A   Mathematical details

This appendix contains the detailed derivations of many of the results discussed in the main text. Results are derived in the order they appear in the text.

Derivation of equations (8) through (10):

Since all factor inputs as well as the price can be costlessly adjusted and capital is rented rather than owned, in every period the firm decides on the factor inputs, k_{it} and l_{it}, its price, p_{it}, and its output level, y_{it}, that maximize its flow profits (6), subject to the technological constraint reflected by the production function (5) and the market constraint represented by the demand function (4).

The associated Lagrangian is

$$
(72) \qquad \mathcal{L}_{it} \;=\; [p_{it}y_{it} - w_t l_{it} - (r_t + \delta)k_{it}]
$$
$$
- \lambda_{dt}\left[y_{it} - \left(\frac{1}{p_{it}}\right)^{\frac{1}{1-\theta}} y_t \right] - \lambda_{yt}\left[y_{it} - a_{it}k_{it}^{\alpha}l_{it}^{1-\alpha} \right]
$$

The first order necessary conditions implied by this objective function are

$$
(73) \qquad \text{w.r.t } y_{it} \quad : \qquad p_{it} = \lambda_{dt} + \lambda_{yt}
$$

$$
(74) \qquad \text{w.r.t. } p_{it} \quad : \qquad y_{it} = \frac{\lambda_{dt}}{1-\theta}\frac{1}{p_{it}}y_{it}
$$

$$
(75) \qquad \text{w.r.t. } l_{it} \quad : \qquad w_t = \lambda_{yt}\alpha\frac{y_{it}}{l_{it}}
$$

$$
(76) \qquad \text{w.r.t. } k_{vt} \quad : \qquad (r_t + \delta) = \lambda_{yt}(1-\alpha)\frac{y_{it}}{k_{it}}
$$

Condition (74) implies that

$$
(77) \qquad \lambda_{dt} = (1-\theta)p_{it}
$$

Combining this with (73) yields that

$$
(78) \qquad \lambda_{yt} = \theta p_{it}
$$

When we substitute this into the optimal factor demand conditions, (75) and (76), we find

$$
(79) \qquad w_t l_{it} = \theta(1-\alpha)p_{it}y_{it} \text{ and } (r_t + \delta)k_{it} = \alpha\theta p_{it}y_{it}
$$

We can use these results to solve for the optimal capital labor ratio, which satisfies

$$
(80) \qquad \left(\frac{k_{it}}{l_{it}}\right) = \left(\frac{w_t}{\theta(1-\alpha)p_{it}a_{it}}\right)^{\frac{1}{\alpha}} = \left(\frac{\theta\alpha p_{it}a_{it}}{r_t + \delta}\right)^{\frac{1}{1-\alpha}}
$$

The price level, p_{it}, that equates the capital labor ratio for the optimal capital and labor demand decisions, and thus solves the above equation, is

$$
(81) \qquad p_{it} = \frac{1}{\theta}mc_{it}
$$

where mc_{it} is the marginal production cost of the producer of intermediate i and equals the unit production cost implied by the Cobb-Douglas production technology, i.e.

$$(82) \qquad mc_{it} = \frac{1}{a_{it}} \left[\frac{w_t}{1-\alpha} \right]^{1-\alpha} \left[\frac{r_t + \delta}{\alpha} \right]^{\alpha}$$

and the resulting profit level equals

$$(83) \qquad \pi_{it} = \frac{1-\theta}{\theta} mc_{it} = (1-\theta) p_{it} y_{it}$$

Hence, the resulting value of the firm is given by

$$(84) \qquad V_{it} = \int_t^\infty e^{-\int_t^s r_j dj} \pi_{is} ds = (1-\theta) \int_t^\infty e^{-\int_t^s r_j dj} p_{is} y_{is} ds$$

which corresponds to (10).

Derivation of equations (11) through (15):

Combining the factor demand equations with the demand equations for each of the intermediates, we obtain

$$(85) \qquad w_t l_{it} = (1-\alpha) \theta y_{it}^\theta y_t^{1-\theta} \text{ and } (r_t + \delta) k_{it} = \alpha \theta y_{it}^\theta y_t^{1-\theta}$$

Integrating the factor demand equations over the set of available intermediate goods yields that

$$(86) \qquad w_t l_t = w_t \int_{-\infty}^t l_{it} di = (1-\alpha) \theta y_t^{1-\theta} \int_{-\infty}^t y_{it}^\theta di = (1-\alpha) \theta y_t$$

and

$$(87) \qquad (r_t + \delta) k_t = (r_t + \delta) \int_{-\infty}^t k_{it} di = \alpha \theta y_t^\theta \int_{-\infty}^t p_{it} y_{it}^{1-\theta} di = \alpha \theta y_t$$

In addition, we find that the relative factor demands, across intermediates, satisfy

$$(88) \qquad \frac{k_{it}}{k_t} = \frac{l_{it}}{l_t} = \left(\frac{y_{it}}{y_t} \right)^\theta$$

Substituting this into the production function of the intermediate goods suppliers yields

$$(89) \qquad y_{it} = a_{it} k_{it}^\alpha l_{it}^{1-\alpha} = a_{it} \left(\frac{y_{it}}{y_t} \right)^\theta k_t^\alpha l_t^{1-\alpha} = \left[\frac{a_{it} k_t^\alpha l_t^{1-\alpha}}{y_t^\theta} \right]^{\frac{1}{1-\theta}}$$

This means that

$$(90) \qquad y_t^\theta = \left(\int_{-\infty}^t y_{it}^\theta di \right) = \left[\frac{1}{y_t^{\frac{\theta}{1-\theta}}} \right] \left[\int_{-\infty}^t a_{it}^{\frac{\theta}{1-\theta}} di \right] \left[k_t^\alpha l_t^{1-\alpha} \right]^{\frac{\theta}{1-\theta}}$$

which yields that

$$(91) \qquad y_t = z_t k_t^\alpha l_t^{1-\alpha}, \text{ where } z_t = \left[\int_{-\infty}^t a_{it}^{\frac{\theta}{1-\theta}} di \right]^{\frac{1-\theta}{\theta}}$$

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which is the aggregate production function representation that we use.

Applying the CES price aggregator, we obtain that

$$(92) \qquad 1 = p_t = \left[\int_{-\infty}^{t} \left(\frac{1}{p_{it}} \right)^{\frac{\theta}{1-\theta}} di \right]^{-\frac{1-\theta}{\theta}} = \frac{1}{z_{it}} \left[\frac{w_t}{1-\alpha} \right]^{1-\alpha} \left[\frac{r_t + \delta}{\alpha} \right]^{\alpha}$$

this implies that

$$(93) \qquad p_{it} = p_{it}/p_t = z_t/a_{it}$$

which allows us to write the relative output levels, factor demands, and prices in terms of relative productivity levels, as in (14).

Derivation of equations (23) and (25):

First of all, in order to see why the maximization problem (22) actually boils down to choosing x_{tt}, it is worthwhile to rewrite the implementation cost function, i.e. (21), as

$$(94) \qquad C_t^{implement}(x_{tt}) = (1 - \theta) \xi z_t^{\frac{1}{1-\alpha}} \left(\frac{\bar{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} [- \ln(1 - x_{tt}) - x_{tt}]$$

Secondly, it is worthwhile to rewrite the expression for the value function, i.e. (20) as

$$(95) \qquad
\begin{aligned}
V_{tt}(x_{tt}) &= (1 - \theta) \bar{a}_t^{\frac{\theta}{1-\theta}} \int_t^\infty e^{-\int_t^s r_j dj} \left(1 - e^{-\lambda(s-t)} \right) \left(\frac{1}{z_s} \right)^{\frac{\theta}{1-\theta}} y_s ds + \\
&\quad (1 - \theta) \bar{a}_t^{\frac{\theta}{1-\theta}} x_{tt} \int_t^\infty e^{-\int_t^s r_j dj} e^{-\lambda(s-t)} \left(\frac{1}{z_s} \right)^{\frac{\theta}{1-\theta}} y_s ds \\
&= (1 - \theta) \left(\frac{\bar{a}_t}{\bar{z}_t} \right)^{\frac{\theta}{1-\theta}} \left(\frac{\bar{z}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} \times \\
&\quad \left[\int_t^\infty e^{-\int_t^s r_j dj} \left(1 - e^{-\lambda(s-t)} \right) \left(\frac{z_t}{z_s} \right)^{\frac{\theta}{1-\theta}} y_s ds + \right. \\
&\quad \left. x_{vt} \int_t^\infty e^{-\int_t^s r_j dj} e^{-\lambda(s-t)} \left(\frac{z_t}{z_s} \right)^{\frac{\theta}{1-\theta}} y_s ds \right]
\end{aligned}$$

then we can write the objective that maximized in the firms implementation decision as

$$(96) \qquad V_{tt}(x_{tt}) - C_t^{implement}(x_{tt}) = (1 - \theta) \left(\frac{\bar{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} \left[(b_{0t} + b_{xt} x_t) y_t - \xi z_t^{\frac{1}{1-\alpha}} [- \ln(1 - x_{tt}) - x_{tt}] \right]$$

where

$$(97) \qquad b_{0t} = \int_t^\infty e^{-\int_t^s r_j dj} \left(1 - e^{-\lambda(s-t)} \right) \left(\frac{z_t}{z_s} \right)^{\frac{\theta}{1-\theta}} \frac{y_s}{y_t} ds$$

$$(98) \qquad b_{xt} = \int_t^\infty e^{-\int_t^s r_j dj} e^{-\lambda(s-t)} \left(\frac{z_t}{z_s} \right)^{\frac{\theta}{1-\theta}} \frac{y_s}{y_t} ds$$

The resulting necessary, and in this case sufficient, first order condition for the optimal choice of x_{tt} is

$$(99) \qquad 0 = b_{xt} y_t - \xi z_t^{\frac{1}{1-\alpha}} \left(\frac{x_{tt}}{1 - x_{tt}} \right)$$

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which yields that

$$(100) \qquad x_{tt} = \frac{y_t b_{xt}}{z_t^{\frac{1}{1-\alpha}} \xi + y_t b_{xt}}$$

which is equation (23) in the main text.

The value of the firm, net of implementation costs, is given by

$$(101) \quad
\begin{aligned}
V_t^* &= (1-\theta) \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}} \left[(b_{0t} + b_{xt} x_t) y_t - \xi z_t^{\frac{1}{1-\alpha}} \left[-\ln(1-x_{tt}) - x_{tt} \right] \right] \\
&= (1-\theta) \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}} \left[b_{0t} y_t + \frac{y_t b_{xt}}{z_t^{\frac{1}{1-\alpha}} \xi + y_t b_{xt}} \left(y_t b_{xt} + z_t^{\frac{1}{1-\alpha}} \xi \right) + \xi z_t^{\frac{1}{1-\alpha}} \ln\left(\frac{z_t^{\frac{1}{1-\alpha}} \xi}{z_t^{\frac{1}{1-\alpha}} \xi + y_t b_{xt}} \right) \right] \\
&= (1-\theta) \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}} \left[(b_{0t} + b_{xt}) y_t - \xi z_t^{\frac{1}{1-\alpha}} \ln\left(1 + \frac{y_t}{z_t^{\frac{1}{1-\alpha}}} \frac{b_{xt}}{\xi} \right) \right] \\
&= (1-\theta) z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}} \left[(b_{0t} + b_{xt}) \frac{y_t}{z_t^{\frac{1}{1-\alpha}}} - \xi \ln\left(1 + \frac{y_t}{z_t^{\frac{1}{1-\alpha}}} \frac{b_{xt}}{\xi} \right) \right]
\end{aligned}$$

which is equation (25).

Derivation of equation (36):

We rewrite

$$(102) \qquad z_t^{\frac{\theta}{1-\theta}} = \int_{-\infty}^{t} a_{vt}^{\frac{\theta}{1-\theta}} dv = \int_{-\infty}^{t} \overline{a}_v^{\frac{\theta}{1-\theta}} x_{vt} dv$$

such that the application of Leibniz rule yields

$$(103) \qquad \left(z_t^{\frac{\theta}{1-\theta}} \right)^{\bullet} = \int_{-\infty}^{t} \overline{a}_v^{\frac{\theta}{1-\theta}} \dot{x}_{vt} dv + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}$$

Since

$$(104) \qquad \dot{x}_{vt} = \lambda (1 - x_{vt})$$

we can write

$$(105) \qquad
\begin{aligned}
\left(z_t^{\frac{\theta}{1-\theta}} \right)^{\bullet} &= \lambda \int_{-\infty}^{t} \overline{a}_v^{\frac{\theta}{1-\theta}} dv - \lambda \int_{-\infty}^{t} \overline{a}_v^{\frac{\theta}{1-\theta}} x_{vt} dv + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt} \\
&= \lambda \left(\overline{z}_t^{\frac{\theta}{1-\theta}} - z_t^{\frac{\theta}{1-\theta}} \right) + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}
\end{aligned}$$

which is (36).

Derivation of transformed equilibrium dynamics:

In terms of the 10 transformed equilibrium variables

$$\{ y_t^*, i_t^*, c_t^*, k_t^*, x_{tt}, g_t, b_{0t}^*, b_{xt}^*, \chi_t, \overline{\chi}_t \}$$

we can rewrite the resource constraint, consumption Euler equation, capital accumulation equation, production function, and capital factor demand equation as

$$(106) \qquad y_t^* = c_t^* + i_t^* + (1 - \theta) \xi \chi_t \overline{\chi}_t \left[\xi \left(- \ln \left(1 - x_{tt} \right) - x_{tt} \right) + \phi g_t \right]$$

$$(107) \qquad \frac{\dot{c}_t^*}{c_t^*} = \sigma \left(\alpha \theta \frac{y_t^*}{k_t^*} - \delta - \rho \right) - \frac{1}{1 - \alpha} g_t$$

$$(108) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1 - \alpha} g_t$$

$$(109) \qquad y_t^* = k_t^{*\alpha}$$

Moreover, we can write

$$(110) \qquad \begin{aligned} \frac{\dot{\chi}_t}{\chi_t} &= \frac{\left(\overline{z}_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{\overline{z}_t^{\frac{\theta}{1-\theta}}} - \frac{\left(z_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{z_t^{\frac{\theta}{1-\theta}}} \\ &= \overline{\chi}_t + \lambda \left(1 - \chi_t \right) - \chi_t \overline{\chi}_t x_{tt} \\ &= \lambda \left(1 - \chi_t \right) + \left(1 - \chi_t x_{tt} \right) \overline{\chi}_t \end{aligned}$$

while

$$(111) \qquad \begin{aligned} \frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} &= \frac{\left(\overline{a}_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{\overline{a}_t^{\frac{\theta}{1-\theta}}} - \frac{\left(\overline{z}_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{\overline{z}_t^{\frac{\theta}{1-\theta}}} \\ &= \frac{\theta}{1 - \theta} g_t - \overline{\chi}_t \end{aligned}$$

Furthermore, we also derive differential equations for the coefficients b_{0t}^* and b_{xt}^*. These yield

$$(112) \qquad \dot{b}_{0t}^* = \left\{ \alpha \theta \frac{y_t^*}{k_t^*} - \delta + \frac{\left(z_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{z_t^{\frac{\theta}{1-\theta}}} - \frac{\dot{y}_t}{y_t} \right\} b_{0t}^* - 1$$

and

$$(113) \qquad \dot{b}_{xt}^* = \left\{ \alpha \theta \frac{y_t^*}{k_t^*} - \delta + \lambda + \frac{\left(z_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{z_t^{\frac{\theta}{1-\theta}}} - \frac{\dot{y}_t}{y_t} \right\} b_{xt}^* - 1$$

However, because

$$(114) \qquad \frac{\dot{y}_t}{y_t} = \frac{\dot{y}_t^*}{y_t^*} + \frac{1}{1 - \alpha} g_t$$

and

$$(115) \qquad \frac{\left(z_t^{\frac{\theta}{1-\theta}} \right)^{\cdot}}{z_t^{\frac{\theta}{1-\theta}}} = -\lambda \left(1 - \chi_t \right) + \chi_t \overline{\chi}_t x_{tt}$$

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we can write

$$（116） \quad \dot{b}_{0t}^* = \left\{ \alpha\theta\frac{y_t^*}{k_t^*} - \delta - \lambda\left(1-\chi_t\right) + \chi_t\overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1-\alpha}g_t \right\} b_{0t}^* - 1$$

$$（117） \quad \dot{b}_{xt}^* = \left\{ \alpha\theta\frac{y_t^*}{k_t^*} - \delta + \lambda\chi_t + \chi_t\overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1-\alpha}g_t \right\} b_{xt}^* - 1$$

Finally

$$（118） \quad g_t = \frac{1}{\phi}\left[b_{0t}^* y_t^* - \xi\ln\left(1 + \frac{y_t^* b_{xt}^*}{\xi}\right)\right]$$

and

$$（119） \quad x_{tt} = \frac{b_{xt}^* y_t^*}{\xi + b_{xt}^* y_t^*}$$

which completes the system of 10 equilibrium equations.

Derivation of steady state:

The steady state values have to satisfy

$$（120） \quad \widetilde{y}^* = \widetilde{c}^* + \widetilde{i}^* + (1-\theta)\widetilde{\overline{\chi}}\widetilde{\chi}\left[\xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right) + \phi\widetilde{g}\right]$$

$$（121） \quad \widetilde{y}^* = \widetilde{k}^{*\alpha}$$

$$（122） \quad \widetilde{x}^* = \frac{\widetilde{b}_x^*\widetilde{y}^*}{\xi + \widetilde{b}_x^*\widetilde{y}^*}$$

$$（123） \quad \widetilde{g} = \frac{1}{\phi}\left[\widetilde{b}_0^*\widetilde{y}^* - \xi\ln\left(1 + \frac{\widetilde{b}_x^*\widetilde{y}^*}{\xi}\right)\right]$$

as well as

$$（124） \quad 0 = \sigma\left(\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta - \rho\right) - \frac{1}{1-\alpha}\widetilde{g}$$

$$（125） \quad 0 = \left\{\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta - \lambda\left(1-\widetilde{\chi}\right) + \widetilde{\chi}\widetilde{\overline{\chi}}\widetilde{x}^* - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_0^* - 1$$

$$（126） \quad 0 = \left\{\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta + \lambda\widetilde{\chi} + \widetilde{\chi}\widetilde{\overline{\chi}}\widetilde{x}^* - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_x^* - 1$$

$$（127） \quad 0 = \frac{\widetilde{i}^*}{\widetilde{k}^*} - \delta - \frac{1}{1-\alpha}\widetilde{g}$$

$$（128） \quad 0 = \lambda\left(1-\widetilde{\chi}\right) + \left(1 - \widetilde{\chi}\widetilde{x}^*\right)\widetilde{\overline{\chi}}$$

$$（129） \quad 0 = \frac{\theta}{1-\theta}\widetilde{g} - \widetilde{\overline{\chi}}$$

Our approach to solving this system is to derive a non-linear equation for $\widetilde{g}$ and show that it has a unique solution.

We obtain that

$$（130） \quad \widetilde{\overline{\chi}} = \frac{\theta}{1-\theta}\widetilde{g}$$

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Because

(131)
$$0 = \lambda\left(1 - \widetilde{\chi}\right) + \left(1 - \widetilde{\chi}\widetilde{x}\right)\widetilde{\widetilde{\chi}}$$

we can write

(132)
$$\widetilde{\widetilde{\chi}} = -\lambda\left(1 - \widetilde{\chi}\right) + \widetilde{\chi}\widetilde{\widetilde{\chi}}\widetilde{x}$$

This simplifies the equations

(133)
$$0 = \left\{\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta + \frac{\theta}{1-\theta}\widetilde{g} - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_0^* - 1$$

(134)
$$0 = \left\{\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta + \lambda + \frac{\theta}{1-\theta}\widetilde{g} - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_x^* - 1$$

From the Euler equation, we obtain that

(135)
$$\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*} = \rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}$$

which allows us to write

(136)
$$\widetilde{b}_0^* = \frac{1}{\rho + \psi\widetilde{g}}$$

while

(137)
$$\widetilde{b}_x^* = \frac{1}{\rho + \lambda + \psi\widetilde{g}}$$

where

(138)
$$\psi = \left[\frac{\theta}{1-\theta} + \left(\frac{1}{\sigma} - 1\right)\frac{1}{1-\alpha}\right]$$

and we will assume that $\psi > 0$. As sufficient condition for this to hold is that $\frac{\theta}{1-\theta} > \frac{1}{1-\alpha}$. Furthermore, we can also solve

(139)
$$\widetilde{\chi} = \frac{\lambda + \widetilde{\widetilde{\chi}}}{\lambda + \widetilde{x}\widetilde{\widetilde{\chi}}}$$

The production function equation, which, combined with the Euler equation, implies that

(140)
$$\alpha\theta\left(\widetilde{k}^*\right)^{\alpha-1} = \rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}$$

This yields that

(141)
$$\widetilde{k}^* = \left[\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]^{\frac{1}{1-\alpha}}$$

and thus

(142)
$$\widetilde{y}^* = \left[\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}}$$

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The capital accumulation equation implies that

$$(143) \qquad \widetilde{i}^* = \left(\delta + \frac{1}{1-\alpha} \widetilde{g} \right) \widetilde{k}^*$$

The solutions for $\widetilde{y}^*$, $\widetilde{b}_0^*$, and $\widetilde{b}_x^*$ now allow us to write the steady state optimal implementation decision as

$$(144) \qquad \widetilde{x} = \frac{(\alpha\theta)^{\frac{\alpha}{1-\alpha}}}{(\alpha\theta)^{\frac{\alpha}{1-\alpha}} + \xi \left(\rho + \lambda + \psi\widetilde{g} \right) \left(\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma} \right)^{\frac{\alpha}{1-\alpha}}}$$

and the steady state R&D free entry condition, which determines the steady state growth rate of the economy, as

$$(145) \qquad \widetilde{g} = \frac{1}{\phi} \left\{ \frac{1}{\rho + \psi\widetilde{g}} \left[\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}} \right]^{\frac{\alpha}{1-\alpha}} - \xi \ln \left(1 + \frac{1}{\xi \left(\rho + \lambda + \psi\widetilde{g} \right)} \left[\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}} \right]^{\frac{\alpha}{1-\alpha}} \right) \right\}$$

Proof of existence and uniqueness of steady state:

In order to prove existence and uniqueness of the steady state, we have to show that the above equation, (145), has a unique positive root, $\widetilde{g}$.

To see this, note that the partial derivative with respect to $\widetilde{g}$ of the left hand side is one. The partial derivative of the right hand side satisfies

$$(146) \qquad 0 > \frac{1}{\phi} \left[\frac{\partial \left(\widetilde{b}_0^* \widetilde{y}^* \right)}{\partial \widetilde{g}} - \frac{1}{1 + \frac{\widetilde{b}_x^* \widetilde{y}^*}{\xi}} \frac{\partial \left(\widetilde{b}_x^* \widetilde{y}^* \right)}{\partial \widetilde{g}} \right]$$

$$= -\frac{\psi}{\phi} \left[\left(\widetilde{b}_0^* \right)^2 \widetilde{y}^* - \frac{1}{1 + \frac{\widetilde{b}_x^*}{\xi}} \left(\widetilde{b}_\psi^* \right)^2 \widetilde{y}^* \right]$$

$$- \frac{\alpha}{(1-\alpha)^2 \sigma} \frac{1}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}} \frac{1}{\phi} \left[\widetilde{b}_0^* \widetilde{y}^* - \frac{\widetilde{b}_x^* \widetilde{y}^*}{1 + \frac{\widetilde{b}_x^* \widetilde{y}^*}{\xi}} \right]$$

which is negative.

At $\widetilde{g} = 0$

$$(147) \qquad 0 < \frac{1}{\phi} \left[\frac{1}{\rho} \left[\frac{\alpha\theta}{\rho + \delta} \right]^{\frac{\alpha}{1-\alpha}} - \xi \ln \left(1 + \frac{1}{\xi \left(\rho + \lambda \right)} \left[\frac{\alpha\theta}{\rho + \delta} \right]^{\frac{\alpha}{1-\alpha}} \right) \right]$$

at $\widetilde{g} \to \infty$, the right hand side will go to zero, because $\widetilde{b}_0^*, \widetilde{b}_x^*, \widetilde{y}^* \to 0$. As a consequence, there is a unique $\widetilde{g}$ that solves the steady state equation.

We have to make sure, though, that at this $\widetilde{g}$ the household's objective function is bounded. This is always the case when $\sigma \leq 1$. However, when $\sigma > 1$, then this requires that

$$(148) \qquad \rho + \left(\frac{1}{\sigma} - 1 \right) \frac{1}{1-\alpha} \widetilde{g} > 0 \text{ that is } \widetilde{g} < (1-\alpha) \frac{\sigma}{\sigma - 1} \rho = g^*$$

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This imposes some parameter restrictions on ξ and ϕ. Namely, it has to hold that

$$(149) \qquad g^* > \frac{1}{\phi}\left[\frac{1}{\frac{\theta}{1-\theta}g^*}\left[\frac{\alpha\theta}{\delta+\frac{\theta}{1-\theta}g^*}\right]^{\frac{\alpha}{1-\alpha}} - \xi\ln\left(1+\frac{1}{\xi\left(\frac{\theta}{1-\theta}g^*+\lambda\right)}\right)\left[\frac{\alpha\theta}{\delta+\frac{\theta}{1-\theta}g^*}\right]^{\frac{\alpha}{1-\alpha}}\right]$$

which yields the restriction that invention can not be too cheap in order to lead to an excessive growth that. In particular, the R&D cost parameter has to satisfy

$$(150) \qquad \phi > \frac{1}{g^*}\left[\frac{1}{\frac{\theta}{1-\theta}g^*}\left[\frac{\alpha\theta}{\delta+\frac{\theta}{1-\theta}g^*}\right]^{\frac{\alpha}{1-\alpha}} - \xi\ln\left(1+\frac{1}{\xi\left(\frac{\theta}{1-\theta}g^*+\lambda\right)}\left[\frac{\alpha\theta}{\delta+\frac{\theta}{1-\theta}g^*}\right]^{\frac{\alpha}{1-\alpha}}\right)\right] \qquad \text{when } \sigma > 1$$

Derivation of comparative statics results, Table 1:

For the derivation of the comparative statics, it is useful to consider the steady state R&D condition

$$(151) \qquad 0 = \phi\widetilde{g} - \left[\widetilde{b}_0^*\widetilde{y}^* - \xi\ln\left(1+\frac{\widetilde{b}_x^*\widetilde{y}^*}{\xi}\right)\right]$$

The partial derivatives of the steady state growth rate can be determined by the application of the implicit function theorem to (151). The partial of (151) with respect to $\widetilde{g}$ gives us

$$(152) \qquad 0 \;<\; \Delta_{\widetilde{g}}^{(151)}$$

$$(153) \qquad =\; \phi + \psi\left[\widetilde{b}_0^{*2} - (1-\widetilde{x}^*)\widetilde{b}_x^{*2}\right]\widetilde{y}^*$$

$$(154) \qquad + \left[\widetilde{b}_0^* - (1-\widetilde{x})\widetilde{b}_x^*\right]\left(\frac{\alpha}{(1-\alpha)^2\sigma}\right)\left(\frac{\widetilde{y}^*}{\rho+\delta+\frac{\widetilde{g}}{(1-\alpha)\sigma}}\right)$$

The partial of (151) with respect to λ equals

$$(155) \qquad \Delta_{\lambda}^{(151)} = -(1-\widetilde{x})\widetilde{b}_x^{*2}\widetilde{y}^* < 0$$

Similarly, the partial of the right hand side of (151) with respect to ξ equals

$$(156) \qquad \Delta_{\xi}^{(151)} = -\ln(1-\widetilde{x}) - \widetilde{x} > 0$$

Finally, the partial of the left hand side of (151) with respect to ϕ is

$$(157) \qquad \Delta_{\phi}^{(151)} = \widetilde{g}$$

These results imply that

$$(158) \qquad \frac{d\widetilde{g}}{d\lambda} = -\left[\Delta_{\widetilde{g}}^{(151)}\right]^{-1}\Delta_{\lambda}^{(151)} > 0$$

while

$$(159) \qquad \frac{d\widetilde{g}}{d\xi} = -\left[\Delta_{\widetilde{g}}^{(151)}\right]^{-1}\Delta_{\xi}^{(151)} < 0$$

and

(160)
$$\frac{d\widetilde{g}}{d\phi} = -\left[\Delta_{\widetilde{g}}^{(151)}\right]^{-1} \Delta_{\phi}^{(151)} < 0$$

which completes the first row of the table.

For the partials of $\widetilde{x}^*$ it is first worthwhile to consider

(161)
$$\Delta_{\widetilde{g}}^{(\widetilde{x})} = \frac{\partial \widetilde{x}}{\partial \widetilde{g}}$$

$$= -\widetilde{x}(1-\widetilde{x})\left[\psi \widetilde{b}_x^* + \frac{\alpha}{(1-\alpha)^2 \sigma} \frac{1}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right] < 0$$

and

(162)
$$\Delta_{\lambda}^{(\widetilde{x})} = \frac{\partial \widetilde{x}}{\partial \lambda} = -\widetilde{x}(1-\widetilde{x})\widetilde{b}_x^* < 0$$

(163)
$$\Delta_{\xi}^{(\widetilde{x})} = \frac{\partial \widetilde{x}}{\partial \xi} = -\widetilde{x}(1-\widetilde{x})\frac{1}{\xi} < 0$$

as well as

(164)
$$\Delta_{\phi}^{(\widetilde{x})} = 0$$

This allows us to write

(165)
$$\frac{d\widetilde{x}}{d\lambda} = \Delta_{\lambda}^{(\widetilde{x})} + \Delta_{\widetilde{g}}^{(\widetilde{x})}\frac{d\widetilde{g}}{d\lambda} < 0$$

(166)
$$\frac{d\widetilde{x}}{d\phi} = \Delta_{\widetilde{g}}^{(\widetilde{x})}\frac{d\widetilde{g}}{d\phi} > 0$$

The most complicated derivative here is

(167)
$$\frac{d\widetilde{x}}{d\xi} = \Delta_{\xi}^{(\widetilde{x})} + \Delta_{\widetilde{g}}^{(\widetilde{x})}\frac{d\widetilde{g}}{d\xi}$$

$$= \Delta_{\xi}^{(\widetilde{x})} - \Delta_{\widetilde{g}}^{(\widetilde{x})}\left[\Delta_{\widetilde{g}}^{(151)}\right]^{-1} \Delta_{\xi}^{(151)}$$

(168)
$$= -\widetilde{x}(1-\widetilde{x})\frac{1}{\xi}$$

(169)
$$+ \frac{\left[\psi \widetilde{b}_x^* + \frac{\alpha}{(1-\alpha)^2 \sigma} \frac{1}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]}{\phi + \psi \left[\widetilde{b}_0^{*2} - (1-\widetilde{x}^*)\widetilde{b}_x^{*2}\right]\widetilde{y}^* + \left[\widetilde{b}_0^* - (1-\widetilde{x})\widetilde{b}_x^*\right]\left(\frac{\alpha}{(1-\alpha)^2\sigma}\right)\left(\frac{\widetilde{y}^*}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right)}$$

(170)
$$\times \widetilde{x}(1-\widetilde{x})\left[-\ln(1-\widetilde{x}) - \widetilde{x}\right]$$

This expression is negative, whenever

(171)
$$\phi + \psi \left[\widetilde{b}_0^{*2} - (1-\widetilde{x}^*)\widetilde{b}_x^{*2}\right]\widetilde{y}^* + \left[\widetilde{b}_0^* - (1-\widetilde{x})\widetilde{b}_x^*\right]\left(\frac{\alpha}{(1-\alpha)^2\sigma}\right)\left(\frac{\widetilde{y}^*}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right)$$

$$> \left[\psi \widetilde{b}_x^* + \frac{\alpha}{(1-\alpha)^2 \sigma} \frac{1}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]\xi\left[-\ln(1-\widetilde{x}) - \widetilde{x}\right]$$

But it turns out that

$$(172) \qquad \xi\left[-\ln\left(1-\widetilde{x}\right)-\widetilde{x}\right] \;<\; \xi\left[\frac{\widetilde{x}}{(1-\widetilde{x})}-\widetilde{x}\right] = \xi\frac{\widetilde{x}}{1-\widetilde{x}}\widetilde{x} = \xi\left(\frac{\widetilde{b}_x^*\widetilde{y}^*}{\xi}\right)\widetilde{x} = \widetilde{b}_x^*\widetilde{y}^*\widetilde{x}$$

$$< \;\left(\widetilde{b}_0^* - \widetilde{b}_x^*\right)\widetilde{y}^* + \widetilde{b}_x^*\widetilde{y}^*\widetilde{x} = \left[\widetilde{b}_0^* - (1-\widetilde{x})\widetilde{b}_x^*\right]\widetilde{y}^*$$

Thus, we obtain that

$$(173) \qquad \psi\left[\widetilde{b}_0^{*2} - (1-\widetilde{x}^*)\widetilde{b}_x^{*2}\right]\widetilde{y}^* \;>\; \psi\widetilde{b}_x^*\left[\widetilde{b}_0^* - (1-\widetilde{x}^*)\widetilde{b}_x^*\right]\widetilde{y}^*$$

$$> \; \psi\widetilde{b}_x^*\xi\left[-\ln\left(1-\widetilde{x}\right)-\widetilde{x}\right]$$

and

$$(174) \qquad \left[\widetilde{b}_0^* - (1-\widetilde{x})\widetilde{b}_x^*\right]\left(\frac{\alpha}{(1-\alpha)^2\,\sigma}\right)\left(\frac{\widetilde{y}^*}{\rho+\delta+\frac{\widetilde{g}}{(1-\alpha)\sigma}}\right)$$

$$> \;\left(\frac{\alpha}{(1-\alpha)^2\,\sigma}\right)\left(\frac{1}{\rho+\delta+\frac{\widetilde{g}}{(1-\alpha)\sigma}}\right)\xi\left[-\ln\left(1-\widetilde{x}\right)-\widetilde{x}\right]$$

and, since $\phi > 0$, inequality (171) holds. Thus

$$(175) \qquad \frac{d\widetilde{x}}{d\xi} < 0$$

Derivation of equations (54) and (55):

The market share satisfies

$$(176) \qquad s_{it} = \left(\frac{a_{it}}{z_t}\right)^{\frac{\theta}{1-\theta}} = \left(\frac{a_{it}}{\overline{a}_i}\right)^{\frac{\theta}{1-\theta}}\left(\frac{\overline{a}_i}{\overline{a}_t}\right)^{\frac{\theta}{1-\theta}}\overline{\chi}_t\chi_t$$

Such that in steady state

$$(177) \qquad s_{ii} = \widetilde{x}\widetilde{\overline{\chi}}\widetilde{\chi}$$

Moreover, on the balanced growth path

$$(178) \qquad x_{it} = \left(\frac{a_{it}}{\overline{a}_i}\right)^{\frac{\theta}{1-\theta}} = \left[\left(1 - e^{-\lambda(t-i)}\right) + e^{-\lambda(t-i)}\widetilde{x}\right]$$

while

$$(179) \qquad \left(\frac{\overline{a}_i}{\overline{a}_t}\right)^{\frac{\theta}{1-\theta}} = e^{-\frac{\theta}{1-\theta}\widetilde{g}(t-i)}$$

Combining these two results, we find that, on the balanced growth path,

$$(180) \qquad s_{it} = \begin{cases} 0 & \text{for } t < i \\ \left[\left(1 - e^{-\lambda(t-i)}\right) + e^{-\lambda(t-i)}\widetilde{x}\right]e^{-\frac{\theta}{1-\theta}\widetilde{g}(t-i)}\widetilde{\overline{\chi}}\widetilde{\chi} & \text{otherwise} \end{cases}$$

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Taking the partial of this expression with respect to t for $t \geq i$, we obtain that

$$(181) \qquad \dot{s}_{it} = \lambda e^{-\lambda(t-i)}(1-\tilde{x})\widetilde{\widetilde{\chi}}\widetilde{\chi} - \frac{\theta}{1-\theta}\tilde{g}s_{it}$$

$$= \lambda e^{-\lambda(t-i)} - \left[\lambda + \frac{\theta}{1-\theta}\tilde{g}\right]s_{it}$$

The learning effect dominates at the introduction of the intermediate good whenever

$$(182) \qquad 0 < \left.\frac{\partial s_{it}}{\partial t}\right|_{t=i} = \lambda - \left[\lambda + \frac{\theta}{1-\theta}\tilde{g}\right]\tilde{x}$$

this is the case whenever the implementation cost is so large that

$$(183) \qquad \tilde{x} < \frac{\lambda}{\lambda + \frac{\theta}{1-\theta}\tilde{g}}$$

Under this condition, the path of the market share is initially increasing and subsequently decreasing.

Derivation of equation (57) :

We obtain that

$$(184) \qquad S_{it} = z_{it}^{\frac{\theta}{1-\theta}} \Big/ z_t^{\frac{\theta}{1-\theta}}, \text{ where } z_{it}^{\frac{\theta}{1-\theta}} = \int_i^t a_{\iota t}^{\frac{\theta}{1-\theta}} d\iota$$

For $z_{it}^{\frac{\theta}{1-\theta}}$ we obtain that it satisfies

$$(185) \qquad z_{it}^{\frac{\theta}{1-\theta}} = \int_i^t a_{\iota t}^{\frac{\theta}{1-\theta}} d\iota = \int_i^t \overline{a}_\iota^{\frac{\theta}{1-\theta}} x_{\iota t} d\iota$$

such that

$$(186) \qquad \left(z_{it}^{\frac{\theta}{1-\theta}}\right)^{\cdot} = \int_i^t \overline{a}_\iota^{\frac{\theta}{1-\theta}} \dot{x}_{\iota t} d\iota + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}$$

$$= \lambda \int_i^t \overline{a}_\iota^{\frac{\theta}{1-\theta}} d\iota - \lambda \int_i^t \overline{a}_\iota^{\frac{\theta}{1-\theta}} x_{\iota t} d\iota + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}$$

$$= \lambda \int_i^t \overline{a}_\iota^{\frac{\theta}{1-\theta}} d\iota - \lambda z_{it}^{\frac{\theta}{1-\theta}} + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}$$

Which simplifies on the balanced growth path to

$$(187) \qquad \left(z_{it}^{\frac{\theta}{1-\theta}}\right)^{\cdot} = \overline{a}_i^{\frac{\theta}{1-\theta}} e^{\frac{\theta}{1-\theta}\tilde{g}(t-i)}\left[\lambda\frac{1-\theta}{\theta\tilde{g}} + \tilde{x}\right] - \lambda\frac{1-\theta}{\theta\tilde{g}}\overline{a}_i^{\frac{\theta}{1-\theta}} - \lambda z_{it}^{\frac{\theta}{1-\theta}}$$

and can be combined with the initial condition that $z_{ii}^{\frac{\theta}{1-\theta}} = 0$.

The particular solution to this differential equation yields that

$$(188) \qquad z_{it}^{\frac{\theta}{1-\theta}} = \int_i^t e^{-\lambda(t-s)}\left[\overline{a}_i^{\frac{\theta}{1-\theta}} e^{\frac{\theta}{1-\theta}\tilde{g}(t-i)}\left[\lambda\frac{1-\theta}{\theta\tilde{g}} + \tilde{x}\right] - \lambda\frac{1-\theta}{\theta\tilde{g}}\overline{a}_i^{\frac{\theta}{1-\theta}}\right]ds$$

$$= \overline{a}_t^{\frac{\theta}{1-\theta}}\int_i^t e^{-\lambda(t-s)}\left[e^{-\frac{\theta}{1-\theta}\tilde{g}(t-s)}\left[\lambda\frac{1-\theta}{\theta\tilde{g}} + \tilde{x}\right] - \lambda\frac{1-\theta}{\theta\tilde{g}}e^{-\frac{\theta}{1-\theta}\tilde{g}(t-i)}\right]ds$$

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such that

$$
(189) \qquad z_{it}^{\frac{\theta}{1-\theta}} = \overline{a}_t^{\frac{\theta}{1-\theta}} \left\{ \left[\frac{\lambda \frac{1-\theta}{\theta \widetilde{g}} + \widetilde{x}}{\lambda + \frac{\theta}{1-\theta} \widetilde{g}} \right] \left[1 - e^{-\left[\lambda + \frac{\theta}{1-\theta} \widetilde{g}\right](t-i)} \right] - \frac{1-\theta}{\theta \widetilde{g}} e^{-\frac{\theta}{1-\theta} \widetilde{g}(t-i)} \left[1 - e^{-\lambda(t-i)} \right] \right\}
$$

On the balanced growth path

$$
(190) \qquad z_t^{\frac{\theta}{1-\theta}} = \overline{a}_t^{\frac{\theta}{1-\theta}} \left[\widetilde{\chi}\widetilde{\chi} \right]^{-1} = \overline{a}_t^{\frac{\theta}{1-\theta}} \left[\frac{\lambda \frac{1-\theta}{\theta \widetilde{g}} + \widetilde{x}}{\lambda + \frac{\theta}{1-\theta} \widetilde{g}} \right]
$$

Hence,

$$
\begin{aligned}
(191) \qquad S_{it} &= \left(\frac{z_{it}}{z_t} \right)^{\frac{\theta}{1-\theta}} \\
&= \left[1 - e^{-\left[\lambda + \frac{\theta}{1-\theta} \widetilde{g}\right](t-i)} \right] - \widetilde{\chi} e^{-\frac{\theta}{1-\theta} \widetilde{g}(t-i)} \left[1 - e^{-\lambda(t-i)} \right] \\
&= 1 - e^{-\frac{\theta}{1-\theta} \widetilde{g}(t-i)} \left[\widetilde{\chi} + (\widetilde{\chi} - 1) e^{-\lambda(t-i)} \right]
\end{aligned}
$$

which is equation (57).

Approximate transitional dynamics:

The approximate transitional dynamics of this economy can be derived by log-linearizing the 10 dynamic equilibrium equations. Throughout our derivation, we will denote percentage differences from the steady state by ˆ. That is, $\widehat{y}_t^*$ is the percentage deviation of detrended output, y_t^*, from its steady state value, $\widetilde{y}^*$.

The dynamics can be approximated by log-linearizing the equations

$$
(192) \qquad y_t^* = c_t^* + i_t^* + (1-\theta)\overline{\chi}_t \chi_t \left[\xi \left(-\ln(1 - x_{tt}) - x_{tt} \right) + \phi g_t \right]
$$

$$
(193) \qquad y_t^* = k_t^{*\alpha}
$$

$$
(194) \qquad x_{tt} = \frac{b_{xt}^* y_t^*}{\xi + b_{xt}^* y_t^*}
$$

$$
(195) \qquad g_t = \frac{1}{\phi} \left[b_{0t}^* y_t^* - \xi \ln\left(1 + \frac{b_{xt}^* y_t^*}{\xi} \right) \right]
$$

and for the dynamic jump variables

$$
(196) \qquad \frac{\dot{c}_t^*}{c_t^*} = \sigma \left(\alpha\theta \frac{y_t^*}{k_t^*} - \delta - \rho \right) - \frac{1}{1-\alpha} g_t
$$

$$
(197) \qquad \dot{b}_{0t}^* = \left\{ \alpha\theta \frac{y_t^*}{k_t^*} - \delta - \lambda(1 - \chi_t) + \chi_t \overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1-\alpha} g_t \right\} b_{0t}^* - 1
$$

$$
(198) \qquad \dot{b}_{xt}^* = \left\{ \alpha\theta \frac{y_t^*}{k_t^*} - \delta + \lambda\chi_t + \chi_t \overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1-\alpha} g_t \right\} b_{xt}^* - 1
$$

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while the state variables evolve according to

(199)
$$\frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1-\alpha}g_t$$

(200)
$$\frac{\dot{\chi}_t}{\chi_t} = \lambda(1-\chi_t) + (1-\chi_t x_{tt})\overline{\chi}_t$$

(201)
$$\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} = \frac{\theta}{1-\theta}g_t - \overline{\chi}_t$$

Which yields

(202)
$$\widehat{y}_t^* = \alpha\widehat{k}_t^*$$

as well as

(203)
$$\begin{aligned}
\widehat{y}_t^* &= \frac{\widetilde{c}^*}{\widetilde{y}^*}\widehat{c}_t^* + \frac{\widetilde{i}^*}{\widetilde{y}^*}\widehat{i}_t^* + \\
&\quad (1-\theta)\frac{\widetilde{\chi\overline{\chi}}}{\widetilde{y}^*}\left[\xi\left(-\ln(1-\widetilde{x})-\widetilde{x}\right) + \phi\widetilde{g}\right]\widehat{\overline{\chi}}_t + \\
&\quad (1-\theta)\frac{\widetilde{\chi\overline{\chi}}}{\widetilde{y}^*}\left[\xi\left(-\ln(1-\widetilde{x})-\widetilde{x}\right) + \phi\widetilde{g}\right]\widehat{\chi}_t + (1-\theta)\xi\frac{\widetilde{\chi\overline{\chi}}}{\widetilde{y}^*}\widetilde{x}\frac{\widetilde{x}}{1-\widetilde{x}}\widehat{x}_{tt} \\
&\quad + (1-\theta)\phi\frac{\widetilde{\chi\overline{\chi}}}{\widetilde{y}^*}\widetilde{g}\widehat{g}_t
\end{aligned}$$

also

(204)
$$\widehat{x}_{tt} = (1-\widetilde{x})\widehat{b}_{xt}^* + (1-\widetilde{x})\widehat{y}_t^*$$

as well as

(205)
$$\widehat{g}_t = \left(\frac{1}{\phi\widetilde{g}}\right)\widetilde{b}_0^*\widetilde{y}^*\widehat{b}_{0t}^* - \left(\frac{\xi}{\phi\widetilde{g}}\right)\widetilde{x}^*\widehat{b}_{xt}^* + \left[\left(\frac{1}{\phi\widetilde{g}}\right)\widetilde{b}_0^*\widetilde{y}^* - \left(\frac{\xi}{\phi\widetilde{g}}\right)\widetilde{x}^*\right]\widehat{y}_t^*$$

such that

(206)
$$\dot{\widehat{c}}_t^* = \sigma\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{y}_t^* - \sigma\alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{k}_t^* - \frac{\widetilde{g}}{1-\alpha}\widehat{g}_t$$

and

(207)
$$\begin{aligned}
\dot{\widehat{b}}_{0t}^* &= \alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{y}_t^* - \alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{k}_t^* + \left(\lambda + \widetilde{x}\widetilde{\chi}\right)\widetilde{\chi}\widehat{\chi}_t + \widetilde{x}\widetilde{\chi}\widetilde{\chi}\widehat{\overline{\chi}}_t + \widetilde{x}^*\widetilde{\chi}\widetilde{\chi}\widehat{x}_{tt} \\
&\quad - \alpha\dot{\widehat{k}}_t^* - \frac{1}{1-\alpha}\widetilde{g}\widehat{g}_t + \frac{1}{\widetilde{b}_0^*}\widehat{b}_{0t}^*
\end{aligned}$$

also

(208)
$$\begin{aligned}
\dot{\widehat{b}}_{xt}^* &= \alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{y}_t^* - \alpha\theta\frac{\widetilde{y}^*}{\widetilde{k}^*}\widehat{k}_t^* + \left(\lambda + \widetilde{x}^*\widetilde{\chi}\right)\widetilde{\chi}\widehat{\chi}_t + \widetilde{x}\widetilde{\chi}\widetilde{\chi}\widehat{\overline{\chi}}_t + \widetilde{x}\widetilde{\chi}\widetilde{\chi}\widehat{x}_{tt} \\
&\quad - \alpha\dot{\widehat{k}}_t^* - \frac{1}{1-\alpha}\widetilde{g}\widehat{g}_t + \frac{1}{\widetilde{b}_x^*}\widehat{b}_{xt}^*
\end{aligned}$$

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It also yields that

$$(209) \qquad \dot{\widehat{k}}_t^* = \frac{\widetilde{i}^*}{\widetilde{k}^*}\widehat{i}_t^* - \frac{\widetilde{i}^*}{\widetilde{k}^*}\widehat{k}_t^* - \frac{1}{1-\alpha}\widetilde{g}\hat{g}_t$$

and

$$(210) \qquad \dot{\widehat{\chi}}_t = -\left(\lambda + \widetilde{x}\widetilde{\chi}\right)\widetilde{\chi}\widehat{\chi}_t + (1 - \widetilde{x}\widetilde{\chi})\widetilde{\widetilde{\chi}}\widehat{\widetilde{\chi}}_t - \widetilde{x}\widetilde{\chi}\widetilde{\widetilde{\chi}}\widehat{x}_{tt}$$

and, finally

$$(211) \qquad \dot{\widehat{\widetilde{\chi}}}_t = \frac{\theta}{1-\theta}\widetilde{g}\hat{g}_t - \widetilde{\widetilde{\chi}}\widehat{\widetilde{\chi}}_t$$

which are the 10 log-linearized equilibrium equations.

Calibration:

We choose σ, ρ, and δ similar to Cooley and Prescott (1995).

The demand elasticity parameter, θ, and the capital elasticity of output, α, are pinned down by the observations that labor costs in the U.S. make up 60% corporate income (value added) if intangible capital is taken into account and that that 15% of value added is paid to intangible capital. The theoretical counterparts of these conditions are

$$(212) \qquad (1-\alpha)\theta = 0.60 \text{ and } (1-\theta) = 0.15$$

which yields that $\theta = 0.85$ and $\alpha = 0.29$.

This leaves λ, ϕ, and ξ to be calibrated. We do so by using the following three moment conditions. First of all, we choose our parameters to be consistent with the 2.1% average post-war growth rate of U.S. real GDP per capita. The theoretical equivalent of this observation is that

$$(213) \qquad \frac{\widetilde{g}}{1-\alpha} = 0.021, \text{ which yields } \widetilde{g} = 0.016$$

Secondly, we choose our parameter to match that R&D expenditures make up 2.5% of GDP. In terms of our model, this implies that

$$(214) \qquad (1-\theta)\widetilde{\chi}\widetilde{\widetilde{\chi}}\frac{\phi\widetilde{g}}{\widetilde{y}^*} = 0.025$$

Finally, we choose our parameters to be consistent with evidence on learning by doing in the U.S. manufacturing sector from Bahk and Gort (1993).

Bahk and Gort (1993) provide evidence on how the level of output (shipments as well as value added) at the plant level, y_{it}, depends on labor inputs measured by employees, l_{it}, and by wages, w_{it}, capital inputs, k_{it}, a trend, t, as well as of the plant's cumulative output level, $X_{it} = \int_{t-age_{it}}^{t} y_{it'}dt'$. The latter dependence is meant to measure learning by doing effects.

For a sample of 1281 plants for the period 1973-1986, they find the relationship[12]

$$(215) \qquad \ln y_{it} \; = \; 1.55 + 0.612 \ln l_{it} + 0.690 \ln w_{it} + 0.282 \ln k_{it}$$

$$+ 0.020 t + 0.028 \ln X_{it} + e_{it}$$

the average age of the plants in the sample is between 6 and 7 years old. We interpret this to mean that for these plants a one percent increase in cumulative output seems to lead to a 0.028 percent increase in total factor productivity.

We calibrate our model such that we find a learning by doing effect of a similar magnitude in our model. Bahk and Gort (1993) claim that most of their identification comes from the cross-sectional dimension of their dataset. In line with this claim, we consider the estimation of a cross-sectional regression of the form

$$(216) \qquad \ln a_{it} = \beta_0 + \beta_x \ln X_{it} + e_{it}$$

for all firms $i \in [t-13, t-1]$. We then choose our parameters such that the ordinary least squares estimate of β_x equals 0.028.

This calibration method requires us to derive the OLS estimate of β_x. The first step in this derivation is to consider that, in the steady state,

$$(217) \qquad \ln a_{it} = \ln \overline{a}_t - \widetilde{g}\,(t-i) + \frac{1-\theta}{\theta} \ln \left[1 - (1-\widetilde{x})\, e^{-\lambda(t-i)} \right]$$

Moreover, the level of output of intermediate i at time t is given by

$$(218) \qquad y_{it} = \left[1 - (1-\widetilde{x})\, e^{-\lambda(t-i)} \right]^{\frac{1}{\theta}} e^{-\left(\frac{1}{1-\theta} - \frac{1}{1-\alpha} \right) \widetilde{g}(t-i)} \widetilde{\chi \chi} y_i$$

Such that

$$(219) \qquad \begin{aligned} X_{it} \; &= \; \widetilde{\chi \chi} y_i \int_i^t \left[1 - (1-\widetilde{x})\, e^{-\lambda(t'-i)} \right]^{\frac{1}{\theta}} e^{-\left(\frac{1}{1-\theta} - \frac{1}{1-\alpha} \right) \widetilde{g}(t'-i)} dt' \\ &= \; \widetilde{\chi \chi} y_t \exp^{-\frac{\widetilde{g}}{1-\alpha}(t-i)} \int_i^t \left[1 - (1-\widetilde{x})\, e^{-\lambda(t'-i)} \right]^{\frac{1}{\theta}} e^{-\left(\frac{1}{1-\theta} - \frac{1}{1-\alpha} \right) \widetilde{g}(t'-i)} dt' \end{aligned}$$

When we define the sample means

$$(220) \qquad \overline{\ln a} = \frac{1}{12} \int_{t-13}^{t-1} \ln a_{it} di \;\text{ and }\; \overline{\ln X} = \frac{1}{12} \int_{t-13}^{t-1} \ln X_{it} di$$

then the OLS estimate of β_x equals

$$(221) \qquad \widehat{\beta}_x = \left(\int_{t-13}^{t-1} \left(\ln X_{it} - \overline{\ln X} \right)^2 di \right)^{-1} \left(\int_{t-13}^{t-1} \left(\ln X_{it} - \overline{\ln X} \right) \left(\ln a_{it} - \overline{\ln a} \right) di \right)$$

and thus $\widehat{\beta}_x = 0.028$ is our last moment condition that we use to calibrate our parameters.

---

[12] Bahk and Gort (1993), Table 1, row (ii).

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Neoclassical model we use for transitional dynamics comparison:

In the Neoclassical model that we use for comparison growth is exogenous and equal to $\widetilde{g}$. Implementation is free such that $\widetilde{x} = \widetilde{\chi} = 1$ and $\widetilde{\overline{\chi}} = \frac{\theta}{1-\theta}\widetilde{g}$. Profits made are paid back to households.

The stationary equilibrium dynamics of the resulting model are given by the Euler equation

$$(222) \qquad \frac{\dot{c}_t^*}{c_t^*} = \sigma\left(\alpha\theta\frac{y_t^*}{k_t^*} - \delta - \rho\right) - \frac{1}{1-\alpha}\widetilde{g}$$

the resource constraint

$$(223) \qquad y_t^* = c_t^* + i_t^*$$

the production function

$$(224) \qquad y_t^* = k_t^{*\alpha}$$

as well as the capital accumulation equation

$$(225) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1-\alpha}\widetilde{g}$$

The planner's problem:

The social planner in this economy would choose a path for

$$(226) \qquad \{c_s, y_s, i_s, k_s, x_{ss}^*, g_s, z_s, \overline{a}_s, \overline{z}_s\}_{s=t}^{\infty}$$

to maximize the present discounted value of the representative household's stream of utility that equals

$$(227) \qquad \int_t^{\infty} e^{-\rho s}\frac{\sigma}{\sigma-1}c_s^{\frac{\sigma-1}{\sigma}}\, ds$$

subject to the resource constraint that

$$(228) \qquad y_t = c_t + i_t + (1-\theta)z_t^{\frac{1}{1-\alpha}}\left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}\{\xi\left[-\ln\left(1-x_{tt}^*\right) - x_{tt}^*\right] + \phi g_t\}$$

the final goods production function

$$(229) \qquad y_t = z_t k_t^{\alpha}$$

the capital accumulation constraint

$$(230) \qquad \dot{k}_t = i_t - \delta k_t$$

the law of motion of potential productivity of the newest intermediate

$$(231) \qquad \left(\overline{a}_t^{\frac{\theta}{1-\theta}}\right)^{\cdot} = \frac{\theta}{1-\theta}\overline{a}_t^{\frac{\theta}{1-\theta}}g_t$$

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the law of motion of average potential productivity

(232)
$$\left(\overline{z}_t^{\frac{\theta}{1-\theta}}\right)^{\cdot} = \overline{a}_t^{\frac{\theta}{1-\theta}}$$

and the law of motion of average productivity

(233)
$$\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\cdot} = \lambda\left(\overline{z}_t^{\frac{\theta}{1-\theta}} - z_t^{\frac{\theta}{1-\theta}}\right) + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}^*$$

This means that we can write the dynamic Lagrangian associated with this problem as

(234)
$$\int_t^\infty e^{-\rho(s-t)} H_s ds$$

Where the current value Hamiltonian, H_t, is given by

(235) $\quad H_t \quad = \quad \dfrac{\sigma}{\sigma-1} c_s^{\frac{\sigma-1}{\sigma}}$

$$+\mu_{rt}\left[y_t - c_t - i_t - (1-\theta)\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-1}\overline{a}_t^{\frac{\theta}{1-\theta}}\left[\xi\left(-\ln\left(1-x_{tt}\right)-x_{tt}\right)+\phi g_t\right]\right]$$

$$+\mu_{yt}\left[y_t - \left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1-\theta}{\theta}}k_t^\alpha\right] + \mu_{kt}\left[i_t - \delta k_t\right]$$

$$+\mu_{\overline{a}t}\left[\frac{\theta}{1-\theta}\overline{a}_t^{\frac{\theta}{1-\theta}}g_t\right] + \mu_{\overline{z}t}\left[\overline{a}_t^{\frac{\theta}{1-\theta}}\right] + \mu_{zt}\left[\lambda\left(\overline{z}_t^{\frac{\theta}{1-\theta}} - z_t^{\frac{\theta}{1-\theta}}\right) + \overline{a}_t^{\frac{\theta}{1-\theta}} x_{tt}\right]$$

This yields the following first order necessary conditions for an interior solution

w.r.t. y_t :	$\mu_{rt} = -\mu_{yt}$
w.r.t. c_t :	$\mu_{rt} = c_t^{-\frac{1}{\sigma}}$
w.r.t. i_t :	$\mu_{rt} = \mu_{kt}$
w.r.t. g_t :	$\mu_{rt}(1-\theta)\phi z_t^{\frac{1}{1-\alpha}}\left(\dfrac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}} = \dfrac{\theta}{1-\theta}\mu_{\overline{a}t}\overline{a}_t^{\frac{\theta}{1-\theta}}$
w.r.t. x_{tt} :	$\mu_{rt}(1-\theta)\xi z_t^{\frac{1}{1-\alpha}}\left(\dfrac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}\dfrac{x_{tt}}{1-x_{tt}} = \mu_{zt}\overline{a}_t^{\frac{\theta}{1-\theta}}$
w.r.t. k_t :	$-\alpha\mu_{yt}\dfrac{y_t}{k_t} - \delta\mu_{kt} = \rho\mu_{kt} - \dot{\mu}_{kt}$
w.r.t. $\overline{z}_t^{\frac{\theta}{1-\theta}}$:	$\lambda\mu_{zt} = \rho\mu_{\overline{z}t} - \dot{\mu}_{\overline{z}t}$

w.r.t. $\overline{a}_t^{\frac{\theta}{1-\theta}}$:
$$-\mu_{rt}(1-\theta)\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-1}\left[\xi\left(-\ln\left(1-x_{tt}\right)-x_{tt}\right)+\phi g_t\right]$$
$$+\tfrac{\theta}{1-\theta}\mu_{\overline{a}t}g_t + \left(\mu_{\overline{z}t} + x_{tt}\mu_{zt}\right) = \rho\mu_{\overline{a}t} - \dot{\mu}_{\overline{a}t}$$

w.r.t. $z_t^{\frac{\theta}{1-\theta}}$:
$$-\mu_{rt}(1-\theta)\left(\tfrac{1}{1-\alpha}\tfrac{1-\theta}{\theta}-1\right)\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-2}\overline{a}_t^{\frac{\theta}{1-\theta}}\left[\xi\left(-\ln\left(1-x_{tt}\right)-x_{tt}\right)+\phi g_t\right]$$
$$-\mu_{yt}\tfrac{1-\theta}{\theta}\dfrac{y_t}{z_t^{\frac{\theta}{1-\theta}}} - \lambda\mu_{zt} = \rho\mu_{zt} - \dot{\mu}_{zt}$$

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The planner's stationary transformed allocation dynamics:

We will write the above system in a set of transformed variables that are stationary along the resource allocation path chosen by the social planner. These transformed variables are detrended consumption, output, capital, and investment

$$(236) \qquad c_t^* = \frac{c_t}{z_t^{\frac{1}{1-\alpha}}}, \ y_t^* = \frac{y_t}{z_t^{\frac{1}{1-\alpha}}}, \ k_t^* = \frac{k_t}{z_t^{\frac{1}{1-\alpha}}}, \text{ and } i_t^* = \frac{i_t}{z_t^{\frac{1}{1-\alpha}}}$$

as well as the potential productivity gap and implementation gap

$$(237) \qquad \overline{\chi}_t = \left(\frac{\overline{a}_t}{\overline{z}_t}\right)^{\frac{\theta}{1-\theta}} \text{ and } \chi_t = \left(\frac{\overline{z}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}$$

The implementation level, x_{tt}, and the growth rate, g_t. As well as three transformed costate variables.

$$(238) \qquad \mu_{zt}^* = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{\mu_{zt} \overline{a}_t^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}-\frac{\theta}{1-\theta}} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{\mu_{zt} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}}}{\mu_{zt} z_t^{\psi}}$$

$$= \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}}}{\mu_{zt} \left[\left(\frac{z_t}{\overline{a}_t}\right)^{\frac{\theta}{1-\theta}}\right]^{\frac{1-\theta}{\theta}\psi} \overline{a}_t^{\psi}} = \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}}}{\mu_{zt} \left[\frac{1}{\overline{\chi}_t \chi_t}\right]^{\frac{1-\theta}{\theta}\psi} \overline{a}_t^{\psi}}$$

$$= \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}} \left(\overline{\chi}_t \chi_t\right)^{\eta}}{\mu_{zt} \overline{a}_t^{\psi}}, \text{ where } \eta = \frac{1-\theta}{\theta}\psi$$

and

$$(239) \qquad \mu_{\overline{z}t}^* = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{\mu_{\overline{z}t} \overline{a}_t^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}-\frac{\theta}{1-\theta}}}{\mu_{\overline{z}t}} = \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}} \left(\overline{\chi}_t \chi_t\right)^{\eta}}{\mu_{\overline{z}t} \overline{a}_t^{\psi}}$$

$$(240) \qquad \mu_{\overline{a}t}^* = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{\mu_{\overline{a}t} \overline{a}_t^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\,\mu_{rt} z_t^{\frac{1}{1-\alpha}-\frac{\theta}{1-\theta}}}{\mu_{\overline{a}t}} = \frac{(1-\theta)\,(c_t^*)^{-\frac{1}{\sigma}} \left(\overline{\chi}_t \chi_t\right)^{\eta}}{\mu_{\overline{a}t} \overline{a}_t^{\psi}}$$

where it is worth noting that

$$(241) \qquad \frac{\dot{\mu}_{zt}^*}{\mu_{zt}^*} = -\frac{\dot{\mu}_{zt}}{\mu_{zt}} - \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} + \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} + \eta\frac{\dot{\chi}_t}{\chi_t} - \psi g_t$$

$$(242) \qquad \frac{\dot{\mu}_{\overline{z}t}^*}{\mu_{\overline{z}t}^*} = -\frac{\dot{\mu}_{\overline{z}t}}{\mu_{\overline{z}t}} - \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} + \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} + \eta\frac{\dot{\chi}_t}{\chi_t} - \psi g_t$$

$$(243) \qquad \frac{\dot{\mu}_{\overline{a}t}^*}{\mu_{\overline{a}t}^*} = -\frac{\dot{\mu}_{\overline{a}t}}{\mu_{\overline{a}t}} - \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} + \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} + \eta\frac{\dot{\chi}_t}{\chi_t} - \psi g_t$$

This allows us to write the system of equations as

$$(244) \qquad y_t^* = (k_t^*)^\alpha$$

$$(245) \qquad y_t^* = c_t^* + i_t^* + (1 - \theta)\chi_t \overline{\chi}_t \left[\xi\left(-\ln\left(1 - x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$

$$(246) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1-\alpha}g_t$$

$$(247) \qquad \frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} = \frac{\theta}{1-\theta}g_t - \overline{\chi}_t$$

$$(248) \qquad \frac{\dot{\chi}_t}{\chi_t} = \lambda\left(1 - \chi_t\right) + \left(1 - \chi_t x_{tt}\right)\overline{\chi}_t$$

and the transformed optimality conditions

$$(249) \qquad \mu_{\bar{a}t}^* = \frac{1}{\phi}\frac{\theta}{1-\theta}$$

and

$$(250) \qquad \frac{1}{\xi\mu_{\bar{z}t}^*} = \frac{x_{tt}}{1 - x_{tt}}$$

The transformed Euler equation

$$(251) \qquad \frac{\dot{c}_t^*}{c_t^*} = \sigma\left(\alpha\frac{y_t^*}{k_t^*} - \delta - \rho\right) - \frac{1}{1-\alpha}g_t$$

as well as

$$(252) \qquad \lambda\frac{\mu_{\bar{z}t}^*}{\mu_{\bar{z}t}^*} = \rho - \frac{\dot{\mu}_{\bar{z}t}}{\mu_{\bar{z}t}}$$

$$= \rho + \frac{\dot{\mu}_{\bar{z}t}^*}{\mu_{\bar{z}t}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \eta\frac{\dot{\chi}_t}{\chi_t} + \psi g_t$$

Furthermore, we can write

$$(253) \qquad -\mu_{rt}\left(1 - \theta\right)\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta} - 1}\left[\xi\left(-\ln\left(1 - x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$

$$+\frac{\theta}{1-\theta}\mu_{\bar{a}t}g_t + \left(\mu_{\bar{z}t} + x_{tt}\mu_{zt}\right) = \rho\mu_{\bar{a}t} - \dot{\mu}_{\bar{a}t}$$

as

$$(254) \qquad -\frac{\mu_{rt}(1-\theta)z_t^{\frac{1}{1-\alpha}}\left(\frac{a_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{a_t^{\frac{\theta}{1-\theta}}\mu_{\bar{a}t}}\left[\xi\left(-\ln\left(1 - x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$

$$+\frac{\theta}{1-\theta}g_t + \left(\frac{\mu_{\bar{z}t}}{\mu_{\bar{a}t}} + x_{tt}\frac{\mu_{zt}}{\mu_{\bar{a}t}}\right) = \rho - \frac{\dot{\mu}_{\bar{a}t}}{\mu_{\bar{a}t}}$$

such that

$$(255) \qquad -\mu_{\bar{a}t}^*\left[\xi\left(-\ln\left(1 - x_{tt}\right) - x_{tt}\right) + \phi g_t\right] + \frac{\theta}{1-\theta}g_t + \left(\frac{\mu_{\bar{a}t}^*}{\mu_{\bar{z}t}^*} + x_{tt}\frac{\mu_{\bar{a}t}^*}{\mu_{\bar{z}t}^*}\right)$$

$$= \rho + \frac{\dot{\mu}_{\bar{a}t}^*}{\mu_{\bar{a}t}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \eta\frac{\dot{\chi}_t}{\chi_t} + \psi g_t$$

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However, because $\mu_{\bar{a}t}^* = \frac{1}{\phi}\frac{\theta}{1-\theta}$, this can be written as

(256)
$$\frac{\theta}{1-\theta}\frac{1}{\phi}\left[\left(\frac{1}{\mu_{\bar{z}t}^*} + x_{tt}\frac{1}{\mu_{zt}^*}\right) - \xi\left(-\ln\left(1-x_{tt}\right) - x_{tt}\right)\right]$$
$$= \rho + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \eta\frac{\dot{\chi}_t}{\chi_t} + \psi g_t$$

Finally

(257)
$$-\mu_{rt}\left(1-\theta\right)\left(\frac{1}{1-\alpha}\frac{1-\theta}{\theta} - 1\right)\left(z_t^{\frac{\theta}{1-\theta}}\right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-2}a_t^{\frac{\theta}{1-\theta}}\left[\xi\left(-\ln\left(1-x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$
$$-\mu_{yt}\frac{1-\theta}{\theta}\frac{y_t}{z_t^{\frac{\theta}{1-\theta}}} - \lambda\mu_{zt} = \rho\mu_{zt} - \dot{\mu}_{zt}$$

Can be written as

(258)
$$\frac{1-\theta}{\theta}\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{(1-\theta)\mu_{rt}z_t^{\frac{1}{1-\alpha}}\left(\frac{\bar{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{a_t^{\frac{\theta}{1-\theta}}\mu_{zt}}\left(\frac{\bar{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}\left[\xi\left(-\ln\left(1-x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$
$$+ \frac{1}{\theta}\frac{(1-\theta)\mu_{rt}z_t^{\frac{1}{1-\alpha}}\left(\frac{\bar{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}}{a_t^{\frac{\theta}{1-\theta}}\mu_{zt}}y_t^* - \lambda = \rho - \frac{\dot{\mu}_{zt}}{\mu_{zt}}$$

Further transformation of this equation yields

(259)
$$\frac{1-\theta}{\theta}\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\mu_{zt}^*\overline{\chi}_t\chi_t\left[\xi\left(-\ln\left(1-x_{tt}\right) - x_{tt}\right) + \phi g_t\right]$$
$$+ \frac{1}{\theta}\mu_{zt}^*y_t^* - \lambda = \rho + \frac{\dot{\mu}_{zt}^*}{\mu_{zt}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \eta\frac{\dot{\chi}_t}{\chi_t} + \psi g_t$$

which we can simplify to

(260)
$$\frac{1}{\theta}\left[(1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\overline{\chi}_t\chi_t\left[\xi\left(-\ln\left(1-x_{tt}\right) - x_{tt}\right) + \phi g_t\right] + y_t^*\right]\mu_{zt}^*$$
$$= \rho + \lambda + \frac{\dot{\mu}_{zt}^*}{\mu_{zt}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \eta\frac{\dot{\chi}_t}{\chi_t} + \psi g_t$$

The planner's steady state:

The planner's equilibrium variables are

(261)
$$\{y_t^*, c_t^*, i_t^*, x_{tt}, g_t, k_t^*, \mu_{\bar{z}t}^*, \mu_{zt}^*, \chi_t, \overline{\chi}_t\}$$

which in steady state have to satisfy

(262) $\quad \widetilde{y}^* = \left(\widetilde{k}^*\right)^\alpha$

(263) $\quad \widetilde{y}^* = \widetilde{c}^* + \widetilde{i}^* + (1-\theta)\widetilde{\chi}\widetilde{\overline{\chi}}\left[\xi\left[-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right] + \phi\widetilde{g}\right]$

(264) $\quad 0 = \dfrac{\widetilde{i}^*}{\widetilde{k}^*} - \delta - \dfrac{1}{1-\alpha}\widetilde{g}$

(265) $\quad 0 = \dfrac{\theta}{1-\theta}\widetilde{g} - \widetilde{\overline{\chi}}$

(266) $\quad 0 = \lambda\left(1-\widetilde{\chi}\right) + (1-\widetilde{\chi}\widetilde{x})\widetilde{\overline{\chi}}$

as well as

$$(267) \qquad \frac{1}{\xi \widetilde{\mu}_z^*} = \frac{\widetilde{x}}{1 - \widetilde{x}}$$

$$(268) \qquad 0 = \sigma \left(\alpha \frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta - \rho \right) - \frac{1}{1 - \alpha} \widetilde{g}$$

$$(269) \qquad \lambda \frac{\widetilde{\mu}_{\widetilde{z}}^*}{\widetilde{\mu}_z^*} = \rho + \psi \widetilde{g}$$

$$(270) \qquad \rho + \psi \widetilde{g} = \frac{\theta}{1 - \theta} \frac{1}{\phi} \left[\left(\frac{1}{\widetilde{\mu}_{\widetilde{z}}^*} + \widetilde{x} \frac{1}{\widetilde{\mu}_z^*} \right) - \xi \left(-\ln \left(1 - \widetilde{x} \right) - \widetilde{x} \right) \right]$$

$$(271) \qquad \rho + \lambda + \psi \widetilde{g} = \frac{1}{\theta} \left[(1 - \theta) \left(\frac{\theta}{1 - \theta} - \frac{1}{1 - \alpha} \right) \widetilde{\chi} \widetilde{\chi} \left[\xi \left(-\ln \left(1 - \widetilde{x} \right) - \widetilde{x} \right) + \phi \widetilde{g} \right] + \widetilde{y}^* \right] \widetilde{\mu}_z^*$$

The solution of this steady state is the following. From the Euler equation we obtain that it must be the case that the steady state level of capital satisfies

$$(272) \qquad \alpha \left(\widetilde{k}^* \right)^{\alpha - 1} = \rho + \delta + \frac{\widetilde{g}}{(1 - \alpha) \sigma}$$

which yields that the steady state level of capital is

$$(273) \qquad \widetilde{k}^* = \left[\frac{\alpha}{\rho + \delta + \frac{\widetilde{g}}{(1 - \alpha)\sigma}} \right]^{\frac{1}{1 - \alpha}}$$

From the production function, we now know that the steady state level of output must equal

$$(274) \qquad \widetilde{y}^* = \left[\frac{\alpha}{\rho + \delta + \frac{\widetilde{g}}{(1 - \alpha)\sigma}} \right]^{\frac{\alpha}{1 - \alpha}} = \left(\frac{1}{\theta} \right)^{\frac{\alpha}{1 - \alpha}} \left[\frac{\alpha \theta}{\rho + \delta + \frac{\widetilde{g}}{(1 - \alpha)\sigma}} \right]^{\frac{\alpha}{1 - \alpha}}$$

Because $\theta < 1$ one can see from this that, even if the growth rates in the decentralized and planner's allocation would be the same, the markup distortion would result in less output in the decentralized equilibrium.

As a function of the growth rate and implementation level the potential productivity gap and the implementation gap in steady state again equal

$$(275) \qquad \widetilde{\widetilde{\chi}} = \frac{\theta}{1 - \theta} \widetilde{g}$$

and

$$(276) \qquad \widetilde{\chi} = \frac{\lambda + \widetilde{\widetilde{\chi}}}{\lambda + \widetilde{x} \widetilde{\widetilde{\chi}}}$$

Also

$$(277) \qquad \widetilde{i}^* = \left(\delta + \frac{1}{1 - \alpha} \widetilde{g} \right) \widetilde{k}^*$$

and

$$(278) \qquad \widetilde{c}^* = \widetilde{y}^* - \widetilde{i}^* - (1 - \theta) \widetilde{\chi} \widetilde{\chi} \left[\xi \left(-\ln \left(1 - \widetilde{x} \right) - \widetilde{x} \right) + \phi \widetilde{g} \right]$$

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What is different is the optimality conditions that determine the growth rate and the implementation decisions. Remember that in the decentralized equilibrium, it was the case that in steady state

$$(279) \qquad \frac{\widetilde{y}^*}{\xi\left(\rho + \lambda + \psi\widetilde{g}\right)} = \frac{\widetilde{x}}{1 - \widetilde{x}}$$

In the planner's steady state

$$(280) \qquad \frac{1}{\xi\widetilde{\mu}_z^*} = \frac{\widetilde{x}}{1 - \widetilde{x}}$$

and

$$(281) \qquad \frac{1}{\widetilde{\mu}_z^*} = \frac{\frac{1}{\theta}\left[(1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\widetilde{\chi}\widetilde{\chi}\left[\xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right) + \phi\widetilde{g}\right] + \widetilde{y}^*\right]}{\left(\rho + \lambda + \psi\widetilde{g}\right)}$$

$$(282) \qquad = \frac{\widetilde{y}^*}{\left(\rho + \lambda + \psi\widetilde{g}\right)}\left(\frac{1}{\theta}\right)\frac{\left[(1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\widetilde{\chi}\widetilde{\chi}\left[\xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right) + \phi\widetilde{g}\right] + \widetilde{y}^*\right]}{\widetilde{y}^*}$$

Moreover, we can write

$$
\begin{aligned}
(283) \qquad \rho + \psi\widetilde{g} &= \frac{\theta}{1-\theta}\frac{1}{\phi}\left[\left(\frac{1}{\widetilde{\mu}_z^*} + \widetilde{x}\frac{1}{\widetilde{\mu}_z^*}\right) - \xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right)\right] \\
&= \frac{\theta}{1-\theta}\frac{1}{\phi}\left[\frac{1}{\widetilde{\mu}_z^*}\left(\frac{\widetilde{\mu}_z^*}{\widetilde{\mu}_{\overline{z}}^*} + \widetilde{x}\right) - \xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right)\right] \\
&= \frac{\theta}{1-\theta}\frac{1}{\phi}\left[\frac{1}{\widetilde{\mu}_z^*}\left(\frac{\lambda}{\rho + \psi\widetilde{g}} + \widetilde{x}\right) - \xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right)\right] \\
&= \frac{\theta}{1-\theta}\frac{1}{\phi}\left[\xi\frac{\widetilde{x}}{1-\widetilde{x}}\left(\frac{\lambda}{\rho + \psi\widetilde{g}} + \widetilde{x}\right) - \xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right)\right] \\
&= \frac{\theta}{1-\theta}\frac{\xi}{\phi}\left[\left(\frac{\lambda + \rho + \psi\widetilde{g}}{\rho + \psi\widetilde{g}}\right)\frac{\widetilde{x}}{1-\widetilde{x}} + \ln\left(1-\widetilde{x}\right)\right]
\end{aligned}
$$

Thus, the planner's steady state growth rate and implementation levels satisfy the following two conditions

$$(284) \qquad \frac{\frac{1}{\theta}\left[(1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\widetilde{\chi}\widetilde{\chi}\left[\xi\left(-\ln\left(1-\widetilde{x}\right) - \widetilde{x}\right) + \phi\widetilde{g}\right] + \left[\frac{\alpha}{\rho+\delta+\frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}}\right]}{\xi\left(\rho + \lambda + \psi\widetilde{g}\right)} = \frac{\widetilde{x}}{1 - \widetilde{x}}$$

and

$$(285) \qquad \rho + \psi\widetilde{g} = \frac{\theta}{1-\theta}\frac{\xi}{\phi}\left[\left(\frac{\lambda + \rho + \psi\widetilde{g}}{\rho + \psi\widetilde{g}}\right)\frac{\widetilde{x}}{1-\widetilde{x}} + \ln\left(1-\widetilde{x}\right)\right]$$

The decentralized equilibrium with subsidies on R&D, implementation, and capital inputs:
The three optimality conditions in the decentralized equilibrium that are distorted by the three subsidies are the ones associated with capital demand of the firms, the innovation decision, and the implementation choice.

Given the subsidy on capital inputs, the firm's flow profits equal

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$$(286) \qquad \pi_{it} = p_{it}y_{it} - w_t l_{it} - (1 - s_k)(r_t + \delta)k_{it}$$

and the firm chooses its capital inputs to equate the after tax rental rate of capital to the marginal revenue of capital. The optimal capital demand condition under this subsidy is

$$(287) \qquad (1 - s_k)(r_t + \delta) = \alpha \theta \frac{y_{it}}{k_{it}}$$

The price is still set at a constant markup over marginal costs

$$(288) \qquad p_{it} = \frac{1}{\theta} mc_{it}$$

However, marginal costs are now measured including the capital input subsidy, s_k. That is,

$$(289) \qquad mc_{it} = \frac{1}{a_{it}} \left[\frac{w_t}{1 - \alpha} \right]^{1-\alpha} \left[\frac{(1 - s_k)(r_t + \delta)}{\alpha} \right]^{\alpha}$$

This again allows us to write the aggregate production function (11), where the aggregate capital input now satisfies

$$(290) \qquad (1 - s_k)(r_t + \delta) = \alpha \theta \frac{y_{it}}{k_{it}}$$

what this subsidy does is to lower the rental rate of capital in equilibrium to correct for the underusage of capital in the decentralized equilibrium that is the result of the monopolistic competition between intermediate goods suppliers.

The implementation subsidy reduces the after tax cost of implementation from ξ to $(1 - s_x)\xi$. As a result, the optimal implementation level is now determined by

$$(291) \qquad y_t b_{xt} = z_t^{\frac{1}{1-\alpha}} (1 - s_x) \xi \frac{x_{tt}}{1 - x_{tt}}$$

The resulting value of the firm is given by

$$
\begin{aligned}
(292) \qquad V_t^* &= (1 - \theta) \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} \left[(b_{0t} + b_{xt}x_t) y_t - (1 - s_x) \xi z_t^{\frac{1}{1-\alpha}} [-\ln(1 - x_{tt}) - x_{tt}] \right] \\
&= (1 - \theta) \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} \left[\left(\frac{b_{0t} + b_{xt}}{b_{xt}} \right) b_{xt}y_t + (1 - s_x) \xi z_t^{\frac{1}{1-\alpha}} \ln(1 - x_{tt}) \right] \\
&= (1 - \theta) \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} (1 - s_x) \xi z_t^{\frac{1}{1-\alpha}} \left[\left(\frac{b_{0t} + b_{xt}}{b_{xt}} \right) \frac{x_{tt}}{1 - x_{tt}} + \ln(1 - x_{tt}) \right]
\end{aligned}
$$

Similarly, the R&D subsidy reduces the after tax cost of doing R&D from ϕ to $(1 - s_g)\phi$. The resulting optimal innovation decision is

$$(293) \qquad (1 - s_g)(1 - \theta)\phi \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} z_t^{\frac{1}{1-\alpha}} g_t = V_t^*$$

which yields that

$$(294) \qquad g_t = \frac{(1 - s_x)}{(1 - s_g)} \frac{\xi}{\phi} \left[\left(\frac{b_{0t} + b_{xt}}{b_{xt}} \right) \frac{x_{tt}}{1 - x_{tt}} + \ln(1 - x_{tt}) \right]$$

In terms of the transformed variables that are constant along the balanced growth path, we can now rewrite the dynamics as

$$(295) \qquad y_t^* = c_t^* + i_t^* + (1 - \theta) \overline{\chi}_t \chi_t \left[\xi \left(-\ln(1 - x_{tt}) - x_{tt} \right) + \phi g_t \right]$$

where the lump-sum taxation that balances the government's budget makes this resource constraint hold.

$$(296) \qquad y_t^* = k_t^{*\alpha}$$

$$(297) \qquad y_t^* b_{xt}^* = (1 - s_x) \xi \frac{x_{tt}}{1 - x_{tt}}$$

$$(298) \qquad g_t = \frac{(1 - s_x)}{(1 - s_g)} \frac{\xi}{\phi} \left[\left(\frac{b_{0t}^*}{b_{xt}^*} \right) \frac{x_{tt}}{1 - x_{tt}} + \ln(1 - x_{tt}) \right]$$

and for the dynamic jump variables

$$(299) \qquad \frac{\dot{c}_t^*}{c_t^*} = \sigma \left(\frac{\alpha \theta}{(1 - s_k)} \frac{y_t^*}{k_t^*} - \delta - \rho \right) - \frac{1}{1 - \alpha} g_t$$

$$(300) \qquad \dot{b}_{0t}^* = \left\{ \frac{\alpha \theta}{(1 - s_k)} \frac{y_t^*}{k_t^*} - \delta - \lambda(1 - \chi_t) + \chi_t \overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1 - \alpha} g_t \right\} b_{0t}^* - 1$$

$$(301) \qquad \dot{b}_{xt}^* = \left\{ \frac{\alpha \theta}{(1 - s_k)} \frac{y_t^*}{k_t^*} - \delta + \lambda \chi_t + \chi_t \overline{\chi}_t x_{tt} - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1 - \alpha} g_t \right\} b_{xt}^* - 1$$

while the state variables evolve according to

$$(302) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1 - \alpha} g_t$$

$$(303) \qquad \frac{\dot{\chi}_t}{\chi_t} = \lambda(1 - \chi_t) + (1 - \chi_t x_{tt}) \overline{\chi}_t$$

$$(304) \qquad \frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} = \frac{\theta}{1 - \theta} g_t - \overline{\chi}_t$$

The resulting steady state conditions of the decentralized equilibrium with the distortionary subsidies are

$$(305) \qquad \widetilde{y}^* = \widetilde{c}^* + \widetilde{i}^* + (1 - \theta) \widetilde{\overline{\chi}} \widetilde{\chi} \left[\xi \left(-\ln(1 - \widetilde{x}) - \widetilde{x} \right) + \phi \widetilde{g} \right]$$

$$(306) \qquad \widetilde{y}^* = \widetilde{k}^{*\alpha}$$

$$(307) \qquad \widetilde{b}_x^* \widetilde{y}^* = (1 - s_x) \xi \frac{\widetilde{x}}{1 - \widetilde{x}}$$

$$(308) \qquad \widetilde{g} = \frac{(1 - s_x)}{(1 - s_g)} \frac{\xi}{\phi} \left[\left(\frac{\widetilde{b}_0^*}{\widetilde{b}_x^*} \right) \frac{\widetilde{x}}{1 - \widetilde{x}} + \ln(1 - \widetilde{x}) \right]$$

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as well as

$$(309) \quad 0 = \sigma\left(\frac{\alpha\theta}{(1-s_k)}\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta - \rho\right) - \frac{1}{1-\alpha}\widetilde{g}$$

$$(310) \quad 0 = \left\{\frac{\alpha\theta}{(1-s_k)}\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta - \lambda\left(1-\widetilde{\chi}\right) + \widetilde{\chi}\widetilde{\chi}\widetilde{x}^* - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_0^* - 1$$

$$(311) \quad 0 = \left\{\frac{\alpha\theta}{(1-s_k)}\frac{\widetilde{y}^*}{\widetilde{k}^*} - \delta + \lambda\widetilde{\chi} + \widetilde{\chi}\widetilde{\chi}\widetilde{x}^* - \frac{1}{1-\alpha}\widetilde{g}\right\}\widetilde{b}_x^* - 1$$

$$(312) \quad 0 = \frac{\widetilde{i}^*}{\widetilde{k}^*} - \delta - \frac{1}{1-\alpha}\widetilde{g}$$

$$(313) \quad 0 = \lambda\left(1-\widetilde{\chi}\right) + \left(1-\widetilde{\chi}\widetilde{x}^*\right)\widetilde{\widetilde{\chi}}$$

$$(314) \quad 0 = \frac{\theta}{1-\theta}\widetilde{g} - \widetilde{\widetilde{\chi}}$$

This yields that

$$(315) \quad \widetilde{b}_0^* = \frac{1}{\rho + \psi\widetilde{g}}, \text{ and } \widetilde{b}_x^* = \frac{1}{\rho + \lambda + \psi\widetilde{g}}$$

while

$$(316) \quad \widetilde{\chi} = \frac{\lambda + \widetilde{\widetilde{\chi}}}{\lambda + \widetilde{x}^*\widetilde{\widetilde{\chi}}} \text{ and } \widetilde{\widetilde{\chi}} = \frac{\theta}{1-\theta}\widetilde{g}$$

Furthermore,

$$(317) \quad \widetilde{k}^* = \left[\frac{1}{1-s_k}\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]^{\frac{1}{1-\alpha}}, \text{ and } \widetilde{y}^* = \left[\frac{1}{1-s_k}\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}}$$

and then there are the optimal implementation and innovation conditions

$$(318) \quad \frac{1}{(1-s_x)\xi\left(\rho + \lambda + \psi\widetilde{g}\right)}\widetilde{y}^* = \frac{\widetilde{x}}{1-\widetilde{x}}$$

and

$$(319) \quad \widetilde{g} = \frac{(1-s_x)}{(1-s_g)}\frac{\xi}{\phi}\left[\left(\frac{\rho + \lambda + \psi\widetilde{g}}{\rho + \psi\widetilde{g}}\right)\frac{\widetilde{x}}{1-\widetilde{x}} + \ln\left(1-\widetilde{x}\right)\right]$$

In order to consider which combination of subsidies supports the planner's solution in the decentralized equilibrium, realize that the planner's solution satisfies

$$(320) \quad \widetilde{y}^{*sp} = \left[\frac{\alpha}{\rho + \delta + \frac{\widetilde{g}^{sp}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}}$$

Hence, conditional on the other subsidies supporting the planner's growth rate, the optimal capital input subsidy is such that

$$(321) \quad \left[\frac{1}{1-s_k}\frac{\alpha\theta}{\rho + \delta + \frac{\widetilde{g}^{sp}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}} = \left[\frac{\alpha}{\rho + \delta + \frac{\widetilde{g}^{sp}}{(1-\alpha)\sigma}}\right]^{\frac{\alpha}{1-\alpha}}$$

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Hence, the capital input subsidy is such that $s_k = 1 - \theta$.

Next, note that the planner's optimal implementation condition is

$$(322) \quad \frac{\widetilde{x}^{sp}}{1 - \widetilde{x}^{sp}} = \frac{1}{\theta}\left[1 + (1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{\widetilde{\chi}\widetilde{\chi}}{\widetilde{y}^{*sp}}\left[\xi\left(-\ln\left(1-\widetilde{x}^{sp}\right) - \widetilde{x}^{sp}\right) + \phi\widetilde{g}^{sp}\right]\right]\frac{\widetilde{y}^{*sp}}{\xi\left(\rho + \lambda + \psi\widetilde{g}^{sp}\right)}$$

Thus, to support the optimal implementation level, conditional on supporting $\widetilde{y}^{*sp}$ and $\widetilde{g}^{sp}$, the implementation subsidy has to satisfy

$$(323) \quad \frac{1}{(1-s_x)} = \frac{1}{\theta}\left[1 + (1-\theta)\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{\widetilde{\chi}\widetilde{\chi}}{\widetilde{y}^{*sp}}\left[\xi\left(-\ln\left(1-\widetilde{x}^{sp}\right) - \widetilde{x}^{sp}\right) + \phi\widetilde{g}^{sp}\right]\right] > 1$$

Finally, we consider the optimal R&D subsidy in this economy. The planner's optimal innovation condition is

$$(324) \quad \rho + \psi\widetilde{g}^{sp} = \frac{\theta}{1-\theta}\frac{\xi}{\phi}\left[\left(\frac{\lambda + \rho + \psi\widetilde{g}^{sp}}{\rho + \psi\widetilde{g}^{sp}}\right)\frac{\widetilde{x}^{sp}}{1-\widetilde{x}^{sp}} + \ln\left(1-\widetilde{x}^{sp}\right)\right]$$

Hence, to support the planner's growth rate and implementation level in the decentralized equilibrium, the R&D subsidy has to satisfy

$$(325) \quad \begin{aligned} \widetilde{g}^{sp} &= \frac{(1-s_x)}{(1-s_g)}\frac{\xi}{\phi}\left[\left(\frac{\rho + \lambda + \psi\widetilde{g}^{sp}}{\rho + \psi\widetilde{g}^{sp}}\right)\frac{\widetilde{x}^{sp}}{1-\widetilde{x}^{sp}} + \ln\left(1-\widetilde{x}^{sp}\right)\right] \\ &= \frac{(1-s_x)}{(1-s_g)}\frac{1-\theta}{\theta}\left(\rho + \psi\widetilde{g}^{sp}\right) \end{aligned}$$

such that

$$(326) \quad (1-s_g) = (1-s_x)\left(1-s_g'\right), \text{ where } \left(1-s_g'\right) = \frac{1-\theta}{\theta}\left(\frac{\rho}{\widetilde{g}^{sp}} + \psi\right)$$

and thus

$$(327) \quad s_g' = \frac{1-\theta}{\theta}\left(\frac{1}{1-\alpha}\left(\frac{\sigma-1}{\sigma}\right) - \frac{\rho}{\widetilde{g}^{sp}}\right)$$

A version of the model without implementation:

Decentralized equilibrium with subsidies:

If there is no implementation cost, the value of the firm equals

$$(328) \quad \begin{aligned} V_{it} &= (1-\theta)\overline{a}_i^{\frac{\theta}{1-\theta}}\int_t^\infty e^{-\int_t^s r_j\,dj}\left(\frac{1}{z_s}\right)^{\frac{\theta}{1-\theta}}y_s\,ds \\ &= (1-\theta)b_{0t}^* z_t^{\frac{1}{1-\alpha}}\left(\frac{\overline{a}_i}{z_t}\right)^{\frac{\theta}{1-\theta}}\frac{y_t}{z_t^{\frac{1}{1-\alpha}}} \end{aligned}$$

where

$$(329) \quad b_{0t}^* = \int_t^\infty e^{-\int_t^s r_j\,dj}\left(\frac{z_s}{z_t}\right)^{\frac{\theta}{1-\theta}}\frac{y_t}{y_s}\,ds$$

The firm's optimal capital demand condition is affected by the capital subsidy and equals

(330)
$$(1 - s_k)(r_t + \delta) = \alpha\theta\frac{y_t}{k_t}$$

The resulting R&D free entry condition equals

(331)
$$V_{tt} = (1 - \theta)(1 - s_g)\phi z_t^{\frac{1}{1-\alpha}}\left(\frac{\bar{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}g_t$$

which yields

(332)
$$b_{0t}^*\frac{y_t}{z_t^{\frac{1}{1-\alpha}}} = \phi g_t, \text{ such that } g_t = \frac{1}{(1-s_g)\phi}b_{0t}^*\frac{y_t}{z_t^{\frac{1}{1-\alpha}}}$$

Since there is full implementation in this case, there is implementation gap and thus

(333)
$$\chi_t = 1$$

The resulting system of equilibrium equations is

(334)
$$y_t = c_t + i_t + (1 - \theta)z_t^{\frac{1}{1-\alpha}}\left(\frac{\bar{a}_t}{z_t}\right)^{\frac{\theta}{1-\theta}}\phi g_t$$

(335)
$$\frac{\dot{c}_t}{c_t} = \sigma(r_t - \rho)$$

(336)
$$y_t = z_t k_t^\alpha$$

(337)
$$z_t = \bar{z}_t$$

(338)
$$g_t = \frac{1}{\phi}b_{0t}^*\frac{y_t}{z_t^{\frac{1}{1-\alpha}}}$$

(339)
$$\dot{\bar{a}}_t = g_t\bar{a}_t$$

(340)
$$\dot{k}_t = i - \delta k_t$$

(341)
$$z_t^{\frac{\theta}{1-\theta}} = \bar{a}_t^{\frac{\theta}{1-\theta}}$$

(342)
$$r_t + \delta = \alpha\theta\frac{y_t}{k_t}$$

which, in terms of the transformed variables, yields

$$(343) \qquad y_t^* = c_t^* + i_t^* + (1-\theta)\overline{\chi}_t\phi g_t$$

$$(344) \qquad \frac{\dot{c}_t}{c_t} = \sigma\left(\frac{\alpha\theta}{1-s_k}\frac{y_t^*}{k_t^*} - \delta - \rho\right) - \frac{1}{1-\alpha}g_t$$

$$(345) \qquad y_t^* = (k_t^*)^\alpha$$

$$(346) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1-\alpha}g_t$$

$$(347) \qquad g_t = \frac{1}{(1-s_g)\phi}b_{0t}^*y_t^*$$

$$(348) \qquad \frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} = \frac{\theta}{1-\theta}g_t - \overline{\chi}_t$$

$$(349) \qquad \dot{b}_{0t}^* = \left\{\frac{\alpha\theta}{1-s_k}\frac{y_t^*}{k_t^*} - \delta + \overline{\chi}_t - \frac{\dot{y}_t^*}{y_t^*} - \frac{1}{1-\alpha}g_t\right\}b_{0t}^* - 1$$

The resulting steady state is of the form

$$(350) \qquad \frac{\alpha\theta}{1-s_k}\frac{\widetilde{y}^*}{\widetilde{k}^*} = \frac{1}{(1-\alpha)\sigma}\widetilde{g} + \delta + \rho$$

The steady state potential productivity gap again equals

$$(351) \qquad \widetilde{\overline{\chi}} = \frac{\theta}{1-\theta}\widetilde{g}$$

The steady state present discounted value coefficient is

$$(352) \qquad \widetilde{b}_0^* = \frac{1}{\rho + \psi\widetilde{g}}$$

The capital stock in steady state satisfies

$$(353) \qquad \widetilde{k}^* = \left[\frac{1}{1-s_k}\frac{\alpha\theta}{\rho + \delta + \frac{1}{(1-\alpha)\sigma}\widetilde{g}}\right]^{\frac{1}{1-\alpha}}$$

such that the steady state level of output equals

$$(354) \qquad \widetilde{y}^* = \left[\frac{1}{1-s_k}\frac{\alpha\theta}{\rho + \delta + \frac{1}{(1-\alpha)\sigma}\widetilde{g}}\right]^{\frac{\alpha}{1-\alpha}}$$

The resulting steady state free entry condition into R&D yields

$$(355) \qquad \widetilde{g} = \frac{1}{(1-s_g)\phi}\frac{\widetilde{y}^*}{\rho + \psi\widetilde{g}}$$

In order to see which subsidy scheme actually supports the planner's allocation, we have to solve for it.

Planner's problem:

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The current value Hamiltonian associated with the planner's problem without implementation is

$$
(356) \qquad H_t \;=\; \frac{\sigma}{\sigma-1} c_t^{\frac{\sigma-1}{\sigma}} + \mu_{rt} \left\{ y_t - c_t - i_t - (1-\theta) \left[\overline{z}_t^{\frac{\theta}{1-\theta}} \right]^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}} \left(\frac{\overline{a}_t}{\overline{z}_t} \right)^{\frac{\theta}{1-\theta}} \phi g_t \right\}
$$
$$
+ \mu_{yt} \left\{ y_t - \left(\overline{z}_t^{\frac{\theta}{1-\theta}} \right)^{\frac{1-\theta}{\theta}} k_t^\alpha \right\} + \mu_{kt} \left\{ i_t - \delta k_t \right\} + \mu_{\overline{a}t} \left\{ \frac{\theta}{1-\theta} \overline{a}_t^{\frac{\theta}{1-\theta}} g_t \right\} + \mu_{\overline{z}t} \overline{a}_t^{\frac{\theta}{1-\theta}}
$$

which yields the first order necessary conditions

$$
(357) \qquad \text{w.r.t. } y_t \quad : \qquad \mu_{rt} = -\mu_{yt}
$$

$$
(358) \qquad \text{w.r.t. } c_t \quad : \qquad \mu_{rt} = c_t^{-\frac{1}{\sigma}}
$$

$$
(359) \qquad \text{w.r.t. } i_t \quad : \qquad \mu_{rt} = \mu_{kt}
$$

$$
(360) \qquad \text{w.r.t. } g_t \quad : \qquad \mu_{rt}(1-\theta)\phi z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}} = \frac{\theta}{1-\theta} \mu_{\overline{a}t} \overline{a}_t^{\frac{\theta}{1-\theta}}
$$

$$
(361) \qquad \text{w.r.t. } k_t \quad : \qquad -\alpha\mu_{yt} \frac{y_t}{k_t} - \delta\mu_{kt} = \rho\mu_{kt} - \dot{\mu}_{kt}
$$

$$
(362) \qquad \text{w.r.t. } \overline{a}_t^{\frac{\theta}{1-\theta}} \quad : \qquad -\mu_{rt}(1-\theta) \left(z_t^{\frac{\theta}{1-\theta}} \right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-1} \phi g_t +
$$
$$
\frac{\theta}{1-\theta} \mu_{\overline{a}t} g_t + \mu_{\overline{z}t} = \rho\mu_{\overline{a}t} - \dot{\mu}_{\overline{a}t}
$$

$$
(363) \qquad \text{w.r.t. } \overline{z}_t^{\frac{\theta}{1-\theta}} \quad : \qquad -\mu_{rt}(1-\theta) \left(\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-1 \right) \left(z_t^{\frac{\theta}{1-\theta}} \right)^{\frac{1}{1-\alpha}\frac{1-\theta}{\theta}-2} \overline{a}_t^{\frac{\theta}{1-\theta}} \phi g_t
$$
$$
-\mu_{yt} \frac{1-\theta}{\theta} \frac{y_t}{z_t^{\frac{\theta}{1-\theta}}} = \rho\mu_{\overline{z}t} - \dot{\mu}_{\overline{z}t}
$$

This system can be rewritten in the transformed variables

$$
(364) \qquad c_t^* = \frac{c_t}{z_t^{\frac{1}{1-\alpha}}},\; y_t^* = \frac{y_t}{z_t^{\frac{1}{1-\alpha}}},\; k_t^* = \frac{k_t}{z_t^{\frac{1}{1-\alpha}}},\; i_t^* = \frac{i_t}{z_t^{\frac{1}{1-\alpha}}},\; \text{and } \overline{\chi}_t = \left(\frac{\overline{a}_t}{\overline{z}_t} \right)^{\frac{\theta}{1-\theta}}
$$

as well as the transformed costate variables

$$
(365) \qquad \mu_{\overline{z}t}^* \;=\; \frac{(1-\theta)\mu_{rt} z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}}}{\mu_{\overline{z}t} \overline{a}_t^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\mu_{rt} z_t^{\frac{1}{1-\alpha}-\frac{\theta}{1-\theta}}}{\mu_{\overline{z}t}} = \frac{(1-\theta)(c_t^*)^{-\frac{1}{\sigma}} \overline{\chi}_t^\eta}{\mu_{\overline{z}t} \overline{a}_t^{-\psi}}
$$

$$
(366) \qquad \mu_{\overline{a}t}^* \;=\; \frac{(1-\theta)\mu_{rt} z_t^{\frac{1}{1-\alpha}} \left(\frac{\overline{a}_t}{z_t} \right)^{\frac{\theta}{1-\theta}}}{\mu_{\overline{a}t} \overline{a}_t^{\frac{\theta}{1-\theta}}} = \frac{(1-\theta)\mu_{rt} z_t^{\frac{1}{1-\alpha}-\frac{\theta}{1-\theta}}}{\mu_{\overline{a}t}} = \frac{(1-\theta)(c_t^*)^{-\frac{1}{\sigma}} \overline{\chi}_t^\eta}{\mu_{\overline{a}t} \overline{a}_t^{-\psi}}
$$

where it is worth noting that

$$
(367) \qquad \frac{\dot{\mu}_{\overline{z}t}^*}{\mu_{\overline{z}t}^*} \;=\; -\frac{\dot{\mu}_{\overline{z}t}}{\mu_{\overline{z}t}} - \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} + \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \psi g_t
$$

$$
(368) \qquad \frac{\dot{\mu}_{\overline{a}t}^*}{\mu_{\overline{a}t}^*} \;=\; -\frac{\dot{\mu}_{\overline{a}t}}{\mu_{\overline{a}t}} - \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} + \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} - \psi g_t
$$

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The transformed system reads

$$(369) \qquad y_t^* = c_t^* + i_t^* + (1-\theta)\overline{\chi}_t\phi g_t$$

$$(370) \qquad \frac{\dot{c}_t}{c_t} = \sigma\left(\frac{\alpha\theta}{1-s_k}\frac{y_t^*}{k_t^*} - \delta - \rho\right) - \frac{1}{1-\alpha}g_t$$

$$(371) \qquad y_t^* = (k_t^*)^\alpha$$

$$(372) \qquad \frac{\dot{k}_t^*}{k_t^*} = \frac{i_t^*}{k_t^*} - \delta - \frac{1}{1-\alpha}g_t$$

$$(373) \qquad \frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} = \frac{\theta}{1-\theta}g_t - \overline{\chi}_t$$

and the optimality condition

$$(374) \qquad \mu_{\overline{a}t}^* = \frac{1}{\phi}\frac{\theta}{1-\theta}$$

and

$$(375) \qquad \frac{1-\theta}{\theta}\left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\mu_{\overline{z}t}^*\overline{\chi}_t\phi g_t + \frac{1}{\theta}\mu_{\overline{z}t}^*y_t^* = \rho + \frac{\dot{\mu}_{\overline{z}t}^*}{\mu_{\overline{z}t}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} + \psi g_t$$

as well as

$$(376) \qquad -\mu_{\overline{a}t}^*\phi g_t + \frac{\theta}{1-\theta}g_t + \frac{\mu_{\overline{a}t}^*}{\mu_{\overline{z}t}^*} = \rho + \frac{\dot{\mu}_{\overline{a}t}^*}{\mu_{\overline{a}t}^*} + \frac{1}{\sigma}\frac{\dot{c}_t^*}{c_t^*} - \eta\frac{\dot{\overline{\chi}}_t}{\overline{\chi}_t} + \psi g_t$$

The steady state of this system

$$(377) \qquad \widetilde{\mu_{\overline{a}}^*} = \frac{1}{\phi}\frac{\theta}{1-\theta} \text{ and } \widetilde{\overline{\chi}} = \frac{\theta}{1-\theta}\widetilde{g}$$

as well as

$$(378) \qquad \widetilde{\mu_{\overline{z}}^*} = \frac{1}{\phi}\frac{\theta}{1-\theta}\frac{1}{\rho + \psi\widetilde{g}}$$

and

$$(379) \qquad \left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{\theta}{1-\theta}\frac{\widetilde{g}^2}{\rho + \psi\widetilde{g}} + \frac{1}{\phi}\frac{1}{1-\theta}\frac{\widetilde{y}^*}{\rho + \psi\widetilde{g}} = \rho + \psi\widetilde{g}$$

The steady state levels of the capital stock and output equal

$$(380) \qquad \widetilde{k}^* = \left[\frac{\alpha}{\rho + \delta + \frac{1}{(1-\alpha)\sigma}\widetilde{g}}\right]^{\frac{1}{1-\alpha}} \text{ and } \widetilde{y}^* = \left[\frac{\alpha}{\rho + \delta + \frac{1}{(1-\alpha)\sigma}\widetilde{g}}\right]^{\frac{\alpha}{1-\alpha}}$$

Subsidies to support planner's steady state:

Just like in the case with implementation, the optimal capital input subsidy is the one which offsets the distortion due to monopolistic competition. This is $s_k = 1-\theta$. The optimal R&D subsidy solves the following

combination of the planner's optimal innovation condition and the decentralized free entry condition into R&D.

$$(381) \qquad \left[1 - \left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{\theta}{1-\theta}\left(\frac{\widetilde{g}}{\rho+\psi\widetilde{g}}\right)^2\right](\rho+\psi\widetilde{g}) = \frac{1}{1-\theta}(1-s_g)\widetilde{g}$$

Solving this yields that the R&D subsidy that support's the planner's steady state allocation satisfies

$$(382) \qquad (1-s_g) = (1-\theta)\left[1 - \left(\frac{\theta}{1-\theta} - \frac{1}{1-\alpha}\right)\frac{\theta}{1-\theta}\left(\frac{\widetilde{g}}{\rho+\psi\widetilde{g}}\right)^2\right]\left(\frac{\rho}{\widetilde{g}}+\psi\right)$$

Welfare analysis:

The representative household's welfare in the steady state for a given level of consumption, $\widetilde{c}^*$, and growth rate, $\widetilde{g}$, is given by

$$(383) \qquad W = \frac{\sigma}{\sigma-1}\int_t^\infty e^{-\rho(s-t)}c_s^{\frac{\sigma-1}{\sigma}}ds, \text{ where } c_s = \widetilde{c}^*e^{\frac{1}{1-\alpha}\widetilde{g}(s-t)}$$

which yields

$$(384) \qquad W = \begin{cases} \frac{\sigma}{\sigma-1}\widetilde{c}^*\left[\frac{1}{\rho-\frac{1}{1-\alpha}\frac{\sigma-1}{\sigma}\widetilde{g}}\right] & \text{for } \sigma \neq 1 \\[2ex] \frac{1}{\rho}\left[\ln\widetilde{c}^* + \frac{1}{1-\alpha}\frac{\widetilde{g}}{\rho}\right] & \text{for } \sigma = 1 \end{cases}$$