Risk and Return in Segmented Markets with Expertise*

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Abstract

Complex assets appear to earn persistent high average returns, and to display high Sharpe ratios. Despite this, investor participation is very limited. We provide an explanation for these facts using a model of the pricing of complex securities by risk-averse investors who are subject to asset-specific risk in a dynamic model of industry equilibrium. Investor expertise varies, and the investment technology of investors with more expertise is subject to less asset-specific risk. Expert demand lowers equilibrium required returns, reducing participation, and leading to endogenously segmented markets. Amongst participants, portfolio decisions and realized returns determine the joint distribution of financial expertise and financial wealth. This distribution, along with participation, then determines market-level risk bearing capacity. We show that more complex assets deliver higher equilibrium returns to expert participants. Moreover, we explain why complex assets can have lower overall participation despite higher market-level alphas and Sharpe ratios. Finally, we show how complexity affects the size distribution of complex asset investors in a way that is consistent with the size distribution of hedge funds.

Key Words: segmented markets, slow moving capital, risky arbitrage, hedge funds, industry equilibrium, firm size distribution, financial expertise, intellectual capital, intermediary asset pricing.

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1 Introduction

Complex investment strategies, such as those employed by hedge funds and other sophisticated investors, appear to generate persistent alphas, high Sharpe ratios[1], and to feature limited participation, despite free entry. We develop an industry equilibrium model of the complex asset management industry which explains these facts, and generates additional testable predictions about the industrial organization of complex asset markets which we show are consistent with data on hedge funds. Investing in a complex asset requires an investment not only in the asset itself, but also in a technology with which to manage the asset. We argue that this joint investment in complex asset strategies exposes investors to asset-specific, idiosyncratic risk, and that variation in expertise across investors leads to variation in the asset-specific risk investors face. Thus, we define a complex asset as one that imposes idiosyncratic risk on investors, and argue that more complex assets impose more asset-specific risk. We use our model to characterize how the equilibrium pricing of complex assets is determined by the endogenous joint distribution of expertise and financial wealth. In equilibrium, this joint distribution is in turn determined by the deep parameters which describe preferences, endowments, and technologies in our model economies, and which proxy for asset complexity.

Our model economy is populated by a continuum of agents who choose to be either non-experts who can invest only in the risk free asset, or experts who can invest in both the risk free and risky assets. Investors who choose to be experts make an initial investment in expertise, which represents the investor's personnel, data, hedging and risk management technologies, back office operations and trade clearing processes, relationships with dealers, and relationships with clients.

The acquisition and management of complex assets require a joint investment in the asset (or strategy) and in an implementation technology which requires financial expertise. All expert investors in the market earn a common equilibrium return that clears the market. However, their returns are subject to asset-specific (or strategy-specific) shocks. Expertise improves an investor's specific implementation technology and shrinks the asset-specific volatility of the returns to the risky asset, implying that more expert investors face a higher Sharpe ratio. Thus, expertise may be interpreted as the ability to implement complex strategies better either by developing a superior model or infomration technology, hiring better employees, or by gathering superior information.

[1]See Sharpe [1966].

By definition, true "alpha" must be due to idiosyncratic, not systematic, risk. In our stationary model, all risk is asset-specific and idiosyncratic. This is, of course, a useful assumption technically. However, we argue that emphasizing the role of idiosyncratic risk in asset pricing is also realistic, as argued in Merton [1987]. There is a growing literature that documents the importance of idiosyncratic risk in complex asset strategies. Pontiff [2006] investigates the role of idiosyncratic risk faced by arbitrageurs in a review of the literature and argues that "The literature demonstrates that idiosyncratic risk is the single largest cost faced by arbitrageurs". Greenwood [2011] states that "Arbitrageurs are specialized and must be compensated for idiosyncratic risk," and lists this first as the key friction investors in complex strategies face. To paraphrase Emanuel Derman, if you are using a model, you are short volatility, since you will lose money when your model is wrong.[2]

Idiosyncratic risk is likely to be particularly important in markets for complex assets. Complex assets expose their owners to idiosyncratic risk through several channels. First, any investment in a complex asset requires a joint investment in the front and back office infrastructure necessary to implement the srategy. Second, their constituents tend to be significantly heterogeneous, so that no two investors hold exactly the same asset. Third, the risk management of complex assets typically requires a hedging strategy that will be subject to the individual technological constraints of the investor. Fourth, firms which manage complex assets may be exposed to key person risk due to the importance of specialized traders, risk managers, and marketers. Finally, complex assets may introduce or amplify idiosyncratic risk on the liability side of the balance sheet, through the fact that they are difficult for outside investors to understand, but tend to be funded with external finance.[3] We provide one specific micro-foundation for the idiosyncratic risk complex assets impose on investors in strategies involving a long position in an underlying asset, and a short position in an imperfectly correlated, investor specific, tracking portfolio. Thus, an additional contribution of our paper is to provide a precise explanation for the idiosyncratic risk that the prior literature has argued is important for understanding complex asset returns.

We assume that funds cannot be reallocated across individual risk-averse investors. Clearly, since the risk in our economy is idiosyncratic, pooling this risk would eliminate the risk premium that experts require to hold it. Complex assets tend to be held in managed accounts. For

[2]Derman [2016]

[3]Broadly interpreted, these risks may come either from the asset side, or from the liability side, since funding stability likely varies with expertise. However, we abstract from the microfoundations of risks from the liability side of funds' balance sheets, and model risk on the asset side.

incentive reasons, these managers cannot hedge their own exposure to their particular portfolio. In fact, Panageas and Westerfield [2009] and Drechsler [2014] provide important results for the portfolio choice of hedge fund managers who earn fees based on assets under management and portfolio performance. In particular, they show that these managers behave like constant relative risk aversion investors. This motivates why we endow expert investors in our model with CRRA preferences.

In our model, expertise varies in the cross-section but is fixed for each agent over time. This allows us to solve our model analytically, including the joint stationary wealth and expertise distribution, in closed form, up to the equilibrium fixed point for expected returns.[4] The deep preference and technology parameters determine the joint distribution of wealth and expertise and the resulting equilibrium alpha and Sharpe ratio. More complex assets, characterized by a higher required expertise to achieve a given lower level of risk (or, equivalently, a larger amount of fundamental risk) have higher alphas, and under natural conditions on the distribution of expertise, lower participation and higher Sharpe ratios. The complex asset market is endogenously segmented, since expert demand lowers required returns. Although alphas and Sharpe ratios of participants may look attractive, they are not representative of what investors with less expertise can achieve. As a result, participation is naturally limited, and elevated excess returns with modest average market-level risk persist. We focus on differences in complexity arising from variation in the amount of idiosyncratic risk the asset class imposes on investors, for example because more complex assets impose more model-specific risk. However, we also show that other proxies for complexity display similar comparative statics. In particular, assets which have higher costs of maintaining expertise, or require expertise which is more scarce, also have higher alphas.

The equilibrium stationary wealth distribution of participants is Pareto conditional on each expertise level. The decay parameter depends on investors' portfolio choice and exposure to the risky complex asset. In particular, because investors with higher expertise choose a higher exposure to the risky asset, both the drift and the volatility of their wealth will be greater, leading to a fatter tailed distribution at higher expertise levels. Our model predicts, under natural conditions on the distribution of expertise (for example using a log-normal distribution), that more complex assets will have less concentrated wealth distributions, a fact that is consistent with data on strategy-level hedge fund data. This result is driven by the fact

[4] We use a numerical algorithm to solve the market clearing fixed point problem. However, the solution is straightforward given our analytical solution for policy functions and distribution over individual states.

that more complex assets will have a higher threshold for expertise required for participation. Although there are likely to be many agents with lower levels of expertise, the distribution across expertise at higher levels is flatter, leading to a less concentrated wealth distribution for more complex assets. We provide evidence for this ancillary prediction using data from Hedge Fund Research on size distributions across different strategies.

The paper proceeds as follows. In Section 2 we review the related literature. Section 3 contains the construction and analysis of our dynamic model, and finally Section 5 concludes. Most proofs appear in the Appendix. In separate work (Eisfeldt et al. [2015]), we study a discrete time dynamic model with stochastic expertise, which we use to study the impact of unanticipated aggregate shocks and to develop quantitative results. In particular, using intuition developed in this paper, we show that expertise can act as an excess capacity-like barrier to entry, leading to interesting dynamics for market excess returns and volatility following shocks to investor wealth and to fundamental asset volatility.

2 Literature

Our paper contributes to a large and growing literature on segmented markets and asset pricing. Relative to the existing literature, we provide a model with endogenous entry, a continuous distribution of heterogeneous expertise, and a rich distribution of expert wealth that is determined in stationary equilibrium. Thus, we have segmented markets, but allow for a participation choice. Our market has limited risk bearing capacity, determined in part by expert wealth, but in addition to the amount of wealth, the efficiency of the wealth distribution also matters for asset pricing.

We group the existing literature into three main categories, namely financial constraints and limits to arbitrage, intermediary asset pricing, and segmented market models with alternative microfoundations to agency theory. Although our model is not one of arbitrage per se, our study shares the goal of understanding the returns to complex assets and strategies. Our model also shares the features of segmented markets and trading frictions with the limits to arbitrage literature. Gromb and Vayanos [2010b] provide a recent survey of the theoretical literature on limits to arbitrage, starting with the early work by Brennan and Schwartz [1990] and Shleifer and Vishny [1997].[5] Shleifer and Vishny [1997] emphasize that arbitrage is conducted by a

[5] See also Aiyagari and Gertler [1999], Froot and O'Connell [1999], Basak and Croitoru [2000], Xiong [2001], Gromb and Vayanos [2002], Yuan [2005], Gabaix, Krishnamurthy, and Vigneron [2007], Mitchell, Pedersen, and Pulvino [2007], Acharya, Shin, and Yorulmazer [2009], Kondor [2009], Duffie [2010], Gromb and Vayanos

fraction of investors with specialized knowledge, but similar to He and Krishnamurthy [2012], they focus on the effects of the agency frictions between arbitrageurs and their capital providers. Although we do not explicitly model risks to the liability side of investors' balance sheets, one can interpret the shocks agents in our model face to include idiosyncratic redemptions.[6]

Recently, the broader asset pricing impact of financially constrained intermediaries has been studied in the literature on intermediary asset pricing following He and Krishnamurthy [2012] and He and Krishnamurthy [2013].[7] This literature applies results from the literature on asset pricing with heterogenous agents, following Dumas [1989], to segmented markets with financial constraints.[8] In doing so, the intermediary asset pricing literature both connects to empirical applications, and to the asset price dynamics which are the focus of the limits to arbitrage literature. Finally, several papers develop alternative microfoundations to agency theory for segmented markets. Allen and Gale [2005] provide an overview of their theory of asset pricing based on "cash-in-the-market". Plantin [2009] develops a model of learning by holding. Duffie and Strulovici [2012] develop a theory of capital mobility and asset pricing using search foundations. Glode, Green, and Lowery [2012] study asset price dynamics in a model of financial expertise as an arms race in the presence of adverse selection. Kurlat [2013] studies an economy with adverse selection in which buyers vary in their ability to evaluate the quality of assets on the market, and, like us, emphasizes the distribution of expertise on the equilibrium price of the asset. Garleanu, Panageas, and Yu [2014] derive market segmentation endogenously from differences in participation costs. Kacperczyk, Nosal, and Stevens [2014] construct a model of consumer wealth inequality from differences in investor sophistication.

Our model is an example of an "industry equilibrium" model in the spirit of Hopenhayn [1992a] and Hopenhayn [1992b]. These models study the important effects of firm dynamics, entry and exit in the heterogeneous agent framework developed in Bewley [1986]. This literature focuses in large part on explaining firm growth, and moments describing the firm size distribution. Recent progress in the firm dynamics literature using continuous time techniques to solve for policy functions and stationary distributions include Miao [2005], Luttmer [2007],

[2010a], Hombert and Thesmar [2011], Edmond and Weill [2012], Mitchell and Pulvino [2012], Pasquariello [2013], and Kondor and Vayanos [2014].

[6]For other models of risks stemming from redemptions and fund outflows and the resulting asset pricing implications, see Berk and Green [2004], and Liu and Mello [2011].

[7]See also, for example, Adrian and Boyarchenko [2013]. For empirical applications, see for example, Adrian, Etula, and Muir [Forthcoming] and Muir [2014].

[8]For closely related work on asset pricing with heterogeneous risk aversion and segmented markets, see also Basak and Cuoco [1998], Kogan and Uppal [2001], Chien, Cole, and Lustig [2011], and Chien, Cole, and Lustig [forthcoming].

Gourio and Roys [2014], Moll [Forthcoming], and Achdou, Han, Lasry, Lions, and Moll [2014]. We draw on results in these papers as well as discrete time models of firm dynamics, as in recent work by Clementi and Palazzo [2014], which emphasizes the role of selection in explaining the observed relationships between firm age, size, and productivity. We also draw on work in the city size literature in Gabaix [1999] and the literature on the consumer wealth distribution with idiosyncratic risk and fiscal policy in Benhabib et al. [2014].

We use the the hedge fund industry, and in particular the asset backed fixed income (ABFI) segment, for some motivating empirical moments describing size and performance. As such, we draw from the literature on hedge funds performance and compensation.[9] In particular, we use ABFI funds as one example of a complex strategy using the evidence in Duarte, Longstaff, and Yu [2006]. They provide evidence that MBS strategies are relatively complex and earn higher returns even in comparison to other sophisticated fixed income arbitrage strategies. Several papers provide evidence for the importance of idiosyncratic risk in the hedge fund returns, following the idea in Merton [1987] that idiosyncratic risk will be priced when there are costs associated with learning about or hedging a specific asset.[10] Relatedly, Fung and Hsieh [1997] find that hedge fund returns have low and sometimes negative correlation with asset class returns. Our model features investors with constant relative risk aversion (CRRA) preferences. While we do this for tractability and parsimony to retain our focus on the effects of the joint wealth and expertise distribution, Panageas and Westerfield [2009] show that hedge fund compensation contracts with long horizons lead to portfolio choice which aligns perfectly with that of a CRRA investor. Drechsler [2014] extends these results to include variation in managers' outside options and shows the CRRA result holds as long as such reservation values are neither too high nor too low. These results extend the analysis of the impact of high-water marks in Goetzmann et al. [2003].

[9] Fung, Hsieh, Naik, and Ramadorai [2008] is a well known paper describing performance. Jagannathan, Malakhov, and Novikov [2010] carefully correct for selection bias and smoothed returns in a study of hedge fund performance persistence. Carlson and Steinman [2007] consider the relationship between hedge fund survival and market conditions. In a related spirit to our work, Getmansky [2012] empirically studies the effects of size and competition on hedge fund returns.

[10] See Titman and Tiu [2011] and Lee and Kim [2014]. Jurek and Stafford [Forthcoming] emphasize that scarce and specialized knowledge may drive both hedge fund returns and put pricing.

3 Model

3.1 Preferences, Endowments, & Technologies

We study a model with a continuum of investors of measure one, with CRRA utility functions over consumption:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Investment Technology Investors are endowed with a level of expertise which varies in the cross section, but is fixed for each agent over time. Each individual investor is born with a fixed expertise level, x, drawn from a distribution with pdf $\lambda(x)$, and cdf $\Lambda(x)$.

Investors can choose to be experts, and have access to the complex risky asset, or non-experts, who can only invest in the risk free asset. Each investor's complex risky asset delivers a stochastic return which follows a geometric Brownian motion:

$$\frac{dP(t,s)}{P(t,s)} = [r_f + \alpha(s)]\, ds + \sigma(x)\, dB(t,s) \tag{1}$$

where $\alpha(s)$ is the common excess return on the risky asset and $B(t,s)$ is a standard Brownian motion which is investor specific and i.i.d. in the cross section. For parsimony, we suppress the dependence of the Brownian shock on investor i in our notation. The volatility of the risky technology $\sigma(x)$ decreases in the investor's level of expertise x, i.e. $\frac{\partial \sigma(x)}{\partial x} < 0$. For now, we focus on describing the equilibrium for a single asset, and we suppress the positive dependence of $\sigma(x)$ on the fundamental volatility of the asset class σ_ν. Below, we describe comparative statics across assets with varying complexity, with more complex assets characterized by a higher σ_ν, or "fundamental volatility". We refer to $\sigma(x)$ as "effective volatility", meaning the remaining fundamental volatility the investor faces after expertise has been applied. We require that $\lim_{x\to\infty} \sigma(x) = \underline{\sigma} > 0$. The lower bound on volatility, $\underline{\sigma}$, represents complex asset risk that cannot be eliminated even by the agents with the greatest expertise, and it guarantees that the growth rate of wealth is finite.

One interpretation of the return process in Equation **??** is that in order to invest in the risky asset and to earn the common market clearing return, an investor must also jointly invest in a technology with a zero mean return and an idiosyncratic shock. This technology represents each investor's specific hedging and financing technologies, as well as the unique features of their particular asset. According to its general definition, α cannot be generated by bearing systematic risk. However, capturing α is risky because it requires a model and execution, and each investor's model and execution technology is unique. For example, hedging portfolios tend to vary substantially across different investors in the same asset class.[11] We present an example of an explicit micro-foundation for equation (1) based on a long position in a fundamental asset and a short position in a tracking portfolio in the Appendix.

To be an expert, an investor must pay the entry cost F_{nx} to set up their specific technology for investing in the complex risky asset. Experts must also pay a maintenance cost, F_{xx} to maintain continued access to the risky technology. We specify that both the entry and maintenance costs are proportional to wealth:

$$\begin{aligned} F_{nx} &= f_{nx} w, \\ F_{xx} &= f_{xx} w, \end{aligned}$$

which yields value functions which are homogeneous in wealth.

Optimization We first describe the Bellman equations for non-experts and experts respectively, and characterize their value functions, as well as the associated optimal policy functions. With the value functions of experts and non-experts in hand, we then characterize the entry decision.

We begin with non-experts, who can only invest in the risk free asset. Let $w(t,s)$ denote the wealth of investors at time s with initial wealth W_t at time t. The riskless asset delivers a fixed return of r_f. All investors choose consumption, and an optimal stopping, or entry time according to the Bellman Equation:

$$\begin{aligned} V^n(w(t,s),x) &= \max_{c^n(t,s),\tau} E\left[\int_t^\tau e^{-\rho(s-t)} u(c^n(t,s))\,ds + e^{-\rho(\tau-t)} V^x(w(t,s)-F_{nx},x)\right] && (2) \\ \text{s.t. } dw(t,s) &= (r_f w(t,s) - c^n(t,s))\,ds && (3) \end{aligned}$$

where ρ is their subjective discount factor, $c(t,s)$ is consumption at time s, F_{nx} is the entry cost, and τ is the optimal entry date.

Under the assumptions of linear entry and maintenance costs, and expertise which is fixed over time, the optimal entry date in a stationary equilibrium will be either immediately, or never.

[11] For example, for MBS, there is no agreed upon method to hedge mortgage duration risk, though most all active investors do so. Some hedge according to empirical durations, using various estimation periods and rebalancing periods. Others hedge according to the sensitivity of MBS prices yield curve shifts using their own (widely varying) proprietary model of MBS prepayments and prices.

Thus, assuming an initial stationary equilibrium, investors who choose an infinite stopping time are then non-experts, and investors who choose a stopping time $\tau = t$ are experts.[12]

Experts allocate their wealth between current consumption, a risky asset, and a riskless asset. They also choose an optimal stopping time to exit the market.

$$\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} u\left(c^x\left(t,s\right)\right) ds + e^{-\rho(T-t)} V^n\left(w\left(t,s\right),x\right)\right] &(4)\\
\text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f + \theta\left(x,t,s\right)\alpha\left(t,s\right)\right) - c^x\left(t,s\right) - F_{xx}\right] ds &(5)\\
&\quad + w\left(t,s\right)\theta\left(x,t,s\right)\sigma\left(x\right) dB\left(t,s\right),
\end{aligned}$$

where $\alpha\left(s\right)$ is the equilibrium excess return on the risky asset, $\theta\left(x,t,s\right)$ is the portfolio allocation to the risky asset by investors with expertise level x at time s, $c\left(t,s\right)$ is consumption, F_{xx} is the maintenance cost, and T is the optimal exit date. We include exit for completeness. However, exit will not occur in this homogeneous model with fixed expertise.

The following proposition states the analytical solutions for the value and policy functions in our model. We prove this Proposition by guess and verify in the Appendix.

Proposition 3.1 ***Value and Policy Functions:*** *The value functions are given by*

$$\begin{aligned}
V^x\left(w\left(t,s\right),x\right) &= y^x\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} &(6)\\
V^n\left(w\left(t,s\right),x\right) &= y^n\left(x,t,s\right)\frac{w\left(t,s\right)^{1-\gamma}}{1-\gamma} &(7)
\end{aligned}$$

where $y^x\left(x\right)$ and $y^n\left(x\right)$ are given by:

$$\begin{aligned}
y^x\left(x\right) &= \left[\frac{\left(\gamma-1\right)\left(r_f - f_{xx}\right)+\rho}{\gamma} + \frac{\left(\gamma-1\right)\alpha^2}{2\gamma^2\sigma^2\left(x\right)}\right]^{-\gamma} \quad \textit{and} &(8)\\
y^n\left(x\right) &= \left[\frac{\left(\gamma-1\right)r_f+\rho}{\gamma}\right]^{-\gamma}. &(9)
\end{aligned}$$

[12] Outside of a stationary equilibrium, because α is not constant, both entry and exit are possible.

The optimal policy functions $c^x(x,t,s)$, $c^n(t,s)$, *and* $\theta(x)$ *are given by:*

$$\begin{aligned} c^x(x,t,s) &= [y^x(x)]^{-\frac{1}{\gamma}} w(t,s), && (10)\\ c^n(t,s) &= [y^n(x)]^{-\frac{1}{\gamma}} w(t,s) \;\; \textit{and} && (11)\\ \theta(x,t,s) &= \frac{\alpha(t,s)}{\gamma\sigma^2(x)}. && (12) \end{aligned}$$

Furthermore, the wealth of experts evolves according to the law of motion:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s) \tag{13}$$

Finally, investors choose to be experts if the excess return earned per unit of wealth exceeds the maintenance cost per unit of wealth:

$$\frac{\alpha^2(t,s)}{2\sigma^2(x)\,\gamma} \geq f_{xx}. \tag{14}$$

We define $\underline{x}$ as the lowest level of expertise amongst participating investors, for which Equation (14) holds with equality. Note that the law of motion for wealth is a sort of weighted average of the return to the risky and riskless assets, as determined by portfolio choice, net of consumption. The drift and volatility of investors' wealth are increasing in the allocation to the risky asset. This mechanism has important implications for the wealth distribution in the stationary equilibrium of our model.

3.2 The Distribution(s) of Expert Wealth

The total amount of wealth allocated to the complex risky asset, as well as the distribution of expert wealth across expertise levels, are key aggregate state variables for the the first and second moments of the equilibrium returns to the complex risky asset. Once the entry decision has been made, given that we do not clear the market for the riskless asset, the wealth of non-experts is irrelevant for the returns to the complex risky asset. We solve for the cross-sectional distribution of expert wealth in a stationary equilibrium of our model. Given that expertise is fixed over time for each investor, constructing the wealth distribution at each expertise level is sufficient to obtain the cross-sectional joint distribution of wealth and expertise.

First, we note that in order to construct a stationary equilibrium given that experts' wealth on average grows over time, it is convenient to study the ratio $z(t,s)$ of individual wealth to the mean wealth of agents with highest expertise, $\mathbb{E}\left[w|_{\bar{x}}\left(t,s\right)\right]$.

$$z\left(t,s\right)\equiv\frac{w\left(t,s\right)}{\mathbb{E}\left[w|_{\bar{x}}\left(t,s\right)\right]}.$$

Next, note that the law of motion for the mean wealth of agents with a given level of expertise x is given by

$$d\mathbb{E}[w|_{x}\left(t,s\right)]\equiv\left[g\left(x\right)\right]dt.$$

where $g\left(x\right)$ will be determined in equilibrium. Define the average growth rate amongst agents with the "highest" level of expertise as $g(\bar{x})\equiv\sup_{x}g(x)$. Then, the ratio $z\left(t,s\right)$ follows a geometric Brownian motion given by

$$\frac{dz\left(t,s\right)}{z\left(t,s\right)}=\left(\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{\left(\gamma+1\right)\alpha^2\left(t,s\right)}{2\gamma^2\sigma^2\left(x\right)}-g\left(\bar{x}\right)\right)dt+\frac{\alpha\left(t,s\right)}{\gamma\sigma\left(x\right)}dB\left(t,s\right),\tag{15}$$

where $\frac{r_f-f_{xx}-\rho}{\gamma}+\frac{(\gamma+1)\alpha^2(t)}{2\gamma^2\sigma^2(x)}-g\left(\bar{x}\right)<0$ represents the negative drift, or growth rate.

Let the cross-sectional p.d.f. of expert investors' wealth and expertise at time t be denoted by $\phi^x\left(z,x,t\right)$. Without additional assumptions, the relative wealth of lower expertise agents will shrink to zero. Two methods are commonly used to generate a stationary distribution. The first, for example used in Benhabib et al. [2014], is to employ a life cycle model, or Poisson elimination of agents. The second, employed by Gabaix [1999], is to introduce a reflecting barrier at a minimum wealth share, $z_{\min}$.[13] We adopt the assumption of a minimum wealth share because it leads to a more elegant expression for the wealth distribution. Moreover, for asset pricing, only the higher ends of the wealth distribution are quantitatively relevant, so this elegance comes at a low cost. We will show that the stationary distribution of wealth at each expertise level will be a Pareto distribution.[14] Note that the reflecting barrier at $z_{\min}$ implies that the growth rate of any individual agent, even those with the highest level of expertise, will grow more slowly than the mean wealth of the highest expertise agents.

Since the reflecting boundary mainly affects low wealth investors, decisions near the bound-

[13] Gabaix [1999] constructs a model of the city size distribution, and thus his share variable represents relative population shares. See also the Appendix of that paper for a related method of constructing a stationary distribution using a Kesten [1973] process, which introduces a random shock with a positive mean to normalized city size.

[14] Adopting the assumption of Poisson death with a fixed initial wealth, for example, would instead lead to a double Pareto distribution, with a cutoff at the initial value of wealth. For example, see Benhabib et al. [2014] for the wealth distribution under the alternative assumption of Poisson elimination in a closely related model. This is also the assumption we adopt in our quantitative study in Eisfeldt et al. [2015]. The alternative, initializing agents according to the stationary distribution involves solving a challenging fixed point problem.

ary matter little for equilibrium pricing. However, we adopt an interpretation of exit and entry at $z_{\min}$ which ensures that policies are not distorted there. Then, since both time and state variables are continuous in our model, if policies are not distorted at $z_{\min}$, then they will not be distorted elsewhere. The strategy we employ is to ensure that the value at $z_{\min}$ from adopting the optimal policy functions under non-reflecting wealth share dynamics is equal to the value of adopting those policies given that with some probability the investor will be punished by being forced to exit, and with some probability the investor will be rewarded by being able to infuse funds themselves, or by receiving new external funds. In the case of exit, we assume the investor is replaced by a new entrant with wealth share $z_{\min}$ and the same level of expertise x as the exiting agent.[15]

We derive the Kolmogorov forward equations describing the evolution of the wealth distribution, taking $\alpha(t)$ as given, as follows:[16]

$$\begin{aligned}
\partial_t \phi^x(z,x,t) = &- \partial_z\left[\left(\left(r_f + \theta(x,t)\,\alpha(t,s)\right) - \left[y^x(x)\right]^{-\frac{1}{\gamma}} - f_{xx} - g(\bar{x})\right) z\phi^x(z,x,t)\right] \\
&+ \frac{1}{2}\partial_{zz}\left(\left[z\theta(x,t)\,\sigma(x)\right]^2 \phi^x(z,x,t)\right) \\
&= -\partial_z\left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2(t,s)}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right) z\phi^x(z,x,t)\right] \\
&+ \frac{1}{2}\partial_{zz}\left[\left(z\frac{\alpha(t,s)}{\gamma\sigma(x)}\right)^2 \phi^x(z,x,t)\right].
\end{aligned} \tag{16}$$

We then study the stationary distribution of wealth shares, in which $\partial_t \phi^x(z,x,t) = 0$. We take as given, for now, that $\alpha(t,s)$ will be constant, as in the stationary equilibrium we define in the following section. This will be true given a stationary distribution over investors' individual state variables. A stationary distribution of wealth shares $\phi^x(z,x)$ satisfies the following equation:

$$\begin{aligned}
0 \;=\; &-\partial_z\left[\left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x})\right) z\phi^x(z,x)\right] \\
&+\frac{1}{2}\partial_{zz}\left[\left(z\frac{\alpha}{\gamma\sigma(x)}\right)^2 \phi^x(z,x)\right].
\end{aligned} \tag{17}$$

We use guess and verify to show that the stationary distribution of wealth shares at each level of expertise is given by a Pareto distribution with an expertise specific tail parameter.

[15] We discuss the interpretation we adopt in detail in the Appendix.

[16] See Dixit and Pindyck [1994] for a heuristic derivation, or Karlin and Taylor [1981] for more detail.

This tail parameter, which we denote by β, is determined by the drift and volatility of the expertise specific law of motion for wealth shares. Intuitively, the larger the drift and volatility of the expertise specific wealth process, the fatter the tail of the wealth distribution at that level of expertise will be.

Proposition 3.2 *The stationary distribution of wealth shares $\phi^x(z,x)$ has the following form:*

$$\phi(z,x) \propto C(x) z^{-\beta(x)-1},$$

where

$$\begin{aligned}
\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), \\
C(x) &= \frac{1}{\int z^{-\beta} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.
\end{aligned}$$

See the Appendix for the Proof, where we also show that, in the stationary distribution, $\beta > 1$, which ensures a finite integral, and confirms that the distribution satisfies stationarity. The following Corollary, which we also prove in the Appendix, gives the tail parameters for the highest expertise agents, as well as all other investors.

Corollary 3.1 *For the highest expertise agents, we have*

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}} = C_1 \frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

where $\bar{z}$ is mean of normalized wealth of experts with highest expertise,

$$\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z,\bar{x}) dz = z_{\min}\left[1 + \frac{1}{\beta(\bar{x}) - 1}\right]$$

and

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})} \frac{1}{1 - z_{\min}/\bar{z}}$$

For all other expertise levels, we have

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}}\right) \frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma > 1. \tag{18}$$

The parameter β controls the tail of each expertise specific wealth distribution. The smaller is β, the more slowly the distribution decays, and the fatter is the upper tail. Clearly, β is an increasing function of risk aversion, γ, and an increasing function of expertise level volatility, $\sigma(x)$. The dependence of the tail parameter on expertise is given by $\frac{\sigma^2(x)}{\sigma^2(\bar{x})}$. Since expertise-specific effective volatility $\sigma(x)$ is decreasing in x, the wealth distribution of experts with a higher level of fixed expertise has a fatter tail. Investors with higher expertise allocate more wealth to the risky asset, which increases the mean and volatility of their wealth growth rate. Both a higher drift, and a wider distribution of shocks, lead to a fatter upper tail for wealth. Moreover, equation (18) shows that if the relation between expertise and effective volatility is steeper, then the difference in the size of the right tails of the wealth distribution across expertise levels increases. In equilibrium, variation in effective volatilities in complex asset markets will be driven both by the functional form for effective volatility, and by participation decisions which determine how different effective volatilities of participating agents will be. We can also measure the degree of wealth inequality within each expertise level as $\frac{1}{\beta(x)}$. High expertise levels exhibit greater size "inequality", and again, if the relation between expertise and effective volatility is steeper, indicating a more complex asset, then the difference in size inequality within expertise levels increases.

It is intuitive that investing more in the risky asset leads to a fatter tailed wealth distribution. However, perhaps surprisingly, as Lemma 3.1 illustrates, not every parameter which increases difference in the fraction of wealth allocated to the risky asset leads to an increase in the degree of fat tails of the expertise specific wealth distributions. We show in Lemma 3.1 that, while differences in portfolio choice driven by differences in effective volatilities lead to greater differences in decay parameters, this is not true for variation in portfolio choice driven by higher excess returns or lower risk aversion. This result offers a unique prediction for our model of complexity as differences in risk vs. risk aversion. See the Appendix for the proof.

Lemma 3.1 ***Relation Between*** $\theta(x)$ ***and*** $\beta(x)$

Consider two level of expertise, $x_{\min}$ *and* $x_{\max}$*, we have*

$$\theta(x_{\max}) - \theta(x_{\min}) = \frac{\alpha}{\gamma} \frac{\sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\sigma^2(x_{\max})\,\sigma^2(x_{\min})},$$

and

$$\beta(x_{\max}) - \beta(x_{\min}) = 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right) \frac{\sigma^2(x_{\max})\,\sigma^2(x_{\min})}{\alpha^3}\left[\theta(x_{\min}) - \theta(x_{\max})\right].$$

If a larger difference in portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller. If it is due to an increase in the difference in effective volatilities, then the difference in β's is larger.

3.3 Aggregation and Stationary Equilibrium

In this section, we define a stationary equilibrium, and state the condition which determines the market clearing α in a stationary equilibrium. Slightly abusing notation by suppressing dependence on the distribution of wealth and expertise, or equivalently on α, we define aggregate investment in the complex risky asset to be I, given each sum of expertise level investment $I(x)\ \forall x$, where:

$$I = \int \lambda(x) I(x)\, dx. \tag{19}$$

We first define a stationary equilibrium. In order to ensure that the supply of the complex risky asset does not become negligible as investor wealth grows, we assume that the supply grows proportionally to the mean wealth of the highest expertise investors. That is, we assume:

$$S(t) = Sg(\bar{x})\, t.$$

For convenience, we assume that the support of expertise is bounded above by $\bar{x}$, although most of our results only require that $\sigma(x)$ satisfies $\lim_{x\to\infty} \sigma(x) = \underline{\sigma} > 0$.

Definition 3.1 *A stationary equilibrium consists of a market clearing α, policy functions for all investors, and a stationary distribution over investor types $i \in \{x, n\}$, expertise levels x, and wealth shares z, $\phi(i, z, x, t)$, such that given an initial wealth distribution, an expertise distribution $\lambda(x)$, and parameters $\{\gamma, \rho, S, r_f, f_{nx}, f_{xx}, \sigma_\nu\}$ the economy satisfies:*

1. *Investor optimality: Investors choose participation in the complex risky asset market according to Equation (14), as well as optimal consumption and portfolio choices $\{c^n(t), c(x,t), \theta(x,t)\}_{t=0}^{\infty}$ according to Equations (10)-(10), such that their utilities are maximized.*

2. *Market clearing: The equilibrium market clearing α is determined by equating supply and demand:*

$$S(t) = \int \lambda(x,t)\, \theta(x,t)\, W(x,t)\, dx.$$

In a stationary equilibrium, we have:

$$I \equiv \int \lambda(x) I(x) \, dx = S, \tag{20}$$

Define $Z(x)$ to be the total expertise level wealth share,

$$Z(x) = z_{\min} \left(1 + \frac{1}{\beta(x) - 1} \right).$$

Then, define $I(x)$ to be the detrended total expertise level investment in the complex risky asset, namely,

$$I(x) \quad = \quad \frac{\alpha}{\gamma \sigma^2(x)} Z(x). \tag{21}$$

3. *The distribution over all individual state variables is stationary, i.e. $\partial_t \phi(i, z, x, t) = 0$.*

4 Results

4.1 Analytical Asset Pricing Results

With policy functions, stationary distributions, and the equilibrium definition in hand, we turn to our asset pricing results. We focus on the definition of a more complex asset as one that introduces more idiosyncratic risk. Comparing across assets, we use σ_ν to denote the fundamental volatility of the asset before expertise is applied, so that the risk in each investor's asset is $\sigma(\sigma_\nu, x)$, and is increasing in the first argument, and decreasing in the second. We provide a specific example below, but begin with any general function satisfying two these properties. Importantly, we describe natural conditions under which more complex assets, or assets which introduce more idiosyncratic risk, have lower participation despite higher α's and higher Sharpe ratios.

We begin by studying comparative statics over the equilibrium market clearing α. Although we focus on comparative statics over fundamental volatility, we also provide results for the market clearing α for changes other parameters which might proxy for asset complexity, such as the cost of maintaining expertise, or investor risk aversion. Next, we analyze individual Sharpe ratios. We emphasize heterogeneity across investors with different levels of expertise in changes in the risk return tradeoff as fundamental volatility changes. Because the other parameters which proxy for complexity do not change investor specific volatility, the results for individual Sharpe ratios are the same as those for α. Finally, we study market level Sharpe ratios, with a focus on the effects of the intensive and extensive margins of participation by investors with heterogeneous expertise.

Investor Demand, Aggregate Demand, and Equilibrium α We first describe the comparative statics for demand conditional on investors' expertise levels in Lemma 4.1.

Lemma 4.1 *Using Equation (21) for investor demand conditional on expertise, x, we have following comparative statics, $\forall x$:*

1. $\frac{\partial I(x)}{\partial \sigma^2(x)} < 0$
2. $\frac{\partial I(x)}{\partial \sigma_\nu} < 0$
3. $\frac{\partial I(x)}{\partial \alpha} > 0$
4. $\frac{\partial I(x)}{\partial \gamma} < 0$
5. $\frac{\partial I(x)}{\partial f_{xx}} < 0$

Demand for the risky asset at each level of expertise is increasing in the squared investor specific Sharpe ratio, and it is increasing in α. Demand is decreasing in effective variance, fundamental volatility, risk aversion, and the maintenance cost.

With expertise level total demands in hand, we can construct comparative statics for aggregate demand. We cannot express the equilibrium excess return in closed form. However, the following Proposition shows that the equilibrium excess return, α, and aggregate demand, I, form a bijection. This uniqueness result in turn ensures that α can be numerically solved for as the unique fixed point to Equation (20).

Proposition 4.1 *Aggregate market demand for the complex risky asset is an increasing function of the excess return, α, and α and I form a bijection. Mathematically,*

$$\frac{\partial I}{\partial \alpha} > 0.$$

Proposition 4.2 provides comparative statics over the aggregate demand for the complex risky asset, I. Using the result in Proposition 4.1, these comparative statics also hold for α.

Proposition 4.2 *Using the market clearing condition, we have that the following comparative statics hold:*

1. $\frac{\partial I}{\partial \sigma_\nu} < 0$, *thus* α *is an increasing function of fundamental risk*

2. $\frac{\partial I}{\partial \gamma} < 0$, *thus* α *is an increasing function of risk aversion*

3. $\frac{\partial I}{\partial f_{xx}} < 0$, *thus* α *is an increasing function of the maintenance cost.*

Demand for the risky asset is decreasing in fundamental volatility, risk aversion, and the maintenance cost. As a result, α is increasing in fundamental volatility, risk aversion, and the maintenance cost. We argue that an increase in these parameters proxies for greater asset complexity, and thus that our model predicts that α will be higher in more complex asset markets.

We now turn to the effect of the efficiency of the joint distribution of wealth and expertise on equilibrium pricing. In particular, we demonstrate that the equilibrium required excess return on the complex risky asset is decreasing in the amount of wealth commanded by agents with higher levels of expertise. The wealth distribution at each expertise level is a Pareto distribution with an expertise specific tail parameter. By shifting the distribution of expertise rightward, leading to a new distribution with a relatively larger fraction of higher expertise investors, relatively more wealth will reside with agents with higher expertise. Thus, with any rightward shift, the joint distribution of wealth and expertise becomes more efficient. Moreover, because the wealth distribution at higher expertise levels exhibits fatter right tails, there is an additional direct effect on overall wealth from a rightward shift in the distribution of expertise. Accordingly, Proposition 4.3 shows that if the density of experts shifts to the right, then demand for the complex risky asset will increase, and the required equilibrium excess return will decrease. The equilibrium excess return is decreasing in the amount of wealth which resides in the hands of agents with higher expertise. Note that this result can also be interpreted to state that in asset markets in which higher levels of expertise are more widespread, or less rare, equilibrium required returns will be lower. We argue that the scarcity of relevant expertise is increasing with asset complexity, again implying a higher α in more complex markets. The proof appears in the Appendix.

Proposition 4.3 *If* $\frac{\partial \sigma(x)}{\partial x} < 0$, *and* Λ_1 *exhibits first-order stochastic dominance over* Λ_2, $I(\Lambda_1) \geq I(\Lambda_2)$. *As a result* $\alpha(\Lambda_1) < \alpha(\Lambda_2)$.

Investor Specific Sharpe ratios, Investor Participation, and Market Level Sharpe ratios With the analysis of equilibrium excess returns in hand, we now turn to the equilibrium risk-return tradeoff at the investor and market level as described by the investor-specific, and market level Sharpe ratios. We emphasize the variation across indivdual Sharpe ratios as a function of expertise; all investors face a common market clearing α, but their effective risk varies. For the market level Sharpe ratio, two effects are present. First, there is the effect of any changes on parameters on the individual Sharpe ratios of participants. Second, there is a selection effect, or the effect on participation. We provide a natural condition under which participation declines as the asset becomes more complex. We focus on the equally weighted market-level equilibrium Sharpe ratio in our analysis. In addition to offering cleaner comparative statics because it does not depend on investor portfolio choices and market shares, the equally weighted Sharpe ratio represents the expected Sharpe ratio that an investor who could pay a cost to draw from the expertise distribution above the entry cutoff would earn. In that sense, it is the "expected Sharpe ratio". Note that the Sharpe ratio for non-experts is not defined.

Investor-specific Sharpe ratios: We define the investor-specific Sharpe Ratio as:

$$SR\left(x\right) = \frac{\alpha}{\sigma\left(x\right)}.$$

We provide results for how investor-specific Sharpe ratios change as fundamental volatility changes under the three possible cases for the elasticity of investor specific risk with respect to fundamental volatility in Proposition 4.4. The sign of this elasticity is a key determinant of our Sharpe ratio results.

Proposition 4.4 *The comparative statics for the investor-specific Sharpe ratios with respect to fundamental volatility depend on which of the three possible cases for the elasticity of investor-specific risk with respect to fundamental volatility applies:*

1. *Case 1, Constant Elasticity: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_{\nu}}$ *is a constant, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_{\nu}}}{\partial x} = 0,$$

we must have that $SR\left(x\right)$ *is either an increasing or a decreasing function of fundamental risk for all expertise levels.*

2. *Case 2, Increasing Elasticity: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}$ *is an increasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} > 0,$$

there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$*. Further,* x^* *exists if for any small* $\varepsilon < 10^{-6}$

$$(0, \varepsilon) \subseteq \left\{ \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \text{ for all } x \right\} \subseteq [0, \infty).$$

3. *Case 3 Decreasing Elasticity: If* $\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}$ *is a decreasing function of expertise, that is*

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0,$$

then there is a cutoff level x^**, such that for all* $x < x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$*; and for all* $x > x^*$*, we have* $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$.

Proposition 4.4 demonstrates that the effect of an increase in fundamental volatility on individual Sharpe ratios varies in the cross section, except in Case 1. The intuition is that the change in investors' Sharpe ratios depends on the percentage change in α relative to the percentage change in effective volatility. The change in α is aggregate, the same for all investors. So, the changes in individual Sharpe ratios with respect to changes in fundamental volatility depend on the percentage changes in effective volatility relative to the percentage change in fundamental volatility. If this elasticity is the same for all investors (Case 1), then the percentage change in α relative to the percentage change in effective volatility is the same for all investors. On the other hand, if the elasticity of effective volatility with respect to fundamental volatility is increasing in expertise (Case 2), then Sharpe ratios increase below a cutoff level of expertise and decrease above as fundamental volatility increases. This case is interesting if one interprets the increase in fundamental volatility as coming from a change in the asset which hurts incumbent higher expertise agents worse than potential new entrants. Finally, if this elasticity is declining in expertise, so that higher expertise investors can weather an increase in fundamental volatility better (Case 3), then Sharpe ratios increase above a cutoff level of expertise and decrease below. We focus our analysis on this case, because it is the only case which leads to the empirically plausible implication that more complex assets, with higher fundamental volatilities, have *lower* participation despite having persistently elevated excess returns. Thus, we argue that the decreasing elasticity case is the most relevant for describing a long-run, stationary equilibrium in a complex asset market.

Investor Participation: Before turning to the market-level Sharpe ratio, we describe investor participation. There are two key inputs into the market level risk return tradeoff. First, incumbents' individual Sharpe ratios change. Second, as equilibrium α changes, participation also changes. This selection effect plays a key role in determining comparative static results in general equilibrium. We show in the Appendix that participation increases with fundamental volatility in Cases 1 and 2 of Proposition 4.4. This is intuitive because α must increase with fundamental volatility σ_ν in order to clear the market. If all elasticities of $\sigma(x)$ with respect to σ_ν are the same, or if they are lower for lower expertise investors, then participation will increase with fundamental volatility. Thus, we focus on Case 3, and provide a natural condition under which participation declines as the asset becomes more complex and fundamental volatility increases.

Proposition 4.5 *Define the entry cutoff $\underline{x}$ as in Equation (14). We have*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$$

if the following conditions hold:

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$, *(Case 3 of Proposition 4.4) and*
2. $l^{\sigma_\nu}_{\text{sup}} > \left(1 + \frac{1}{1+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}\right) E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | x \geq \underline{x}\right]$,

   where $l^{\sigma_\nu}_{\text{sup}}$ is defined to be the highest elasticity of all participating investors' effective volatility with respect to fundamental volatility.

The first condition, namely that the elasticity of effective volatility with respect to fundamental volatility is decreasing in expertise, is necessary for participation to decline as complexity, and fundamental volatility, increase. The second condition gives a sufficient condition which states that the elasticity of the lowest expertise agent who participates, i.e. the agent with the highest sensitivity of effective volatility to fundamental volatility, must be sufficiently different from the average. Intuitively, what is necessary for participation to decline as fundamental volatility increases is that there is enough variation in the effect of the change in fundamental volatility across agents with high and low expertise so that α does not need to increase enough to satisfy the marginal investor or entice lower expertise investors to participate. We argue that it is natural for more complex assets, in addition to exposing investors to more risk overall, to pose a larger difference in risk across investors with different levels of expertise. Under the conditions in Proposition 4.5 our model generates higher persistent α's and lower participation, despite free entry, as fundamental volatility and asset complexity increase.

Equilibrium Market-Level Sharpe Ratio We define the equally weighted market equilibrium Sharpe ratio as:

$$SR^{ew} = E\left[\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right].$$

We focus on comparative statics for the equally weighted market equilibrium Sharpe ratio for simplicity.[17]

Proposition 4.6 *The equally weighted market Sharpe Ratio is increasing with fundamental risk in general equilibrium, i.e.,*

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0,$$

if:

1. *Participation increases,* $\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$ *or,*
2. *Participation decreases,* $\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0$ *and* $l_{\text{sup}}^{\sigma_\nu} > \frac{d\Lambda(\underline{x})}{E\left[1 - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} | x > \underline{x}\right]}$, *where we restrict the the average elasticity of participants to be less than 1, so that the denominator is positive.*

Condition 1 of Proposition 4.6 shows that the equally weighted market Sharpe ratio increases with fundamental volatility if participation increases. However, we argue that the more relevant case is in Condition 2, which covers the case when participation is lower when assets have higher fundamental volatility and are more complex. Note that the restriction that when fundamental risk is increased by 1%, the average effective vol is increased less by 1% is easily satisfied, as expertise reduces fundamental volatility. Thus, for the equally weighted market Sharpe ratio to increase with fundamental volatility while participation declines, the model first requires that agents with more expertise are less sensitive to increases in volatility (a necessary

[17] See the Appendix for the definition of the value-weighted market equilibrium Sharpe ratio.

condition for participation to decline with fundamental volatility). The second condition is a sufficient condition that if there are many investors around the entry threshold, these investors do not have such low Sharpe ratios that the market Sharpe ratio is overwhelmed by their participation.[18] Note also the similarity between the second conditions in Propositions 4.5 and 4.6. Both require the elasticity of the lowest participating investor to be sufficiently different from the average. Thus, another intuitive statement of the requirement in Condition 2 in Proposition 4.6 is that the average elasticity will be very different from that of the threshold investors if there are relatively few investors at the threshold.

We argue that the declining elasticity case of Proposition 4.4 is the most natural in a stationary equilibrium for complex assets with limited participation. Moreover, it seems reasonable to assume a distribution for expertise which does not put too much weight on investors near the threshold. For example, we show below that a log-normal distribution easily delivers the relevant result. Under the conditions in Proposition 4.6, our model delivers a rational explanation for why more complex assets have a higher α, a higher equally-weighted equilibrium market Sharpe ratio, but low participation, despite free entry. Intuitively, as in a standard industrial organization model, the superior volatility reduction technologies of more expert investors provide them with an excess of (risk-bearing) capacity, which serves to reduce the entry incentives of newcomers despite attractive conditions for incumbents.

4.2 Numerical Examples

This section presents complementary numerical results and comparative statics for Case 3 from Proposition 4.4, in which the elasticity of effective volatility with respect to fundamental volatility declines with expertise. Results for the other cases are available upon request. The model generates closed form policy functions and wealth distributions conditional on expertise levels. To provide intuition for the effects of equilibrium pricing, we provide the comparative statics in both partial equilibrium and general equilibrium. In partial equilibrium, the excess return is given exogenously, and held fixed, while aggregate demand (and hence implicitly supply) varies. In general equilibrium, the excess return is computed endogenously given a fixed supply of the risky asset. Because α and I form a bijection (Proposition 4.1 provides conditions for which they are one to one and onto), for any given supply of the complex risky asset, we can solve for the market equilibrium α in the following steps:

[18] We also note that the value weighted Sharpe ratio puts less weight on agents at the threshold, as they have a lower share of wealth and a smaller share of their wealth allocated to the risky asset.

1. Choose an upper and a lower bound for α, namely α_1 and α_2, $(\alpha_1 > \alpha_2)$.

2. Let $\alpha = \frac{\alpha_1+\alpha_2}{2}$, and compute the total demand for the risky asset

$$\int \lambda(x)\, I(x)\, dx$$

3. If $S - \int \lambda(x)\, I(x)\, dx < -10^{-4}$, let $\alpha_1 = \alpha$ and back to step 1; if $S - \int \lambda(x)\, I(x)\, dx > 10^{-4}$, let $\alpha_2 = \alpha$ and back to step 1; otherwise, STOP.

We provide results under specific parametric assumptions. Specifically, we specify that:

$$\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0, \sigma(x) = a + x^{-b}\sigma_\nu^2. \text{ [19]}$$

Our baseline parameters are summarized in Table 1. The time interval is one quarter. The risk-free rate is 1%. The discount factor is 1%. The maintenance cost is also 1%. The coefficient of relative risk-aversion is 5. The log-normal distribution of expertise has a mean of 0 and volatility of 5. The minimum wealth share is set to 0.05. The fundamental standard deviation of the risky asset return is 20%. We set $a = 0.0112$ and $b = 1$. This implies that the highest expertise investors can eliminate 47% of fundamental risk, and face an effective standard deviation of 10.6%.

Figure 1 studies the effects of changes in fundamental volatility, with more complex assets characterized by higher fundamental volatility. Starting in the top row, as fundamental volatility increases, demand for the risky asset in partial equilibrium decreases, implying a higher α in general equilibrium. The left hand side of the second row displays the entry cut-off, which is increasing in fundamental volatility, consistent with our result in Proposition 4.5. Accordingly, participation, graphed on the right hand side of the second row, declines. We note that participation declines by less in general equilibrium, due to the positive effect of fundamental volatility on α, but still the decline is nearly as large as in partial equilibrium given our parametric assumptions. Finally, the third row plots the equally weighted standard deviation of the risky asset returns, which are increasing in both partial and general equilibrium. The effect is magnified in general equilibrium because participation declines by more, and hence there is more positive selection to higher expertise investors, since α is held constant in partial equilibrium. Finally, the bottom right panel of Figure 1 shows that despite the fact that the

[19] x^{-b} can be replaced by any function $f(x)$ as long as $\frac{\partial f(x)}{\partial x} < 0$.

equally weighted standard deviation is increasing, the larger, positive effect of the increase in α in general equilibrium implies that the equally weighted Sharpe ratio increases, consistent with Proposition 4.6. Thus, the numerical example confirms the model's ability to generate persistently higher α's and larger Sharpe ratios, but lower participation despite free entry, for more complex assets characterized by higher fundamental volatility.

Finally, we present results on the size distribution of funds in our model, and in the data, across asset classes which are more and less complex. Although in the model, it is easy to define a complex asset as one with a higher fundamental volatility, fundamental volatility (before expertise is applied) is unobservable in the data. Thus, we use the implication of our model that Sharpe ratios are higher in more complex asset classes. We use the subset of the Hedge Fund Research (HFR) data which describes Relative Value fund performance, as these strategies are likely to involve long-short positions as in the micro-foundation for our return process. We compute "pseudo" Sharpe ratios as the ratio of the average industry level return to the time series average of the cross section standard deviation of returns. We then rank strategies from most to least complex by these pseudo Sharpe ratios. This ranking is essentially unchanged if we instead use the cross sectional average of time series standard deviations of returns by fund in the denominator. We note also that the time series average of the cross section standard deviation of returns and the cross sectional average of time series standard deviations of returns by fund are very similar supports the structure of our stationary model.

The top panel of Figure 2 displays the relative concentration of wealth across strategies in the HFR Relative Value data by plotting the cumulative wealth shares by wealth decile. Although the relationship is not quite monotonic, on average the more complex, higher Sharpe ratio strategies display lower concentration. The bottom panel of Figure 2 plots the relative concentration of wealth in the model across strategies with varying levels of complexity, given by the level of fundamental volatility. The model generates the pattern seen in the data; more complex strategies have less wealth concentration. This might seem surprising given that in our model high expertise agents have fatter tailed wealth distributions, and have Sharpe ratios (and hence portfolio allocations to the risky asset) which increase with fundamental volatility. The reason more complex assets have less concentrated wealth distributions in the model are twofold. First, participation is limited, so agents who are in the market cannot have too different of individual Sharpe ratios. Second, our specification for effective volatility has a positive second derivative, therefore there are essentially decreasing returns to expertise. As a result, agents who participate in the most complex asset classes are not as different from each other as those in less complex asset markets, which results in a less concentrated wealth distribution.[20]

5 Conclusion

We study the equilibrium returns to complex risky assets in segmented markets with expertise. We show that required returns increase with asset complexity, as proxied for by higher fundamental volatility, higher costs of maintaining expertise, and by exertise being scarce in the population. We emphasize heterogeneity in the risk-return tradeoff faced by investors with different levels of expertise. Accordingly, we show that in our model, under reasonable conditions, improvements in market level Sharpe ratios can be accompanied by lower market participation, consistent with empirical observations. Finally, we describe the implications of our model for the industrial organization of markets for complex risky assets. Markets for more complex assets have a less concentrated size distribution, which we show is consistent with data on relative value hedge fund strategies.

[20] Our model does a better job matching the upper end of the wealth distribution than the lower end. This is a well-known problem in models of the wealth distribution featuring a Pareto distribution. See **?** for a review of the literature, and specifically Table 1 for the errors in six prominent models for the low end of the wealth distribution.

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Table 1: Parameter Values: Numerical Example

Parameter	Symbol	Value	Target
Discount factor	ρ	0.01	Annual interest rate
Risk-free rate	r_f	0.01	Annual interest rate
Risk aversion	γ	5	Data/mean portfolio choice
Entry cost	fnx	0.03	
Maintenance cost	fxx	0.01	
Risky asset supply	S	0.52	$\alpha = 5.5\%$
Volatility of risky asset return	σ_ν	20%	
Mean of expertise process	μ_x	0	
Volatility of expertise process	σ_x	5	
Constant in σ_x^2	a	0.0112	
Slope of σ_x^2	b	1	
Minimum wealth share	z_{min}	0.05	

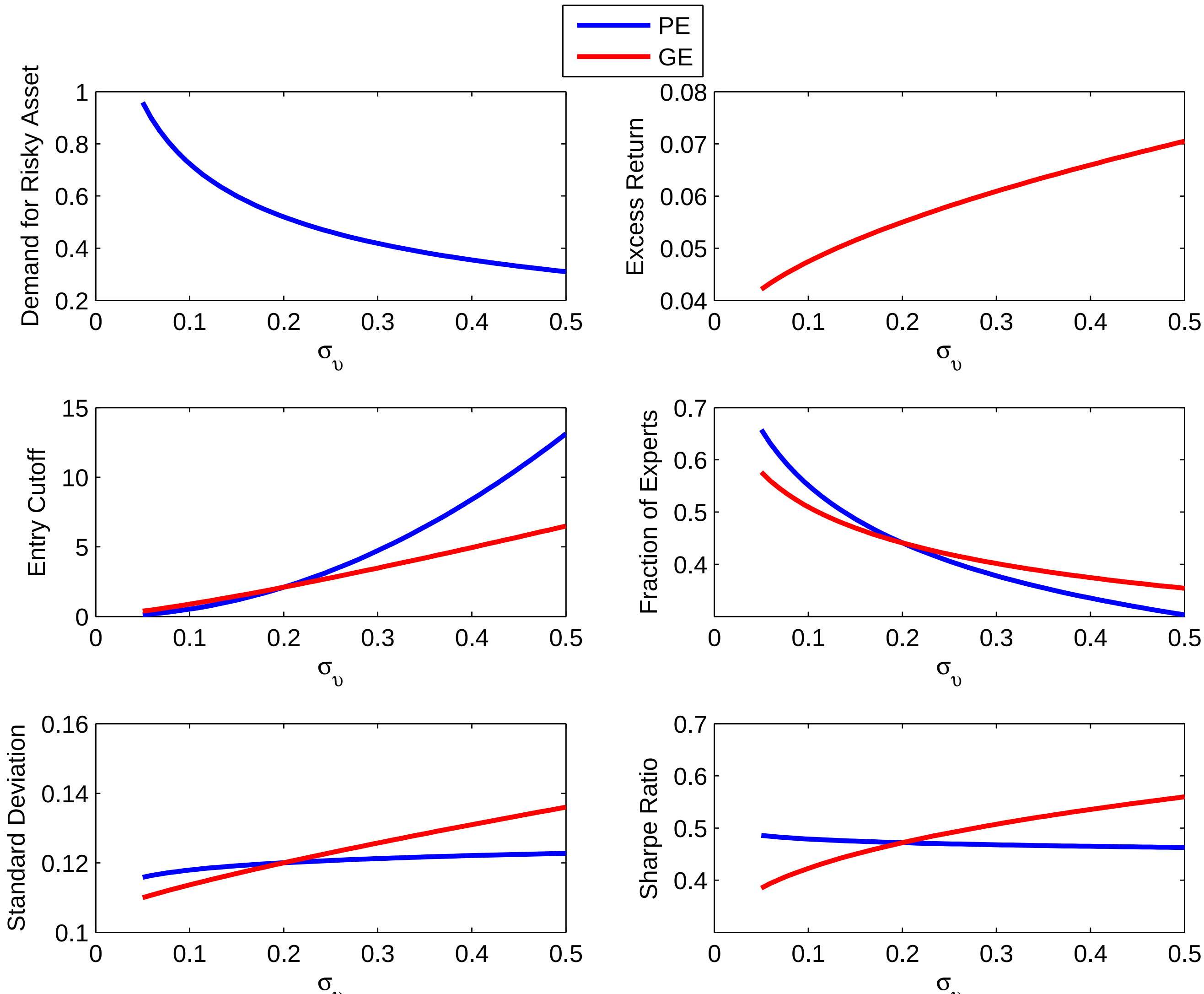

Figure 1: Case 3 Model comparative statics: fundamental risk

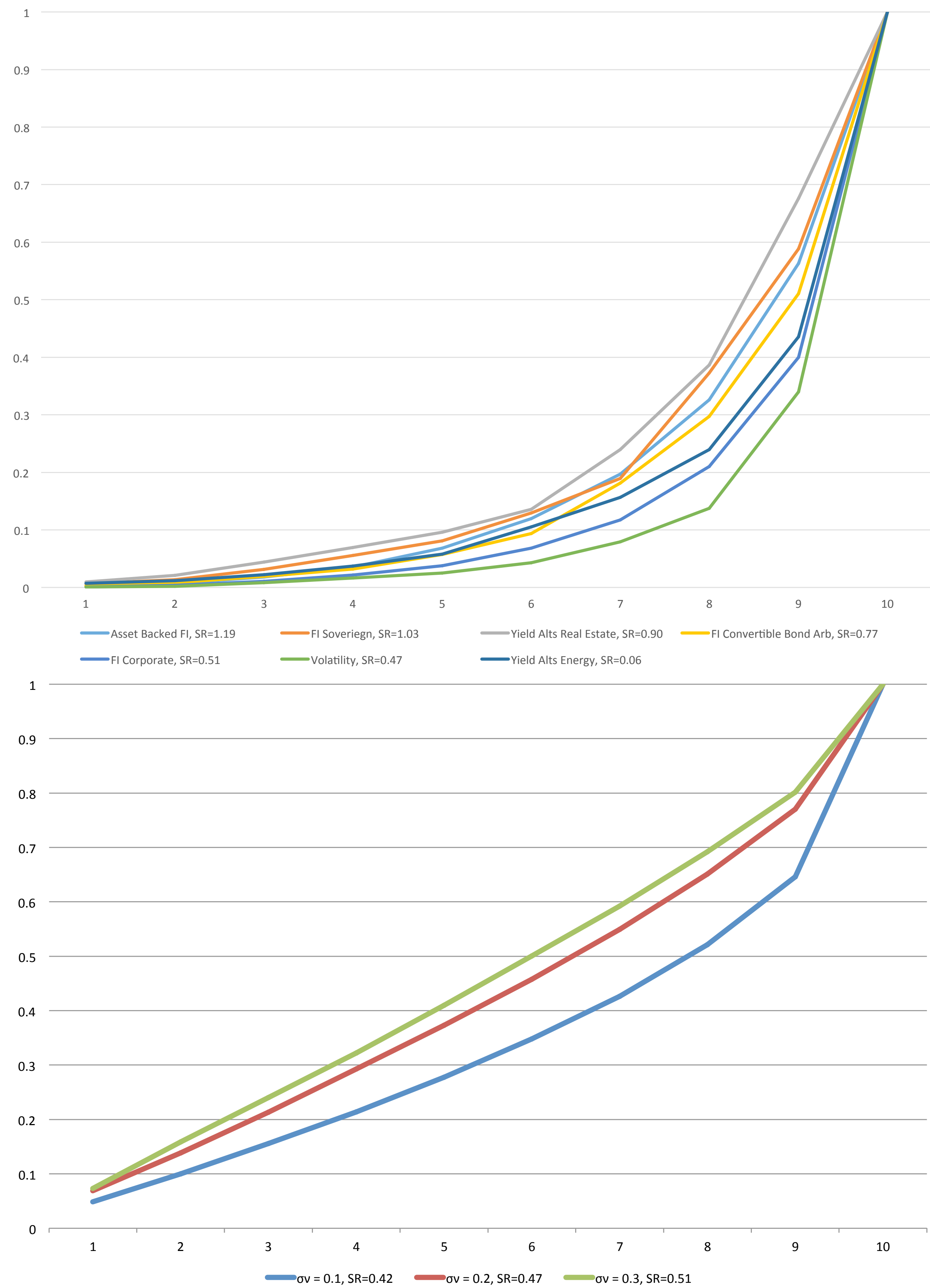

Figure 2: Cumulative wealth shares in the data (top) and model (bottom) across asset classes. Complex assets have higher Sharpe ratios, and (on average) lower concentration. FI = Fixed Income. Data is from HFR Relative Value strategies, excluding multi-strategy.

A Appendix

Complex Asset Return Process: Hedging with Tracking Portfolios. We construct an example motivation for the return process in Equation (1) based on executing an arbitrage opportunity via a long position in an underlying asset and a short position in a hedging or tracking portfolio. We interpret the α as the "mispricing" of the complex asset, and it is equal to the equilibrium return it earns because investors must bear idiosyncratic model risk to invest in the long-short strategy. There is an underlying complex asset, such as an MBS or convertible bond, which returns:

$$\frac{dU(t,s)}{U(t,s)} = [r_f + \alpha(s) + a(s)]\, dt + \sigma^U dB^U(t,s).$$

Investors have heterogeneous access to, or knowledge of, tracking or hedging portfolios. The value of each agent's "best" tracking portfolio per unit of the underlying asset evolves according to $\frac{dT_i(t,s)}{T_i(t,s)}$. Thus, each agent takes a unit short position in their tracking portfolio for each unit long position they hold in the underlying asset $U(t,s)$.[21] Tracking portfolio returns evolve according to:[22]

$$\frac{dT_i(t,s)}{T_i(t,s)} = a(s)dt + \vartheta\sigma^U dB^U(t,s) - \sigma\left(x,\vartheta\right) dB_i^*(t,s),$$

where

$$\sigma^2\left(x,\vartheta\right) = \left(\frac{\vartheta}{h\left(x\right)}\right)^2 - \left(\vartheta\sigma^U\right)^2.$$

Note that, by definition, if the tracking portfolio returns are not perfectly correlated with the underlying asset returns (in which case there would exist a risk-less arbitrage opportunity), then the tracking portfolio will introduce independent risk. We assume this risk is uncorrelated across investors. Because each investor has their own model and strategy implementation, tracking portfolios introduce investor-specific shocks. We then use the fact that any Brownian shock which is partially correlated with the underlying Brownian shock $dB^U(t,s)$ can be decomposed into a linear combination of a correlated shock and an independent shock. We denote this independent, investor-specific shock $dB_i^*(t,s)$. Our assumption for the amount of idiosyncratic risk the tracking portfolio introduces implies that this risk is larger the lower is ϑ, the loading on the underlying asset's Brownian shock, which is intuitive. Here, the effect of expertise on risk is captured by $h(x)$, with $h'(x) > 0$. Within an asset class, investors with higher expertise have superior models and tracking portfolios, hence they face lower risk. Across asset classes, more complex assets are characterized by more imperfect models and tracking portfolios, and hence more complex assets impose more risk on investors. For example, one can interpret a more complex asset as one for which $h(x)$ is lower for all agents.

[21]We do not clear the market for tracking portfolios. We instead argue that it is realistic to assume that demand for the tracking portfolio from hedging the complex asset is "small" relative to total demand.

[22]Note we leave r_f out of the tracking portfolio return for parsimonious (and familiar) expressions for expert portfolio returns but this is without loss of generality. The equilibrium excess return will simply increase by r_f if the net asset's drift is decreased by r_f.

Given these returns to the underlying asset and tracking portfolio, we have

$$\begin{aligned} corr\left(dB^F(t,s), dB_i^*(t,s)\right) &= 0, \\ corr\left(\frac{dU(t,s)}{U(t,s)}, \frac{dT_i(t,s)}{T_i(t,s)}\right) &= h(x)\,\sigma^U. \end{aligned}$$

Thus, investors with more expertise have tracking portfolios with a higher correlation with the underlying asset, as is intuitive. The linear form ensures that the idiosyncratic risk introduced by the tracking portfolio will remain even if there is no underlying risk, which is also intuitive, and consistent with our assumptions.

Returns for the net asset evolve according to

$$\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)]\,dt + (1-\vartheta)\sigma^U dB^U(t,s) + \sigma(x,\vartheta)\,dB_i^*(t,s)$$

We have for the net asset, then:

$$\begin{aligned} E\left(\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)}\right) &= r_f + \alpha(s) \\ Var\left(\frac{dU(t,s)}{U(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)}\right) &= [1-\vartheta]^2\left(\sigma^U\right)^2 + \left(\frac{\vartheta}{h(x)}\right)^2 - \left(\vartheta\sigma^U\right)^2 \\ &= \left(\frac{\vartheta}{h(x)}\right)^2 + (1-2\vartheta)\left(\sigma^U\right)^2 \end{aligned}$$

Since we abstract from aggregate risk, we study the case in which ϑ goes to one, which implies that, given our assumptions, the underlying Brownian risk drops out as follows. Taking $\vartheta \to 1$, we have:

$$\frac{dF(t,s)}{F(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)]\,dt + \sigma(x,\vartheta)\,dB_i^*(t,s).$$

We thus micro-found the return process in Equation (1) with the volatility of the independent shock given by:

$$\sigma(x,\vartheta) = \left(\frac{1}{h(x)}\right)^2 - \left(\sigma^U\right)^2,$$

where we note that $dB^U(t,s)$ drops out, leaving only the fixed parameter σ^U and a term which is decreasing in expertise.[23]

[23] We note that if one instead takes $\sigma^F \to 0$, we have $\frac{dF(t,s)}{F(t,s)} - \frac{dT_i(t,s)}{T_i(t,s)} = [r_f + \alpha(s)]\,dt + \sigma(\vartheta)\,dB_i^*(t,s)$, where $\sigma^2(x,\vartheta) = \left(\frac{\vartheta}{h(x)}\right)^2$, which is also yields a micro-foundation consistent with our assumptions, and no aggregate risk.

Note that we can generate the example functional form from Section 4.2 by assuming the following for $h(x)$:

$\sigma^2(x) = a + x^{-b}\sigma_\nu^2$, implies $\left(\frac{1}{h(x)}\right)^2 = a + x^{-b}\sigma_\nu^2 + \left(\sigma^U\right)^2$.

Proof. Proposition 3.1. We prove this Proposition by guess and verify. First, we write the HJB equations of our model

$$\begin{aligned}
\max_{c^x(t,s),\theta(x,t,s)} 0 &= u\left(c^x(t,s)\right) + V_w^x\left[w(t,s)\left(r_f + \theta(x,t,s)\,\alpha(t,s)\right) - c^x(t,s) - f_{xx}w(t,s)\right] \\
&\quad + \frac{\theta^2(x)\,\sigma^2(x)\,w(t,s)^2}{2}V_{ww}^x - \rho V^x \\
\max_{c^n(t,s)} 0 &= u\left(c^n(t,s)\right) + V_w^n\left(r_f w(t,s) - c^n(t,s)\right) - \rho V^n
\end{aligned}$$

The first order conditions are

$$\begin{aligned}
u'\left(c^x(t,s)\right) &= V_w^x, \\
u'\left(c^n(t,s)\right) &= V_w^n, \\
V_w^x\alpha(t,s) + \theta(x,t,s)\,\sigma^2(x)\,w(t,s)\,V_{ww}^x &= 0.
\end{aligned}$$

Next, we guess that

$$\begin{aligned}
V^x\left(w(t,s),x\right) &= y^x(x,t,s)\,\frac{w(t,s)^{1-\gamma}}{1-\gamma}, \\
V^n\left(w(t,s),x\right) &= y^n(x,t,s)\,\frac{w(t,s)^{1-\gamma}}{1-\gamma}.
\end{aligned}$$

Thus

$$\begin{aligned}
c^x &= \left[y^x(x,t,s)\right]^{-\frac{1}{\gamma}} w(t,s), \\
c^n &= \left[y^n(x,t,s)\right]^{-\frac{1}{\gamma}} w(t,s),
\end{aligned}$$

and portfolio choice is given by

$$\theta(x,t,s) = \frac{\alpha(t,s)}{\gamma\sigma^2(x)}.$$

Plugging these choices into the HJB equations, we get

$$\begin{aligned}
0 &= \left[y^x(x,t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s)\left(r_f + \frac{\alpha^2(t,s)}{\gamma\sigma^2(x)} - \left[y^x(x,t,s)\right]^{-\frac{1}{\gamma}} - f_{xx}\right)(1-\gamma) \\
&\quad - \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)}y^x(x,t,s)(1-\gamma) - \rho y^x(x,t,s) \\
&= \gamma\left[y^x(x,t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^x(x,t,s)\left(r_f + \frac{\alpha^2(t,s)}{2\gamma\sigma^2(x)} - f_{xx}\right)(1-\gamma) - \rho y^x(x,t,s), \\
0 &= \gamma\left[y^n(x,t,s)\right]^{-\frac{1-\gamma}{\gamma}} + y^n(x,t,s)(1-\gamma)\,r_f - \rho y^n(x,t,s).
\end{aligned}$$

Rearranging the equations, we solve for $y^x(x,t,s)$ and $y^n(x,t,s)$,

$$\begin{aligned} y^x(x,t,s) &= \left[\frac{(\gamma-1)(r_f - f_{xx}) + \rho}{\gamma} + \frac{(\gamma-1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right]^{-\gamma}, \\ y^n(x,t,s) &= \left[\frac{(\gamma-1)r_f + \rho}{\gamma}\right]^{-\gamma}. \end{aligned}$$

Given all policy functions, we get the experts' wealth growth rates:

$$\frac{dw(t,s)}{w(t,s)} = \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\alpha^2(t,s)}{2\gamma^2\sigma^2(x)}\right)dt + \frac{\alpha(t,s)}{\gamma\sigma(x)}dB(t,s)$$

Finally, given homogeneity of the value functions in wealth, the participation cutoff is constructed by direct comparison between $y^x(x,t,s)$ and $y^n(x,t,s)$. ■

Proof of equivalence of policy functions under the reflecting barrier $\mathbf{z_{min}}$

Interpretation of $z_{\min}$: We assume that one of two things can happen to an investor at $z_{\min}$. With probability q, the investor is eliminated from the market, and replaced with a new agent with wealth share $z_{\min}$ and the same expertise as the exiting agent. Note that elimination in isolation would cause the incumbent agent to be conservative, to avoid $z_{\min}$. With probability $1-q$, the agent is rewarded by being able to infuse funds themselves, or by receiving new external funds, and the wealth share reflects. Note that this reward in isolation would cause the agent to risk shift, to take advantage of limited liability at $z_{\min}$. We require that $E[V^x(z,x)_{true}] = qE[V^x(z,x)_{die}] + (1-q)E[V^x(z,x)_{reflect}]$, conditional on the optimal policies under the true wealth share dynamics. Since the value under the true, non-reflecting, dynamics lies between the punishment value of dying and the reward value of reflection, we conjecture that there exists some probability, conditional on parameters, that this is the case. For simplicity, we assume that $V^x(z,x)_{die} = 0$. It seems quite realistic that investors face uncertainty about what will happen to them as their assets fall below a threshold level. Will they be liquidated, or rescued? Note that our proof offers a technical contribution, since in Gabaix [1999] cities do not choose size, unlike the case for our investors, who choose their savings and portfolio allocations.

We show that the optimal policies in the model with reflecting barrier z_{min} are equivalent to those in the original model under our assumptions of a zero value at death, which is traded off with the positive value of reflection. Our proof assumes an optimal exit date. This is without loss of generality in a stationary equilibrium with no entry or exit.

Model 1:

$$\begin{aligned} V^x\left(w\left(t,s\right),x\right) &= \max_{c^x(t,s),T,\theta(x,t,s)} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)}u\left(c^x\left(t,s\right)\right)ds + e^{-\rho(T-t)}V^n\left(w\left(t,s\right),x\right)\right] \\ \text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(x,t,s\right)\alpha\left(t,s\right)\right)-c^x\left(t,s\right)-F_{xx}\right]ds \\ &\quad + w\left(t,s\right)\theta\left(x,t,s\right)\sigma\left(x\right)dB\left(t,s\right), \end{aligned}$$

Model 2:

$$\begin{aligned} V^y\left(w\left(t,s\right),y\right) &= \max_{c^y(t,s),T,\theta(y,t,s)} \max\left\{V^x\left(w\left(t,s\right),y\right),\mathbb{E}\left[\begin{array}{c} \int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds \\ +\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},y\right)\end{array}\right]\right\} \\ \text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^y\left(t,s\right)-F_{yy}\right]ds \\ &\quad + w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right) \end{aligned}$$

Assume $F_{xx}=F_{yy}$. They are both linear in wealth. By definition, we have

$$V^y\left(w\left(t,s\right),x\right)=\left(1-q\right)V^y\left(w_{\min},x\right), \text{ for } w\left(t,s\right)\leq w_{\min}.$$

Define

$$q\left(w\left(t,s\right),w_{\min}\right)=1-\left[\frac{w\left(t,s\right)}{w_{\min}}\right]^{1-\gamma}, \text{ for } w\left(t,s\right)\leq w_{\min}.$$

Therefore, we have

$$V^x\left(w\left(t,s\right),x\right)=\left(1-q\right)V^x\left(w_{\min},x\right), \text{ for } w\left(t,s\right)\leq w_{\min}.$$

It suffices to show that

$$V^y\left(w\left(t,s\right),x\right)=V^x\left(w\left(t,s\right),x\right), \text{ for all } x \text{ and } w\left(t,s\right),$$

when agent's wealth hits $w_{\min}$ before he/she exits the market. That is

$$\begin{aligned} V^y\left(w\left(t,s\right),y\right) &= \max_{c^y(t,s),\theta(y,t,s)} \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},y\right)\right] \\ \text{s.t. } dw\left(t,s\right) &= \left[w\left(t,s\right)\left(r_f+\theta\left(y,t,s\right)\alpha\left(t,s\right)\right)-c^y\left(t,s\right)-F_{yy}\right]ds \\ &\quad + w\left(t,s\right)\theta\left(y,t,s\right)\sigma\left(y\right)dB\left(t,s\right) \end{aligned}$$

First,

$$\begin{aligned} V^y\left(w_{\min},x\right) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^y\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},x\right)\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)}u\left(c^x\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^y\left(w_{\min},x\right)\right], \end{aligned}$$

that is,

$$\begin{aligned} & E \int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds \\ \leq \quad & E \int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds \\ = \quad & \frac{1}{1-\mathbb{E}\left[(1-q)\, e^{-\rho(s'-t)}\right]} V^y\left(w_{\min}, x\right). \end{aligned}$$

Second,

$$\begin{aligned} V^x\left(w_{\min}, x\right) &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds + e^{-\rho(s'-t)} V^x\left(w\left(t, s'\right), x\right)\right] \\ &= \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds + (1-q)\, e^{-\rho(s'-t)} V^x\left(w_{\min}, x\right)\right] \\ &\geq \mathbb{E}\left[\int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds + (1-q)\, e^{-\rho(s'-t)} V^x\left(w_{\min}, x\right)\right], \end{aligned}$$

that is,

$$\begin{aligned} & E \int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds \\ \leq \quad & E \int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds \\ = \quad & \frac{1}{1-\mathbb{E}\left[(1-q)\, e^{-\rho(s'-t)}\right]} V^x\left(w_{\min}, x\right). \end{aligned}$$

Therefore, we must have

$$E \int_t^{s'} e^{-\rho(s-t)} u\left(c^y(t,s)\right) ds = E \int_t^{s'} e^{-\rho(s-t)} u\left(c^x(t,s)\right) ds,$$

and

$$V^y\left(w_{\min}, x\right) = V^x\left(w_{\min}, x\right).$$

Next,

$$
\begin{aligned}
V^{y}\left(w\left(t,s\right),x\right) &= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right] \\
&\geq \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{x}\left(w_{\min},x\right)\right] \\
&= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds+e^{-\rho(s'-t)}V^{x}\left(w\left(t,s'\right),x\right)\right] \\
&= V^{x}\left(w\left(t,s\right),x\right), \text{ for all } w\left(t,s\right)
\end{aligned}
$$

with equality iff $c^{x}\left(t,s\right)=c^{y}\left(t,s\right)$ and $\theta^{x}\left(x,t,s\right)=\theta^{y}\left(x,t,s\right)$.

Lastly,

$$
\begin{aligned}
V^{x}\left(w\left(t,s\right),x\right) &= \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{x}\left(t,s\right)\right)ds+e^{-\rho(s'-t)}V^{x}\left(w\left(t,s'\right),x\right)\right] \\
&\geq \mathbb{E}\left[\int_{t}^{s'} e^{-\rho(s-t)}u\left(c^{y}\left(t,s\right)\right)ds+\left(1-q\right)e^{-\rho(s'-t)}V^{y}\left(w_{\min},x\right)\right] \\
&= V^{y}\left(w\left(t,s\right),x\right), \text{ for all } w\left(t,s\right)
\end{aligned}
$$

with equality iff $c^{x}\left(t,s\right)=c^{y}\left(t,s\right)$ and $\theta^{x}\left(x,t,s\right)=\theta^{y}\left(x,t,s\right)$.

Therefore,

$$
\begin{aligned}
V^{y}\left(w\left(t,s\right),x\right) &= V^{x}\left(w\left(t,s\right),x\right), \text{ for all } x \text{ and } w\left(t,s\right). \\
c^{x}\left(t,s\right) &= c^{y}\left(t,s\right), \\
\theta^{x}\left(x,t,s\right) &= \theta^{y}\left(x,t,s\right).
\end{aligned}
$$

Proof. Proposition 3.2 We prove this Proposition by guess and verify. We guess that:

$$\phi(z,x) = C(x) z^{-\beta(x)-1},$$

Then, we have

$$\begin{aligned}
0 = & -\partial_z \left(z^{-\beta(x)} \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \right) \\
& + \frac{1}{2}\partial_{zz} \left(z^{1-\beta(x)} \frac{\alpha^2}{\gamma^2\sigma^2(x)} \right) \\
= & \ \beta(x) \left(\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{(\gamma+1)\,\alpha^2}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right) \\
& - \frac{1}{2}\beta(x)\left(1-\beta(x)\right) \left[\frac{\alpha}{\gamma\sigma(x)} \right]^2 \\
= & \ \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2 (\gamma + \beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right]
\end{aligned}$$

Thus

$$\begin{aligned}
\beta(x) &= C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma \geq 1, \\
C_1 &= 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right), \\
C(x) &= \frac{1}{\int z^{-\beta-1} dz} = \frac{C_1 \frac{\sigma^2(x)}{\alpha^2} - \gamma}{z_{\min}^{-C_1 \frac{\sigma^2(x)}{\alpha^2} + \gamma}}.
\end{aligned}$$

Note there are two roots of equation

$$0 = \beta(x) \left[\frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2 (\gamma + \beta(x))}{2\gamma^2\sigma^2(x)} - g(\bar{x}) \right].$$

We only take the root that is larger than 1 to ensure the mean wealth has a finite mean. ∎

Proof. Corollary 3.1. For the highest expertise agents, we have

$$\bar{z} = \int_{z_{\min}}^{\infty} z\phi(z,\bar{x})dz = \int_{z_{\min}}^{\infty} C z^{-\beta(\bar{x})} dz = z_{\min}\left[1 + \frac{1}{\beta(\bar{x}) - 1}\right].$$

This gives us another expression of $\beta(\bar{x})$,

$$\beta(\bar{x}) = \frac{1}{1 - z_{\min}/\bar{z}}.$$

Also, we know

$$\beta(\bar{x}) = 2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma$$

Therefore, we have

$$2\gamma\left(f_{xx} + \rho - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(\bar{x})}{\alpha^2} - \gamma = \frac{1}{1 - z_{\min}/\bar{z}},$$

Rearrange the above equation, we get

$$g(\bar{x}) = \frac{r_f - f_{xx} - \rho}{\gamma} + \frac{\alpha^2}{2\gamma\sigma^2(\bar{x})} + \frac{\alpha^2}{2\gamma^2\sigma^2(\bar{x})}\frac{1}{1 - z_{\min}/\bar{z}}.$$

Plug $g(\bar{x})$ into $\beta(x)$, we derive

$$\beta(x) = \left(\gamma + \frac{z_{\min}/\bar{z}}{1 - z_{\min}/\bar{z}}\right)\frac{\sigma^2(x)}{\sigma^2(\bar{x})} - \gamma.$$

■

Proof. Lemma 3.1

$$\begin{aligned}
\text{Recall that:}\theta(x) &= \frac{\alpha}{\gamma\sigma^2(x)} \\
\beta(x) &= 2\gamma\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x)}{\alpha^2} - \gamma
\end{aligned}$$

Consider two levels of expertise, $x_{\min}$ and $x_{\max}$, we have

$$\begin{aligned}
\theta(x_{\max}) - \theta(x_{\min}) &= \frac{\alpha}{\gamma}\left[\frac{1}{\sigma^2(x_{\max})} - \frac{1}{\sigma^2(x_{\min})}\right] \\
&= \frac{\alpha}{\gamma}\frac{\sigma^2(x_{\min}) - \sigma^2(x_{\max})}{\sigma^2(x_{\max})\sigma^2(x_{\min})},
\end{aligned}$$

and

$$\begin{aligned}
\beta(x_{\max}) - \beta(x_{\min}) &= 2\gamma\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{1}{\alpha^2}\left[\sigma^2(x_{\max}) - \sigma^2(x_{\min})\right] \\
&= 2\gamma^2\left(f_{xx} + r - r_f + \gamma g(\bar{x})\right)\frac{\sigma^2(x_{\max})\sigma^2(x_{\min})}{\alpha^3}\left[\theta(x_{\min}) - \theta(x_{\max})\right].
\end{aligned}$$

If a larger dispersion of portfolio choice is due to either a higher excess return or a lower risk aversion, the dispersion in β is smaller, since:

$$\begin{aligned}
\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\alpha} &< 0, \text{ and } \frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\alpha}>0 \\
\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\gamma} &> 0, \text{ and } \frac{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}{\partial\gamma}<0
\end{aligned}$$

Consider the case where $\sigma^2\left(x_{\max}\right)\sigma^2\left(x_{\min}\right)$ is a constant, then

$$\frac{\partial\left[\beta\left(x_{\max}\right)-\beta\left(x_{\min}\right)\right]}{\partial\left[\theta\left(x_{\min}\right)-\theta\left(x_{\max}\right)\right]}=2\gamma^2\left(f_{xx}+r-r_f+\gamma g\left(\bar{x}\right)\right)\frac{\sigma^2\left(x_{\max}\right)\sigma^2\left(x_{\min}\right)}{\alpha^3}>0.$$

A larger dispersion in portfolio choice, resulting from a larger difference between effective volatility, implies a larger dispersion of tail distribution. The condition on the product of the effective variances is not necessary, however, as can be seen by simple algebra. ■

Proof. Proof of Lemma 4.1 Direct calculation. We use 1 to denote a positive sign.

First,

$$\begin{aligned}
\log I\left(x\right) &= \log\frac{\alpha}{\gamma\sigma^2\left(x\right)}+\log Z\left(x\right) \\
&= \log\alpha-\log\gamma-\log\sigma^2\left(x\right)+\log Z\left(x\right).
\end{aligned}$$

We have

$$\begin{aligned}
sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^2\left(x\right)}\right) &= sign\left(\frac{\partial\log I\left(x\right)}{\partial\sigma^2\left(x\right)}\right) \\
&= sign\left(-1-\frac{1}{Z\left(x\right)}\frac{z_{\min}}{\left(\beta\left(x\right)-1\right)^2}C_1\frac{1}{\alpha^2}\right) \\
&= -1
\end{aligned}$$

Second, for each level of expertise, we have

$$\begin{aligned}
& sign\left(\frac{\partial I\left(x\right)}{\partial\sigma_\nu}\right) \\
&= sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^2\left(x\right)}\frac{\partial\sigma^2\left(x\right)}{\partial\sigma_\nu}\right) \\
&= sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^2\left(x\right)}\right)sign\left(\frac{\partial\sigma^2\left(x\right)}{\partial\sigma_\nu}\right) \\
&= -1.
\end{aligned}$$

Third, for each level of expertise, we have

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial \alpha}\right) &= sign\left(\frac{\partial \log I(x)}{\partial \alpha}\right) \\ &= sign\left(1+\frac{2}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}C_1\frac{\sigma^2(x)}{\alpha^3}\right) \\ &= 1 \end{aligned}$$

Fourth, for each level of expertise:

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial \gamma}\right) &= sign\left(\frac{\partial \log I(x)}{\partial \gamma}\right) \\ &= sign\left(-1-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\left(\frac{\sigma^2(x)}{\alpha^2}\left(\frac{C_1}{\gamma}+2\gamma g(\overline{x})\right)-1\right)\right) \\ &\leq sign\left(-1-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\left(\frac{\sigma^2(x)}{\alpha^2}\frac{C_1}{\gamma}-1\right)\right) \\ &= -1 \end{aligned}$$

Lastly, for each level of expertise:

$$\begin{aligned} sign\left(\frac{\partial I(x)}{\partial f_{xx}}\right) &= sign\left(\frac{\partial \log I(x)}{\partial f_{xx}}\right) \\ &= sign\left(-\frac{1}{Z(x)}\frac{z_{\min}}{(\beta(x)-1)^2}\frac{\sigma^2(x)}{\alpha^2}2\gamma\right) \\ &= -1 \end{aligned}$$

■

Proof. Proof of Proposition 4.1 For each level of expertise, we have

$$sign\left(\frac{\partial I(x)}{\alpha}\right) = 1, \text{ for all } x \text{ such that } \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx}$$

And when α is higher, more experts enter. Thus

$$\frac{\partial I}{\partial \alpha} > 0.$$

■

Proof. Proof of Proposition 4.2 Direct calculation. We use 1 to denote a positive sign.

$$\begin{aligned}
& sign\left(\frac{\partial I\left(x\right)}{\partial\sigma_{\nu}}\right) \\
= \; & sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\frac{\partial\sigma^{2}\left(x\right)}{\partial\sigma_{\nu}}\right) \\
= \; & sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\right) sign\left(\frac{\partial\sigma^{2}\left(x\right)}{\partial\sigma_{\nu}}\right).
\end{aligned}$$

We also have

$$sign\left(\frac{\partial I\left(x\right)}{\partial\sigma^{2}\left(x\right)}\right) = -1$$

Thus for each level of expertise, when fundamental risk is higher, the demand for the complex risky asset is smaller. And when σ_{ν} is higher, fewer experts enter the complex risky asset market. Thus

$$\frac{\partial I}{\partial\sigma_{\nu}} < 0.$$

Next, for each level of expertise:

$$sign\left(\frac{\partial I\left(x\right)}{\partial\gamma}\right) = -1,$$

Lastly, for each level of expertise:

$$sign\left(\frac{\partial I\left(x\right)}{\partial f_{xx}}\right) = -1,$$

Therefore:

$$\frac{\partial I}{\partial\gamma} < 0 \text{ and } \frac{\partial I}{\partial f_{xx}} < 0$$

■

Proof. Proof of Proposition 4.3 We have

$$sign\left(\frac{\partial I\left(x\right)}{\partial x}\right) = sign\left(\frac{\partial I\left(x\right)}{\partial\sigma\left(x\right)}\frac{\partial\sigma\left(x\right)}{\partial x}\right) = 1$$

And

$$\begin{aligned}
I\left(\Lambda_{1}\right) - I\left(\Lambda_{2}\right) &= \int\left[\lambda_{1}\left(x\right)-\lambda_{2}\left(x\right)\right] I\left(x\right)dx \\
&= -I\left(x\right)\left[\Lambda_{1}\left(\underline{x}\right)-\Lambda_{2}\left(\underline{x}\right)\right] - \int\frac{\partial I(x)}{\partial x}\left[\Lambda_{1}\left(x\right)-\Lambda_{2}\left(x\right)\right]dx \\
&> 0
\end{aligned}$$

■

Proof. Proof of Proposition 4.4. Given

$$\frac{\partial SR(x)}{\partial \sigma_\nu} = \frac{\frac{\partial \alpha}{\partial \sigma_\nu}\sigma(x) - \alpha \frac{\partial \sigma(x)}{\partial \sigma_\nu}}{\sigma^2(x)}$$

we have

$$\frac{\partial SR(x)}{\partial \sigma_\nu} > 0 \text{ iff } \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} < \frac{\partial \log \alpha}{\partial \log \sigma_\nu}.$$

If $\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}$ is a constant, we must have either $\frac{\partial \log \alpha}{\partial \log \sigma_\nu} > \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}$ for all x or $\frac{\partial \log \alpha}{\partial \log \sigma_\nu} < \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}$ for all x.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_\nu} = \frac{\partial \log \alpha}{\partial \log \sigma_\nu},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$.

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} > 0$, and assume there is a cutoff x^* such that

$$\frac{\partial \log \sigma(x^*)}{\partial \log \sigma_\nu} = \frac{\partial \log \alpha}{\partial \log \sigma_\nu},$$

then for all $x < x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} > 0$; and for all $x > x^*$, we have $\frac{\partial SR(x)}{\partial \sigma_\nu} < 0$. ■

Value Weighted Equilibrium Sharpe ratio The market value weighted Sharpe ratio can be written as

$$\begin{aligned} SR^{vw} &= E\left[\frac{\theta Z(x)}{I}\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\ &= E\left[\frac{\theta Z(x)}{I}\frac{\alpha}{\sigma(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \\ &= \frac{\alpha}{\gamma I} E\left[\frac{Z(x)}{\sigma^3(x)} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] \end{aligned}$$

Participation: Intermediate results and proofs We begin by describing results for bounds on the elasticity of α with respect to changes in fundamental volatility, and the implications of these bounds for participation. First, we show that the percentage change in α has to be large enough to at least satisfy the investors whose risk-return tradeoff deteriorates the least as fundamental volatility increases.

Lemma A.1 *In the equilibrium, we have*

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l_{\inf}^{\sigma_\nu},$$

where $l_{\inf}^{\sigma_\nu}$ is the lowest elasticity of all participating investors' effective volatility with respect to fundamental volatility

$$l_{\inf}^{\sigma_\nu} \equiv \inf\left\{\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right\}.$$

Proof. Proof of Lemma A.1 Proof by contradiction. Suppose σ_ν is increased by 1%, but the equilibrium α is increased by less than $l_{\inf}^{\sigma_\nu}$%, that is

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \leq l_{\inf}^{\sigma_\nu}$$

We have

1. Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l_{\inf}^{\sigma_\nu}$, $\underline{x}$ is higher.
2. Less investment in the complex risky asset:

$$\begin{aligned}
& \frac{\partial \log I(x)}{\partial \sigma_\nu} \\
= \; & -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{z_{\min}}{Z(x)}\frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} + \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu}\right] \\
= \; & -\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu} + \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu}\right] \\
< \; & 0\ , \text{ for all } x.
\end{aligned}$$

Therefore, in the new equilibrium, the total demand for risky asset is less than the total supply. Contradiction. It must be that

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > \inf\left\{\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{2\sigma^2(x)\gamma} \geq f_{xx}\right\}.$$

■

We can also put an upper bound on the percentage change in α relative to the percentage change in fundamental volatility. The change will not be greater than twice the elasticity of the agent with the highest elasticity, which we prove by contradiction.

Lemma A.2 *In the equilibrium, we have*

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < 2l_{\sup}^{\sigma_\nu},$$

where $l_{\sup}^{\sigma_\nu}$ is the highest elasticity of all participating investors' effective volatility with respect to fundamental volatility,

$$l_{\sup}^{\sigma_\nu} \equiv \sup\left\{\frac{\partial\log\sigma(x)}{\partial\log\sigma_\nu}|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right\}.$$

Proof. Proof of Lemma A.2 Proof by contradiction. Suppose σ_ν is increased by 1%, but the equilibrium α is increased by more than $2l_{\sup}^{\sigma_\nu}$ %, that is

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \geq 2l_{\sup}^{\sigma_\nu}$$

We have

1. More participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \geq 2l_{\sup}^{\sigma_\nu} > l_{\sup}^{\sigma_\nu}$, $\underline{x}$ is lower.
2. More investment in the complex risky asset:

$$\begin{aligned}
&\frac{\partial\log I(x)}{\partial\sigma_\nu}\\
=\;&\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}+\frac{1}{\sigma_\nu}\left[1+\frac{z_{\min}}{Z(x)}\frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]\right\}\\
=\;&\frac{1}{\sigma_\nu}\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]\right\}\\
=\;&\frac{1}{\sigma_\nu}\left\{\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\left[2+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right\}\\
=\;&\frac{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}{\sigma_\nu}\left\{\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{2+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right\}\\
\geq\;&\frac{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}{\sigma_\nu}\left\{2-\frac{2+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}\right\}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\\
=\;&\frac{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}{\sigma_\nu}\left\{1-\frac{1}{1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}}\right\}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\\
>\;&0,
\end{aligned}$$

Therefore

$$\frac{\partial\log I(x)}{\partial\sigma_\nu} > 0$$

Therefore, in the new equilibrium, the total demand for risky asset is more than the total supply. Contradiction. It must be that

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < 2l_{\sup}^{\sigma_\nu}.$$

■

The following lemma describes bounds on the percentage change in α for a given percentage change in fundamental volatility for the case of decreasing elasticities of effective volatility with respect to fundamental volatility (Case 3 of Proposition 4.4). We show that the percentage change in α for a given percentage change in fundamental volatility will be greater than the highest elasticity of effective volatility with respect to fundamental volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is less than a constant times the average elasticity over participating investors. The constant will be near one if β is close to one, which it will be as it is the tail parameter from a Pareto distribution. Note we derive a sufficient condition which is based on the wealth distribution of the highest expertise agents, as using the entire distribution, a mixture of Pareto distributions, is more complicated but would yield similar intuition. We also show the converse: The percentage change in α for a given percentage change in fundamental volatility will be less than the highest elasticity of effective volatility with respect to fundamental volatility (displayed by the participating investor with the lowest expertise) if that highest elasticity is less than a constant near one times the average elasticity over participating investors. Case 3 of Proposition 4.4 is the only case which yields a decline in participation as fundamental volatility increases. It does so under natural conditions, related to these bounds. We show below that participation increases if Condition 1 of Lemma A.3 holds, but decreases if Condition 2 holds. Intuitively, participation will increase if the change in α is large enough to satisfy lower expertise investors in Case 3, but will decrease otherwise. Lemma A.3 provides bounds on the percentage change in α for a given percentage change in fundamental volatility for Case 3. We provide a sufficient condition for participation to decline as fundamental volatility increases below.

Lemma A.3 *When* $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} \le 0$, *in the equilibrium, we have,*

1. $$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l_{\sup}^{\sigma_\nu} \text{ if } l_{\sup}^{\sigma_\nu} < \left(1 + \frac{1}{1 + \frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}\right) E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{\sigma^2(x)} \ge 2\gamma f_{xx}\right].$$

   and

2. $$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l_{\sup}^{\sigma_\nu} \text{ if } l_{\sup}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}\right) E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{\sigma^2(x)} \ge 2\gamma f_{xx}\right],$$

Proof. Proof of Lemma A.3 In case 3, we have $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$.

First, we show that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\text{sup}}^{\sigma_\nu}$ if

$$l_{\text{sup}}^{\sigma_\nu} < \left(1 + \frac{1}{1 + \frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}\right) E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right].$$

Proof by contradiction. Assume $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \leq l_{\text{sup}}^{\sigma_\nu} < \frac{2+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}{1+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}} E\left[\frac{\partial \log\sigma(x)}{\partial\log\sigma_\nu}|x \geq \underline{x}\right]$, We have

- Less participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu}$, $\underline{x}$ is higher.
- Less investment in the complex risky asset:

$$\begin{aligned}
&\frac{\partial \log I(x)}{\partial\sigma_\nu} \\
= \;& -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{z_{\min}}{Z(x)}\frac{2(\beta(x)+\gamma)}{(\beta(x)-1)^2}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right] \\
= \;& -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]
\end{aligned}$$

  Thus,

$$\begin{aligned}
\frac{\partial I}{\partial\sigma_\nu} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial\sigma_\nu} d\Lambda(x) - I(x)\, d\Lambda(x)\,|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial\sigma_\nu} \\
&< E\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]\right\} \\
&< E\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1 + \frac{1}{\beta(\overline{x})}\frac{2(\beta(\overline{x})+\gamma)}{\beta(\overline{x})-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu} + l_{\text{sup}}^{\sigma_\nu}\right]\right\} \\
&= \frac{1 + \frac{1}{\beta(\overline{x})}\frac{2(\beta(\overline{x})+\gamma)}{\beta(\overline{x})-1}}{\sigma_\nu} l_{\text{sup}}^{\sigma_\nu} - \frac{2 + \frac{1}{\beta(\overline{x})}\frac{2(\beta(\overline{x})+\gamma)}{\beta(\overline{x})-1}}{\sigma_\nu} E\left\{\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}\right\} \\
&< 0
\end{aligned}$$

Therefore, in the new equilibrium, the total demand for the complex risky asset is less than the total supply. Contradiction. Therefore, it must be that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\text{sup}}^{\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu}.$$

Second, we show that $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\text{sup}}^{\sigma_\nu}$ if

$$l_{\text{sup}}^{\sigma_\nu} > \left(1 + \frac{1}{1 + \frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}\right) E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | \frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right].$$

Proof by contradiction. Assume $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} \geq l_{\sup}^{\sigma_\nu} > \frac{2+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}{1+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}} E\left[\frac{\partial\log\sigma(x)}{\partial\log\sigma_\nu}|x \geq \underline{x}\right]$, We have

- More participation: because $\frac{\alpha^2}{2\sigma^2(\underline{x})\gamma} = f_{xx}$ and $\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} > l_{\sup}^{\sigma_\nu}$, $\underline{x}$ is lower.
- More investment in the complex risky asset:

$$
\begin{aligned}
&\frac{\partial \log I(x)}{\partial \sigma_\nu} \\
= & -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right] \\
> & -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu} + \frac{1}{\sigma_\nu}\left[1+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+l_{\sup}^{\sigma_\nu}\right]
\end{aligned}
$$

Next

$$
\begin{aligned}
\frac{\partial I}{\partial\sigma_\nu} &= \int_{\underline{x}}^{\infty}\frac{\partial I(x)}{\partial\sigma_\nu}d\Lambda(x) - I(x)\,d\Lambda(x)\,|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}}\frac{\partial\underline{x}}{\partial\sigma_\nu} \\
&> E\left\{-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}+\frac{1}{\sigma_\nu}\left[1+\frac{1}{\beta(\underline{x})}\frac{2(\beta(\underline{x})+\gamma)}{\beta(\underline{x})-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+l_{\sup}^{\sigma_\nu}\right]\right\} \\
&= \frac{1+\frac{1}{\beta(\underline{x})}\frac{2(\beta(\underline{x})+\gamma)}{\beta(\underline{x})-1}}{\sigma_\nu}l_{\sup}^{\sigma_\nu} - \frac{2+\frac{1}{\beta(\underline{x})}\frac{2(\beta(\underline{x})+\gamma)}{\beta(\underline{x})-1}}{\sigma_\nu}E\left\{\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}\right\} \\
&> 0
\end{aligned}
$$

Therefore, in the new equilibrium, the total demand for risky asset is more than total supply. Contradiction. Therefore, it must be that

$$\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} < l_{\sup}^{\sigma_\nu} = \frac{\partial\sigma(\underline{x})/\sigma(\underline{x})}{\partial\sigma_\nu/\sigma_\nu}.$$

■

We now show conditions under which participation increases, i.e. under which the cutoff level of expertise for participation $\underline{x}$ declines, as fundamental volatility increases. In particular, we show that participation increases with fundamental volatility in Cases 1 and 2 of Proposition 4.4, but only under a tight restriction in Case 3. In Case 3, participation only increases if the elasticity of the effective volatility of the lowest expertise investor is not too different from that of the average participating investor. In other words, participation increases if there is very little difference across expertise levels in the effect of changes in fundamental volatility on effective volatility, so that elasticities are nearly constant, as in Case 1. Notice that the condition restricting the differences in elasticities across investors is the same as Condition 1 in Lemma A.3 which bounds the change in α from below. Thus, participation will increase only if the change in α is large enough, which will be the case if all participating investors face similar changes to their effective volatility as fundamental volatility changes. We discuss the more empirically relevant case, when elasticities vary more across high expertise and low expertise agents, and participation thus declines, in the text.

Proposition A.1 *Define the entry cutoff* $\underline{x}$*,*

$$\underline{x} = \sigma^{-1}\left(\frac{\alpha}{\sqrt{2\gamma f_{xx}}}\right),$$

where $\sigma^{-1}(\cdot)$ *is the inverse function of* $\sigma(x)$*. We have that participation increases with fundamental volatility,*

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$$

if the following conditions hold

1. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} \geq 0$, *(Proposition 4.4 Cases 1 and 2) or*

2. $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$, *(Proposition 4.4 Case 3) and* $l_{\sup}^{\sigma_\nu} < \frac{2+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}{1+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}} E\left[\frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu} | x \geq \underline{x}\right]$.

Proposition A.1 shows that participation increases in Cases 1 and 2 as fundamental volatility increases. The reason is that demand for the complex asset by incumbent experts declines, and new wealth must be brought into the market to clear the fixed supply. However, in Case 3, it is possible that because higher expertise agents' risk-return tradeoff deteriorates by less as fundamental volatility increases, that participation declines. This can be seen in the condition for increased participation in Case 3, which requires a very small difference between the highest and lowest elasticities, since $\beta \approx 1$, and we confirm this formally in Proposition 4.5.

Proof. Proof of Proposition A.1 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} > 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} = 2\left(\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} - \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_\nu / \sigma_\nu}\right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} > 0 \text{ iff } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_\nu / \sigma_\nu}$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} \geq 0$, from Proposition A.1 we have

$$\frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} > l_{\inf}^{\sigma_\nu} = \frac{\partial \sigma(\underline{x}) / \sigma(\underline{x})}{\partial \sigma_\nu / \sigma_\nu}.$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$ and $l^{\sigma_\nu}_{\sup} < \frac{2+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}{1+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}} E\left[l^{\sigma_\nu}|x \geq \underline{x}\right]$, from Lemma A.3, we know

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} > l^{\sigma_\nu}_{\sup} = \frac{\partial \sigma\left(\underline{x}\right)/\sigma\left(\underline{x}\right)}{\partial \sigma_\nu/\sigma_\nu}.$$

■

Proof. Proof of Proposition 4.5 First,

$$\frac{\partial \underline{x}}{\partial \sigma_\nu} > 0 \text{ iff } \frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} < 0.$$

We have

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} = 2\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma\left(\underline{x}\right)/\sigma\left(\underline{x}\right)}{\partial \sigma_\nu/\sigma_\nu}\right)$$

Therefore

$$\frac{\partial \log \frac{\alpha^2}{\sigma^2(\underline{x})}}{\partial \log \sigma_\nu} < 0 \text{ iff } \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l^{\sigma_\nu}_{\sup} = \frac{\partial \sigma\left(\underline{x}\right)/\sigma\left(\underline{x}\right)}{\partial \sigma_\nu/\sigma_\nu}.$$

If $\frac{\partial \frac{\partial \log \sigma(x)}{\partial \log \sigma_\nu}}{\partial x} < 0$ and $l^{\sigma_\nu}_{\sup} > \frac{2+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}{1+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}} E\left[l^{\sigma_\nu}|x \geq \underline{x}\right]$, from Lemma A.3, we know

$$\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l^{\sigma_\nu}_{\sup} = \frac{\partial \sigma\left(\underline{x}\right)/\sigma\left(\underline{x}\right)}{\partial \sigma_\nu/\sigma_\nu}.$$

■

We note that the conditions in Proposition A.1 and Proposition 4.5 are sufficient, but not necessary. As discussed in the main text, we use the tail parameters for the highest and lowest expertise levels since the entire wealth distribution is a mixture of Pareto distributions (a complicated object). These conditions are also not overlapping, because

$$\frac{2+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}}{1+\frac{2}{\beta(\overline{x})}\frac{\beta(\overline{x})+\gamma}{\beta(\overline{x})-1}} < \frac{2+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}{1+\frac{2}{\beta(\underline{x})}\frac{\beta(\underline{x})+\gamma}{\beta(\underline{x})-1}}.$$

Proof. Proof of Proposition 4.6

We first consider the case in which participation increases. There are two subcases, with slightly different proof strategies:

1. $\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$ and $\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \geq l^{\sigma_\nu}_{\sup}$,

2. $\frac{\partial \underline{x}}{\partial \sigma_\nu} < 0$ and $\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} < l^{\sigma_\nu}_{\sup}$,

First, we show that, for Case 1,

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0 \text{ if } \frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \text{ and } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} \geq l_{\text{sup}}^{\sigma_\nu}.$$

Suppose

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} < 0.$$

We have

$$\begin{aligned} & \frac{\partial SR^{ew}}{\partial \sigma_\nu} \\ = \; & E\left[\frac{1}{\sigma(x)}\frac{\partial \alpha}{\partial \sigma_\nu} - \frac{\alpha}{\sigma^2(x)}\frac{\partial \sigma(x)}{\partial \sigma_\nu}|x \geq \underline{x}\right] - \frac{\alpha}{\sigma(x)} d\Lambda(x)\left.\right|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ = \; & \frac{\alpha}{\sigma_\nu} E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}\right)|x \geq \underline{x}\right] - \frac{\alpha}{\sigma(x)} d\Lambda(x)\left.\right|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ < \; & 0, \end{aligned}$$

Therefore

$$\frac{\sigma(\underline{x})}{\sigma_\nu} E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}\right)|x \geq \underline{x}\right] < d\Lambda(x)\left.\right|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_\nu} < 0.$$

But

$$E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}\right)|x \geq \underline{x}\right] \geq 0 \text{ because } \frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} \geq l_{\text{sup}}^{\sigma_\nu}.$$

Second, we show that, for Case 2,

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} > 0 \text{ if } \frac{\partial \underline{x}}{\partial \sigma_\nu} < 0 \text{ and } \frac{\partial \alpha / \alpha}{\partial \sigma_\nu / \sigma_\nu} < l_{\text{sup}}^{\sigma_\nu}.$$

Suppose

$$\frac{\partial SR^{ew}}{\partial \sigma_\nu} < 0.$$

We have

$$\begin{aligned} & \frac{\partial SR^{ew}}{\partial \sigma_\nu} \\ = \; & E\left[\frac{1}{\sigma(x)}\frac{\partial \alpha}{\partial \sigma_\nu} - \frac{\alpha}{\sigma^2(x)}\frac{\partial \sigma(x)}{\partial \sigma_\nu}|x \geq \underline{x}\right] - \frac{\alpha}{\sigma(x)} d\Lambda(x)\left.\right|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ = \; & \frac{\alpha}{\sigma_\nu} E\left[\frac{1}{\sigma(x)}\left(\frac{\partial \alpha/\alpha}{\partial \sigma_\nu/\sigma_\nu} - \frac{\partial \sigma(x)/\sigma(x)}{\partial \sigma_\nu/\sigma_\nu}\right)|x \geq \underline{x}\right] - \frac{\alpha}{\sigma(x)} d\Lambda(x)\left.\right|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial \sigma_\nu} \\ < \; & 0. \end{aligned}$$

Therefore, we must have

$$E\left[\frac{1}{\sigma(x)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)|x\geq\underline{x}\right]<0.$$

Next,

$$\begin{aligned}&\frac{\partial\log I(x)}{\partial\sigma_\nu}\\ =&\ -\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu}+\frac{1}{\sigma_\nu}\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]\\ <&\ \frac{1}{\sigma_\nu}\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right].\end{aligned}$$

So,

$$\begin{aligned}\frac{\partial I}{\partial\sigma_\nu} &= \int_{\underline{x}}^{\infty}\frac{\partial I(x)}{\partial\sigma_\nu}d\Lambda(x)-I(x)\,d\Lambda(x)\,|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}}\frac{\partial\underline{x}}{\partial\sigma_\nu}\\ &< E\left\{\frac{I(x)\sigma(x)}{\sigma_\nu}\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]\frac{1}{\sigma(x)}\left[-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}+\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\right]\right\}\end{aligned}$$

Define

$$J(x)=\frac{I(x)\sigma(x)}{\sigma_\nu}\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right].$$

It is straightforward to show that

$$J'(x)>0.$$

In Case 2 we have $\frac{\partial\frac{\partial\log\sigma(x)}{\partial\log\sigma_\nu}}{\partial x}>0$ and $E\left[\frac{1}{\sigma(x)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)|x\geq\underline{x}\right]<0$. Therefore,

$$E\left[\frac{J(x)}{\sigma(x)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)|x\geq\underline{x}\right]<0.$$

Therefore

$$\frac{\partial I}{\partial\sigma_\nu}<0,$$

Contradiction. We must have

$$\frac{\partial SR^{ew}}{\partial\sigma_\nu}>0.$$

Last, we show that if participation is increasing, SR^{ew} increases as long as a condition on the distribution of expertise holds. In particular, we require that, if there are many investors around the cutoff level of expertise, that their effective volatility does not increase by so much that it drives the market Sharpe ratio down.

$$\frac{\partial SR^{ew}}{\partial\sigma_\nu}>0\text{ if }\frac{\partial\underline{x}}{\partial\sigma_\nu}>0\text{ and }E\left[1-\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}|x>\underline{x}\right]>d\Lambda(\underline{x})\frac{1}{l_{\text{sup}}^{\sigma_\nu}}.$$

$$
\begin{aligned}
&\frac{\partial SR^{ew}}{\partial \sigma_\nu} \\
= &\frac{\alpha}{\sigma_\nu} E\left[\frac{1}{\sigma(x)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)|\frac{\alpha^2}{\sigma^2(x)} \geq 2\gamma f_{xx}\right] - \frac{\alpha}{\sigma(x)} d\Lambda(x)\,|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial\sigma_\nu}
\end{aligned}
$$

Next

$$
\begin{aligned}
\frac{\partial I}{\partial \sigma_\nu} &= \int_{\underline{x}}^{\infty} \frac{\partial I(x)}{\partial\sigma_\nu} d\Lambda(x) - I(x)\, d\Lambda(x)\,|_{\sigma^2(x)=\frac{\alpha^2}{2\gamma f_{xx}}} \frac{\partial \underline{x}}{\partial\sigma_\nu} \\
&= E\left\{\frac{I(x)}{\sigma_\nu}\left(1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right)\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} - \frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right)\right\} \\
&\quad - E\left[\frac{I(x)}{\sigma_\nu}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right] - I(\underline{x})\, d\Lambda(\underline{x})\, \frac{\partial \underline{x}}{\partial\sigma_\nu},
\end{aligned}
$$

We also have $\sigma^2(\underline{x}) = \frac{\alpha^2}{2\gamma f_{xx}}$, thus

$$
\frac{\partial \underline{x}}{\partial\sigma_\nu} = \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}\frac{1}{l_{\sup}^{\sigma_\nu}}.
$$

Since $\frac{\partial I}{\partial\sigma_\nu} = 0$, we have

$$
\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} = \frac{E\left[\left(2 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]}{E\left(1 + \frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right) - I(\underline{x})\, d\Lambda(\underline{x})\, \frac{1}{l_{\sup}^{\sigma_\nu}}}.
$$

Furthermore,

$$\begin{aligned}
& \frac{\partial SR^{ew}}{\partial \sigma_\nu}\frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\alpha} \\
= \;& \frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\sigma_\nu}E\left[\frac{1}{\sigma\left(x\right)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right)|\frac{\alpha^2}{\sigma^2\left(x\right)}\geq 2\gamma f_{xx}\right]-I\left(\underline{x}\right)d\Lambda\left(\underline{x}\right)\frac{\partial\underline{x}}{\partial\sigma_\nu} \\
= \;& \frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\sigma_\nu}E\left[\frac{1}{\sigma\left(x\right)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right)|\frac{\alpha^2}{\sigma^2\left(x\right)}\geq 2\gamma f_{xx}\right]-\int_{\underline{x}}^{\infty}\frac{\partial I\left(x\right)}{\partial\sigma_\nu}d\Lambda\left(x\right) \\
= \;& E\left[\frac{1}{\sigma_\nu}\frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right)\right]+E\left[\frac{I\left(x\right)}{\sigma_\nu}\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right] \\
& -E\left[\frac{I\left(x\right)}{\sigma_\nu}\left(1+\frac{1}{\beta\left(x\right)}\frac{2\left(\beta\left(x\right)+\gamma\right)}{\beta\left(x\right)-1}\right)\left(\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right)\right] \\
= \;& \frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu}E\left[\frac{1}{\sigma_\nu}\frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}-\frac{I\left(x\right)}{\sigma_\nu}\left(1+\frac{1}{\beta\left(x\right)}\frac{2\left(\beta\left(x\right)+\gamma\right)}{\beta\left(x\right)-1}\right)\right] \\
& +E\left\{\left[\left(2+\frac{1}{\beta\left(x\right)}\frac{2\left(\beta\left(x\right)+\gamma\right)}{\beta\left(x\right)-1}\right)\frac{I\left(x\right)}{\sigma_\nu}-\frac{1}{\sigma_\nu}\frac{I\left(\underline{x}\right)\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}\right]\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right\}
\end{aligned}$$

Therefore, $\frac{\partial SR^{ew}}{\partial\sigma_\nu}>0$ iff

$$\begin{aligned}
\frac{\partial\alpha/\alpha}{\partial\sigma_\nu/\sigma_\nu} &= \frac{E\left[\left(2+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]}{E\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]-I\left(\underline{x}\right)d\Lambda\left(\underline{x}\right)\frac{1}{l_{\text{sup}}^{\sigma_\nu}}} \\
&< \frac{E\left[\left(2+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right)\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]-E\left[\frac{I(\underline{x})\sigma(\underline{x})}{\sigma(x)}\frac{\partial\sigma(x)/\sigma(x)}{\partial\sigma_\nu/\sigma_\nu}\right]}{E\left[1+\frac{1}{\beta(x)}\frac{2(\beta(x)+\gamma)}{\beta(x)-1}\right]-E\left[\frac{I(\underline{x})\sigma(\underline{x})}{\sigma(x)}\right]},
\end{aligned}$$

It suffices to show that

$$E\left[\frac{\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}|x>\underline{x}\right]<E\left[\frac{\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}\right]-d\Lambda\left(\underline{x}\right)\frac{1}{l_{\text{sup}}^{\sigma_\nu}}.$$

This is true because

$$\begin{aligned}
& E\left[\frac{\sigma\left(\underline{x}\right)}{\sigma\left(x\right)}\left(1-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}\right)|x>\underline{x}\right] \\
> \;& E\left[1-\frac{\partial\sigma\left(x\right)/\sigma\left(x\right)}{\partial\sigma_\nu/\sigma_\nu}|x>\underline{x}\right] \\
> \;& d\Lambda\left(\underline{x}\right)\frac{1}{l_{\text{sup}}^{\sigma_\nu}}.
\end{aligned}$$

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