Efficient Procurement Auctions with Increasing Returns

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For procuring from sellers with decreasing returns, there are known efficient dynamic auction formats. In this paper, we design an efficient dynamic procurement auction for the case where goods are homogeneous and bidders have increasing returns. Our motivating example is the procurement of vaccines, which often exhibit large fixed costs and small constant marginal costs. The auctioneer names a price and bidders report the interval of quantities that they are willing to sell at that price. The process repeats with successively lower prices, until the efficient outcome is discovered. We demonstrate an equilibrium that is efficient and generates VCG prices.

The auction literature provides us with a number of prescriptions for effective auction design. First, truthful revelation of information is fostered by making bidders' payments as independent as possible of their own bids. Second, when bidders' values are interdependent, the auction should utilize a dynamic structure that permits the revelation of value information during the auction. Third, at the same time, the auction process should avoid requesting or disclosing information that is unnecessary for determining the outcome. Fourth, bidder participation and desirable outcomes are facilitated by simple, transparent and fast auction designs.[1]

For selling a single item, the English auction adheres to all of these design principles. For more general settings, these prescriptions point us toward dynamic auctions that iteratively converge to the Vickrey-Clarke-Groves (VCG) outcome.[2] Dynamic implementations of the VCG mechanism in various environments have received and continue to receive a great deal of attention in the literature (see Demange et al. (1986), Gul and Stacchetti (2000), Parkes and Ungar (2000),

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[1] See, for example, the closely-related discussion in the first paragraph of Ausubel (2004).

[2] See Vickrey (1961), Clarke (1971) and Groves (1973).

Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002 and 2006), Ausubel (2004 and 2006), de Vries et al. (2007), Mishra and Parkes (2007), and Lamy (2012)).

For general private-values environment, Mishra and Parkes (2007) construct a class of ascending-price combinatorial auctions that terminate at the VCG outcome. These auctions are quite complex, implying that the auctioneer has to sacrifice a number of desirable properties of the English auction in order to implement the VCG outcome in general settings. However, simpler auction designs are known for more restrictive settings. For example, Demange et al. (1986) develop a dynamic Vickrey auction for the unit-demand case, and Ausubel (2004) does the same for environments with homogeneous items and nonincreasing marginal values.

In this article, we study procurement settings with homogeneous goods. For the case of convex cost functions, a *descending clock auction with "clinching"*[3] is a simple dynamic auction that implements the VCG outcome. However, many important procurement markets exhibit economies of scale and production limits, resulting in concave cost functions with capacity constraints. For such settings, we provide a relatively simple dynamic auction that implements the VCG outcome.

Our motivating example for studying this setting is the procurement of vaccines. The largest buyer of vaccines worldwide is an international organization called Global Alliance for Vaccines and Immunisation (GAVI), which was launched in 2000 with the mission to increase access to immunization in poor countries. As of this writing in 2016, GAVI is assisting 73 low-income countries in obtaining vaccines, resulting in half a billion additional children being vaccinated to date. As the largest buyer, GAVI shapes the world vaccine market by ensuring persistent demand that attracts new suppliers and reduces immunization costs. UNICEF, which serves as a procurement agent for GAVI, is responsible for procuring hundreds of millions of doses of vaccines annually.[4]

Manufacturing vaccines is a highly specialized industry with large barriers to entry. New entry into the vaccine market may require making significant investments in R&D, performing clinical trials, obtaining regulatory approvals and building production facilities. It typically takes about 10 years and costs more than $1 billion to bring a new vaccine to market. The production line for a vaccine is capable of producing the raw vaccine for a fixed number of doses; in addition, marginal costs are associated with the fill/finish process. Suppliers cannot adjust their production in response to sudden demand changes. Production lines of a multi-vaccine supplier are not fungible in the sense that the production facility for one vaccine cannot easily be converted to produce a different vaccine. As a result, it would typically take years for a manufacturer to expand its capacity and to get the required regulatory approvals.

[3]A reverse version of the ascending clock auction with "clinching" from Ausubel (2004).

[4]For an overview of UNICEF vaccine procurement, see http://www.unicef.org/supply/index_vaccines.html (last accessed on 10/11/2016).

The global vaccine market is dominated by a handful of large multinational firms, though smaller vaccine manufacturers from developing countries have recently begun to play a larger role. The fixed capital expenditures associated with R&D, clinical trials, regulatory approvals and plant equipment constitute a significant proportion of the total production costs and are largely independent of the number of doses that is ultimately produced. In particular, the average cost of producing a vaccine is decreasing up to a predetermined maximum capacity. Typically, UNICEF's demand for a particular vaccine is so large that it cannot be fulfilled by a single supplier. Moreover, due to concerns about supply security and future procurements, UNICEF appears to prefer to have multiple suppliers for a given vaccine even if the short-run procurement costs would have been lower with fewer suppliers.[5] Given the available information on vaccine production, it seems appropriate to model the cost structure of a vaccine manufacturer as consisting of a large fixed cost and a small constant marginal cost, up to a predetermined capacity limit. In this article, we consider the more general setting of a concave cost function, again up to a predetermined capacity.

In a typical descending clock procurement auction, the auctioneer quotes a unit price and asks each bidder for its supply (i.e., its optimal quantity) at that price. With convex cost functions, bidders would gradually decrease their desired supply in response to the descending price, converging to an efficient market clearing. However, when bidders have concave cost functions and the auctioneer quotes a unit price, it is optimal for the bidder either to produce at its capacity limit or to produce nothing, and it is never optimal for the bidder to produce any intermediate quantity. But, then, the standard descending clock format can only discover one point on the cost curve — the cost associated with producing at maximum capacity — so the auctioneer never learns the costs of other quantities as the price goes down.

Consider an example with three suppliers, each characterized by a cost function with a fixed cost, a constant marginal cost, and a capacity. The auctioneer wants to procure a total quantity of 4 of the fully divisible good, and each supplier can produce no more than 3. The cost functions of suppliers are $c_1(q) = 25 + q$, $c_2(q) = 27 + q$, and $c_3(q) = 24 + 3q$. Due to concavity, the cost-minimizing assignment is a 3 - 1 split of the award between two suppliers, e.g., one supplier would produce a quantity of 3 and another supplier would produce a quantity of 1. Hence, to identify the optimal split, the auctioneer needs to collect costs for producing 1 and 3 from each supplier.

Suppose that all suppliers bid truthfully. In a standard descending clock auction, a truthful bidder would offer 3 units (its capacity) until the bidder drops out. Therefore, the auctioneer would learn the suppliers' costs for producing $q = 3$, but not for $q = 1$. Specifically, a standard descending auction would terminate at a price of 10 (when the aggregate supply falls below demand) with the award for $q = 3$ going to supplier 1 and the award for $q = 1$ being unassigned (see the

[5]Supply security considerations are not explicitly studied in this paper.

Table 1—Example with Fixed Costs and Constant Marginal Costs

	Supplier 1	*Supplier 2*	*Supplier 3*	
Total Costs:	$c_1(q) = 25 + q$	$c_2(q) = 27 + q$	$c_3(q) = 24 + 3\,q$	
Lowest Profitable Quantity given $p(t)$:	$q_1(t) = \frac{25}{p(t)-1}$	$q_2(t) = \frac{27}{p(t)-1}$	$q_3(t) = \frac{24}{p(t)-3}$	
Price $p(t)$		Bidding Intervals (new approach)		Standard Agg. Supply
$p(t_0) = 30$	$s_1 = \left[\frac{25}{29}, 3\right]$	$s_2 = \left[\frac{27}{29}, 3\right]$	$s_3 = \left[\frac{24}{27}, 3\right]$	9
$p(t_1) = 28$	$s_1 = \left[\frac{25}{27}, 3\right]$	$s_2 = [1, 3]$	$s_3 = \left[\frac{24}{25}, 3\right]$	9
$p(t_2) = 27$	$s_1 = \left[\frac{25}{26}, 3\right]$	$s_2 = \left[1\frac{1}{26}, 3\right]$	$s_3 = [1, 3]$	9
$p(t_3) = 26$	$s_1 = [1, 3]$	$s_2 = \left[1\frac{2}{25}, 3\right]$	$s_3 = \left[1\frac{1}{23}, 3\right]$	9
...	...	...	...	...
$p(t_4) = 11$	$s_1 = \left[2\frac{1}{2}, 3\right]$	$s_2 = \left[2\frac{7}{10}, 3\right]$	$s_3 = []$	6
$p(t_5) = 10$	$s_1 = \left[2\frac{7}{9}, 3\right]$	$s_2 = []$	$s_3 = []$	3
$p(t_6) = 9\frac{2}{3}$	$s_1 = \left[2\frac{23}{26}, 3\right]$	$s_2 = []$	$s_3 = []$	(N/A)

last column of Table 1). Furthermore, at this point, there is no good way for the auctioneer either to select whom to assign the $q = 1$ award or to determine the corresponding payment. Also, with the auction terminated at price 10, it has not been proven that supplier 1 should be assigned 3 rather than 1 (the auction only proved that supplier 1 should not be assigned zero). To summarize, standard auction designs based upon eliciting one-point supplies are, in general, ill-suited for determining optimal assignments in this setting.

We propose a new bidding procedure. Given the current price, instead of asking which quantity a bidder ***prefers*** to supply, the auctioneer requests all quantities that the bidder is ***willing*** to supply. With decreasing average costs, the minimum quantity that a bidder is willing to supply should gradually increase as the clock price decreases (while the maximum quantity is equal to the capacity, until that becomes unprofitable). Then the auctioneer can ask for a connected ***interval of quantities*** that would be profitable for the bidder to supply at a given price.

In our example, given a unit price of $p(t)$ and assuming truthful bidding, supplier i would be willing to supply any quantity q such that $p(t)\,q \geq c_i(q)$. For example, when the unit price is 26, supplier 1 is willing to supply any quantity $q \in [1, 3]$; and supplier 1's bidding interval reduces to $[2.5, 3]$ when the clock price drops to 11. Table 1 presents the detailed auction dynamics for this example.

Suppose that the auctioneer initializes the auction at $p(t_0) = 30$. Due to the interval bidding approach, the auctioneer learns the costs of each supplier for producing $q = 1$ by the time the price drops to 26. By the time the price drops to 10, the auctioneer knows that the optimal assignment is either (1, 3, 0) or (3, 0, 1): the current total cost of (1, 3, 0) is 56 and the current total cost of (3, 0, 1) is 57. Hence, the auctioneer needs to find out whether supplier 1 is willing to produce $q = 3$ for 29, which would reduce the total cost of (3, 0, 1) to 56. By allowing the price to drop a little further, to $9\frac{2}{3}$, the auctioneer confirms that the assignment (3, 0, 1) is indeed efficient.

The auctioneer uses the cost information generated by the interval bidding approach to reconstruct the suppliers' cost functions. In general, the auctioneer would stop the auction before all cost information is revealed, since the efficient assignment and corresponding payments can be found using only partial cost information (due to the concavity assumption). The efficient assignment and the payments are calculated by solving the standard winner determination problems using the partially-reconstructed cost functions as inputs.

The proposed auction design has a number of desirable properties. The auction uses linear and anonymous prices to elicit costs — making it simple, intuitive and fast. At each price, bidders reveal cost information about quantities that are no longer profitable. If the auctioneer discloses this information to bidders, the format can potentially yield price discovery, reducing bidders' cost uncertainties (if costs are interdependent). At the same time, winning bidders in general do not reveal their costs for the awarded quantities. Therefore, the format strikes a balance between price discovery and privacy preservation. Finally, if the auctioneer uses the VCG outcome, then a fully efficient assignment is supported as an equilibrium.

Our analysis is strongly influenced by the general dynamic implementation of the VCG mechanism from Mishra and Parkes (2007) and the follow-up analysis in Lamy (2012). However, we do not explicitly use their concept of *universal competitive equilibrium* (UCE) price. For our setting, finding a UCE price vector is equivalent to partially reconstructing the cost functions of the suppliers so that the VCG outcome can be found. Other related articles include Mishra and Parkes (2009) and Mishra and Veeramani (2007), who develop Vickrey-Dutch auctions and compare their privacy preservation properties with their standard "English-like" counterparts.

The article is organized as follows. Section I provides a model of the environment, and Section II formally describes the auction procedure with interval bidding. The main results are established in Section III. Several implementation issues are discussed in Section IV. Section V presents a detailed example that illustrates the main elements of the new bidding procedure. Section VI concludes. Appendix A contains results pertaining to the implementation of *core* outcomes. Most of the proofs are relegated to Appendix B.

I. Model

An auctioneer wishes to procure D units of an indivisible homogeneous good from a set of suppliers $N = \{1, 2, ..., n\}$.[6] Supplier i can produce any quantity from the set $S_i = \{0, 1, ..., \bar{s}_i\}$ where $\bar{s}_i$ is the maximum production capacity of supplier i. Production capabilities of supplier i are fully characterized by a cost function $c_i(q),\ q \in S_i$. A supplier's cost for producing zero units is zero, $c_i(0) = 0$. We assume that all suppliers have increasing cost functions with non-increasing marginal costs (e.g. concave), i.e., $c_i(q) - c_i(q-1) \geq c_i(q+1) - c_i(q)$ for all $q \in \{1, ..., \bar{s}_i - 1\}$ and for all $i \in N$. Supplier i realizes a net payoff $p_i - c_i(q_i)$ when it receives a payment p_i in exchange for supplying q_i units of the good.

An economy that includes only suppliers from set $M \subseteq N$ is denoted as $E(M)$. Let $N_{-i} = N \backslash \{i\}$ denote the set of all suppliers in N excluding supplier i. The *main economy* is $E(N)$ and the *marginal economy* for supplier i is $E(N_{-i})$.

It is assumed that the auctioneer has an alternative source to procure any quantity of the good at a per unit cost of $\bar{c}$.[7] This assumption ensures that the auctioneer can always procure D units of the good in any economy $E(M)$ even if the total maximum capacity of suppliers in M is not sufficient to meet the full demand. For purely expositional purposes, we assume that $c_i(q) \leq \bar{c}\, q$ for all $q \in S_i$ and all $i \in N$.[8]

We say that assignment $x = (x_1, ..., x_n)$ is *feasible* for the economy $E(M)$ if $x_i \in S_i$ for all $i \in M$, $x_i = 0$ for all $i \notin M$ and $\sum_M x_i \leq D$. Denote $X(M)$ a set of feasible assignments for the economy $E(M)$.

Assignment x is *efficient* for the economy $E(M)$ if it is a feasible assignment that minimizes the total cost of producing D units of the good:

$$TC(M) = \min_{x \in X(M)} \left[\sum_M c_i(x_i) + \bar{c} \cdot (D - \sum_M x_i) \right] \tag{1}$$

Proposition 1 shows that efficient assignments in this environment tend to be asymmetric, allocating either their maximum capacity or zero to the majority of suppliers.

PROPOSITION 1: *For any economy $E(M)$, $M \subseteq N$, there exists an efficient assignment $x = (x_1, ..., x_n)$ such that at most one supplier $i \in M$ is assigned a positive quantity that is strictly less than its capacity, i.e., $0 < x_i < \bar{s}_i$.*

A *Vickrey outcome* is an assignment $x = (x_1, ..., x_n)$ and a payment vector

[6]The assumption of indivisible goods is purely for expositional convenience and general usefulness for practical auction designs. With divisible goods, an appropriately modified auction design has the same properties.

[7]This assumption is equivalent to having a reserve price of $\bar{c}$, the maximum price the auctioneer is willing to pay per unit of the good.

[8]If $c_i(q) > \bar{c}\, q$ for some q of supplier i, this cost information is irrelevant for the auctioneer since it is not efficient to assign quantity q to supplier i in any economy.

$p^V(x) = (p_1^V(x), ..., p_n^V(x))$ such that x is an efficient assignment for $E(N)$ and $p_i^V(x) = c_i(x_i) + [TC(N_{-i}) - TC(N)]$. The Vickrey payment to supplier i consists of (1) the cost compensation for producing x_i units and (2) a nonnegative bonus that reflects the social value of supplier i to the main economy. The corresponding Vickrey payoff of supplier i is $\pi_i^V = p_i^V(x) - c_i(x_i) = TC(N_{-i}) - TC(N) \geq 0$.

II. Auction Procedure with Interval Bidding

Our auction procedure utilizes a standard descending clock price initialized at $p(0) = \bar{c}$. Let $p(t)$ denote a continuous descending price path on $[0, T]$ where T is the termination time at which one of the auction closing conditions (specified later) is met.

The auctioneer has several ways to elicit cost information from bidders using the interval bidding approach. The most natural one is to ask suppliers to name all possible production levels that they would be willing to provide in exchange for a per unit payment $p(t)$. Then supplier i who at time t excluded a previously acceptable quantity q from its report has just revealed its cost for q to be $p(t)\,q$. We refer to this approach as *average cost elicitation.*

Under the average cost elicitation approach, supplier i is said *to bid according to cost function* $c(.)$ *on set* S if the set of acceptable quantities is $s_i(t) = \{q \in S : c(q) < p(t)\,q\}$ at every time $t \in [0, T]$; and supplier i is said to *bid truthfully* if it bids according to its true concave cost function $c_i(.)$ on its true feasible set S_i. Throughout this section, we assume that all suppliers bid truthfully.

LEMMA 1: *If supplier i bids truthfully according to its concave cost function $c_i(.)$, then for all $t, t' \in [0, T]$:*

(a) *$s_i(t)$ is a connected set;*

(b) *$s_i(t') \subseteq s_i(t)$ for all $t' > t$;*

(c) *If $s_i(t)$ is nonempty, then $\bar{s}_i \in s_i(t)$.*

PROOF:

Concavity of $c_i(.)$ implies non-increasing average costs. Given the descending clock price trajectory $p(t)$, a supplier who is bidding according to a concave cost function would submit an interval that includes all quantities from its feasible set that are above some threshold level, trivially implying properties (a)-(c).

A. Reconstructing Cost Functions

The auctioneer infers the maximum capacity of supplier i, $\bar{s}_i$, by noting its highest acceptable quantity at $t = 0$. Let $q_i(t)$ denote the highest unacceptable quantity of supplier i in set S_i at time t. For any $q \leq q_i(t)$, denote $t_i(q) = \{\min\ t' \in [0, t] : q \notin s_i(t')\}$, the time supplier i removed quantity q from its bidding interval and $\tilde{c}_i(q) = p(t_i(q))\, q$ the associated revealed cost for q units. The revealed cost for producing zero units is set to zero, $\tilde{c}_i(0) = 0$.

At time t, the revealed marginal cost for the highest unacceptable unit $q_i(t)$ is $mc_i^-(t) = \tilde{c}_i(q_i(t)) - \tilde{c}_i(q_i(t)-1)$. Because of the concavity of the cost function, the auctioneer can infer that the marginal cost of supplier i for the lowest acceptable unit, $q_i(t)+1$, is bounded from above by $mc_i^-(t)$. Furthermore, since supplier i is willing to supply $q_i(t)+1$ units at price $p(t)$ per unit, the auctioneer can infer that $c_i(q_i(t)+1) \leq p(t)\,[q_i(t)+1]$, and thus the marginal cost of supplier i for the lowest acceptable unit, $q_i(t)+1$, is also bounded from above by $mc_i^+(t) = p(t)\,[q_i(t)+1] - \tilde{c}(q_i(t))$. We denote the lower of these two upper bounds on the revealed marginal cost for the lowest acceptable unit of supplier i at time t as $mc_i(t) = \min\{mc_i^+(t), mc_i^-(t)\}$.

The auctioneer approximates the cost function for supplier i as follows:

$$\hat{c}_i(q,t) = \begin{cases} \tilde{c}_i(q) & q \leq q_i(t) \\ \tilde{c}_i(q_i(t)) + mc_i(t)\,[q - q_i(t)] & q_i(t) < q \leq \bar{s}_i \end{cases} . \tag{2}$$

The approximation error at time t for supplier i and quantity q is given by:

$$\delta_i(q,t) = \hat{c}_i(q,t) - c_i(q). \tag{3}$$

The approximation process is illustrated in Figure 1. Suppose that at time t, supplier i drops quantity q from its bidding interval (e.g., $q_i(t) = q$) and continues to include quantity $q_i(t)+1$ in its bidding interval until time t' where $t < t^* < t'$ and t^* is determined by the equation $mc_i^-(t^*) = mc_i^+(t^*)$. Note that at time t, supplier i revealed its cost for quantity $q_i(t)$ to be $\tilde{c}_i(q_i(t)) = p(t)\, q_i(t)$ and the associated marginal cost to be $mc_i^-(t)$ (the slope of segment AB).

If supplier i were to drop $q_i(t)+1$ at time $t_1 \in (t, t^*)$, the implied cost of $q_i(t)+1$ would be too high to produce a concave cost function (segment BC). To maintain the concave approximation, the auctioneer approximates costs for $q_i(t)+1$ at time t_1 to be $\tilde{c}_i(q_i(t)) + mc_i^-(t_1)$ (segment BD). Note that a supplier who bids truthfully according to a concave cost function would never drop quantity $q_i(t)+1$ from its bidding interval on (t, t^*).[9] However, if supplier i were to drop this quantity at $t_2 \in [t^*, t')$, the implied cost of $q_i(t)+1$ would be fully consistent with a concave cost function and the auctioneer would approximate the cost for $q_i(t)+1$ to be $\tilde{c}_i(q_i(t)) + mc_i^+(t_2)$ (segment BE).

Lemma 2 below shows that $\hat{c}_i(.,t)$ is a well-behaved approximation of the cost function $c_i(.)$. It weakly converges towards $c_i(.)$ from above and maintains the concave shape at all times. Also both the approximation error $\delta_i(.,t)$ and the reduction in the approximation error over time are monotonic functions of quantity.

[9]This property is used to motivate the activity rule AR2 in Proposition 3.

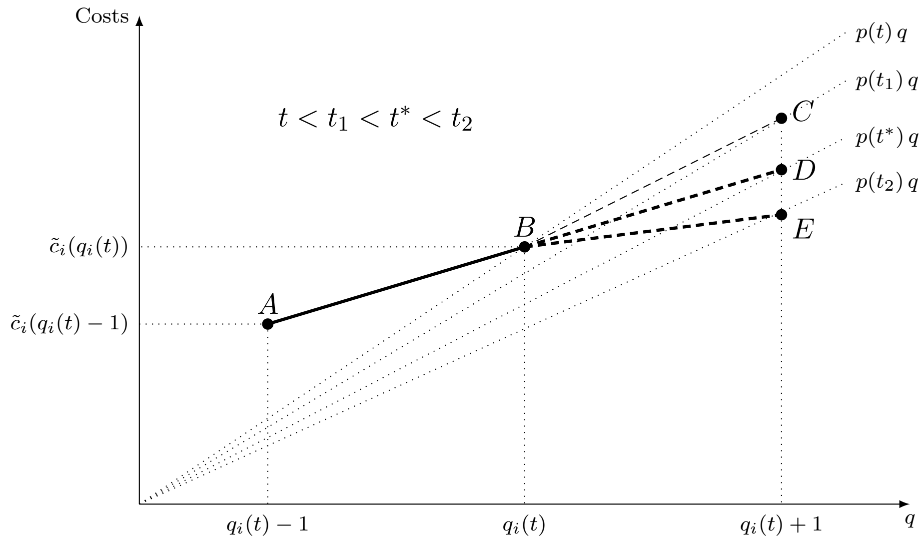

FIGURE 1. *Cost Approximation using the Interval Bidding Approach*

Note: Points A, B and D lie on a straight line.

LEMMA 2: *If supplier i bids truthfully according to its concave cost function $c_i(.)$, then for all $q, q' \in S_i$ and all $t, t' \in [0, T]$:*

(a) $\hat{c}_i(q,t) \geq c_i(q)$ *for all* $q \in S_i$ *and* $\hat{c}_i(q,t) = c_i(q)$ *for all* $q \leq q_i(t)$*;*

(b) $\hat{c}_i(q,t)$ *is increasing and concave in q and weakly decreasing in t;*

(c) $\delta_i(q,t)$ *is weakly increasing in q;*

(d) *for* $t' \geq t$ *and* $q' \geq q$

$$\delta_i(q,t) - \delta_i(q,t') \leq \delta_i(q',t) - \delta_i(q',t').$$

B. *Closing Rule and Auction Outcome*

Given the current clock price $p(t)$, supplier i exits the auction when it is no longer profitable to supply its maximum capacity $\bar{s}_i$. A supplier who wishes to exit submits an empty bidding interval $s_i(t) = \{\}$ which implies that $q_i(t) = \bar{s}_i$. Denote by $A(M,t) = \{i \in M : q_i(t) < \bar{s}_i\}$ the set of *active* suppliers from set $M \subseteq N$ who are still willing to supply their maximum capacity at time t, and denote $I(M,t)$ a complimentary set of *inactive* suppliers from set M.

Utilizing current estimates of the cost functions $\hat{c}_i(.,t)$ for all suppliers in M, the auctioneer finds a tentative assignment for economy $E(M)$ denoted as $\hat{x}(M,t) =$

$(\hat{x}_1(M,t),...,\hat{x}_n(M,t))$ by minimizing the total cost of procurement:

$$\widehat{TC}(M,t) = \min_{x \in X(M)} \left[\sum_M \hat{c}_i(x_i,t) + \bar{c}\,(D - \sum_M x_i) \right] \tag{4}$$

If there are several assignments that minimize (4), the auctioneer selects the one that maximizes the total number of units assigned to active suppliers in $A(M,t)$.

An aggregate supply is traditionally defined as the sum of quantities desired by suppliers at a given price. In our setting, the desired quantity of each supplier is either its maximum capacity or zero, resulting in a very lumpy aggregate supply that in general does not provide enough information to make the decision about closing the auction. Instead, we introduce an alternative definition of the aggregate supply that is suitable for this setting. Let $AS(M,t)$ be the aggregate supply for economy $E(M)$ at time t calculated as the sum of (1) the maximum capacities of active suppliers in $A(M,t)$;[10] (2) the tentative assignments for inactive suppliers in $I(M,t)$ and (3) the units procured from the alternative source (if any):

$$AS(M,t) = \sum_{A(M,t)} \bar{s}_i + \sum_{I(M,t)} \hat{x}_i(M,t) + \left[D - \sum_M \hat{x}_i(M,t) \right] \tag{5}$$

The rationale for the aggregate supply defined in (5) is simple. Aggregate supply at $p(t)$ should reflect the current level of competition between *all* suppliers. In a setting with concave cost functions, sometimes an inactive supplier can create competition for active suppliers due to a better fit. The aggregate supply should account for such competition; and the second term in (5) reflects such competition from inactive suppliers. Lemma 3 below summarizes several properties of the aggregate supply $AS(M,t)$.

LEMMA 3: *If all suppliers bid truthfully according to their concave cost functions, then for any $M \subseteq N$ and for all $t \in [0,T]$:*

(a) $AS(M,t) = D + \sum_{A(M,t)} \left[\bar{s}_i - \hat{x}_i(M,t) \right]$;

(b) $AS(M,t) \geq D$;

(c) *If $AS(M,t) = D$, then $AS(M,t') = D$ for all $t' \geq t$;*

(d) *If $\sum_{A(M,t)} \bar{s}_i = D$, then $AS(M,t) = D$.*

According to property (a) of Lemma 3, aggregate supply for economy $E(M)$ equals demand, $AS(M,t) = D$, when all actively bidding suppliers in $A(M,t)$ have

[10]This term is the usual aggregate supply since all active bidders desire their maximum capacity at $p(t)$.

been assigned their maximum capacities in the tentative assignment $\hat{x}(M,t)$. We say that economy $E(M)$ *is cleared* at t if its aggregate supply $AS(M,t)$ equals demand D.

The traditional Walrasian notion of market clearing might also apply here – by property (d), if at any time t, the maximum supply of active suppliers in $A(M,t)$ equals demand D, then $AS(M,t) = D$ and economy $E(M)$ is cleared. However, the existence of Walrasian clearing price is not guaranteed in our environment with concave cost functions.

In Proposition 2, we prove that clearing an economy is equivalent to finding an efficient assignment for this economy.

PROPOSITION 2: *If all suppliers bid truthfully according to their concave cost functions and economy $E(M)$ clears at time t, the tentative assignment $\hat{x}(M,t)$ is an efficient assignment for $E(M)$ and*

$$\widehat{TC}(M,t) = TC(M) + \sum_{A(M,t)} \delta_i(\hat{x}_i(M,t),t) \tag{6}$$

The setting with concave costs permits bidder complementarities and, as a result, the aggregate supply $AS(M,t)$ may be nonmonotonic in t.[11] Note that potential nonmonotonicity of the aggregate supply does not affect clearing – by property (c) of Lemma 3, once an economy $E(M)$ is cleared ($AS(M,t) = D$), it stays cleared until the end.

The aggregate supply $AS(M,t)$ can also be nonmonotonic in M, so the main economy $E(N)$ might sometimes clear before some of its marginal economies.[12] To recover the Vickrey outcome, the auctioneer needs to know efficient assignments for the main economy and for all marginal economies. Hence, the auctioneer must continue to collect information about cost functions until the main economy and all marginal economies clear.[13] For supplier i, define a tentative Vickrey payment at time t for quantity q as follows:

$$\hat{p}_i^V(q,t) = \hat{c}_i(q,t) + [\widehat{TC}(N_{-i},t) - \widehat{TC}(N,t)]. \tag{7}$$

The first set of auction closing rules is as follows:

Closing Rule 1: *The auctioneer stops the clock price ($T := t$) once all economies in the set $\{E(N), E(N_{-1}), ..., E(N_{-n})\}$ have been cleared. Supplier i is awarded its tentative allocation for the main economy $x_i = \hat{x}_i(N,T)$ and receives payment $\hat{p}_i^V(x_i,T)$.*

[11] An example of nonmonotonic aggregate supply is included with the proof for Lemma 3.

[12] For example, $AS(N,t_4) = 6$ and $AS(N_{-4},t_4) = 8$ in the illustrative example used in Section V.

[13] In general, once the main economy is cleared, the auctioneer needs additional cost information only from a subset of active bidders. It is possible to modify our auction procedure to minimize unnecessary cost elicitation.

III. Main Results

THEOREM 1: *If all suppliers bid truthfully according to their concave cost functions, the interval bidding auction procedure with Closing Rule 1 implements the Vickrey outcome.*

PROOF:

By Proposition 2, $x = \hat{x}(N,T)$ is an efficient assignment for $E(N)$ and $x' = \hat{x}(N_{-i},T)$ is an efficient assignment for $E(N_{-i})$. Further, if bidder j is active at time T, then $x_j = x'_j = \bar{s}_j$. The tentative Vickrey payment of bidder i equals its Vickrey payment since:

$$\begin{aligned}\hat{p}_i^V(x_i,T) &= \hat{c}_i(x_i,T) + [\widehat{TC}(N_{-i},T) - \widehat{TC}(N,T)] \\ &= \hat{c}_i(x_i,T) + [TC(N_{-i}) - TC(N)] - \delta_i(x_i,T) \\ &= c_i(x_i) + [TC(N_{-i}) - TC(N)] = p_i^V(x)\end{aligned}$$

So far we have been assuming that all suppliers bid truthfully. When the Vickrey outcome is implemented through a direct revelation mechanism, it is weakly dominant for suppliers to report their true costs. A dynamic implementation of the Vickrey outcome, such as ours, should preserve good incentives for suppliers provided they are sufficiently constrained in their action space at each stage of the dynamic game – a requirement that each supplier bids according to some increasing concave cost function. This requirement can be enforced by constraining bidders with appropriate activity rules.

PROPOSITION 3: *Supplier i bids according to an increasing concave cost function if and only if its bidding interval $s_i(t) = \{\underline{s}_i(t), ..., \bar{s}_i(t)\}$ is constrained by the activity rules AR1-AR3:*

> ***AR1:*** *$\underline{s}_i(t)$ is weakly increasing in t and $\bar{s}_i(t) = \bar{s}_i(0)$ for all t such that $s_i(t) \neq \{\}$;*
>
> ***AR2:*** *Supplier i is not allowed to increase its $\underline{s}_i(t)$ when $mc_i^+(t) > mc_i^-(t)$;*[14]
>
> ***AR3:*** *Supplier i becomes inactive ($s_i(t) = \{\}$) at time t if $p(t)\,\underline{s}_i(t) = \tilde{c}_i(q_i(t))$.*

Proposition 3 provides a complete characterization of bidding in accordance with an increasing concave function at all times. Therefore, AR1-AR3 together form the strictest set of activity rules that always permit truthful bidding in our setting. Intuitively, AR1 ensures that supplier i bids according to a cost function

[14] If supplier i wants to increase its $\underline{s}_i(t)$ by more than one unit at $p(t)$, $\underline{s}_i(t)$ is increased in one-unit increments provided that AR2 stays satisfied after each increase (since both $mc_i^+(t)$ and $mc_i^-(t)$ are updated after each increase in $\underline{s}_i(t)$).

with nonincreasing average costs, and AR2 ensures that this cost function is concave.[15] Finally, AR3 ensures that the underlying cost function is nondecreasing. Additionally, AR3 ensures that supplier i will be inactive by the time $p(t) = 0$, so the auction cannot run indefinitely.

The truthful bidding assumption used in Lemmas 1 - 3 is made solely for expositional convenience. All lemmas (with appropriate changes to notation) stay true and the interval bidding auction procedure is well-defined as long as all suppliers bid according to some concave cost functions, i.e., when their bidding is constrained by AR1-AR3. However, the truthful bidding assumption is necessary for Proposition 2 and Theorem 1. The next theorem provides a game-theoretic justification for this assumption. Note that ex post equilibrium is the standard solution concept in the literature on dynamic implementations of the VCG mechanism.[16]

THEOREM 2: *If all suppliers have concave cost functions and their bidding is constrained by activity rules AR1-AR3, then truthful bidding by all suppliers is an ex post equilibrium.*

PROOF:

Suppose that all suppliers in N_{-i} bid truthfully. By Proposition 3, any deviation by supplier i from its true cost function $c_i(.)$ is equivalent to truthful bidding according to some other concave cost function $c_i'(.)$. By Theorem 1, the outcome of the interval bidding auction in that case would correspond to an outcome of the VCG mechanism when the submitted cost functions are $\{c_i'(.), c_{-i}(.)\}$. But the VCG mechanism is strategy-proof, and so supplier i's best response is to bid truthfully according to $c_i(.)$.

Continuing the clock auction after the main economy has been cleared can potentially run into some problems in applications. If bidders are aware that the efficient allocation has been identified and cannot be altered, their incentives can be compromised.[17] Next, we study the properties of an auction procedure with interval bidding that stops collecting information once the main economy is cleared.

Closing Rule 2: *The auctioneer stops the clock price ($T := t$) once the main economy $E(N)$ has been cleared. Supplier i is awarded its tentative allocation for the main economy $x_i = \hat{x}_i(N, T)$ and receives payment $\hat{p}_i^V(x_i, T)$.*

THEOREM 3: *If all suppliers bid truthfully according to their concave cost functions, the interval bidding auction procedure with Closing Rule 2 implements an*

[15] See Figure 1, its surrounding text and footnote 9 for the intuition regarding AR2.

[16] See Bikhchandani and Ostroy (2006), de Vries et al. (2007) and Mishra and Parkes (2007). Several papers have used a stronger ex post perfect equilibrium concept. However, they either established the equilibrium result for auction procedures that did not employ activity rules (see Gul and Stacchetti (2000) and Ausubel(2006)) or for simpler settings where bidding as close as possible to truthfully can be argued to be the best response at every stage of the auction (see Ausubel (2004)).

[17] For example, bidders can deviate from truthful bidding in order to reduce payments made by the auctioneer to their competitors.

efficient assignment $x = \hat{x}(N, T)$ *and payments* $\hat{p}_i^V(x_i, T)$ *such that:*

$$c_i(x_i) \leq \hat{p}_i^V(x_i, T) \leq p_i^V(x) \qquad \textit{for all} \quad i \in N. \tag{8}$$

Theorem 3 shows that stopping the auction when the main economy is cleared is a viable alternative – when using current cost approximations to determine Vickrey payments, the auctioneer neither underpays suppliers nor overcompensates them. However, the auctioneer risks paying too little to suppliers, potentially compromising their incentives for truthful revelation of their costs.

We finish this section by highlighting a curious property of the interval bidding procedure. Mishra and Parkes (2007) show that stopping the auction once the main economy is cleared (e.g. Closing Rule 2) is sufficient to implement a Vickrey outcome when the underlying value/cost functions satisfy the well-known "bidders are substitutes" condition. Intuitively, bidders are substitutes when their values/costs are sufficiently similar to each other. To be more specific, in our setting, *suppliers are substitutes* (SAS) if

$$TC(N/M) - TC(N) \;\geq\; \sum_{i \in M} \big[TC(N/i) - TC(N)\big] \qquad \text{for all} \quad M \subseteq N. \tag{9}$$

Proposition 4 below demonstrates that the SAS condition is no longer sufficient for the interval bidding procedure to produce the same result. The key difference that is responsible for this limitation is the way in which approximations of values/costs are constructed under the interval bidding procedure.[18]

PROPOSITION 4: *If suppliers are substitutes (SAS), the interval bidding auction procedure with Closing Rule 2 in general does not yield a Vickrey outcome.*

PROOF:

The proof is by an example with $D = 4$ and three suppliers that is provided in Table 2. It can be verified that cost functions satisfy the SAS condition. The main economy clears at t_1 when the clock price equals $p(t_1) = 16$. However, the tentative Vickrey payments for Supplier 1 is equal to 36 which is lower than its actual Vickrey payment of 38 (note that $AS(N_{-1}, t_1) = AS(N_{-2}, t_1) = 5 > 4 = AS(N, t_1)$).

A detailed example illustrating the mechanics of the auction with interval bidding is provided in Section V. Results on implementing core outcomes using the interval bidding procedure can be found in Appendix A.

[18]Most of the dynamic combinatorial auction literature, including Mishra and Parkes (2007), uses the semi-truthful approximations. The approximation of the cost function $\hat{c}(., t)$ utilized by the interval bidding procedure is not semi-truthful. Stated using our terms, an approximation of the cost function $\hat{c}(., t)$ is semi-truthful if $\hat{c}(q, t) = \min\{\bar{c}\,q, c(q) + \alpha(t)\}$ for all $q \in S_i$ and all $t \in [0, T]$.

Table 2—Proof by Example for Proposition 4 ($D = 4$)

	Supplier 1	*Supplier 2*	*Supplier 3*
Costs:	$c_1 = (20, 30)$	$c_2 = (20, 30)$	$c_3 = (28, 42, 48)$
Efficient Assignment:	2	2	0
Vickrey Payments:	38	38	0
p(0) = 30	$s_1 = \{1, 2\}$ $\hat{c}_1 = (30, 60)$ $\hat{p}_1^V(2) = 60$	$s_2 = \{1, 2\}$ $\hat{c}_2 = (30, 60)$ $\hat{p}_2^V(2) = 60$	$s_3 = \{1, 2, 3\}$ $\hat{c}_3 = (30, 60, 90)$ $\hat{p}_3^V(3) = 90$
$p(t_1) = 16$ *efficiency proven*	$s_1 = \{2\}$ $\hat{c}_1 = (20, 32)$ $\hat{p}_1^V(2) = 36$	$s_2 = \{2\}$ $\hat{c}_2 = (20, 32)$ $\hat{p}_2^V(2) = 36$	$s_3 = \{\}$ $\hat{c}_3 = (28, 42, 48)$ $\hat{p}_3^V(0) = 0$

Note: $c_1 = (20, 30)$ indicates that supplier 1 can produce 1 unit of the good at a cost of 20, and 2 units of the good at a cost of 30.

IV. Implementing the Interval Bidding Auction

In this section we discuss a few issues related to implementing the auction with interval bidding.

A. *Privacy Preservation*

Dynamic auctions are favored over sealed-bid ones for several reasons. One of them is "privacy preservation" – the ability to determine an optimal outcome while relying only on partial information about suppliers' costs. The notion of privacy preservation is trivial for a single-item English procurement auction: a winner only reveals that its cost is lower than the lowest cost among its rivals. For multiple items, the meaning of privacy preservation is unclear – which part of its cost function would a winner like to keep private?

Under the interval bidding procedure, suppliers start to reveal their cost functions from low quantities towards the high quantities. Then, if the auction stops without fully revealing the cost function of a given supplier, this supplier wins its maximum quantity while revealing its costs only for low quantities. Hence, the interval bidding approach preserves private cost information for quantities closest to the suppliers' winnings.

The interval bidding procedure can be sometimes excessive in terms of revealing cost information: winners can end up revealing more information than is necessary to establish their winnings.[19] This is a result of using a simple elicitation process based on anonymous and linear price path. It is possible to reduce the excess elicitation of unrelated information via simple changes to the design that would allow stopping the auction for one set of suppliers and continuing it for others.

[19]The interval bidding procedure violates the *minimality property* advocated in Lamy (2012) for dynamic auctions.

B. Activity Rules

Activity rules AR1-AR3 from Proposition 3 are needed to ensure that all suppliers bid according to acceptable cost functions. AR2 can be somewhat counter intuitive: as the clock price descends, a supplier can be precluded from changing its bidding interval until the clock price catches up with the current cost approximation of its cost function. This is a natural consequence of using average cost elicitation to reconstruct a cost function with nonincreasing marginal costs. Average costs are higher than marginal costs for concave cost functions; therefore, average costs are revealed at a higher clock price.[20]

One can consider using *marginal* cost elicitation instead of *average* cost elicitation to relax the need for AR2. Under marginal cost elicitation, supplier i reduces its bidding interval when the marginal cost of its current lowest acceptable alternative equals to the current clock price, i.e., when $c_i(q_i(t)+1) - c_i(q_i(t)) = p(t)$.

The marginal approach is sufficiently similar to the average approach that the majority of the results in the paper continue to hold.[21] However, relying on marginal elicitation in the environment with nonicreasing marginal costs is ill-founded. The marginal approach works very well when the marginal costs should be "equalized" across different winners at the efficient assignment, e.g., when cost functions are convex. But there is no value in trying to equalize marginal costs across suppliers when the efficient assignment does not satisfy this property (see Proposition 1). In comparison, the average cost elicitation identifies suppliers who can produce their maximum capacities at the lowest average costs which is the relevant information for finding an efficient assignment.

C. Information Policy

It is common in dynamic auctions to provide bidders with a current aggregate measure of competition. If needed, the auctioneer can report aggregate supply $AS(M,t)$, as defined in (5), to suppliers so they can track the progress of the auction.

When utilizing Closing Rule 2, the public reporting of $AS(N,t)$ does not cause any concerns. In contrast, when utilizing Closing Rule 1, public reporting of $AS(N,t)$ may be problematic since $AS(M,t)$ can be nonmonotonic in M. If some marginal economies have not been cleared by the time $AS(N,t) = D$, suppliers will immediately realize that their future bids have no effect on their payoffs, compromising their incentives. A reasonable alternative in this case is to report the maximum aggregate supply across the main economy and all marginal economies, i.e., $\max\{AS(N,t), AS(N_{-1},t), ..., AS(N_{-n},t)\}$, instead of $AS(N,t)$.

[20]Consider a supplier with a cost function $c(1) = 10$, $c(2) = 14$. The marginal cost for the second unit is 4, whereas the average cost for producing 2 units is 7.

[21]There are a few exceptions. For example, property (d) in Lemma 3 does not hold for the marginal approach since it is not guaranteed that an active supplier from $A(N,t)$ would want to supply its maximum capacity at the current price.

Nonmonotonicity of $AS(M,t)$ in t can be somewhat inconvenient as well, but it is not critical for the informational purposes. Possible increases in $AS(M,t)$ indicate that there exist bidder complementarities between active and inactive suppliers.

D. Dynamic Vickrey Pricing

Another advantage of dynamic auctions over the sealed-bid alternatives is their ability to provide information about prospective winnings and payments to each bidder as the auction progresses. For auctions that implement Vickrey outcomes, providing information about bidders' tentative payments is especially important since in general, the clock price is a misleading indicator of the per unit final payment.[22]

For our auction design, a natural feedback would be to report the suppliers' tentative assignments and payments that are calculated using the current approximations of the cost functions. This approach works for inactive suppliers, but they are no longer bidding in the auction. At the same time, active suppliers might be frustrated since not all of them can be assigned their maximum capacities until the main economy has been cleared.

A less confusing approach would be to report payments for $\bar{s}_i$ for each active supplier in $A(N,t)$. Using the current cost functions, for each active supplier i, the auctioneer solves for (1) $\widehat{TC}(N_{-i},t)$; and (2) $\widehat{TC}(N,t)$ with the extra constraint that $x_i = \bar{s}_i$. A tentative payment for $\bar{s}_i$ for supplier i is then given by $\hat{c}_i(\bar{s}_i,t) + [\widehat{TC}(N_{-i},t) - \widehat{TC}(N,t)]$.[23] Some caution is needed when providing payment information to bidders since it might reveal additional information to suppliers.

V. An Illustration of the Auction with Interval Bidding

We illustrate our auction procedure using an example with four suppliers. Two suppliers, 1 and 2, can produce up to 3 units each, and suppliers 3 and 4 can produce up to 2 units each. The auctioneer wishes to buy 6 units of the good ($D = 6$). Cost information and auction dynamics for this example are provided in Table 3. For completeness, we also report aggregate supply $AS(N,t)$ and tentative Vickrey prices (see Information Policy and Dynamic Vickrey Pricing in Section IV).

[22] In general, the per unit payment to a supplier is higher than the final clock price $p(T)$, but the opposite relationship is possible in certain scenarios. However, for truthful suppliers, all payments are always individually rational (see Theorems 1 and 3). Situations when the clock price $p(T)$ overestimates the final payment (in per unit terms) are possible due to the concavity restriction placed on the current cost approximation $\hat{c}_i(.,t)$. Enforcing concavity results in the most precise approximation of true costs that frequently outpaces the speed of the falling clock price. Due to superior precision, it is possible to construct specific examples where the auctioneer learns all required information too early for the clock price to reach a low enough level to bound all payments from below.

[23] It is possible that the current tentative payment for $\bar{s}_i$ is less than the current reported cost $\hat{c}_i(\bar{s}_i,t)$. This can happen when $\hat{x}_i(N,t) < \bar{s}_i$.

TABLE 3—AN ILLUSTRATIVE EXAMPLE OF THE AUCTION WITH INTERVAL BIDDING ($D = 6$)

	Supplier 1	*Supplier 2*	*Supplier 3*	*Supplier 4*	
Costs:	$c_1 = (20, 30, 35)$	$c_2 = (40, 50, 60)$	$c_3 = (20, 30)$	$c_4 = (25, 40)$	
Efficient Assignment:	3	0	2	1	
Vickrey Payments:	60	0	35	30	
Clock Price	Bidding Intervals / Cost Approximations				$AS(N, t)$
p(0) = 50	$s_1 = \{1, 2, 3\}$ $\hat{c}_1 = (50, 100, 150)$ $\hat{p}_1^V(3) = 150$	$s_2 = \{1, 2, 3\}$ $\hat{c}_2 = (50, 100, 150)$ $\hat{p}_2^V(3) = 150$	$s_3 = \{1, 2\}$ $\hat{c}_3 = (50, 100)$ $\hat{p}_3^V(2) = 100$	$s_4 = \{1, 2\}$ $\hat{c}_4 = (50, 100)$ $\hat{p}_4^V(2) = 100$	10
$p(t_1) = 40$	$s_1 = \{1, 2, 3\}$ $\hat{c}_1 = (40, 80, 120)$ $\hat{p}_1^V(3) = 120$	$s_2 = \{2, 3\}$ $\hat{c}_2 = (40, 80, 120)$ $\hat{p}_2^V(3) = 120$	$s_3 = \{1, 2\}$ $\hat{c}_3 = (40, 80)$ $\hat{p}_3^V(2) = 80$	$s_4 = \{1, 2\}$ $\hat{c}_4 = (40, 80)$ $\hat{p}_4^V(2) = 80$	10
$p(t_2) = 25$	$s_1 = \{1, 2, 3\}$ $\hat{c}_1 = (25, 50, 75)$ $\hat{p}_1^V(3) = 75$	$s_2 = \{3\}$ $\hat{c}_2 = (40, 50, 60)$ $\hat{p}_2^V(3) = 75$	$s_3 = \{1, 2\}$ $\hat{c}_3 = (25, 50)$ $\hat{p}_3^V(2) = 50$	$s_4 = \{2\}$ $\hat{c}_4 = (25, 50)$ $\hat{p}_4^V(2) = 50$	10
$p(t_3) = 20$	$s_1 = \{2, 3\}$ $\hat{c}_1 = (20, 40, 60)$ $\hat{p}_1^V(3) = 60$	$s_2 = \{\}$ $\hat{c}_2 = (40, 50, 60)$ $\hat{p}_2^V(0) = 0$	$s_3 = \{2\}$ $\hat{c}_3 = (20, 40)$ $\hat{p}_3^V(2) = 35$	$s_4 = \{\}$ $\hat{c}_4 = (25, 40)$ $\hat{p}_4^V(2) = 40$	7
$p(t_4) = 17.5$ *efficiency proven*	$s_1 = \{2, 3\}$ $\hat{c}_1 = (20, 35, 50)$ $\hat{p}_1^V(3) = 60$	$s_2 = \{\}$ $\hat{c}_2 = (40, 50, 60)$ $\hat{p}_2^V(0) = 0$	$s_3 = \{2\}$ $\hat{c}_3 = (20, 35)$ $\hat{p}_3^V(2) = 35$	$s_4 = \{\}$ $\hat{c}_4 = (25, 40)$ $\hat{p}_4^V(1) = 25$	6
$p(t_5) = 15$ *Vickrey payments*	$s_1 = \{3\}$ $\hat{c}_1 = (20, 30, 40)$ $\hat{p}_1^V(3) = 60$	$s_2 = \{\}$ $\hat{c}_2 = (40, 50, 60)$ $\hat{p}_2^V(0) = 0$	$s_3 = \{\}$ $\hat{c}_3 = (20, 30)$ $\hat{p}_3^V(2) = 35$	$s_4 = \{\}$ $\hat{c}_4 = (25, 40)$ $\hat{p}_4^V(1) = 30$	6

Note: $c_1 = (20, 30, 35)$ indicates that supplier 1 can produce 1, 2 or 3 units of the good at a cost of 20, 30, or 35 correspondingly

The auctioneer starts a descending price clock at 50. The current clock price is interpreted as a per unit payment for supplying the good. At each price, suppliers reply with quantities they are willing to supply at the current clock price. In our example, when the clock price is above 40, all four suppliers are willing to supply any feasible quantity. However, at price of 40, it is not profitable for Supplier 2 to supply 1 unit of the good, but it is still profitable to supply 2 or 3 units. Supplier 2 communicates this information by reducing its bidding interval from $s_2 = \{1, 2, 3\}$ to $s_2 = \{2, 3\}$ at $p(t_1) = 40$. The auctioneer keeps track of all reductions in bidding intervals as the clock price decreases, and uses them to dynamically reconstruct suppliers' cost functions $\hat{c}_i$ for all $i \in \{1, 2, 3, 4\}$.

There are several interesting moments in this example. The first one occurs at t_3 when the clock price reaches 20. At this price, both Supplier 2 and Supplier 4 become inactive, driving the usual aggregate supply (a sum of quantities that suppliers want to deliver at the current price) below the demand. However, the efficient assignment cannot be established at this point. Observe that assignment (3, 0, 2, 1), the one where still active suppliers 1 and 3 receive their maximum capacities, results in a total cost of 125. At the same time, assignment (3, 0, 1, 2) can be procured at a cost of 120 resulting in $AS(N, t_3) = 7$. Hence, the auctioneer has to continue the auction to find out whether Supplier 3 should be awarded 2 units. At t_4, when the clock price reaches 17.5, $AS(N, t_4) = 6$ and the optimality of assignment (3, 0, 2, 1) is proven.

However, $AS(N_{-4}, t_4) = 8$, so the auction should be continued until $E(N_{-4})$ is cleared. When the clock price reaches $p(t_5) = 15$, Supplier 3 becomes inactive leading to the clearing of $E(N_{-4})$. The auctioneer solves for the Vickrey outcome using the reconstructed cost functions, awarding (3, 0, 2, 1) for payments (60, 0, 35, 30).

VI. Conclusion

We have designed an efficient procurement auction for environments with homogeneous goods where suppliers have nonincreasing marginal costs and capacity constraints. Potential applications include procurement settings where the underlying production process exhibits increasing returns, such as the manufacturing of vaccines. The auction design is based on a novel interval bidding approach: each supplier is asked to report all quantities that it is willing to supply at the current price, not just its optimal supply. Due to nonincreasing marginal costs, the supplier's report always constitutes a connected interval of acceptable quantities. The new bidding procedure allows the auctioneer to collect cost information via a linear and anonymous price clock, resulting in a fast and simple auction. The auction terminates with the Vickrey outcome, ensuring that truthful bidding by all suppliers constitutes an ex post equilibrium. Moreover, privacy is preserved in the sense that the winners are not required to reveal the costs of producing their winning quantities.

Our interval bidding approach can be adapted to other settings. By the usual arguments, this approach can be used to sell homogeneous items to buyers with nondecreasing marginal values. In this setting, all results of the paper hold with obvious changes to the notation. Also, the interval bidding procedure and results continue to be valid for the setting with homogeneous goods that are fully divisible subject to appropriately modifying the process of reconstructing cost functions (e.g., replacing formula (2) by its continuous analog).

The interval bidding approach may also be a useful building block in constructing efficient auctions for other settings of practical relevance. One of the most interesting settings is the procurement of homogeneous goods from suppliers with U-shaped marginal cost curves (i.e., decreasing marginal costs up to some quantity and then increasing marginal costs thereafter) — a cost structure commonly used in economics. To accommodate this setting, the auctioneer can use a combination of the standard supply elicitation approach and our new interval bidding approach: at each clock price, the supplier indicates both its optimal supply and the minimum quantity that is still profitable to produce. With this information, the auctioneer can reconstruct cost functions in a similar fashion, using them to calculate aggregate supply and to determine the stopping time for the auction.

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Appendix A: Core Payments

In this appendix we provide results pertaining to implementation of the core outcomes using the interval bidding approach. A *core outcome* is an assignment vector $x = (x_1, ..., x_n)$ and a payment vector $p^C = (p_1^C, ..., p_n^C)$ such that x is an efficient assignment for the $E(N)$ and p^C belongs to the set of core payments $CP(x)$:

$$\text{(A1)} \qquad CP(x) = \left\{ \begin{array}{c} p^C \in R^n \quad \text{such that for all} \quad M \subseteq N \\ \sum\limits_{N \setminus M} c_i(x_i) \;\leq\; \sum\limits_{N \setminus M} p_i^C \;\leq\; TC(M) - \sum\limits_{M} c_i(x_i) \end{array} \right\}$$

A *supplier optimal core outcome* is a core outcome in which the sum of payments to suppliers, $\sum_N p_i^C$, is maximized. Denote $SOCP(x)$ a set of all supplier optimal core payments for assignment x. It is well-known in the literature that the set of supplier optimal core payments $SOCP(x)$ coincides with the Vickrey payments $p^V(x)$ when suppliers are substitutes.[24]

Analogously to the tentative Vickrey payments defined in (7), define a set of tentative core payments at time t for assignment x as follows:

$$\text{(A2)} \qquad \widehat{CP}(x,t) = \left\{ \begin{array}{c} p^C \in R^n \quad \text{such that for all} \quad M \subseteq N \\ \sum\limits_{N \setminus M} \hat{c}_i(x_i,t) \;\leq\; \sum\limits_{N \setminus M} p_i^C \;\leq\; \widehat{TC}(M,t) - \sum\limits_{M} \hat{c}_i(x_i,t) \end{array} \right\}$$

Analogously to Closing Rule 2, Closing Rule 3 terminates the auction once the main economy has been cleared and pays suppliers a payment vector from the set of tentative core payments. Theorem 4 below shows that any payment vector from the set of tentative core payments also belongs to the set of core payments.

Closing Rule 3: *The auctioneer stops the clock price ($T := t$) once the main economy has been cleared. Suppliers are awarded the tentative allocation for the main economy $x = \hat{x}(N,T)$ and receive a payment vector p^C that belongs to $\widehat{CP}(x,T)$.*[25]

THEOREM 4: *If all suppliers bid truthfully according to their concave cost functions, the interval bidding auction procedure with Closing Rule 3 implements an efficient assignment $x = \hat{x}(N,T)$ and payment vector p^C such that:*

$$\text{(A3)} \qquad p^C \in CP(x).$$

[24] See Bikhchandani and Ostroy (2002).

[25] In general, such payment vector is not uniquely defined. The auctioneer would need to specify a rule that selects one set of payments consistent with the constraints. For example, the auctioneer might select a payment vector that maximizes the sum of payments made to suppliers.

Lamy (2012) demonstrated that the dynamic procedure developed by Mishra and Parkes (2007) for general preferences terminated when the main economy clears generates enough information to find at least one bidder optimal core outcome. Similarly to Proposition 4 in Section III, Proposition 5 shows that the interval bidding procedure with Closing Rule 3 cannot in general yield a supplier optimal outcome.

PROPOSITION 5: *The interval bidding auction procedure with Closing Rule 3 in general does not yield a supplier optimal core outcome.*

PROOF:

The costs functions used to prove Proposition 4 satisfy SAS. It can be also shown that the approximations of cost functions at t_1 also satisfy SAS. Given that the Vickrey payments coincide with the supplier optimal core payments, the inability to implement Vickrey payments (proved in Proposition 4) implies Proposition 5.

Appendix B: Proofs

PROOF OF PROPOSITION 1:

Suppose that assignment $x = (x_1, ..., x_n)$ is efficient for the economy $E(M)$ and there are two suppliers, i and j, such that $0 < x_i < \bar{s}_i$ and $0 < x_j < \bar{s}_j$. From the efficiency of assignment x and the concavity of $c_i(.)$ and $c_j(.)$, we have the following:

$$\begin{array}{rcl} c_i(x_i+1) - c_i(x_i) \ \geq & c_j(x_j) - c_j(x_j-1) & \geq \ c_j(x_j+1) - c_j(x_j) \\ & \text{and} & \\ c_j(x_j+1) - c_j(x_j) \ \geq & c_i(x_i) - c_i(x_i-1) & \geq \ c_i(x_i+1) - c_i(x_i) \end{array}$$

In other words, taking 1 unit from supplier j and giving it to supplier i is also an efficient assignment for $E(M)$. Proposition 1 follows by iterating this argument.

PROOF OF LEMMA 2:

(a): For $q \leq q_i(t)$, $\hat{c}_i(q,t) = c_i(q)$ by construction. For $q_i(t) < q < \bar{s}_i$, using the concavity of $c_i(.)$, we have

$$\begin{aligned} c_i(q) \ &\leq c_i(q_i(t)) + [c_i(q_i(t)+1) - c_i(q_i(t))](q - q_i(t)) \\ &\leq \tilde{c}_i(q_i(t)) + [p(t)(q_i(t)+1) - \tilde{c}_i(q_i(t))](q - q_i(t)) \\ &= \tilde{c}_i(q_i(t)) + mc_i^+(t)(q - q_i(t)) \end{aligned}$$

and

$$\begin{aligned} c_i(q) \ &\leq c_i(q_i(t)) + [c_i(q_i(t)) - c_i(q_i(t)-1)](q - q_i(t)) \\ &= \tilde{c}_i(q_i(t)) + [\tilde{c}_i(q_i(t)) - \tilde{c}_i(q_i(t)-1)](q - q_i(t)) \\ &= \tilde{c}_i(q_i(t)) + mc_i^-(t)(q - q_i(t)) \end{aligned}$$

and $c_i(q) \leq \tilde{c}_i(q_i(t)) + mc_i(t)(q - q_i(t)) = \hat{c}_i(q,t)$ follows.

(b): Monotonicity in q is by construction. To show concavity, note that for any $q \leq q_i(t) - 1$, $\hat{c}_i(q,t) = c_i(q)$ and $c_i(.)$ is a concave function. For any $q \geq q_i(t)+1$, $\hat{c}_i(q,t)$ is linear in q. For $q = q_i(t)$, $\hat{c}_i(q,t) - \hat{c}_i(q-1,t) = mc_i^-(t) \geq mc_i(t) = \hat{c}_i(q+1,t) - \hat{c}_i(q,t)$.

To show monotonicity in t, observe that for $t' > t$ and $q \leq q_i(t')$, $\hat{c}_i(q,t') - \hat{c}_i(q,t) = c_i(q) - \hat{c}_i(q,t) \leq 0$.

For $t' > t$ and $q > q_i(t')$,

$$\begin{aligned} \hat{c}_i(q,t') - \hat{c}_i(q,t) \ &= c_i(q_i(t')) - c_i(q_i(t)) + mc_i(t')[q - q_i(t')] - mc_i(t)[q - q_i(t)] \\ &\leq mc_i(t)[q_i(t') - q_i(t)] + mc_i(t')[q - q_i(t')] - mc_i(t)[q - q_i(t)] \\ &= [mc_i(t') - mc_i(t)][q - q_i(t')] \\ &\leq 0 \end{aligned}$$

(c): For any $q \leq q_i(t)$, $\delta_i(q,t) = 0$. For $q > q_i(t)$,

$$\begin{aligned} \delta_i(q+1,t) - \delta_i(q,t) \ &= mc_i(t) - [c_i(q+1) - c_i(q)] \\ &\geq [c_i(q_i(t)+1) - c_i(q_i(t))] - [c_i(q+1) - c_i(q)] \\ &\geq 0 \end{aligned}$$

(d): The inequality in part (d) is equivalent to:

$$\hat{c}_i(q,t) - \hat{c}_i(q',t) \leq \hat{c}_i(q,t') - \hat{c}_i(q',t') \tag{B1}$$

For any $q \in S_i$ and any $t' \geq t$, $\hat{c}_i(q,t) \geq \hat{c}_i(q,t')$ by property (b). Then for any $q \leq q_i(t)$, $\hat{c}_i(q,t) = c_i(q)$ and $\hat{c}_i(q,t') = c_i(q)$, and inequality (B1) follows.

For $q > q_i(t)$,

$$\begin{aligned} \hat{c}_i(q,t) - \hat{c}_i(q',t) &= mc_i(t)[q-q'] \\ \hat{c}_i(q,t') - \hat{c}_i(q',t') &\geq mc_i(t')[q-q'] \end{aligned}$$

and inequality (B1) follows from:

$$\begin{aligned} \hat{c}_i(q,t') - \hat{c}_i(q',t') &\geq mc_i(t')[q-q'] \\ &\geq mc_i(t)[q-q'] \\ &= \hat{c}_i(q,t) - \hat{c}_i(q',t) \end{aligned}$$

PROOF OF LEMMA 3:

(a): by a trivial rearrangement of terms in (5).

(b): (b) follows from (a).

(c): If $\sum_M \bar{s}_i \leq D$, then $AS(M,t) = D$ for all $t \in [0,T]$. If $\sum_M \bar{s}_i > D$, then the auctioneer would never use the alternative source for procurement. For $t' > t$, let $x = \hat{x}(M,t)$ and $x' = \hat{q}(M,t')$ denote assignments that minimize the costs of procurement. Since $AS(M,t) = D$, then $x_i = \bar{s}_i$ for all $i \in A(M,t)$.

$$\begin{aligned} 0 &\leq \sum_M \hat{c}_i(x'_i,t) - \sum_M \hat{c}_i(x_i,t) \\ &= \sum_{A(M,t)} [\hat{c}_i(x'_i,t) - \hat{c}_i(\bar{s}_i,t)] + \sum_{I(M,t)} [\hat{c}_i(x'_i,t) - \hat{c}_i(x_i,t)] \\ &\leq \sum_{A(M,t)} [\hat{c}_i(x'_i,t') - \hat{c}_i(\bar{s}_i,t')] + \sum_{I(M,t)} [\hat{c}_i(x'_i,t') - \hat{c}_i(x_i,t')] \\ &= \sum_M \hat{c}_i(x'_i,t') - \sum_M \hat{c}_i(x_i,t') \end{aligned}$$

The second inequality follows from (1) property (d) of Lemma 2 for suppliers in $A(M,t)$; and (2) no cost updating between t and t' for bidders in $I(M,t)$. The inequality shows that $x = \hat{x}(M,t)$ also solves cost minimization problem at t', but then $AS(M,t') = D$.

A possibility of nonmonotonic aggregate supply $AS(M,t)$ in t is demonstrated with an example in Table B1 where the auctioneer demands five units ($D = 5$).

(d): If at time t, $\sum_{A(M,t)} \bar{s}_i = D$, then for all inactive suppliers $\hat{c}_i(q,t) = c_i(q) \geq p(t)\,q$ for all $q \in S_i$. For all active suppliers $\hat{c}_i(q,t) \leq p(t)\,q$ for all $q \geq q_i(t)$. Since $\hat{c}_i(.,t)$ is concave in q for all $i \in M$ by Lemma 2(b), the tentative assignment $\hat{x}(M,t)$ for economy $E(M)$ is as follows: $\hat{x}_i(M,t) = \bar{s}_i$ for all $i \in A(M,t)$ and $\hat{x}_i(M,t) = 0$ for all $i \in I(M,t)$ by Proposition 1. But then $AS(M,t) = D$ by

TABLE B1—EXAMPLE OF NONMOTONIC $AS(N,t)$ $(D=5)$

	Supplier 1	*Supplier 2*	*Supplier 3*	*Supplier 4*	
Costs:	$c_1 = (12, 20, 21)$	$c_2 = (12, 20, 21)$	$c_3 = (11)$	$c_4 = (7)$	
Clock Price $p(t)$:					AS(N,t)
$p(0) = 15$	$s_1 = \{1, 2, 3\}$ $\hat{c}_1 = (15, 30, 45)$	$s_2 = \{1, 2, 3\}$ $\hat{c}_2 = (15, 30, 45)$	$s_3 = \{1\}$ $\hat{c}_3 = (15)$	$s_4 = \{1\}$ $\hat{c}_4 = (15)$	8
$p(t_1) = 12$	$s_1 = \{2, 3\}$ $\hat{c}_1 = (12, 24, 36)$	$s_2 = \{2, 3\}$ $\hat{c}_2 = (12, 24, 36)$	$s_3 = \{1\}$ $\hat{c}_3 = (12)$	$s_4 = \{1\}$ $\hat{c}_4 = (12)$	8
$p(t_2) = 11$	$s_1 = \{2, 3\}$ $\hat{c}_1 = (12, 22, 32)$	$s_2 = \{2, 3\}$ $\hat{c}_2 = (12, 22, 32)$	$s_3 = \{\}$ $\hat{c}_3 = (11)$	$s_4 = \{1\}$ $\hat{c}_4 = (11)$	7
$p(t_3) = 10$	$s_1 = \{3\}$ $\hat{c}_1 = (12, 20, 28)$	$s_2 = \{3\}$ $\hat{c}_2 = (12, 20, 28)$	$s_3 = \{\}$ $\hat{c}_3 = (11)$	$s_4 = \{1\}$ $\hat{c}_4 = (10)$	7
$p(t_4) = 8$	$s_1 = \{3\}$ $\hat{c}_1 = (12, 20, 24)$	$s_2 = \{3\}$ $\hat{c}_2 = (12, 20, 24)$	$s_3 = \{\}$ $\hat{c}_3 = (11)$	$s_4 = \{1\}$ $\hat{c}_4 = (8)$	8

Note: $c_1 = (12, 20, 21)$ indicates that supplier 1 can produce 1, 2 or 3 units of the good at a cost of 12, 20, or 21 correspondingly

property (a) of Lemma 3.

PROOF OF PROPOSITION 2:

Suppose that $\hat{x} = \hat{x}(M, t)$ is not efficient for economy $E(M)$, and there exists an efficient assignment x' such that

$$\sum_M c_i(x_i') + \bar{c}\,[D - \sum_M x_i'] < \sum_M c_i(\hat{x}_i) + \bar{c}\,[D - \sum_M \hat{x}_i]$$

Note that $\hat{x}_i = \bar{s}_i$ for all $i \in A(M,t)$. Then by Lemma 2(c), $\delta_i(x_i', t) \leq \delta_i(\hat{x}_i, t)$ for all $i \in A(M,t)$. But then

$$\begin{aligned}
\widehat{TC}(M,t) &= \sum_M \hat{c}_i(\hat{x}_i, t) + \bar{c}\,[D - \sum_M \hat{x}_i] \\
&= \sum_M c_i(\hat{x}_i) + \sum_{A(M,t)} \delta_i(\hat{x}_i, t) + \bar{c}\,[D - \sum_M \hat{x}_i] \\
&> \sum_M c_i(x_i') + \sum_{A(M,t)} \delta_i(x_i', t) + \bar{c}\,[D - \sum_M x_i'] \\
&= \sum_M \hat{c}_i(x_i', t) + \bar{c}\,[D - \sum_M x_i']
\end{aligned}$$

which is a contradiction to $\hat{x}$ solving the cost minimization problem (4) at time t.

Given that $\hat{x}(M,t)$ is efficient

$$\begin{aligned}\widehat{TC}(M,t) &= \textstyle\sum_M \hat{c}_i(\hat{x}_i,t) + \bar{c}\,[D - \sum_M \hat{x}_i] \\ &= \textstyle\sum_M c_i(\hat{x}_i) + \sum_M \delta_i(\hat{x}_i,t) + \bar{c}\,[D - \sum_M \hat{x}_i] \\ &= TC(M) + \textstyle\sum_M \delta_i(\hat{x}_i,t) \\ &= TC(M) + \textstyle\sum_{A(M,t)} \delta_i(\hat{x}_i,t)\end{aligned}$$

PROOF OF PROPOSITION 3:

AR1: Weakly increasing $\underline{s}_i(t)$ results in a weakly increasing $t_i(q)$ for $q \leq q_i(t)$. The implied average cost function for $q \leq q_i(t)$ is $\tilde{c}_i(q)/q = p(t_i(q))$. Then, given decreasing $p(t)$, the implied average cost function is weakly decreasing in q. For the converse, AR1 is satisfied by Lemma 1.

AR2: Suppose that supplier i increases its $\underline{s}_i(t)$ by 1 unit at t. Then $\tilde{c}_i(q_i(t)) = p(t)q_i(t)$. If $mc_i^+(t) \leq mc_i^-(t)$ at time t, then $\tilde{c}_i(q_i(t)) - \tilde{c}_i(q_i(t)-1) \leq \tilde{c}_i(q_i(t)-1) - \tilde{c}_i(q_i(t)-2)$ implying nonincreasing marginal costs. If $mc_i^+(t) > mc_i^-(t)$ at time t, then $\tilde{c}_i(q_i(t)) - \tilde{c}_i(q_i(t)-1) > \tilde{c}_i(q_i(t)-1) - \tilde{c}_i(q_i(t)-2)$ implying increasing marginal costs. For the converse, AR2 is trivially satisfied.

AR3: If $p(t)\,\underline{s}_i(t) = \tilde{c}_i(q_i(t))$ at t, then $mc_i(t) = 0$. But then the only weakly increasing cost function consistent with the bidding of supplier i is $\tilde{c}_i(q) = p(t_i(q))q$ for all $q \leq q_i(t)$ and $\tilde{c}_i(q) = \tilde{c}_i(q_i(t))$ for all $q > q_i(t)$. For the converse, AR3 is trivially satisfied.

PROOF OF THEOREM 3:

Theorem 4 is a generalization of Theorem 3. The proof for Theorem 3 can be produced by applying the proof for Theorem 4 to $M = N_{-i}$.

PROOF OF THEOREM 4:

Since $E(N)$ clears at t, $x = \hat{x}(N,t)$ is an efficient allocation for $E(N)$ by Proposition 2. Suppose that economy $E(M)$ clears at $t' \geq t$, then for any $s \geq t'$:

$$\begin{aligned}&\widehat{TC}(M,s) - \sum_M \hat{c}_i(x_i,s) = \\ &= TC(M) + \sum_M [\hat{c}_i(\hat{x}_i(M,s),s) - c_i(\hat{x}_i(M,s))] - \sum_M \hat{c}_i(x_i,s) \\ &= TC(M) + \sum_{A(M,s)} [\hat{c}_i(x_i,s) - c_i(x_i)] - \sum_{A(M,s)} \hat{c}_i(x_i,s) - \sum_{I(M,s)} c_i(x_i) \\ &= TC(M) - \sum_M c_i(x_i)\end{aligned}$$

where the second equality holds since $x_i = \hat{x}_i(N,t) = \hat{x}_i(M,s) = \bar{s}_i$ for all $i \in A(M,s)$, and $\hat{c}_i(x_i,s) = c_i(x_i)$ for all $i \in I(M,s)$.

Now we demonstrate that $\widehat{TC}(M,s) - \sum_M \hat{c}_i(x_i,s)$ is weakly increasing in s on $[t,t']$. Function $\widehat{TC}(M,s) - \sum_M \hat{c}_i(x_i,s)$ is continuous in s. Hence, we only need to consider $s, s' \in [t, t']$ such that $s < s'$ and $\hat{x}(M, s) = \hat{x}(M, s')$.[26]

$$\begin{aligned}
\widehat{TC}(M,s) - \widehat{TC}(M,s') &= \sum_M [\hat{c}_i(\hat{x}_i(M,s),s) - \hat{c}_i(\hat{x}_i(M,s'),s')] \\
&= \sum_{A(M,s)} [\hat{c}_i(\hat{x}_i(M,s),s) - \hat{c}_i(\hat{x}_i(M,s'),s')] \\
&\leq \sum_{A(M,s)} [\hat{c}_i(\bar{s}_i,s) - \hat{c}_i(\bar{s}_i,s')] \qquad \text{(by Lemma 2(d))} \\
&= \sum_{A(M,s)} [\hat{c}_i(x_i,s) - \hat{c}_i(x_i,s')] + \sum_{I(M,s)} [\hat{c}_i(x_i,s) - \hat{c}_i(x_i,s')] \\
&= \sum_M \hat{c}_i(x_i,s) - \sum_M \hat{c}_i(x_i,s')
\end{aligned}$$

This implies that all upper bounds on core payments in $\widehat{CP}(x,t)$ are weakly increasing in t.

For the lower bounds, by property (b) of Lemma 2 :

$$\sum_{N \setminus M} \hat{c}_i(x_i, t') \leq \sum_{N \setminus M} \hat{c}_i(x_i, t) \qquad \forall t' \geq t \qquad \forall M \subseteq N,$$

implying that all lower bounds for core payments in $\widehat{CP}(x,t)$ are weakly decreasing in t.

Since both the lower bounds and upper bounds of $\widehat{CP}(x,t)$ are weakly expanding with t, and there exists $t' \geq t$ such that $\widehat{CP}(x,t') = CP(x)$, it follows that

$$p^C \in \widehat{CP}(x,t) \subseteq CP(x).$$

[26]If $\hat{x}(M,s) \neq \hat{x}(M,s')$, the interval $[s,s']$ can be partitioned into finitely many intervals $[s_1,s_2],[s_2,s_3],...,[s_{k-1},s_k]$ where $s_1 = s$ and $s_k = s'$ such that $\hat{x}(M,z) = \hat{x}(M,z')$ for any $z,z' \in [s_{l-1},s_l]$ and any $l = 2,...,k$.