000,0.0005)	(0.0000)	(-1.1102,-0.0003)	(0.0001)
Without 07–09	-0.0012	-0.0008	0.0086	-0.0002	0.0004	-0.0001	-0.0005	-0.0000
	(-0.0015,-0.0010)	(0.0001)	(0.0033,0.7173)	(0.0001)	(0.0002,0.0006)	(0.0000)	(-0.2058,0.0000)	(0.0001)
LogIncPc-Weighted	-0.0015	-0.0009	-0.0071	-0.0001	0.0003	-0.0001	-0.0003	0.0000
	(-0.0019,-0.0013)	(0.0001)	(-0.8764,-0.0032)	(0.0001)	(0.0001,0.0006)	(0.0000)	(-0.3137,-0.0000)	(0.0001)
Unweighted	-0.0015	-0.0009	-0.0063	-0.0001	0.0003	-0.0001	-0.0002	-0.0000
	(-0.0019,-0.0013)	(0.0001)	(-0.5803,-0.0028)	(0.0001)	(0.0001,0.0006)	(0.0000)	(-0.0238,0.0001)	(0.0001)
Five Lags	-0.0017	-0.0009	-0.0264	-0.0001	0.0005	-0.0001	-0.0002	-0.0000
	(-0.0021,-0.0014)	(0.0001)	(-1.2066,-0.0300)	(0.0001)	(0.0002,0.0008)	(0.0000)	(-0.0515,0.0002)	(0.0001)
Six Lags	-0.0022	-0.0009	-0.0048	-0.0001	0.0003	-0.0001	-0.0000	-0.0000
	(-0.0028,-0.0018)	(0.0001)	(-0.9013,-0.0017)	(0.0001)	(0.0000,0.0005)	(0.0000)	(-0.0012,0.0015)	(0.0001)
State-Year Clustering	-0.0014	-0.0009	-0.0089	-0.0001	0.0003	-0.0001	-0.0020	-0.0000
	(-0.0034,-0.0006)	(0.0002)	(-0.4093,-0.0071)	(0.0001)	(-0.0002,0.0050)	(0.0001)	(-0.6033,-0.0008)	(0.0001)
No Clustering	-0.0014	-0.0009	-0.0089	-0.0001	0.0003	-0.0001	-0.0020	-0.0000
	(-0.0017,-0.0011)	(0.0001)	(-2.5220,-0.0038)	(0.0001)	(0.0001,0.0006)	(0.0000)	(-0.1220,-0.0006)	(0.0001)
Small Disc	-0.0014	-0.0009	0.0085	-0.0001	0.0003	-0.0001	-0.0009	-0.0000
	(-0.0017,-0.0011)	(0.0001)	(0.0037,0.4186)	(0.0001)	(0.0001,0.0006)	(0.0000)	(-0.2428,-0.0002)	(0.0001)
Large Disc	-0.0014	-0.0009	-0.0034	-0.0001	0.0003	-0.0001	-0.0254	-0.0000
	(-0.0017,-0.0011)	(0.0001)	(-0.1251,-0.0009)	(0.0001)	(0.0001,0.0005)	(0.0000)	(-0.5171,-0.5171)	(0.0001)

In parentheses: 95% confidence intervals (for ILS) and standard errors (for OLS) bootstrapped from 1000 samples with resampling at the county level, unless otherwise noted.
Regressions weighted by each county's log population in 2002, unless otherwise noted.
Years: 1970–2019, unless otherwise noted.

E Results for Other Temperature Variables

This appendix reports results for the other weather variables. Surprisingly, extreme cold appears to increase output and income per capita. This may be a case where precipitation is an important omitted variable, as more very cold days could displace wetter or snowier days. Cool days are also generally beneficial whereas warm days are generally harmful. In all cases, Wald tests tend to reject the hypothesis that a given region's lagged effects on either dependent variable are jointly equal to zero.

Table A-11 reports the projected effects of climate change by Census region and weather variable. The bottom panel reports the projected changes in each weather variable. Changes in all four weather variables tend to be harmful because there are fewer cold and cool days but more warm and hot days.

Table A-5: Estimating effects of days below 20 degF on county output per capita.

	Output p.c.		Income p.c.	
	Regression (21)	Regression (22)	Regression (21)	Regression (22)
Midwest				
Contemporary	0.00072***	0.00038***	0.00070***	0.00067***
	(0.00016)	(0.00012)	(0.000074)	(0.000062)
Lag 1	0.00070***		0.00013*	
	(0.00018)		(0.000073)	
Lag 2	0.00049**		0.00037***	
	(0.00022)		(0.000067)	
Northeast				
Contemporary	0.00029	0.0000049	0.00042***	0.00019***
	(0.00019)	(0.00013)	(0.000064)	(0.000047)
Lag 1	0.00068***		0.00049***	
	(0.00023)		(0.000069)	
Lag 2	0.00041*		0.00028***	
	(0.00023)		(0.000076)	
South				
Contemporary	0.00088***	0.00053***	0.00035***	0.00026***
	(0.00028)	(0.00020)	(0.000095)	(0.000086)
Lag 1	0.0010***		0.00049***	
	(0.00034)		(0.000100)	
Lag 2	0.00057		0.00012	
	(0.00037)		(0.000088)	
West				
Contemporary	0.00046***	0.00017	0.00012**	0.00014***
	(0.00018)	(0.00014)	(0.000059)	(0.000048)
Lag 1	0.00059***		0.000081	
	(0.00023)		(0.000076)	
Lag 2	0.000079		-0.00013*	
	(0.00026)		(0.000075)	
Wald test of the null that the first two lags are jointly zero				
p-value Lags 0, Midwest	0.00065		0.00000026	
p-value Lags 0, Northeast	0.0083		9.7e-11	
p-value Lags 0, South	0.011		0.0000040	
p-value Lags 0, West	0.0053		0.0090	

Standard errors in parentheses
Standard errors clustered by county.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table A-6: Indirect least squares estimates for the effects of days below 20 degF, from Proposition 7 and regression (21).

	Direct Effects	Ex-Post Adaptation	Total Effects	$\hat{\phi}_{2r1}/\hat{\phi}_{1r1}$
		Output p.c.		
Midwest	0.0023	0.00023	0.0025	0.69
	(-0.0033,0.011)	(-0.00067,0.0015)	(-0.0039,0.012)	(0.17,1.26)
Northeast	0.0015	0.00018	0.0017	0.59
	(0.00015,0.0038)	(0.000025,0.00057)	(0.00018,0.0043)	(-0.14,1.01)
South	0.0026	0.00026	0.0029	0.57
	(-0.014,0.0090)	(-0.00086,0.0015)	(-0.013,0.011)	(-0.31,1.18)
West	0.0010	0.000086	0.0011	0.13
	(0.00023,0.0025)	(0.0000059,0.00027)	(0.00025,0.0028)	(-2.06,0.78)
		Income p.c.		
Midwest	0.00062	-0.000012	0.00060	2.78
	(0.00023,0.00081)	(-0.00014,-0.000000070)	(0.00012,0.00079)	(-4.00,17.6)
Northeast	0.0013	0.00013	0.0014	0.56
	(0.00081,0.0018)	(0.000070,0.00021)	(0.00087,0.0020)	(0.30,0.80)
South	0.00089	0.000081	0.00097	0.24
	(0.00050,0.0013)	(0.000042,0.00013)	(0.00054,0.0014)	(-0.13,0.67)
West	0.00015	0.0000044	0.00015	-1.62
	(-0.0000092,0.00035)	(3.9e-09,0.000025)	(-0.000013,0.00036)	(-145.7,3.08)

95% confidence intervals (in parentheses) bootstrapped from 1000 samples with resampling at the county level.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.

Table A-7: Estimating effects of days in 20–50 degF on county output per capita.

	Output p.c.		Income p.c.	
	Regression (21)	Regression (22)	Regression (21)	Regression (22)
Midwest				
Contemporary	0.00044***	0.00025***	0.00048***	0.00054***
	(0.00012)	(0.000085)	(0.000051)	(0.000043)
Lag 1	0.00051***		-0.000029	
	(0.00014)		(0.000051)	
Lag 2	0.00038**		0.00032***	
	(0.00017)		(0.000050)	
Northeast				
Contemporary	-0.00015	0.000052	0.00014***	0.000065
	(0.00013)	(0.000096)	(0.000052)	(0.000043)
Lag 1	-0.000044		0.00018***	
	(0.00015)		(0.000057)	
Lag 2	0.000030		0.00015**	
	(0.00017)		(0.000065)	
South				
Contemporary	0.00022***	0.00021***	0.00017***	0.00018***
	(0.000077)	(0.000063)	(0.000029)	(0.000025)
Lag 1	-0.00011		-0.000018	
	(0.000098)		(0.000029)	
Lag 2	0.000092		0.000028	
	(0.000095)		(0.000029)	
West				
Contemporary	0.000084	-0.000021	0.000031	0.000061**
	(0.000094)	(0.000076)	(0.000032)	(0.000027)
Lag 1	0.00035***		-0.0000091	
	(0.00010)		(0.000041)	
Lag 2	-0.000061		-0.00010**	
	(0.00010)		(0.000041)	
Wald test of the null that the first two lags are jointly zero				
p-value Lags 0, Midwest	0.0012		1.2e-12	
p-value Lags 0, Northeast	0.84		0.0076	
p-value Lags 0, South	0.023		0.34	
p-value Lags 0, West	0.00020		0.014	

Standard errors in parentheses
Standard errors clustered by county.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table A-8: Indirect least squares estimates for the effects of days in 20–50 degF, from Proposition 7 and regression (21).

	Direct Effects	Ex-Post Adaptation	Total Effects	$\hat{\phi}_{2r2}/\hat{\phi}_{1r2}$
		Output p.c.		
Midwest	0.0017	0.00019	0.0019	0.74
	(-0.0033,0.014)	(-0.00049,0.0021)	(-0.0038,0.016)	(0.16,1.37)
Northeast	-0.00018	-0.0000036	-0.00018	-0.68
	(-0.0014,0.0043)	(-0.00021,0.00049)	(-0.0015,0.0050)	(-2437.6,0.84)
South	0.00016	-0.0000087	0.00015	-0.81
	(-0.00024,0.00036)	(-0.000052,-0.000000034)	(-0.00028,0.00036)	(-37.0,3.61)
West	0.00035	0.000039	0.00038	-0.18
	(0.000015,0.00076)	(0.0000092,0.000082)	(0.000018,0.00084)	(-1.27,0.38)
		Income p.c.		
Midwest	0.00048	-0.00000037	0.00048	-10.8
	(0.00037,0.00058)	(-0.0000038,-1.8e-11)	(0.00037,0.00058)	(-2067.3,-2.58)
Northeast	0.00079	0.000098	0.00089	0.88
	(-0.00034,0.036)	(-0.000055,0.0054)	(-0.00041,0.041)	(0.24,1.87)
South	0.00016	-0.00000097	0.00016	-1.58
	(0.000066,0.00049)	(-0.000010,0.000052)	(0.000058,0.00054)	(-514092.4,0.82)
West	0.000032	0.00000014	0.000032	11.2
	(-0.00088,0.00010)	(-0.00038,0.0000017)	(-0.0020,0.00010)	(4.73,777.8)

95% confidence intervals (in parentheses) bootstrapped from 1000 samples with resampling at the county level.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.

Table A-9: Estimating effects of days in 70–80 degF on county output per capita.

	Output p.c.		Income p.c.	
	Regression (21)	Regression (22)	Regression (21)	Regression (22)
Midwest				
Contemporary	-0.00012	-0.000033	-0.00045***	-0.00031***
	(0.00011)	(0.000092)	(0.000040)	(0.000033)
Lag 1	0.00013		-0.00017***	
	(0.00013)		(0.000042)	
Lag 2	0.00078***		0.00022***	
	(0.00015)		(0.000037)	
Northeast				
Contemporary	-0.00027**	-0.00027***	-0.000079**	-0.000031
	(0.00011)	(0.000062)	(0.000032)	(0.000019)
Lag 1	0.000059		-0.00011**	
	(0.00015)		(0.000052)	
Lag 2	-0.000033		-0.00017***	
	(0.00017)		(0.000056)	
South				
Contemporary	-0.000042	0.000016	0.000061***	0.000032*
	(0.000049)	(0.000038)	(0.000022)	(0.000016)
Lag 1	-0.000042		0.000043*	
	(0.000066)		(0.000025)	
Lag 2	-0.00017**		-0.000015	
	(0.000087)		(0.000027)	
West				
Contemporary	0.00011	-0.0000087	-0.000088**	-0.00012***
	(0.00012)	(0.000098)	(0.000041)	(0.000031)
Lag 1	0.000032		0.000017	
	(0.00012)		(0.000051)	
Lag 2	-0.0000095		0.00015***	
	(0.00014)		(0.000055)	
Wald test of the null that the first two lags are jointly zero				
p-value Lags 0, Midwest	7.9e-09		2.0e-24	
p-value Lags 0, Northeast	0.59		0.0083	
p-value Lags 0, South	0.065		0.0086	
p-value Lags 0, West	0.93		0.011	

Standard errors in parentheses
Standard errors clustered by county.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table A-10: Indirect least squares estimates for the effects of days in 70–80 degF, from Proposition 7 and regression (21).

	Direct Effects	Ex-Post Adaptation	Total Effects	$\hat{\phi}_{2r3}/\hat{\phi}_{1r3}$
		Output p.c.		
Midwest	-0.00015	-0.0000041	-0.00015	5.92
	(-0.00033,0.000051)	(-0.000046,-9.9e-09)	(-0.00034,0.000054)	(-9.11,682.3)
Northeast	-0.00023	0.0000052	-0.00023	-0.56
	(-0.0083,0.00083)	(-0.0013,0.00016)	(-0.0089,0.0010)	(-442.3,0.95)
South	-0.000028	0.0000022	-0.000025	4.07
	(-0.0040,0.000073)	(-0.000063,0.000040)	(-0.0046,0.000086)	(1.39,371.2)
West	0.00014	0.0000034	0.00014	-0.29
	(-0.00089,0.0018)	(-0.00012,0.00033)	(-0.0010,0.0021)	(-2212.7,1.43)
		Income p.c.		
Midwest	-0.00051	-0.000010	-0.00052	-1.31
	(-0.00064,-0.00040)	(-0.000020,-0.0000033)	(-0.00066,-0.00041)	(-2.87,-0.62)
Northeast	0.00018	0.000039	0.00022	1.56
	(-0.0010,0.020)	(-0.00014,0.0031)	(-0.0012,0.023)	(0.90,5.59)
South	0.000090	0.0000043	0.000094	-0.35
	(0.000022,0.00023)	(0.000000024,0.000018)	(0.000023,0.00025)	(-10.9,2.15)
West	-0.000090	-0.00000033	-0.000091	8.90
	(-0.00017,0.0027)	(-0.0000051,0.00089)	(-0.00017,0.0055)	(2.82,437.2)

95% confidence intervals (in parentheses) bootstrapped from 1000 samples with resampling at the county level.
Regressions weighted by each county's log population in 2002.
Years: 2002–2019 for output and 1970–2019 for income.

Table A-11: Projected end-of-century percentage change in output and income per capita due to climate change.

	Midwest		Northeast		South		West	
	ILS	OLS	ILS	OLS	ILS	OLS	ILS	OLS
Output p.c. (%)								
<20 degF	-1.26	-0.19	-0.21	-0.00060	-0.23	-0.042	-0.12	-0.018
	(-6.23,1.99)	(-0.32,-0.078)	(-0.53,-0.022)	(-0.032,0.034)	(-0.84,1.05)	(-0.075,-0.012)	(-0.30,-0.027)	(-0.049,0.011)
20–50 degF	-0.67	-0.089	0.022	-0.0065	-0.14	-0.20	-0.12	0.0070
	(-5.65,1.36)	(-0.15,-0.028)	(-0.62,0.19)	(-0.028,0.017)	(-0.33,0.26)	(-0.30,-0.093)	(-0.27,-0.0060)	(-0.039,0.059)
70–80 degF	-0.016	-0.0034	-0.029	-0.034	0.013	-0.0083	0.032	-0.0020
	(-0.035,0.0056)	(-0.022,0.014)	(-1.14,0.13)	(-0.050,-0.019)	(-0.043,2.33)	(-0.045,0.027)	(-0.23,0.49)	(-0.044,0.047)
>80 degF	-2.33	-0.96	-0.26	-0.051	0.68	-0.32	0.19	0.044
	(-3.59,-1.47)	(-1.28,-0.63)	(-6.20,0.34)	(-0.11,-0.00077)	(-0.067,1.74)	(-0.59,-0.080)	(-0.025,0.48)	(-0.043,0.15)
All Days	-4.28	-1.25	-0.48	-0.092	0.32	-0.56	-0.024	0.031
	(-12.0,1.98)	(-1.65,-0.88)	(-6.50,1.66)	(-0.17,-0.012)	(-0.98,1.96)	(-0.81,-0.31)	(-0.60,0.45)	(-0.10,0.18)
Income p.c. (%)								
<20 degF	-0.31	-0.34	-0.17	-0.023	-0.077	-0.020	-0.016	-0.015
	(-0.40,-0.059)	(-0.40,-0.28)	(-0.25,-0.11)	(-0.035,-0.011)	(-0.11,-0.043)	(-0.033,-0.0069)	(-0.039,0.0014)	(-0.026,-0.0051)
20–50 degF	-0.17	-0.19	-0.11	-0.0080	-0.15	-0.16	-0.010	-0.020
	(-0.21,-0.13)	(-0.22,-0.17)	(-5.14,0.051)	(-0.018,0.0033)	(-0.50,-0.053)	(-0.21,-0.12)	(-0.033,0.66)	(-0.038,-0.0028)
70–80 degF	-0.054	-0.032	0.028	-0.0039	-0.047	-0.016	-0.021	-0.028
	(-0.068,-0.042)	(-0.040,-0.026)	(-0.15,2.98)	(-0.0086,0.00070)	(-0.12,-0.011)	(-0.031,-0.00022)	(-0.038,1.28)	(-0.044,-0.015)
>80 degF	-1.65	-1.03	-1.67	-0.017	0.60	-0.24	-0.50	-0.0082
	(-1.99,-1.37)	(-1.19,-0.85)	(-293.0,-0.92)	(-0.038,0.0046)	(0.19,1.20)	(-0.36,-0.13)	(-60.5,-0.20)	(-0.054,0.044)
All Days	-2.18	-1.59	-1.93	-0.052	0.33	-0.44	-0.55	-0.071
	(-2.69,-1.81)	(-1.82,-1.40)	(-293.1,-1.04)	(-0.080,-0.023)	(-0.088,0.84)	(-0.57,-0.31)	(-60.5,-0.23)	(-0.12,-0.011)
Projected Weather Changes by End of Century (Days per Year)								
<20 degF	-5.1		-1.2		-0.8		-1.1	
20–50 degF	-3.6		-1.2		-9.2		-3.2	
70–80 degF	1.0		1.3		-5.0		2.3	
>80 degF	12.1		1.9		21.0		2.6	

95% confidence intervals (in parentheses) bootstrapped from 1000 samples with resampling at the county level.
Projected temperature change averages 30 CMIP6 models, for RCP 4.5 in 2081–2100 relative to 1995–2014.
ILS columns multiply projected temperature changes by indirect least squares estimates, from Proposition 7 and (21).
OLS columns multiply projected temperature changes by $\hat{\theta}_{rk}$ from (22).

F Formal Analysis and Proofs

F.1 Deriving equation (7)

With A_t defined implicitly from the first-order condition $\pi_A = 0$, approximate A_t around $w_t = C$ and use the quadratic form for payoffs:

$$A_t = \bar{A} + \frac{\bar{\pi}_{wA}}{-\bar{\pi}_{AA}}(w_t - C). \tag{A-2}$$

Therefore,

$$E_0[A_t] = \bar{A}.$$

Approximating the payoff function around $w_t = C$, we have:

$$\begin{aligned} E_0[\pi(w_t, A_t, S_t; K)] =& \bar{\pi} + \bar{\pi}_w \underbrace{(E_0[w_t] - C)}_{=0} + \bar{\pi}_A \underbrace{(E_0[A_t] - \bar{A})}_{=0} \\ &+ \frac{1}{2}\bar{\pi}_{ww} E_0[(w_t - C)^2] + \frac{1}{2}\bar{\pi}_{AA} E_0[(A_t - \bar{A})^2] + \bar{\pi}_{wA} Cov_0[A_t, w_t], \end{aligned} \tag{A-3}$$

for $t > 2$. Differentiating with respect to C, and using that these assumptions imply $\bar{A} = E_0[A_t]$, we have the expression given in the text. $\bar{\pi}_A = 0$ and $\bar{\pi}_K = 0$ follow from the first-order conditions given just before equation (7).

F.2 Proof of Proposition 1

Following the derivation of equation (A-3) and applying the first-order condition, we have:

$$\begin{aligned} \Delta\pi_{jt} =& \bar{\pi}_w(w_{jt} - w_{j(t-1)}) + \frac{1}{2}\bar{\pi}_{ww}[(w_{jt} - C)^2 - (w_{j(t-1)} - C)^2] \\ &+ \frac{1}{2}\bar{\pi}_{AA}[(A_{jt} - \bar{A})^2 - (A_{j(t-1)} - \bar{A})^2] \\ &+ \bar{\pi}_{wA}[(w_{jt} - C)(A_{jt} - \bar{A}) - (w_{j(t-1)} - C)(A_{j(t-1)} - \bar{A})]. \end{aligned}$$

Substitute from (A-2):

$$\Delta\pi_{jt} = \bar{\pi}_w(w_{jt} - w_{j(t-1)}) + \frac{1}{2}\left[\bar{\pi}_{ww} + \bar{\pi}_{wA}\frac{\bar{\pi}_{wA}}{-\bar{\pi}_{AA}}\right][(w_{jt} - C)^2 - (w_{j(t-1)} - C)^2].$$

The covariance between a mean-zero normally distributed random variable and a higher power of a mean-zero normally distributed random variable is zero. The $w_t - C$ and $w_{t-1} - C$ are mean-zero normally distributed random variables. Therefore:

$$Cov[\Delta\pi_{jt}, \Delta w_{jt}] = \bar{\pi}_w Var[\Delta w_{jt}].$$

And in this setting without unobservables,

$$Cov[\Delta\pi_{jt} - \mu_j^\pi, \Delta w_{jt} - \mu_j^w] = Cov[\Delta\pi_{jt}, \Delta w_{jt}].$$

The result follows.

F.3 Proof that there is a unique maximizer in the deterministic model ($\zeta = 0$)

With $\zeta = 0$, rewrite payoffs as a function of S_t and S_{t+1} by using $A_t = h^{-1}(S_{t+1} - gS_t)$: $\tilde{\pi}(S_t, S_{t+1}) \triangleq \pi(C, A_t, S_t; K)$. If the payoff function is strictly concave and bounded, then there is a unique maximizer by Theorem 4.8 in Stokey and Lucas (1989). Strict concavity requires that $\tilde{\pi}_{S_t S_t} < 0$ and $\tilde{\pi}_{S_t S_t}\tilde{\pi}_{S_{t+1}S_{t+1}} - (\tilde{\pi}_{S_t S_{t+1}})^2 > 0$. We have:

$$\begin{aligned}
\tilde{\pi}_{S_t S_t}\tilde{\pi}_{S_{t+1}S_{t+1}} - (\tilde{\pi}_{S_t S_{t+1}})^2 =&(1/h')^4 \left[(h')^2\pi_{SS} + 2h'\pi_{AS} + \pi_{AA} - (h''/h')\pi_A\right]\left[\pi_{AA} - (h''/h')\pi_A\right] \\
&- (1/h')^4 \left[h'\pi_{AS} + \pi_{AA} - (h''/h')\pi_A\right]^2 \\
=&[1/h']^2\Big(\pi_{SS}\left[\pi_{AA} - (h''/h')\pi_A\right] - [\pi_{AS}]^2\Big).
\end{aligned}$$

This is strictly positive if and only if inequality (2) holds. By the inequality of arithmetic and geometric means, inequality (2) in turn implies

$$h'\pi_{AS} < \frac{1}{2}\left(-\pi_{AA} + (h''/h')\pi_A\right) - \frac{1}{2}(h')^2\pi_{SS},$$

which is equivalent to $\tilde{\pi}_{S_t S_t} < 0$. We have therefore established that inequality (2) implies that payoffs are strictly concave in S_t and S_{t+1}.

F.4 Proof that deterministic model ($\zeta = 0$) has a unique steady state and is saddle-path stable

Fix $\zeta = 0$, in which case $w_t = f_{1,t} = f_{2,t} = C$ at all times t.

The first-order condition for the deterministic model is:

$$0 = \pi_A(C, A_t, S_t; K) + \beta h'(A_t)\, V_S(S_{t+1}, C, C, C; 0, K).$$

This implies:

$$V_S(S_{t+1}, C, C, C; 0, K) = \frac{-\pi_A(C, A_t, S_t; K)}{\beta h'(A_t)}.$$

The envelope theorem yields:

$$V_S(S_{t+1}, C, C, C; 0, K) = \pi_S(C, A_{t+1}, S_{t+1}; K) + \beta\, g\, V_S(S_{t+2}, C, C, C; 0, K).$$

Advancing the first-order condition by one timestep and substituting in, we have the Euler equation:

$$-\pi_A(C, A_t, S_t; K) = \beta\, h'(A_t)\, \pi_S(C, A_{t+1}, S_{t+1}; K) + \beta\, h'(A_t)\, g\, \frac{-\pi_A(C, A_{t+1}, S_{t+1}; K)}{h'(A_{t+1})}. \tag{A-4}$$

The steady state (denoted with a bar) is defined by the following pair of equations:

$$\begin{aligned}-\pi_A(C,\bar{A},\bar{S};K) =&\beta\, h'(\bar{A})\, \pi_S(C,\bar{A},\bar{S};K) - \beta\, g\, \pi_A(C,\bar{A},\bar{S};K),\\ \bar{S} =& g\bar{S} + h(\bar{A}).\end{aligned}$$

The second implies:

$$\bar{S} = \frac{h(\bar{A})}{1-g}. \tag{A-5}$$

Substituting into the first equation and rearranging, we have:

$$-(1-\beta g)\pi_A\left(C,\bar{A},\frac{h(\bar{A})}{1-g};K\right) - \beta h'(\bar{A})\pi_S\left(C,\bar{A},\frac{h(\bar{A})}{1-g};K\right) = 0. \tag{A-6}$$

By (4), the left-hand side is negative as A_t goes to $-\infty$, and by (5), the left-hand side is positive as A_t goes to ∞. The derivative of the left-hand side of (A-6) with respect to $\bar{A}$ is

$$-(1-\beta g)\bar{\pi}_{AA} - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \beta h''(\bar{A})\bar{\pi}_S - (1-\beta g)\frac{h'(\bar{A})}{1-g}\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}.$$

Substituting for $\beta\bar{\pi}_S$ from (A-4), this becomes:

$$(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \left[\frac{1-\beta g}{1-g} + \beta\right]h'(\bar{A})\bar{\pi}_{AS}.$$

This expression is strictly positive if and only if

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1-(1+\beta)g+\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta-2\beta g}. \tag{A-7}$$

From (6), we have

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1-2g(1+\beta)+3\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta-2\beta g}.$$

The right-hand side of this last inequality is weakly less than the right-hand side of inequality (A-7). Therefore inequality (A-7) holds, which in turn implies that the left-hand side of (A-6) strictly increases in $\bar{A}$. Then, by (4) and (5), (A-6) defines a unique $\bar{A} \in (-\infty,\infty)$ and (A-5) then defines the unique $\bar{S} \in (-\infty,\infty)$.

The Euler equation (A-4) implicitly defines $A^*_{t+1}(A_t, S_t)$. Using the implicit function theorem:

$$\frac{\partial A_{t+1}}{\partial S_t} = \frac{h'(A_{t+1})\left[-\frac{\pi_{AS}(C,A_t,S_t;K)}{h'(A_t)} - \beta g \pi_{SS}(C, A_{t+1}, S_{t+1}; K) + \beta g^2 \frac{\pi_{AS}(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right]}{\beta h'(A_{t+1})\pi_{AS}(C, A_{t+1}, S_{t+1}; K) + \beta g\left(-\pi_{AA}(C, A_{t+1}, S_{t+1}; K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)},$$

$$\begin{aligned}\frac{\partial A_{t+1}}{\partial A_t} =& \frac{h'(A_{t+1})\left[-\beta h'(A_t)\,\pi_{SS}(C, A_{t+1}, S_{t+1}; K) + \frac{-\pi_{AA}(C,A_t,S_t;K)}{h'(A_t)} + h''(A_t)\frac{\pi_A(C,A_t,S_t;K)}{[h'(A_t)]^2}\right]}{\beta h'(A_{t+1})\pi_{AS}(C, A_{t+1}, S_{t+1}; K) + \beta g\left(-\pi_{AA}(C, A_{t+1}, S_{t+1}; K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)} \\ &+ \frac{h'(A_{t+1})\beta g h'(A_t)\frac{\pi_{AS}(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}}{\beta h'(A_{t+1})\pi_{AS}(C, A_{t+1}, S_{t+1}; K) + \beta g\left(-\pi_{AA}(C, A_{t+1}, S_{t+1}; K) + h''(A_{t+1})\frac{\pi_A(C,A_{t+1},S_{t+1};K)}{h'(A_{t+1})}\right)}.\end{aligned}$$

Approximate A_{t+1} around the steady state:

$$\begin{aligned}A_{t+1} \approx& \bar{A} + \frac{-(1-\beta g^2)\bar{\pi}_{AS} - \beta g h'(\bar{A})\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}(S_t - \bar{S}) \\ &+ \frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\,\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}(A_t - \bar{A}).\end{aligned}$$

Linearize the dynamic system around the steady state:

$$\begin{bmatrix} A_{t+1} - \bar{A} \\ S_{t+1} - \bar{S} \end{bmatrix} \approx \begin{bmatrix} \frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\,\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} & \frac{-(1-\beta g^2)\bar{\pi}_{AS} - \beta g h'(\bar{A})\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} \\ h'(\bar{A}) & g \end{bmatrix} \begin{bmatrix} A_t - \bar{A} \\ S_t - \bar{S} \end{bmatrix}.$$

The determinant is $1/\beta$, which is > 1. Therefore both eigenvalues have the same sign. The characteristic equation is

$$0 = z^2 - \left[\frac{-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} + \beta g h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\,\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} + g\right] z + \frac{1}{\beta}.$$

This is a parabola that opens up. At $z = 1$, its value is:

$$-\frac{(1-g)(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - (1+\beta-2\beta g)h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\,\bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}.$$

By inequality (A-7), the numerator is positive. If the denominator is positive, then the expression is negative, so there is one root $\in (0, 1)$ and one root > 1, making the system saddle-path stable. If the denominator is negative, then the analogous expression for $z = -1$ is negative, so there is one root $\in (-1, 0)$ and one root < -1, making the system again saddle-path stable.

F.5 Optimal actions in the stochastic system

The first-order condition is:

$$0 = \pi_A(w_t, A_t, S_t; K) + \beta h'(A_t)\, E_t[V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K)].$$

This implies:

$$E_t[V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K)] = \frac{-\pi_A(w_t, A_t, S_t; K)}{\beta h'(A_t)}.$$

The envelope theorem yields:

$$V_S(S_{t+1}, w_{t+1}, f_{1,t+1}, f_{2,t+1}; \zeta, K) = \pi_S(w_{t+1}, A_{t+1}, S_{t+1}; K) + \beta g E_{t+1}[V_S(S_{t+2}, w_{t+2}, f_{1,t+2}, f_{2,t+2}; \zeta, K)].$$

Advancing the first-order condition by one timestep and substituting in, we have the stochastic Euler equation:

$$\frac{-\pi_A(w_t, A_t, S_t; K)}{h'(A_t)} = \beta\, E_t[\pi_S(w_{t+1}, A_{t+1}, S_{t+1}; K)] + \beta\, g\, E_t\left[\frac{-\pi_A(w_{t+1}, A_{t+1}, S_{t+1}; K)}{h'(A_{t+1})}\right]. \tag{A-8}$$

For $\zeta = 0$, the weather in period $t+2$ matches the forecast $f_{2,t}$ and the weather is always C after period $t+2$. So we are back to the deterministic system in period $t+3$. Consider some distant time T at which the world ends. We will work backwards from there, solving for time $t+3$ policy as $T \to \infty$. Once we have that, we solve for time $t+2$ policy given $w_{t+2} = f_{2,t}$ and $f_{1,t+2} = f_{2,t+2} = C$; then we solve for time $t+1$ policy given $w_{t+1} = f_{1,t}$, $f_{1,t+1} = f_{2,t}$, and $f_{2,t+1} = C$; and finally we solve for time t policy given w_t, $f_{1,t}$, and $f_{2,t}$.

Write A_t as $A(S_t, w_t, f_{1,t}, f_{2,t}; \zeta)$ and define $\tilde{A}_t \triangleq A(S_t, w_t, f_{1,t}, f_{2,t}; 0)$. At time T, we have a static problem. The first-order condition is $\pi_A = 0$. Note that $\partial \tilde{A}_T / \partial S_T = \pi_{AS}/[-\pi_{AA}]$. Using the time $T-1$ Euler equation, first-order approximate $\tilde{A}_{T-1}$ around $S_{T-1} = \bar{S}$. This approximation is exact when payoffs are quadratic and $(S_{T-1} - \bar{S})^2$ is small. We thereby obtain $\tilde{A}_{T-1}$ as a function of S_{T-1}:

$$\tilde{A}_{T-1} = \bar{A} + \frac{\beta g \bar{\pi}_{SS} + (1 - \beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\frac{\bar{\pi}_{AS}}{-\bar{\pi}_{AA}}}{\chi_{T-1}}(S_{T-1} - \bar{S}),$$

where

$$\begin{aligned}\chi_{T-1} \triangleq & \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} \\ & - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A}) \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right]\frac{\bar{\pi}_{AS}}{-\bar{\pi}_{AA}}.\end{aligned}$$

Denote the coefficient on $S_t - \bar{S}$ in $\tilde{A}_t$ as Z_t. Stepping backwards through time, we find the following relationships:

$$
\begin{aligned}
Z_t =& \frac{\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}}{\chi_t}, \\
\chi_t =& \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} \\
& - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A}) \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}.
\end{aligned}
$$

Consider the fate of Z_t and χ_t as the terminal time T recedes to infinity. The steady state is:

$$
\begin{aligned}
\bar{Z} =& \frac{\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] \bar{Z}}{\bar{\chi}}, \\
\bar{\chi} =& \frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g \bar{\pi}_{AS} \\
& - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A}) \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] \bar{Z}.
\end{aligned}
\tag{A-9}
$$

Substitute $\bar{\chi}$ into $\bar{Z}$ and rearrange:

$$
\begin{aligned}
0 =& \bar{Z}^2 - \frac{(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta [h'(\bar{A})]^2 \bar{\pi}_{SS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)} \frac{1}{h'(\bar{A})}\bar{Z} \\
& - \frac{1}{[h'(\bar{A})]^2} \frac{-\beta g [h'(\bar{A})]^2 \bar{\pi}_{SS} - (1-\beta g^2) h'(\bar{A})\bar{\pi}_{AS}}{\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)}.
\end{aligned}
$$

From the quadratic formula, the solution is

$$
\begin{aligned}
\bar{Z} =& \left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta [h'(\bar{A})]^2 \bar{\pi}_{SS} \pm \sqrt{discrim}\right] \\
& \left[2h'(\bar{A})\left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right)\right]^{-1},
\end{aligned}
$$

where the discriminant is

$$
\begin{aligned}
discrim =& \left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta [h'(\bar{A})]^2 \bar{\pi}_{SS}\right)^2 \\
& + 4\left(-\beta g [h'(\bar{A})]^2 \bar{\pi}_{SS} - (1-\beta g^2) h'(\bar{A})\bar{\pi}_{AS}\right) \left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g \left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]\right).
\end{aligned}
\tag{A-10}
$$

The proof of Lemma 2 will show that (6) implies that *discrim* is positive.

In order to analyze stability, linearize the difference equations. Substituting χ_t into Z_t, we find:

$$Z_t = \left[\beta g \bar{\pi}_{SS} + (1-\beta g^2)\frac{\bar{\pi}_{AS}}{h'(\bar{A})} + \left[\beta g \bar{\pi}_{AS} + \beta g^2 \left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}\right]$$
$$\left[\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{-\bar{\pi}_A}{[h'(\bar{A})]^2} - \beta h'(\bar{A})\bar{\pi}_{SS} + \beta g h'(\bar{A})\frac{\bar{\pi}_{AS}}{h'(\bar{A})}\right.$$
$$\left. - \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g h'(\bar{A})\left(\frac{-\bar{\pi}_{AA}}{h'(\bar{A})} + h''(\bar{A})\frac{\bar{\pi}_A}{[h'(\bar{A})]^2}\right)\right] Z_{t+1}\right]^{-1}.$$

Linearizing and evaluating at the steady state:

$$\left.\frac{\partial Z_t}{\partial Z_{t+1}}\right|_{\bar{Z}} = \left[2\beta g h'(\bar{A})\bar{\pi}_{AS} + (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} \pm \sqrt{discrim}\right]$$
$$\left[2\beta g h'(\bar{A})\bar{\pi}_{AS} + (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - \left(\pm\sqrt{discrim}\right)\right]^{-1}. \tag{A-11}$$

The terms outside the square root are positive if

$$-2\beta g h'(\bar{A})\bar{\pi}_{AS} < (1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}. \tag{A-12}$$

The following lemma establishes that those terms are in fact positive:

Lemma 1. *Inequality* (2) *implies inequality* (A-12).

Proof. By the inequality of arithmetic and geometric means, inequality (2) implies

$$-h'(A_t)\pi_{AS} < \frac{1}{2}\left(-\pi_{AA} + \frac{h''(A_t)}{h'(A_t)}\pi_A\right) - \frac{1}{2}[h'(A_t)]^2\pi_{SS}.$$

Multiplying both sides by β and using inequality (3) and $1+\beta g^2 > \beta$, this inequality implies

$$-\beta h'(A_t)\pi_{AS} < \frac{1}{2}(1+\beta g^2)\left(-\pi_{AA} + \frac{h''(A_t)}{h'(A_t)}\pi_A\right) - \frac{1}{2}\beta[h'(A_t)]^2\pi_{SS}.$$

Using $g < 1$, this last inequality in turn implies inequality (A-12).

□

Because the terms outside the square root in (A-11) are positive, the numerator and denominator are both larger when the square root is added rather than subtracted.

The stable steady state (with eigenvalue <1 in magnitude) is therefore the one with a negative sign in the numerator of (A-11). The steady state of interest is therefore

$$\bar{Z} = \left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - \sqrt{discrim}\right] \left[2h'(\bar{A})\left(\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right)\right]^{-1}. \tag{A-13}$$

Substituting into $\bar{\chi}$, we find:

$$\bar{\chi} = \frac{1}{2h'(\bar{A})}\left[(1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2\beta g h'(\bar{A})\bar{\pi}_{AS} + \sqrt{discrim}\right]. \tag{A-14}$$

From Lemma 1 and inequality (2), $h'(\bar{A})\,\bar{\chi} > 0$. Lemma 2 below will imply that $\lim_{g\to 0}\bar{Z} \propto \bar{\pi}_{AS}$.

Now return to the case in which $\zeta = 0$ from some time t onward. We have derived an expression for $\tilde{A}_t$ as $T \to \infty$. Using this,

$$\tilde{A}_{t+3} = \bar{A} + \bar{Z}(S_{t+2} - \bar{S}).$$

At time $t+2$, the relevant Euler equation is:

$$0 = \frac{\pi_A(f_{2,t}, \tilde{A}_{t+2}, S_{t+2}; K)}{h'(\tilde{A}_{t+2})} + \beta\pi_S(C, \tilde{A}_{t+3}, S_{t+3}; K) + \beta g\,\frac{-\pi_A(C, \tilde{A}_{t+3}, S_{t+3}; K)}{h'(\tilde{A}_{t+3})},$$

where we recognize that $w_{t+2} = f_{2,t}$. A first-order approximation to $\tilde{A}_{t+2}$ around $S_{t+2} = \bar{S}$ and $f_{2,t} = C$ is exact when $(S_{t+2} - \bar{S})^2$ is small and payoffs are quadratic. We thereby obtain

$$\tilde{A}_{t+2} = \bar{A} + \bar{Z}(S_{t+2} - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(f_{2,t} - C).$$

When $(S_{t+1} - \bar{S})^2$ is small and payoffs are quadratic, approximating A_{t+1} around $S_{t+1} = \bar{S}$, $w_{t+1} = f_{1,t} = C$, $f_{1,t+1} = f_{2,t} = C$, and $\zeta = 0$ in a version of the stochastic Euler equation (A-8) advanced by one timestep yields:

$$A_{t+1} = \bar{A} + \bar{Z}(S_{t+1} - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C),$$

where

$$\beta\Gamma \triangleq \beta h'(\bar{A})\bar{\pi}_{wS} - \beta g\bar{\pi}_{wA} + \beta\overbrace{\left[h'(\bar{A})\bar{\pi}_{AS} + g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right]}^{\triangleq\Psi}\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}. \tag{A-15}$$

When $(S_t - \bar{S})^2$ is small and payoffs are quadratic, approximating A_t around $S_t = \bar{S}$, $w_t = C$, $f_{1,t} = C$, $f_{2,t} = C$, and $\zeta = 0$ in the stochastic Euler equation (A-8) yields:

$$A_t = \bar{A} + \bar{Z}(S_t - \bar{S}) + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_t - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C). \tag{A-16}$$

Throughout, the terms with ζ drop out due to the expectation operator in the stochastic Euler equation, and the terms with ζ^2 drop out due being close to the steady state and payoffs being quadratic.

F.6 Evolution of expected actions and states

For $t \geq 2$,

$$E_0[A_t] = \bar{A} + \bar{Z}(E_0[S_t] - \bar{S}).$$

Approximate S_t around $A_{t-1} = \bar{A}$ and $S_{t-1} = \bar{S}$:

$$S_t \approx \bar{S} + h'(\bar{A})(A_{t-1} - \bar{A}) + g(S_{t-1} - \bar{S}).$$

We then have:

$$E_0[A_t] = \bar{A} + \bar{Z}h'(\bar{A})(E_0[A_{t-1}] - \bar{A}) + \bar{Z}g\,(E_0[S_{t-1}] - \bar{S}]).$$

Repeatedly substituting, we find:

$$E_0[A_t] = \bar{A} + [\bar{Z}h'(\bar{A}) + g]^{x-1}\left[\bar{Z}h'(\bar{A})(E_0[A_{t-x}] - \bar{A}) + \bar{Z}g(E_0[S_{t-x}] - \bar{S}])\right]$$

for $x \in \{1, ..., t-1\}$. Analogously,

$$E_0[S_t] = \bar{S} + [\bar{Z}h'(\bar{A}) + g]^{x-1}\left[h'(\bar{A})(E_0[A_{t-x}] - \bar{A}) + g(E_0[S_{t-x}] - \bar{S}])\right].$$

We have geometric series. The following lemma establishes that the common ratio is less than 1 in magnitude.

Lemma 2. (6) *implies* $|\bar{Z}h'(\bar{A}) + g| < 1$.

Proof. From (A-9) and (A-14),

$$\bar{Z}h'(\bar{A}) + g = \frac{\overbrace{2g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) + 2h'(\bar{A})\bar{\pi}_{AS}}^{=2\Psi}}{(1+\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2\beta g h'(\bar{A})\bar{\pi}_{AS} + \sqrt{discrim}}, \tag{A-17}$$

where the numerator is equal to 2Ψ by (A-15). Recalling that inequality (2) implies inequality (A-12) (Lemma 1), the denominator is clearly positive. Rewrite (A-17) as:

$$\begin{aligned}\bar{Z}h'(\bar{A})+g =&\left[g\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)+h'(\bar{A})\bar{\pi}_{AS}\right]\\&\Bigg[\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{-\bar{\pi}_A}{h'(\bar{A})}\right)-\beta[h'(\bar{A})]^2\bar{\pi}_{SS}+\beta g h'(\bar{A})\bar{\pi}_{AS}\\&\quad+\frac{1}{2}\sqrt{discrim}-\frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)-\beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right)\Bigg]^{-1}.\end{aligned} \tag{A-18}$$

We desire to show $\bar{Z}h'(\bar{A})+g<1$ if $\bar{Z}h'(\bar{A})+g>0$ and to show $\bar{Z}h'(\bar{A})+g>-1$ if $\bar{Z}h'(\bar{A})+g<0$.

First consider $\bar{Z}h'(\bar{A})+g>0$. The first line of the denominator in (A-18) is positive and is larger than the numerator. The second line in the denominator is positive if and only if

$$\left\{-\beta g[h'(\bar{A})]^2\bar{\pi}_{SS}-(1-\beta g^2)h'(\bar{A})\bar{\pi}_{AS}\right\}\left\{\beta h'(\bar{A})\bar{\pi}_{AS}+\beta g\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]\right\}>0.$$

The expression contained in the second curly braces is positive because it is proportional to $\bar{Z}h'(\bar{A})+g$. The expression contained in the first curly braces is positive if $h'(\bar{A})\bar{\pi}_{AS}\leq 0$. In this case, the inequality does hold and the second line of the denominator reinforces the first. So $\bar{Z}h'(\bar{A})+g<1$ if $\bar{Z}h'(\bar{A})+g>0$ and $h'(\bar{A})\bar{\pi}_{AS}\leq 0$.

If, instead, $\bar{Z}h'(\bar{A})+g>0$ with $h'(\bar{A})\bar{\pi}_{AS}>0$, the second line of the denominator in (A-18) can be negative if $h'(\bar{A})\bar{\pi}_{AS}$ is sufficiently large. So we seek the largest value of $h'(\bar{A})\bar{\pi}_{AS}$ compatible with $\bar{Z}h'(\bar{A})+g\leq 1$. Rearranging the inequality $\bar{Z}h'(\bar{A})+g<1$, we find:[44]

$$\begin{aligned}1 >&\bar{Z}h'(\bar{A})+g\\ \Leftrightarrow&\sqrt{discrim}>[2g-1-\beta g^2]\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)+\beta[h'(\bar{A})]^2\bar{\pi}_{SS}+2(1-\beta g)h'(\bar{A})\bar{\pi}_{AS}.\end{aligned}$$

The right-hand side is positive in the region of interest, around where the inequality binds. Squaring both sides, this inequality becomes:

$$\begin{aligned}0 <&g(1-g)(1-\beta g)\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2-\beta g\left(-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS}\\&-\beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS}+[1-2g(1+\beta)+3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]\\&-[1+\beta-2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2.\end{aligned} \tag{A-19}$$

[44]Doing so, it is easy to see that $discrim>0$ if $\bar{Z}h'(\bar{A})+g<1$ which validates one half of an earlier claim (i.e., only for the case with $\bar{Z}h'(\bar{A})+g>0$) once we establish that $\bar{Z}h'(\bar{A})+g<1$.

This is a quadratic in $h'(\bar{A})\bar{\pi}_{AS}$. It opens down. So the acceptable values of $h'(\bar{A})\bar{\pi}_{AS}$ will be in an intermediate range (if they exist). We already saw that the inequality must hold for small positive values of $h'(\bar{A})\bar{\pi}_{AS}$, so it should be the case that any roots are on either side of zero with the y-intercept strictly positive (as is easy to verify).[45] So $\bar{Z}h'(\bar{A}) + g < 1$ only if $h'(\bar{A})\bar{\pi}_{AS}$ is less than the positive root. Observe that the product of the constant and the quadratic coefficient is negative. Therefore, from the quadratic formula, inequality (A-19) holds if

$$h'(\bar{A})\bar{\pi}_{AS} < \frac{[1 - 2g(1+\beta) + 3\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1 + \beta - 2\beta g}.$$

Indeed, this holds by (6). Therefore $\bar{Z}h'(\bar{A})+g < 1$ if $\bar{Z}h'(\bar{A})+g > 0$ and $h'(\bar{A})\bar{\pi}_{AS} > 0$.

Finally, consider the case with $\bar{Z}h'(\bar{A})+g < 0$. It must be true that $h'(\bar{A})\bar{\pi}_{AS} < 0$. Rearranging the inequality $\bar{Z}h'(\bar{A}) + g > -1$, we find:[46]

$$\begin{aligned} 1 >& -\left[\bar{Z}h'(\bar{A}) + g\right] \\ \Leftrightarrow \sqrt{discrim} >& [-2g - 1 - \beta g^2]\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) + \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + 2(-1-\beta g)h'(\bar{A})\bar{\pi}_{AS}. \end{aligned}$$

The right-hand side must be positive in the region where $h'(\bar{A})\bar{\pi}_{AS}$ is sufficiently large in magnitude to make this inequality bind. Squaring both sides, this becomes:

$$\begin{aligned} 0 <& -g(1+g)(1+\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2 + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS} \\ &+ \beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS} + [-1 - 2g(1+\beta) - 3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] \\ &- [1 + \beta + 2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2. \end{aligned} \tag{A-20}$$

This quadratic opens down. The y-intercept is strictly negative. The derivative at the y-intercept is:

$$\beta[h'(\bar{A})]^2\bar{\pi}_{SS} + [-1 - 2g(1+\beta) - 3\beta g^2]\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] < 0.$$

So both roots are negative. The root that is closer to zero does not bind the inequality of ultimate interest. (Indeed, $\bar{Z}h'(\bar{A}) + g$ is not even negative for $h'(\bar{A})\bar{\pi}_{AS}$ close to

[45] We also saw that the inequality must hold for negative values of $h'(\bar{A})\bar{\pi}_{AS}$, so readers may be confused by the fact that there is a negative root as well. But observe that sufficiently negative $h'(\bar{A})\bar{\pi}_{AS}$ is incompatible with $\bar{Z}h'(\bar{A}) + g > 0$.

[46] Doing so, it is easy to see that $discrim > 0$ if $\bar{Z}h'(\bar{A})+g > -1$ which validates the remaining half of an earlier claim (i.e., now for the case with $\bar{Z}h'(\bar{A})+g < 0$) once we establish that $\bar{Z}h'(\bar{A})+g > -1$.

0.) Observe that the product of the constant and the quadratic coefficient is positive. Therefore, from the quadratic formula, inequality (A-20) holds if

$$h'(\bar{A})\bar{\pi}_{AS} > \frac{[-1-2g(1+\beta)-3\beta g^2]\left[-\bar{\pi}_{AA}+h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right]+\beta[h'(\bar{A})]^2\bar{\pi}_{SS}}{1+\beta+2\beta g}.$$

Indeed, this holds by (6). Therefore $\bar{Z}h'(\bar{A})+g>-1$ if $\bar{Z}h'(\bar{A})+g<0$.

□

We therefore have, when $(S_0-\bar{S})^2$ is not too large and payoffs are quadratic,

$$\lim_{t\to\infty} E_0[A_t]=\bar{A} \text{ and } \lim_{t\to\infty} E_0[S_t]=\bar{S}.$$

F.7 Deriving equation (10)

Expand π_t around $w_t=C$, $A_t=\bar{A}$, and $S_t=\bar{S}$:

$$\begin{aligned}\pi_t =&\bar{\pi}+\bar{\pi}_w(w_t-C)+\bar{\pi}_A(A_t-\bar{A})+\bar{\pi}_S(S_t-\bar{S})\\&+\frac{1}{2}\bar{\pi}_{ww}(w_t-C)^2+\frac{1}{2}\bar{\pi}_{AA}(A_t-\bar{A})^2+\frac{1}{2}\bar{\pi}_{SS}(S_t-\bar{S})^2\\&+\bar{\pi}_{wA}(w_t-C)(A_t-\bar{A})+\bar{\pi}_{wS}(w_t-C)(S_t-\bar{S})+\bar{\pi}_{AS}(A_t-\bar{A})(S_t-\bar{S}),\end{aligned} \tag{A-21}$$

where higher order terms vanish because payoffs are quadratic. Appendix F.6 showed that

$$\lim_{t\to\infty} E_0[A_t]=\bar{A} \text{ and } \lim_{t\to\infty} E_0[S_t]=\bar{S}$$

if $(S_0-\bar{S})^2$ is not too large and payoffs are quadratic. Using these and $E_0[w_t]=C$ for $t>1$, we find:

$$\begin{aligned}\lim_{t\to\infty} E_0[\pi_t] =&\bar{\pi}+\frac{1}{2}\bar{\pi}_{ww}trace(\Sigma)\zeta^2+\frac{1}{2}\bar{\pi}_{AA}E_0[(A_t-\bar{A})^2]+\frac{1}{2}\bar{\pi}_{SS}E_0[(S_t-\bar{S})^2]\\&+\bar{\pi}_{wA}E_0[(w_t-C)(A_t-\bar{A})]+\bar{\pi}_{wS}E_0[(w_t-C)(S_t-\bar{S})]+\bar{\pi}_{AS}E_0[(A_t-\bar{A})(S_t-\bar{S})].\end{aligned}$$

Differentiating and using the quadratic nature of payoffs, we find

$$\lim_{t\to\infty}\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} =\bar{\pi}_w+\bar{\pi}_A\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}+\bar{\pi}_S\frac{\mathrm{d}\bar{S}}{\mathrm{d}C}+\bar{\pi}_K\frac{\mathrm{d}K}{\mathrm{d}C}.$$

Using the quadratic nature of payoffs, the first-order condition for the infrastructure problem is:

$$0=\sum_{t=0}^{\infty}\beta^t\left\{\bar{\pi}_K+\bar{\pi}_{AK}E[A_t^*-\bar{A}]+\bar{\pi}_{SK}E[S_t-\bar{S}]\right\}.$$

From analysis in Appendix F.6 and using $S_0=\bar{S}$ and that expectations in the infrastructure problem do not contain specialized forecasts, the expected actions and stock are here just $\bar{A}$ and $\bar{S}$. The first-order condition for the infrastructure problem becomes

$$\bar{\pi}_K = 0.$$

From equation (A-5),

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}C} = \frac{h'(\bar{A})}{1-g}\frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

Therefore

$$\lim_{t\to\infty} \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} = \bar{\pi}_w + \left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right] \frac{\mathrm{d}\bar{A}}{\mathrm{d}C}.$$

F.8 Deriving equation (13)

Implicitly differentiating equation (A-6), we have:

$$\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} =& \frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS}}{-(1-\beta g)\bar{\pi}_{AA} - \beta h''(\bar{A})\bar{\pi}_S - \beta h'(\bar{A})\frac{h'(\bar{A})}{1-g}\bar{\pi}_{SS} - \frac{1-\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}} \\ &+ \frac{(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}}{-(1-\beta g)\bar{\pi}_{AA} - \beta h''(\bar{A})\bar{\pi}_S - \beta h'(\bar{A})\frac{h'(\bar{A})}{1-g}\bar{\pi}_{SS} - \frac{1-\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS} - \beta h'(\bar{A})\bar{\pi}_{AS}} \frac{\mathrm{d}K}{\mathrm{d}C}.\end{aligned}$$

Substitute for $\beta\bar{\pi}_S$ from equation (A-6):

$$\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} =& \frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS}}{(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \frac{1+\beta-2\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS}} \\ &+ \frac{(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}}{(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \frac{\beta}{1-g}[h'(\bar{A})]^2\bar{\pi}_{SS} - \frac{1+\beta-2\beta g}{1-g}h'(\bar{A})\bar{\pi}_{AS}} \frac{\mathrm{d}K}{\mathrm{d}C}.\end{aligned} \tag{A-22}$$

The denominator is strictly positive if and only if inequality (A-7) holds, which we saw indeeds hold when (6) holds. Therefore

$$\frac{\mathrm{d}\bar{A}}{\mathrm{d}C} \propto (1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + [(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}]\frac{\mathrm{d}K}{\mathrm{d}C}.$$

We have established the result we sought. For later use, note that if $(S_0 - \bar{S})^2$ is not too large and payoffs are quadratic, then, from equation (12),

$$\lim_{t\to\infty} \frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} = \bar{\pi}_w - \frac{1-\beta}{\beta}\bar{\pi}_A \frac{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + [(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}]\frac{\mathrm{d}K}{\mathrm{d}C}}{D}, \tag{A-23}$$

where

$$D \triangleq (1-g)(1-\beta g)\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} - (1+\beta-2\beta g)h'(\bar{A})\bar{\pi}_{AS} \tag{A-24}$$

and $D > 0$ from (6). And observe that, from equations (A-14) and (A-24),

$$\begin{aligned} h'(\bar{A})\bar{\chi} =& D + [1+\beta(1-g)]\left\{g\left[-\bar{\pi}_{AA} - h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] + h'(\bar{A})\bar{\pi}_{AS}\right\} \\ &+ \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right). \end{aligned} \tag{A-25}$$

F.9 Proof of Proposition 2

Expanding S_t around $\bar{A}$ and $\bar{S}$, we have, from Taylor's theorem,

$$S_t = \bar{S} + h'(\bar{A})(A_{t-1} - \bar{A}) + g(S_{t-1} - \bar{S}) + higherorderterms1_t,$$

where $higherorderterms1_t$ is a linear function of terms with $(A_{t-1} - \bar{A})^{\alpha_1}(S_{t-1} - \bar{S})^{\alpha_2}$ for $\alpha_1, \alpha_2 \in \mathbb{Z}_+$ and $\alpha_1 + \alpha_2 > 1$. Substituting for S_t then S_{t-1} and so on, equation (A-21) becomes:

$$\pi_t =\bar{\pi} + \bar{\pi}_w(w_t - C) + \bar{\pi}_A(A_t - \bar{A}) + \bar{\pi}_S h'(\bar{A})\sum_{i=0}^{\infty} g^i(A_{t-1-i} - \bar{A}) + higherorderterms2_t, \tag{A-26}$$

where $higherorderterms2_t$ is a linear function of terms with $(w_t - C)^{\alpha_1}(A_{t-1-k} - \bar{A})^{\alpha_2}(S_{t-1-k} - \bar{S})^{\alpha_3}$ for $k \geq 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_+$, and $\alpha_1 + \alpha_2 + \alpha_3 > 1$. Using that payoffs are quadratic and substituting for S_t and then for A_{t_1} and S_{t-1} and so on, equation (A-16) becomes:

$$\begin{aligned} A_t =& \bar{A} + \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_t - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t} - C) + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t} - C) \\ &+ \bar{Z}h'(\bar{A})\sum_{i=0}^{\infty}[\bar{Z}h'(\bar{A}) + g]^i\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}(w_{t-1-i} - C) + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{1,t-1-i} - C)\right. \\ &\left.\qquad + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}(f_{2,t-1-i} - C)\right] \\ &+ higherorderterms3_t, \end{aligned}$$

where $higherorderterms3_t$ is a linear function of terms with $(w_{t-k} - C)^{\alpha_1}(f_{1,t-k} - C)^{\alpha_2}(f_{2,t-k} - C)^{\alpha_3}$ for $k > 0$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_+$, and $\alpha_1 + \alpha_2 + \alpha_3 > 1$. Using this and its analogues in (A-26), we find

$$\begin{aligned}\Delta\pi_t =& \left[\bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right]\Delta w_t + \bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{1,t} + \bar{\pi}_A \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{2,t} \\ &+ \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \\ &\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\Delta w_{t-1} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{1,t-1} + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{2,t-1}\right] \\ &+ \sum_{i=2}^{\infty}\left\{\bar{\pi}_A \bar{Z} h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A}) g^{i-1} + \bar{\pi}_S \bar{Z}[h'(\bar{A})]^2 \sum_{j=1}^{i-1}[\bar{Z}h'(\bar{A}) + g]^{i-j-1} g^{j-1}\right\} \\ &\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\Delta w_{t-i} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{1,t-i} + \frac{\beta\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\Delta f_{2,t-i}\right] \\ &+ \Delta higherorderterms4_t, \end{aligned} \tag{A-27}$$

where $higherorderterms4_t$ is a linear function of $higherorderterms2_t$ and $higherorderterms3_t$.

The vector of estimated coefficients is

$$\begin{bmatrix}\hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\Lambda}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\gamma}}\end{bmatrix} = E[X^{\intercal}X]^{-1}E[X^{\intercal}\boldsymbol{\pi}],$$

where $\hat{\boldsymbol{\alpha}}$ is a $J \times 1$ vector stacking the $\hat{\alpha}_j$; $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{\gamma}}$ are $I \times 1$ vectors stacking the $\hat{\Lambda}_i$, $\hat{\lambda}_i$, and $\hat{\gamma}_i$; $\boldsymbol{\pi}$ is a $JT \times 1$ vector with rows π_{jt}; and X is a $JT \times (J + 3I)$ matrix with the final $3I$ columns of each row being

$$\begin{bmatrix}\Delta w_{jt} & \dots & \Delta w_{j(t-I)} & \Delta f_{j1,t} & \dots & \Delta f_{j1,t-I} & \Delta f_{j2,t} & \dots & \Delta f_{j2,t-I}\end{bmatrix}.$$

By the Frisch-Waugh Theorem,

$$\begin{bmatrix}\hat{\boldsymbol{\Lambda}} \\ \hat{\boldsymbol{\lambda}} \\ \hat{\boldsymbol{\gamma}}\end{bmatrix} = E[\tilde{X}^{\intercal}\tilde{X}]^{-1}E[\tilde{X}^{\intercal}\tilde{\boldsymbol{\pi}}],$$

where $\tilde{X}$ is a $JT \times 3I$ matrix with rows

$$\begin{bmatrix}\Delta w_{jt} - \mu_j^w & \dots & \Delta w_{j(t-I)} - \mu_j^w & \Delta f_{j1,t} - \mu_j^{f1} & \dots & \Delta f_{j1,t-I} - \mu_j^{f1} & \Delta f_{j2,t} - \mu_j^{f2} & \dots & \Delta f_{j2,t-I} - \mu_j^{f2}\end{bmatrix}$$

and $\tilde{\boldsymbol{\pi}}$ is similarly demeaned $\boldsymbol{\pi}$. Observe that:

$$E[\tilde{X}^\intercal \tilde{\boldsymbol{\pi}}] = JT \begin{bmatrix} Cov[\Delta w_{jt} - \mu_j^w, \Delta \pi_{jt} - \mu_j^\pi] \\ \vdots \\ Cov[\Delta w_{j(t-I)} - \mu_j^w, \Delta \pi_{jt} - \mu_j^\pi] \\ Cov[\Delta f_{j1,t} - \mu_j^{f1}, \Delta \pi_{jt} - \mu_j^\pi] \\ \vdots \\ Cov[\Delta f_{j1,t-I} - \mu_j^{f1}, \Delta \pi_{jt} - \mu_j^\pi] \\ Cov[\Delta f_{j2,t} - \mu_j^{f2}, \Delta \pi_{jt} - \mu_j^\pi] \\ \vdots \\ Cov[\Delta f_{j2,t-I} - \mu_j^{f2}, \Delta \pi_{jt} - \mu_j^\pi] \end{bmatrix}.$$

Following the proof of Proposition 1, $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{\gamma}}$ are independent of $\Delta higherorderterms4_t$ because the ϵ are normally distributed. From here, drop the j subscript to save on unnecessary notation.

Observe that $Cov[\Delta w_{t-k}, \Delta w_{t-k-j}] = Cov[\Delta w_{t-k}, \Delta f_{1,t-k-j}] = Cov[\Delta w_{t-k}, \Delta f_{2,t-k-j}] = 0$ for $j > 3$, that $Cov[\Delta f_{1,t-k}, \Delta w_{t-k-j}] = Cov[\Delta f_{1,t-k}, \Delta f_{1,t-k-j}] = Cov[\Delta f_{1,t-k}, \Delta f_{2,t-k-j}] = 0$ for $j > 2$, and that $Cov[\Delta f_{2,t-k}, \Delta w_{t-k-j}] = Cov[\Delta f_{2,t-k}, \Delta f_{1,t-k-j}] = Cov[\Delta f_{2,t-k}, \Delta f_{2,t-k-j}] = 0$ for $j > 1$. It is obvious from standard regression results on omitted variables bias (and verifiable through tedious algebra) that, for $i < I-2$, the probability limits of $\hat{\Lambda}_i$, $\hat{\lambda}_i$, and $\hat{\gamma}_i$ are identical to the coefficients on, respectively, Δw_{t-i}, $\Delta f_{1,t-i}$, and $\Delta f_{2,t-i}$ in equation (A-27). We then find:

$$\begin{aligned}
&\lim_{I\to\infty} \text{plim} \sum_{i=0}^{I-3} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right] \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
&+ \sum_{i=2}^{\infty} \left\{\bar{\pi}_A \bar{Z} h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A}) g^{i-1} + \bar{\pi}_S \bar{Z}[h'(\bar{A})]^2 \sum_{j=1}^{i-1} [\bar{Z}h'(\bar{A}) + g]^{i-j-1} g^{j-1}\right\} \\
&\qquad \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \left[\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right] \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
&+ \sum_{i=2}^{\infty} \left\{\bar{\pi}_A \bar{Z} h'(\bar{A})[\bar{Z}h'(\bar{A}) + g]^{i-1} + \bar{\pi}_S h'(\bar{A})(\bar{Z}h'(\bar{A}) + g)^{i-1}\right\} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
=&\bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] + \frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right],
\end{aligned}$$

where we used Lemma 2 to establish that the common ratio is less than 1 in magnitude. Substituting for $\bar{\pi}_S$ from equation (A-6),

$$\frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z} h'(\bar{A}) + g]} = -\frac{1}{\beta}\bar{\pi}_A \frac{1 - \beta[\bar{Z} h'(\bar{A}) + g]}{1 - [\bar{Z} h'(\bar{A}) + g]}.$$

Then, using equations (A-14) and (A-17),

$$\begin{aligned}
&\lim_{I\to\infty} \operatorname{plim} \sum_{i=0}^{I-2} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] \\
=&\bar{\pi}_w \\
&- \frac{1-\beta}{\beta}\bar{\pi}_A \left\{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right]\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right\} \\
&\left\{\frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_{AA}}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS} + \sqrt{discrim}\right]\right. \\
&\qquad \left. - g(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_{AA}}{h'(\bar{A})}\right) - (1-\beta g)h'(\bar{A})\bar{\pi}_{AS}\right\}^{-1} \\
=&\bar{\pi}_w \\
&- \frac{1-\beta}{\beta}\bar{\pi}_A \left\{(1-\beta g)\bar{\pi}_{wA} + \beta h'(\bar{A})\bar{\pi}_{wS} + \left[\beta h'(\bar{A})\bar{\pi}_{AS} + \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)\right]\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}\right\} \\
&\left\{D + \beta(1-g)\Psi + \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right]\right\}^{-1},
\end{aligned}$$

where the last equality uses equation (A-24). Using equations (A-6), (A-22), and (A-23),

$$\lim_{I\to\infty} \operatorname{plim} \sum_{i=0}^{I-2} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] = \bar{\pi}_w + \omega\left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right),$$

where

$$\Omega \triangleq \frac{\beta\Psi \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}}{D/(1-g)}, \tag{A-28}$$

$$\begin{aligned}
\omega \triangleq D &\left\{D\right. \\
&\quad + \beta(1-g)\Psi \\
&\quad \left. + \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left[(1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}\right]\right\}^{-1},
\end{aligned} \tag{A-29}$$

and, from equation (16),

$$\Psi \triangleq h'(\bar{A})\bar{\pi}_{AS} + g \underbrace{\left(-\bar{\pi}_{AA} + \frac{h''(\bar{A})}{h'(\bar{A})}\bar{\pi}_A\right)}_{>0 \text{ by } (3)}.$$

D, from equation (A-24), is positive if and only if inequality (A-7) holds, which we saw indeeds hold by (6). Observe that, from (A-22), the denominator of $\mathrm{d}\bar{A}/\,\mathrm{d}C$ is $D/(1-g)$.

Analyze ω by considering the divergence between the terms in curly braces in (A-29) and D. First, if $\beta\Psi = 0$, then the second line in curly braces is zero and, from equation (A-10), so is the third line in curly braces. Therefore $\omega = 1$ if $\beta\Psi = 0$.

Next, if $\beta\Psi < 0$, then the second line in curly braces is strictly negative. Further, $h'(\bar{A})\bar{\pi}_{AS}$ must be weakly negative. From equation (A-10), $\beta\Psi < 0$ and $h'(\bar{A})\bar{\pi}_{AS} \leq 0$ imply that the third line in curly braces is negative. Using (A-25), the denominator of ω is strictly greater than $h'(\bar{A})\bar{\chi}$ when $\Psi < 0$. From equation (A-14), Lemma 1, and inequality (2), $h'(\bar{A})\,\bar{\chi} > 0$. Therefore the denominator of ω is strictly positive when $\Psi < 0$. And because the combined terms in curly braces in (A-29) are strictly less than D, we have established that $\omega > 1$ if $\beta\Psi < 0$.

If $\beta\Psi > 0$, then the second line in curly braces in (A-29) is strictly positive. From equation (A-10), the third line in curly braces is positive if $\beta\Psi > 0$ and $h'(\bar{A})\bar{\pi}_{AS}$ is not too much greater than 0. In that case, $\omega < 1$.

Finally, consider $\beta\Psi > 0$ with $h'(\bar{A})\bar{\pi}_{AS}$ strictly positive and sufficiently large to make the third line in curly braces negative. Consider whether that line can be so negative as to overwhelm the positive second line in curly braces and make $\omega > 1$. Those final two lines in curly braces are strictly positive with $h'(\bar{A})\bar{\pi}_{AS} > 0$ if and only if

$$\sqrt{discrim} > [1 + \beta g^2 - 2\beta g]\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right) - 2\beta(1-g)h'(\bar{A})\bar{\pi}_{AS} - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}.$$

Squaring both sides, this inequality holds if and only if

$$\begin{aligned} 0 < & g(1-g)(1-\beta g)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)^2 - \beta g\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right)[h'(\bar{A})]^2\bar{\pi}_{SS} \\ & - \beta[h'(\bar{A})]^2\bar{\pi}_{SS}h'(\bar{A})\bar{\pi}_{AS} + [1 - 2g(1+\beta) + 3\beta g^2]h'(\bar{A})\bar{\pi}_{AS}\left[-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})}\right] \\ & - [1 + \beta - 2\beta g]\left(h'(\bar{A})\bar{\pi}_{AS}\right)^2. \end{aligned}$$

This last inequality is identical to inequality (A-19), which we saw holds by (6). Therefore $\omega < 1$ if $\beta\Psi > 0$.

F.10 Proof of Corollary 3

First consider $I' > 1$. As described in the proof of Proposition 2, the probability limits of $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ are, for $i < I-2$, identical to the coefficients on Δw_{t-i} and Δf_{t-i} in equation (A-27). Using Lemma 2 to establish that the common ratio is not equal to 1, we obtain:

$$\begin{aligned}\text{plim}\sum_{i=0}^{I'}\left[\hat{\Lambda}_i+\hat{\lambda}_i\right] =&\bar{\pi}_w+\bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right]+\left[\bar{\pi}_A\bar{Z}h'(\bar{A})+\bar{\pi}_S h'(\bar{A})\right]\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right]\\&+\sum_{i=2}^{I'}\left\{\bar{\pi}_A\bar{Z}h'(\bar{A})[\bar{Z}h'(\bar{A})+g]^{i-1}+\bar{\pi}_S h'(\bar{A})(\bar{Z}h'(\bar{A})+g)^{i-1}\right\}\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right]\\=&\bar{\pi}_w+\bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right]\\&+\left[1-[\bar{Z}h'(\bar{A})+g]^{I'}\right]\frac{\bar{\pi}_A\bar{Z}h'(\bar{A})+\bar{\pi}_S h'(\bar{A})}{1-[\bar{Z}h'(\bar{A})+g]}\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].\end{aligned}$$

Substituting for $\bar{\pi}_S$ from equation (A-6),

$$\begin{aligned}&\bar{\pi}_A+\left[1-[\bar{Z}h'(\bar{A})+g]^{I'}\right]\frac{\bar{\pi}_A\bar{Z}h'(\bar{A})+\bar{\pi}_S h'(\bar{A})}{1-[\bar{Z}h'(\bar{A})+g]}\\=&-\frac{1-\beta}{\beta}\bar{\pi}_A\frac{1}{1-[\bar{Z}h'(\bar{A})+g]}\left\{1-[\bar{Z}h'(\bar{A})+g]^{I'}\frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right\}.\end{aligned}$$

For $I' = 1$, we have:

$$\text{plim}\sum_{i=0}^{1}\left[\hat{\Lambda}_i+\hat{\lambda}_i\right]=\bar{\pi}_w+\bar{\pi}_A\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right]+\left[\bar{\pi}_A\bar{Z}h'(\bar{A})+\bar{\pi}_S h'(\bar{A})\right]\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}+\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].$$

Substituting for $\bar{\pi}_S$ from equation (A-6) and rearranging, we find:

$$\bar{\pi}_A+[\bar{\pi}_A\bar{Z}h'(\bar{A})+\bar{\pi}_S h'(\bar{A})]=-\frac{1-\beta}{\beta}\bar{\pi}_A\frac{1}{1-[\bar{Z}h'(\bar{A})+g]}\left[1-[\bar{Z}h'(\bar{A})+g]\frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right].$$

Using these results and following the analysis of Proposition 2, we have, for $I' \geq 1$,

$$\text{plim}\sum_{i=0}^{I'}\left[\hat{\Lambda}_i+\hat{\lambda}_i\right]=\bar{\pi}_w+\omega_{I'}\left[\bar{\pi}_A+\bar{\pi}_S\frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial\bar{A}(K,C)}{\partial C}+\Omega\right),$$

where

$$\omega_{I'}\triangleq\left\{1-[\bar{Z}h'(\bar{A})+g]^{I'}\frac{1-\beta[\bar{Z}h'(\bar{A})+g]}{1-\beta}\right\}\omega. \tag{A-30}$$

and where Ω and ω are as in Proposition 2. From equation (A-17), $\bar{Z}h'(\bar{A}) + g \propto \Psi$. Thus $\bar{Z}h'(\bar{A}) + g = 0$ if $g = \bar{\pi}_{AS} = 0$. In that case, $\omega_{I'} = \omega$ for all $I' \geq 1$. If $\Psi > 0$, then, using Lemma 2, the combined terms in curly braces in (A-30) are strictly positive, strictly less than 1, and strictly increasing in I'. In that case, following the analysis in Proposition 2, $\omega_{I'} \in (0, \omega)$ and $\omega_{I'}$ increases in I'. If $\Psi < 0$, then the combined terms in curly braces in (A-30) are strictly greater than 1 for I' odd. The statement of the corollary follows from the analysis of Proposition 2.

F.11 Proof of Corollary 4

It is obvious that

$$\lim_{\beta \to 0} \Omega = 0$$

From equation (A-10),

$$\lim_{\beta \to 0} discrim = \left((1 - \beta g^2) \left(-\bar{\pi}_{AA} + h''(\bar{A}) \frac{\bar{\pi}_A}{h'(\bar{A})} \right) - \beta [h'(\bar{A})]^2 \bar{\pi}_{SS} \right)^2 .$$

Using (A-29), this implies

$$\lim_{\beta \to 0} \omega = 1.$$

Also observe that

$$\lim_{\beta \to 0} \hat{\lambda}_i = 0$$

because

$$\lim_{\beta \to 0} \beta \Gamma = 0.$$

The corollary follows from Proposition 2.

F.12 Proof of Corollary 5

First, observe that, with $\pi_{AS} = 0$,

$$\lim_{g \to 0} \Omega = 0$$

because

$$\lim_{g \to 0} \Psi = 0.$$

From equation (A-10),

$$\lim_{g \to 0} discrim = \left((1 - \beta g^2) \left(-\bar{\pi}_{AA} + h''(\bar{A}) \frac{\bar{\pi}_A}{h'(\bar{A})} \right) - \beta [h'(\bar{A})]^2 \bar{\pi}_{SS} \right)^2 .$$

Using (A-29), this implies

$$\lim_{g\to 0} \omega = 1.$$

Finally, recall that Corollary 3 established that $\omega_{I'} = \omega$ when $\Psi = 0$. The corollary follows from Proposition 2.

F.13 Proof of Proposition 6

Following the proof of Proposition 2 and using equation (A-27), we have:

$$\begin{aligned}
\text{plim}\,\hat{\Lambda}_0 =& \bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\Lambda}_1 =& \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\Lambda}_2 =& [\bar{Z} h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_0 =& \bar{\pi}_A \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_1 =& \left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}},\\
\text{plim}\,\hat{\lambda}_2 =& [\bar{Z} h'(\bar{A}) + g]\left\{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})\right\} \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.
\end{aligned}$$

Observe that

$$\text{plim}\,\frac{\hat{\lambda}_0}{\hat{\lambda}_1} = \frac{\bar{\pi}_A}{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}$$

and

$$\text{plim}\,\frac{\hat{\Lambda}_2}{\hat{\Lambda}_1} = \bar{Z} h'(\bar{A}) + g. \tag{A-31}$$

Substitute for $\bar{\pi}_S h'(\bar{A})$ from (A-6):

$$\text{plim}\,\frac{\hat{\lambda}_0}{\hat{\lambda}_1} = \frac{1}{\bar{Z} h'(\bar{A}) + g - \frac{1}{\beta}}.$$

This is strictly less than 0 by Lemma 2. And, using equations (A-17) and (A-31),

$$\Psi \propto \text{plim}\,\frac{\hat{\Lambda}_2}{\hat{\Lambda}_1}.$$

Substituting for $\mathrm{d}\bar{A}/\,\mathrm{d}C$ from Appendix F.8, equation (12) becomes:

$$\lim_{t\to\infty}\frac{\mathrm{d}E_0[\pi_t]}{\mathrm{d}C} = \overbrace{\bar{\pi}_w}^{\text{direct effects}} \overbrace{-\frac{1-\beta}{\beta}\bar{\pi}_A\frac{\bar{\pi}_{wA}}{D}}^{\text{ex-post adaptation}} \overbrace{-\frac{1-\beta}{\beta}\bar{\pi}_A\frac{\beta[h'(\bar{A})\bar{\pi}_{wS} - g\,\bar{\pi}_{wA}]}{D}}^{\text{ex-ante adaptation}} \underbrace{-\frac{1-\beta}{\beta}\frac{\bar{\pi}_A}{D}\left[(1-\beta g)\bar{\pi}_{AK} + \beta h'(\bar{A})\bar{\pi}_{SK}\right]\frac{\mathrm{d}K}{\mathrm{d}C}}_{\text{interactions with long-lived infrastructure}}, \tag{A-32}$$

with $D > 0$ itself a function of cross-partials.[47]

Rearranging the foregoing results, we find:

$$\bar{\pi}_A\frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}} = \operatorname{plim}\hat{\lambda}_0, \qquad \bar{\pi}_A\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} = \operatorname{plim}\hat{\Lambda}_1\frac{\hat{\lambda}_0}{\hat{\lambda}_1}, \qquad \bar{\pi}_w = \operatorname{plim}\left(\hat{\Lambda}_0 - \hat{\Lambda}_1\frac{\hat{\lambda}_0}{\hat{\lambda}_1}\right).$$

Using these results and labeling pieces as in (A-32), we can calculate the overall effect of climate:

$$\begin{aligned}
&\operatorname{plim}\left(\hat{\Lambda}_0 - \hat{\Lambda}_1\frac{\hat{\lambda}_0}{\hat{\lambda}_1} - \frac{1-\beta}{\beta}\left[\hat{\Lambda}_1\frac{\hat{\lambda}_0}{\hat{\lambda}_1} + \hat{\lambda}_0\right]\right)\\
&=\bar{\pi}_w - \frac{1-\beta}{\beta}\bar{\pi}_A\left\{\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\,\bar{\chi}}\right\}\\
&=\bar{\pi}_w - \frac{D}{h'(\bar{A})\bar{\chi}}\frac{1-\beta}{\beta}\bar{\pi}_A\left(\frac{\bar{\pi}_{wA}}{D} + \frac{\beta[h'(\bar{A})\bar{\pi}_{wS} - g\,\bar{\pi}_{wA}]}{D} + \frac{\Omega}{1-g}\right)\\
&=\bar{\pi}_w + \frac{D}{h'(\bar{A})\bar{\chi}}\left[\bar{\pi}_A + \bar{\pi}_S\frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial\bar{A}(K,C)}{\partial C} + \Omega\right).
\end{aligned} \tag{A-33}$$

The second line uses foregoing results to express the calculation in terms of model primitives. The third line substitutes for Γ. Substituting $\mathrm{d}\bar{A}/\,\mathrm{d}C$ and also $\bar{\pi}_S$ from the Euler equation (11), the final line indicates how close we get to the true effect of climate from (10).

Consider the bias Ω. Following the proof of Proposition 2 and using equation (A-27), we have:

$$\operatorname{plim}\hat{\gamma}_0 = \bar{\pi}_A\frac{\beta\,\Psi}{h'(\bar{A})\bar{\chi}}\frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}.$$

Therefore

$$\operatorname{plim}\frac{\hat{\gamma}_0}{\hat{\lambda}_0} = \frac{\beta\,\Psi}{h'(\bar{A})\bar{\chi}}.$$

[47]See equations (A-23) and (A-24) in Appendix F.8. Note that D absorbs the $1-g$ in the denominator of (12).

From equation (A-28),

$$\bar{\pi}_A \frac{\Omega}{1-g} = \bar{\pi}_A \frac{h'(\bar{A})\bar{\chi}}{D} \frac{\hat{\gamma}_0}{\hat{\lambda}_0} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}.$$

Substituting from foregoing results for $\bar{\pi}_A \bar{\pi}_{wA}/[h'(\bar{A})\,\bar{\chi}]$, we find:

$$\frac{D}{h'(\bar{A})\bar{\chi}} \bar{\pi}_A \frac{\Omega}{1-g} = \text{plim}\, \frac{\hat{\gamma}_0}{\hat{\lambda}_0} \hat{\Lambda}_1 \frac{\hat{\lambda}_0}{\hat{\lambda}_1}.$$

Combining this with (A-33) and defining $\tilde{\omega} \triangleq D/[h'(\bar{A})\bar{\chi}]$ yields equation (19) in the proposition.

Using equation (A-25), we have:

$$\begin{aligned}\frac{D}{h'(\bar{A})\bar{\chi}} =& D\Bigg\{ D \\ &+ [1+\beta(1-g)]\Psi \\ &+ \frac{1}{2}\sqrt{discrim} - \frac{1}{2}\left((1-\beta g^2)\left(-\bar{\pi}_{AA} + h''(\bar{A})\frac{\bar{\pi}_A}{h'(\bar{A})} \right) - \beta[h'(\bar{A})]^2 \bar{\pi}_{SS} \right) \Bigg\}^{-1}.\end{aligned}$$

Comparing to equation (A-29), we here have a coefficient of $[1+\beta(1-g)]$ on Ψ instead of $\beta(1-g)$. The analysis of $\Psi \leq 0$ is as in the case of $\beta\Psi \leq 0$ from before, except now $\beta = 0$ does not bring the second line in curly braces to zero. For $\Psi > 0$, note that it is now even harder for the third line in curly braces to overwhelm the second line, so if that could not happen for ω with $\beta\Psi > 0$, then it cannot happen here either for $\Psi > 0$.

F.14 Proof of Proposition 7

Following the proof of Proposition 2 and using equation (A-27), we have:

$$\begin{aligned}\text{plim}\,\hat{\phi}_0 =& \bar{\pi}_w + \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}, \\ \text{plim}\,\hat{\phi}_1 =& \left\{ \bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A}) \right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}, \\ \text{plim}\,\hat{\phi}_2 =& [\bar{Z}h'(\bar{A}) + g] \left\{ \bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A}) \right\} \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}.\end{aligned}$$

Substitute for $\bar{\pi}_S h'(\bar{A})$ from (A-6):

$$\text{plim}\,\hat{\phi}_1 = \left\{ \bar{Z}h'(\bar{A}) + g - \frac{1}{\beta} \right\} \bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}}$$

Observe that

$$\text{plim}\,\frac{\hat{\phi}_2}{\hat{\phi}_1} = \bar{Z}h'(\bar{A}) + g. \tag{A-34}$$

From Lemma 2,

$$\left|\text{plim}\,\frac{\hat{\phi}_2}{\hat{\phi}_1}\right| < 1.$$

And using equations (A-17) and (A-34),

$$\Psi \propto \text{plim}\,\frac{\hat{\phi}_2}{\hat{\phi}_1}.$$

Rearranging the foregoing results, we have:

$$\bar{\pi}_A \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} = \text{plim}\,\frac{\hat{\phi}_1}{\frac{\hat{\phi}_2}{\hat{\phi}_1} - \frac{1}{\beta}}, \qquad \bar{\pi}_w = \text{plim}\left(\hat{\phi}_0 - \frac{\hat{\phi}_1}{\frac{\hat{\phi}_2}{\hat{\phi}_1} - \frac{1}{\beta}}\right).$$

The proposition follows the proof of Proposition 6 from here.

F.15 Proof of Proposition A-1

Let there be N aggregated timesteps in total. We seek

$$\begin{bmatrix} \hat{\breve{\Lambda}} \\ \hat{\breve{\lambda}} \end{bmatrix} = E[\tilde{X}^\intercal \tilde{X}]^{-1} E[\tilde{X}^\intercal \tilde{\breve{\boldsymbol{\pi}}}], \tag{A-35}$$

where, guided by previous proofs, $\tilde{X}$ is a $JN \times 2$ matrix with rows

$$\begin{bmatrix} \Delta\breve{w}_{js} - \mu_j^w & \Delta\breve{f}_{j1,s} - \mu_j^{f1} \end{bmatrix}$$

and $\tilde{\breve{\boldsymbol{\pi}}}$ is demeaned $\breve{\boldsymbol{\pi}}$. Observe that:

$$E[\tilde{X}^\intercal \tilde{\breve{\boldsymbol{\pi}}}] = JN \begin{bmatrix} Cov[\Delta\breve{w}_{js} - \mu_j^w, \Delta\breve{\pi}_{js} - \mu_j^\pi] \\ Cov[\Delta\breve{f}_{j1,s} - \mu_j^{f1}, \Delta\breve{\pi}_{js} - \mu_j^\pi] \end{bmatrix}.$$

From here, drop the j subscript to avoid excess notation. Observe that

$$\Delta\breve{w}_{js} = \frac{1}{\delta} \sum_{t=s}^{s+\delta-1} \Delta w_t$$

and analogously for other variables. After applying the Frisch-Waugh Theorem to partial out the effects of forecasts, correlations between payoffs and weather within a timestep are controlled by the coefficients on weather in equation (A-27), with weather earlier in the timestep appearing as a lag for payoffs later in the timestep. In addition, variation in Δw_s (i.e., the first difference within the long timestep) also picks up the effect of $\Delta f_{1,s-1}$ (i.e., the last forecast issued in the previous long timestep) because the latter variable is missing from $\Delta \breve{f}_{1,s}$. We then have:

$$\text{plim}\, \hat{\mathring{\Lambda}} = \text{plim} \sum_{i=0}^{\delta-1} \frac{\delta - i}{\delta} \hat{\Lambda}_i + \text{plim}\, \frac{1}{\delta} \sum_{i=0}^{\delta-1} \frac{\Sigma_{22}}{\Sigma_{11} + \Sigma_{22}} \hat{\lambda}_{i+1}.$$

Analogously, we find:

$$\text{plim}\, \hat{\mathring{\lambda}} = \text{plim} \sum_{i=0}^{\delta-1} \frac{\delta - i}{\delta} \hat{\lambda}_i.$$

Therefore:

$$\begin{aligned}
\text{plim}\left(\hat{\mathring{\Lambda}} + \hat{\mathring{\lambda}}\right) =& \text{plim}\left(\sum_{i=0}^{\delta-1} \frac{\delta - i}{\delta}\left[\hat{\Lambda}_i + \hat{\lambda}_i\right] + \frac{1}{\delta}\sum_{i=0}^{\delta-1} \frac{\Sigma_{22}}{\Sigma_{11} + \Sigma_{22}} \hat{\lambda}_{i+1}\right) \\
=& \text{plim}\left(\hat{\Lambda}_0 + \hat{\lambda}_0 + \sum_{i=1}^{\delta-1} \frac{\delta - i}{\delta} \hat{\Lambda}_i + \sum_{i=1}^{\delta-1} \frac{\delta - i + \frac{\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}}{\delta} \hat{\lambda}_i + \frac{\frac{\Sigma_{22}}{\Sigma_{11}+\Sigma_{22}}}{\delta} \hat{\lambda}_\delta\right).
\end{aligned} \tag{A-36}$$

The coefficients on $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ are each $\in [0,1]$. In the proof of Corollary 3 (Appendix F.10), we established that, for $I \geq I' + 3$,

$$\begin{aligned}
\text{plim} \sum_{i=0}^{I'} \left[\hat{\Lambda}_i + \hat{\lambda}_i\right] =& \bar{\pi}_w + \bar{\pi}_A \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
&+ \left[1 - [\bar{Z}h'(\bar{A}) + g]^{I'}\right] \frac{\bar{\pi}_A \bar{Z} h'(\bar{A}) + \bar{\pi}_S h'(\bar{A})}{1 - [\bar{Z}h'(\bar{A}) + g]} \left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right] \\
=& \bar{\pi}_w - \frac{1-\beta}{\beta} \bar{\pi}_A \frac{1}{1 - [\bar{Z}h'(\bar{A}) + g]} \left\{1 - [\bar{Z}h'(\bar{A}) + g]^{I'} \frac{1 - \beta[\bar{Z}h'(\bar{A}) + g]}{1-\beta}\right\} \\
&\left[\frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].
\end{aligned}$$

In equation (A-36), each of the coefficients $\hat{\Lambda}_i$ and $\hat{\lambda}_i$ is weighted by a fraction. Using $I' = \delta$ in the previous expression, there exist $x_1 \in (0,1)$ and $x_2 \in (0,1)$ such that, for $\Psi > 0$,

$$\begin{aligned}
\text{plim}\left(\hat{\mathring{\Lambda}} + \hat{\mathring{\lambda}}\right) =& \bar{\pi}_w - \frac{1-\beta}{\beta} \bar{\pi}_A \frac{1}{1 - [\bar{Z}h'(\bar{A}) + g]} \left\{1 - [\bar{Z}h'(\bar{A}) + g]^{I'} \frac{1 - \beta[\bar{Z}h'(\bar{A}) + g]}{1-\beta}\right\} \\
&\left[x_1 \frac{\bar{\pi}_{wA}}{h'(\bar{A})\,\bar{\chi}} + x_2 \frac{\beta\Gamma}{h'(\bar{A})\bar{\chi}}\right].
\end{aligned}$$

Following the proof of Corollary F.10, there exists $x \in [x_1, x_2]$ (or $x \in [x_2, x_1]$ if $x_2 < x_1$) such that, for $\Psi > 0$,

$$\text{plim}\left(\mathring{\hat{\Lambda}} + \mathring{\hat{\lambda}}\right) = \bar{\pi}_w + x\,\omega_\delta \left[\bar{\pi}_A + \bar{\pi}_S \frac{h'(\bar{A})}{1-g}\right]\left(\frac{\partial \bar{A}(K,C)}{\partial C} + \Omega\right).$$

The statement of the proposition follows from defining $\breve{\omega} \triangleq x\,\omega_\delta$.

References from the Appendix

Burke, Marshall and Kyle Emerick (2016) "Adaptation to climate change: Evidence from US agriculture," *American Economic Journal: Economic Policy*, Vol. 8, No. 3, pp. 106–140.

Colacito, Riccardo, Bridget Hoffmann, and Toan Phan (2019) "Temperature and growth: A panel analysis of the United States," *Journal of Money, Credit and Banking*, Vol. 51, No. 2-3, pp. 313–368.

Dell, Melissa, Benjamin F. Jones, and Benjamin A. Olken (2012) "Temperature shocks and economic growth: Evidence from the last half century," *American Economic Journal: Macroeconomics*, Vol. 4, No. 3, pp. 66–95.

Ghanem, Dalia and Aaron Smith (2021) "What are the benefits of high-frequency data for fixed effects panel models?" *Journal of the Association of Environmental and Resource Economists*, Vol. 8, No. 2, pp. 199–234.

Hsiang, Solomon M. (2016) "Climate econometrics," *Annual Review of Resource Economics*, Vol. 8, pp. 43–75.

Stokey, Nancy and Robert E. Lucas, Jr. (1989) *Recursive Methods in Economic Dynamics*, Cambridge Mass.: Harvard University Press.