Appendix to "Natural Expectations, Macroeconomic Dynamics, and Asset Pricing," by Andreas Fuster, Benjamin Hebert, and David Laibson, in NBER Macroeconomics Annual 2011, edited by Daron Acemoglu and Michael Woodford, (University of Chicago Press, 2012)

Appendices

A Matrix Representation of Consumption Model

In this section, we solve for the optimal consumption policy for beliefs based on an ARIMA(p,1,0) model:

$$\Delta d_t = \Phi(L)\Delta d_t + \sigma_\varepsilon \varepsilon_t$$

We can represent this system as an AR(1) system with evolution operator Φ:

$$\begin{bmatrix} \Delta d_t \\ \Delta d_{t-1} \\ ... \\ \Delta d_{t-p+1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & ... & \phi_p \\ 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} \Delta d_{t-1} \\ \Delta d_{t-2} \\ ... \\ \Delta d_{t-p} \end{bmatrix} + \begin{bmatrix} \sigma_\varepsilon \\ 0 \\ ... \\ 0 \end{bmatrix} \varepsilon_t$$

There is also the foreign debt variable, b_t, which evolves as

$$b_{t+1} = c_t + Rb_t - d_t - y.$$

Here we analyze a slightly more general version of the model, which includes constant labor income y. We also assume that R is constant. Define the AR(1) representation state vector:

$$z_t = \begin{bmatrix} b_t & c_{t-1} & 1 & y & d_t & \Delta d_t & \Delta d_{t-1} & ... & \Delta d_{t-p+1} \end{bmatrix}'$$

We use CARA utility with habits, as in Alessie and Lusardi (1997):

$$u(c_t, c_{t-1}) = -\frac{1}{\alpha}\exp(-\alpha(c_t - \gamma c_{t-1}))$$

Now, guess a linear policy function, $c_t = P'z_t$. Because the policy function is linear, we can define the AR(1) evolution operator:

$$M = \bar{M} + NP' = \begin{bmatrix} R & 0 & 0 & -1 & -1 & \vec{0} \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 1 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 1 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 1 & e'_{1,p}\Phi \\ \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \Phi \end{bmatrix} + (e_{1,p+5} + e_{2,p+5})P',$$

where $e_{i,n}$ is the $i'th$ basis vector of length n. This satisfies

$$z_t = Mz_{t-1} + C\varepsilon_t,$$

where $C = \left(e_{5,p+5} + e_{6,p+5}\right)\sigma_\varepsilon$, i.e. C is a column vector that applies the random shock to d_t and Δd_t and is otherwise empty. An alternative evolution equation is

$$z_t = \bar{M}z_{t-1} + Nc_{t-1} + C\varepsilon_t.$$

We guess that the value function is of the form

$$V(z_t) = -\frac{\Psi}{\alpha}\exp(-\alpha(c_t - \gamma c_{t-1}))$$

Define $\tilde{P} = P - \gamma e_{2,p+5}$, and plug into the Bellman equation:

$$-\frac{\Psi}{\alpha}e^{-\alpha\tilde{P}'z_t} = -\frac{1}{\alpha}e^{-\alpha\tilde{P}'z_t} + \delta E\left[-\frac{\Psi}{\alpha}e^{-\alpha\tilde{P}'z_{t+1}}\right]$$

The expectation for $\tilde{P}'z_{t+1}$ is $\tilde{P}'Mz_t$, and the variance is $C'\tilde{P}\tilde{P}'C$. So the Bellman equation simplifies to:

$$-\frac{\Psi}{\alpha}e^{-\alpha\tilde{P}'z_t} = -\frac{1}{\alpha}e^{-\alpha\tilde{P}'z_t} - \delta\frac{\Psi}{\alpha}e^{-\alpha(\tilde{P}'Mz_t - \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C)}$$

Now, without worrying about optimality, we solve the Bellman equation. Dividing through by common terms,

$$\Psi = 1 + \delta\Psi e^{-\alpha(\tilde{P}'(M-I)z_t - \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C)}$$

For this equation to be solved for all z_t, it must be that for some constant κ,

$$\tilde{P}'(M-I)z_t = \kappa$$

Next, we need to derive an optimality condition for c_t. The first-order condition is that

$$\exp(-\alpha\tilde{P}'z_t) + \delta E[\Psi\exp(-\alpha\tilde{P}'z_{t+1})\tilde{P}'N] = 0$$

Expanding the expectation, and noting that $\tilde{P}'N$ is a scalar constant,

$$\exp(-\alpha\tilde{P}'z_t) + \delta\Psi\tilde{P}'N\exp(-\alpha(\tilde{P}'Mz_t - \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C) = 0$$

Dividing through,

$$0 = 1 + \delta\Psi\tilde{P}'N\exp(-\alpha(\tilde{P}'(M-I)z_t - \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C)$$

Again, note that if $\tilde{P}(M-I)z_t = \kappa$ for all t, this equation can be satisfied for some constant Ψ. Combining the two equations, we can see that

$$0 = 1 + \tilde{P}'N(\Psi - 1)$$

Solving,

$$\Psi = 1 - \frac{1}{\tilde{P}'N}$$

At this point, we will try to guess P and show that our guess satisfies the equations above. For some constants K and Q,

$$P = \begin{bmatrix} -(R-1)(1-\frac{\gamma}{R}) \\ \frac{\gamma}{R} \\ Q \\ \frac{R-\gamma}{R} \\ \frac{R-\gamma}{R} \\ \left[Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi\right]' \end{bmatrix}$$

We can first solve for $\tilde{P}'N$.

$$\tilde{P}'N = (P - \gamma e_{2,p+5})N = -(R-1)(1-\frac{\gamma}{R}) + \frac{\gamma}{R} - \gamma = 1 - R$$

Therefore,

$$\Psi = 1 - \frac{1}{1-R} = \frac{R}{R-1}$$

Returning to the first order condition,

$$0 = 1 - R\delta e^{-\alpha(\tilde{P}'(M-I)z_t - \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C)}$$

Next, we need to confirm that $\tilde{P}'(M-I)z_t = \kappa$ for all t.

$$M - I = \bar{M} - I + NP' = \begin{bmatrix} R-1 & 0 & 0 & -1 & -1 & \vec{0} \\ 0 & -1 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & e'_{1,p}\Phi \\ \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \Phi - I \end{bmatrix} + NP'$$

$$NP' = \begin{bmatrix} -(R-1)(1-\frac{\gamma}{R}) & \frac{\gamma}{R} & Q & \frac{R-\gamma}{R} & \frac{R-\gamma}{R} & Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi \\ -(R-1)(1-\frac{\gamma}{R}) & \frac{\gamma}{R} & Q & \frac{R-\gamma}{R} & \frac{R-\gamma}{R} & Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & 0 \end{bmatrix}$$

$$M - I = \begin{bmatrix} (R-1)\frac{\gamma}{R} & \frac{\gamma}{R} & Q & \frac{-\gamma}{R} & \frac{-\gamma}{R} & Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi \\ -(R-1)(1-\frac{\gamma}{R}) & \frac{-R+\gamma}{R} & Q & \frac{R-\gamma}{R} & \frac{R-\gamma}{R} & Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & \vec{0}' \\ 0 & 0 & 0 & 0 & 0 & e'_{1,p}\Phi \\ \vec{0} & \vec{0} & \vec{0} & \vec{0} & \vec{0} & \Phi - I \end{bmatrix}$$

$$\tilde{P}'(M-I) = \begin{bmatrix} 0 \\ 0 \\ -Q(R-1) \\ 0 \\ 0 \\ \begin{bmatrix} -(R-1)Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi + \frac{R-\gamma}{R}e'_{1,p}\Phi + \\ + Ke'_{1,p}(I - \frac{1}{R}\Phi)^{-1}\frac{1}{R}\Phi(\Phi - I) \end{bmatrix}' \end{bmatrix}'$$

We need to solve for K so that the last element becomes zero. Simplifying,

$$0 = -RKe'_{1,p}\left(I - \frac{1}{R}\Phi\right)^{-1}\frac{1}{R}\Phi + \frac{R-\gamma}{R}e'_{1,p}\Phi + RKe'_{1,p}\left(I - \frac{1}{R}\Phi\right)^{-1}\frac{1}{R}\Phi\frac{1}{R}\Phi$$

Noting that

$$\left(I - \frac{1}{R}\Phi\right)^{-1} \frac{1}{R}\Phi\frac{1}{R}\Phi = \left(I - \frac{1}{R}\Phi\right)^{-1} \frac{1}{R}\Phi - \frac{1}{R}\Phi$$

We can simplify to

$$0 = \frac{R-\gamma}{R} e'_{1,p}\Phi - RKe'_{1,p}\frac{1}{R}\Phi$$

Solving,

$$K = \frac{R-\gamma}{R} = \left(1 - \frac{\gamma}{R}\right)$$

For this value of K, and any value of Q, $\tilde{P}(M-I)z_t = \kappa = -Q(R-1)$ for all t. To solve for Q, we can rewrite the FOC, replacing for our value of Ψ,

$$0 = 1 - \exp\left(\alpha\left[\frac{1}{\alpha}\ln(R\delta) + Q(R-1) + \frac{\alpha}{2}C'\tilde{P}\tilde{P}'C\right]\right)$$

$$\tilde{P}'C = \sigma_\varepsilon\left(1 - \frac{\gamma}{R}\right) e'_{1,p}\left(I - \frac{1}{R}\Phi\right)^{-1} e_{1,p} = \sigma_c$$

$$Q = \frac{1}{R-1}\left[-\frac{1}{\alpha}\ln(R\delta) - \frac{\alpha}{2}\sigma_c^2\right]$$

We have now fully solved for the linear policy function, and shown that it is optimal. In the body of the paper we impose the additional restriction, $y_t = 0$.

B Asset Pricing

Next, we derive a price for the dividend stream. Our timing convention is that the price at time t does not include the dividend at time t. To calculate the equilibrium price of the Lucas tree, we consider the asset allocation problem (as opposed to the planner's problem). We then solve for the asset price that leads the representative agent to hold one unit of the equity tree.

Start with the Bellman Equation:

$$\begin{aligned} V\left(c_{t-1}, w_t, p_t, \vec{d}_t\right) = \sup_{\theta_t, c_t} u(c_t, c_{t-1}) \quad + \quad & E_t\delta V\Big(c_t,\ (w_t - c_t - \theta_t p_t)\, R + \\ & + \theta_t\,(d_{t+1} + p_{t+1})\,,\ p_{t+1},\ \vec{d}_{t+1}\Big), \end{aligned}$$

or alternatively

$$V\left(c_{t-1}, x_t, p_t, \vec{d}_t\right) = \frac{-R}{\alpha(R-1)}\exp\left(-\alpha\left[c_t - \gamma c_{t-1}\right]\right),$$

where

$$c_t = \frac{\gamma}{R}c_{t-1} + \left(1 - \frac{\gamma}{R}\right)x_t - \psi.$$

Consumption is a weighted average of lagged consumption and the (risk-neutral) annuity value of future dividends, x_t, shifted down by an additive constant ψ. Let's write x_t so that we allow the agent to buy more or less of the risky tree. Buying more means raising θ above unity.

$$x_t = \frac{R-1}{R}\left[-Rb_t - (\theta - 1)\,p_t + d_t + \theta\sum_{s=1}^{\infty}\frac{E_t d_{t+s}}{R^s}\right]$$

$$\psi = \frac{1}{R-1}\left[\frac{1}{\alpha}\ln(R\delta) + \frac{\alpha}{2}Var_t(\Delta c_{t+1})\right].$$

In equilibrium, $\theta = 1$ (supply equals demand). The associated first-order condition is:

$$\frac{dV}{d\theta} = 0 \text{ at } \theta = 1$$

We can expand this derivative:

$$\frac{dV}{d\theta} = \frac{\partial V}{\partial c_t}\left[\frac{\partial c_t}{\partial x_t}\frac{\partial x_t}{\partial \theta} + \frac{\partial c_t}{\partial \psi}\frac{\partial \psi}{\partial \theta}\right] = 0,$$

which implies that the asset price, p_t, is chosen such that

$$\frac{\partial c_t}{\partial x_t}\frac{\partial x_t}{\partial \theta} + \frac{\partial c_t}{\partial \psi}\frac{\partial \psi}{\partial \theta} = 0 \text{ at } \theta = 1 \tag{4}$$

Let's evaluate each of these partial derivatives in turn:

$$\begin{aligned}
\frac{\partial c_t}{\partial x_t} &= \left(1 - \frac{\gamma}{R}\right) \\
\frac{\partial x_t}{\partial \theta} &= \frac{R-1}{R}\left[-p_t + \sum_{s=1}^{\infty}\frac{E_t d_{t+s}}{R^s}\right] \\
\frac{\partial c_t}{\partial \psi} &= -1 \\
\frac{\partial \psi}{\partial \theta_t} &= \frac{1}{R-1}\frac{\alpha}{2}\frac{\partial Var_t(\Delta c_{t+1})}{\partial \theta} \\
&= \frac{1}{R-1}\frac{\alpha}{2}2\theta Var_t(\Delta c_{t+1})
\end{aligned}$$

Now we are ready to use our equilibrium condition (equation (4)):

$$\begin{aligned}
\frac{\partial c_t}{\partial x}\frac{\partial x}{\partial \theta} &= -\frac{\partial c_t}{\partial \psi}\frac{\partial \psi}{\partial \theta} \quad \text{(evaluated at } \theta = 1\text{)} \\
\left(1 - \frac{\gamma}{R}\right)\frac{R-1}{R}\left[-p_t + \sum_{s=1}^{\infty}\frac{E_t d_{t+s}}{R^s}\right] &= \frac{1}{R-1}\frac{\alpha}{2}2Var_t(\Delta c_{t+1})
\end{aligned}$$

Rearrange to get

$$\begin{aligned}
-p_t + \sum_{s=1}^{\infty}\frac{E_t d_{t+s}}{R^s} &= \frac{R\alpha \times Var_t(\Delta c_{t+1})}{\left(1 - \frac{\gamma}{R}\right)(R-1)^2} \\
p_t &= \sum_{s=1}^{\infty}\frac{E_t d_{t+s}}{R^s} - \frac{R\alpha \times Var_t(\Delta c_{t+1})}{\left(1 - \frac{\gamma}{R}\right)(R-1)^2}
\end{aligned}$$

We can also express the expectation of the discounted stream of dividends using our earlier matrix notation, implying that

$$p_t = \frac{1}{R-1}d_t + \frac{R}{R-1}e'_{1,p}\left(I - \frac{1}{R}\Phi\right)^{-1}\frac{1}{R}\Phi\Delta\vec{d}_t - \frac{R\alpha}{(1-\frac{\gamma}{R})(R-1)^2}\sigma_c^2$$

B.1 Equity Premium

The mathematical expectation of the equity premium does not exist in our economy, since equity prices are not bounded below by zero. Instead of characterizing the expected equity premium, we characterize the equity premium conditional on a history in which all dividends take on their expected value. Specifically, assume that $d_t = d$ and $\Delta\vec{d}_t = 0$. The average quarterly return conditional on this history is $\frac{d}{p_t}$, where

$$p_t = \frac{d}{R-1} - \frac{R\alpha}{(1-\frac{\gamma}{R})(R-1)^2}\sigma_c^2.$$

Hence,

$$\frac{d}{p} = \frac{d}{\frac{d}{R-1} - \frac{R\alpha}{\left(1-\frac{\gamma}{R}\right)(R-1)^2}\sigma_c^2}$$

Therefore the annualized equity premium when dividends have a flat history (i.e., $\Delta\vec{d}_t = 0$) is given by

$$4 \times \left[\frac{d}{\frac{d}{R-1} - \frac{R\alpha}{\left(1-\frac{\gamma}{R}\right)(R-1)^2}\sigma_c^2} - (R-1)\right]$$

C Rational Expectations Investors

C.1 Excess Gains with Natural Expectations

We start with the definition of the price in the natural expectations framework:

$$p_t = \frac{1}{R-1}d_t + \frac{R}{R-1}e'_{1,p}\left(I - \frac{1}{R}\Phi\right)^{-1}\frac{1}{R}\Phi\Delta\vec{d}_t - \frac{R\alpha}{(1-\frac{\gamma}{R})(R-1)^2}\sigma_c^2$$

We would like to understand the gains process, defined as

$$g_{t+1} = p_{t+1} + d_{t+1} - Rp_t$$

We replace the Φ matrix and σ_c with their 'hat' equivalents, to denote misspecification. Expanding the definitions of p_t and p_{t+1}, we find that

$$\begin{aligned} g_{t+1} = d_{t+1} + \frac{1}{R-1}(d_{t+1} - Rd_t) + \\ \frac{R}{R-1}e'_{1,p}\left(I - \frac{1}{R}\hat{\Phi}\right)^{-1}\frac{1}{R}\hat{\Phi}\left(\Delta\vec{d}_{t+1} - R\Delta\vec{d}_t\right) + \frac{R\alpha}{(1-\frac{\gamma}{R})(R-1)}\hat{\sigma}_c^2 \end{aligned}$$

Define

$$\mu = \frac{R\alpha}{(1-\frac{\gamma}{R})(R-1)}\hat{\sigma}_c^2$$

Using the definition of the AR process driving d_t, we can rewrite this as

$$\begin{aligned} g_{t+1} =& \frac{R}{R-1}e'_{1,p}\left(\Phi\Delta\vec{d}_t + \sigma_e\varepsilon_{t+1}e_{1,p}\right) + \\ &+ \frac{R}{R-1}e'_{1,p}\left(I - \frac{1}{R}\hat{\Phi}\right)^{-1}\hat{\Phi}\left(\frac{1}{R}\Phi\Delta\vec{d}_t - \Delta\vec{d}_t + \frac{1}{R}\sigma_\varepsilon\varepsilon_{t+1}e_{1,p}\right) + \mu \end{aligned}$$

Regrouping terms,

$$\begin{aligned} g_{t+1} =& \frac{R}{R-1}e'_{1,p}\left(\Phi - \hat{\Phi}\left(I - \frac{1}{R}\hat{\Phi}\right)^{-1}\left(I - \frac{1}{R}\Phi\right)\right)\Delta\vec{d}_t + \\ &+ \frac{R}{R-1}\sigma_\varepsilon e'_{1,p}\left(I - \frac{1}{R}\hat{\Phi}\right)^{-1}e_{1,p}\varepsilon_{t+1} + \mu \end{aligned}$$

We can define the vector M and constant σ_g so that the equation above is

$$g_{t+1} = M\Delta\vec{d}_t + \sigma_g\varepsilon_{t+1} + \mu$$

C.2 Budget Constraint

Let w_t be the agent's wealth in period t before consumption is chosen. Assume the agent can hold either risk-free assets with return R, or a risky asset. There are no shorting/leverage constraints. The agent's budget constraint is

$$\begin{aligned} w_{t+1} &= (w_t + y - c_t - \theta_t p_t)R + \theta_t(d_{t+1} + p_{t+1}) \\ &= (w_t + y - c_t)R + \theta_t g_{t+1}, \end{aligned}$$

where the choice variables are consumption (c_t) and dollar amount in the risky asset (θ_t). We can rewrite the budget constraint in terms of the evolution of g_t,

$$w_{t+1} = R(w_t + y - c_t) + \theta_t M\Delta\vec{d}_t + \theta_t\sigma_g\varepsilon_{t+1} + \theta_t\mu.$$

It will be helpful to solve for an inter-temporal budget constraint. The transversality condition is

$$\lim_{k\to\infty} E_t[R^{-k}w_{t+k}] = 0$$

Seeing that

$$\begin{aligned} E_t[w_{t+2}] &= E_t[R(w_{t+1} - c_{t+1} + y) + \theta_{t+1}M\Delta\vec{d}_{t+1} + \theta_{t+1}\mu] \\ &= E_t[R^2(w_t - c_t + y) + R(-c_{t+1} + y)\theta_{t+1}M\Delta\vec{d}_{t+1} + \\ &\quad + \theta_{t+1}\mu + R\theta_t M\Delta\vec{d}_t + R\mu] \end{aligned}$$

We conjecture that

$$E_t[R^{-k}w_{t+k}] = w_t + \frac{1}{R}\sum_{j=0}^{k-1} R^{-j} E_t[\theta_{t+j} M\Delta\vec{d}_{t+j} + \theta_{t+j}\mu - Rc_{t+j} + Ry]$$

This holds trivially for $k = 1$ and $k = 2$. Assume it holds for $k > 1$:

$$\begin{aligned} E_t[R^{-k-1}w_{t+k+1}] &= E_t\big[R^{-k}w_{t+k} - R^{-k}c_{t+k} + R^{-k}y + \\ &\quad + R^{-k-1}\theta_{t+k}M\Delta\vec{d}_{t+k} + R^{-k-1}\theta_{t+k}\mu\big] \\ &= w_t + \frac{1}{R}\sum_{j=0}^{k} R^{-j}E_t[\theta_{t+j}M\Delta\vec{d}_{t+j} + \theta_{t+j}\mu - Rc_{t+j} + Ry] \end{aligned}$$

By induction, it holds in the limit, and therefore the inter-temporal budget constraint is

$$\sum_{j=0}^{\infty} E_t[R^{-j}c_{t+j}] = w_t + \frac{R}{R-1}y + \sum_{j=0}^{\infty} E_t[R^{-j-1}\theta_{t+j}M\Delta\vec{d}_{t+j} + R^{-j-1}\theta_{t+j}\mu]$$

We can rewrite the sum of c_{t+j} in terms of $\hat{c}_{t+j} = c_{t+j} - \gamma c_{t+j-1}$.

$$\begin{aligned} \sum_{j=0}^{\infty} E_t[R^{-j}c_{t+j}] &= \sum_{j=0}^{\infty} E_t[R^{-j}(\hat{c}_{t+j} + \gamma c_{t+j-1})] \\ &= \gamma c_{t-1} + \sum_{j=0}^{\infty} E_t[R^{-j}\hat{c}_{t+j}] + \frac{\gamma}{R}\sum_{j=1}^{\infty} E_t[R^{-j+1}c_{t+j-1}] \\ &= \gamma c_{t-1} + \sum_{j=0}^{\infty} E_t[R^{-j}\hat{c}_{t+j}] + \frac{\gamma}{R}\sum_{k=0}^{\infty} E_t[R^{-k}c_{t+k}] \end{aligned}$$

We can then solve to see that

$$\sum_{j=0}^{\infty} E_t[R^{-j}c_{t+j}] = \frac{1}{1-\frac{\gamma}{R}}(\gamma c_{t-1} + \sum_{j=0}^{\infty} E_t[R^{-j}\hat{c}_{t+j}])$$

Rewriting the inter-temporal budget constraint,

$$\begin{aligned} \sum_{j=0}^{\infty} E_t[R^{-j}\hat{c}_{t+j}] = -\gamma c_{t-1} + \left(1 - \frac{\gamma}{R}\right)\Bigg(w_t + \frac{R}{R-1}y \\ + \sum_{j=0}^{\infty} E_t[R^{-j-1}\theta_{t+j}M\Delta\vec{d}_{t+j} + R^{-j-1}\theta_{t+j}\mu]\Bigg) \end{aligned} \tag{5}$$

C.3 Utility and Value Functions

The agent has flow utility of the form

$$u(c_t, c_{t-1}) = -\frac{1}{\alpha}e^{-\alpha(c_t - \gamma c_{t-1})}$$

It is convenient to define

$$\hat{c}_t = c_t - \gamma c_{t-1}$$

The state is captured entirely by w_t, c_{t-1}, and $\Delta\vec{d}_t$. The Bellman equation is

$$V(w_t, c_{t-1}, \Delta\vec{d}_t) = \max_{\hat{c}_t, \theta_t} u(\hat{c}_t) + \delta E_t[V(w_{t+1}, \hat{c}_t + \gamma c_{t-1}, \Delta\vec{d}_{t+1})]$$

The first order conditions are

$$u'(\hat{c}_t) = \delta E_t \left[R \frac{\partial V_{t+1}}{\partial w_{t+1}} - \frac{\partial V_{t+1}}{\partial c_t} \right] \tag{6}$$

and

$$\delta E_t \left[\frac{\partial V_{t+1}}{\partial w_{t+1}} \left(M\Delta\vec{d}_t + \mu + \sigma_g \varepsilon_{t+1} \right) \right] = 0$$

The envelope condition for w_t is

$$\frac{\partial V_t}{\partial w_t} = \delta E_t \left[\frac{\partial V_{t+1}}{\partial w_{t+1}} R \right] \tag{7}$$

From the inter-temporal budget constraint (5),

$$\frac{\partial V_t}{\partial c_{t-1}} = \frac{-\gamma}{1 - \frac{\gamma}{R}} \frac{\partial V_t}{\partial w_t}$$

We use this to derive the Euler equations, which will be verified after deriving a solution. Rewriting (6), and then using (7),

$$\begin{aligned} u'(\hat{c}_t) &= \delta E_t \left[R \frac{\partial V_{t+1}}{\partial w_{t+1}} + \frac{\gamma}{1 - \frac{\gamma}{R}} \frac{\partial V_{t+1}}{\partial w_{t+1}} \right] \\ \left(1 - \frac{\gamma}{R}\right) u'(\hat{c}_t) &= \delta E_t \left[R \frac{\partial V_{t+1}}{\partial w_{t+1}} \right] = \frac{\partial V_t}{\partial w_t} \end{aligned}$$

Advancing time by one unit, and taking expectations,

$$\left(1 - \frac{\gamma}{R}\right) E_t \left[u'(\hat{c}_{t+1}) \right] = E_t \left[\frac{\partial V_{t+1}}{\partial w_{t+1}} \right]$$

The consumption Euler equation, assuming $\delta R = 1$, is therefore

$$u'(\hat{c}_t) = E_t \left[u'(\hat{c}_{t+1}) \right]$$

The asset Euler equation is

$$0 = E_t \left[\left(1 - \frac{\gamma}{R}\right) u'(\hat{c}_{t+1}) \left(M\Delta\vec{d}_t + \mu + \sigma_g \varepsilon_{t+1} \right) \right] \tag{8}$$

C.4 Guess and Check

We guess that

$$\hat{c}_t = D c_{t-1} + \left(1 - \frac{\gamma}{R}\right) \left(\frac{R-1}{R} w_t + A' \Delta\vec{d}_t + \Delta\vec{d}_t' B \Delta\vec{d}_t + q \right), \tag{9}$$

where A is a vector and B is a matrix. From this guess, it follows that

$$\hat{c}_{t+1} = Dc_t + \left(1 - \frac{\gamma}{R}\right)\left(\frac{R-1}{R}w_{t+1} + A'\Delta\vec{d}_{t+1} + \Delta\vec{d}'_{t+1}B\Delta\vec{d}_{t+1} + q\right) \tag{10}$$

From the budget constraint and the evolution of $\Delta\vec{d}_t$,

$$\begin{aligned}\hat{c}_{t+1} &= Dc_t + \\ &\left(1 - \frac{\gamma}{R}\right)\left[\begin{array}{c} \frac{R-1}{R}\left(R(w_t - c_t) + \theta_t M\Delta\vec{d}_t + \theta_t\mu + \theta_t\sigma_g\varepsilon_{t+1}\right) + A'\Phi\Delta\vec{d}_t + \\ + A'e_{1,p}\sigma_\varepsilon\varepsilon_{t+1} + \left(\Phi\Delta\vec{d}_t + e_{1,p}\sigma_\varepsilon\varepsilon_{t+1}\right)' B\left(\Phi\Delta\vec{d}_t + e_{1,p}\sigma_\varepsilon\varepsilon_{t+1}\right) + q \end{array}\right]\end{aligned}$$

Taking expectations,

$$E_t[\hat{c}_{t+1}] = Dc_t + \left(1 - \frac{\gamma}{R}\right)\left[\begin{array}{c} \frac{R-1}{R}\left(R(w_t - c_t) + \theta_t M\Delta\vec{d}_t + \theta_t\mu\right) + A'\Phi\Delta\vec{d}_t + \\ + \Delta\vec{d}'_t\Phi' B\Phi\Delta\vec{d}_t + \sigma_e^2 e'_{1,p}Be_{1,p} + q \end{array}\right]$$

We can then write

$$\begin{aligned}\hat{c}_{t+1} &= E[\hat{c}_{t+1}] + \\ &\left(1 - \frac{\gamma}{R}\right)\left[\begin{array}{c} \left(\frac{R-1}{R}\theta_t\sigma_g + \sigma_\varepsilon A'e_{1,p} + \sigma_\varepsilon e'_{1,p}B\Phi\Delta\vec{d}_t + \sigma_\varepsilon\Delta\vec{d}'_t\Phi' Be_{1,p}\right)\varepsilon_{t+1} + \\ -\sigma_\varepsilon^2 e'_{1,p}Be_{1,p} + \sigma_\varepsilon^2 e'_{1,p}Be_{1,p}\varepsilon_{t+1}^2 \end{array}\right]\end{aligned}$$

Define the following constants:

$$\begin{aligned} k_t &= \left(1 - \frac{\gamma}{R}\right)\left(\frac{R-1}{R}\theta_t\sigma_g + \sigma_\varepsilon A'e_{1,p} + \sigma_\varepsilon e'_{1,p}B\Phi\Delta\vec{d}_t + \sigma_\varepsilon\Delta\vec{d}'_t\Phi' Be_{1,p}\right) \\ \lambda &= \left(1 - \frac{\gamma}{R}\right)\sigma_\varepsilon^2 e'_{1,p}Be_{1,p}. \end{aligned}$$

We can rewrite $u'(\hat{c}_{t+1})$ as

$$u'(\hat{c}_{t+1}) = e^{-\alpha\left(E[\hat{c}_{t+1}] - \lambda + k_t\varepsilon_{t+1} + \lambda\varepsilon_{t+1}^2\right)},$$

and compute expectations to get

$$E[u'(\hat{c}_{t+1})] = e^{-\alpha(E[\hat{c}_{t+1}] - \lambda)} \times \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\alpha k_t x - \alpha\lambda x^2}e^{-\frac{x^2}{2}}dx$$

Define another constant,

$$s = \sqrt{\frac{1}{1 + 2\alpha\lambda}}$$

We can complete the square on the integral as follows:

$$-\alpha k_t x - \alpha\lambda x^2 - 0.5x^2 = -\frac{x^2 + 2\alpha k_t s^2 x}{2s^2} = -\frac{(x + \alpha k_t s^2)^2}{2s^2} + \frac{\alpha^2}{2}k_t^2 s^2$$

$$E[u'(\hat{c}_{t+1})] = e^{-\alpha\left(E[\hat{c}_{t+1}] - \lambda - \frac{1}{2}\alpha k_t^2 s^2\right)} s\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi s^2}}e^{-\frac{\left(x + \alpha k_t s^2\right)^2}{2s^2}}dx$$

We therefore conclude that

$$E[u'(\hat{c}_{t+1})] = e^{-\alpha\left(E[\hat{c}_{t+1}]-\lambda-\frac{1}{2}\alpha k_t^2 s^2-\frac{1}{2\alpha}\ln s^2\right)}$$

Next, consider the integral from the second Euler equation.

$$E[u'(\hat{c}_{t+1})\sigma_g \varepsilon_{t+1}] = e^{-\alpha\left(E[\hat{c}_{t+1}]-\lambda-\frac{1}{2}\alpha k_t^2 s^2\right)} \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sigma_g x e^{-\frac{\left(x+\alpha k_t s^2\right)^2}{2s^2}} dx$$

For the integral, use the following transformation:

$$u = \frac{x+\alpha k_t s^2}{s}$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sigma_g x e^{-\frac{\left(x+\alpha k_t s^2\right)^2}{2s^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sigma_g (s\,u - \alpha k_t s^2) e^{-\frac{u^2}{2}} s du = -\alpha \sigma_g k_t s^3$$

We combine (8) and the equation above to see that

$$e^{-\alpha\left(E_t[\hat{c}_{t+1}]-\lambda-\frac{1}{2}\alpha k_t^2 s^2-\frac{1}{2\alpha}\ln s^2\right)}(\mu + M\Delta\vec{d}_t) = e^{-\alpha\left(E_t[\hat{c}_{t+1}]-\lambda-\frac{1}{2}\alpha k_t^2 s^2\right)} \alpha \sigma_g k_t s^3$$

Simplifying,

$$\mu + M\Delta\vec{d}_t = \alpha \sigma_g k_t s^2$$

C.5 Solving for the Policy Rules

Next, we use consumption Euler equation to see that

$$\hat{c}_t = E_t[\hat{c}_{t+1}] - \lambda - \frac{1}{2}\alpha k_t^2 s^2 - \frac{1}{2\alpha}\ln s^2$$

From our conjecture about $\hat{c}_t$ (equation (10)), we can expand $\hat{c}_{t+1}$:

$$\hat{c}_t + \lambda + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \left(1-\frac{\gamma}{R}\right)\left(\frac{R-1}{R}E_t[w_{t+1}] + A' E_t[\Delta\vec{d}_{t+1}] + E_t[\Delta\vec{d}'_{t+1} B \Delta\vec{d}_{t+1}] + q\right) + Dc_t$$

or

$$\hat{c}_t + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \left(1-\frac{\gamma}{R}\right)\left(\frac{R-1}{R}E_t[w_{t+1}] + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}'_t \Phi' B \Phi \Delta\vec{d}_t + q\right) + Dc_t$$

Expanding using the budget constraint,

$$\hat{c}_t - Dc_t + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \left(1-\frac{\gamma}{R}\right)\left(\frac{R-1}{R}E_t[g_{t+1}]\theta_t + (R-1)(w_t - c_t) + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}'_t \Phi' B \Phi \Delta\vec{d}_t + q\right)$$

Simplifying, and replacing c_t with $\hat{c}_t + \gamma c_{t-1}$,

$$\begin{aligned}
&\hat{c}_t \left(1 - D + \left(1 - \frac{\gamma}{R}\right)(R-1)\right) + \\
&\qquad + \left(\left(1 - \frac{\gamma}{R}\right)(R-1) - D\right)\gamma c_{t-1} + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \\
&\qquad \left(1 - \frac{\gamma}{R}\right)\left(\frac{R-1}{R}(M\Delta\vec{d}_t + \mu)\theta_t + (R-1)w_t + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}_t'\Phi' B\Phi\Delta\vec{d}_t + q\right)
\end{aligned}$$

Let $Q = 1 - D + (1 - \frac{\gamma}{R})(R-1)$. Simplifying, and expanding the terms of $\hat{c}_t$,

$$\begin{aligned}
&((Q-1)\gamma + QD)c_{t-1} + Q\left(1 - \frac{\gamma}{R}\right)\left(\frac{R-1}{R}w_t + A'\Delta\vec{d}_t + Q\Delta\vec{d}_t' B\Delta\vec{d}_t + Qq\right) + \\
&\qquad + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \\
&\qquad \left(1 - \frac{\gamma}{R}\right)\left((R-1)w_t + \frac{R-1}{R}(M\Delta\vec{d}_t + \mu)\theta_t + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}_t'\Phi' B\Phi\Delta\vec{d}_t + q\right)
\end{aligned}$$

We guess that

$$D = \frac{\gamma}{R} - \gamma,$$

and see that this simplifies nicely:

$$\begin{aligned}
Q &= R \\
(Q-1)\gamma + QD &= 0
\end{aligned}$$

Therefore, noting that w_t cancels,

$$\begin{aligned}
&R\left(1 - \frac{\gamma}{R}\right)\left(A'\Delta\vec{d}_t + \Delta\vec{d}_t' B\Delta\vec{d}_t + q\right) + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \\
&\qquad \left(1 - \frac{\gamma}{R}\right)\left(\frac{R-1}{R}(M\Delta\vec{d}_t + \mu)\theta_t + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}_t'\Phi' B\Phi\Delta\vec{d}_t + q\right)
\end{aligned}$$

Using the asset allocation Euler result, we see that

$$\frac{1}{2}\alpha k_t^2 s^2 = \frac{1}{2\alpha\sigma_g^2 s^2}\left(\mu^2 + 2\mu M\Delta\vec{d}_t + \Delta\vec{d}_t' M' M\Delta\vec{d}_t\right)$$

Expanding using the definition of s,

$$\frac{1}{2}\alpha k_t^2 s^2 = \frac{1 + 2\alpha\lambda}{2\alpha\sigma_g^2}\left(\mu^2 + 2\mu M\Delta\vec{d}_t + \Delta\vec{d}_t' M' M\Delta\vec{d}_t\right)$$

We also need to replace θ_t using the asset allocation Euler equation:

$$\begin{aligned}
&\mu + M\Delta\vec{d}_t = \alpha\sigma_g s^2 k_t = \\
&\qquad \left(1 - \frac{\gamma}{R}\right)\alpha\sigma_g s^2\left(\frac{(R-1)}{R}\theta_t\sigma_g + \sigma_\varepsilon A' e_{1,p} + \sigma_e e_{1,p}' B\Phi\Delta\vec{d}_t + \sigma_\varepsilon\Delta\vec{d}_t'\Phi' B e_{1,p}\right)
\end{aligned}$$

Solving,

$$\theta_t = \frac{R}{R-1}\frac{1}{\sigma_g}\left(\frac{\mu + M\Delta\vec{d}_t}{(1-\frac{\gamma}{R})\alpha\sigma_g s^2} - \sigma_\varepsilon A' e_{1,p} - \sigma_\varepsilon e'_{1,p} B\Phi\Delta\vec{d}_t - \sigma_e \Delta\vec{d}_t' \Phi' B e_{1,p}\right)$$

To replace the term in the consumption equation,

$$\begin{aligned}\frac{R-1}{R}(M\Delta\vec{d}_t + \mu)\theta_t \quad = \quad & \frac{1}{(1-\frac{\gamma}{R})\,\alpha\sigma_g^2 s^2}\left(\mu^2 + 2\mu M\Delta\vec{d}_t + \Delta\vec{d}_t' M' M\Delta\vec{d}_t\right) - \\ & - \frac{\mu\sigma_\varepsilon}{\sigma_g} A' e_{1,p} - \frac{\sigma_\varepsilon}{\sigma_g}\left(A' e_{1,p} M + \mu e'_{1,p}(B+B')\Phi\right)\Delta\vec{d}_t - \\ & - \frac{\sigma_\varepsilon}{\sigma_g}\Delta\vec{d}_t' M' e'_{1,p} B\Phi\Delta\vec{d}_t - \frac{\sigma_\varepsilon}{\sigma_g}\Delta\vec{d}_t'\Phi' B e_{1,p} M\Delta\vec{d}_t\end{aligned}$$

We can now substitute all of these results:

$$\begin{aligned}R\left(1-\frac{\gamma}{R}\right)\left(A'\Delta\vec{d}_t + \Delta\vec{d}_t' B\Delta\vec{d}_t + q\right) + \frac{1}{2}\alpha k_t^2 s^2 + \frac{1}{2\alpha}\ln s^2 = \alpha k_t^2 s^2 + \\ \left(1-\frac{\gamma}{R}\right)\begin{bmatrix} -\frac{\mu\sigma_\varepsilon}{\sigma_g} A' e_{1,p} - \frac{\sigma_\varepsilon}{\sigma_g}(A' e_{1,p} M + \mu e'_{1,p}(B+B')\Phi)\Delta\vec{d}_t - \\ -\frac{\sigma_\varepsilon}{\sigma_g}\Delta\vec{d}_t' M' e'_{1,p} B\Phi\Delta\vec{d}_t - \frac{\sigma_\varepsilon}{\sigma_g}\Delta\vec{d}_t'\Phi' B e_{1,p} M\Delta\vec{d}_t + \\ + A'\Phi\Delta\vec{d}_t + \Delta\vec{d}_t'\Phi' B\Phi\Delta\vec{d}_t + q\end{bmatrix}\end{aligned}$$

The final version of the system is:

$$\begin{aligned}R\left(1-\frac{\gamma}{R}\right)(A'\Delta\vec{d}_t + \Delta\vec{d}_t' B\Delta\vec{d}_t + q) + \frac{1}{2\alpha}\ln s^2 = \\ \frac{1+2\alpha\lambda}{2\alpha\sigma_g^2}(\mu^2 + 2\mu M\Delta\vec{d}_t + \Delta\vec{d}_t' M' M\Delta\vec{d}_t) + \\ \left(1-\frac{\gamma}{R}\right)\begin{bmatrix} -\frac{\mu\sigma_\varepsilon}{\sigma_g} A' e_{1,p} - \frac{\sigma_\varepsilon}{\sigma_g}\left(A' e_{1,p} M + \mu e'_{1,p}(B+B')\Phi\right)\Delta\vec{d}_t + \\ + \Delta\vec{d}_t'(\Phi - \frac{\sigma_\varepsilon}{\sigma_g} e_{1,p} M)' B\Phi\Delta\vec{d}_t + \Delta\vec{d}_t'\Phi' B(\Phi - \frac{\sigma_\varepsilon}{\sigma_g} e_{1,p} M)\Delta\vec{d}_t - \\ - \Delta\vec{d}_t'\Phi' B\Phi\Delta\vec{d}_t + A'\Phi\Delta\vec{d}_t + q\end{bmatrix}\end{aligned} \tag{11}$$

This equation must hold for all value of $\Delta\vec{d}_t$, so we use term matching. Note that λ is actually the upper left element of B, scaled by $(1-\frac{\gamma}{R})\sigma_\varepsilon^2$. Beginning with the second order terms,

$$\begin{aligned}\Delta\vec{d}_t'\Bigg(RB - \frac{1+2\alpha\lambda}{2\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g^2} M' M - (\Phi - \frac{\sigma_\varepsilon}{\sigma_g} e_{1,p} M)' B\Phi \\ - \Phi' B(\Phi - \frac{\sigma_\varepsilon}{\sigma_g} e_{1,p} M) + \Phi' B\Phi\Bigg)\Delta\vec{d}_t = 0\end{aligned}$$

Define

$$\Lambda = \Phi - \hat{\Phi}$$

and note that

$$
\begin{aligned}
M &= \frac{R}{R-1} e_{1,p}' \left[\Phi - \hat{\Phi} \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1} \left(I - \frac{1}{R}\Phi \right) \right] \\
&= \frac{R}{R-1} e_{1,p}' \left[\Lambda + \hat{\Phi} - \hat{\Phi} \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1} \left(I - \frac{1}{R}\hat{\Phi} - \frac{1}{R}\Lambda \right) \right] \\
&= \frac{R}{R-1} e_{1,p}' \left[\Lambda + \hat{\Phi} \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1} \frac{1}{R}\Lambda \right] \\
&= \frac{R}{R-1} e_{1,p}' \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1} \Lambda
\end{aligned}
$$

Because Λ has non-zeros only in the top row, and $e_{1,p}e_{1,p}'$ has non-zeros only in the upper left element, which is one,

$$\Lambda = e_{1,p}e_{1,p}'\Lambda$$

Therefore,

$$
\begin{aligned}
e_{1,p}M &= \frac{R}{R-1} e_{1,p}e_{1,p}' \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1} e_{1,p}e_{1,p}'\Lambda \\
&= \frac{\sigma_g}{\sigma_\varepsilon}\Lambda
\end{aligned}
$$

and

$$\Phi - \frac{\sigma_\varepsilon}{\sigma_g} e_{1,p}M = \Phi - \Lambda = \hat{\Phi}$$

The equation for B can be rewritten:

$$\Delta\vec{d}_t' \left(RB - \frac{1+2\alpha\lambda}{2\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g^2} M'M - \hat{\Phi}'B\Phi - \Phi'B\hat{\Phi} + \Phi'B\Phi \right) \Delta\vec{d}_t = 0$$

Substituting for λ and regrouping terms,

$$\Delta\vec{d}_t' \left(RB - \frac{1}{2\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g^2} M'M - \frac{\sigma_\varepsilon^2}{\sigma_g^2} M'e_{1,p}'Be_{1,p}M - \hat{\Phi}'B\hat{\Phi} + \Lambda'B\Lambda \right) \Delta\vec{d}_t = 0$$

By our earlier result relating $e_{1,p}M$ and Λ, and assuming that this holds for all $\Delta\vec{d}_t$,

$$B = \frac{1}{2R\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g^2} M'M + \frac{1}{R}\hat{\Phi}'B\hat{\Phi}$$

This is a discrete time Lyapunov equation. We can apply the standard convergence results to see that, because the eigenvalues of $\frac{1}{\sqrt{R}}\hat{\Phi}$ are entirely less than 1 (no unit root), convergence is certain. Therefore,

$$B = \frac{1}{2R(1-\frac{\gamma}{R})\alpha\sigma_g^2} \sum_{k=0}^{\infty} R^{-k} \hat{\Phi}'^{k} M'M\hat{\Phi}^k$$

From this, we can easily solve for

$$\lambda = \left(1 - \frac{\gamma}{R}\right) \sigma_\varepsilon^2 e_{1,p}' B e_{1,p}$$

Moving on to the first order terms in equation (11), and using the symmetry of B,

$$\left(R\left(1 - \frac{\gamma}{R}\right) A' - \frac{1+2\alpha\lambda}{2\alpha\sigma_g^2} 2\mu M - \left(1 - \frac{\gamma}{R}\right) A'\Phi \right.$$
$$\left. + \left(1 - \frac{\gamma}{R}\right) \frac{\sigma_\varepsilon}{\sigma_g} (e_{1,p}' A M + 2\mu e_{1,p}' B \Phi) \right) \Delta\vec{d}_t = 0$$

Subsituting for λ, and regrouping,

$$\left(RA' - \frac{1}{2\left(1 - \frac{\gamma}{R}\right)\alpha\sigma_g^2} 2\mu M - \frac{2\mu\sigma_\varepsilon}{\sigma_g}\frac{\sigma_\varepsilon}{\sigma_g} e_{1,p}' B e_{1,p} M - A'\Phi \right.$$
$$\left. + \frac{\sigma_\varepsilon}{\sigma_g} A' e_{1,p} M + \frac{2\mu\sigma_e}{\sigma_g} e_{1,p}' B \Phi \right) \Delta\vec{d}_t = 0$$

Again using the relation between $e_{1,p}M$ and Λ,

$$\left(RA'(I - \frac{1}{R}\hat{\Phi}) - \frac{1}{2\left(1 - \frac{\gamma}{R}\right)\alpha\sigma_g^2} 2\mu M + \frac{2\mu\sigma_\varepsilon}{\sigma_g} e_{1,p}' B \hat{\Phi} \right) \Delta\vec{d}_t = 0$$

This is solved by inversion:

$$A' = \frac{2\mu}{R\sigma_g} \left(\frac{1}{2(1 - \frac{\gamma}{R})\alpha\sigma_g} M + \sigma_\varepsilon e_{1,p}' B \hat{\Phi} \right) \left(I - \frac{1}{R}\hat{\Phi} \right)^{-1}$$

Finally, we solve for the constants:

$$(R-1)\left(1 - \frac{\gamma}{R}\right) q - \frac{1+2\alpha\lambda}{2\alpha\sigma_g^2}\mu^2 + \left(1 - \frac{\gamma}{R}\right) \frac{\mu\sigma_\varepsilon}{\sigma_g} A' e_{1,p} + \frac{1}{2\alpha} \ln s^2 = 0$$

Consequently,

$$q = \frac{1+2\alpha\lambda}{2(R-1)\left(1 - \frac{\gamma}{R}\right)\alpha\sigma_g^2}\mu^2 - \frac{\mu\sigma_\varepsilon}{\sigma_g(R-1)} A' e_{1,p} + \frac{1}{2\alpha(R-1)\left(1 - \frac{\gamma}{R}\right)} \ln(1 + 2\alpha\lambda).$$

Thus, we have proved that our guess for c_t (equation (9)) was correct, with A, B, D and q as solved for in this section.

Next, it is worth simplifying the expression for asset allocation:

$$\theta_t = \frac{R}{R-1}\frac{1}{\sigma_g} \left(\frac{\mu + M\Delta\vec{d}_t}{\left(1 - \frac{\gamma}{R}\right)\alpha\sigma_g s^2} - \sigma_e A' e_{1,p} - 2\sigma_\varepsilon e_{1,p}' B \Phi \Delta\vec{d}_t \right)$$

Applying the standard substitution for s and then λ, we can write the optimal asset allocation as

$$\theta_t = \frac{R}{R-1}\frac{1}{\sigma_g}\left(\frac{\mu + M\Delta\vec{d}_t}{\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g} + \frac{2\mu\sigma_\varepsilon^2}{\sigma_g}e_{1,p}'Be_{1,p} - \sigma_\varepsilon A'e_{1,p} - 2\sigma_\varepsilon e_{1,p}'B\hat{\Phi}\Delta\vec{d}_t\right)$$

The average value of θ (assuming $\Delta\vec{d}_t = 0$) is

$$\bar{\theta} = \frac{R}{R-1}\frac{\mu}{\left(1-\frac{\gamma}{R}\right)\alpha\sigma_g^2} + \frac{R}{R-1}\frac{2\mu\sigma_\varepsilon^2}{\sigma_g^2}e_{1,p}'Be_{1,p} - \frac{R}{R-1}\frac{\sigma_\varepsilon}{\sigma_g}A'e_{1,p}$$

Using results from appendices A and B, we can show that,

$$\begin{aligned}
\mu &= \frac{R\alpha}{\left(1-\frac{\gamma}{R}\right)(R-1)}\hat{\sigma}_c^2 \\
\hat{\sigma}_c &= \frac{R-1}{R}\left(1-\frac{\gamma}{R}\right)\frac{\widehat{\sigma}_\varepsilon}{\sigma_\varepsilon}\sigma_g.
\end{aligned}$$

Using these results, we can rewrite the average asset allocation as

$$\bar{\theta} = \frac{\widehat{\sigma}_\varepsilon^2}{\sigma_\varepsilon^2} + \frac{R}{R-1}\frac{2\mu\sigma_\varepsilon^2}{\sigma_g^2}e_{1,p}'Be_{1,p} - \frac{R}{R-1}\frac{\sigma_\varepsilon}{\sigma_g}A'e_{1,p}.$$