Online Appendix D: Additional Proofs

D.1 Bounds on $\bar{U}$ ensuring non-negative equilibrium wages.

We make explicit here the restrictions on $\bar{U}$ ensuring that: (i) $z_H(t) \geq 0$, for all t;[36] (ii) firms want to keep L types on board. We then provide sufficient conditions for all of them to hold jointly. In Region III, $z_H(t) = \bar{U} + \hat{y}_L(t)\Delta\theta - u(y^*) - \theta_H y^*$ is decreasing, so it suffices that $\lim_{t\to+\infty} z_H(t) = z_H^m \geq 0$. In Regions I and II, $z_H(t) = U_L(t) - \theta_L \hat{y}_H(t) - u(\hat{y}_H(t))$; its variations with t are ambiguous, but since $U(t)$ is (weakly) declining toward $\bar{U}$ and $\hat{y}_H(t)$ strictly decreasing from $\hat{y}_H^c(0) = y_H^c$, it is bounded below by $\bar{U} - \theta_L y_H^c - u(y_H^c)$. Combining this with (16), we require:

$$\bar{U}_{\min} \equiv \max\left\{u(y^*) + \theta_H y^* - y_L^m \Delta\theta,\ u(y_H^c) + \theta_L y_H^c\right\} \leq \bar{U} \leq w(y_L^m) + \theta_L B - (q_H/q_L) y_L^m \Delta\theta \equiv \bar{U}_{\max}.$$

This defines a nonempty interval for $\bar{U}$ as long as $\bar{U}_{\min} < \bar{U}_{\max}$, which can be insured in at least two ways. First, for q_L close enough to 1 (thus also satisfying the requirement of (30)) y_L^m is close to y^*, so $\bar{U}_{\min} \approx\ u(y_H^c) + \theta_L y_H^c < w(y_H^c) + \theta_L y_H^c < w(y^*) + \theta_L B \approx \bar{U}_{\max}$, ensuring the result.

Alternatively, one can slightly modify firms' technology so that the revenue generated by each worker of type θ becomes instead $Aa + B(\theta + b) + \bar{d}$, where $\bar{d}$ is a constant reflecting some other "basic" activity, performed at a fixed (e.g., perfectly monitored) level by all employees, and for which their compensation is therefore part of the fixed wage z. This augments total surplus $w(y)$ by the same amount $\bar{d}$, which can be made large enough to ensure that $\bar{U}_{\min} < \bar{U}_{\max}$. ■

D.2 General optimization program

Let $\widehat{\mathcal{C}} \equiv (\hat{U}_H, \hat{U}_L, \hat{y}_H, \hat{y}_L)$ denote the (presumptive) symmetric-equilibrium strategies and payoffs, given in Proposition 4, and played by the other firm. For all $u \in \mathbb{R}$, let $\mathcal{X}(u) \equiv \min\{\max\{u, 0\}, 2t\}$. The firm's general problem is to choose (U_H, U_L, y_H, y_L) to solve the program:

$$\max \Big\{ q_H \mathcal{X}(U_H + t - \hat{U}_H)[w(y_H) + \theta_H B - U_H$$
$$+ q_L\, \mathcal{X}(U_L + t - \hat{U}_L) \mathbf{1}_{\{U_L \geq \bar{U}\}}[w(y_L) + \theta_L B - U_L] \Big\} \tag{D.1}$$

subject to:

$$U_H \geq U_L + y_L \Delta\theta \tag{D.2}$$
$$U_L \geq U_H - y_H \Delta\theta \tag{D.3}$$
$$y_L \geq 0. \tag{D.4}$$

Note that the objective function (D.1) is not everywhere differentiable, nor (as we shall see), is it globally concave. Note also that if either $U_L \leq \hat{U}_L - t$ or $U_L < \bar{U}$, the firm employs zero (measure of) low types, in which case it clearly must sell to a positive measure of H agents, requiring $U_H > \max\{\hat{U}_H - t, \bar{U}\}$. We first rule out such "exclusion" of low-skill workers, and likewise for high-skill ones. We then show that is also not optimal to "corner" the market on either type.

[36] Recall that $z_L(t) \geq z_H(t)$ everywhere by incentive compatibility. As to the bonus rates $yi(t)$, they all are bounded below by y_L^m, which is positive since we have assumed that $w'(0) > (q_H/q_L)\Delta\theta$.

D.2.1 No exclusion

Lemma 9 *There exists $\bar{q}_L \in [q_L^*, 1)$, independent of t, such that, for all $q_L \geq \bar{q}_L$, it is strictly suboptimal not to employ a positive measure of L-type agents. In particular, $U_L \geq \bar{U}$.*

Proof. Selling only to H agents under some contract (y_H, U_H) is less profitable than sticking to the symmetric strategy $(\hat{y}_H, \hat{U}_H)$ if

$$\begin{aligned} q_H \pi_H &\equiv q_H \chi(U_H - \hat{U}_H + t)\,[w(y_H) + B\theta_H - U_H] \\ &\leq q_H t[w(\hat{y}_H) + B\theta_H - \hat{U}_H] + q_L t[w(\hat{y}_L) + B\theta_L - \hat{U}_L] \equiv q_H \hat{\pi}_H + q_L \hat{\pi}_L \equiv \hat{\pi}. \end{aligned} \tag{D.5}$$

For any $t > 0$, $\hat{\pi}_L > 0$, so the inequality is satisfied for q_H low enough, or equivalently q_L/q_H large enough. To ensure a lower bound independent of t, however, the ratio $(\pi_H - \hat{\pi}_H)/\hat{\pi}_L$ must remain bounded above as t tends to zero, even though $\lim_{t \to 0} \hat{\pi}_L = 0$. We will in fact show a stronger property, namely that $\pi_H(t) < \hat{\pi}_H(t)$ for t small enough.

Observe first that to exclude the L types, it must be that $U_L \leq \max\{\bar{U}, \hat{U}_L - t\}$. For all $t < t_1$ we have $\hat{U}_L > \bar{U}$, so for small t the relevant constraint is $U_L \leq \hat{U}_L - t$. The firm thus solves:

$$\begin{array}{ll} \max \Big\{ \chi(U_H - \hat{U}_H + t)\,[w(y_H) + B\theta_H - U_H] \Big\}, & \text{subject to:} \\ U_H \geq U_L + y_L \Delta\theta & (\mu_H) \\ U_L \geq U_H - y_H \Delta\theta & (\mu_L) \\ U_L \;\leq\; \hat{U}_L - t & (\varphi) \\ y_L \geq 0 & (\psi). \end{array}$$

To have a positive share of the H types it must be that $U_H - \hat{U}_H > -t > U_L - \hat{U}_L$, therefore $U_H - U_L > \hat{U}_H - \hat{U}_L = \hat{y}_H \Delta\theta$, implying $y_H > \hat{y}_H$.[37] Consider now the first-order conditions:

$$\begin{array}{l} -2t \leq \mu_L - \mu_H \leq w(y_H) + B\theta_H - 2U_H + \hat{U}_H - t, \\ \text{with equality for } U_H - \hat{U}_H > t \text{ and } U_H - \hat{U}_H < t, \text{ respectively;} \\ -\mu_H + \mu_L - \varphi = 0, \\ \chi(U_H - \hat{U}_H + t) w'(y_H) + \mu_L \Delta\theta = 0, \\ \psi - \mu_H \Delta\theta = 0. \end{array}$$

If $\mu_L = 0$, the third condition and $\chi(U_H - \hat{U}_H + t) > 0$ imply $y_H = y^* \leq \hat{y}_H$, a contradiction. Therefore $\mu_L > 0$, so that $U_H - U_L = y_H \Delta\theta$, with $\hat{y}_H < y_H$. Next, it cannot be that $\psi > 0$, otherwise $y_L = 0$ and $U_H - U_L = y_L \Delta\theta$ so $y_H = y_L = 0$, another contradiction. Hence $\mu_H = 0$, so $\varphi = \mu_L > 0, U_L = \hat{U}_L - t$. Since $\hat{U}_H - \hat{U}_L = \hat{y}_H \Delta\theta$ for $t \leq t_2$ this implies $U_H - \hat{U}_H + t = (y_H - \hat{y}_H)\Delta\theta$, which furthermore cannot exceed $2t$, since $-2t < \mu_L - \mu_H$. Thus, $\chi(U_H - \hat{U}_H + t) = U_H - \hat{U}_H + t$. Next, eliminating μ_L,

[37] We ignore the constraint $U_H \geq \bar{U}$, since the result is trivial if it does not hold ($\pi_H(t) = 0$).

$$w(y_H) + B\theta_H - 2U_H + \hat{U}_H - t + (U_H - \hat{U}_H + t)w'(y_H)/\Delta\theta \geq 0, \tag{D.6}$$

with equality for $U_H - \hat{U}_H < t$.

We also have, from (36) and (38)-(39) with $\hat{y}_L = y^*$, a similar condition (with equality) for $\hat{y}_H$:

$$w(\hat{y}_H) + B\theta_H - \hat{U}_H - t + tw'(\hat{y}_H)/\Delta\theta = 0. \tag{D.7}$$

Subtracting and replacing $U_H - \hat{U}_H + t$ by $(y_H - \hat{y}_t)\Delta\theta$ yields:

$$\begin{aligned} \Upsilon(y_H; \hat{y}_H, t) \equiv w(y_H) - w(\hat{y}_H) - 2\left[(y_H - \hat{y}_H)\Delta\theta - t\right] \\ +(y_H - \hat{y}_H)w'(y_H) - tw'(\hat{y}_H)/\Delta\theta \geq 0, \end{aligned} \tag{D.8}$$

with equality for $U_H - \hat{U}_H < t$. It cannot be that $U_H - \hat{U}_H = t$, moreover, otherwise $(y_H - \hat{y}_H)\Delta\theta = 2t$ and $\Upsilon(y_H; \hat{y}_H, t) = w(y_H) - w(\hat{y}_H) - 2t + [2w'(y_H) - w'(\hat{y}_H)](t/\Delta\theta) < 0$, a contradiction. Therefore (D.8) is an equality, and since $\partial\Upsilon/\partial y_H = 2w'(y_H) - 2\Delta\theta + y_H w''(y_H) + t\left[w'(y_H) - w'(\hat{y}_H)\right] < 0$, it uniquely defines y_H as a function $y_H = Y(\hat{y}_H, t)$. Taken now as a function of t, $y_H(t) = Y(\hat{y}_H(t), t)$ tends to $Y(\hat{y}_H(0), 0) = \hat{y}_H(0) = y^c_H$, as can be seen from taking limits in (D.8) as an equality. A Taylor expansion of $\Upsilon(y_H(t); \hat{y}_H(t), t) = 0$ then yields

$$\begin{aligned} 2\left[\Delta\theta - w'(y^c_H)\right](y_H(t) - \hat{y}_H(t)) &= t\left[2 - w'(y^c_H)/\Delta\theta\right] + \mathcal{O}(t^2) \Rightarrow \\ y_H(t) - \hat{y}_H(t) &= \omega t + \mathcal{O}(t^2), \end{aligned} \tag{D.9}$$

where $\omega \equiv [2 - w'(y^c_H)/\Delta\theta] / [2\Delta\theta - 2w'(y^c_H)] \in (0, 1)$. Turning now to the associated profit margins, we have from (D.7) and (D.6) (now known to be an equality) respectively,

$$\begin{aligned} w(\hat{y}_H) + B\theta_H - \hat{U}_H &= t[1 - w'(\hat{y}_H)/\Delta\theta], \\ w(y_H) + B\theta_H - U_H &= (U_H - \hat{U}_H + t)[1 - w'(y_H)/\Delta\theta]. \end{aligned}$$

Consequently, as $t \to 0$,

$$\frac{\pi_H(t)}{\hat{\pi}_H(t)} = \frac{(U_H - \hat{U}_H + t)^2}{t^2} \frac{1 - w'(y_H(t))/\Delta\theta}{1 - w'(\hat{y}_H(t))/\Delta\theta} \to (\omega\Delta\theta)^2 < 1,$$

which concludes the proof. ■

We now rule out excluding high-skill workers.

Lemma 10 *It is always strictly suboptimal not to employ a positive measure of H-type agents.*

Proof. If a firm, say Firm 0, employs no H agent it must sell to a positive measure of L agents and reap strictly positive profits from their contract (y_L, U_L). Furthermore, the optimal level of y_L is clearly y^*. Thus, it must be that $\bar{U} \leq U_L$ and $\hat{U}_L - t < U_L < w(y^*) + B\theta_L$.

In Region III, let the firm deviate and offer the single contract (y_L, U_L). By taking it, an agent of type H gets $\tilde{U}_H = U_L + y^*\Delta\theta > \hat{U}_L - t + y^*\Delta\theta \geq \hat{U}_H - t$, so it is preferred by a positive measure of them to going to work for Firm 1, as well as to the outside option ($\tilde{U}_H > \bar{U}$). Each of these workers then generates profits $w(y^*) + B\theta_H - \tilde{U}_H = w(y^*) + B\theta_L - U_L + (B - y^*)\Delta\theta > 0$. Therefore, a contract excluding H workers could not in fact have been optimal.

In Regions I and II, we will show that there always exists a contract $(\tilde{y}_H, \tilde{U}_H)$ that can be offered alongside with (y_L, U_L) so as to attract a positive measure of H types, not be strictly preferred by any L type, and generate positive profits. Note first that if $U_L \geq \hat{U}_L$, we can simply choose $(\tilde{y}_H, \tilde{U}_H) = (\hat{y}_H, \hat{U}_H)$, that is, the same contract as offered by Firm 1. Indeed, since $U_L \geq \hat{U}_L \geq \hat{U}_H - \hat{y}_H\Delta\theta = \tilde{U}_H - \tilde{y}_H\Delta\theta$, the L types employed at Firm 0 (weakly) prefer their original contract, (y_L, U_L). For the H types, clearly $\tilde{U}_H = \hat{U}_H > \bar{U}$ and getting it from Firm 0 is preferable to getting it from Firm 1 for all such agents located at $x < 1/2$. Such a deviation is thus strictly profitable.

Suppose from now on that $U_L < \hat{U}_L$ and consider the contract $(\tilde{y}_H, \tilde{U}_H) \equiv (\hat{y}_H, U_L + \hat{y}_H\Delta\theta)$. The L types have no reason to switch (they are indifferent), while for the H types we have $\tilde{U}_H = U_L + \hat{y}_H\Delta\theta > \hat{U}_L + \hat{y}_H\Delta\theta - t = \hat{U}_H - t$, so a positive measure of them prefer this new offer to what they could get at Firm 1. Furthermore, since $\tilde{U}_H \geq U_L + y^*\Delta\theta$, they also prefer it to the L types' contract at Firm 0. The firm can thus offer the incentive-compatible menu $\{(y_L, U_L), (\tilde{y}_H, \tilde{U}_H)\}$ and attract a positive measure of H agents, on which it makes unit profit

$$\begin{aligned} w(\hat{y}_H) + B\theta_H - \tilde{U}_H &= w(\hat{y}_H) + B\theta_H - \hat{y}_H\Delta\theta - U_L \\ &> w(\hat{y}_H) + B\theta_H - \hat{y}_H\Delta\theta - \hat{U}_L = w(\hat{y}_H) + B\theta_H - \hat{U}_H > 0. \end{aligned}$$

The deviation is therefore profitable, which concludes the proof. ■

D.2.2 A key property

By Lemmas 9 and 10, at an optimum it must be that $X_H \equiv \mathcal{X}(U_H + t - \hat{U}_H) > 0$ and $X_L \equiv \mathcal{X}(U_L + t - \hat{U}_L)\mathbf{1}_{\{U_L \geq \bar{U}\}} > 0$. This, in turn, implies:

Lemma 11 *At any optimum, it must be that either:*
(i) $y_L = y^* \leq y_H$ *and* $U_H - U_L = y_H\Delta\theta$, *with multiplier* $\mu_H = 0$ *on (D.2), or*
(ii) $y_L \leq y^* = y_H$ *and* $U_H - U_L = y_L\Delta\theta$, *with multiplier* $\mu_L = 0$ *on (D.3).*

Proof. Consider the sub-problem of maximizing (D.1) over (y_H, y_L), while keeping (U_H, U_L) and therefore $(X_H > 0,\ X_L > 0)$ fixed. This is a differentiable and concave problem; denoting by μ_H and μ_L the multipliers on the high and low type's incentive constraints, the first-order conditions are:

$$0 = q_H X_H w'(y_H) + \mu_L\Delta\theta, \tag{D.10}$$

$$0 = q_L X_L w'(y_L) - \mu_H\Delta\theta + \psi. \tag{D.11}$$

Once again it cannot be that $\mu_H > 0$ and $\mu_L > 0$, otherwise (D.2)-(D.3) and (D.10) imply that $y_L = y_H > y^*$ and so $\psi = 0$, yielding a contradiction in (D.11). Suppose first that $\mu_H = 0$, implying that $\psi = 0$ and $y_L = y^*$. If (D.3) were not binding, we would have $\mu_L = 0$, hence

$y_H = y^* = y_L$ and $U_L > U_H - y_H\Delta\theta = U_H - y_L\Delta\theta \geq U_L$, a contradiction. Thus it must be that $y_H\Delta\theta = U_H - U_L \geq y_L\Delta\theta = y^*\Delta\theta$, which corresponds to case (i). If $\mu_H > 0$, then (D.2) is binding and μ_L must equal 0, hence $y_H = y^*$. Furthermore, $y_L\Delta\theta = U_H - U_L \leq y_H\Delta\theta = y^*\Delta\theta$, which corresponds to case (ii). ■

D.2.3 No cornering.

Lemma 12 *At an optimum,* $X_H \equiv U_H + t - \hat{U}_H$ *and* $X_L \equiv U_L + t - \hat{U}_L$ *must both lie in* $(0, 2t]$.

Proof. The fact that $X_H > 0$ and $X_L > 0$ was established previously. Suppose first that $\min\{U_H + t - \hat{U}_H, U_L + t - \hat{U}_L\} > 2t$. Note that this implies $U_L > \hat{U}_L + t > \bar{U}$. The firm can then reduce both U_H and U_L slightly while keeping the full market of both types, $X_H = X_L = 1$ and not violating any constraint; this increases profits, a contradiction.

Suppose next that $U_H + t - \hat{U}_H \leq 2t < U_L + t - \hat{U}_L$, which again implies $U_L > \bar{U}$; furthermore, one must also have $U_H - U_L \leq \hat{U}_H - \hat{U}_L$. The chosen allocation must thus solve

$$\max\left\{q_H\chi(U_H + t - \hat{U}_H)[w(y_H) + \theta_H B - U_H] + q_L(2t)[w(y_L) + \theta_L B - U_L]\right\},$$

subject again to (D.2)-(D.3), plus the participation constraint $U_L \geq \bar{U}$, which in this particular case is not binding. Maximizing over U_L thus yields the first-order condition

$$0 = -2tq_L - \mu_H + \mu_L, \tag{D.12}$$

which must hold in addition to (D.10)-(D.11) with $X_L = 1$. Clearly, it cannot be that $\mu_L = 0$. Therefore, $\mu_H = 0 < \mu_L = 2tq_L$, implying that (D.10) becomes $q_H(X_H/2t)w'(y_H) + q_L\Delta\theta = 0$. Furthermore, $y_H\Delta\theta = U_H - U_L \leq \hat{U}_H - \hat{U}_L \leq y_H^c\Delta\theta$, so $y_H \leq y_H^c$. But then the interim-efficiency condition (23) implies that $q_H w'(y_H) + q_L\Delta\theta > 0$, a contradiction since $X_H \leq 2t$.

Suppose now that $U_L + t - \hat{U}_L \leq 2t < U_H + t - \hat{U}_H$. The allocation must be a solution to

$$\max\left\{q_H(2t)[w(y_H) + \theta_H B - U_H] + q_L\chi(U_L + t - \hat{U}_L)[w(y_L) + \theta_L B - U_L]\right\},$$

subject to (D.2)-(D.3) and the constraint $U_L \geq \bar{U}$, with associated multiplier $\nu \geq 0$. Maximizing over U_H thus yields the first-order condition

$$0 = -2tq_H + \mu_H - \mu_L. \tag{D.13}$$

This precludes $\mu_H = 0$, so $\mu_L = 0 < \mu_H = 2tq_H$, $y_H = y^*$ and $q_L X_L w'(y_L) = 2tq_H\Delta\theta - \psi \equiv 2tq_L w'(y_L^m) - \psi$. If $\psi > 0$ then $y_L = 0 < y_L^m$, and if $\psi = 0$ then $(X_L/2t)\, w'(y_L) = w'(y_L^m)$ so $y_L < y_L^m$, as $X_L \leq 2t$. But we also have $y_L\Delta\theta = U_H - U_L > \hat{U}_H - \hat{U}_L > y_L^m\Delta\theta$, a contradiction. ■

D.2.4 Proof of global optimality

The objective function in (D.16) is not globally concave, as can be seen computing the Hessian. The proof of global optimality will therefore require several steps. First, we will show that for any

$\mathcal{C} = (U_H, U_L, y_H, y_L)$ to be an optimum, it must lie in either the following subspaces:

$$S_H \equiv \{(U_H, U_L, y_H, y_L) \, | y_L = y^* \leq y_H \leq y_H^c \text{ and } U_H - U_L = y_H \Delta\theta\}, \tag{D.14}$$
$$S_L \equiv \{(U_H, U_L, y_H, y_L) \, | y_H = y^* \geq y_L \geq y_L^m \text{ and } U_H - U_L = y_L \Delta\theta\}. \tag{D.15}$$

We will then show that the program is strictly concave on S_H and on S_L separately, which implies that $\widehat{\mathcal{C}} = (\hat{U}_H, \hat{U}_L, \hat{y}_H, \hat{y}_L)$ achieves a maximum over all feasible allocations in the subspace to which it belongs, namely S_H for $t \leq t_2$ (Regions I and II), or S_L for $t \geq t_2$ (Region III). Finally, we will show that the global optimum can never lie in the other subspace than the one to which $\widehat{\mathcal{C}}$ belongs, concluding the proof.

Lemma 13 *A global optimum $C = (U_H, U_L, y_H, y_L)$ must lie in S_H or in S_L.*

Proof. Let S_H' be denote the superset of S_H obtained by omitting the inequality $y_H \leq y_H^c$ from (D.14), and similarly let S_L' denote the superset of S_L obtained by omitting the inequality $y_L \geq y_L^m$ from (D.15). By Lemma 11, an optimum must belong to S_H' or S_L'. Furthermore, given no exclusion nor strict cornering (Lemmas 9, 10 and 12), solving (D.1)-(D.3) is equivalent to solving the smooth program

$$\max q_H(U_H + t - \hat{U}_H)[w(y_H) + \theta_H B - U_H] + q_L(U_L + t - \hat{U}_L)[w(y_L) + \theta_L B - U_L] \, , \tag{D.16}$$

subject to:

$$\begin{array}{ll} X_H \equiv U_H + t - \hat{U}_H \leq 2t & (\tau_H) \\ X_L \equiv U_L + t - \hat{U}_L \leq 2t & (\tau_L) \\ U_H \geq U_L + y_L\Delta\theta & (\mu_H) \\ U_L \geq U_H - y_H\Delta\theta & (\mu_L) \\ U_L \geq \bar{U} & (\nu) \\ y_L \geq 0 & (\psi). \end{array}$$

The first-order conditions are:

$$q_H \left[w(y_H) + B\theta_H - 2U_H + \hat{U}_H - t \right] + \mu_H - \mu_L - \tau_H = 0, \tag{D.17}$$
$$q_L \left[w(y_L) + B\theta_L - 2U_L + \hat{U}_L - t \right] + \mu_L - \mu_H + \nu - \tau_L = 0, \tag{D.18}$$
$$q_H \left(U_H - \hat{U}_H + t \right) w'(y_H) + \mu_L \Delta\theta = 0, \tag{D.19}$$
$$q_L \left(U_L - \hat{U}_L + t \right) w'(y_L) - \mu_H \Delta\theta + \psi = 0 \tag{D.20}$$

and we also know that $X_H > 0$ and $X_L > 0$ at an optimum.

Case A. Consider first $C \in S_H'$. We have $y_L = y^*$ (so $\psi = 0$) and $\mu_H = 0$, so eliminating μ_L :

$$w(y_H) + B\theta_H - 2U_H + \hat{U}_H - t + (U_H - \hat{U}_H + t)\frac{w'(y_H)}{\Delta\theta} - \frac{\tau_H}{q_H} = 0, \tag{D.21}$$
$$w(y^*) + B\theta_L - 2U_L + \hat{U}_L - t - \frac{q_H}{q_L}(U_H - \hat{U}_H + t)\frac{w'(y_H)}{\Delta\theta} + \frac{\nu}{q_L} - \frac{\tau_L}{q_L} = 0. \tag{D.22}$$

Subtracting and using $U_H - U_L = y_H \Delta\theta$ and $\hat{U}_H - \hat{U}_L = \hat{y}\Delta\theta$ (with $\hat{y} = \hat{y}_H$ in Regions I and II and $\hat{y} = \hat{y}_L$ in Region III) yields

$$w(y_H) - w(y^*) + (B + \hat{y} - 2y_H)\Delta\theta = (U_H - \hat{U}_H + t)\left(1 + \frac{q_H}{q_L}\right)\frac{-w'(y_H)}{\Delta\theta} + \frac{\nu}{q_L} + \frac{\tau_H}{q_H} - \frac{\tau_L}{q_L}.$$

Next, subtracting $w(y_H^c) - w(y^*) + (B - y_H^c)\Delta\theta = 0$, we have

$$\begin{aligned} & w(y_H) - w(y_H^c) - (y_H - y_H^c)\,\Delta\theta - (y_H - \hat{y})\Delta\theta \\ = \; & (U_H - \hat{U}_H + t)\left(-1 - \frac{q_H}{q_L}\right)\frac{w'(y_H)}{\Delta\theta} + \frac{\nu}{q_L} + \frac{\tau_H}{q_H} - \frac{\tau_L}{q_L}, \end{aligned}$$

or

$$w(y_H) - w(y_H^c) - (2y_H - y_H^c - \hat{y})\,\Delta\theta = (U_H - \hat{U}_H + t)\frac{-w'(y_H)}{q_L\Delta\theta} + \frac{\nu}{q_L} + \frac{\tau_H}{q_H} - \frac{\tau_L}{q_L}. \qquad \text{(D.23)}$$

If $y_H > y_H^c \geq \hat{y}$ the left-hand side is negative, while the right-hand side is positive, since $U_H - U_L > \hat{U}_H - \hat{U}_L$ implies that $U_L - \hat{U}_L < U_H - \hat{U}_H \leq t$, so $\tau_L = 0$. Hence, a contradiction, from which we conclude that $y_H \leq y_H^c$, so that $C \in S_H$.

Case B. Consider now $C \in S_L'$. We have $y_H = y^*$ and $\mu_L = 0$, so eliminating μ_H :

$$\begin{aligned} w(y^*) + B\theta_H - 2U_H + \hat{U}_H - t + \frac{q_L}{q_H}(U_L - \hat{U}_L + t)\frac{w'(y_L)}{\Delta\theta} + \frac{\psi}{q_H\Delta\theta} - \frac{\tau_H}{q_H} &= 0, &\text{(D.24)} \\ w(y_L) + B\theta_L - 2U_L + \hat{U}_L - t - (U_L - \hat{U}_L + t)\frac{w'(y_L)}{\Delta\theta} + \frac{\nu}{q_L} - \frac{\psi}{q_L\Delta\theta} - \frac{\tau_L}{q_L} &= 0. &\text{(D.25)} \end{aligned}$$

If $y_L < y_L^m$ then $U_H - U_L = y_L\Delta\theta < \hat{y}\Delta\theta = \hat{U}_H - \hat{U}_L$ so $U_H - \hat{U}_H < U_L - \hat{U}_L \leq 2t$, hence $\tau_H = 0$.

Suppose first that $U_L > \bar{U}$; then $\nu = 0$ and from the two above equations we have

$$\begin{aligned} w(y^*) + B\theta_H - 2U_H + \hat{U}_H - t &< 0 < w(y_L) + B\theta_L - 2U_L + \hat{U}_L - t \iff \\ w(y^*) - w(y_L) + (B - y_L)\,\Delta\theta + (\hat{y} - y_L)\Delta\theta &< 0, \end{aligned}$$

a contradiction since this last expression is clearly positive. Therefore, $U_L = \bar{U}$. Next, for $y_L < y_L^m$ we have $w'(y_L) > w'(y_L^m) = (q_H/q_L)\,\Delta\theta$, hence, by (D.24):

$$\begin{aligned} -\psi/q_H\Delta\theta &> w(y^*) + B\theta_H - 2U_H + \hat{U}_H - t + U_L - \hat{U}_L + t \\ &= w(y^*) + B\theta_L - \bar{U} + (B - y_L)\Delta\theta - [U_H - U_L - (\hat{U}_H - \hat{U}_L)] \\ &= w(y^*) + B\theta_L - \bar{U} + (B - y_L)\Delta\theta + (\hat{y} - y_L)\,\Delta\theta. \end{aligned}$$

This last expression is strictly positive, however, since $y_L < y_L^m \leq \hat{y} < B$, where $\hat{y} = \hat{y}_H$ when t is in Region I or II and $\hat{y} = \hat{y}_L$ when t is in Region III. Hence another contradiction, from which we conclude that $y_L \geq y_L^m$, so that $C \in S_L$. ■

Lemma 14 *The objective function in (D.16) is strictly concave over S_H and over S_L.*

In passing, note that this result implies that the symmetric solution $\widehat{\mathcal{C}} \equiv (\hat{U}_H, \hat{U}_L, \hat{y}_H, \hat{y}_L)$ always satisfies the *local* second-order conditions for a maximum of the program (D.16).[38]

Proof. First, over S_H, the objective function becomes

$$\begin{aligned}\phi(U_H, y_H) \equiv\; & q_H(U_H - \hat{U}_H + t)\left[w(y_H) + \theta_H B - U_H\right] \\ & + q_L(U_H - y_H\Delta\theta - \hat{U}_L + t)\left[w(y^*) + \theta_L B - U_H + y_H\Delta\theta\right], \end{aligned} \tag{D.26}$$

for which the Hessian is

$$H(\phi) = \begin{bmatrix} -2 & q_H w'(y_H) + 2q_L\Delta\theta \\ q_H w'(y_H) + 2q_L\Delta\theta & q_H(U_H - \hat{U}_H + t)w''(y_H) - 2q_L\Delta\theta^2 \end{bmatrix}$$

and its determinant equals

$$\begin{aligned} & -q_H^2 w'(y_H)^2 - 4q_H q_L w'(y_H)\Delta\theta - 4q_L^2\Delta\theta^2 + 4q_L\Delta\theta^2 - 2q_H w''(y_H)\left(U_H - \hat{U}_H + t\right) \\ = \; & -q_H w'(y_H)\left[q_H w'(y_H) + 4q_L\Delta\theta\right] + 4q_H q_L\Delta\theta^2 - 2q_H w''(y_H)\left(U_H - \hat{U}_H + t\right), \end{aligned}$$

which is positive since $y_H \leq y_H^c$ implies that $q_H w'(y_H) + 4q_L\Delta\theta \geq q_H w'(y_H^c) + 4q_L\Delta\theta > 0$, by (23).

Next, over S_L, the objective function becomes

$$\begin{aligned}\phi(U_L, y_L) \equiv\; & q_H(U_L + y_L\Delta\theta - \hat{U}_H + t)[w(y^*) + \theta_H B - U_L - y_L\Delta\theta\,] \\ & + q_L(U_L + t - \hat{U}_L)[w(y_L) + \theta_L B - U_L], \end{aligned} \tag{D.27}$$

for which the Hessian is

$$H(\phi) = \begin{bmatrix} -2 & q_L w'(y_L) - 2q_H\Delta\theta \\ q_L w'(y_L) - 2q_H\Delta\theta & q_L(U_L - \hat{U}_L + t)w''(y_L) - 2q_H\Delta\theta^2 \end{bmatrix}$$

and its determinant equals:

$$\begin{aligned} & -q_L^2 w'(y_L)^2 + 4q_H q_L w'(y_L)\Delta\theta - 4q_H^2\Delta\theta^2 + 4q_H\Delta\theta^2 - 2q_L w''(y_L)(U_L - \hat{U}_L + t) \\ = \; & q_L w'(y_L)\left[-q_L w'(y_L) + 4q_H\Delta\theta\right] + 4q_H q_L\Delta\theta^2 - 2q_L w''(y_L)(U_L - \hat{U}_L + t), \end{aligned}$$

which is positive since $y_L \geq y_L^m$ implies $q_L w'(y_L) \leq q_L w'(y_L^m) < 4q_H\Delta\theta$, by (15). ■

Proposition 19 *The unique global optimum to (D.1)-(D.3) is the allocation $\widehat{\mathcal{C}} \equiv (\hat{U}_H, \hat{U}_L, \hat{y}_H, \hat{y}_L)$ characterized in Proposition 4, which is therefore an equilibrium (the unique symmetric one) of the game between the two firms.*

[38] This can also be shown directly, by computing the appropriate bordered Hessians, given each of the constraints binding in Regions I, II and III respectively. The proof is available upon request.

Proof. By Lemmas 9 and 10, the global solution $\mathcal{C} = (U_H, U_L, y_H, y_L)$ to (D.1)-(D.3) is also the global solution to (D.16) and satisfies the associated first-order condition (D.17)-(D.20), with $X_H \equiv U_H - \hat{U}_H + t$ and $X_L \equiv U_H - \hat{U}_H + t$ both in $(0, 2t]$. By Proposition 4, $\widehat{\mathcal{C}} \equiv (\hat{U}_H, \hat{U}_L, \hat{y}_H, \hat{y}_L)$ solves these conditions (with $\hat{X}_H = \hat{X}_L = t$), is the unique candidate for a symmetric equilibrium, and is such that $\widehat{\mathcal{C}} \in S_H$ when t is in Regions I and II, while $\widehat{\mathcal{C}} \in S_L$ when t is in Region III. Furthermore, by Lemma 14, the objective function is strictly concave over each of these subspaces, so in each case $\widehat{\mathcal{C}}$ maximizes the program over the one to which it belongs. By Lemma 14, moreover, the global optimum $\mathcal{C}$ must also belong to S_H or S_L. Two cases therefore remain to consider.

Case A: t lies in Region I or II, so that $\widehat{\mathcal{C}} \in S_H$. If $\mathcal{C} \in S_H$ as well, they must coincide. If $\mathcal{C} \in S_L$ then $y_H = y^*$, $\mu_L = 0$ and

$$U_H - U_L = y_L \Delta\theta \ \leq \hat{y}_H \Delta\theta = \hat{U}_H - \hat{U}_L. \tag{D.28}$$

Subcase A1. If the inequality is strict then

$$U_H - \hat{U}_H \ < U_L - \hat{U}_L. \tag{D.29}$$

Note that this requires $\tau_H = 0$, otherwise $t = U_H - \hat{U}_H \ < U_L - \hat{U}_L \leq t$, a contradiction. Next, subtracting from (D.17) its counterpart for $\widehat{\mathcal{C}}$, and likewise for (D.18), we have:

$$\begin{aligned} q_H \left[w(y^*) + B\theta_H - 2U_H + \hat{U}_H - t \right] + \mu_H &= q_H \left[w(\hat{y}_H) + B\theta_H - \hat{U}_H - t \right] - \hat{\mu}_L, \\ q_L \left[w(y_L) + B\theta_L - 2U_L + \hat{U}_L - t \right] - \mu_H + \nu - \tau_L &= q_L \left[w(y^*) + B\theta_L - \hat{U}_L - t \right] + \hat{\mu}_L + \hat{\nu}. \end{aligned}$$

The first equation implies that $w(y^*) - w(\hat{y}_H) \leq 2(U_H - \hat{U}_H)$, hence $U_L - \hat{U}_L > 0$ by (D.29). Thus $U_L > \bar{U}$, implying $\nu = 0$. It then follows from the second equation above that $w(y_L) + B\theta_L - 2U_L + \hat{U}_L \geq w(y^*) + B\theta_L - \hat{U}_L$, hence $2(U_L - \hat{U}_L) \leq w(y_L) - w(y^*) \leq 0$, which contradicts $U_L > \hat{U}_L$.

Subcase A2. Equation (D.28) is therefore an equality, implying that $y_L = \hat{y}_H = y^* = y_H$ (and $\psi = 0$). Thus $U_H - U_L = y_H \Delta\theta$ and $y_L = y^*$, implying that $\mathcal{C} \in S_H$, so it must coincide with $\widehat{\mathcal{C}}$. Note that $\widehat{\mathcal{C}} \in S_H \cap S_L$ can only occur at $t = t_2$.

Case B: t lies in Region III, so that $\widehat{\mathcal{C}} \in S_L$. If $\mathcal{C} \in S_L$ as well, they must coincide. If $\mathcal{C} \in S_H$ then $y_L = y^*$, $\mu_H = 0$ and $U_H - U_L = y_H \Delta\theta \ \geq \hat{y}_L \Delta\theta = \hat{U}_H - \hat{U}_L$. Therefore:

$$U_H - \hat{U}_H \ \geq U_L - \hat{U}_L = U_L - \bar{U} \geq 0. \tag{D.30}$$

Subtracting from (D.17) its counterpart for $\widehat{\mathcal{C}}$, we now have:

$$q_H \left[w(y_H) + B\theta_H - 2U_H + \hat{U}_H - t \right] - \mu_L - \tau_H = q_H \left[w(y^*) + B\theta_H - \hat{U}_H - t \right] + \hat{\mu}_H,$$

Therefore $w(y_H) - w(y^*) \geq 2(U_H - \hat{U}_H)$, which together with (D.30) requires that $U_H = \hat{U}_H, U_L = \hat{U}_L$ and $y_H = y^* = \hat{y}_L$, so that $\mathcal{C} = \widehat{\mathcal{C}}$. Here again it must be that $t = t_2$, which corresponds to the only intersection of S_H and S_L. ■