Information aggregation, learning, and non-strategic behavior in voting environments

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Abstract

A presumed benefit of group decision-making is to select the best alternative by aggregating privately dispersed information. In reality, people often learn to make decisions based on previous experience. When the consequences of unchosen past alternatives (i.e., counterfactuals) are not observed, learning takes place from a biased sample. We investigate the extent to which information aggregation is precluded in such a learning environment. We apply the notion of a behavioral equilibrium (Esponda, 2008) to a benchmark voting game in order to formalize the assumption that players fail to account for selection bias. We present a dynamic framework that provides explicit learning foundations for our solution concept and clarifies the nature of the selection problem and our behavioral assumptions. We then characterize equilibrium in games with a large number of players, provide necessary and sufficient conditions for information to be aggregated (and, therefore, for biases to be inconsequential in large games), and characterize optimal voting rules. Our results provide a more nuanced view of the benefits of using group decision-making for the purpose of information aggregation.

1 Introduction

One rationale for elections is that better outcomes are chosen by aggregating information that is dispersed in the population. We study settings where members of a group, such as

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a committee, have a particular objective (to elect or hire the best candidate, to choose the best treatment for a patient) and obtain private information (campaign advertising, personal interviews, physical exams) about how best to achieve their objective. We deviate from the literature by assuming that people learn to make decisions from past experience. In this context, counterfactuals are usually not observed and, consequently, learning suffers from a selection problem. For example, when deciding between two political parties, voters will consider the past performance of each party. While voters can judge the elected party’s performance in office, they do not observe whether the losing party would have performed better or worse had it been elected. Our objective is to evaluate the extent to which group decision-making aggregates information in a learning environment with unobserved counterfactuals.

The setup is a standard voting environment (e.g., Fedderson and Pesendorfer, 1997) with a non-standard behavioral assumption. Voters simultaneously decide which of two alternatives to support. The best alternative depends on the state of the world, and votes are cast after observing private signals that are correlated with the state. The outcome of the election is determined by a particular voting rule (e.g., majority voting). We assume that voters naively take information at face value, thus failing to account for the possibility that the sample from which they learn is biased. We model this behavior using the notion of a behavioral equilibrium (Esponda, 2008), which builds on the idea of a self-confirming equilibrium (Battigalli (1987), Fudenberg and Levine (1993), Dekel, Fudenberg, and Levine (2004)).

We provide three main contributions. First, we present a dynamic learning model that clarifies our behavioral assumptions and provides a foundation for (behavioral) equilibrium. The framework is an adaption of the model by Fudenberg and Kreps (1993). Players repeatedly face the same voting environment and update their beliefs about the desirability of each alternative by observing how previously chosen alternatives have fared. Our main result is that, when players do not account for the possibility that they learn from a biased sample, a steady state of the dynamic model is an equilibrium of the stage game with naive voters.

Our second contribution is to develop an approach that allows us to characterize all equilibria of the voting game with a large number of players. We are able to find necessary and sufficient conditions for equilibrium by slightly perturbing players’ payoffs and by keeping track of the average strategy that each type of player follows. The key insight leading to our characterization result is that, in the perturbed game, the probability that a player is pivotal (i.e., decides the election) goes to zero as the number of players increases.

Third, we use the characterization result to investigate the extent to which information

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1In the case of doctors deciding whether or not to perform surgery, the consequences of treating an untreated patient and of not treating a treated patient are both unobservable.
aggregation (i.e., efficiency) obtains in equilibrium with sufficiently many players. On the one hand, a source of bias disappears as players become negligible because their biased decisions have a negligible impact on their own learning. On the other hand, the aggregate biased decisions of all other players do have an impact on each player’s learning. We show that information may or may not be aggregated and provide necessary and sufficient conditions on the primitives (including the voting rule) under which information is aggregated as the number of players goes to infinity. We also characterize the voting rules that maximize social welfare and, in particular, provide a new rationale for optimality of majority voting in symmetric settings where players have sufficiently accurate signals.

The results that information may not be aggregated and that institutional details (e.g., voting rules) matter are in stark contrast to the well-known result, due to Fedderson and Pesendorfer (1997), that information is aggregated when voters play a Nash equilibrium under any non-unanimous voting rule. By implicitly assuming that players have correct beliefs about the consequences of both their equilibrium and off-equilibrium (or counterfactual) choices, the Nash solution concept assumes that players can perfectly account for selection. The issue of selection may be of first-order importance when counterfactuals are not (perfectly) observed in a decentralized learning problem. The difference in results highlights the importance of understanding which behavioral assumptions are appropriate in different contexts. While the Nash assumption is sensible in certain settings (see Section 4.5 for a discussion of alternative behavioral assumptions), our behavioral assumption can always be viewed as a modeling device for understanding how serious the selection problem can be.

Our results also provide guidance to a planner who must determine whether to promote decentralized learning in committees, as opposed to, for example, promoting coordinated learning through randomized trials. We show that the welfare loss from sample-selection issues is less of a concern when the two alternatives result in similar payoffs when adopted in the states of the world in which they are best; surprisingly, the costs from choosing the wrong alternative play a relatively minor role. For example, suppose that voters choose between two political parties, A and B. Party A is actually best if the underlying (unobservable) state of the economy is strong, while party B is best if the economy is weak. Voters get imperfect signals correlated with the state of the economy. In this case, decentralized learning will lead to inefficient outcomes. Roughly, the intuition is that, if learning were to lead to the efficient outcome, where party A is elected in a strong economy and party B in a weak economy, then voters would always observe party A doing better than party B (since it is easier to govern in a strong economy). Hence, all voters would prefer to vote for party A, thus contradicting the hypothesis that the right party is chosen in its corresponding state of the world. In equilibrium,
party A will have to be occasionally elected into office in a weak economy; this mistake will then reduce party A’s popularity and provide incentives for voters to choose both parties in equilibrium.

This paper relates to several strands of literature: voting (Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997, 1998)); learning in games (Fudenberg and Levine (1998, 2009)); information aggregation in both auctions (Milgrom (1979, 1981b), Pesendorfer and Swinkels (2000), Perry and Reny (2006)) and elections (Feddersen and Pesendorfer (1997, 1998)); and equilibrium concepts with boundedly rational players (Eyster and Rabin (2005), Jehiel and Koessler (2008), Esponda (2008)). We deviate from the voting literature by proposing an alternative behavioral assumption that provides a complementary view of the information-aggregation question.2,3 We also adapt the learning setup in Fudenberg and Kreps (1993) to players that observe private information, learn expected payoffs for each of the two alternatives (rather than learning the strategies of other players), and face a selection problem while learning. We explicitly account for the possibility of correlation in strategies and show that the correlation vanishes with time when payoffs are slightly perturbed.4 Finally, in contrast to most of the information-aggregation literature, we characterize all equilibria (not just equilibria in symmetric strategies) and provide both necessary and sufficient conditions for (epsilon) equilibrium.

In Section 2, we present an example that illustrates the motivation for our behavioral assumption, the relationship to other assumptions in the literature, and the intuition for some of our results. In Section 3, we present the voting stage game and the notion of a (naive) behavioral equilibrium. In Section 4, we introduce the dynamic setting and use it to interpret and justify the solution concept. In Section 5, we present the setup for games with many players, and in Section 6, we characterize equilibrium as the number of players goes to infinity. In Section 7, we apply these results to provide necessary and sufficient conditions

2The seminal contribution by Feddersen and Pesendorfer (1997) sparked a literature that maintains the Nash assumption but qualifies results on aggregation when some of the assumptions in the benchmark model are relaxed (e.g., costly information acquisition: Persico (2004), Martinelli (2006), Oliveros (2007), Gershkov and Szentes (2009); costly voting: Krishna and Morgan (2008); non-monotone preferences: Bhattacharya (2008)).

3Eyster and Rabin (2005) apply their notion of (partially) cursed equilibrium to voting games, thus capturing a convex combination of the sincere and Nash approaches discussed in Section 2. Costinot and Kartik (2007) study voters who are level-k thinkers (Stahl and Wilson (1995), Nagel (1995)) and show that, under homogeneous preferences, the optimal voting rule is the same regardless of whether players are sincere, Nash, level-k thinkers, or mixtures among all of these.

4There is an alternative literature on learning and experimentation by multiple agents (Bolton and Harris (1999), Keller, Rady and Cripps (2005); Strulovici (forthcoming) in a voting context). That literature studies learning in an equilibrium context, while we study learning as a justification for equilibrium.
for information aggregation and to characterize optimal voting rules. We briefly conclude in Section 8.

2 Motivation and examples

A group of $n$ players chooses between alternatives A and B. Alternative A provides a payoff of 2 in state of the world $\omega_A$ and 0 in state $\omega_B$, while B provides a safe payoff of 1 in both states. These payoffs are summarized in Figure 1(a): A is best in state $\omega_A$ and B is best in state $\omega_B$. Before casting their vote, players observe private signals $s \in \{a,b\}$ that are independently drawn, conditional on the state; in particular,

$$\Pr (a \mid \omega_A) = \Pr (b \mid \omega_B) = q > 1/2.$$

Hence, signal $a$ is more favorable about $\omega_A$ than signal $b$ and vice versa for state $\omega_B$. After observing their signals, players simultaneously cast their vote for one of the two alternatives. The team adopts A if and only if the proportion of votes in favor of A is higher than some threshold $\rho$. We later generalize this setup by allowing for heterogeneity in preferences and information structure among players.

The literature has focused on two different behavioral assumptions. In the first case, players know the primitives of the game and vote for the best alternative given their information. In our example, players would vote for A after observing signal $a$ and for B after observing $b$. A well known result, dating back to Condorcet (1785), states that, if signals are sufficiently precise (i.e., $q > 1/2$), then such sincere voting under majority rule ($\rho = 1/2$) selects the best alternative with probability that goes to 1 as the group size increases—i.e., information is aggregated. Figure 2(a) illustrates the argument. By the law of large numbers, the proportion of players that observe signal $a$ and, therefore, vote for A, concentrates around $q$, conditional on state $\omega_A$, and around to $1 - q$, conditional on $\omega_B$. Consequently, if $1 - q < 1/2 < q$
Figure 2: Comparison of sincere, Nash, and naive voting

is the proportion required to choose \( A \), then the probability that \( A \) is chosen converges to 1, conditional on state \( \omega_A \), and to 0, conditional on \( \omega_B \). This behavioral assumption raises the question of how players learn to make the right choice with probability greater than one-half. In particular, how do players learn to associate signals \( a \) and \( b \) with states \( \omega_A \) and \( \omega_B \), respectively?

Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997,1998) emphasize a different concern: Sincere voting does not necessarily constitute a Nash equilibrium of the voting game. In a Nash equilibrium, voters may sometimes vote against the best alternative, given their private information alone. The reason is that a vote is relevant only if it changes the outcome of the election, so voters should choose the alternative that is optimal, conditional on the information that they can infer from the hypothetical fact that they are pivotal. Figure 2(a) illustrates this argument for voting rule \( \rho > q \). If all players were voting sincerely, then a player’s vote would be pivotal with vanishing probability. However, conditional on the event that a vote is pivotal, it is much more likely that the state is \( \omega_A \) rather than \( \omega_B \)—a simple consequence of the central limit theorem and a comparison of the vanishing tails of two normal distributions. Therefore, a player should ignore her private information and vote as if the state were \( \omega_A \), thus indicating that sincere voting does not constitute a Nash equilibrium.

Despite sincere voting not necessarily being a Nash equilibrium, Feddersen and Pesendorfer (1997) show that information is aggregated under any non-unanimous voting rule when voters play a Nash equilibrium. To see some intuition, suppose that both alternatives were chosen
with positive probability but that information were not aggregated in equilibrium, as depicted in Figure 2(b): In state $\omega_B$, B is correctly chosen with probability that goes to 1; however, in state $\omega_A$, both A and B are chosen with non-negligible probability. Again, the information that a vote is pivotal suggests that the state is $\omega_A$. But then, no one would want to vote for B, contradicting the assumption that this case can arise in equilibrium.\(^5\)

The previous reasoning, however, relies on players being sophisticated enough to realize that there is information to be inferred from other players’ votes, and that they should, therefore, condition their choice on the hypothetical event that their vote is pivotal. In addition, players must be able to make correct inferences from being pivotal. In a static context, these inferences may require players to have correct beliefs about the primitives and about the strategies being followed by other players. In a learning context, making correct inferences will be difficult—see Section 4.5 for further discussion.

Our alternative behavioral assumption is motivated by thinking of a learning environment in which players play the same stage game every period and learn to make decisions by observing the outcome of previous choices; it may be viewed as the analog of sincere voting, but when the primitives of the game must be learned. Figure 3 depicts a particular history of past outcomes observed by a player after playing the game for 8 periods. The player remembers the observed signals, the outcome of the election, and the observed payoff in each

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\(^5\)Alternatively, for the special case of identical preferences, McLennan (1998) shows that the fact that this is a symmetric game of common interest implies that whenever sincere voting leads to information aggregation (e.g., under majority rule), there is a symmetric Nash equilibrium under which information is aggregated.
period. Suppose that in period 9, the player observes signal $a$. We postulate the following behavior. First, the player forms beliefs about the expected benefit of outcome A.\textsuperscript{6} These beliefs are given by the average observed payoff obtained from A when the observed signal was $a$, which in this case is $(0 + 2 + 2)/3 = 4/3$.\textsuperscript{7} Second, the player votes for the alternative that she believes has the highest expected payoffs: in this case $4/3 > 1$ and, therefore, she votes for A.

The learning rule does not take into account two sources of sample selection. The first source is exogenous: Estimates are likely to be biased upwards if alternatives tend to be chosen when they are most likely to be successful—which is to be expected if players are using their private information to make decisions. In Figure 3, counterfactual payoffs for A are not observed in periods 3 and 7, but the fact that A was not chosen makes it likely that counterfactual payoffs would have been lower, on average, than observed payoffs for A. The second source is endogenous: A player’s vote affects the sample that she will observe. For example, suppose that the player was pivotal in period 1. Then, had she voted for B instead of A, B would have been the outcome, and no payoff would have been observed for A in period 1. If all other votes were unchanged, then in period 9, the player would have even stronger beliefs of $(2 + 2)/2 = 2$ in favor of A. In both the exogenous and endogenous cases, the underlying source of the bias is that other players use their private information to make decisions. Failing to account for selection in a learning environment is, then, analogous to failing to account for the informational content of other players’ actions.\textsuperscript{8,9}

Consider, again, the example in Figure 1(a) with voting rule $\rho > q$. We argued that sincere voting was not a Nash equilibrium; a related argument shows that sincere voting cannot be a naive equilibrium either. If it were a naive equilibrium, then A would be chosen with probability that goes to zero as the number of players increases. However, beliefs about the benefits from choosing A would come from those instances where A is chosen—an event that

\textsuperscript{6}For simplicity, we assume that the payoff from alternative B is known; the general model allows for learning about both of the alternatives.

\textsuperscript{7}The important aspect of the belief-formation process is that players consistently estimate observed mean payoffs; in particular, players could also start with a prior and apply Bayesian updating based on observed payoff outcomes.

\textsuperscript{8}See Kagel and Levin (2002) and Charness and Levin (2006) for experimental evidence of this type of naivete in auction-like contexts, where some players do not take into account the information that can be inferred from being the winner of the auction. In a voting context, Guarnaschelli et al. (2000), Ali et al. (2008) and Battaglini et al. (forthcoming) find that voters sometimes vote against their signal, and interpret it as evidence of strategic voting. In our learning context, naive voters may also vote against their signal. These experiments do not take place in a learning context and, therefore, cannot provide a direct test of our behavioral assumptions.

\textsuperscript{9}See Aragones et al. (2005), Al-Najjar (forthcoming), Al-Najjar and Pai (2009), and Schwartzstein (2009) for theoretical foundations of related forms of naivete.
is much more likely to happen when the state is $\omega_A$ rather than $\omega_B$. Therefore, players would mostly observe a payoff of 2 from alternative A, thus concluding that A is the best choice and contradicting that sincere voting is an equilibrium. This example highlights that what Nash and naive behavior have in common is that beliefs are endogenously restricted by the strategies being followed by all players.

Nash and naive behavior, however, could be fundamentally different. We argued that the situation depicted in Figure 2(b) cannot be a Nash equilibrium and that information must always be aggregated. However, in our example, information cannot be aggregated in a naive equilibrium. Suppose that information were close to being aggregated in a naive equilibrium: Then, players would almost always observe that alternative A has a payoff of 2, thus contradicting the assumption that any of them would ever vote for B. In fact, a naive equilibrium will have the features of the situation in Figure 2(b). In order to induce players to vote for B, the committee must make enough mistakes so that a payoff of 0 is frequently observed for A, thus counterbalancing the high payoff of 2 that is observed when A is chosen in the right state.

While the example illustrates lack of information aggregation in a naive equilibrium, there are cases where naive behavior yields information aggregation. An example is given by the payoff structure in Figure 1(b), where there are now 3 states of the world.\textsuperscript{10} If information were aggregated, then players would observe payoffs of 4 and 2 for alternative A and a payoff of 3 for B. Now suppose that there are two signals, and that the weighted average of 4 and 2 is higher than 3 conditional on one of the signals and lower than 3 conditional on the other. Unlike the example in Figure 1(a), players now have incentives to make both choices in equilibrium, and information will be aggregated provided that the voting rule is chosen appropriately. The rest of the paper formalizes and generalizes the arguments in this section and provides additional insights into the nature of the information-aggregation problem.

3 Voting framework

3.1 Voting stage game

A group of $n$ players must choose between two alternatives, $x_i \in X_i = \{A, B\}$. A state $\omega$ is drawn from a finite set $\Omega$ according to a probability distribution $p \in \Delta(\Omega)$. Player $i$'s utility

\textsuperscript{10}Naturally, the assumption is that counterfactuals are not observed, so players do not know the structure of payoffs in Figure 1(b); otherwise, they could infer a counterfactual payoff of 1 after observing a payoff of 4 for alternative A.
is
\[ u_i(o(x), \omega) + 1 \{ o(x) = B \} v_i, \]
where \( \omega \) is the state of the world, \( v_i \in V_i \) is a privately-observed payoff perturbation à la Harsanyi (1973) and Selten (1975), and \( o(x) \in \{ A, B \} \) is the alternative chosen by the committee, given votes \( x = (x_1, \ldots, x_n) \in X \equiv \prod_{i=1}^n X_i \).

In addition to their idiosyncratic payoff shock \( v_i \), each player also observes a signal \( s_i \) from a finite set \( S_i \); let \( S \equiv \prod_{i=1}^n S_i \). Each signal \( s_i \) is drawn independently, conditional on the state \( \omega \), with probability \( q_i(s_i \mid \omega) > 0 \). After observing their private payoff-shock \( v_i \) and signal \( s_i \), players simultaneously submit a vote for either \( A \) or \( B \). Votes are aggregated according to a threshold voting rule: The committee chooses alternative \( A \) if and only if \( k > 0 \) or more people vote for \( A \); otherwise, it chooses alternative \( B \).

Utility is uniformly bounded—i.e., \( |u_i(o, \omega)| < K < \infty \) for all \( i = 1, \ldots, n \), \( o \in \{ A, B \} \), and \( \omega \in \Omega \). Moreover, the perturbation \( v_i \) is independently drawn from an absolutely continuous distribution \( F_i \) that satisfies \( F_i(-2K) > 0 \) and \( F_i(2K) < 1 \).

Let \( Y_i = \{ A, B \}^{#S_i} \) be the set of signal-contingent actions of player \( i \). An action plan is a function \( \phi_i : V_i \to Y_i \) such that \( \phi_i(v_i)(s_i) = A \) for \( v_i < -2K \) and \( \phi_i(v_i)(s_i) = B \) for \( v_i > 2K \), for all \( s_i \in S_i \). An action plan indicates player \( i \)'s signal-contingent action as a function of her payoff perturbation; the restriction is motivated by the bound on utility.

For each action plan \( \phi_i \), there is an associated (mixed) strategy \( \alpha_i \in A_i \), where
\[ A_i = \{ \alpha_i : F_i(-2K) \leq \alpha_i(s_i) \leq F_i(2K) \ \forall s_i \in S_i \} \]
is the set of player \( i \)'s strategies, and
\[ \alpha_i(s_i) = \Pr \{ v_i : \phi_i(v_i)(s_i) = A \} \]
is the probability that player \( i \) votes for \( A \) after observing signal \( s_i \). Each strategy profile \( \alpha = (\alpha_1, \ldots, \alpha_n) \in A \equiv A_1 \times \ldots \times A_n \) induces a distribution over outcomes \( P(\alpha) \in \Delta(Z) \), where \( Z \equiv X \times S \times \Omega \) and
\[ P(\alpha)(x, s, \omega) = p(\omega) \prod_{i=1}^n \alpha_i(s_i)^{1(x_i = A)} (1 - \alpha_i(s_i))^{1(x_i = B)} q_i(s_i \mid \omega). \tag{1} \]
Whenever an expectation \( E_P \) has a subscript \( P \), this means that the probabilities are taken with respect to the distribution \( P \).
3.2 The role of payoff perturbations

The independent payoff perturbations play several roles in this paper. The first is as a refinement criterion (in the spirit of Selten (1975)). Without payoff perturbations, existence of equilibrium (both Nash and our solution concept) is trivial for two reasons. First, there are strategy profiles for which no profitable deviation exists simply because no unilateral deviation may affect the chosen outcome.\(^{11}\) Second, in our context, beliefs about the desirability of alternatives will be determined endogenously in equilibrium. But if an alternative is never chosen, then equilibrium beliefs may be arbitrary, hence justifying the decision not to choose the alternative in the first place. The perturbations imply that each alternative is chosen with strictly positive probability, thus providing the experimentation necessary to pin down beliefs in equilibrium. The following result formalizes the perfection role discussed above. The proof is straightforward and, therefore, omitted.

**Lemma 1.** Let \(\alpha \in \mathcal{A}\). Then, for all \(i\) and \(s_i \in S_i\):

\begin{align*}
(i) & \quad P(\alpha_i', \alpha_{-i}) (o = A \mid s_i) - P(\alpha_i'', \alpha_{-i}) (o = A \mid s_i) > 0 \text{ for all } \alpha_i' > \alpha_i''. \\
(ii) & \quad P(\alpha) (o = A \mid s_i) \in (0, 1). 
\end{align*}

Part (i) says that player \(i\)'s vote affects the outcome of the election; i.e., the probability of being pivotal is strictly greater than zero. Part (ii) says that alternatives \(A\) and \(B\) are chosen with strictly positive probability, so that in equilibrium, beliefs about \(A\) and \(B\) will not be arbitrary. Hence, payoff perturbations play the role of trembles and rule out the type of trivial equilibria described above.

A second role of payoff perturbations is to guarantee that behavior in the dynamic game does not converge to *correlated* strategy profiles. A third role arises when we take the number of players to infinity. Here, payoff perturbations guarantee that the variance of the probability of voting for an alternative stays bounded away from zero. We can then apply a version of the central limit theorem to show that the probability that players are pivotal (i.e., that their vote decides the election) goes to zero. This result is crucial for characterizing equilibrium in games with many players.\(^{12}\)

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\(^{11}\)For example, it is an equilibrium for everyone to vote for the same alternative (no matter how bad this alternative may be) whenever \(k < n\). The analogous refinement in Fedderson and Pesendorfer (1997) is to restrict attention to strategies that are not weakly dominated.

\(^{12}\)Later, we consider sequences of equilibrium outcomes where the perturbation vanishes. Here, perturbations can be seen as playing the standard Harsanyi (1973) role of purifying mixed strategies.
3.3 Definition of equilibrium

A naive (or, more generally, behavioral) equilibrium (Espóna, 2008) combines the idea of a self-confirming equilibrium (Battigalli (1987), Fudenberg and Levine (1993), Dekel, Fudenberg, and Levine (2004)) with an information-processing bias. Players choose strategies that are optimal, given their beliefs about the consequences of following each possible strategy. In contrast to a Nash equilibrium, these beliefs are not necessarily restricted to being correct, but, rather, to being consistent with the information feedback players receive. This information is, in turn, endogenously generated by the equilibrium strategies followed by all players. Our feedback assumption is that players observe only the realized payoff of the alternative that the committee chooses, but not the counter-factual payoff of the other alternative.\(^{13}\)

A naive equilibrium requires beliefs to be *naive-consistent*, meaning that this information is not correctly processed by players. In particular, players do not take into account that other players’ actions may be correlated with the true state of nature. In this way, we formalize the idea that players do not take into account the informational content of other players’ actions, or, equivalently, the sample selection problem.

To gain some intuition for the solution concept, suppose that player \(i\) repeatedly faces a sequence of stage games where players use strategies \(\alpha\) every period. Then, under the assumption that the payoff to alternative \(A\) is observed only whenever \(A\) is chosen, player \(i\) will come to observe that, conditional on observing signal \(s_i\), alternative \(A\) yields in expectation

\[
E^A_i(P(\alpha), s_i) \equiv E_{P(\alpha)}[u_i(A, \omega) \mid o = A, s_i].
\]

Similarly, alternative \(B\) yields in expectation

\[
E^B_i(P(\alpha), s_i) \equiv E_{P(\alpha)}[u_i(B, \omega) \mid o = B, s_i].
\]

A naive player who observes \(v_i\) and \(s_i\) believes that expected utility is maximized by voting for \(A\) whenever \(\Delta_i(P(\alpha), s_i) - v_i > 0\) and voting for \(B\) otherwise,\(^{14}\) where

\[
\Delta_i(P, s_i) \equiv E^A_i(P, s_i) - E^B_i(P, s_i)
\]

is well-defined by Lemma 1.

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\(^{13}\)The assumption that counterfactuals are not observed guarantees that players’ naive model of the world is consistent with their feedback (see Espóna (2008) for further discussion).

\(^{14}\)Implicitly, we assume that players (correctly) believe that they can be pivotal with strictly positive probability.
Definition 1. A strategy profile $\alpha \in A$ is a (naive) equilibrium of the stage game if for every player $i = 1, ..., n$ and for every $s_i \in S_i$,

$$\alpha_i(s_i) = F_i(\Delta_i(P(\alpha), s_i)),$$

We refer to $P(\alpha) \in \Delta(Z)$ as a (naive) equilibrium distribution.

In equilibrium, players best respond to beliefs that are endogenously determined by both their own strategy and those of other players and that are consistent with observed equilibrium outcomes. Naive players, however, do not account for the correlation between others’ votes and the state of the world (conditional on their own private information). In particular, naive players fail to realize that a change in their own strategies may also affect their beliefs about alternatives $A$ and $B$.

Theorem 1. There exists a (naive) equilibrium of the stage game.

Proof. Let $\Phi : A \rightarrow A$ be given by $\Phi(\alpha) = (F_i(\Delta_i(P(\alpha), s_i))_{s_i \in S_i})_{i=1,\ldots,n}$. First, note that $\Phi(\alpha) \in A$ for all $\alpha \in A$. Second, $A$ is a convex and compact subset of a Euclidean space. Third, $P(\cdot)$ is linear, hence continuous, implying that $\Delta_i(P(\cdot), s_i)$ (which is well-defined by Lemma 1) is continuous and, by continuity of $F_i$, that $\Phi$ is continuous. Therefore, by Brouwer’s fixed point theorem, there exists a fixed point of $\Phi$, which is also an equilibrium of the stage game.

$\square$

4 Learning foundation for equilibrium

4.1 A model of learning

We present a dynamic framework in order to clarify and justify the solution concept. A dynamic game is a repetition of the stage game described in Section 3, in which the state and the signals are drawn independently across time periods from the same distribution. The main result is that if behavior stabilizes in the dynamic game, then it stabilizes to an equilibrium of the stage game.

We adapt the learning model of Fudenberg and Kreps (1993) to our context. A group of $n$ players play the stage game described in Section 3 for each discrete time period $t = 1, 2, ....$ At time $t$, the state is denoted by $\omega_t \in \Omega$, the signals by $s_t = (s_1t, ..., s_nt)$, and the votes by
\[ x_t = (x_{1t}, ..., x_{nt}) \]. The outcome of the election at time \( t \) is determined by a threshold voting rule \( k \) and denoted by \( o_t \in \{A, B\} \). Utility is given by

\[ u_i(o_t, \omega_t) + 1 \{o_t = B\} v_{it}, \]

where \( v_{it} \) is the payoff perturbation drawn independently (across players and time) from \( F_i \).

Let \( h^t = (z_1, ..., z_{t-1}) \) denote the history of the game up to time \( t - 1 \), where \( z_t = (x_t, s_t, \omega_t) \in Z \) is the time-\( t \) outcome. Let \( \mathcal{H}^t \) denote the set of all time-\( t \) histories and let \( \mathcal{H} \) be the set of infinite histories. At each round of play, players privately collect feedback about past outcomes. For each player \( i \), \( Z_i \equiv X \times S_i \times U_i \) is the set of outcomes that player \( i \) may observe at any given period, where \( U_i \) is the range of her utility functions \( u_i(A, \cdot) \) and \( u_i(B, \cdot) \). Let \( h^t_i = (z_{i1}, ..., z_{it-1}) \) denote player \( i \)'s private history up to time \( t - 1 \), where \( z_{it} = (x_t, s_{it}, u_i(o_t, \omega_t)) \in Z_i \) is the privately observed outcome at time \( t \). Note that payoff perturbations are not part of the history, implicitly assuming that players understand that the perturbations are independent payoff shocks that are unrelated to the learning problem. Let \( \mathcal{H}^t_i \) denote the set of all time-\( t \) private histories and let \( \mathcal{H}_i \) be the set of infinite private histories for player \( i \). By convention, if \( t = 1 \), then all these sets are singleton sets consisting of the null history.

We complete the specification of the dynamic game by introducing assessment (i.e., belief-updating) and policy rules. An assessment rule for player \( i \) is a sequence \( \mu_i = (\mu_{i1}, ..., \mu_{it}, ...) \) such that \( \mu_{it} : \mathcal{H} \rightarrow \mathbb{R}^{\#S_i} \) is measurable with respect to the player \( i \)'s time-\( t \) private history. The interpretation is that the \( s_i \)-coordinate of \( \mu_{it}(h), \mu_{it}(h)(s_i) \), is player \( i \)'s beliefs–given her private \( t - 1 \)-period history in \( h \)–about the difference in expected utility between alternatives \( A \) and \( B \) conditional on \( s_i \).

A policy rule for player \( i \) is a history-dependent sequence of action plans \( \phi^H_i = (\phi^H_{i1}, ..., \phi^H_{it}, ...) \), where \( \phi^H_{it} : \mathcal{H} \times V_i \rightarrow Y_i \equiv \{A, B\}^{\#S_i} \) is measurable with respect to player \( i \)'s time-\( t \) private history and her time-\( t \) payoff perturbation. The interpretation is that \( \phi_{it}(h, v_i)(s_i) \) is player \( i \)'s vote at time \( t \), conditional on observing private history \( h^t_i \), perturbation \( v_i \), and signal \( s_i \).

The measurability restrictions on assessment and policy rules imply that players’ decisions may depend on the observed payoff outcomes but not on the (unobserved) state of the world, thus capturing the assumption that players do not observe counter-factual payoffs.\(^{15}\)

Given a policy rule profile \( \phi^H = (\phi^H, ..., \phi^H_n) \), let \( P^{\phi^H}(\cdot | h^t) \) denote the probability distri-

\(^{15}\) Implicitly, we are also assuming that players know neither the functional form of the utility function nor the structure of the state space; otherwise, players would be able to make inferences about counter-factual payoffs by observing realized payoffs.
bution over histories, conditional on history up to time \( t, h^t \in H^t \)--which we can construct by Kolmogorov’s extension theorem.

4.2 The selection problem: naive assessments

We place the following restrictions on assessment and policy rules.

**Definition 2.** An assessment rule \( \mu_i \) is **non-strategic and empirical** if

\[
\mu_{it}(h)(s_i) = \frac{\sum_{\tau=1}^{t-1} 1 \{ o_\tau = A, s_{i\tau} = s_i \} u_{i\tau}}{\sum_{\tau=1}^{t-1} 1 \{ o_\tau = A, s_{i\tau} = s_i \} - \sum_{\tau=1}^{t-1} 1 \{ o_\tau = B, s_{i\tau} = s_i \}}
\]

for every \( h \in H, s_i \in S_i, \) and \( t \geq 2 \) such that the denominators are greater than zero, where \( o_\tau, s_{i\tau}, \) and \( u_{i\tau} \) are time-\( \tau \) elements of \( h \).

In words, the definition assumes that players believe that the difference in expected payoffs from A and B, conditional on an observed signal, is given by the **observed** empirical average difference in payoffs--the key here is that only the payoff to the chosen alternative is observed. Hence, players take the information they see at face value. In particular, they make no attempts to account for sample selection that arises because counter-factual payoffs are not observed.

The learning model is completed by assuming that players vote for the alternative that maximizes their current period’s **perceived** expected utility.\(^{16}\)

**Definition 3.** A policy rule \( \phi^H_i \) is myopic relative to an assessment rule \( \mu_i \) if for every \( h \in H, s_i \in S_i, \) and \( t \geq 1, \)

\[
\phi_{it}(h,v_{it})(s_i) = \begin{cases} A & \text{if } \mu_{it}(h)(s_i) - v_{it} \geq 0 \\ B & \text{otherwise} \end{cases}
\]

The assumption of myopia is for simplicity. The results remain true if it is replaced with the definition of asymptotic myopia (as in Fudenberg and Kreps (1993)), thus allowing players to be forward-looking and to experiment.\(^{17}\)

\(^{16}\)Implicitly, we are assuming that players believe (correctly) that their vote affects the outcome with a strictly positive probability. Also, the assumption that A is played if indifferent is only for simplicity and may be replaced with any tie-breaking rule.

\(^{17}\)The reason why naive players have incorrect beliefs is not due to the lack of experimentation, but, rather, to their failure to account for the selection problem (Esponda (2008)). Indeed, even with myopic players, there may exist perpetual “experimentation” in the sense that the proportion of time that the committee chooses either alternative is bounded away from zero in the limit as \( t \) grows to infinity.
4.3 Stability and equilibrium

Our objective is to relate distributions over outcomes of the dynamic game as \( t \to \infty \) to equilibrium distributions over outcomes of the stage game. Define the sequence of random variables \( \mathcal{P}_t : \mathcal{H} \to \Delta(Z) \), where

\[
\mathcal{P}_t(h)(x, s, \omega) = \frac{1}{t-1} \sum_{\tau=1}^{t-1} 1 \{ x_{\tau}(h) = x, s_{\tau}(h) = s, \omega_{\tau}(h) = \omega \}
\]

is the frequency distribution over outcomes in the dynamic game. We look at the frequency distribution in order to allow for the possibility that play in the dynamic game is correlated.

We focus attention on frequency distributions that eventually stabilize around a steady-state distribution over outcomes. The following definition of stability accounts for the probabilistic nature and possible multiplicity of steady states.

**Definition 4.** \( P^* \in \Delta(Z) \) is a stable outcome distribution of the dynamic game under policy rules \( \phi^H \) if for all \( \varepsilon > 0 \) there exists \( t_\varepsilon \) such that

\[
P^{\phi^H}(\|\mathcal{P}_t - P^*\| < \varepsilon \text{ for all } t \geq t_\varepsilon) > 0.
\]

The definition of stability captures the idea that after a finite number of periods, there is a strictly positive probability that the frequency distribution over outcomes \( \mathcal{P}_t \) remains forever close to \( P^* \).

**Theorem 2.** Suppose that \( P^* \) is a stable outcome distribution of the dynamic game under policy rules \( \phi^H \) that are myopic relative to assessment rules \( \mu \) that are non-strategic and empirical. Then, \( P^* \) is a (naive) equilibrium distribution of the stage game.

Theorem 2 is our key justification for focusing on equilibria of the stage game: Any profile that is not an equilibrium generates an outcome distribution that is not stable in the dynamic game. In particular, correlated strategy profiles do not generate stable outcome distributions in our environment. As our proof makes clear, this result follows from the assumption that payoff perturbations are independent across players and time.

---

18The statement is true because \( t_\varepsilon \) is not history-dependent for those histories in the history set that, according to the definition, has positive probability. But note that this definition is weaker than requiring that \( \mathcal{P}_t \) converges to \( P^* \) with strictly positive probability.

19Of course, a converse to Theorem 2 does not necessarily hold; that is, there may exist equilibria that do not generate stable outcome distributions.
4.4 Proof of Theorem 2

Throughout the proof, we fix a stable outcome distribution \( P^* \) and policy rules \( \phi^H \) that are myopic relative to assessment rules \( \mu \) which are non-strategic and empirical. The proof compares “strategies” in the dynamic game with strategies in the stage game. To define the former, let the vector-valued random variable \( \alpha^H_t = (\alpha^H_{1t}, ..., \alpha^H_{nt}) : \mathcal{H} \to A_1 \times \ldots \times A_n \) denote a time-\( t \) strategy profile, where

\[
\alpha^H_{it}(h)(s_i) = \int 1 \{ \phi^H_{it}(h, v_i)(s_i) = A \} dF_i
\]

is the probability that player \( i \) votes for \( A \) when observing signal \( s_i \), conditional on history \( h^t \).

Finally, let \( \alpha^* = (\alpha^*_1, ..., \alpha^*_n) \in A_1 \times \ldots \times A_n \) be such that

\[
\alpha^*_i(s_i) = F_i(\Delta_i(P^*, s_i))
\]

is the probability that player \( i \) votes for \( A \) if she optimally responds to beliefs \( \Delta_i(P^*, s_i) \).

The proof follows from two claims. In Claim 2.1, we show that stability of \( P^* \) implies that beliefs eventually remain close to \( \Delta_i(P^*, s_i) \), thus implying that time-\( t \) strategies \( \alpha^H_t \) eventually remain close to \( \alpha^* \in A_1 \times \ldots \times A_n \). The key of the proof is that players’ payoff perturbations are independently drawn from an atom-less distribution, implying that if beliefs settle down, then strategies must also settle down, not just in an average sense, but actually in a per-period sense. In particular, Claim 2.1 implies that any correlation in players’ strategies induced by a common history eventually vanishes. In Claim 2.2, we show that the fact that strategies remain close to \( \alpha^* \) implies that \( P^* = P(\alpha^*) \), where \( P(\cdot) \) was previously defined in (1). Both claims rely on a straightforward generalization of a powerful technical result by Fudenberg and Kreps (1993, Lemma 6.2); this result allows us to apply the law of large numbers in a context where a sequence of random variables is not independently distributed, but where the distributions conditional on past history are eventually very close to some common distribution.

Theorem 2 follows immediately from Claim 2.2 since, then, for all \( i \) and \( s_i \),

\[
\alpha^*_i(s_i) = F_i(\Delta_i(P(\alpha^*), s_i))
\]

implying that \( \alpha^* \) is an equilibrium of the stage game and, therefore, that \( P^* = P(\alpha^*) \) is an equilibrium distribution.

**Claim 2.1** For all \( \varepsilon > 0 \), there exists \( H_\varepsilon \) with \( P^{\phi^H}(H_\varepsilon) > 0 \) such that for all \( h \in H_\varepsilon \),
there exists $t_{\epsilon,h}$ such for all $t \geq t_{\epsilon,h}$, all $i$, and all $s_i$, $|\alpha^H_{it}(h)(s_i) - \alpha_i^*(s_i)| < \epsilon$ and $|\overline{P}_t(h)(z) - P^*(z)| < \epsilon$ for all $z \in Z$.

**Proof.** By continuity of $F_i$, it suffices to show that for all $\epsilon > 0$, there exist $\gamma(\epsilon) > 0$ with $\lim_{\epsilon \to 0} \gamma(\epsilon) = 0$ and $H_\epsilon$ with $P^\phi(H_\epsilon) > 0$ such that for all $h \in H_\epsilon$, there exists $t_{\epsilon,h}$ such for all $t \geq t_{\epsilon,h}$, all $i$, and all $s_i$,

$$F_i(\Delta_i(P^*,s_i) - \gamma(\epsilon)) \leq \alpha^H_{it}(h)(s_i) \leq F_i(\Delta_i(P^*,s_i) + \gamma(\epsilon))$$

(5)

and

$$|\overline{P}_t(h)(z) - P^*(z)| < \epsilon$$

(6)

or all $z \in Z$.

For each $o \in \{A,B\}$, let

$$Z_{ois_i\omega} \equiv \{(x',s',\omega') \in Z : o(x') = o, s'_i = s_i, \omega' = \omega\}$$

and $Z_{ois_i} \equiv \cup_{\omega \in \Omega} Z_{ois_i\omega}$. Let $\mu_{oi}(h)(s_i)$ denote the $o \in \{A,B\}$ term of $\mu_{it}(h)(s_i)$ in expression (3), which can be written as

$$\mu_{oi}(h)(s_i) = \frac{\sum_{\tau=0}^{t-1} \{o_\tau = o, s_{i,\tau} = s_i\} u_i(A,\omega_i)}{\sum_{\tau=0}^{t-1} \{o_\tau = o, s_{i,\tau} = s_i\}}$$

$$= \sum_{\omega \in \Omega} \frac{\overline{P}_t(h)(Z_{ois_i\omega})u_i(o,\omega)}{\overline{P}_t(h)(Z_{ois_i})},$$

provided that $\overline{P}_t(h)(Z_{ois_i}) > 0$.

Because $P^*$ is stable, for all $\epsilon > 0$, there exists $t_{\epsilon}$ and $H_\epsilon^*$ with $P^\phi(H_\epsilon^*) > 0$ such that for all $h \in H_\epsilon^*$ and $t \geq t_{\epsilon}^*$, equation (6) holds for all $(x,s,\omega) \in Z$. In addition, (6) implies that

$$|\overline{P}_t(h)(Z^*) - P^*(Z^*)| < \epsilon \times \#Z$$

(7)

for all $Z^* \subset Z$. Next, let $\eta = \min \left\{ P^\phi(H_\epsilon^*), .5K_p \right\} > 0$, where $K_p$ is defined by equation (33) in the Appendix; by Lemma 5 in the Appendix, for all $h \in H \setminus H^o$ (where $H^o$ has zero measure) there exists $t_{\eta,h}$ such that for all $t \geq t_{\eta,h}$, all $o \in \{A,B\}$,

$$\overline{P}_t(h)(Z_{ois_i}) > K_p - \eta \geq .5K_p.$$  

(8)
Let $H_\varepsilon = H_\varepsilon^* \cap H \setminus H^o$, and note that by our choice of $H_\varepsilon^*$, $P^{\phi_H}(H_\varepsilon) > 0$. Therefore, for all $\varepsilon > 0$, there exists $H_\varepsilon$ with $P^{\phi_H}(H_\varepsilon) > 0$ such that for all $h \in H_\varepsilon$ and $t \geq t_{\varepsilon,h} \equiv max\{t^*_\varepsilon, t_{\eta,h}\}$, all $i, s_i$

$$|\mu_{it}(h)(s_i) - \Delta_i(P^*, s_i)| \leq \gamma(\varepsilon) \equiv \frac{\varepsilon \times \#Z \times (0.5K_p)^2}{2K(1 + \#\Omega) + 0.5 \times \varepsilon \times \#Z} K_p \quad \varepsilon \to 0$$

(9)

where the inequality follows from (7), (8), the facts that $|u_i(o, \omega)| < K$ and $\#\Omega < \infty$, and simple algebra that uses the fact that $E_0^o(P^*, s_i) = \sum_{\omega \in \Omega} P^*(Z_{oisi\omega} u_i(o, \omega))$. Then, equation (9) and the definition of the policy function imply that

$$\phi^H_{it}(h, v_i)(s_i) = \begin{cases} A & \text{if } v_i \leq \Delta E_i(P^*, s_i) - \gamma(\varepsilon) \\ B & \text{if } v_i > \Delta E_i(P^*, s_i) + \gamma(\varepsilon) \end{cases}$$

so that (5) holds by (4).

\[ \square \]

Claim 2.2 $P^* = P(\alpha^*)$

Proof. Note that for each $z \in Z$,

$$P^{\phi_H}(z_t = z \mid h^t) = P(\alpha^H_t(h))(z).$$

Then, Claim 2.1 and the fact that $P$ is continuous in $\alpha$ imply that for all $\varepsilon > 0$, there exists $H_\varepsilon$ with $P^{\phi_H}(H_\varepsilon) > 0$ such that for all $h \in H_\varepsilon$, there exists $\hat{t}_{\varepsilon,h}$ such that for all $t \geq \hat{t}_{\varepsilon,h}$, $P^{\phi_H}(z_t = z \mid h^t) - P(\alpha^*)(z) < \varepsilon$

and

$$|P_t(h)(z) - P^*(z)| < \varepsilon$$

(10)

for all $z \in Z$.

Then, by Lemma 4 in the Appendix applied to all the singleton sets of $Z$,

$$\limsup_{t \to \infty} P_t(z) \leq P(\alpha^*)(z) + \varepsilon \quad \text{and} \quad \liminf_{t \to \infty} P_t(z) \geq P(\alpha^*)(z) - \varepsilon$$

(11)

for all $z \in Z$, almost surely on $H_\varepsilon$.
By the triangle inequality, for any $t$,

$$||P^* - P(\alpha^*)|| \leq ||P^* - P_t(h)|| + ||P_t(h) - P(\alpha^*)||$$

(12)

for any $h \in H_\varepsilon$; we pick one $h \in H_\varepsilon$ (outside the measure zero set). By equation (11), the second summand in the RHS is less than $\varepsilon$ for all $t$ sufficiently large; by equation (10), the first summand of the RHS is also less than $\varepsilon$ for all $t$ sufficiently large. Hence, $||P^* - P(\alpha^*)|| \leq \varepsilon$; since this holds for all $\varepsilon > 0$, then we obtain the desired result by taking $\varepsilon \to 0$.

4.5 Alternative behavioral rules

A player who is sophisticated and understands the selection problem may account for it by conditioning her learning on past periods in which her vote was pivotal. If her vote is random and independent of the state of the world (conditional on her private information), then the subsample where she is pivotal will not be biased. The reason is that whether or not she observes the payoff of an alternative in those periods in which she is pivotal depends only on whether or not she votes for the alternative, which is independent of the state of the world.

There are a few reasons, however, why even a sophisticated player who understands selection may not behave as above. First, the procedure is not applicable when only the outcome of the election, and not the number of votes for each alternative, is observed. Second, even if the pivotal event were observable, the previous argument relies on a player’s vote being independent of other players’ votes, conditional on her private information. While this condition may hold in steady state, voting strategies are likely be correlated as players are learning to play the game. Hence, sophisticated players will initially face an endogenous selection bias, and the pivotal learning rule will not be optimal along the path of play.\(^\text{20}\) Alternatively, the strategy of randomizing their votes over a few periods of time will not necessarily lead to unbiased learning if other players are concurrently learning to play the game. An exception is when players coordinate their randomization, as in the ubiquitous practice of conducting randomized trials to overcome the selection problem. Third, the inferences that a player makes, conditional on being pivotal, are not suitable for replication, either when facing a similar decision problem with a different group of people (who may have learned to make decisions in a different way) or when facing a decision on her own.

A different kind of behavioral rule may also lead to Nash equilibrium behavior. Suppose

\(^{20}\)Of course, this bias disappears as the number of players increases and a player becomes negligible; however, the proportion of the sample that can be used to make inferences also goes to zero.
that players are pretty unsophisticated and don’t have a good understanding of the problem at hand. All they know is that they must choose either A or B every period, and they keep track of their utility (but not of the payoffs of alternatives A and B). Suppose that players then vote for the alternative yielding the highest expected utility. Under certain conditions, this kind of reinforcement learning may lead to (locally) first-best outcomes. For our motivating examples, however, it seems dubious that voters would not seek to learn the expected payoffs of the alternatives. This is particularly true whenever players interact in other settings, either by themselves or as members of others committees, where this knowledge would be useful.\footnote{One point of this example is that the relationship between sophistication and optimality is not monotone. Our players have a good understanding of the problem they face and are able to learn from the data at hand, but optimality is precluded because they fail to account for selection or (more generally) because they do not perfectly account for it.}

5 Voting framework: a large number of players

We now present the framework for analyzing games in which the number of players goes to infinity. Consider the stage game in Section 3, where we now represent heterogeneity in preferences and information by indexing the utility function $u_\theta$, the precision of the signals $q_\theta$, the finite set of signals $S_\theta$, the strategy set $A_\theta$, and the distribution over the perturbation $F_\theta$ by a type $\theta$. The lowest signal for a player for type $\theta$ is denoted by $s^L_\theta$ and the highest signal by $s^H_\theta$, so that $s^L_\theta < s^H_\theta$. We assume that $\theta$ is randomly drawn from a finite set $\Theta$ according to a probability distribution $\phi \in \Delta(\Theta)$. The threshold voting rule is now $k = \rho n$, where $0 < \rho < 1$.

The following assumptions on the primitives are maintained through the remainder of the paper.

C1. For each $\theta \in \Theta$: (i) $u_\theta(A, \cdot)$ is nondecreasing and $u_\theta(B, \cdot)$ is nonincreasing, and one of them holds strictly; (ii) $\sup_{\theta = \{A,B\}, \omega \in \Omega} |u_\theta(o, \omega)| < K < \infty$.

C2. There exists $z > 0$ such that for all $\theta$, $\omega' > \omega$, and $s^L_\theta > s^H_\theta$:

$$\frac{q_\theta(s^L_\theta|\omega')}{q_\theta(s^L_\theta|\omega)} - \frac{q_\theta(s^H_\theta|\omega')}{q_\theta(s^H_\theta|\omega)} = z(\omega' - \omega).$$

C3. $\Omega = [-1,1]$ and $G$ is an absolutely continuous probability distribution over $\Omega$ with density $g$.

C4. (i) $\inf_{\Omega} g(\omega) > 0$; (ii) there exists $d > 0$ such that $q_\theta(s_\theta|\omega) > d$ for all $\theta, s_\theta, \omega$; (iii) for all $\theta, s_\theta$, $q_\theta(s_\theta | \cdot)$ is continuous; (iv) for all $\theta, \eta$, $F^\eta_\theta$ is absolutely continuous and $\inf_{x \in [-2K,2K]} f^\eta_\theta(x) > 0$. 

\footnotetext{One point of this example is that the relationship between sophistication and optimality is not monotone. Our players have a good understanding of the problem they face and are able to learn from the data at hand, but optimality is precluded because they fail to account for selection or (more generally) because they do not perfectly account for it.}
C5. For all $\theta$, $u_\theta(A, \cdot)$ and $u_\theta(B, \cdot)$ are both continuously differentiable.

C6. For all $\theta$, $u_\theta(A, 1) - Eu_\theta(B, \omega | s^L_\theta) > 0$ and $Eu_\theta(A, \omega | s^H_\theta) - u_\theta(B, -1) < 0$.

Assumptions C1 and C2 provide a standard ordering between states, information, and players’ preferences, as well as a uniform bound on the utility function. In particular, C2 requires that the strict MLRP (monotone likelihood ration property) holds. We actually need a slight strengthening of strict MLRP: There must be a uniform bound on the rate at which the likelihood ratio changes.

Assumption C3 departs from our previous assumption of a finite state space, but its only purpose is to simplify the statement of our characterization result; $\Omega = [-1, 1]$ is chosen for simplicity and can be replaced by any compact, real-valued interval. Assumption C4 requires densities to be uniformly bounded (in particular, “strong signals” (Milgrom, 1979) are ruled out) and, for simplicity in the statement of results, continuity of $q_\theta(s_\theta | \cdot)$. Assumption C5 is for convenience but can be relaxed. Assumption C6 guarantees that, as the perturbation vanishes, there exist equilibria where the probability of choosing each alternative is bounded away from zero for any voting rule.

We now introduce sequences of voting games. We build such sequences by independently drawing infinite sequences of types $\xi = (\theta_1, \theta_2, ..., \theta_n, ...) \in \Xi$ according to the full-support probability distribution $\phi \in \Delta(\Theta)$; we denote the distribution over $\Xi$ by $\Phi$. We interpret each sequence of types as describing an infinite number of $n$-player games by letting the first $n$ elements of $\xi$ represent the types of the $n$ players.

Let $\alpha$ denote a strategy mapping from sequences of types $\Xi$ to sequences of strategy profiles—i.e., for all $\xi \in \Xi$, let $\alpha(\xi) = (\alpha^1(\xi), ..., \alpha^n(\xi), ...)$, where

$$\alpha^n(\xi) = (\alpha^1_n(\xi), ..., \alpha^n_n(\xi)) \in \prod_{i=1}^n A_{\theta_i}$$

is the strategy profile that is played in the $n$-player game with types $\theta_1, ..., \theta_n$. Let $P^n(\alpha(\xi))$
be the probability distribution over $X^{(n)} \times S^{(n)} \times \Omega$ induced by the strategy profile $\alpha^n(\xi)$ in the $n$-player game, where $X^{(n)} \equiv \prod_{i=1}^n X_i$ and $S^{(n)} \equiv \prod_{i=1}^n S_i$.

We define three properties of strategy mappings. The first property requires that, for large enough $n$, players play strategies that constitute an $\varepsilon$ equilibrium. Our notion of equilibrium will require this property to hold for all $\varepsilon > 0$. This condition is slightly weaker than requiring that strategies constitute an equilibrium. This condition allows us to obtain a full characterization of equilibrium. In particular, our result that an equilibrium is a fixed point of a particular correspondence remains true under the stronger requirement that strategies constitute an equilibrium. But the converse result, that any fixed point is also an equilibrium, relies on the notion of $\varepsilon$ equilibrium.

**Definition 5.** A strategy mapping $\alpha$ is an $\varepsilon$-equilibrium mapping if there exists $n_\varepsilon$ such that for all $n \geq n_\varepsilon$, $i = 1, \ldots, n$, and $s_i \in S_i$,

$$|\alpha^n_i(\xi)(s_i) - F_i(\Delta_i(P^n(\alpha(\xi)), s_i))| \leq \varepsilon$$

for all $\xi \in \Xi$.

The second property requires that the probabilities of choosing A and B remain bounded away from zero as the number of players increases. We will restrict attention to studying equilibrium strategies that satisfy this property since we know that the cases where one of the alternatives is chosen with probability 1 always constitute a trivial equilibrium.\footnote{Note that a fixed payoff perturbation does not preclude an alternative from being chosen with probability that goes to 1 as the number of players increases.}

**Definition 6.** A strategy mapping $\alpha$ is $\Xi'$-asymptotically interior if

$$\liminf_{n \to \infty} P^n(\alpha(\xi)) (o = A) > 0$$

and

$$\limsup_{n \to \infty} P^n(\alpha(\xi)) (o = A) < 1$$

a.s. $-\Xi'$.

The final property specifies that, as the number of players increases, the probability that the committee chooses A goes to 1 for states above a cutoff and goes to zero for states below it. We will show that equilibria can be characterized by this convenient property.
Definition 7. A strategy mapping $\alpha$ is $\Xi'$-asymptotically $c$-cutoff if there exists $c \in (-1, 1)$ such that

$$\lim_{n \to \infty} P^n(\alpha(\xi)) (o = A \mid \omega) = \begin{cases} 1 & \text{for } \omega > c \\ 0 & \text{for } \omega < c \end{cases}$$

a.s. $-\Xi'$.

In addition to characterizing the equilibrium $c$-cutoff, our objective is to characterize the entire profile of equilibrium strategies. A complete characterization of equilibrium strategies is cumbersome due to the nature of the equilibrium object: As the number of players increases, the dimension of $\alpha^n$ also increases. We overcome this inconvenience by characterizing the limit, as the number of players increases, of the average strategy chosen by each type of player. But, unlike most of the related literature, we do not a priori restrict players of the same type to following the same strategy.

For a given strategy mapping $\alpha$ and a sequence of types $\xi \in \Xi$, let $\sigma^n(\xi; \alpha) = (\sigma^n_\theta(\xi; \alpha))_{\theta \in \Theta} \in \mathcal{A}^n \equiv \prod_{\theta \in \Theta} A_\theta$ denote the average strategy profile played by players of each type $\theta$ in the $n$-player game with types $(\theta_1, ..., \theta_n)$ and strategy profile $\alpha^n(\xi)$. Formally,

$$\sigma^n_\theta(\xi; \alpha)(s_i) = \frac{\sum_{i=1}^n \{\theta_i(\xi) = \theta\} \alpha^n_i(\xi)(s_i)}{\sum_{i=1}^n \{\theta_i(\xi) = \theta\}} \in A_\theta$$

whenever $\sum_{i=1}^n \{\theta_i(\xi) = \theta\} > 0$, and arbitrary otherwise. We say that an average strategy profile $\sigma$ is increasing if for each type $\theta \in \Theta$, $s_\theta' > s_\theta$ implies $\sigma_\theta(s_\theta') > \sigma_\theta(s_\theta)$.

Definition 8. An average strategy profile $\sigma \in \mathcal{A}^n$ is a limit $\varepsilon$-equilibrium if there exists $\alpha$ and $\Xi'$ with $\Phi(\Xi') > 0$ such that:

1. $\alpha$ is an $\varepsilon$-equilibrium mapping
2. $\alpha$ is $\Xi'$-asymptotically interior
3. $\lim_{n \to \infty} \|\sigma^n(\xi; \alpha) - \sigma\| = 0$ for all $\xi \in \Xi'$

If, in addition, $\alpha$ is $\Xi'$-asymptotically $c$-cutoff, then $\sigma$ is a $c$-cutoff limit $\varepsilon$-equilibrium.

Definition 9. An average strategy profile $\sigma \in \mathcal{A}^n$ is a $[c$-cutoff] limit equilibrium if it is a $[c$-cutoff] limit $\varepsilon$-equilibrium for all $\varepsilon > 0$. 

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6 Characterization of equilibrium in large games

We first characterize equilibrium for a fixed perturbation structure as the number of players goes to infinity, and we then characterize equilibrium as the perturbation vanishes.

6.1 Limit equilibrium

The intuition behind the characterization results can be grasped by thinking about a voting game with a continuum of players. For a given average strategy profile \( \sigma \in A^* \), we interpret

\[
\kappa (\sigma \mid \omega) = \sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_{\theta} (s_\theta \mid \omega) \sigma_{\theta} (s_\theta)
\]

as the proportion of players that vote for A conditional on \( \omega \).

For any \( c \in (-1, 1) \), let

\[
v_\theta (s_\theta; c) \equiv E \left( u_\theta (A, \omega) \mid \omega > c, s_\theta \right) - E \left( u_\theta (B, \omega) \mid \omega < c, s_\theta \right)
\]

denote the expected difference in observed utility of type \( \theta \) from alternatives A and B, conditional on signal \( s_\theta \) and conditional on observing the payoff of A whenever \( \omega > c \) and the payoff of B whenever \( \omega < c \).

We will show that if \( \sigma \) is a limit equilibrium, there exists \( c(\sigma) \in (-1, 1) \) such that

\[
\kappa (\sigma \mid \omega) \begin{cases} > \\ < \end{cases} \rho \text{ if } \omega \begin{cases} > \\ < \end{cases} c(\sigma),
\]

so that alternative A is chosen in states \( \omega > c(\sigma) \) and alternative B is chosen in states \( \omega < c(\sigma) \). We can, therefore, interpret \( v_\theta (s; c(\sigma)) \) as type \( \theta \)'s belief about the difference in expected payoff from A and B in a limit equilibrium \( \sigma \). A limit equilibrium average strategy of type \( \theta \) will, therefore, satisfy

\[
\sigma_{\theta} (s_\theta) = F_{\theta} (v_\theta (s_\theta; c(\sigma))).
\]

Finally, \( c(\sigma) \) will be the solution to \( \kappa(\sigma \mid c(\sigma)) = \rho \), implying that \( c(\sigma) = c^* \) for any limit

\footnote{Of course, if the game were really one of a continuum of players, then each player would be pivotal with probability zero and anything would constitute an equilibrium.}
equilibrium $\sigma$, where $c^*$ is the unique solution to\(^{27}\)
\[
\sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_\theta (s_\theta \mid c^*) F_\theta(v_\theta(s_\theta; c^*)) = \rho. \tag{19}
\]

Therefore, equilibrium strategies and outcomes can be characterized from knowledge of the primitives by using (18) and (19) above. The remainder of the section shows that the above claims, inspired by thinking about a game with a continuum of players, are formally correct in the limit as the number of players goes to infinity.

Lemma 2. There exists $\varepsilon > 0$ and $\gamma(\varepsilon)$ with $\lim_{\varepsilon \to 0} \gamma(\varepsilon) = 0$ such that for all $\varepsilon < \varepsilon$: if $\sigma$ is a limit $\varepsilon$-equilibrium, then (i) $\sigma$ is increasing and is a $c(\sigma)$-cutoff limit $\varepsilon$-equilibrium, where
\[
\kappa(\sigma \mid c(\sigma)) = \rho, \tag{20}
\]
and (ii) for all $\theta \in \Theta$ and $s_\theta \in S_\theta$,
\[
|\sigma_\theta(s_\theta) - F_\theta(v_\theta(s_\theta; c(\sigma)))| \leq \gamma(\varepsilon) \tag{21}
\]

We now provide a discussion and proof of Lemma 2, relegating some of the details to the Appendix. The proof relies on the following Lemma.

Lemma 2.1. Suppose that there exists $\alpha$ and $\Xi'$ with $\Phi(\Xi') > 0$ such that $\alpha$ is $\Xi'$-asymptotically interior and for all $\xi \in \Xi'$
\[
\lim_{n \to \infty} ||\sigma^n(\xi; \alpha) - \sigma|| = 0,
\]
where $\sigma$ is increasing. Then, $\alpha$ is $\Xi'$-asymptotically $c(\sigma)$-cutoff, where $\kappa(\sigma \mid c(\sigma)) = \rho$, and for all $i, s_i$,
\[
\lim_{n \to \infty} \Delta_i(P^n(\alpha(\xi)), s_i) = v_{\theta_i}(s_i; c(\sigma)) \tag{22}
\]
almost surely in $\Xi'$.

Proof. See the Appendix. \(\square\)

\(^{27}\) The expression in the LHS of equation (19) is increasing in $c^*$; see Claim 4.1 in the appendix.
The intuition of the proof is as follows. The assumption that \( \sigma^n(\xi; \alpha) \) converges to \( \sigma \) implies, for a given \( \omega \), that the probability that a randomly chosen player votes for A converges to \( \kappa(\sigma|\cdot) \). By standard asymptotic arguments, the proportion of votes for A becomes concentrated around \( \kappa(\sigma|\omega) \). So, for states where \( \kappa (\sigma \mid \omega) > \rho \), the probability that the outcome is A converges to 1. Similarly, for states where \( \kappa (\sigma \mid \omega) < \rho \), the probability that the outcome is A converges to 0. Finally, we cannot determine what happens to the probability of choosing A for boundary states such that \( \kappa (\sigma \mid \omega) = \rho \), but this is irrelevant since, by the assumption that \( \sigma \) is increasing, there is, at most, one (measure zero) boundary state.

The main challenge of the proof of Lemma 2 is being able to apply Lemma 2.1 by first showing that, indeed, players’ equilibrium strategies are increasing in games with sufficiently many players. This challenge would not arise if voters were playing a Nash equilibrium since, under assumptions C1 and C2, players’ strategies would always be increasing (recall that payoffs are perturbed in such a way that no one votes for an alternative with probability 1). What is different in our setting is that players’ beliefs about which alternative is best does not depend only on a player’s signal and the strategies of other players, but also on a player’s own strategy. To understand the main issue, fix a player and a signal and suppose that she votes for A with probability close to 1. Then, most often, A is the outcome of the election whenever at least \( k - 1 \) or more of the other players have voted for A. Now suppose that the player votes for B with probability close to 1. Then, most often, A is the outcome of the election whenever at least \( k \) or more of the other players have voted for A. If players choose nondecreasing strategies, by MLRP, the second event conveys more favorable information about A. Therefore, the difference in expected payoffs will be decreasing in a player’s own strategy. This effect goes in a direction that is opposite from the effect that a higher signal makes voting for A more desirable. The key of the next result is that for \( n \) sufficiently large, the second effect dominates the first.

**Lemma 2.2.** There exists \( \varepsilon \) such that for all \( \varepsilon < \varepsilon \): If \( \sigma \) is a limit \( \varepsilon \)-equilibrium, then it is increasing.

**Proof.** See the appendix.

The key of the proof is to show that the probability that a player becomes pivotal goes to zero as \( n \) increases. Given this result, the effect of a player’s own strategy on her own learning must eventually vanish and become dominated by the effect of her signal (provided a uniform version of the strict MLRP holds). The proof that players become pivotal with
vanishing probability relies on the assumption that there is a payoff perturbation that bounds away from zero the probability that each individual player votes for A and B. The randomness in players’ votes allows us to apply the central limit theorem to show that the proportion of players that votes for A has a limiting distribution that is continuous, and, hence, the probability that there is any specific number of votes for A must go to zero.\footnote{It is easy to see how the result that the probability of being pivotal vanishes would fail if the variance were zero: For example, suppose that \( n \) is even, voting is by majority rule, and half of the players vote for A and half vote for B. Then, each player is pivotal with probability 1, for all \( n \).}

Proof of Lemma 2: Let \( \varepsilon \leq \varepsilon \), where \( \varepsilon \) is defined by Lemma 2.2. Suppose that \( \sigma \) is a limit \( \varepsilon \)-equilibrium with corresponding \( \varepsilon \)-equilibrium mapping \( \alpha \) and convergence in a set \( \Gamma' \). By Lemma 2.2, \( \sigma \) is increasing. Therefore, all the hypotheses of Lemma 2.1 are satisfied, implying that \( \lim_{n \to \infty} \Delta_i(P^n(\alpha(\xi)), s_i) = v_{\theta_i}(s_i; c(\sigma)) \) a.s.-\( \Gamma' \) and, by continuity of \( F_{\theta_i} \) (assumption C5(i)), that \( \lim_{n \to \infty} F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i)) = F_{\theta_i}(v_{\theta_i}(s_i; c(\sigma))) \) a.s.-\( \Gamma' \). Therefore, there exists \( n_\varepsilon \) such that for all \( n \geq n_\varepsilon \), all \( i, s_i \)

\[
|\alpha_i^n(\xi)(s_i) - F_{\theta_i}(v_{\theta_i}(s_i; c(\sigma)))| \leq |\alpha_i^n(\xi)(s_i) - F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i))| + |F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i)) - F_{\theta_i}(v_{\theta_i}(s_i; c(\sigma)))| \leq 2\varepsilon
\]
a.s.-\( \Gamma' \), where for the first term in the RHS, we have used the fact that \( \alpha \) is an \( \varepsilon \)-equilibrium mapping. Moreover, the previous inequality and equation (16) imply that for all \( n \geq n_\varepsilon \), all \( \theta, s_\theta \),

\[
|\sigma^n_\theta(\xi; \alpha)(s_\theta) - F_\theta(v_\theta(s_\theta; c(\sigma)))| \leq 2\varepsilon.
\]

Finally, the previous result and the fact that \( \lim_{n \to \infty} \sigma^n(\xi; \alpha) = \sigma \) for all \( \xi \in \Gamma' \) imply that there exists \( n'_\varepsilon \geq n_\varepsilon \) such that for \( n \geq n'_\varepsilon \), all \( \theta, s_\theta \),

\[
|\sigma_\theta(s_\theta) - F_\theta(v_\theta(s_\theta; c(\sigma)))| \leq |\sigma_\theta(s_\theta) - \sigma^n_\theta(\xi; \alpha)(s_\theta)| + |\sigma^n_\theta(\xi; \alpha)(s_\theta) - F_\theta(v_\theta(s_\theta; c(\sigma)))| \leq 3\varepsilon.
\]

Lemma 2 then follows by letting \( \gamma(\varepsilon) = 3\varepsilon \). \( \square \)
To conclude this section, we use Lemma 2 to show that the set of limit equilibria has a convenient characterization.

**Theorem 3.** If $\sigma$ is a limit equilibrium, then it is an increasing, $c^*$-cutoff limit equilibrium, where $c^*$ solves equation (19), and equation (18) is satisfied for all $\theta \in \Theta$ and $s_\theta \in S_\theta$. If, on the other hand, $\sigma$ satisfies equation (18) for all $\theta \in \Theta$ and $s_\theta \in S_\theta$, where $c(\sigma) = c^* \in (-1, 1)$ solves equation (19), then $\sigma$ is a limit equilibrium.

**Proof.** Part 1. Let $\sigma$ be a limit equilibrium, so that $\sigma$ is a limit $\varepsilon$-equilibrium for all $\varepsilon > 0$. Lemma 2 implies that (a) $\sigma$ is increasing, (b) $\sigma$ is a $c(\sigma)$-cutoff limit $\varepsilon$-equilibrium for all $\varepsilon > 0$, where equation (20) is satisfied, and (c) for all $\varepsilon \geq \varepsilon > 0$, for all $\theta \in \Theta$ and $s_\theta \in S_\theta$, $|\sigma_\theta(s_\theta) - F_\theta(v_\theta(s_\theta;c(\sigma)))| \leq \gamma(\varepsilon)$. Since the LHS of the inequality in part (c) does not depend on $\varepsilon$ and $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, then $|\sigma_\theta(s_\theta) - F_\theta(v_\theta(s_\theta;c(\sigma)))| = 0$, thus establishing equation (18). Equation (19) follows by replacing equation (18) into equations (17) and (20).

Part 2. Consider the strategy mapping $\alpha$ defined by letting players of type $\theta$ always play $\sigma_\theta$-i.e., $\alpha_i(\xi)(s_i) = \sigma_{\theta_i}(s_{\theta_i})$ for all $\xi$, all $i$. First, note that $\sigma^n = \sigma$ converges trivially to $\sigma$, and $\sigma$ is increasing because it satisfies equation (18) and by claim 3.1(ii) and assumption C4(iv). Therefore, we can follow the proof leading to equation (36) in the Appendix to obtain that $\lim_{n \to \infty} P^n(\xi)(\omega = A|\omega) = \mathbb{1}\{\omega < c^*\}$ a.s.-$\Xi$. The dominated convergence theorem and the fact that $c^* \in (-1, 1)$ implies that $\lim_{n \to \infty} P^n(\xi)(\omega = A) \in (0, 1)$, and, thus, $\alpha$ is $\Xi$-asymptotically interior. Therefore, we can apply Lemma 2.1 to obtain $\lim_{n \to \infty} \Delta_i(P^n(\alpha(\xi)), s_i) = v_{\theta_i}(s_i;c^*)$ a.s.-$\Xi$. By continuity of $F_{\theta_i}$ (assumption C5(i)), $\lim_{n \to \infty} F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i)) = F_{\theta_i}(v_{\theta_i}(s_i;c^*))$. Therefore, for $\varepsilon > 0$, there exists a $n_\varepsilon$ such that for $n \geq n_\varepsilon$, all $i$, $s_i$

$$|\alpha^n_i(\xi)(s_i) - F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i))| = |\sigma_{\theta_i}(s_i) - F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i))|$$

$$= |F_{\theta_i}(v_{\theta_i}(s_i;c^*)) - F_{\theta_i}(\Delta_i(P^n(\alpha(\xi)), s_i))| < \varepsilon$$

a.s.-$\Xi$.

**6.2 Vanishing perturbations**

Up to this point, we have defined equilibrium in games where players’ payoffs are independently perturbed every period. While the perturbation may have a real interpretation in some contexts, we now consider sequences of equilibria where the perturbation goes to zero. To do so, we now index games by a parameter $\eta$ that affects the distribution $F^n$ from which
perturbations are drawn. We assume that $F^n_\eta$ satisfies the same assumptions as in Section 3 for all $\eta$ and, in addition, require that perturbations vanish as $\eta$ goes to zero.

**Definition 10.** A family of perturbations $\{F^n_\eta\}_\eta$, where $F^n_\eta = \{F^n_\eta^\theta\}_{\theta \in \Theta}$, is feasible if it satisfies the following assumptions for all $\theta \in \Theta$ and $\eta$: $F^n_\eta^\theta$ is absolutely continuous, $F^n_\eta^\theta(-2K) > 0$ and $F^n_\eta^\theta(2K) < 1$, and

$$
\lim_{\eta \to 0} F^n_\eta^\theta(v) = \begin{cases} 0 & \text{if } v < 0 \\ 1 & \text{if } v > 0 \end{cases}
$$

(23)

By Theorem 3, all limit equilibria can be characterized by the same cutoff, which solves equation (19). Let $c^n$ denote the limit equilibrium cutoff that solves equation (19) when perturbations are drawn from $F^n$.

**Definition 11.** $c$ is a perfect limit equilibrium cutoff if it is the limit of a sequence of limit equilibrium cutoffs $\{c^n_\eta\}$ for some feasible family $\{F^n_\eta\}_\eta$.

The final result of this section characterizes the set of perfect equilibrium cutoffs. Define

$$
c_\theta(s_\theta) = \arg \min_{c \in \Omega} |v_\theta(s_\theta; c)|,
$$

(24)

and note that there is a unique solution $c_\theta(s_\theta)$ that is decreasing in $s_\theta$ (because $\Omega$ is compact, $v_\theta(s_\theta; \cdot)$ is continuous, and strict MLRP holds). For each cutoff outcome $c \in \Omega$,

$$
\kappa(c) \equiv \sum_{\theta \in \Theta} \phi_\theta q_\theta (\{s \in S_\theta : c_\theta(s) < c\} | c)
$$

may be interpreted as the proportion of players that vote for A conditional on state $c$, as the perturbation vanishes.\(^{29}\)

**Lemma 3.** $\kappa : \Omega \to [0, 1]$ is weakly increasing and satisfies

$$
\kappa(c) = \begin{cases} 0 & \text{for } c \begin{cases} < \min_\theta c_\theta(s_\theta^H) \\ > \max_\theta c_\theta(s_\theta^L) \end{cases} \\ 1 & \text{otherwise} \end{cases},
$$

where $-1 < \min_\theta c_\theta(s_\theta^H) < \max_\theta c_\theta(s_\theta^L) < 1$.

\(^{29}\)The interpretation is correct unless $c$ is one of the cutoffs $c_\theta(s_\theta)$ for some $\theta, s_\theta$. 30
Figure 4: Characterization of perfect equilibrium cutoffs.

**Proof.** See the Appendix.

Figure 4 depicts the function $\pi$ for two different sets of primitives of a game. The function in panel (a) is strictly increasing and corresponds to an example with only one type, while the function in panel (b) has a flat segment and corresponds to an example with two types. In panel (a), the perfect equilibrium cutoff is given by the state $c^*$ where $\pi$ intersects the voting rule $\rho$. In panel (b), the perfect equilibrium cutoff lies in the segment where $\pi$ intersects $\rho$; the particular point in the segment depends on the particular family of perturbation that we take.

**Theorem 4.** For a game with voting rule $\rho$, the set of perfect equilibrium cutoffs is given by\(^3\)

$$C_{eqm}(\rho) \equiv \left[ \inf_c \{\pi(c) \geq \rho\}, \sup_c \{\pi(c) \leq \rho\} \right].$$

**Proof.** See the Appendix.

\(^3\)By convention, let $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

7 Information aggregation

In this section, we consider a voting stage game with a large number of players where the perturbation goes to zero. We maintain assumptions C1-C5 and apply the characterization results in Section 6 to obtain necessary and sufficient conditions for information aggregation.
and to characterize optimal voting rules. In particular, we provide a new rationale for major- 
votin rule. We then present examples that illustrate the results and provide additional insights 
into the conditions for information aggregation. Finally, we discuss how our results extend to 
the case where naive players coexist with Nash players.

We carry out our analysis from the perspective of a social planner who wants to maximize 
players’ welfare. Let $W(c)$ denote aggregate welfare when the outcome follows a cutoff rule $c$, 
so that alternative A is chosen for $\omega > c$ and B is chosen for $\omega < c$. We assume that 
$W(c') > W(c)$ for $0 < c' > c$ and $W(c') < W(c)$ for $c' > c > 0$. Thus, $W(c)$ is single-peaked 
at $c = 0$ and strictly decreases as $c$ either increases or decreases away from $c = 0$.\(^{31}\)

**Definition 12.** A voting rule $\rho^\star$ is **optimal** if there exists $c^\star \in C_{eqm}(\rho^\star)$ such that 

$$W(c^\star) \geq W(c)$$

for all $c \in \cup_{0<\rho<1}C_{eqm}(\rho)$. A voting rule $\rho^\star$ **aggregates information** if $0 \in C_{eqm}(\rho^\star)$. **Informa-
**tion is said to be **aggregated** if there exists a voting rule $\rho^\star$ that aggregates information.

Feddersen and Pesendorfer (1997) show that if the solution concept is Nash equilibrium 
and if the planner’s preferences coincide with the preferences of the median (or any other 
percentile) voter, then the first-best outcome can be achieved with majority voting rule (or 
the corresponding percentile voting rule).\(^{32}\) In our context, information may or may not 
be aggregated, depending on the primitives. The next result, which follows immediately 
from Theorem 4 and the characterization of $\pi$ in Lemma 3, provides necessary and sufficient 
conditions on the primitives such that there exists a voting rule that aggregates information.

**Proposition 1.** Information is aggregated by naive voters if and only if 

$$\min_\theta c_\theta(s^H_\theta) \leq 0 \leq \max_\theta c_\theta(s^L_\theta). \quad (25)$$

\(^{31}\)The welfare function $W(c)$ is fairly general and consistent, for example, with the objective of maximizing 
a weighted average of players’ utility. The assumption that $c = 0$ is the optimal cutoff is only for simplicity; 
the important assumption is that the optimal cutoff is interior.

\(^{32}\)Feddersen and Pesendorfer (1997) state their main result in terms of what they call full information 
equivalence, meaning that for any voting rule $\rho$, the (Nash equilibrium) outcome of an election coincides 
with the outcome that would be chosen by the $\rho$-median voter if the state were known by all voters. In our 
context, full-information equivalence need not hold; therefore, we focus on finding rules that achieve the “best” 
outcome—hence the need to introduce the notion of a planner. Of course, Proposition 1 can be reinterpreted 
as providing conditions such that full information equivalence obtains given rule $\rho$ by replacing the optimal 
cutoff 0 with the cutoff of the $\rho$-median voter.
What makes information aggregation difficult is that players’ beliefs do not depend on their equilibrium strategies once we assume that the outcome is the first-best outcome. In contrast, in a Nash equilibrium, beliefs depend on the event that a player is pivotal; even conditional on the first-best outcome, the pivotal event conveys information that depends on players’ equilibrium strategies.

To see intuitively why (25) is necessary, suppose that \( \max_{\theta} c_{\theta}(s^L_{\theta}) < 0 \), as in Figure 5(a). If information were aggregated, then even after observing their lowest signal, all types would prefer to vote for A. But the fact that no one votes for B contradicts the assumption that information is aggregated in the first place.

The next result, also an immediate implication of Theorem 4 and Lemma 3, provides a characterization of optimal voting rules. Let \( \theta^0 \equiv \arg \max_{\theta} c_{\theta}(s^L_{\theta}) \) and \( \theta_0 \equiv \arg \min_{\theta} c_{\theta}(s^H_{\theta}) \).

**Proposition 2.** Suppose that information is not aggregated. If \( \max_{\theta} c_{\theta}(s^L_{\theta}) < 0 \), then voting rule \( \rho \) is optimal if and only if

\[
\rho \geq 1 - \phi_{\theta^0} \cdot q_{\theta^0} \left( s^L_{\theta^0} \mid c_{\theta^0}(s^L_{\theta^0}) \right);
\]

if \( \min_{\theta} c_{\theta}(s^H_{\theta}) > 0 \), then voting rule \( \rho \) is optimal if and only if

\[
\rho \leq \phi_{\theta_0} \cdot q_{\theta_0} \left( s^H_{\theta_0} \mid c_{\theta_0}(s^H_{\theta_0}) \right).
\]
Suppose that information is aggregated. Then, voting rule \( \rho \) is optimal if and only if

\[
\rho \in \left[ \lim_{c \to 0^-} \overline{\kappa}(c), \lim_{c \to 0^+} \overline{\kappa}(c) \right].
\]

To understand part of the intuition behind Proposition 2, suppose that \( \max_{\theta} c_{\theta}(s^L_{\theta}) < 0 \), so that information is not aggregated (see Figure 5(a)). As argued above, the reason why information is not aggregated is that, if it were, everyone would prefer to vote for A, irrespective of their signal. How can we provide incentives so that some type votes for B with positive probability? Clearly, we do so by having the committee occasionally make a mistake and choose A in states of the world where B would have been best; such mistakes make B more attractive to players. But mistakes carry a welfare cost. The lowest level of this mistake that still provides incentives for some type to play B is the mistake that makes the type with the highest \( c_{\theta}(s^L_{\theta}) \), defined as type \( \theta^0 \), indifferent between A and B when she observes her lowest signal. Given such indifference, there is at least a proportion \( 1 - \phi_{\theta^0} \cdot q_{\theta^0}(s^L_{\theta^0} \mid c_{\theta^0}(s^L_{\theta^0})) \) of players who would vote for A conditional on \( c_{\theta^0}(s^L_{\theta^0}) \) being an equilibrium cutoff. But then, the voting rule must be higher than the previous proportion if B is to be the outcome with positive probability. In addition, voting rules that require a lower proportion to choose A also require a larger mistake in order to induce more people to vote for B, so that both A and B are chosen in equilibrium. Since larger mistakes are associated with lower welfare, such voting rules are not optimal.

Our final result provides a novel justification for optimality of majority rule: If information is sufficiently accurate, then majority rule is optimal in symmetric settings where there is only one type of player.

**Definition 13.** Information is *sufficiently accurate* if there exist signals \( s \neq s' \) such that

\[
q(s \mid \omega) > 1/2 \quad \text{for } \omega > 0
\]

and

\[
q(s' \mid \omega) > 1/2 \quad \text{for } \omega < 0.
\]

The notion of signals being sufficiently accurate can be related to Condorcet’s initial praise for majority rule. Condorcet (1785) argued that, if each player votes for the right alternative with probability greater than one-half, then, as the number of players increases, the probability...
that the committee makes the right decision goes to 1. Translated to the voting context, the behavioral assumption in Condorcet’s result is true whenever signals are sufficiently accurate and players vote for A after observing signal s and vote for B given $s'$. In our case, voting behavior is derived endogenously in equilibrium, and it is not necessarily true that players vote in the previous manner or that information gets aggregated. Nevertheless, majority rule is still optimal.

**Proposition 3.** Consider a symmetric voting game where information is sufficiently accurate. Then, majority rule is optimal.

**Proof.** Strict MLRP and the assumption that information is sufficiently accurate imply that the signals that satisfy (26) and (27) are the highest $s^H = s$ and lowest $s^L = s'$ signals, respectively. First, consider the case where information is aggregated, so that 0 is a perfect cutoff equilibrium and Proposition 1 implies $c(s^H) \leq 0 \leq c(s^L)$. Let $c > 0 \geq c(s^H)$: then, $\pi(c) \geq q(s^H | c) > 1/2$, where the inequality follows from (26). Similarly, let $c < 0 \leq c(s^L)$: then, $\pi(c) \leq 1 - q(s^L | c) < 1/2$, where the inequality follows from (27). Proposition 2 then implies that $\rho = 1/2$ is optimal.

Finally, consider the case where information is not aggregated. If $c(s^H) > 0$, then, by (26), $q(s^H | c(s^H)) > 1/2$. Proposition 2 then implies that $\rho = 1/2$ is optimal. Similarly, if $c(s^L) < 0$, then, by (27), $1 - q(s^L | c(s^L)) < 1/2$. Proposition 2 then implies that $\rho = 1/2$ is optimal. \hfill \Box

### 7.1 Examples

The following examples illustrate Propositions 1-3 and provide additional insights into how the payoff and information structure relates to information aggregation. For simplicity, we discuss only examples where all players are symmetric (i.e., there is only one type); our results can also be applied to obtain additional insights when players are asymmetric.\(^{33}\)

First, suppose that

$$\inf_{\omega > 0} u(A, \omega) > \sup_{\omega < 0} u(B, \omega), \quad (28)$$

\(^{33}\)For example, information aggregation may be aggregated in a status quo setup when players strongly disagree about the states in which one alternative is better than the other. Thus, diversity of preferences may facilitate information aggregation. In addition, optimal voting rules will be biased against the preferences of the largest types. But if types with opposite preferences are similar in size, majority rule may again be optimal.
so that alternative A dominates B when restricted to states of the world where each alternative is best. Then, \( v(0, s) > 0 \) for all \( s \), implying that \( c(s^L) < 0 \) and, therefore, by Proposition 1, that information cannot be aggregated: If it were, then no one would like to vote for B.\(^{34}\)

For the remainder of this section, we consider a less extreme example where information aggregation is determined not only by the relative payoffs of making correct choices, but also by the informativeness of the signals. The state \( \omega \in [-1, 1] \) is drawn from the uniform distribution and there are two signals, \( \{s^L, s^H\} \), with

\[
q(s^H|\omega) = \begin{cases} 
(0.5 + r_1\omega)^{1/r_2} & \text{if } \omega > 0 \\
(0.5 + r_1\omega)^{r_2} & \text{if } \omega < 0 
\end{cases}
\]  

(29)

Utility functions are

\[
u_A(\omega) = \begin{cases} 
\omega^3 & \text{if } \omega \geq 0 \\
\omega^3 - h & \text{if } \omega < 0 
\end{cases}
\]

and \( u_B(\omega) = -0.5\omega^3 \). Hence, alternative A does better than B, on average, but (28) does not hold.

We will vary the parameters \( r_1 \in (0, 0.5) \), \( r_2 \in [1, \infty) \) and \( h \geq 0 \) in order to emphasize different points. Suppose that the social planner has the same preferences as the players, so that the first-best cutoff is \( c^* = 0 \) and first-best welfare is consequently given by \( W^{FB} = W(0) \). The (percentage) loss function \( L(c) = (W^{FB} - W(c))/W^{FB} \) measures the percentage by which welfare deviates from the first best.

(i) Correct payoffs and informativeness of signals. Let \( r_2 = 1 \), so that \( q(s^H | \cdot) \) is linear and continuous. At one extreme, \( r_1 \approx 0 \), and the signal is almost uninformative about the state. Since

\[
E(u(A, \omega) \mid \omega > 0) > E(u(B, \omega) \mid \omega < 0),
\]

(30)

information cannot be aggregated. At the other extreme, \( r_1 \approx 0.5 \) and signals are fairly informative. Conditional on \( \omega > 0 \), signal \( s^L \) puts a larger weight on states near 0; conditional on \( \omega < 0 \), signal \( s^L \) puts a higher weight on states near -1. Therefore, we may expect

\[
E(u(A, \omega) \mid \omega > 0, s^L) < E(u(B, \omega) \mid \omega < 0, s^L),
\]

implying that \( c(s^L) > 0 \) and, by Proposition 1, that information gets aggregated. In fact, there

\(^{34}\)An example that satisfies (28) is the case where B is a status quo option with a payoff that does not depend on the state of the world—i.e., \( u(A, \omega) > u(B) \) for all \( \omega > 0 \).
exists $r_1^* = .41$, which is the solution of $c(s^L, r_1^*) = 0$ (see equation 24), such that: for $r_1 < r_1^*$, $c(s^L, r_1) < 0$ and information is not aggregated; for $r_1 > r_1^*$, $c(s^L, r_1) > 0$ and information is aggregated. In the second case, signal $s^L$ puts a larger weight on states such that B is more successful than A (conditional on making correct choices) and, therefore, makes players willing to vote for B under $s^L$. This example suggests that information aggregation obtains, provided that correct payoffs are not too far from each other, that correct payoffs vary in intensity depending on the state, and that there are signals that detect this variation.

(ii) Optimal voting rules. Let’s continue to suppose that $r_2 = 0$ and let’s now fix $h = 0$. Consider, first, the case where $r_1 < r_1^*$, so that information is not aggregated. Figure 5(a) illustrates that the best possible equilibrium outcome is $c(s^L) < 0$, and this outcome is obtained with voting rules $\rho \geq 1-q(s^H \mid c(s^L))$. In particular, (29) and the fact that $c(s^L) < 0$ imply that majority rule, $\rho = 1/2$, aggregates information.

Consider, next, the case where $r_1 > r_1^*$, so that information is aggregated. Since $\kappa$ is continuous and $\kappa(0) = q(s^H \mid 0) = 1/2$, Proposition 2 (see, also, Figure 5(b)) implies that majority rule is the unique optimal voting rule. Taken together, these two cases illustrate optimality of majority rule in symmetric environments (Proposition 3).

Next, we show that choosing the wrong voting rule can substantially reduce welfare in those cases where there exists a rule that aggregates information. By Theorem 4 (see, also, Figure 5(a)), the worst equilibrium outcome is given by $c(s^H) < 0$. We now compute (loss of) welfare under this worst outcome for the two extreme cases $r_1 \approx 0$ and $r_1 = 0.5$. In the first case, the signal is not informative and $c(s^H) \approx c(s^L) \approx -.33$; therefore, all voting rules lead to similar equilibrium welfare loss of $L(-.33) = .26$, or 26% of the first-best welfare. In the case where $r_1 = 0.5$, we obtain $c(s^H) = -.63$ and $L(-.63) = .95$, so that a welfare loss of 95% results from choosing the worst voting rules (compared to no welfare loss from choosing the voting rule that aggregates information).

(iii) Type I errors. So far, the magnitude of the type I error has not played an explicit role. One may conjecture that in cases where information is not aggregated, a large payoff penalty for errors translates into a higher equilibrium cost of making wrong decisions. Nevertheless, we show that any effect of a larger type I error gets mitigated in equilibrium. The idea is that, by making mistakes costlier, a larger type I error makes it easier to provide incentives to those who obtain the lowest signal to vote for B. Thus, a higher cost of making mistakes is mitigated by a corresponding lower probability of making mistakes in equilibrium. To illustrate, suppose

---

35Note that $h$ does not affect the threshold of information aggregation, $r_1^*$. 6
that $r_2 = 1$ and $r_1 = .05$. Then, $L(c(s^L; h = 0)) = .23$ and $\lim_{h \to \infty} L(c(s^L; h)) = .29$. Hence, despite the cost of the type I error going to infinity, welfare loss in an optimal equilibrium increases only from 23% to 29%.

(iv) Between vs. within informativeness. We compare two notions of informativeness of a signal. First, fix $r_1 \approx 0$ and note that as $r_2$ increases, the signals become increasingly good at distinguishing between the events that A is best and B is best—i.e., $\{\omega > 0\}$ and $\{\omega < 0\}$; in the limit as $r_2$ approaches infinity, the signals become fully revealing. Second, fix $r_2 = 1$ and note that as $r_1$ increases, the signals are never fully revealing, but, within each of the events $\{\omega > 0\}$ and $\{\omega < 0\}$, they increasingly distinguish the high from the low states. Above, we showed that in this second case, there is a cutoff $r_1^*$ above which information is aggregated. We now show that in the first case, even for very large values of $r_2$, information fails to aggregate. To see this, let $r_1 \approx 0$ and take $r_2 \to \infty$. Then, $q(s_L|\omega) \approx 1$ for $\omega < 0$ and $q(s_L|\omega) \approx 0$ for $\omega > 0$; within each of these two events the signal function is almost flat and, therefore, pretty uninformative. Therefore, $E(u(A, \omega)|\omega > 0, s_L) \approx E(u(A, \omega)|\omega > 0)$ and $E(u(B, \omega)|\omega < 0, s_L) \approx E(u(B, \omega)|\omega < 0)$. Equation (30) then implies that information cannot be aggregated.\footnote{The above is true if $r_1 = 0$; for $r_1 > 0$ but small, $r_2$ has to be substantially large for information to be aggregated.} This example reinforces the point made in (i) above: For information aggregation to obtain, the key is not so much to have signals that are very good at distinguishing whether A or B is the right alternative, but, rather, to have signals that sufficiently distinguish between states where an alternative is best by a wide margin and states where it is best by a narrow margin.

7.2 Coexistence of naive and Nash players

We now illustrate how our results extend in the presence of a small fraction of Nash players who both understand the selection problem and can somehow perfectly account for it. Of course, as discussed in Section 4.5, the presence of Nash players may or may not be justified, depending on the setting.

First, consider a case where information is not aggregated in the presence of naive players, as in Figure 6(a). If a fraction $\gamma \approx 0$ of players is Nash and the remaining fraction $1 - \gamma$ is naive, the $\pi(c)$ function shifts proportionally downward by $(1 - \gamma)$ for $c < 0$ and remains at 1 for $c > 0$. The reason is that naive players behave as usual, but Nash players now vote conditional on the belief that they are pivotal. Being pivotal at a hypothetical cutoff equilibrium $c < 0$ implies that they can almost perfectly infer that the state is lower than
zero; hence, for $c < 0$, Nash players vote for B irrespective of their signal. Similarly, for $c > 0$, Nash players always vote for A. The implications are the following. For most voting rules $\rho$, equilibrium with naive players is robust to a small introduction of Nash players. However, for rules $\rho > (1 - \gamma)$, equilibrium shifts from $c(s^L)$ to $c^* = 0$. We know this is true because it is a particular result in Fedderson and Pesendorfer (1997): We can interpret the naive players as a large (exogenous) fraction of partisans who always vote B; a small fraction of (Nash) players who vote informatively is then sufficient to aggregate information. Therefore, rules $\rho > (1 - \gamma)$ now aggregate information. This result, however, is weaker than that obtained by Fedderson and Pesendorfer (1997) when all players are Nash: When both Nash and naive players coexist, their full information equivalence result holds only for rules $\rho > (1 - \gamma)$, rather than for all voting rules.\footnote{Again, the key intuition is that the behavior of the partisans (i.e., naive players) is now exogenous and will not adjust in the presence of different rules (beyond what is determined by the original $\pi$ function).} By a similar argument, if $c(s^H) > 0$, then information is aggregated for rules $\rho < \gamma$. If the planner is uncertain about whether $c(s^L) < 0$ or $c(s^H) > 0$, then majority rule may remain optimal.

Second, consider a case where information is aggregated in the presence of naive players, as in Figure 6(b). Again, with a fraction $\gamma$ of Nash players, the $\pi$ function will shift downwards for $c < 0$ and upwards for $c > 0$. In particular, the figure shows that the result that majority rule is optimal in symmetric settings with sufficiently accurate signals remains true in the presence of Nash players.
8 Conclusion

We have studied the information-aggregation properties of group decision-making when people learn in a decentralized fashion and fail to account for sample selection issues. We provided a learning foundation for the notion of a behavioral equilibrium (Esponda, 2008) applied to voting games and then used that notion to fully characterize all equilibria as the number of players becomes large. We provided necessary and sufficient conditions in order for information to be aggregated, showing that biases at the individual level may not necessarily disappear in large populations. We also characterized optimal voting rules and provided a new rationale for optimality of majority voting. Overall, a more nuanced view emerges about the benefits of using elections or committees in order to aggregate information.

While we have focused on the benchmark voting context, we hope that our approach leads to further work in both the areas of learning and information aggregation. Our players are in a learning environment where their actions affect what they learn. In several economic contexts, players must make inferences about the primitives of the environment and the actions of other players; disentangling these two sources of uncertainty is likely to present challenges. In particular, players may need to have a model of how other players learn, how other players think that other players learn, and so on. While it is tempting to close the model by making an equilibrium assumption, the purpose of a dynamic learning model is to close the model without such an assumption. This paper constitutes a step in this direction, and this step is tractable because players learn using a mis-specified model that fails to account for the informational content of other players’ actions.

9 Appendix

9.1 Dynamics

The following 2 lemmas are used in the proof of Theorem 2.

**Lemma 4.** (cf. Fudenberg and Kreps, Lemma 6.2, 1993) Let \((z_t)_t\) be a sequence of random variables with range on a finite set \(Z\). Fix a set-function \(\pi : 2^Z \to [0,1]\) (not necessarily a probability measure) and fix \(\epsilon \in \mathbb{R}\). Let \(H_\epsilon\) be a subset of infinite histories such that for all \(h \in H_\epsilon\) there exists \(t_{\epsilon,h}\) such that for all \(t \geq t_{\epsilon,h}\), the distribution of each \(z_t\) conditional on
By the strong law of large numbers, finitely many of them (roughly speak, moreover, by construction the probability over \( FK 93 \), let the elements as \( \omega \).

Therefore, it suffices to show the result for any (arbitrary) subset of \( \omega \).

Proof. First note that \( \#Z < \infty \) and thus any subset of \( Z \subset 2^Z \) also has finitely many elements. Therefore, it suffices to show the result for any (arbitrary) subset \( Z' \in Z \) since there are only finitely many of them (roughly speak, \( Z' \) is what \( a \) is in FK 93). Since \( Z \) is finite we can order the elements as \( (z_1, ..., z_\#Z) \), and WLOG we set the first \( \#Z' \) to be the elements of \( Z' \). Just as FK 93, let \( (\omega_t)_t \) be an independent sequence of uniform random variables and let \( y_t : \Omega \to Z \) be a new random variable.

As in FK 93, we construct \( (y_t(\omega_t))_t \) as follows. For \( t = 1 \), \( y_1(\omega_1) = z_m \) iff \( \sum_{n=1}^{m-1} \pi_1(z_n) \leq \omega_1 \leq \sum_{n=1}^{m} \pi_1(z_n) \). For \( t = \tau \), let \( y_\tau(\omega_\tau) = z_m \) iff \( \sum_{n=1}^{m-1} \pi_\tau(z_n|y_1, ..., y_{\tau-1}) \leq \omega_\tau \leq \sum_{n=1}^{m} \pi_\tau(z_n|y_1, ..., y_{\tau-1}) \).

Moreover, by construction the probability over \( h' \) coincides with the probability over \( (\omega_t)_t \); we thus can use both interchangeably. In particular, the set of \( \omega \) for which \( y_t(\omega_t) \in Z' \) is the set of \( \{ \omega : \omega_t \leq \sum_{n=1}^{\#Z'} \pi_t(z_n|y_1, ..., y_{t-1}) = \pi_t(Z'|y_1, ..., y_{t-1}) \} \) (recall that \( Z' \) consists of the first \( \#Z' \) elements in \( Z \)).

Under equation 31 the latter set includes the set \( \{ \omega : \omega_t \leq \pi(Z') - \varepsilon \} \); thus \( 1\{ \omega : \omega_t \leq \pi(Z') - \varepsilon \} \leq \pi(Z'|y_1, ..., y_{t-1}) = 1\{ \omega : y_t(\omega_t) \in Z' \} = 1\{ z_\tau \in Z' \} \). Let \( \nu_t(r, \omega) \) be the number of times \( \omega_t \leq r \). Then

\[
\nu_t(\pi(Z') - \varepsilon) \leq \sum_{\tau=1}^{t} 1\{z_\tau \in Z' \}.
\]

By the strong law of large numbers, \( \lim_{t \to \infty} \nu_t(\pi(Z') - \varepsilon) = \pi(Z') - \varepsilon \) a.s.-\( H_\varepsilon \). Therefore it

\[\tag{31}\]

where \( Z \subset 2^Z \) is a set of subsets of \( Z \).\(^{38}\)

Then

\[
\liminf_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} 1\{z_\tau \in Z' \} \geq \pi(Z') - \varepsilon \quad \tag{32}
\]

for all \( Z' \in Z \), almost surely on \( H_\varepsilon \). Moreover, if (31) is replaced by \( \max_{Z' \in Z} \pi_t(Z') - \pi(Z') < \varepsilon \), then the conclusion in (32) is replaced by \( \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} 1\{z_\tau \in Z' \} \leq \pi(Z') + \varepsilon \).

\(^{38}\)If \( H_\varepsilon \) has zero probability, the lemma is taken to be vacuous.
must follow that
\[
\liminf_{t \to \infty} t^{-1} \sum_{\tau=1}^{t} 1\{z_{\tau} \in Z'\} \geq \pi(Z') - \varepsilon.
\]

Similarly, under equation 31, the set \(\{\omega : \omega_t \leq \pi_t(Z'|y_1, \ldots, y_{t-1})\}\) is included in the set \(\{\omega : \omega_t \leq \pi(Z') + \varepsilon\}\). By a similar argument as before,
\[
\limsup_{t \to \infty} t^{-1} \sum_{\tau=1}^{t} 1\{z_{\tau} \in Z'\} \leq \pi(Z') + \varepsilon.
\]

\[\Box\]

**Lemma 5.** There exists \(H'\) with \(P^\pi(H') = 1\), such that for all \(\eta > 0\) and for all \(h \in H'\) there exists \(t_{\eta,h}\) such that for all \(t \geq t_{\eta,h}\) and all \(o \in \{A,B\}\), \(\overline{P}_t(h)(Z_{ois_i}) > K_p - \eta\), where

\[
K_p \equiv \min_{i,s_i} \left\{ \sum_{\omega \in \Omega} q_i(s_i | \omega)p(\omega) \times \min \left\{ (F_i(-2K))^n, (1 - F_i(2K))^n \right\} \right\}. \tag{33}
\]

**Proof.** By restriction on action plans, for all \(i, s_i\), for all \(h\), and for all \(t\)

\[
F_i(-2K) \leq a^H_{it}(h)(s_i) \leq F_i(2K).
\]

Hence, for all \(i, s_i\), for all \(h\), and for all \(t\),

\[
P( z_t \in Z_{Ais_i} | h) \geq (F_i(-2K))^n \sum_{\omega \in \Omega} q_i(s_i | \omega)p(\omega)
\]

\[
\geq K_p,
\]

and, similarly,

\[
P( z_t \in Z_{Bis_i} | h) \geq K_p.
\]

Let \(K_p = \pi(Z_{Ais_i})\) (the case of \(Z_{Bis_i}\) is analogous and thus omitted); then Lemma 4 with \(\varepsilon = 0\) and \(H_\varepsilon = H\) implies that \(\liminf_{t \to \infty} \overline{P}_t(h)(Z_{ois_i}) \geq K_p\) a.s. in \(H\). Therefore, this implies that there exists a \(H' \subseteq H\) with \(P^\pi(H') = 1\) such that for all \(\eta > 0\) and all \(h \in H'\), there exists a \(t_{\eta,h}\) such that for all \(t \geq t_{\eta,h}\), \(\overline{P}_t(h)(Z_{ois_i}) > K_p - \eta\). \(\Box\)
9.2 Limit equilibrium

Let \( x^n_i \in \{A, B\} \) be the vote of agent \( i \) when there are \( n \) players; thus \( \{o(A, x^n_i) = A\} = \left\{ \frac{1}{n} \sum_{i=1}^{n} 1\{x^n_i = A\} \geq \rho - \frac{1}{n} \right\} \). We also let \( \kappa^\alpha_i(\xi \mid \omega) \equiv P^n(x_i = A \mid \omega) \) (we also use the simplified notation of \( \kappa^\alpha_i, \omega \) when \( \xi \) is omitted) be the probability that player \( i = 1, \ldots, n \) votes for \( A \) conditional on the state being \( \omega \), and let \( \kappa^\alpha(\xi \mid \omega) \equiv \frac{1}{n} \sum_{i=1}^{n} \kappa^\alpha_i(\xi \mid \omega) \) (we also use the simplified notation of \( \kappa^\alpha, \omega \) when \( \xi \) is omitted) be the average over all players. Finally, we omit \( \alpha \) from the notation: \( P^n(\xi) \equiv P^n(\alpha(\xi)) \) and \( \sigma^\alpha_\theta(\xi) \) denotes the average strategy profile of type \( \theta \).

9.2.1 Proof of Lemma 2.1

Recall that to show this lemma we assume that: (a) \( \alpha \) is \( \Xi' \) asymptotically interior, (b) \( \lim_{n \to \infty} \sigma^\alpha_\theta(\xi) = \sigma_\theta \) a.s. in \( \Xi' \), and (c) \( \sigma \) is increasing.

The proof relies on the following claims.

**Claim 2.1.1:** \( \kappa(\sigma \mid \cdot) \) is increasing and therefore \( \{\omega : \kappa(\sigma \mid \omega) = \rho\} \) is either empty or a singleton.

**Proof.** We show that \( \kappa(\sigma \mid \cdot) \) is increasing given that \( \sigma_\theta \) is increasing. By Bayes theorem and assumption C3, for all \( \omega' > \omega \), for all \( \theta \), and \( s'_\theta > s_\theta \)

\[
\frac{q_\theta(s'_\theta \mid \omega')}{q_\theta(s'_\theta \mid \omega)} > \frac{q_\theta(s_\theta \mid \omega')}{q_\theta(s_\theta \mid \omega)} \iff \frac{g_\theta(\omega' \mid s'_\theta)}{g_\theta(\omega' \mid s_\theta)} > \frac{g_\theta(\omega \mid s'_\theta)}{g_\theta(\omega \mid s_\theta)}.
\]

(Where \( g_\theta \) is the pdf of \( \omega \) given \( s_\theta \)). Moreover, by Proposition 1 in Milgrom (1981a), \( \sum_{s < s'} q_\theta(s \mid \omega') \) strictly dominates (in a first order stochastic sense) \( \sum_{s < s'} q_\theta(s \mid \omega) \).

Note also that, casting \( S_\theta = \{s^1_\theta, \ldots, s^S_\theta\} \), it follows that

\[
\sum_{s_\theta \in S_\theta} \sigma_\theta(s_\theta) q_\theta(s_\theta \mid \omega) = \sum_{i=1}^{S_\theta} A_\theta(s^i_\theta) \left( \sum_{s \leq s^i_\theta} q_\theta(s \mid \omega) \right),
\]

where \( A_\theta(s^i_\theta) = \sigma_\theta(s^i_\theta) - \sigma_\theta(s^{i+1}_\theta) \) and \( A_\theta(s^S_\theta) = \sigma_\theta(s^S_\theta) \). Hence

\[
\sum_{s_\theta \in S_\theta} \sigma_\theta(s_\theta) \{q_\theta(s_\theta \mid \omega') - q_\theta(s_\theta \mid \omega)\} = \sum_{i=1}^{S_\theta-1} A_\theta(s^i_\theta) \left( \sum_{s \leq s^i_\theta} q_\theta(s \mid \omega') - \sum_{s \leq s^i_\theta} q_\theta(s \mid \omega) \right).
\]

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Since \( \sigma \) is nondecreasing, \( A_\theta(s^\theta) < 0 \), then the expression above is strictly positive. Since \( \phi(\theta) > 0 \) all \( \theta \), the desired result follows from the construction of \( \kappa \).

\[
\text{Claim 2.1.2:} \text{ For all } \omega \in \Omega, \lim_{n \to \infty} \kappa^n(\xi \mid \omega) = \kappa(\sigma \mid \omega) \text{ a.s. in } \Xi'.
\]

\text{Proof.} First, note that

\[
\kappa^n(\xi \mid \omega) = \frac{1}{n} \sum_{i=1}^n \sum_{s \in S^\theta} q_\theta(s \mid \omega) \{ \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\theta_i(\xi) = \theta\} \alpha_i^n(\xi)(s) \}
\]

\[
= \sum_{\theta \in \Theta} \sum_{s \in S^\theta} q_\theta(s \mid \omega) \left\{ \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\theta_i(\xi) = \theta\} \alpha_i^n(\xi)(s) \right\}
\]

\[
= \sum_{\theta \in \Theta} \sum_{s \in S^\theta} q_\theta(s \mid \omega) \left\{ \sigma^\theta_n(\xi)(s) \times \left( \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\theta_i(\xi) = \theta\} \right) \right\}
\]

\[
\to \sum_{\theta \in \Theta} \sum_{s \in S^\theta} q_\theta(s \mid \omega) \sigma_\theta(s) \phi(\theta) = \kappa(\sigma \mid \omega),
\]

where convergence is a.s. in \( \Xi' \) and follows from (i) the assumption that \( \lim_{n \to \infty} \sigma^\theta_n(\xi) = \sigma_\theta \) a.s. in \( \Xi' \), (ii) the strong law of large numbers applied to \( \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\theta_i(\xi) = \theta\} \), and (iii) the fact that \( \mathbb{1}\{\cdot\} \) and \( \sigma^\theta_n \) are uniformly bounded.

\[
\text{Claim 2.1.3:} \lim_{n \to \infty} P^n(\xi(o = A \mid \omega) = \begin{cases} 0 & \text{if } \rho > \kappa(\sigma \mid \omega) \\ 1 & \text{if } \rho < \kappa(\sigma \mid \omega) \end{cases} \text{ a.s. in } \Xi'.
\]

\text{Proof.} It follows that

\[
P^n(\xi(o = A \mid \omega) = \Pr\left(n^{-1} \sum_{i=1}^n \mathbb{1}\{x_i^n = A\} \geq \rho \mid \omega\right)
\]

\[
= \Pr\left(n^{-1/2} \sum_{i=1}^n \left( \mathbb{1}\{x_i^n = A\} - \kappa_i^n(\xi \mid \omega) \right) \geq \sqrt{n}(\rho - \kappa^n(\xi \mid \omega)) \mid \omega \right)
\]

Moreoever, by the Markov inequality,

\[
\Pr\left(n^{-1/2} \sum_{i=1}^n \left( \mathbb{1}\{x_i^n = A\} - \kappa_i^n(\xi \mid \omega) \right) \right) \geq \sqrt{M} \mid \omega \right) \leq (nM)^{-1} \sum_{i=1}^n E \left[ \left( \mathbb{1}\{x_i^n = A\} - \kappa_i^n(\xi \mid \omega) \right)^2 \right] \mid \omega \]

\[
\leq 4M^{-1}.
\]
goes to zero as \( M \to \infty \).

Suppose that \( \rho > \kappa(\sigma \mid \omega) \). By Claim 2.1.2, \( \sqrt{n} (\rho - \kappa^n(\xi \mid \omega)) \to \infty \) a.s. in \( \Xi' \). Therefore, by equations (34) and (35), \( \lim_{n \to \infty} P^n(\xi)(\sigma = A \mid \omega) = 0 \) a.s. in \( \Xi' \). Similarly, if \( \rho < \kappa(\sigma \mid \omega) \) then \( \sqrt{n} (\rho - \kappa^n(\xi \mid \omega)) \to -\infty \) and \( \lim_{n \to \infty} P^n(\xi)(\sigma = A \mid \omega) = 1 \) a.s. in \( \Xi' \).

**Proof of Lemma 2.1.** First, Claim 2.1.3 and the facts that \( \kappa(\sigma \mid \cdot) \) is increasing (Claim 2.1.1) and continuous (by assumption C4(iii)) imply that there exists \( c(\sigma) \in [-1, 1] \) such that \( c(\sigma) \in \arg \min_{c \in [-1, 1]} |\kappa(\sigma \mid c) - \rho| \) and

\[
\lim_{n \to \infty} P^n(\xi)(\sigma = A \mid \omega) = 1\{\omega < c(\sigma)\} \quad \text{a.s. in} \ \Xi. \tag{36}
\]

Suppose that \( c(\sigma) = -1 \). Then \( \lim_{n \to \infty} P^n(\xi)(\sigma = A) > 0 \) a.s. in \( \Xi' \), therefore contradicting that \( \alpha \) is asymptotically interior. A similar argument rules out \( c = 1 \). Therefore, \( c(\sigma) \in (-1, 1) \), implying that \( \alpha \) is \( \Xi' \)-asymptotically \( c(\sigma) \)-cutoff and that \( \kappa(\sigma \mid c(\sigma)) = \rho \).

Second, note that, a.s. in \( \Xi' \)

\[
\lim_{n \to \infty} E_{P^n(\xi)}(u_i(A, \omega) \mid \sigma = A, s_i) = \lim_{n \to \infty} \frac{\int_{\Omega} P^n(\xi)(\sigma = A \mid \omega) q(s_i \mid \omega) u_i(A, \omega) G(\omega) d\omega}{\int_{\Omega} P^n(\xi)(\sigma = A \mid \omega) q(s_i \mid \omega) G(\omega) d\omega} = \frac{\int_{\Omega} \lim_{n \to \infty} P^n(\xi)(\sigma = A \mid \omega) q(s_i \mid \omega) u_i(A, \omega) G(\omega) d\omega}{\int_{\Omega} \lim_{n \to \infty} P^n(\xi)(\sigma = A \mid \omega) q(s_i \mid \omega) G(\omega) d\omega} = \frac{\int_{\Omega} \{\omega > c(\sigma)\} q(s_i \mid \omega) u_i(A, \omega) G(\omega) d\omega}{\int_{\Omega} \{\omega > c(\sigma)\} q(s_i \mid \omega) G(\omega) d\omega} = E(u_{q_i}(A, \omega) \mid \omega > c(\sigma), s_i),
\]

where the expectation is well-defined because C4(ii) and the fact that \( \alpha \) is asymptotically interior imply that the denominator is greater than zero, where the second line follows from the dominated convergence theorem (since \( u_i \) is assumed to be uniformly bounded), and where the third line follows from Claim 2.1.3.

**9.2.2 Proof of Lemma 2.2**

Throughout the proof let \( \Xi' \) be the set in definition 8 and fix \( \xi \in \Xi' \) and a strategy mapping \( \overline{\alpha} \) such that (13), (14), and (15) are satisfied and \( \lim_{n \to \infty} \sigma(\xi; \overline{\alpha}(\xi)) = \sigma \). To simplify notation, we drop \( \xi \) and \( \overline{\alpha} \) from the notation, let \( P^n \equiv P^n(\overline{\alpha}(\xi)) \) and, for each strategy \( \alpha_i \), let \( P^n_{\alpha_i} \equiv P^n(\alpha_i, \overline{\alpha}_{-i}(\xi)) \). The proof relies on the following claims; the proofs of the first three claims appear at the end of this section.
Claim 2.2.1: For all $\delta > 0$ and $\omega \in \Omega$, there exits $n_{\delta,\omega}$ such that for all $n \geq n_{\delta,\omega}$,

$$\left| P_{\alpha_i}^n (o = A \mid \omega, s_i) - P_{\alpha_i'}^n (o = A \mid \omega, s'_i) \right| < \delta$$

uniformly over $i, s_i, s'_i, \alpha_i, \alpha'_i$.

Claim 2.2.2: For all $\delta > 0$ there exist $n_{\delta}$ such that for all $n \geq n_{\delta}$,

$$\left| \Delta_i (P^n, s_i) - \Delta_i (P_{\alpha_i}^n, s_i) \right| < \delta$$

uniformly over $i, s_i, \alpha_i$.

Claim 2.2.3: There exists $c > 0$ and $n_c$ such that for all $n \geq n_c$

$$\Delta_i (P_{\alpha_i}^n, s'_i) - \Delta_i (P_{\alpha_i}^n, s_i) \geq c$$

for all $i$, all $s'_i > s_i$, and $\alpha_i(s'_i) = \alpha_i(s_i)$.

Claim 2.2.*: There exists $c' > 0$ and $n_{c'}$ such that for all $n \geq n_{c'}$

$$\Delta_i (P^n, s'_i) - \Delta_i (P^n, s_i) \geq c'$$

for all $i$ and $s'_i > s_i$.

Proof of Claim 2.2.*. Fix any $\alpha_i$ such that $\alpha_i(s'_i) = \alpha_i(s_i)$. By Claims 2.2.2 and 2.2.3, for all $n \geq \max \{ n_c, n_\delta \}$

$$\Delta_i (P^n, s'_i) - \Delta_i (P^n, s_i) \geq (\Delta_i (P_{\alpha_i}^n, s'_i) - \Delta_i (P_{\alpha_i}^n, s_i) - (\Delta_i (P_{\alpha_i}^n, s_i) + \delta)$$

$$\geq c - 2\delta.$$ 

The claim follows by setting $\delta = c/4$ and $c' = c/2 > 0$. \hfill \qed
Proof of Lemma 2.2. The definition of ε-equilibrium (equation 13) implies that for all \( i, s'_i > s_i, \) \( n \geq n_\varepsilon, \)

\[
\tilde{\alpha}^n_i(s'_i) - \tilde{\alpha}^n_i(s_i) \geq F_i(\Delta_i(P^n, s'_i)) - F_i(\Delta_i(P^n, s_i)) - 2\varepsilon.
\]

\[
+ F_i(\Delta_i(P^n, s_i) + c') - F_i(\Delta_i(P^n, s_i) + c'),
\]

where we have added and subtracted the same term to the RHS. Let \( c' > 0 \) be as defined in Claim 2.2.*. Since \( F_i \) is absolutely continuous, then

\[
F_i(\Delta_i(P^n, s_i) + c') - F_i(\Delta_i(P^n, s_i)) = \int_{\Delta_i(P^n, s_i)}^{\Delta_i(P^n, s_i) + c'} f_i(t) dt \geq d \cdot c' \equiv c'' > 0,
\]

where the inequality follows from C4(iv). Hence, the sum of the second and fourth terms in the RHS of (37) is at least \( c'' > 0 \). By Claim 2.2.*, the sum of the first and last terms in the RHS of (37) is positive. Therefore, for all \( i, s'_i > s_i, \) \( n \geq n_\varepsilon, \)

\[
\tilde{\alpha}^n_i(s'_i) - \tilde{\alpha}^n_i(s_i) \geq c'' - 2\varepsilon > 0.
\]

Since \( \sigma^n_\vartheta(\xi, \alpha) \) are averages of the strategies, then for all \( \theta, s'_\vartheta > s_\vartheta, \) and \( n \geq n_\varepsilon, \) it follows that

\[
\sigma^n_\vartheta(s'_\vartheta) - \sigma^n_\vartheta(s_\vartheta) \geq c'' - 2\varepsilon.
\]

Since \( \lim_{n \to \infty} \sigma^n = \sigma, \) then it follows that \( \sigma_\vartheta(s'_\vartheta) - \sigma_\vartheta(s_\vartheta) \geq c'' - 2\varepsilon > 0, \) thus establishing that limit ε-equilibrium are increasing as long as \( 0 < \varepsilon < \varepsilon' \equiv c''/2 > 0. \)

Proof of Claim 2.2.1. The proof is divided into 3 steps.

Step 1. We first show that the probability of being pivotal goes to zero; i.e., for all \( \omega \in \Omega, \) for all \( i, \lim_{n \to \infty} Piv^n_\omega = 0, \) where

\[
Piv^n_\omega \equiv P^n(\omega(A, x^n_{-i}) = A \mid \omega) - P^n(\omega(B, x^n_{-i}) = A \mid \omega).
\]

By simple algebra,

\[
Piv^n_\omega = P^n\left(\frac{n}{\sqrt{n-1}}K^n_\omega + \frac{\kappa^n_{i\omega} - 1}{V^n_\omega \sqrt{n-1}} \leq \sum_{j \neq i}^{n} Z^n_{j,\omega} < \frac{n}{\sqrt{n-1}}K^n_\omega + \frac{\kappa^n_{i\omega}}{V^n_\omega \sqrt{n-1}} \mid \omega\right),
\]

where \( Z^n_{j,\omega} \equiv \frac{\{1(x^n_{i-1})-\kappa^n_{j,\omega}\}}{V^n_\omega}, \) \( V^n_\omega \equiv \sqrt{\frac{1}{n-1} \sum_{j \neq i} \kappa^n_{j,\omega}(1 - \kappa^n_{j,\omega})}, \) and \( K^n_\omega \equiv \frac{\rho - \kappa^n_{i\omega}}{V^n_\omega}. \) Note that, for a given \( n, \) \( \{Z^n_{j,\omega}\}_{j \neq i} \) are independent, they have zero mean and unit variance. Moreover, by
Step 3 below, \( \liminf_{n \to \infty} V^n_\omega > 0 \), so that

\[
\sum_{j \neq i} E \left( \left| \frac{Z_{j\omega}}{\sqrt{n-1}} \right|^3 \right) \leq \frac{2}{\sqrt{n-1} (V^n_\omega)^3} \to 0 \text{ as } n \to \infty,
\]

Hence by Lindeberg-Feller CLT, it follows that, given \( \omega \), \( \sum_{j \neq i} \frac{Z_{j\omega}}{\sqrt{n-1}} \to N(0,1) \) as \( n \to \infty \).

We divide the remainder of the proof in 3 cases: (a) \( \frac{n}{\sqrt{n-1}} K^n_\omega \to -\infty \), (b) \( \frac{n}{\sqrt{n-1}} K^n_\omega \to K \in (-\infty, \infty) \) or (c) \( \frac{n}{\sqrt{n-1}} K^n_\omega \to \infty \) (if necessary, we take a subsequence that converges, which exists since \( (V^n_\omega(\xi))_n \) and \( (\kappa^n_\omega(\xi))_n \) are uniformly bounded).

We first explore case (a) (case (c) is symmetrical). Note that, since \( \liminf_{n \to \infty} V^n_\omega > 0 \), then \( \frac{\kappa^n_\omega}{V^n_\omega} \to 0 \). Therefore, \( \frac{n}{\sqrt{n-1}} K^n_\omega + \frac{\kappa^n_\omega}{V^n_\omega} \to -\infty \), so that we can take \( n \geq n_{M,\epsilon} \) such that

\[
\sqrt{n}K^n_\omega + \frac{\kappa^n_\omega}{V^n_\omega} \leq -M,
\]

where \( \mathcal{L}_N(-M) < 0.5\epsilon \) (where \( \mathcal{L}_N \) is the standard Gaussian cdf) for any \( \epsilon \). Therefore, for all \( \epsilon > 0 \) there exists \( n_{\epsilon,\omega} \) such that for all \( n \geq \max\{n_{\epsilon,\omega}, n_{M,\epsilon}\} \):

\[
P_{iv}^n \leq P^n \left( \frac{\sum_{j \neq i} Z_{j\omega}}{\sqrt{n-1}} < -M \mid \omega \right)
\leq 0.5\epsilon + \mathcal{L}_N(-M) < \epsilon,
\]

where the first inequality follows from the fact that \( n \geq n_{M,\epsilon} \) and the second follows from CLT and our choice of \( M \).

For case (b) (i.e., \( K \) finite) it follows for all \( \epsilon > 0 \), there exists \( n_{\epsilon,\omega} \) such that for all \( n \geq \max\{n_{\epsilon,\omega}, n_{\delta,\epsilon}\} \):

\[
P_{iv}^n \leq P^n \left( \frac{n}{\sqrt{n-1}} K^n_\omega - \frac{1}{V^n_\omega \sqrt{n-1}} \leq \frac{\sum_{j \neq i} Z_{j\omega}}{\sqrt{n-1}} < \frac{n}{\sqrt{n-1}} K^n_\omega + \frac{1}{V^n_\omega \sqrt{n-1}} \mid \omega \right)
\leq P^n \left( K - \delta < \frac{\sum_{j \neq i} Z_{j\omega}}{\sqrt{n-1}} \leq K + \delta \mid \omega \right)
\leq 0.5\epsilon + \mathcal{L}_N \left( K - \delta < \frac{\sum_{j \neq i} Z_{j\omega}}{\sqrt{n-1}} \leq K + \delta \right) < \epsilon,
\]

where \( \delta \) is such that \( (V^n_\omega \sqrt{n-1})^{-1} < \delta \) for all \( n \geq n_{\delta,\epsilon} \) and \( \mathcal{L}_N(K + \delta) - \mathcal{L}_N(K - \delta) < 0.5\epsilon \). The second inequality follows from the CLT. We showed that for any convergent subsequence \( (K^n_\omega(\xi))_n \), the associated subsequences of probabilities converge to zero, thus this result must hold for the whole sequence.
Step 2. Note that:

\[ P^n_{a_i} (o = A \mid \omega, s_i) = \alpha_i P^n (o(A, x^n_i) = A \mid \omega) + (1 - \alpha_i) P^n (o(B, x^n_i) = A \mid \omega) \]
\[ = P^n (o(B, x^n_i) = A \mid \omega) + \alpha_i \left( P^n (o(A, x^n_i) = A \mid \omega) - P^n (o(B, x^n_i) = A \mid \omega) \right) \]
\[ = P^n (o(B, x^n_i) = A \mid \omega) + \alpha_i \Delta P^n (\omega). \]

Therefore

\[ |P^n_{a_i} (o = A \mid \omega, s_i) - P^n_{a'_i} (o = A \mid \omega, s'_i)| \leq |\alpha_i - \alpha'_i| |\Delta P^n (\omega)|. \]

By step 1, it follows that for all \( n \geq n_{\delta, \omega} \): \(|\Delta P^n (\omega)| \leq \delta\). Since \(|\alpha_i - \alpha'_i| \leq 1\) the desired result follows.

Step 3. We now show that for all \( \omega \in \Omega \),

\[ \liminf_{n \to \infty} \frac{1}{n - 1} \sum_{j \neq i} \kappa^n_{j\omega} (1 - \kappa^n_{j\omega}) > 0. \]  \hspace{1cm} (39)

Fix any \( n \) and \( j \leq n \). By assumption, \( \alpha^n_j (s_j) \in [F_j (-2K), F_j (2K)] \subset (0, 1) \) for all \( s_j \). Therefore, \( 0 < \kappa^n_{j\omega} < 1 \) for all \( \omega \), thus implying equation (39).

Proof of Claim 2.2.2. We prove that

\[ \lim_{n \to \infty} \left( E^A_i (P^n, s_i) - E^A_i (P^n_{a_i}, s_i) \right) = 0; \]

the proof for the \( E^B_i \) terms is similar and therefore omitted. We first show that, for all \( i, s_i, \alpha_i \),

\[ E^A_i (P^n_{a_i}, s_i) = \frac{\int_{\Omega} P^n_{a_i} (o = A \mid \omega, s_i) q(s_i \mid \omega) u_i(A, \omega) G(d\omega)}{\int_{\Omega} P^n_{a_i} (o = A \mid \omega, s_i) q(s_i \mid \omega) G(d\omega)} \]

is well-defined for sufficiently large \( n \). Fix any \( i \). Assumption C4(ii) and the fact that \( \overline{\alpha} \) is asymptotically interior imply that there exists \( \overline{n} \) such that for all \( n \geq \overline{n} \), there exists \( s_i^* \) such that

\[ P^n(o = A, s_i^*) = \int_{\Omega} P^n(o = A \mid \omega, s_i^*) q(s_i^* \mid \omega) G(d\omega) \geq c > 0, \]

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which implies that \( \int_{\Omega} P^n(o = A \mid \omega, s_i^*) G(d\omega) \geq c > 0 \). By Claim 2.2.1, for each \( s_i, \alpha_i \), \( P^n(o = A \mid \omega, s_i^*) - P^n_{\alpha_i}(o = A \mid \omega, s_i) \) converges to zero as \( n \to \infty \). Since both probabilities are bounded by one, then the dominated convergence theorem implies that \( \int_{\Omega} (P^n(o = A \mid \omega, s_i^*) - P^n_{\alpha_i}(o = A \mid \omega, s_i)) G(d\omega) = 0 \) as \( n \to \infty \), uniformly over \( \alpha_i \). Therefore, there exists \( n_{0.5c} \) such that \( \sup_{\alpha_i} |\int_{\Omega} [P^n(o = A \mid \omega, s_i^*) - P^n_{\alpha_i}(o = A \mid \omega, s_i)] G(d\omega)| < 0.5c \) for all \( n \geq n_{0.5c} \). So for all \( n \geq \max \bar{n}, n_{0.5c} \equiv \bar{n}_c \),

\[
\int_{\Omega} P^n_{\alpha_i}(o = A \mid \omega, s_i) q(s_i \mid \omega) G(d\omega) \geq d \int_{\Omega} P^n_{\alpha_i}(o = A \mid \omega, s_i) G(d\omega) > 0.5dc > 0.
\]

Hence, \( E_i^A(P^n_{\alpha_i}, s_i) \) is well defined.

By simple algebra,

\[
|E_i^A(P^n, s_i) - E_i^A(P^n_{\alpha_i}, s_i)| = 
\leq \frac{|\int_{\Omega} (P^n(o = A \mid \omega) - P^n_{\alpha_i}(o = A \mid \omega)) q(s_i \mid \omega) u_i(A, \omega) G(d\omega)| \int_{\Omega} P^n(o = A \mid \omega) q(s_i \mid \omega) G(d\omega)}{\int_{\Omega} P^n(o = A \mid \omega) q(s_i \mid \omega) G(d\omega)}
+ \frac{\int_{\Omega} (P^n(o = A \mid \omega) - P^n_{\alpha_i}(o = A \mid \omega)) q(s_i \mid \omega) G(d\omega) | \int_{\Omega} P^n(o = A \mid \omega) q(s_i \mid \omega) u_i(A, \omega) G(d\omega)}{\int_{\Omega} P^n(o = A \mid \omega) q(s_i \mid \omega) G(d\omega)}
\]

To establish the desired result, it is sufficient to show that each of the two absolute value terms in the numerator of the second and third line converge to zero as \( n \to \infty \). However, this result follows by the dominated convergence theorem since \( |u_i(A, \omega)| < K, q(s_i \mid \omega) \leq 1 \), and pointwise convergence (for each \( \omega \)) is obtained by Claim 2.2.1.

**Proof of Claim 2.2.3.** Throughout this proof, let \( P^n_i(\omega) \equiv P^n_{\alpha_i}(o = A \mid \omega, s_i) = P^n_{\alpha_i}(o = A \mid \omega, s_i) \), where the equality follows by conditional independence and because \( \alpha_i(s_i^*) = \alpha_i(s_i) \).

Let also \( G(\omega \mid o = A, s_i) \) be the conditional conditional distribution function of \( \omega \) given \( o = A \) and \( s_i \) when the number of players is \( n \). Also, we assume that in assumption C1, \( u(A, \cdot) \) is the one that is strictly increasing and show the result only for this part of \( \Delta_i \left( P^n_i(\xi), \cdot \right) \).

**Step 1.** We first show that, for any given \( i \) and \( s_i' > s_i \). There exists \( (\Omega^n)_n \) with \( \Omega^n \subseteq \Omega \) and \( \lim_{n \to \infty} \text{Pr}_G(\Omega^n) \equiv \int_{\Omega^n} G(d\omega) > 0 \) such that for all \( n \geq n_c \) and all \( \omega^* \in \Omega^n \setminus \{-1, 1\} \),

\[
G(\omega^* \mid o = A, s_i) - G(\omega^* \mid o = A, s_i') \geq \beta > 0.
\]
As shown in the proof of Claim 2.2.2, for all \( n \geq n_c \),

\[
\int_{\Omega} P^n_i(\omega) G(d\omega) \geq c
\]

for all \( i, s_i \). For \( a \in (0, 1) \), let

\[
\omega^n_a = \min \left\{ \omega : \int_{\omega' \leq \omega} P^n_i(\omega') G(d\omega') \geq a \cdot c \right\} \in \Omega.
\]

Fix \( n \geq n_c \). Then

\[
c/4 = \int_{\omega_{0.25} \leq \omega \leq \omega_{0.50}} P^n_i(\omega) G(d\omega) \leq G(\omega_{0.50}^n) - G(\omega_{0.25}^n).
\]

Therefore the fact that \( G \) has no mass points (assumption C3) implies that

\[
\omega_{0.50}^n - \omega_{0.25}^n \geq d_1 > 0. \tag{40}
\]

A similar argument establishes that

\[
\omega_{0.75}^n - \omega_{0.50}^n \geq d_2 > 0.
\]

Let \( \Omega^n = [\omega_{0.50}^n - d_3/2, \omega_{0.50}^n + d_3/2] \), where \( d_3 \equiv \min\{d_1, d_2\} > 0 \). Then for all \( \omega^* \in \Omega^n \)

\[
\int_{\omega < \omega^* - d_3/2} P^n_i(\omega) G(d\omega) \geq c/4 \tag{41}
\]

and

\[
\int_{\omega > \omega^* + d_3/2} P^n_i(\omega) G(d\omega) \geq c/4. \tag{42}
\]

In addition, assumption C4(i) and equation 40 imply that \( \Pr_G(\Omega^n) \geq c_g > 0 \), so that \( \lim_{n \to \infty} \Pr_G(\Omega^n) > 0 \). Next, assumption C2 implies that for \( \omega' > \omega \) there exists \( z > 0 \) such that

\[
P^n_i(\omega) P^n_i(\omega') (q_i(s_i' | \omega') q_i(s_i | \omega) - q_i(s_i | \omega') q_i(s_i' | \omega)) = z P^n_i(\omega) P^n_i(\omega') q_i(s_i' | \omega) q_i(s_i | \omega) (\omega' - \omega).
\]

Integrating each side twice, first with respect to \( G(d\omega) \) over \( \omega \leq \omega^* \) and second with respect
to $G(d\omega')$ over $\omega > \omega^*$, we obtain
\[
G(\omega^* | o = A, s_i) - G(\omega^* | o = A, s'_i) = \int_{\omega' > \omega^*} \int_{\omega} P^n_i(\omega') P^n_i(\omega) z(\omega' - \omega) g_i(s_i | \omega) q_i(s_i | \omega) G(d\omega) G(d\omega').
\]
(43)

Thus, the desired result follows from
\[
G(\omega^* | o = A, s_i) - G(\omega^* | o = A, s'_i) \geq z d^2 \int_{\omega > \omega^* + d/2} \int_{\omega < \omega^* + d/2} P^n_i(\omega) P^n_i(\omega) G(d\omega) G(d\omega')
\]
\[
\geq z \cdot d^2 \int_{\omega > \omega^* + d/2} P^n_i(\omega) G(d\omega) \int_{\omega < \omega^* + d/2} P^n_i(\omega) G(d\omega)
\]
\[
\geq z d^2 (c/4)^2 \equiv \beta > 0,
\]
where the last line follows from (41) and (42).

**STEP 2.** Note that $\int_{\Omega} u_i(A, \omega) (G(d\omega | o = A, s'_i) - G(d\omega | o = A, s_i))$; so by integration by parts, the fact that $G(\omega | o = A, s'_i) = G(\omega | o = A, s_i)$ for $\omega \in \{-1, 1\}$, and $u'_i(A, \cdot) > 0$ (assumption C1), MLRP (assumption C2) and similar calculations to the ones in step 1 but over all $\omega$, it follows
\[
- \int_{\Omega} u_i(A, \omega) (G(d\omega | o = A, s'_i) - G(d\omega | o = A, s_i)) \geq \int_{\Omega} u'_i(A, \omega) (G(d\omega | o = A, s'_i) - G(d\omega | o = A, s_i)) d\omega
\]
\[
\geq \beta \int_{\Omega} u'_i(A, \omega) d\omega
\]
the last inequality follows from step 1. The proof is thus established by noting that,
\[
\int_{\Omega} u'_i(A, \omega) d\omega = \int_{\Omega} \frac{u'_i(A, \omega)}{g(\omega)} g(\omega) d\omega
\]
By assumptions C4(i) and C5, $\int_{\Omega} u'_i(A, \omega) G(d\omega) \geq \text{const.} \times \Pr_G(\Omega^n)$. Finally, since $\lim_{n \to \infty} \Pr_G(\Omega^n) > 0$ the desired result follows.

9.3 Vanishing perturbations

9.3.1 Proof of Lemma 3

Claim 3.1 (i) $v_\theta(s_\theta; \cdot)$ is increasing and continuous for all $(\theta, s_\theta)$; (ii) $v_\theta(\cdot; c)$ is increasing for all $c \in \Omega$. 

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Proof. (i) Monotonicity of payoffs (C1(i)) and strict MLRP (C2) imply that $v_\theta(s_\theta; \cdot)$ is increasing. For continuity, it is sufficient to show that $E[u_\theta(A, \omega) \mid \omega > c, s_\theta]$ is continuous (the result for $E[u_\theta(B, \omega) \mid \omega < c, s_\theta]$ is analogous). It follows that $E[u_\theta(A, \omega)\mid \omega > c, s_\theta] = \int_{\omega > c} u_\theta(A, \omega) G(d\omega \mid s_\theta)$ so by assumptions C1 and C3, $\int_{\omega > c} u_\theta(A, \omega) G(d\omega \mid s_\theta)$ and $G(c| s_\theta)$ are continuous.

(ii) For $s'_\theta > s_\theta$, $G(\cdot \mid s_\theta)$ strictly dominates (in a first order stochastic sense) $G(\cdot \mid s'_\theta)$ by the MLRP (C2). Since $u_\theta(A, \cdot)$ is nondecreasing then $E[u_\theta(A, \omega)\mid \omega > c, \cdot] = \int_{\omega > c} u_\theta(A, \omega) G(d\omega \mid s_\theta)$ is nondecreasing; similarly, $E[u_\theta(B, \omega)\mid \omega < c, \cdot] = \int_{\omega < c} u_\theta(A, \omega) G(d\omega \mid s_\theta)$ is nonincreasing. By assumption C1 one of them holds strictly, thus $v_\theta(\cdot; c)$ is increasing.

Proof of Lemma 3. Let $S_\theta(c) \equiv \{s \in S_\theta : c_\theta(s) < c\}$ and define

$$\hat{\kappa}(c \mid \omega) \equiv \sum_{\theta \in \Theta} \phi_\theta q_\theta(S_\theta(c) \mid \omega).$$  \hfill (44)

First, note that $q_\theta(S_\theta(c) \mid \omega)$ is weakly increasing in $c$, because the fact that $c_\theta(\cdot)$ is monotone implies that the set $S_\theta(c)$ becomes weakly larger as $c$ increases. Second, MLRP and the fact that $S_\theta(c)$ is an interval of the form $[s_\theta, s'_\theta]$ for some $s_\theta$ imply that $q_\theta(S_\theta(c) \mid \omega)$ is weakly increasing in $\omega$; the weakly arises because $S_\theta(c)$ may be either $\emptyset$ or $S_\theta$. Finally, if $c < \min_{\theta} c_\theta(s'_\theta)$ then $S_\theta(c) = \emptyset$ for all $\theta$ and therefore $\hat{\kappa}(c \mid \omega) = 0$ for all $\omega$. Similarly, if $c > \max_{\theta} c_\theta(s'_\theta)$ then $S_\theta(c) = S_\theta$ for all $\theta$ and therefore $\hat{\kappa}(c \mid \omega) = 1$ for all $\omega$. The characterization of $\overline{\kappa}(\cdot)$ then follows because $\overline{\kappa}(c) = \hat{\kappa}(c \mid c)$. Finally, note that $v_\theta(s_\theta; 1) = v_\theta(s'_\theta; 1) = u_\theta(A, 1) - E u_\theta(B, \omega \mid s'_\theta) > 0$ for all $\theta, s_\theta$, where the last inequality follows from assumption C6. By continuity of $v_\theta(s_\theta; \cdot)$ (Claim 3.1), it follows that $c_\theta(s'_\theta) < 1$ for all $\theta$. A similar proof establishes that $\min_{\theta} c_\theta(s'_\theta) > -1$. \hfill \Box

9.3.2 Proof of Theorem 4

Let

$$\overline{\kappa}(c) \equiv \sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_\theta(S_\theta \mid c) F_\theta^\eta(v_\theta(s_\theta; c)).$$

Claim 4.1 $\overline{\kappa}(\cdot)$ is increasing and continuous.
Proof. Continuity of $\pi^n(\cdot)$ follows from continuity of $v_\theta(s_\theta; \cdot)$ (Claim 3.1), $F^n_{\theta}$ (assumption C4(iv)), and $q_\theta(s_\theta | \omega)$ (assumption C4(iii)). To show that $\pi^n(\cdot)$ is increasing, it is sufficient to establish that

$$\hat{\pi}^n(\omega_1, \omega_2) = \sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_\theta(s_\theta | \omega_1) F^n_{\theta}(v_\theta(s_\theta; \omega_2))$$

is increasing in $\omega_1$ and $\omega_2$. First, by Claim 3.1 and assumption C4(iv), $F^n_{\theta}(v_\theta(s_\theta; \omega_2))$ is increasing in $\omega_2$. Second, fix $\omega_2$ and let $\sigma_\theta(s_\theta) = F^n_{\theta}(v_\theta(s_\theta; \omega_2))$. Note that by MLRP and because $F^n_{\theta}$ is increasing, then $\sigma_\theta(\cdot)$ is increasing by claim 3.1(ii) and assumption C4(iv). Therefore, we can apply the proof of Claim 2.1.1 to conclude that $\hat{\pi}^n(\cdot, \omega_2)$ is increasing. \hfill \Box

**Claim 4.2** $C^{eqm}(\rho) \subset (-1, 1)$.

**Proof.** Follows immediately from the characterization of $\bar{\pi}(\cdot)$ in Lemma 3. \hfill \Box

**Claim 4.3** Let $\Omega_\theta = \{ \omega : c_\theta(s_\theta) = \omega, s_\theta \in S_\theta \}$, where $c_\theta(s_\theta)$ is defined in equation (24). If $\{F^n\}$ is feasible, then $\lim_{\eta \to 0} \pi^n(\omega) = \bar{\pi}(\omega)$ for all $\omega \in [-1, 1] \setminus \cup_{\theta \in \Theta} \Omega_\theta$.

**Proof.** Take any $\omega \in [-1, 1] \setminus \cup_{\theta \in \Theta} \Omega_\theta$. Then for all $\theta \in \Theta$ and all $s_\theta \in S_\theta$ either $v_\theta(s_\theta; \omega) > 0$ or $< 0$. Thus, for each $\theta \in \Theta$, $\lim_{\eta \to 0} F^n_{\theta}(v_\theta(s_\theta; \omega)) = 1\{v_\theta(s_\theta; \omega) > 0\} = 1\{s_\theta : c_\theta(s) < \omega\}$. So, since $S_\theta$ and $\Theta$ are finite,

$$\lim_{\eta \to 0} \sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_\theta(s_\theta | \omega) F^n_{\theta}(v_\theta(s_\theta; \omega)) = \sum_{\theta \in \Theta} \phi(\theta) \sum_{s_\theta \in S_\theta} q_\theta(s_\theta | \omega) \lim_{\eta \to 0} F^n_{\theta}(v_\theta(s_\theta; \omega))$$

$$= \sum_{\theta \in \Theta} \phi(\theta) q_\theta(\{s \in S_\theta : c_\theta(s) < \omega\})$$

hence the desired result follows. \hfill \Box

**Claim 4.4** (i) If $C^{eqm}(\rho) = \{c\}$ then for all feasible $\{F^n\}$ there exists $\overline{\eta}$ and $\{c^n\}$ such that $\pi^n(c^n) = \rho$ for all $\eta < \overline{\eta}$ and $c^n \to c$; (ii) If $c \in C^{eqm}(\rho)$ and $(C^{eqm}(\rho))^o \neq \emptyset$ (non-empty interior) then there exists a feasible $\{F^n\}$ such that $\pi^n(c) = \rho$ for all $\eta$. 

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Proof. Part (i). Let \( c^0 \equiv \arg\min_{\omega \in [-1, 1]} |\overline{\eta}(\omega) - \rho| \). Suppose, in order to obtain a contradiction, that \( c^0 = 1 \) for all \( \eta \). Then, by continuity of \( \overline{\eta}(\cdot) \), \( \lim_{\eta \to 0} \overline{\eta}(1) < \rho \). Since \( c < 1 \) (by Claim 4.2), then \( 1 \not\in \cup_{\theta \in \Theta} \Omega_\theta \). Claim 4.3 then implies that \( \overline{\eta}(\omega) \leq \overline{\eta}(1) < \rho \) for all \( \omega \), but then it follows that \( c = 1 \), thus contradicting Claim 4.2. Therefore, we can rule out \( c^0 = 1 \) for \( \eta \) sufficiently low. Similarly, we can rule out \( c^0 = -1 \) for \( \eta \) sufficiently low. Therefore, there exists \( \eta \) such that \( c^0 \in (-1, 1) \) and therefore \( \overline{\eta}(c^0) = \rho \) for all \( \eta < \eta \). Finally, consider a subsequence of \( \{c^0\} \) that converges to \( c^* \). It remains to show that \( c^* = c \). Suppose, in order to obtain a contradiction, that \( c^* > c \) (the case \( c < c^* \) is similar). Choose \( c' \in \cup_{\theta \in \Theta} \Omega_\theta \) such that \( c < c' < c^* \) (this is possible because \( \cup_{\theta \in \Theta} \Omega_\theta \) has only a finite number of elements). Let \( \eta \) be such that \( c^0 > c' \) for all \( \eta \). Then \( \overline{\eta}(c') < \rho < \overline{\eta}(c') \) for all \( \eta < \eta \), but this contradicts Claim 4.3.

Part (ii). Since \( (C^{eqm}(\rho))^\circ \neq \{0\} \), there exist types \( 1 \in \Theta \) and \( 2 \in \Theta \) such that \( C^{eqm}(\rho) = \{c_1(s_1^L), c_2(s_2^H)\} \), \( \overline{c}(\cdot) = \rho \) for all \( c \in C^{eqm}(\rho) \), and for all other types \( \theta \), \( c_\theta(s_\theta) \notin C^{eqm}(\rho) \) for all \( s_\theta \in S_\theta \). Therefore, we can partition the type space as follows: \( \Theta = \{1\} \cup \{2\} \cup \{\Theta_\cup\} \cup \{\Theta_+\} \), where \( \theta \in \Theta_\cup \) iff \( c_\theta(s_\theta^H) > c_2(s_2^H) \) and \( \theta \in \Theta_+ \) iff \( c_\theta(s_\theta^H) > c_1(s_1^1) \). Fix \( c \in (C^{eqm}(\rho))^\circ \cup \{c_2(s_2^H)\} \) (the proof for \( c = c_1(s_1^1) \) is similar and therefore omitted), and note that

\[
\overline{v}_\theta(s_\theta; c) \leq 0 \quad \text{for all } \theta \in \Theta_\cup \cup \{2\}, \text{ all } s_\theta \in S_\theta
\]

and

\[
\overline{v}_\theta(s_\theta; c) > 0 \quad \text{for all } \theta \in \Theta_+ \cup \{1\}, \text{ all } s_\theta \in S_\theta,
\]

where the first inequality holds with equality if and only if \( c = c_2(s_2^H) \) and \( \theta = 2 \).

We construct \( \{F^0\} \) as follows. Let \( z_\theta : S_\theta \to (0, 1) \) be an increasing function and let \( z'_\theta : S_\theta \to (0, 1) \) be a decreasing function. Let \( F^0_\theta(v_\theta(s_\theta; c)) = z_\theta(s_\theta)\eta \) for each \( \theta \in \Theta_\cup \) and \( s_\theta \in S_\theta \) as well as for \( \theta = 2 \) and all \( s_\theta \neq s_2^H \). In addition, let \( F^0_\theta(v_\theta(s_\theta; c)) = 1 - z'_\theta(s_\theta)\eta \) for each \( \theta \in \Theta_+ \) and \( s_\theta \in S_\theta \) as well as for \( \theta = 1 \) and all \( s_\theta \neq s_1^L \). Finally, let \( F^0_1(v_1(s_1^1; c)) = 1 - d_1 \eta \), where we leave \( d_1 \) and \( F^0_2(v_2(s_2^H; c)) \) unspecified for the moment. It follows that

\[
\overline{\eta}(c) = \phi(\Theta_\cup) + (B(c) - A(c))\eta - d_1 q_1(s_1^L; c)\phi(\theta_1)\eta + q_2(s_2^H; c)F^0_2(v_2(s_2^H; c)) \phi(\theta_2),
\]

where \( A(c) \) and \( B(c) \) are terms that do not depend on \( \eta \). By the fact that \( \overline{\eta}(c) = \rho \) for all \( c \in C^{eqm}(\rho) \), it follows that \( \phi(\Theta_\cup) = \rho \). We now specify \( F^0_2(v_2(s_2^H; c)) \) to be such that

\[39\]The proof of the case where there is more than one type satisfying each of these restrictions is very similar and therefore omitted.
\[ \eta^\eta(c) = \rho, \text{ i.e.,} \]

\[ F_2^\eta \left( v_2 (s_2^H ; c) \right) = D(c, d_1) \eta, \]

where \( D(c, \cdot) \) is increasing and \( \lim_{d_1 \to \infty} D(c, d_1) = \infty \) for all \( c \). Therefore, we can find \( 1 \leq d_1 < \infty \) such that \( D(c, d_1) \geq 1 \). Pick any such \( d_1 \) for our construction. Finally, let \( \eta \) be small enough such that \( F_2^\eta (v_\theta (s_\theta ; c)) \in (0, 1) \) for all \( \theta, s_\theta \). It then follows by construction that \( \{F_\eta\}_{\eta<\eta} \) is a feasible family of perturbations.

\[ \square \]

**Proof of Theorem 4.** Part 1. Let \( \{c^\eta\} \) be a sequence of limit equilibrium cutoffs that converges to \( c^* \). Suppose, in order to obtain a contradiction, that \( c^* > \sup_{\omega \in [-1,1]} \{\eta^\eta(\omega) \leq \rho\} \). Choose \( c' \notin \cup_{\theta \in \Omega} \{\eta^\eta(\omega) \leq \rho\} \) such that \( \sup_{\omega \in [-1,1]} \{\eta^\eta(\omega) \leq \rho\} < c' < c^* \) (this is possible because \( \cup_{\theta \in \Omega} \Omega \theta \) has only a finite number of elements). Then \( \eta(c') > \rho \) and, by Claim 4.3, \( \eta^\eta(c') > \rho \) for all \( \eta \) small enough. Since \( \eta^\eta(\cdot) \) is increasing (Claim 4.1) and \( c^\eta \to c^* > c' \), it follows that \( \eta^\eta(c^\eta) > \rho \) for all \( \eta \) small enough. But this contradicts that \( c^\eta \) is a limit equilibrium cutoff, according to Theorem 3. A similar proof shows that it cannot be the case that \( c^* < \inf_c \{\eta^\eta(\omega) \geq \rho\} \).

Part 2. Let \( c \in C^{eqm}(\rho) \). By Claim 4.4, there exists a feasible \( \{F_\eta\}_{\eta<\eta} \) such that \( \eta^\eta(c^\eta) = \rho \) and \( c^\eta \to c \). Then, by Theorem 3, \( \{c^\eta\} \) is a sequence of limit equilibrium cutoffs, so that \( c \) is a perfect limit equilibrium cutoff. \[ \square \]

**References**


