Full Substitutability*

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Abstract

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extend earlier models' canonical definitions of substitutability to settings in which an agent can be both a buyer in some transactions and a seller in others, and show that all these definitions are equivalent. We introduce a new class of substitutable preferences that allows us to model intermediaries with production capacity. We then prove that substitutability is preserved under economically important transformations such as trade endowments, mergers, and limited liability. We also show that substitutability corresponds to submodularity of the indirect utility function, the single improvement property, and a no complementarities condition. Finally, we show that substitutability implies the monotonicity conditions known as the Laws of Aggregate Supply and Demand.

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1 Introduction

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, exchange economies with indivisible goods, and trading networks (Kelso and Crawford, 1982; Roth, 1984; Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999, 2000; Milgrom, 2000; Ausubel and Milgrom, 2006; Hatfield and Milgrom, 2005; Sun and Yang, 2006, 2009; Ostrovsky, 2008; Hatfield et al., 2013). Substitutability arises in a number of important applications, including matching with distributional constraints (Abdulkadiroğlu and Sönmez, 2003; Hafalir et al., 2013; Sönmez and Switzer, 2013; Sönmez, 2013; Westkamp, 2013; Ehlers et al., 2014; Echenique and Yenmez, 2015; Kominers and Sönmez, 2014; Kamada and Kojima, 2015), supply chains (Ostrovsky, 2008), markets with horizontal subcontracting (Hatfield et al., 2013), "swap" deals in exchange markets (Milgrom, 2009), and combinatorial auctions for bank securities (Klemperer, 2010; Baldwin and Klemperer, 2015).

The diversity of settings in which substitutability plays a role has led to a variety of different definitions of substitutability, and a number of restrictions on preferences that appear in some definitions but not in others.[1] In this paper, we show how the different definitions of substitutability are related to each other, while dispensing with some of the restrictions in the preceding literature. We consider agents who can simultaneously be buyers in some transactions and sellers in others, which allows us to embed the key substitutability concepts from the matching, auctions, and exchange economy literatures.[2] Our main result shows that all the substitutability concepts are equivalent. We call preferences satisfying these conditions *fully substitutable.*[3]

We introduce a new class of fully substitutable preferences that models the preferences of intermediaries with production capacity. We then prove that full substitutability is preserved under several economically important transformations: trade endowments and obligations, mergers, and limited liability. We show that full substitutability can be recast in terms of submodularity of the indirect utility function, the single improvement property, a "no complementarities" condition, and a condition from discrete convex analysis called $M^\natural$-concavity. Finally, we prove that full substitutability implies two key monotonicity conditions:

[1]For instance, some definitions assume "free disposal"/"monotonicity," under which an agent is always weakly better off with a larger set of goods than with a smaller one, while other definitions do not; some definitions assume that all bundles of goods are feasible for the agent, while others do not; and so on.

[2]While all of the results in our paper consider the preferences of a single agent, and thus do not depend on the details of the agent's setting, for concreteness, notational simplicity, and continuity with prior literature, we state and prove these results in the general trading network setting of Hatfield et al. (2013).

[3]We use the modifier "fully" to highlight the possibility that under such preferences, an agent can be both a buyer in some transactions and a seller in others, whereas under the "gross substitutes" preferences of Kelso and Crawford (1982), an agent can be only a buyer or only a seller.

the Laws of Aggregate Supply and Demand.

All of our results explicitly incorporate economically important features such as indifferences, non-monotonicities, and unbounded utility functions that were not fully addressed in the earlier literature. For example, unbounded utility functions allow us to model firms with technological constraints under which some production plans are infeasible.

1.1 History and Related Literature

For two-sided settings, Kelso and Crawford (1982) introduced the (demand-theoretic) gross substitutability condition, under which substitutability is expressed in terms of changes in an agent's demand as prices change. Roth (1984) introduced a related (choice-theoretic) definition, under which substitutability is expressed in terms of changes in an agent's choice as the set of available options changes. These conditions were subsequently extended and generalized, giving rise to two (mostly) independent literatures.

In two-sided matching models, (choice-theoretic) substitutability guarantees the existence of stable outcomes (Roth, 1984; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2013). Ostrovsky (2008) generalized the classic substitutability conditions to the context of supply chain networks by introducing a pair of related assumptions: same-side substitutability and cross-side complementarity. These assumptions impose two constraints: First, when an agent's opportunity set on one side of the market expands, that agent does not choose any options previously rejected from that side of the market. Second, when an agent's opportunity set on one side of the market expands, that agent does not reject any options previously chosen from the other side of the market. Both Ostrovsky (2008) and Hatfield and Kominers (2012) showed that under same-side substitutability and cross-side complementarity, a stable outcome always exists if the contractual set has a supply chain structure. Moreover, Hatfield and Kominers (2012) showed that same-side substitutability and cross-side complementarity are together equivalent to the assumption of quasisubmodularity of the indirect utility function—an adaptation of submodularity to the setting without transfers.

In exchange economies with indivisible goods, (demand-theoretic) gross substitutability guarantees the existence of core allocations and competitive equilibria (Kelso and Crawford, 1982; Gul and Stacchetti, 1999, 2000). Ausubel and Milgrom (2002) offered a convenient alternative definition of gross substitutability for a setting with continuous prices, in which demand is not guaranteed to be single-valued, and showed that gross substitutability is equivalent to submodularity of the indirect utility function. Sun and Yang (2006) introduced the gross substitutability and complementarity condition for the setting of indivisible object allocation. The gross substitutability and complementarity condition, akin to same-side substitutability and cross-side complementarity, requires that objects can be divided into two groups such that objects in the same group are substitutes and objects in different groups are complements. Sun and Yang (2009) showed that like gross substitutability, the gross substitutability and complementarity condition is equivalent to submodularity of the indirect utility function.

Subsequent to our work, Baldwin and Klemperer (2015) obtained additional insights on the underlying mathematical structure of fully substitutable preferences using the techniques of tropical geometry. Baldwin and Klemperer (2015) study the set of price vectors for which the demand correspondence is multi-valued, and associate them with convex-geometric objects called tropical hypersurfaces. Then, using the normal vectors that determine agents' tropical hypersurfaces, they distinguish among preferences that are strongly substitutable, are gross substitutable, or have complementarities.[4]

The discrete mathematics literature has explored several other concepts that are equivalent to substitutability in certain settings. We provide one point of connection to that literature in Section 6.5, where we establish the equivalence of full substitutability and $M^\natural$-concavity in our setting. Paes Leme (2014) provides a detailed survey that covers the discrete-mathematical substitutability concepts and their algorithmic properties.[5]

1.2 Structure of the Paper

The rest of the paper is organized as follows. In Section 2, we present our framework. In Section 3, we present three definitions of full substitutability, and show that they are all equivalent. In Section 4, we present examples of classes of fully substitutable preferences. In Section 5, we discuss transformations that preserve full substitutability. In Section 6, we provide several alternative characterizations of full substitutability. In Section 7, we show that full substitutability implies the Laws of Aggregate Supply and Demand. Section 8 concludes the main body of the paper.

In Appendix A, we present six additional definitions of full substitutability, which deal explicitly with indifferences in preferences. We discuss the connections of these definitions to those in the earlier literature and to the three definitions in Section 3. In Appendix B, we prove that all six definitions of full substitutability in Appendix A are equivalent, and are also equivalent to the three definitions in Section 3. Appendix C contains the proofs of all other results in the paper.

[4] Our full substitutability concept corresponds to the "strong substitutes demand type" in Baldwin and Klemperer (2015).

[5] Unlike in our paper, the setting of Paes Leme (2014) assumes that all bundles of goods are feasible for the agent. Consequently, not all of the algorithmic results discussed by Paes Leme (2014) can be applied directly in our setting.

2 Model

All results in the paper deal with the preferences of an individual agent, and thus do not depend on the environment in which this agent is located.[6] However, for notational convenience and for continuity with the related literature, we present these results in the trading network setting of Hatfield et al. (2013).

There is an economy with a finite set I of *agents* and a finite set Ω of *trades*. Each trade $\omega \in \Omega$ is associated with a *buyer* $b(\omega) \in I$ and a *seller* $s(\omega) \in I$, with $b(\omega) \neq s(\omega)$. We allow Ω to contain multiple trades associated to the same pair of agents, and allow for the possibility of trades $\omega \in \Omega$ and $\psi \in \Omega$ such that the seller of ω is the buyer of ψ, i.e., $s(\omega) = b(\psi)$, and the seller of ψ is the buyer of ω, i.e., $s(\psi) = b(\omega)$.

A *contract* x is a pair $(\omega, p_\omega) \in \Omega \times \mathbb{R}$ that specifies a trade and an associated price. For a contract $x = (\omega, p_\omega)$, we denote by $b(x) \equiv b(\omega)$ and $s(x) \equiv s(\omega)$ the buyer and the seller associated with the trade ω of x. The set of possible contracts is $X \equiv \Omega \times \mathbb{R}$. A set of contracts $Z \subseteq X$ is *feasible* if it does not contain two or more contracts for the same trade: formally, Z is feasible if $(\omega, p_\omega), (\omega, \hat{p}_\omega) \in Z$ implies that $p_\omega = \hat{p}_\omega$. We call a feasible set of contracts an *outcome*. An outcome specifies a set of trades along with associated prices, but does not specify prices for trades that are not in that set. An *arrangement* is a pair $[\Psi; p]$, with $\Psi \subseteq \Omega$ and $p \in \mathbb{R}^\Omega$. Note that an arrangement specifies prices for *all* the trades in the economy. For any arrangement $[\Psi; p]$, we denote by $\kappa([\Psi; p]) \equiv \cup_{\psi \in \Psi}\{(\psi, p_\psi)\}$ the outcome induced by $[\Psi; p]$.

For a set of contracts $Y \subseteq X$ and agent $i \in I$, we let $Y_{i\rightarrow} \equiv \{y \in Y : i = s(y)\}$ denote the set of contracts in Y in which i is the seller and let $Y_{\rightarrow i} \equiv \{y \in Y : i = b(y)\}$ denote the set of contracts in Y in which i is the buyer; we let $Y_i \equiv Y_{i\rightarrow} \cup Y_{\rightarrow i}$. We use analogous notation with regard to sets of trades $\Psi \subseteq \Omega$. For a set of contracts $Y \subseteq X$, we let $\tau(Y) \equiv \{\omega \in \Omega : (\omega, p_\omega) \in Y \text{ for some } p_\omega \in \mathbb{R}\}$ denote the set of trades associated with contracts in Y.

2.1 Preferences

Each agent i has a *valuation* $u_i : 2^{\Omega_i} \to \mathbb{R} \cup \{-\infty\}$ over the sets of trades in which he is involved, with $u_i(\varnothing) \in \mathbb{R}$.[7] Allowing the utility of an agent to equal $-\infty$ formalizes the idea that an agent, due to technological constraints, may only be able to produce or sell certain outputs contingent upon procuring appropriate inputs; for example, if $\psi, \omega \in \Omega$ with

[6] For example, some *other* agents in the economy may be allowed to form multi-sided contracts, or face uncertainty, or behave strategically, etc.

[7] We assume that trades in $\Omega \setminus \Omega_i$ do not affect i, and abuse notation slightly by writing $u_i(\Psi) \equiv u_i(\Psi_i)$ for $\Psi \subseteq \Omega$.

$b(\psi) = s(\omega) = i$ and agent i cannot sell ω unless he has procured ψ, then $u_i(\{\omega\}) = -\infty$.[8] The assumption that $u_i(\varnothing)$ is finite for each $i \in I$ implies that no agent is obligated to engage in market transactions at highly unfavorable prices; he can always choose a (finite) outside option.

The valuation u_i over bundles of trades gives rise to a quasilinear utility function U_i over bundles of trades and associated transfers. Specifically, for any feasible set of contracts $Y \subseteq X$, we define

$$U_i(Y) \equiv u_i(\tau(Y)) + \sum_{(\omega,p_\omega)\in Y_{i\to}} p_\omega - \sum_{(\omega,p_\omega)\in Y_{\to i}} p_\omega,$$

and, slightly abusing notation, for any arrangement $[\Psi; p]$, we define

$$U_i([\Psi;p]) \equiv u_i(\Psi) + \sum_{\psi\in\Psi_{i\to}} p_\psi - \sum_{\psi\in\Psi_{\to i}} p_\psi.$$

Note that by construction, $U_i([\Psi; p]) = U_i(\kappa([\Psi; p]))$.

The *choice correspondence* of agent i from the set of contracts $Y \subseteq X$ is defined by

$$C_i(Y) \equiv \operatorname*{arg\,max}_{Z\subseteq Y;\ Z \text{ is feasible}} U_i(Z)$$

and the *demand correspondence* of agent i, given a price vector $p \in \mathbb{R}^\Omega$, is defined by

$$D_i(p) \equiv \operatorname*{arg\,max}_{\Psi\subseteq\Omega_i} U_i([\Psi;p]).$$

Note that both choice and demand correspondences can be multi-valued. Also, the choice correspondence may be empty-valued (e.g., if Y is the set of all contracts with prices strictly between 0 and 1), while the demand correspondence always contains at least one element. When the set Y is finite, the choice correspondence is also guaranteed to contain at least one element.

3 Substitutability Concepts

We now introduce three substitutability concepts that generalize the existing definitions from matching, auctions, and exchange economies with indivisible goods. For convenience, in this section, we use the approach of Ausubel and Milgrom (2002) and restrict attention to

[8]In the classical exchange economy literature (Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999), the valuation of an agent i is defined over bundles of objects Ω as $u_i : 2^{\Omega_i} \to \mathbb{R}$, and is normalized such that $u_i(\varnothing) = 0$. While these assumptions are completely innocuous and natural in the context of exchange economies, they immediately rule out the kinds of technological constraints discussed above.

opportunity sets and vectors of prices for which choices and demands are single-valued. In Appendices A and B, we introduce additional definitions that explicitly deal with indifferences and multi-valued correspondences, and prove that those definitions are equivalent to each other and to the definitions given in this section.

3.1 Choice-Language Full Substitutability

First, we define full substitutability in the language of sets and choices, adapting and merging the Ostrovsky (2008) same-side substitutability and cross-side complementarity conditions. In choice language, we say that a choice correspondence C_i is fully substitutable if, when attention is restricted to sets of contracts for which C_i is single-valued, whenever the set of options available to i on one side expands, i rejects a larger set of contracts on that side (same-side substitutability) and selects a larger set of contracts on the other side (cross-side complementarity).

Definition 1. *The preferences of agent i are* choice-language fully substitutable (CFS) *if:*

1. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$;*
2. *for all sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| = |C_i(Y)| = 1$, $Y_{\to i} = Z_{\to i}$, and $Y_{i\to} \subseteq Z_{i\to}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $Y_{i\to} \smallsetminus Y^*_{i\to} \subseteq Z_{i\to} \smallsetminus Z^*_{i\to}$ and $Y^*_{\to i} \subseteq Z^*_{\to i}$.*

3.2 Demand-Language Full Substitutability

Our second definition uses the language of prices and demands, adapting the gross substitutes and complements condition (GSC) of Sun and Yang (2006).[9] We say that a demand correspondence D_i is fully substitutable if, when attention is restricted to prices for which demands are single-valued, a decrease in the price of some inputs for agent i leads to a decrease in his demand for other inputs and to an increase in his supply of outputs, and an increase in the price of some outputs leads to a decrease in his supply of other outputs and an increase in his demand for inputs.

Definition 2. *The preferences of agent i are* demand-language fully substitutable (DFS) *if:*

[9]The definition of full substitutability that corresponds directly to (GSC) is Definition A.4 (DCFS) in Appendix A. See Appendix A for a detailed discussion of the connection between (GSC) and (DCFS).

1. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$, and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.*

3.3 Indicator-Language Full Substitutability

Our third definition is essentially a reformulation of Definition 2, using a convenient vector notation due to Hatfield and Kominers (2012). For each agent i, for any set of trades $\Psi \subseteq \Omega_i$, define the *(generalized) indicator function* $e_i(\Psi) \in \{-1, 0, 1\}^{\Omega_i}$ to be the vector with component $e_{i,\omega}(\Psi) = 1$ for each "upstream" trade $\omega \in \Psi_{\to i}$, $e_{i,\omega}(\Psi) = -1$ for each "downstream" trade $\omega \in \Psi_{i\to}$, and $e_{i,\omega}(\Psi) = 0$ for each trade $\omega \notin \Psi$. The interpretation of $e_i(\Psi)$ is that an agent buys a strictly positive amount of a good if he is the buyer in a trade in Ψ, and "buys" a strictly negative amount if he is the seller of such a trade.

Definition 3. *The preferences of agent i are* indicator-language fully substitutable (IFS) *if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \leq p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, we have $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.*

Definition 3 clarifies the reason for the term "full substitutability"—an agent is more willing to "demand" a trade (i.e., keep an object that he could potentially sell, or buy an object that he does not initially own) if prices of other trades increase.

3.4 Equivalence of the Definitions

The main result of this section is that the three definitions of full substitutability presented are all equivalent. Subsequently, we use the term *full substitutability* to refer to all our substitutability concepts.

Theorem 1. *Choice-language full substitutability (CFS), demand-language full substitutability (DFS), and indicator-language full substitutability (IFS) are all equivalent.*

In Appendix A, we introduce six additional definitions of full substitutability, which explicitly deal with indifferences in preferences and with expanding, contracting, or both expanding and contracting sets of available "options"; we discuss in detail how those definitions relate to various definitions of substitutability considered in the literatures on matching, auctions, and exchange economies. In Appendix B we state and prove Theorem B.1, which says that the six definitions introduced in Appendix A are all equivalent, and are also equivalent to (CFS), (DFS), and (IFS). Theorem 1 thus follows immediately from Theorem B.1.

4 Examples

By construction, fully substitutable preferences include, as a special case, "one-sided" preferences that satisfy the gross substitutability condition of Kelso and Crawford (1982). Gross substitutability has been extensively studied in the literatures on matching, competitive equilibrium, and discrete concave optimization, and a variety of examples and classes of preferences satisfying the gross substitutability condition have been presented.[10]

In this section, we discuss two classes of fully substitutable preferences that involve complementarities between the contracts an agent can form as a buyer and those that he can form as a seller. We start with "intermediary" preferences, under which an intermediary is trying to maximize his profit from matching some of his inputs to some of the requests that he receives. We then discuss "intermediary with production capacity" preferences, under which an intermediary has access to some production capacity, and needs this capacity to transform inputs into outputs.

4.1 "Intermediary" Preferences

We start with "intermediary" preferences, introduced by Hatfield et al. (2013) in the context of used car dealers, but applicable more generally.[11]

Consider an intermediary i who has access to a number of heterogeneous inputs (e.g., used cars, raw diamonds, and temporary workers), formally represented as a set of "upstream" contracts $Y_{\rightarrow i}$. Each element $(\varphi, p_\varphi) \in Y_{\rightarrow i}$ specifies the characteristics of the particular input and the price at which this input is available to intermediary i. The intermediary also has a set of requests (e.g., for used cars, for engagement rings, and for temp services), represented as a set of "downstream" contracts $Y_{i\rightarrow}$. Each element $(\psi, p_\psi) \in Y_{i\rightarrow}$ specifies the characteristics required by the contract's customer and the price that customer is willing to pay.

[10] See, e.g., Kelso and Crawford (1982), Hatfield and Milgrom (2005), Milgrom (2009), Milgrom and Strulovici (2009), Ostrovsky and Paes Leme (2014), and Paes Leme (2014).

[11] A closely related class of preferences was introduced by Sun and Yang (2006, Section 4) in the context of two-sided markets in which agents on one side (firms) have preferences over agents and objects on the other side (workers and machines) that are determined by the productivity of each worker on each machine. Sun and Yang (2006) show that such preferences satisfy the gross substitutes and complements (GSC) condition, with all workers being substitutes for one another, machines being substitutes for one another, and workers and machines being complements.

Some inputs φ and requests ψ are compatible with each other, while others are not.[12,13] For every compatible input–request pair (φ, ψ), there is also a cost $c_{\varphi,\psi}$ of preparing the input φ for resale to satisfy the compatible request ψ.[14,15] Intermediary i's objective is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$ in a way that maximizes his profit, $\sum_{(\varphi,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,\psi})$, where μ denotes the set of compatible input–request pairs that the intermediary selects.

Formally, following Hatfield et al. (2013), define a *matching*, μ, as a set of pairs of trades (φ, ψ) such that φ is an element of $\Omega_{\to i}$ (i.e., an input available to intermediary i), ψ is an element in $\Omega_{i\to}$ (i.e., a request received by i), φ and ψ are compatible, and each trade in Ω_i belongs to at most one pair in μ. Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of pairs involved in μ (i.e., $c(\mu) = \sum_{(\varphi,\psi)\in\mu} c_{\varphi,\psi}$).

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ such that every element of Ξ belongs to exactly one pair in μ.[16] Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)} c(\mu) & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., it is equal to the cost of the cheapest way of matching all requests and inputs in Ξ if such a matching is possible, and is equal to $-\infty$ otherwise.[17] (Note that $u_i(\varnothing) = 0$.) The

[12]For instance, as Hatfield et al. (2013) discuss in the context of used car dealers: "[A] blue Toyota Camry of a particular year and mileage would be compatible with a request for a Toyota Camry with matching year and mileage range, but would not be compatible with a request for a blue Honda Accord or for a blue Camry with the "wrong" year or mileage range." A given raw diamond can only be turned into polished diamonds of certain grades, and thus can only be used for some engagement rings but not others. A particular temp worker is only qualified to perform certain types of jobs.

[13]Note that some requests may have the same customers, so in particular there is no requirement that a customer in the economy only demands one very specific type of object that he buys. For instance, as Hatfield et al. (2013) explain in the used car context: "[A] buyer's preferences can specify, for example, that the value of a Toyota Camry to him is \$2,000 higher than the value of a Honda Accord with the same characteristics, or that each additional 1,000 miles on the car's odometer decreases that car's value by \$150. In other words, each request ψ is detailed enough that the buyer has the same value for any car that matches the request ψ, and the buyer's preferences are represented by a set of requests that he is indifferent over ('I am willing to pay \$15,000 for a Toyota Camry with such-and-such characteristics *or* \$14,500 for a Toyota Camry with so-and-so characteristics *or* \$13,000 for a Honda Accord with such-and-such characteristics *or* . . . ')."

[14]For example, the cost of repairing a car, turning a diamond into an engagement ring, or training a worker to perform a specific set of tasks.

[15]Note that we could formally allow all pairs of inputs and requests to be compatible, and encode incompatibilities by saying that for some pairs (φ, ψ), the cost $c_{\varphi,\psi}$ is infinite.

[16]Of course, $\mathcal{M}(\Xi)$ can be empty; e.g., it is empty if the number of inputs in Ξ is not equal to the number of requests, or if there are some requests in Ξ that are not compatible with any input in Ξ.

[17]Under this valuation function, any set chosen by intermediary i will contain an equal number of offers and requests. In principle, we could consider a more general (yet still fully substitutable) valuation function in which an intermediary has utility for an input that he does not resell. In that case, the intermediary may

utility function of i over feasible sets of contracts is induced by valuation u_i in the standard way formalized in Section 2.1.

Proposition 1. *"Intermediary" preferences are fully substitutable.*

Hatfield et al. (2013) present a rather involved proof of Proposition 1. Sun and Yang (2006) also present an elaborate proof of an analogous result for the two-sided setting (Theorem 4.1 in their paper, with the proof on pp. 1397–1401). The results of the current paper allow us to construct a much simpler and shorter proof, presented in Appendix C. Proposition 1 follows as a special case of Proposition 2, which shows the full substitutability of the new class of preferences that we introduce in the next section: "intermediary with production capacity". Proposition 2, in turn, follows directly from our result on "mergers" of agents with fully substitutable preferences (Theorem 4 of Section 5.2).

4.2 "Intermediary with Production Capacity" Preferences

In the "intermediary" preferences considered in Section 4.1, the intermediary either did not need to use any of his own resources to facilitate the matches between inputs and requests, or when he did, those resources could be expressed in monetary terms: there was a cost $c_{\varphi,\psi}$ of "preparing" input φ for request ψ. In some settings, however, we may want to consider intermediaries who need to rely on specific physical resources that they have in order to turn inputs into outputs, and it is more appropriate to think of these resources as fixed. For example, a manufacturer may have a fixed set of machines, and needs to assign a set of workers to those machines and at the same time needs to decide which outputs to produce on the machines. An agricultural firm may have a fixed set of land lots, and needs to hire workers to work on these lots, and at the same time needs to decide which outputs to produce. A steel manufacturer has access to a variety of inputs (different sources of iron ore and scrap metal) and can produce a variety of outputs (different grades and types of steel products), and needs to assign these inputs and outputs to the fixed number of steel plants that it has. In this section, we formally present a model of such an intermediary and show that the preferences described by this model are fully substitutable.

An intermediary i has access to a number of inputs, formally represented as a set of "upstream" contracts $Y_{\to i}$. Each element $(\varphi, p_\varphi) \in Y_{\to i}$ specifies the characteristics of the particular input and the price at which this input is available to intermediary i. The intermediary also has a set of requests, represented as a set of "downstream" contracts $Y_{i\to}$. Each element $(\psi, p_\psi) \in Y_{i\to}$ specifies the characteristics required by the contract's customer

end up choosing more offers than requests.

and the price that customer is willing to pay. Finally, the intermediary has a set M of "machines"; each machine $m \in M$ can be used to prepare one input contract for one output request.

For each input φ and machine m, there is a cost $c_{\varphi,m} \in \mathbb{R} \cup \{+\infty\}$ of preparing the input to work with the machine (e.g., the cost of training a particular worker, or the cost of transporting iron ore from its source). For each machine m and each request ψ, there is a cost $c_{m,\psi} \in \mathbb{R} \cup \{+\infty\}$ of using this machine to produce the requested output (e.g., the cost of water required to produce a particular agricultural crop on a particular land lot, or the cost of transporting a batch of steel to its destination). Note that we allow both costs to take the value $+\infty$, to enable the possibility that a particular input is not compatible with a particular machine, or a particular machine is not compatible with a particular request. The total cost of preparing input φ for request ψ using machine m is thus $c_{\varphi,m} + c_{m,\psi}$. The objective of intermediary i is to match some of the inputs in $Y_{\to i}$ to some of the requests in $Y_{i\to}$, via some of the machines, in a way that maximizes his profit, $\sum_{(\varphi,m,\psi)\in\mu}(p_\psi - p_\varphi - c_{\varphi,m} - c_{m,\psi})$, where μ denotes the set of input–machine–request triples that the intermediary selects.

Formally, define a *matching*, μ, as a set of triples (φ, m, ψ) such that

1. φ is an element of $\Omega_{\to i}$,
2. m is a machine available to intermediary i,
3. ψ is an element of $\Omega_{i\to}$, and
4. each φ belongs to at most one triple in μ, each m belongs to at most one triple in μ, and each ψ belongs to at most one triple in μ.

Slightly abusing notation, let the cost of matching μ, $c(\mu)$, be equal to the sum of the costs of triples involved in μ, i.e., $c(\mu) = \sum_{(\varphi,m,\psi)\in\mu}(c_{\varphi,m} + c_{m,\psi})$.

For a set of trades $\Xi \subseteq \Omega_i$, let $\mathcal{M}(\Xi)$ denote the set of matchings μ of elements of Ξ and machines available to the intermediary, such that every element of Ξ belongs to exactly one triple in μ. Then the valuation of intermediary i over sets of trades $\Xi \subseteq \Omega_i$ is given by:

$$u_i(\Xi) = \begin{cases} -\min_{\mu\in\mathcal{M}(\Xi)} c(\mu) & \text{if } \mathcal{M}(\Xi) \neq \varnothing \\ -\infty & \text{if } \mathcal{M}(\Xi) = \varnothing, \end{cases}$$

i.e., it is equal to the cost of the cheapest way of satisfying all requests in Ξ using all of the inputs in Ξ and some of the machines, if such a production plan is possible; and is equal to $-\infty$ otherwise. The utility function of intermediary i over feasible sets of contracts is induced by the valuation u_i in the standard way.

Proposition 2. *"Intermediary with production capacity" preferences are fully substitutable.*

The proofs of Proposition 2 and all subsequent results are in Appendix C. The idea of the proof of Proposition 2 is as follows. First, it is immediate that if the intermediary has only one machine, his preferences are fully substitutable. Next, if an intermediary has multiple machines (say, a set M of machines), he can be, in essence, viewed as a "merger" of $|M|$ single-machine agents. Theorem 4 of Section 5.2 shows that this "merger" operation preserves full substitutability.[18]

To conclude this section, we note that if an agent's preferences simultaneously incorporate capacity constraints (as in the "intermediary with production capacity" preferences) and costs that depend directly on how inputs are linked to outputs (as in the "intermediary" preferences), then those preferences may not be fully substitutable. For example: Consider a firm that has exactly one machine, can hire workers Ann and Bob, and has requests for outputs α and β. Suppose Ann can use the machine to produce output α (but not β), while Bob can use the machine to produce output β (but not α). In this case, the preferences of the firm are not fully substitutable: reducing a price of an input (say, Ann) may lead to the firm choosing to drop an output (β).

5 Transformations

In this section, we show that fully substitutable preferences can be transformed and combined in several economically interesting ways that preserve full substitutability. We first consider the possibility that an agent is endowed with the right to execute any trades in a given set and the possibility that an agent has an obligation to execute all trades in a given set. We also examine mergers, where the valuation function of the merged entity is constructed as the convolution of the valuation functions of the merging parties.[19] Finally, we consider a form

[18]One complication in the proof is that the merger operation, as defined in Section 5.2, would allow multiple machines to "buy" the same input, or "sell" the same request, because it would view these input trades and request trades as distinct. To deal with this complication, the proof adds a layer of "input dummies" and a layer of "request dummies" to enforce the constraint that one input can only be used by one machine and the constraint that one request can only be fulfilled by one machine. The "merger" operation then combines these dummies with those for single-machine firms. Note that while we use this "dummy layers and mergers" construction for the specific purpose of proving the full substitutability of "intermediary with production capacity" preferences, it may be useful more generally to incorporate various restrictions (say, incompatibility of some input trades) in agents' preferences while maintaining full substitutability, both in network and two-sided settings.

[19]In two-sided matching settings, the operations of "endowment" and "merger" were used by Hatfield and Milgrom (2005) to construct the class of endowed assignment valuations, starting with singleton preferences and iteratively applying these operations. Hatfield and Milgrom (2005) showed that these operations preserve substitutability (Theorems 13 and 14 of Hatfield and Milgrom (2005)), and thus show that all endowed

of limited liability, where an agent may back out of some agreed-upon trades in exchange for paying an exogenously-fixed penalty.

5.1 Trade Endowments and Obligations

Suppose an agent i is endowed with the right to execute trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ. Let

$$\hat{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv \max_{\Xi \subseteq \Phi} \left\{ u_i(\Psi \cup \Xi) + \sum_{\xi \in \Xi_{i\to}} p_\xi - \sum_{\xi \in \Xi_{\to i}} p_\xi \right\}$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\hat{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i while being endowed with the option of executing any trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

Theorem 2. *If the preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\hat{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$.*

Intuitively, when we endow agent i with access to the trades in Φ at prices p_Φ, we are effectively restricting (1) the set of prices that may change and (2) the set of trades that are required to be substitutes in the demand-theoretic definition of full substitutability (Definition 2). Naturally, this process cannot *create* complementarities among trades in $\Omega \smallsetminus \Phi$, given that under u_i these trades already are substitutes for each other *and* for the trades in Φ. Hence, $\hat{u}_i^{(\Phi,p_\Phi)}$ induces fully substitutable preferences over trades in $\Omega \smallsetminus \Phi$.

Apart from endowments, agents may have obligations, that is, an agent i may be obliged to execute trades in some set $\Phi \subseteq \Omega_i$ at *fixed* prices p_Φ. We now show that if an agent's preferences are initially fully substitutable, they remain fully substitutable when an obligation arises to execute some trades at pre-specified prices. Suppose agent i is obliged to execute trades in $\Phi \subseteq \Omega_i$ at prices p_Φ and that Φ is technologically feasible in the sense that $u_i(\Phi) \neq -\infty$. Let

$$\tilde{u}_i^{(\Phi,p_\Phi)}(\Psi) \equiv u_i(\Psi \cup \Phi) + \sum_{\varphi \in \Phi_{i\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to i}} p_\varphi$$

be a valuation over trades in $\Omega \smallsetminus \Phi$; $\tilde{u}_i^{(\Phi,p_\Phi)}$ represents agent i having a valuation over trades in $\Omega \smallsetminus \Phi$ consistent with u_i while being obliged to execute all trades in the set $\Phi \subseteq \Omega_i$ at prices p_Φ.

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assignment valuation preferences are substitutable. Ostrovsky and Paes Leme (2014) showed that there exist substitutable preferences that cannot be represented as an endowed assignment valuation, and introduced the class of matroid-based valuations, which is obtained by iteratively applying the "endowment" and "merger" operations to weighted-matroid valuations. Since every weighted-matroid valuation is substitutable (Murota, 1996; Murota and Shioura, 1999; Fujishige and Yang, 2003), every matroid-based valuation is also substitutable. It is an open question whether every substitutable valuation is a matroid-based valuation.

Theorem 3. *If the preferences of agent i are fully substitutable, then the preferences induced by the valuation function $\tilde{u}_i^{(\Phi,p_\Phi)}$ are fully substitutable for any $\Phi \subseteq \Omega_i$ and $p_\Phi \in \mathbb{R}^\Phi$ such that $u_i(\Phi) \neq -\infty$.*

The idea of the proof is to note that the demand correspondence of agent i with valuation $\tilde{u}_i^{(\Phi,p_\Phi)}$ does not depend on prices p_Φ—changing these prices simply leads to a shift in the agent's utility function by a fixed amount. Thus, we can assume that the trades that the agent is obliged to *buy* have negative and very large (in absolute magnitude) prices, while the trades that the agent is obliged to *sell* have positive and very large prices. Under those assumptions, "obligations" become "endowments" (because the agent would voluntarily want to execute all of these trades), and thus Theorem 3 follows from Theorem 2.

Combining Theorems 2 and 3, we see that if the preferences of agent i are fully substitutable, then they remain fully substitutable when i is endowed with some trades and obliged to execute others (assuming that the obligation is technologically feasible).

5.2 Mergers

The second transformation we consider is the case when several agents merge. Given a set of agents J, we denote the set of trades that involve only agents in J as $\Omega^J \equiv \{\omega \in \Omega : \{b(\omega), s(\omega)\} \subseteq J\}$. We let the *convolution* of the valuation functions $\{u_j\}_{j \in J}$ be defined as

$$u_J(\Psi) \equiv \max_{\Phi \subseteq \Omega^J} \left\{ \sum_{j \in J} u_j(\Psi \cup \Phi) \right\} \tag{1}$$

for sets of trades $\Psi \subseteq \Omega \smallsetminus \Omega^J$. The convolution u_J represents a "merger" of the agents in J, as it treats the agents in J as able to execute any within-J trades costlessly.

Theorem 4. *For any set of agents $J \subseteq I$, if the preferences of each $j \in J$ are fully substitutable, then the preferences induced by the convolution u_J (defined in* (1)*) are fully substitutable.*

While Theorem 4 is of independent interest, note that we also use it in the proof of Proposition 2, where we show the full substitutability of "intermediary with production capacity" preferences.

Note that substitutability is not preserved following *dissolution/de-mergers*. For example, if agents i and j only trade with each other (i.e., $\Omega_i = \Omega_j$), then the preferences induced by the convolution valuation $u_{\{i,j\}}$ are trivially fully substitutable, even if the preferences of i and j are not.

Note also that while merging agents preserves substitutability, the same cannot be said about merging trades between two agents. For example, consider a simple economy with agents i and j and four trades: set Ω consists of trades χ, φ, ψ, and ω. Agent i is the buyer in all of these trades, and agent j is the seller. The valuation of agent i is as follows:

$$u_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \\ 1 & |\Psi_i| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The preferences of i are clearly fully substitutable. But now consider merging the trades χ and φ into a single trade ξ. The resulting valuation function of i over the subsets of $\tilde{\Omega} \equiv (\Omega \smallsetminus \{\chi, \varphi\}) \cup \{\xi\}$ is given by

$$\tilde{u}_i(\Psi) = \begin{cases} 2 & |\Psi_i| \geq 2 \text{ or } \xi \in \Psi \\ 1 & |\Psi_i| = 1 \text{ and } \xi \notin \Psi \\ 0 & \text{otherwise.} \end{cases}$$

Valuation function $\tilde{u}_i$ is not fully substitutable. To see this, note that for price vector $p = (p_\xi, p_\psi, p_\omega) = (1.7, 0.8, 0.8)$, the unique optimal demand of agent i is $\{\psi, \omega\}$, but for price vector $p' = (p'_\xi, p'_\psi, p'_\omega) = (1.7, 1, 0.8)$, the unique optimal demand of agent i is $\{\xi\}$. That is, under price vector p', agent i no longer demands the trade ω, even though its price remains unchanged while the price of ψ increases and the price of ξ remains unchanged.

5.3 Limited Liability

The final transformation we consider is "limited liability." Specifically, suppose that after agreeing to a trade, an agent is allowed to renege on that trade in exchange for paying a fixed penalty. We show that this transformation preserves substitutability. In addition to being economically interesting, the preservation of substitutability under limited liability is also useful technically; indeed, it enables us to transform unbounded utility functions into bounded ones while preserving substitutability. (The fact that this transformation preserves substitutability simplifies the technical analysis in a number of settings; see, e.g., the proof of Theorem 1 in Hatfield et al. (2013).)

Formally, consider a fully substitutable valuation function u_i for agent i. Take an arbitrary set of trades $\Phi \subseteq \Omega_i$, and for every trade $\varphi \in \Phi$, pick $\Pi_\varphi \in \mathbb{R}$—the penalty for reneging on trade φ. (For mathematical completeness, we allow Π_φ to be negative.) Define the modified valuation function $\hat{u}_i$ as

$$\hat{u}_i(\Psi) \equiv \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}. \tag{2}$$

That is, under valuation $\hat{u}_i$, agent i can "buy out" some of the trades to which he has committed (provided these trades are in the set Φ of trades the agent may renege on), and pay the corresponding penalty for each trade he buys out.

Theorem 5. *For any $\Phi \subseteq \Omega_i$ and $\Pi_\Phi \in \mathbb{R}^\Phi$, if agent i has fully substitutable preferences, then the valuation function $\hat{u}_i$ with limited liability (as defined in* (2)*) induces fully substitutable preferences.*

A common assumption in the earlier literature on two-sided matching and exchange economies (e.g., Kelso and Crawford (1982) and Gul and Stacchetti (1999)) is that buyers' valuation functions are *monotonic*.[20] Intuitively, monotonicity corresponds to the special case of our setting in which an agent has free disposal, in the sense that he can renege on any trade at no cost. More formally, if u_i is fully substitutable, then Theorem 5 implies that we can obtain a fully substitutable and monotonic valuation function $\hat{u}_i$ by allowing the agent to renege on any trade in Ω_i at a per-trade cost of $\Pi_\varphi = 0$, for all $\varphi \in \Omega_i$.

6 Properties Equivalent to Full Substitutability

In this section, we discuss several interesting properties of valuation functions that turn out to be equivalent to full substitutability. While these results are of independent interest, some of them are also useful in applications. For example: The submodularity equivalence we prove in Section 6.1 is used in our proof that substitutability is preserved under trade endowments (Theorem 2). The object-language formulation of full substitutability we develop in Section 6.3 is used in showing the Laws of Aggregate Supply and Demand (Theorem 10); a closely related transformation is used in the proof of the main result of Hatfield et al. (2013). The single-improvement property, which we introduce in Section 6.2, and $M^\natural$-concavity, discussed in Section 6.5, are useful for efficiently computing the choice function of an agent with fully substitutable preferences, because they imply that local search for an optimal bundle eventually reaches a global optimum (Paes Leme, 2014).

[20] Monotonicity of the valuation function u_i requires that, for all Ξ and Ψ such that $\Xi \subseteq \Psi \subseteq \Omega_i$, $u_i(\Psi) \geq u_i(\Xi)$.

6.1 Submodularity of the Indirect Utility Function

A classical approach (see, e.g., the work of Gul and Stacchetti (1999) and Ausubel and Milgrom (2002)) relates substitutability of the utility function to submodularity of the indirect utility function. In particular, every (grossly) substitutable utility function corresponds to a submodular indirect utility function and vice versa.[21]

For price vectors $p, \bar{p} \in \mathbb{R}^{\Omega}$, let the *join* of p and $\bar{p}$, denoted $p \vee \bar{p}$, be the pointwise maximum of p and $\bar{p}$; let the *meet* of p and $\bar{p}$, denoted $p \wedge \bar{p}$, be the pointwise minimum of p and $\bar{p}$.

Definition 4. *The* indirect utility function *of agent i,*

$$V_i(p) \equiv \max_{\Psi \subseteq \Omega_i} \{U_i([\Psi; p])\},$$

is submodular *if, for all price vectors $p, \bar{p} \in \mathbb{R}^{\Omega}$, we have that*

$$V_i(p \wedge \bar{p}) + V_i(p \vee \bar{p}) \leq V_i(p) + V_i(\bar{p}).$$

Theorem 6. *The preferences of an agent are fully substitutable if and only if they induce a submodular indirect utility function.*

6.2 The Single Improvement Property

Gul and Stacchetti (1999) first observed (in the setting of exchange economies) that substitutability is equivalent to the *single improvement property*—an agent's preferences are substitutable if and only if, when an agent does not have an optimal bundle, that agent can make himself better off by adding a single item, dropping a single item, or doing both. Sun and Yang (2009) extended this result to their setting. Baldwin and Klemperer (2015) showed that in their setting the single improvement property is equivalent to requiring that agents have *complete* preferences.

Definition 5. *The preferences of agent i have the* single improvement property *if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that*

1. $U_i([\Psi, p]) < U_i([\Phi, p])$,

[21]Similar correspondences hold in markets without transferable utility: In many-to-many matching with contracts markets without transfers, every substitutable choice function can be represented by a submodular indirect utility function, and every submodular indirect utility function corresponds to a substitutable choice function (Hatfield and Kominers, 2013). In trading networks without transferable utility, every indirect utility function representing a fully substitutable choice function is quasi-submodular (Hatfield and Kominers, 2012).

2. *there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi)$, and*

3. *there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi)$.*

The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one item not currently held (either by making a new purchase, i.e., adding a trade in $\Omega_{\to i} \smallsetminus \Psi$, or by canceling a sale, i.e., removing a trade in $\Psi_{i\to}$),

2. relinquishing one item currently held (either by canceling a purchase, i.e., removing a trade in $\Psi_{\to i}$, or by making a new sale, i.e., adding a trade in $\Omega_{i\to} \smallsetminus \Psi$), or

3. both obtaining one item not currently held and relinquishing one item currently held.

For instance, an agent may buy one more input and commit to provide one additional output as a "single improvement."

Moreover, when the preferences of agent i satisfy the single improvement property, it is easy to find an optimal bundle since, at any non-optimal bundle, a local adjustment can strictly increase the utility of i.

We now generalize the earlier results of Gul and Stacchetti (1999) and Sun and Yang (2009) to our setting.

Theorem 7. *The preferences of an agent are fully substitutable if and only if they have the single improvement property.*

6.3 Object-Language Substitutability

An alternative way of thinking about trades in our setting is to consider each trade as representing the transfer of an underlying object. Under this interpretation, an agent's preferences over trades are fully substitutable if and only if that agent's preferences over objects have the standard Kelso and Crawford (1982) property of gross substitutability. This interpretation allows us to rewrite indicator-language full substitutability to more naturally correspond to the intuitive explanation of the concept given in Section 3.

Formally, we consider each trade $\omega \in \Omega$ as transferring an underlying object from $s(\omega)$ to $b(\omega)$; we denote this underlying object as $\mathfrak{o}(\omega)$. We call the set of all underlying objects $\mathbf{\Omega}$. Hence, after executing the set of trades $\Psi \subseteq \Omega_i$, agent i is left with both the set of objects corresponding to the trades in Ψ where i is a buyer and the set of objects corresponding to trades in $\Omega_i \setminus \Psi$ where i is a seller. We define the set of objects held by agent i after executing the set of trades Ψ as

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \Omega_{i\to} \setminus \Psi_{i\to}\}.$$

Conversely, we define the trade associated with an object $\boldsymbol{\omega}$ as $\mathfrak{t}(\boldsymbol{\omega})$; note that $\mathfrak{t}(\mathfrak{o}(\omega)) = \omega$. We also define the set of trades executed by i for a given set of objects $\boldsymbol{\Psi} \subseteq \boldsymbol{\Omega}_i \equiv \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : i \in \{b(\mathfrak{t}(\boldsymbol{\omega})), s(\mathfrak{t}(\boldsymbol{\omega}))\}$ as

$$\mathfrak{t}_i(\boldsymbol{\Psi}) = \{\omega \in \Omega_{\to i} : \mathfrak{o}(\omega) \in \boldsymbol{\Psi}\} \cup \{\omega \in \Omega_{i\to} : \mathfrak{o}(\omega) \in \boldsymbol{\Omega}_i \setminus \boldsymbol{\Psi}\}.$$

Hence, for a partition of objects $\{\boldsymbol{\Psi}^i\}_{i\in I}$, the set of trades that implements this partition is given by

$$\bigcup_{i\in I} \mathfrak{t}_i(\boldsymbol{\Psi}^i).$$

For notational simplicity, for a set of objects $\boldsymbol{\Psi}$, we let $u_i(\boldsymbol{\Psi}) \equiv u_i(\mathfrak{t}_i(\boldsymbol{\Psi}))$.

Using object language, we can also reformulate indicator-language full substitutability to *object-language full substitutability.*

Definition 6. *The preferences of agent i are* object-language fully substitutable (OFS) *if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$ and $p \le p'$, for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, if $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, then $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi')$ for each $\boldsymbol{\omega} \in \boldsymbol{\Omega}_i$ such that $p_{\mathfrak{t}(\boldsymbol{\omega})} = p'_{\mathfrak{t}(\boldsymbol{\omega})}$.*

Under object-language full substitutability, an increase in the price of object $\boldsymbol{\psi}$ cannot decrease the agent's demand for any object $\boldsymbol{\omega} \neq \boldsymbol{\psi}$. That is, the agent's preferences over objects are grossly substitutable, in the sense of Kelso and Crawford (1982).

We can now interpret the indicator vector $e_{i,\psi}(\Psi)$ as encoding whether the object $\boldsymbol{\psi} = \mathfrak{o}(\psi)$ is transferred under Ψ:

- If $\psi \in \Psi_{\to i}$, then $\boldsymbol{\psi} \in \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = 1$, i.e., i obtains the object associated with ψ.
- If $\psi \in \Psi_{i\to}$, then $\boldsymbol{\psi} \notin \mathfrak{o}_i(\Psi)$ and $e_{i,\psi}(\Psi) = -1$, i.e., i gives up the object associated with ψ.
- Finally, if $\psi \notin \Psi$, then $e_{i,\psi}(\Psi) = 0$, i.e., i neither obtains nor gives up the object associated with ψ.

Additionally, object-language full substitutability helps us define a "no complementarities condition," equivalent to full substitutability, in the next section. Also, it is useful in our proof that fully substitutable preferences satisfy the Laws of Aggregate Supply and Demand (under quasilinear utility).

We can reformulate the definition of the single improvement property in terms of objects.

Definition 7. *The preferences of agent i have the* single improvement property *if for any price vector p and set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$, there exists a set of trades Φ such that*

1. $U^i([\Psi, p]) < U^i([\Phi, p])$,
2. *there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Phi) \setminus \mathfrak{o}_i(\Psi)$, and*
3. *there exists at most one object $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi) \setminus \mathfrak{o}_i(\Phi)$.*

Using object language, we obtain a definition of the single improvement property that exactly matches the intuition provided on page 19. The single improvement property says that, when an agent holds a suboptimal bundle of trades Ψ, that agent can be made be better off by

1. obtaining one object $\boldsymbol{\omega}$ not currently held, i.e., $\boldsymbol{\omega} \notin \mathfrak{o}_i(\Psi)$,
2. relinquishing one object $\boldsymbol{\omega}$ currently held, i.e., $\boldsymbol{\omega} \in \mathfrak{o}_i(\Psi)$, or
3. both obtaining one object and relinquishing one object.

When substitutability is expressed in terms of preferences over trades, it is necessary to treat relationships between "same-side" and "cross-side" contracts differently. Both Sun and Yang (2006) and Ostrovsky (2008) introduced a concept of cross-side complementarity, which requires that agents treat buy-side contracts as complementary with sell-side contracts (as in our Definitions 1 and 2), which might suggest that there is something fundamentally different between how contracts on one side are interdependent with each other versus how contracts on different sides are interdependent. The representation of preferences in the language of object-language substitutability uncovers that cross-side complementarity is not really a complementarity condition *per se*: rather, it corresponds to an underlying substitutability condition over objects—the same one as in the case of same-side substitutability.

The formalization of substitutability in terms of preferences over objects (Definition 6) thus provides a very simple and compact interpretation of full substitutability that does not require treating two sides differently: it simply says that when an agent's object opportunity set shrinks, the agent does not reduce demand for any object that remains in his opportunity set. In particular, in settings with transferable utility, when prices increase, an agent's object opportunity set shrinks; hence, substitutability requires that the agent (weakly) increase his demand for objects whose prices do not rise.

6.4 The No Complementarities Condition

Gul and Stacchetti (1999) proved that substitutability is equivalent to the *no complementarities condition*; we extend this observation here.

Definition 8. *The preferences of agent i satisfy the* no complementarities condition *if, for every price vector p, for any $\Phi, \Psi \in D_i(p)$, and for any $\bar{\mathbf{\Psi}} \subseteq \mathfrak{o}_i(\Psi)$, there exists $\bar{\mathbf{\Phi}} \subseteq \mathfrak{o}_i(\Phi)$ such that $\mathfrak{t}_i((\mathbf{\Psi} \smallsetminus \bar{\mathbf{\Psi}}) \cup \bar{\mathbf{\Phi}}) \in D_i(p)$.*

The no complementarities condition requires that for any pair of optimal bundles of objects, $\mathbf{\Psi}$ and $\mathbf{\Phi}$, and for any $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi}$, there exists a set of objects $\bar{\mathbf{\Phi}} \subseteq \mathbf{\Phi}$ that "perfectly substitute" for the objects in $\bar{\mathbf{\Psi}}$, in the sense that $(\mathbf{\Psi} \smallsetminus \bar{\mathbf{\Psi}}) \cup \bar{\mathbf{\Phi}}$ is optimal.

Theorem 8. *The preferences of an agent are fully substitutable if and only if they satisfy the no complementarities condition.*

The proof of Theorem 8 is an adaptation of the proof of Theorem 1 of Gul and Stacchetti (1999). Gul and Stacchetti (1999) assume that valuation functions are monotone and bounded from below; thus, in our proof of Theorem 8, we must be careful to ensure that non-monotonicities and unboundedness do not invalidate the Gul and Stacchetti (1999) proof strategy.

6.5 $M^\natural$-Concavity over Objects

Reijnierse et al. (2002) and Fujishige and Yang (2003) independently observed that gross substitutability in the Kelso and Crawford (1982) model is equivalent to a classical condition from discrete optimization theory, *$M^\natural$-concavity* (Murota, 2003). In our object-language notation, the condition can be stated as follows.

Definition 9. *The valuation u_i is $M^\natural$-*concave over objects *if for all $\mathbf{\Phi}, \mathbf{\Psi} \in \mathbf{\Omega}_i$, for any $\boldsymbol{\psi} \in \mathbf{\Psi}$,*

$$u_i(\mathbf{\Psi}) + u_i(\mathbf{\Phi}) \le \max\Big\{u_i(\mathbf{\Psi} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\mathbf{\Phi} \cup \{\boldsymbol{\psi}\}), \\ \max_{\boldsymbol{\varphi} \in \mathbf{\Phi}}\{u_i(\mathbf{\Psi} \cup \{\boldsymbol{\varphi}\} \smallsetminus \{\boldsymbol{\psi}\}) + u_i(\mathbf{\Phi} \cup \{\boldsymbol{\psi}\} \smallsetminus \{\boldsymbol{\varphi}\})\}\Big\}.$$

A valuation function is $M^\natural$-concave if, for any sets of objects $\mathbf{\Psi}$ and $\mathbf{\Phi}$, the sum of $u_i(\mathbf{\Psi})$ and $u_i(\mathbf{\Phi})$ is weakly increased when either we move a given object $\boldsymbol{\psi}$ from $\mathbf{\Psi}$ to $\mathbf{\Phi}$ or we swap $\boldsymbol{\psi}$ for some other object $\boldsymbol{\varphi} \in \mathbf{\Phi}$.

Theorem 9. *The preferences of an agent are fully substitutable if and only if the associated valuation function is $M^\natural$-concave over objects.*

This equivalence result follows from Theorem 7 of Murota and Tamura (2003), which shows that $M^\natural$-concavity is equivalent to the single improvement property—and which in turn, by our Theorem 7, implies the equivalence between full substitutability and $M^\natural$-concavity.

7 Laws of Aggregate Supply and Demand

In two-sided matching markets with transfers and quasilinear utility, all fully substitutable preferences satisfy a monotonicity condition called the Law of Aggregate Demand (Hatfield and Milgrom, 2005).[22] The analogues of this condition for the current setting are the *Laws of Aggregate Supply and Demand* for trading networks, first introduced by Hatfield and Kominers (2012).

Definition 10. *The preferences of agent i satisfy the* Law of Aggregate Demand *if for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{\to i}| - |Y^*_{\to i}| \geq |Z^*_{i\to}| - |Y^*_{i\to}|$.*

The preferences of agent i satisfy the Law of Aggregate Supply *if for all finite sets of contracts Y and Z such that $Y_{i\to} \subseteq Z_{i\to}$ and $Y_{\to i} = Z_{\to i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $|Z^*_{i\to}| - |Y^*_{i\to}| \geq |Z^*_{\to i}| - |Y^*_{\to i}|$.*

Intuitively, the choice correspondence C_i satisfies the Law of Aggregate Demand if, whenever the set of options available to i as a buyer expands, the net change in the number of buy-side contracts chosen is at least as great as the net change in the number of sell-side contracts chosen. Similarly, the choice correspondence C_i satisfies the Law of Aggregate Supply if, whenever the set of options available to i as a seller expands, the net change in the number of sell-side contracts chosen is at least as great as the net change in the number of buy-side contracts chosen. These conditions extend the canonical Law of Aggregate Demand (Hatfield and Milgrom (2005); see also Alkan and Gale (2003)) to the current setting, in which each agent can be both a buyer in some trades and a seller in others.

In our setting, full substitutability implies the Laws of Aggregate Supply and Demand.

Theorem 10. *If the preferences of agent i are fully substitutable, then they satisfy the Laws of Aggregate Supply and Demand.*

Theorem 10 generalizes Theorem 7 of Hatfield and Milgrom (2005), who showed the analogous result in the special case when agent i acts only as a buyer. The proof essentially follows from applying the Hatfield and Milgrom (2005) result to the agent's preferences over objects.

[22] In the context of two-sided matching with contracts, the Law of Aggregate Demand is essential for "rural hospitals" and strategy-proofness results (see Hatfield and Milgrom (2005) and Hatfield and Kominers (2013)).

8 Conclusion

Various forms of substitutability are essential for establishing the existence of equilibria and other useful properties in diverse settings such as matching, auctions, and exchange economies with indivisible goods. We extended earlier models' canonical definitions of substitutability to a setting in which an agent can be both a buyer in some transactions and a seller in others, and showed that all these definitions are equivalent. We introduced a new class of substitutable preferences that allows us to model intermediaries with production capacity. We proved that substitutability is preserved under economically important transformations such as trade endowments and obligations, mergers, and limited liability. We also showed that substitutability corresponds to submodularity of the indirect utility function, the single improvement property, gross substitutability under a suitable transformation ("object-language substitutability"), a no complementarities condition, and $M^\natural$-concavity. Finally, we showed that substitutability implies the monotonicity conditions known as the Laws of Aggregate Supply and Demand. All of our results explicitly incorporate economically important features such as indifferences, non-monotonicities, and unbounded utility functions that were not fully addressed in prior work.

In the current paper, we focused on the full substitutability of the preferences of an *individual* agent. In related work, we have explored the properties of *economies* with multiple agents whose preferences are fully substitutable. That work shows that when all agents' preferences are fully substitutable, outcomes that are *stable* (in the sense of matching theory) exist for any underlying network structure (Hatfield et al., 2013, Theorems 1 and 5). Furthermore, full substitutability of preferences guarantees that the set of stable outcomes is essentially equivalent to the set of competitive equilibria with personalized prices (Hatfield et al., 2013, Theorems 5 and 6) and to the set of *chain stable* outcomes (Hatfield et al., 2015, Theorem 1 and Corollary 1), and that all stable outcomes are in the core and are efficient (Hatfield et al., 2013, Theorem 9). Full substitutability also delineates a maximal domain for the existence of equilibria (Hatfield et al., 2013, Theorem 7): for any domain of preferences strictly larger than that of full substitutability, the existence of stable outcomes and competitive equilibria cannot be guaranteed.

Appendix

A Full Substitutability Definitions with Indifferences

In this Appendix, we introduce six alternative definitions of full substitutability, as follows:

- Definitions A.1 and A.2 are analogues of our choice-language definition (Definition 1),
- Definitions A.3 and A.4 are analogues of our demand-language definition (Definition 2), and
- Definitions A.5 and A.6 are analogues of our indicator-language definition (Definition 3).

In contrast to Definitions 1, 2 and 3, which consider single-valued choices and demands, Definitions A.1–A.6 explicitly consider multi-valued correspondences and deal directly with indifferences. By explicitly accounting for indifferences and multi-valued correspondences, we directly generalize the original gross substitutability condition of Kelso and Crawford (1982) to our setting. Moreover, the conditions that explicitly account for indifferences turn out to be useful for proving various results on trading networks, as we discuss below.

Definition A.1, stated in the language of choice functions, and Definition A.3, stated in the language of demand functions, are conceptually related in that in both definitions the set of "options" available on one side expands, while the set of options on the other side remains unchanged.[23] The idea of expanding options on one side originated in the matching literature, where it is natural to consider an expansion in the set of available trades, which in turn induces an expansion in the set of available contracts (see Ostrovsky (2008), Westkamp (2010), Hatfield and Kominers (2012), and Hatfield et al. (2013)). Definition A.1 is the full substitutability concept used by Hatfield et al. (2015) to prove the equivalence of stability and chain stability in trading networks.[24] The equivalence of Definition A.3 (DEFS) to other definitions of full substitutability is used in the proof of Theorem 6 of Hatfield et al. (2013) on the equivalence of stability and competitive equilibrium.

Definition A.2, stated in the language of choice functions, and Definition A.4, stated in the language of demand functions, are related in that in both definitions the set of "options" available on one side contracts, while the set of options on the other side remains unchanged.[25] Definition A.4 (DCFS) is the full substitutability definition that corresponds most directly to the original definition of gross substitutability of Kelso and Crawford (1982) and the definition of Gul and Stacchetti (1999, 2000): When an agent is not a seller in any trade in the economy, the (DCFS) condition directly reduces to those definitions of gross substitutability.

[23]In choice-language, the "options" are the contracts available to choose from. In demand-language, the expansion of the set of "options" corresponds to prices of trades moving in the direction advantageous for the agent: trades in which he is the buyer become cheaper, and trades in which he is the seller become more expensive.

[24]Hatfield et al. (2015) do not assume the quasilinearity of preferences or the continuity of transfers, and thus our equivalence results do not apply to the most general version of their setting.

[25]In demand-language, the contraction of the set of "options" corresponds to prices of trades moving in the direction disadvantageous for the agent: trades in which he is the buyer become more expensive, and trades in which he is the seller become cheaper.

It is also the definition that corresponds to the gross substitutes and complements condition of Sun and Yang (2006, 2009).[26] The equivalence of the (DCFS) condition to other full substitutability conditions (in particular, to the (IFS) and (DFS) conditions that only consider single-valued demands) is used in the proof of Theorem 1 of Hatfield et al. (2013) on the existence of competitive equilibria, in the step of the proof that "transforms" a trading network economy to a Kelso-Crawford two-sided, "many-to-one" matching market. The equivalence of the (DCFS) condition to the "single-valued" substitutability conditions implies that agents' preferences in the "transformed" market satisfy the gross substitutes condition of Kelso and Crawford (1982), making it possible to apply the results of Kelso and Crawford (1982) to the "transformed" market.

In contrast to Definitions A.1–A.4, which consider a change in the set of available options on one side while keeping the options on the other side unchanged, Definitions A.5 and A.6 consider changes in the set of options available on both sides simultaneously (i.e., the set of options on one side expands while the set of options on the other side contracts). This idea is in line with the auction literature, where it is standard to consider the effects of a weak increase (or decrease) of the entire price vector (see, e.g., Ausubel and Milgrom (2006) and Ausubel (2006)). We use Definitions A.5 and A.6 in the proof of Theorem 7 on the equivalence of full substitutability and the single-improvement property.

A.1 Choice-Language Full Substitutability

Our next two definitions are analogues of Definition 1.

[26]Sun and Yang (2006, 2009) studied exchange economies in which agents' preferences satisfy the gross substitutes and complements (GSC) condition. This condition requires that the set of objects can be partitioned into two sets S^1 and S^2 in such a way that whenever the price of one particular object in S^1 increases, each agent's demand for other objects in S^1 increases and each agent's demand for other objects in S^2 decreases; there is a symmetric requirement for the case where the price of an object in S^2 increases.

As discussed in Section IV.B of Hatfield et al. (2013), the framework of Sun and Yang (2006, 2009) can be embedded into the trading networks framework by viewing agents as intermediaries that (1) "buy inputs" from a set of artificial agents who each own one of the objects in S^1 and only care about the price received, and (2) "sell outputs" to a set of artificial agents who each can acquire only one particular object in S^2 and otherwise only care about the price charged. With this embedding, the (GSC) condition for exchange economies maps to the (DCFS) condition of our paper, and thus by our results is also equivalent to other definitions of full substitutability.

Note that while the framework of Sun and Yang (2006, 2009) can be embedded into the trading network framework of Hatfield et al. (2013) (as described above), the reverse is not true, because in the trading network framework, it will usually not be possible to partition the set of trades Ω into two sets Ω^1 and Ω^2 such that all agents' preferences simultaneously satisfy the (GSC) condition with respect to that partition (see Section IV.C of Hatfield et al. (2013) for details). Hence, in the presence of intermediaries, the trading network framework with agents' preferences satisfying the (DCFS) condition is more general than the exchange economy setting with agents' preferences satisfying the (GSC) condition.

Definition A.1. *The preferences of agent i are* choice-language expansion fully substitutable (CEFS) *if:*

1. *for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\rightarrow} = Z_{i\rightarrow}$ and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{\rightarrow i} \smallsetminus Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \smallsetminus Z^*_{\rightarrow i})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;*
2. *for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\rightarrow i} = Z_{\rightarrow i}$ and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for every $Y^* \in C_i(Y)$, there exists $Z^* \in C_i(Z)$ such that $(Y_{i\rightarrow} \smallsetminus Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} \smallsetminus Z^*_{i\rightarrow})$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.*

Definition A.2. *The preferences of agent i are* choice-language contraction fully substitutable (CCFS) *if:*

1. *for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{i\rightarrow} = Z_{i\rightarrow}$ and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{\rightarrow i} \smallsetminus Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \smallsetminus Z^*_{\rightarrow i})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;*
2. *for all finite sets of contracts $Y, Z \subseteq X_i$ such that $Y_{\rightarrow i} = Z_{\rightarrow i}$ and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for every $Z^* \in C_i(Z)$, there exists $Y^* \in C_i(Y)$ such that $(Y_{i\rightarrow} \smallsetminus Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} \smallsetminus Z^*_{i\rightarrow})$ and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$.*

Note that we use Y as the "starting set" in (CEFS) and Z as the "starting set" in (CCFS) to make the two notions more easily comparable. Furthermore, note that in Case 1 of (CEFS) and (CCFS), requiring $Y_{\rightarrow i} \smallsetminus Y^*_{\rightarrow i} \subseteq Z_{\rightarrow i} \smallsetminus Z^*_{\rightarrow i}$ is equivalent to requiring that $Z^* \cap Y_{\rightarrow i} \subseteq Y^*$, and similarly, in Case 2, requiring $Y_{i\rightarrow} \smallsetminus Y^*_{i\rightarrow} \subseteq Z_{i\rightarrow} \smallsetminus Z^*_{i\rightarrow}$ is equivalent to requiring that $Z^* \cap Y_{i\rightarrow} \subseteq Y^*$.

A.2 Demand-Language Full Substitutability

Our next two definitions are analogues of Definition 2.

Definition A.3. *The preferences of agent i are* demand-language expansion fully substitutable (DEFS) *if:*

1. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \subseteq \Psi_{\rightarrow i}$ and $\Psi_{i\rightarrow} \subseteq \Psi'_{i\rightarrow}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$ such that $\{\omega \in \Psi'_{i\rightarrow} : p_\omega = p'_\omega\} \subseteq \Psi_{i\rightarrow}$ and $\Psi_{\rightarrow i} \subseteq \Psi'_{\rightarrow i}$.*

Definition A.4. *The preferences of agent i are* demand-language contraction fully substitutable (DCFS) *if:*

1. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$;*
2. *for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega \leq p'_\omega$ for all $\omega \in \Omega_{i\to}$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$ such that $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ and $\Psi_{\to i} \subseteq \Psi'_{\to i}$.*

Note that we use p as the "starting price vector" in (DEFS) and p' as the "starting price vector" in (DCFS). Also, in Case 1 of (DEFS) and (DCFS), requiring $\{\omega \in \Psi'_{\to i} : p_\omega = p'_\omega\} \subseteq \Psi_{\to i}$ is equivalent to requiring that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'$, and similarly, in Case 2, requiring $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} \subseteq \Psi_{i\to}$ is equivalent to requiring that $\{\omega \in (\Omega_{i\to} \smallsetminus \Psi) : p_\omega = p'_\omega\} \subseteq \Omega_{i\to} \smallsetminus \Psi'$.

A.3 Indicator-Language Full Substitutability

Our next two definitions are analogues of Definition 3.

Definition A.5. *The preferences of agent i are* indicator-language increasing-price fully substitutable (IIFS) *if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, for every $\Psi \in D_i(p)$ there exists $\Psi' \in D_i(p')$, such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.*

Definition A.6. *The preferences of agent i are* indicator-language decreasing-price fully substitutable (IDFS) *if for all price vectors $p, p' \in \mathbb{R}^\Omega$ such that $p \leq p'$, for every $\Psi' \in D_i(p')$ there exists $\Psi \in D_i(p)$, such that $e_{i,\omega}(\Psi) \leq e_{i,\omega}(\Psi')$ for each $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.*

Note that we use p as the "starting price vector" in (IIFS) and $p' \geq p$ as the "starting price vector" in (IDFS).

B Equivalence of Full Substitutability Definitions

In this Appendix, we show that the three definitions in Section 3 and the six definitions in Appendix A are all equivalent. In particular, this implies Theorem 1.

Theorem B.1. *The (CFS), (DFS), (IFS), (CEFS), (CCFS), (DEFS), (DCFS), (IIFS), and (IDFS) conditions are all equivalent.*

Proof. It is immediate that (CEFS) and (CCFS) each imply (CFS), and (IIFS) and (IDFS) both imply (IFS). Below we establish the remaining equivalences by showing that (CFS) $\Rightarrow$ (DFS), (DFS) $\Rightarrow$ (DEFS), (DFS) $\Rightarrow$ (DCFS), (DEFS) $\Rightarrow$ (CEFS), (DCFS) $\Rightarrow$ (CCFS), (DEFS) + (DCFS) $\Rightarrow$ (IDFS) + (IIFS), and (IFS) $\Rightarrow$ (DFS).

(CFS) $\Rightarrow$ (DFS) We first show that Case 1 of (CFS) implies Case 1 of (DFS). For any agent i and price vector $p \in \mathbb{R}^{\Omega}$, let $X_i(p) \equiv \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{\to i},\ \hat{p}_\omega \geq p_\omega\} \cup \{(\omega, \hat{p}_\omega) : \omega \in \Omega_{i\to},\ \hat{p}_\omega \leq p_\omega\}$, in essence denoting the set of contracts available to agent i under prices p.

Let price vectors $p, p' \in \mathbb{R}^{\Omega}$ be such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$. Let $Y = X_i(p)$ and $Z = X_i(p')$. Clearly, $Y_{i\to} = Z_{i\to}$ and $Y_{\to i} \subseteq Z_{\to i}$. Furthermore, it is immediate that $\Psi \in D_i(p)$ if and only if $\kappa([\Psi; p]) \in C_i(Y)$, and similarly, $\Psi' \in D_i(p')$ if and only if $\kappa([\Psi'; p']) \in C_i(Z)$. In particular, we have $|C_i(Y)| = |C_i(Z)| = 1$ and can thus apply (CFS) to the sets Y and Z.

Take the unique $\Psi \in D_i(p)$, let $Y^* = \kappa([\Psi, p])$, and note that $Y^* \in C_i(Y)$. By (CFS), the unique $Z^* \in C_i(Z)$ satisfies $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ and $Y^*_{i\to} \subseteq Z^*_{i\to}$. Let $\Psi' = \tau(Z^*)$ and note that $\Psi' \in D_i(p')$. We show that Ψ' satisfies the conditions in Case 1 of Definition 2.

Note that $Y_{\to i} \smallsetminus Y^*_{\to i} \subseteq Z_{\to i} \smallsetminus Z^*_{\to i}$ implies $\{\omega \in \Omega_{\to i} \smallsetminus \Psi_{\to i} : p_\omega = p'_\omega\} \subseteq \tau(Y_{\to i}) \smallsetminus \tau(Y^*_{\to i}) \subseteq \tau(Z_{\to i}) \smallsetminus \tau(Z^*_{\to i}) \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$. Furthermore, $Y^*_{i\to} \subseteq Z^*_{i\to}$ and $p_\omega = p'_\omega$ for each $\omega \in \Omega_{i\to}$ imply $\Psi'_{i\to} \subseteq \Psi_{i\to}$.

The proof that Case 2 of (CFS) implies Case 2 of (DFS) is analogous.

(DFS) $\Rightarrow$ (DEFS), (DFS) $\Rightarrow$ (DCFS) We first show that Case 1 of (DFS) implies Case 1 of (DEFS). Take two price vectors p, p' such that $p'_\omega \leq p_\omega$ for all $\omega \in \Omega_{\to i}$ and $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to}$, and fix an arbitrary $\Psi \in D_i(p)$. We need to show that there exists a set $\Psi' \in D_i(p')$ that satisfies the conditions of Case 1 of (DEFS).

As the statement is trivial when $D_i(p') = \{\Xi : \Xi \subset \Omega_i\}$, we assume the contrary. In the following, let $\tilde{\Omega}_{\to i} = \{\omega \in \Omega_{\to i} : p'_\omega < p_\omega\}$. Let $\varepsilon_1 = V_i(p') - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(p')} U_i([\Xi; p'])$, and $\varepsilon_2 = \min_{\omega \in \tilde{\Omega}_{\to i}} (p_\omega - p'_\omega)$. Let $\varepsilon = \frac{\min\{\varepsilon_1, \varepsilon_2\}}{2|\Omega_i|}$. Note that by construction, $\varepsilon > 0$.

We now define a price vector q^1 by

$$q^1_\omega = \begin{cases} p_\omega - \varepsilon & \omega \in \Omega_{i\to} \smallsetminus \Psi \text{ or } \omega \in \Psi_{\to i} \\ p_\omega + \varepsilon & \omega \in \Omega_{\to i} \smallsetminus \Psi \text{ or } \omega \in \Psi_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^1) = \{\Psi\}$. Now define q^2 by $q^2_\omega = q^1_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$ and $q^2_\omega = p'_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. We claim that $D_i(q^2) \subseteq D_i(p')$. To see this, fix an arbitrary

$\Phi \in D_i(p')$ and an arbitrary $\Xi \notin D_i(p')$. Then we must have

$$U_i([\Phi; q^2]) \geq U_i([\Phi; p']) - |\Phi|\varepsilon > U_i([\Xi; p']) \geq U_i([\Xi; q^2]),$$

where the first and third inequalities follow directly from the definitions of q^2, and the second inequality follows from $|\Phi|\varepsilon \leq |\Omega_i|\varepsilon_1 < V_i(p') - U_i([\Xi; p']) = U_i([\Phi; p']) - U_i([\Xi; p'])$.

We will now show that the condition in Case 1 of Definition 2 is satisfied for any set of trades $\Psi' \in D_i(q^2)$. Take any such Ψ'. Similar to the above, we define $\delta_1 = V_i(q^1) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^1)} U_i([\Xi; q^1])$, $\delta_2 = V_i(q^2) - \max_{\Xi \subseteq \Omega_i, \Xi \notin D_i(q^2)} U_i([\Xi; q^2])$, and $\delta_3 = \min_{\omega \in \tilde{\Omega}_{\to i}} (q^1_\omega - p'_\omega)$. Let $\delta = \frac{\min\{\delta_1, \delta_2, \delta_3\}}{3|\Omega_i|}$, and define price vector q^3 as

$$q^3_\omega = \begin{cases} q^2_\omega - \delta & \omega \in \Omega_{i\to} \smallsetminus \Psi' \text{ or } \omega \in \Psi'_{\to i} \\ q^2_\omega + \delta & \omega \in \Omega_{\to i} \smallsetminus \Psi' \text{ or } \omega \in \Psi'_{i\to} \\ 0 & \omega \notin \Omega_i. \end{cases}$$

Clearly, we must have $D_i(q^3) = \{\Psi'\}$. Now define q^4 by $q^4_\omega = q^3_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$ and $q^4_\omega = q^1_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$. Similar to the above, we can show that $D_i(q^4) \subseteq D_i(q^1)$, and therefore $D_i(q^4) = \{\Psi\}$. Since $q^3_\omega < q^4_\omega$ for all $\omega \in \tilde{\Omega}_{\to i}$ and $q^3_\omega = q^4_\omega$ for all $\omega \in \Omega \smallsetminus \tilde{\Omega}_{\to i}$, we can now apply Case 1 of (DFS) to conclude that Ψ' satisfies the condition in Case 1 of (DEFS).

The proofs that Case 2 of (DFS) implies Case 2 of (DEFS), and that (DFS) implies (DCFS) are completely analogous.

(DEFS) $\Rightarrow$ (CEFS), (DCFS) $\Rightarrow$ (CCFS) We first prove Case 1 of (CEFS). Take agent i and any sets of contracts $Y, Z \subseteq X_i$ such that $|C_i(Z)| > 0$, $|C_i(Y)| > 0$, $Y_{i\to} = Z_{i\to}$, and $Y_{\to i} \subseteq Z_{\to i}$. Define *usable* and *unusable* trades in Y as follows. Take trade $\omega \in Y_{i\to}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' > r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Similarly, take trade $\omega \in Y_{\to i}$. If there exists real number r such that (i) $(\omega, r) \in Y$ and (ii) for any $r' < r$, $(\omega, r') \notin Y$, then trade ω is usable in Y; otherwise, it is unusable in Y. Note that an unusable trade cannot be a part of any contract involved in any optimal choice in $C_i(Y)$. The definitions of trades usable and unusable in Z are completely analogous.

We now construct preliminary price vectors q and q' as follows. First, for every trade $\omega \notin \Omega_i$, $q_\omega = q'_\omega = 0$. Second, for every trade ω unusable in Y, $q_\omega = 0$, and for every trade ω unusable in Z, $q'_\omega = 0$. Next, for any trade $\omega \in \Omega_{i\to}$ usable in Y, $q_\omega = \max\{r : (\omega, r) \in Y\}$, and similarly, for any trade $\omega \in \Omega_{i\to}$ usable in Z, $q'_\omega = \max\{r : (\omega, r) \in Z\}$. Finally, for any trade $\omega \in \Omega_{\to i}$ usable in Y, $q_\omega = \min\{r : (\omega, r) \in Y\}$ and for any trade $\omega \in \Omega_{\to i}$ usable in Z, $q'_\omega = \min\{r : (\omega, r) \in Z\}$.

We now construct price vectors p and p'. First, for any trade $\omega \notin \Omega_i$, $p_\omega = p'_\omega = 0$. Second, for any trade $\omega \in \Omega_i$ that is usable in both Y and Z, let $p_\omega = q_\omega$ and let $p'_\omega = q'_\omega$. Finally, we need to set prices for trades unusable in Y or Z. We already noted that for any trade ω unusable in set Y, it has to be the case that ω is not involved in any contract in any optimal choice in $C_i(Y)$; and likewise, if ω is unusable in Z, then ω is not involved in any contract in any optimal choice in $C_i(Z)$. Thus, in forming prices p and p', we will need to assign to these trades prices that are so large (or small, depending on which side the trade is on) that the corresponding trades are not demanded by agent i.

Let Π be a very large number. For instance, let

$$\Delta_1 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; q]) - U_i([\Omega_2; q])|,$$

$$\Delta_2 = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i(\Omega_1; q') - U_i(\Omega_2; q')|,$$

and $\Pi = 1 + \Delta_1 + \Delta_2 + \max_{\omega \in \Omega_i} |q_\omega| + \max_{\omega \in \Omega_i} |q'_\omega|$. For all $\omega \in \Omega_{i\to}$ that are unusable in Y (and thus also in Z), let $p_\omega = p'_\omega = -\Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in both Y and Z, let $p_\omega = p'_\omega = \Pi$. For all $\omega \in \Omega_{\to i}$ that are unusable in Y but not in Z, let $p_\omega = \Pi$ and $p'_\omega = q'_\omega$. Finally, for all $\omega \in \Omega_{\to i}$ that are unusable in Z but not in Y, let $p_\omega = p'_\omega = q_\omega$. Note that for any such ω, since $Y \subset Z$, $(\omega, q_\omega) \in Z$; also, as ω is unusable in Z, there are no contracts involving ω in any optimal choice in $C_i(Z)$.

Now, $p'_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$ and $p'_\omega \le p_\omega$ for all $\omega \in \Omega_{\to i}$. Take any $Y^* \in C_i(Y)$, and let $\Psi = \tau(Y^*)$. By construction, $\Psi \in D_i(p)$. By (DEFS), there exists $\Psi' \in D_i(p')$ such that $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$. Let $Z^* = \kappa([\Psi', p'])$. Again, by construction, $Z^* \in C_i(Z)$. We now show that this set of contracts satisfies the conditions in Case 1 of (CEFS).

First, take some $y \in Y_{\to i} \smallsetminus Y^*_{\to i}$ and suppose that contrary to what we want to show, $y \in Z^*_{\to i}$. The latter implies that $y = (\omega, p'_\omega)$ for some trade ω, which, in turn, implies that $p_\omega = p'_\omega$ (because $y = (\omega, p'_\omega) \in Y$ and, since $Y \subset Z$, $(\omega, r) \notin Y$ for any $r < p'_\omega$). But then, by construction, $\{\omega \in (\Omega_{\to i} \smallsetminus \Psi_{\to i}) : p_\omega = p'_\omega\} \subseteq \Omega_{\to i} \smallsetminus \Psi'_{\to i}$, contradicting $y \in Z^*_{\to i}$. Second, since $Y^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi_{i\to}\}$, $Z^*_{i\to} = \{(\omega, p_\omega) : \omega \in \Psi'_{i\to}\}$, and $\Psi_{i\to} \subseteq \Psi'_{i\to}$, it is immediate that $Y^*_{i\to} \subseteq Z^*_{i\to}$. This completes the proof that Case 1 of (DEFS) implies Case 1 of (CEFS).

The proofs that Case 2 of (DEFS) implies Case 2 of (CEFS) and that (DCFS) implies (CCFS) are completely analogous.

(DEFS) + (DCFS) ⇒ (IDFS) + (IIFS) We first show that (DEFS) and (DCFS) jointly imply (IDFS). Take two price vectors p, p' such that $p \leq p'$. Let $\Psi' \in D_i(p')$ be arbitrary. We have to show that there exists a set of trades $\Psi \in D_i(p)$ such that $e_{i,\omega}(\Psi') \geq e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_i$ such that $p_\omega = p'_\omega$.

First, let p^1 be a price vector such that $p^1_\omega = p'_\omega$ for all $\omega \in \Omega_{\to i}$ and $p^1_\omega = p_\omega$ for all $\omega \in \Omega_{i\to}$. By (DCFS) there must exist a $\Psi^1 \in D_i(p^1)$ such that $\{\omega \in \Psi'_{i\to} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{\to i} \subseteq \Psi'_{\to i}$. Now note that $p_\omega = p^1_\omega$ for all $\omega \in \Omega_{i\to}$ and $p_\omega \leq p^1_\omega$ for all $\omega \in \Omega_{\to i}$. By (DEFS), there must exist a $\Psi \in D_i(p)$ such that $\{\omega \in \Psi_{\to i} : p^1_\omega = p_\omega\} \subseteq \Psi^1$ and $\Psi^1_{i\to} \subseteq \Psi_{i\to}$. Combining this with what we know about Ψ^1, we obtain that $\{\omega \in \Psi_{\to i} : p_\omega = p'_\omega\} = \{\omega \in \Psi_{\to i} : p_\omega = p^1_\omega\} \subseteq \Psi^1_{\to i} \subseteq \Psi'_{\to i}$ and $\{\omega \in \Psi'_{i\to} : p_\omega = p'_\omega\} = \{\omega \in \Psi'_{i\to} : p_\omega = p^1_\omega\} \subseteq \Psi^1_{i\to} \subseteq \Psi_{i\to}$. This implies $e_{i,\omega}(\Psi') \geq e_{i,\omega}(\Psi)$ for all $\omega \in \Omega_i$ such that $p'_\omega = p^1_\omega$.

The proof that (DEFS) and (DCFS) jointly imply (IIFS) is completely analogous.

(IFS) ⇒ (DFS) This follows immediately, because the price change conditions in both Cases 1 and 2 of (DFS) are special cases of the price change condition of (IFS). □

C Proofs of the Results in Sections 4, 5, 6, and 7

Proof of Proposition 1

Consider the intermediary i. Let Φ (with a typical element φ) denote the set of potential inputs this intermediary faces, and let Ψ (with a typical element ψ) denote the set of potential requests. The cost of using input φ to satisfy request ψ is given by $c_{\varphi,\psi}$. For convenience, when φ and ψ are incompatible, we simply say that $c_{\varphi,\psi} = +\infty$.

Let us now construct a "synthetic" agent $\hat{i}$ whose preferences will be identical to those of agent i, yet will be represented in the form of "intermediary with production capacity" preferences as defined in Section 4.2. The full substitutability of the preferences of intermediary i will then follow immediately from Proposition 2.

Agent $\hat{i}$ faces the same sets of inputs, Φ, and requests, Ψ, as agent i. Agent $\hat{i}$ also has $|\Phi| \times |\Psi|$ machines, indexed by pairs of inputs and requests: machine $m_{\varphi,\psi}$ "corresponds" to an input–request pair (φ, ψ). The costs of intermediary $\hat{i}$ are as follows (to avoid confusion, we will denote various costs of agent $\hat{i}$ by "$\hat{c}$" with various subindices, while the costs of agent i are denoted by "c" with various subindices):

- For input φ and machine $m_{\varphi,\psi}$ "corresponding" to input φ and some request ψ, the cost $\hat{c}_{\varphi,m_{\varphi,\psi}}$ of using input φ in machine $m_{\varphi,\psi}$ is equal to $c_{\varphi,\psi}$—the cost of using input φ to satisfy request ψ under the original cost structure of agent i.

- For any input $\varphi' \neq \varphi$ and any request ψ, the cost $\hat{c}_{\varphi', m_{\varphi,\psi}}$ is equal to $+\infty$.
- For request ψ and any machine $m_{\varphi,\psi}$ "corresponding" to request ψ and some input φ, the cost $\hat{c}_{m_{\varphi,\psi},\psi}$ of using machine $m_{\varphi,\psi}$ to satisfy request ψ is equal to 0.
- For any request $\psi' \neq \psi$ and any machine $m_{\varphi,\psi}$, the cost $\hat{c}_{m_{\varphi,\psi},\psi'}$ is equal to $+\infty$.

With this construction, the preferences of agents i and $\hat{i}$ over sets of inputs and requests are identical. Moreover, the preferences of agent $\hat{i}$ are those of "intermediary with production capacity" and are thus fully substitutable (by Proposition 2). Therefore, the "intermediary" preferences of agent i are also fully substitutable.

Proof of Proposition 2

Consider first an "intermediary with production capacity" who has exactly one machine at his disposal. It is immediate that the preferences of such an intermediary are fully substitutable.

Next, consider a general "intermediary with production capacity", i, who has a set of machines M (with a typical element m) at his disposal and faces the set of inputs Φ (with a typical element φ) and the set of potential requests Ψ (with a typical element ψ), with costs as described in Section 4.2. We will show that the preferences of intermediary i can be represented as a "merger" of several (specifically, $|M| + |\Phi| + |\Psi|$) agents with fully substitutable preferences, which by Theorem 4 will imply that the preferences of intermediary i are fully substitutable.

Specifically, consider the following set of artificial agents. First, there are $|\Phi|$ "input dummies", with a typical element $\hat{\varphi}$ for a dummy that corresponds to input φ. Second, there are $|M|$ "machine dummies", with a typical element $\hat{m}$ for a dummy that corresponds to machine m. Finally, there are $|\Psi|$ "request dummies", with a typical element $\hat{\psi}$ for a dummy that corresponds to request ψ.

Each input dummy $\hat{\varphi}$ can only buy one trade: input φ. He can also form $|M|$ trades as a seller: one trade with every machine dummy $\hat{m}$. We denote the trade between an input dummy $\hat{\varphi}$ (as the seller) and a machine dummy $\hat{m}$ (as the buyer) by $\omega_{\varphi,m}$. Likewise, each request dummy $\hat{\psi}$ can only sell one trade: request ψ. He can also form $|M|$ trades as a buyer: one trade with every machine dummy $\hat{m}$. We denote the trade between a machine dummy $\hat{m}$ (as the seller) and a request dummy $\hat{\psi}$ (as the buyer) by $\omega_{m,\psi}$. Each machine dummy can thus form $|\Phi|$ trades as the buyer (one with each input dummy) and $|\Psi|$ trades as the seller (one with each request dummy).

The preferences of the agents are as follows. Each input dummy and each request dummy has valuation 0 if the number of trades he forms as the seller is equal to the number of trades he forms as the buyer (this number can thus be equal to either 0 or 1), and $-\infty$ if these numbers are not equal. It is immediate that the preferences of input and request dummies are fully substitutable.

The preferences of each machine dummy $\hat{m}$ are as follows. If it buys no trades and sells no trades, its valuation is 0. If it buys exactly one trade, say $\omega_{\varphi,m}$ for some φ, and sells exactly one trade, say $\omega_{m,\psi}$ for some ψ, then its valuation is $-(c_{\varphi,m} + c_{m,\psi})$—the total cost of preparing input φ for request ψ using machine m in the original construction of the utility function of agent i. In all other cases (i.e., when the machine dummy buys or sells more than two trades, or when the number of trades it buys is not equal to the number of trades it sells), the valuation of the machine dummy is $-\infty$. Note that the preferences of the machine dummy are also fully substitutable.

Consider now the "synthetic" agent $\hat{i}$ obtained as the merger of the $|\Phi|$ input dummies, $|M|$ machine dummies, and $|\Psi|$ request dummies (see Section 5.2 for the details of the "merger" operation). By Theorem 4, the preferences of agent $\hat{i}$ are fully substitutable. Moreover, the valuation of agent $\hat{i}$ over any bundle of inputs and requests is identical to the valuation of agent i over that bundle. Thus, the preferences of agent i are fully substitutable.

Proof of Theorem 2

The indirect utility function for $\hat{u}_i^{(\Phi,p_\Phi)}$ is given by

$$\begin{aligned}
\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) &\equiv \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi) + \sum_{\xi\in\Xi_{\to i}} p_\xi - \sum_{\xi\in\Xi_{\to i}} p_\xi\right\} + \sum_{\psi\in\Psi_{\to i}} p_\xi - \sum_{\psi\in\Psi_{\to i}} p_\xi\right\} \\
&= \max_{\Psi\subseteq\Omega\smallsetminus\Phi}\left\{\max_{\Xi\subseteq\Phi}\left\{u_i(\Psi\cup\Xi) + \sum_{\lambda\in\Xi_{\to i}\cup\Psi_{\to i}} p_\lambda - \sum_{\lambda\in\Xi_{i\to}\cup\Psi_{i\to}} p_\lambda\right\}\right\} \\
&= \max_{\Lambda\subseteq\Omega}\left\{u_i(\Lambda) + \sum_{\lambda\in\Lambda_{\to i}} p_\lambda - \sum_{\lambda\in\Lambda_{i\to}} p_\lambda\right\}.
\end{aligned}$$

Hence, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi}) = V_i(p_{\Omega\smallsetminus\Phi}, p_\Phi)$. Now, $V_i(p)$ is submodular over $\mathbb{R}^\Omega$ by Theorem 6. As a submodular function restricted to a subspace of its domain is still submodular, $\hat{V}_i^{(\Phi,p_\Phi)}(p_{\Omega\smallsetminus\Phi})$ is submodular over $\mathbb{R}^{\Omega\smallsetminus\Phi}$. Hence, by Theorem 6, we see that $\hat{u}_i^{(\Phi,p_\Phi)}$ is fully substitutable.

Proof of Theorem 3

Fix a set of trades $\Phi \subseteq \Omega_i$ such that $u_i(\Phi) \neq -\infty$ and a vector of prices $\overline{p}_\Phi$ for trades in Φ. Let $\tilde{D}_i$ be the demand function for trades in $\Omega \smallsetminus \Phi$ induced by $\tilde{u}_i^{\Phi,\overline{p}_\Phi}$. Fix two price vectors

$p \in \mathbb{R}^{\Omega \smallsetminus \Phi}$ and $p' \in \mathbb{R}^{\Omega \smallsetminus \Phi}$ such that $|\tilde{D}_i(p)| = |\tilde{D}_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\to} \smallsetminus \Phi$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\to i} \smallsetminus \Phi$. Let $\Psi \in \tilde{D}_i(p)$ be the unique demanded set from $\Omega_i \smallsetminus \Phi$ at price vector p and $\Psi' \in \tilde{D}_i(p')$ be the unique demanded set from $\Omega_i \smallsetminus \Phi$ at price vector p'. Note that since $u_i(\Phi) \neq -\infty$, there exists a vector of prices p^*_Φ for trades in Φ such that, for all $\Xi \in D_i((p, p^*_\Phi)) \cup D_i((p', p^*_\Phi))$, we have $\Phi \subseteq \Xi$. Fix an arbitrary $\Xi \in D_i((p, p^*_\Phi))$ and let $\tilde{\Psi} \equiv \Xi \smallsetminus \Phi$.

Claim. *We must have* $\tilde{\Psi} = \Psi$.

Proof. Suppose the contrary. Since $\tilde{\Psi} \cup \Phi = \Xi \in D_i((p, p^*_\Phi))$, we must have

$$\begin{aligned} u_i(\Xi) &= u_i(\tilde{\Psi} \cup \Phi) + \sum_{\psi \in \tilde{\Psi}_{i\to}} p_\psi - \sum_{\psi \in \tilde{\Psi}_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} p^*_\varphi - \sum_{\varphi \in \Phi_{\to i}} p^*_\varphi \\ &\geq u_i(\Psi \cup \Phi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} p^*_\varphi - \sum_{\varphi \in \Phi_{\to i}} p^*_\varphi. \end{aligned} \tag{3}$$

The inequality (3) is equivalent to

$$\begin{aligned} &u_i(\tilde{\Psi} \cup \Phi) + \sum_{\psi \in \tilde{\Psi}_{i\to}} p_\psi - \sum_{\psi \in \tilde{\Psi}_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} \overline{p}_\varphi - \sum_{\varphi \in \Phi_{\to i}} \overline{p}_\varphi \\ &\qquad \geq u_i(\Psi \cup \Phi) + \sum_{\psi \in \Psi_{i\to}} p_\psi - \sum_{\psi \in \Psi_{\to i}} p_\psi + \sum_{\varphi \in \Phi_{i\to}} \overline{p}_\varphi - \sum_{\varphi \in \Phi_{\to i}} \overline{p}_\varphi. \end{aligned} \tag{4}$$

However, the inequality (4) implies that $\tilde{\Psi} \in \tilde{D}_i(p)$; this contradicts the assumption that $\tilde{D}_i(p) = \{\Psi\}$ given that $\tilde{\Psi} \neq \Psi$. $\square$

The preceding claim implies that we must have $D_i((p, p^*_\Phi)) = \{\Xi\} = \{\tilde{\Psi} \cup \Phi\} = \{\Psi \cup \Phi\}$. A similar argument shows that $D_i((p', p^*_\Phi)) = \{\Psi' \cup \Phi\}$. The full substitutability of u_i then implies that $\{\psi \in \Psi'_{\to i} : p_\psi = p'_\psi\} \subseteq \Psi_{\to i}$ and $\Psi_{i\to} \subseteq \Psi'_{i\to}$.

Proof of Theorem 4

We suppose, by way of contradiction, that u_J does not induce fully substitutable preferences over trades in $\Omega \smallsetminus \Omega^J$. By Corollary 1 of Hatfield et al. (2013), there exist fully substitutable preferences $\tilde{u}_i$ for the agents $i \in I \smallsetminus J$ such that no competitive equilibrium exists for the *modified economy* with

1. set of agents $(I \smallsetminus J) \cup \{J\}$,

2. set of trades $\Omega \smallsetminus \Omega^J$,

3. and valuation function for agent J given by u_J.[27]

Now, we consider the *original economy* with

1. set of agents I,

2. set of trades Ω,

3. valuations for $i \in I \smallsetminus J$ given by $\tilde{u}_i$, and

4. valuations for $j \in J$ given by u_j.

Let $[\Psi; p]$ be a competitive equilibrium of this economy; such an equilibrium must exist by Theorem 1 of Hatfield et al. (2013).

Claim. $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ *is a competitive equilibrium of the modified economy.*

Proof. It is immediate that $[\Psi \smallsetminus \Omega^J]_i \in D_i(p_{\Omega \smallsetminus \Omega^J})$ for all $i \in I \smallsetminus J$. Moreover, since Ψ is individually-optimal for each $j \in J$ in the original economy (at prices p),

$$u_j(\Psi) + \sum_{\psi \in \Psi_{j\to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \geq u_j(\Phi) + \sum_{\varphi \in \Phi_{j\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\varphi \tag{5}$$

for every $\Phi \subseteq \Omega$. Summing (5) over all $j \in J$ and simplifying, we obtain

$$\begin{aligned}
\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in \Psi_{j\to}} p_\psi - \sum_{\psi \in \Psi_{\to j}} p_\psi \right) &\geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in \Phi_{j\to}} p_\varphi - \sum_{\varphi \in \Phi_{\to j}} p_\psi \right) \\
\sum_{j \in J} \left(u_j(\Psi) + \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{j\to}} p_\psi - \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{\to j}} p_\psi \right) &\geq \sum_{j \in J} \left(u_j(\Phi) + \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{j\to}} p_\varphi - \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{\to j}} p_\psi \right) \\
\sum_{j \in J} u_j(\Psi) + \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{J\to}} p_\psi - \sum_{\psi \in [\Psi \smallsetminus \Omega^J]_{\to J}} p_\psi &\geq \sum_{j \in J} u_j(\Phi) + \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{J\to}} p_\varphi - \sum_{\varphi \in [\Phi \smallsetminus \Omega^J]_{\to J}} p_\psi. \quad \square
\end{aligned}$$

The preceding claim shows that $[\Psi \smallsetminus \Omega^J; p_{\Omega \smallsetminus \Omega^J}]$ is a competitive equilibrium of the modified economy, contradicting the earlier conclusion that no competitive equilibrium exists in the modified economy. Hence, we see that u_J must be fully substitutable.

[27] Technically, in order to apply Corollary 1 of Hatfield et al. (2013), we must have that for every pair (i, j) of distinct agents in I, there exists a trade ω such that $b(\omega) = i$ and $s(\omega) = j$. For any pair (i, j) of distinct agents in I such that no such trade ω exists, we can augment the economy by adding the requisite trade ω and, if $i \in J$, letting $\bar{u}_i(\Psi \cup \{\omega\}) = u^i(\Psi)$ (and similarly for j). It is immediate that $\bar{u}_i$ is substitutable if and only if u_i is substitutable.

Proof of Theorem 5

The proof of this result is very close to Step 1 of the proof of Theorem 1 of Hatfield et al. (2013). The only differences are that in the Hatfield et al. (2013) results, all trades could be bought out, and the price for buying them out was set to a single large number that was the same for all trades. By contrast, in Theorem 5 of the current paper we allow for the possibility that only a subset of trades can be bought out, and that the prices at which these trades can be bought out can be different, and need not be large. Adapting Step 1 of the proof of Theorem 1 of Hatfield et al. (2013) to the current more general setting is straightforward, but we include the proof for completeness.

Consider the fully substitutable valuation function u_i, and take any trade $\varphi \in \Omega_{i\to} \cap \Phi$. Consider a modified valuation function u_i^φ:

$$u_i^\varphi(\Psi) = \max\left\{u_i(\Psi), u_i(\Psi \smallsetminus \{\varphi\}) - \Pi_\varphi\right\}.$$

That is, the valuation $u_i^\varphi(\Psi)$ allows (but does not require) agent i to pay Π_φ instead of executing one particular trade, φ.

Claim. *The valuation function u_i^φ is fully substitutable.*

Proof. We consider utility U_i^φ and demand D_i^φ corresponding to valuation u_i^φ. We show that D_i^φ satisfies the (IFS) condition (Definition 3). Fix two price vectors p and p' such that $p \le p'$ and $|D_i^\varphi(p)| = |D_i^\varphi(p')| = 1$. Take the unique $\Psi \in D_i^\varphi(p)$ and $\Psi' \in D_i^\varphi(p')$. We need to show that

$$e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi') \text{ for all } \omega \in \Omega_i \text{ such that } p_\omega = p'_\omega. \tag{6}$$

Let price vector q coincide with p on all trades other than φ, and set $q_\varphi = \min\{p_\varphi, \Pi_\varphi\}$. Note that if $p_\varphi < \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p)$. If $p_\varphi > \Pi_\varphi$, then under utility U_i^φ, agent i always wants to execute trade φ at price p_φ, and the only decision is whether to "buy it out" or not at the cost Π_φ; i.e., the agent's effective demand is the same as under price vector q. Thus, $D_i^\varphi(p) = \{\Xi \cup \{\varphi\} : \Xi \in D_i(q)\}$. Finally, if $p_\varphi = \Pi_\varphi$, then $p = q$ and $D_i^\varphi(p) = D_i(p) \cup \{\Xi \cup \{\varphi\} : \Xi \in D_i(p)\}$. We construct price vector q' corresponding to p' analogously.

Now, if $p_\varphi \le p'_\varphi < \Pi_\varphi$, then $D_i^\varphi(p) = D_i(p)$, $D_i^\varphi(p') = D_i(p')$, and thus $e_{i,\omega}(\Psi) \le e_{i,\omega}(\Psi')$ follows directly from (IFS) for demand D_i.

If $\Pi_\varphi \le p_\varphi \le p'_\varphi$, then (since we assumed that D_i^φ was single-valued at p and p') it has to be the case that D_i is single-valued at the corresponding price vectors q and q'. Let $\Xi \in D_i(q)$ and $\Xi' \in D_i(q')$. Then $\Psi = \Xi \cup \{\varphi\}$, $\Psi' = \Xi' \cup \{\varphi\}$, and statement (6) follows from the (IFS) condition for demand D_i, because $q \le q'$.

Finally, if $p_\varphi < \Pi_\varphi \leq p'_\varphi$, then $p = q$, Ψ is the unique element in $D_i(p)$, and Ψ' is equal to $\Xi' \cup \{\varphi\}$, where Ξ' is the unique element in $D_i(q')$. Then for $\omega \neq \varphi$, statement (6) follows from (IFS) for demand D_i, because $p \leq q'$. For $\omega = \varphi$, statement (6) does not need to be checked, because $p_\varphi < p'_\varphi$.

Thus, when $\varphi \in \Omega_{i\to}$, the valuation function u_i^φ is fully substitutable. The proof for the case when $\varphi \in \Omega_{\to i}$ is completely analogous. $\square$

To complete the proof of Theorem 5, it is now enough to note that valuation function $\hat{u}_i(\Psi) = \max_{\Xi \subseteq \Psi \cap \Phi} \left\{ u_i(\Psi \smallsetminus \Xi) - \sum_{\varphi \in \Xi} \Pi_\varphi \right\}$ can be obtained from the original valuation u_i by allowing agent i to "buy out" all of the trades in set Φ, one by one, and since the preceding claim shows that each such transformation preserves substitutability (and Ω_i is finite), the valuation function $\hat{u}_i$ is substitutable as well.

Proof of Theorem 6

We first show that if the preferences of an agent i are fully substitutable, then those preferences induce a submodular indirect utility function. It is enough to show that for any two trades $\varphi, \psi \in \Omega_i$ and any prices $p \in \mathbb{R}^\Omega$, $p_\varphi^{\text{high}} > p_\varphi$, and $p_\psi^{\text{high}} > p_\psi$ we have that[28]

$$\begin{aligned} V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\ \geq V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi). \end{aligned} \quad (7)$$

Suppose that $\varphi, \psi \in \Omega_{\to i}$.[29] There are three cases to consider:

1. Suppose that $\varphi \notin \Phi$ for any $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$. Then, by individual rationality, $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) = 0$$

and so equation (7) is satisfied, as the left side of (7) must be non-negative.

[28] The definition of submodularity given in Definition 4 is equivalent to the pointwise definition given here; see, e.g., Schrijver (2002).

[29] The other three cases—

1. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i\to}$,
2. $\varphi \in \Omega_{\to i}$ and $\psi \in \Omega_{i\to}$, and
3. $\varphi, \psi \in \Omega_{i\to}$—

are analogous.

2. Suppose $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. Then, by individual rationality, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}})$. Hence,

$$V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) = -(p_\varphi - p_\varphi^{\text{high}}) = p_\varphi^{\text{high}} - p_\varphi$$

and so equation (7) is satisfied, as the right side of (7) is (weakly) bounded from above by $p_\varphi^{\text{high}} - p_\varphi$ (with equality in the case that φ is demanded at both $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi)$).

3. Suppose that $\varphi \in \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi)$ and $\varphi \notin \Phi$ for some $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}})$. In this case, as the preferences of i are fully substitutable, there exists a unique price $p_\varphi^\uparrow$ such that there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}})$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \leq p_\varphi^\uparrow \leq p_\varphi^{\text{high}}$. Similarly, let $p_\varphi^\downarrow$ be the unique price at which there exists $\Phi, \bar{\Phi} \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi)$ such that $\varphi \in \Phi$ and $\varphi \notin \bar{\Phi}$; note that $p_\varphi \leq p_\varphi^\downarrow \leq p_\varphi^{\text{high}}$. By the definition of the utility function, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi < p_\varphi^\uparrow$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi^{\text{high}})$ for all $\tilde{p}_\varphi > p_\varphi^\uparrow$; similarly, $\varphi \in \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi < p_\varphi^\downarrow$, and $\varphi \notin \Phi$ for all $\Phi \in D_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, \tilde{p}_\varphi, p_\psi)$ for all $\tilde{p}_\varphi > p_\varphi^\downarrow$.

   Since the preferences of i are fully substitutable, $p_\varphi^\downarrow \leq p_\varphi^\uparrow$. Hence,

$$\begin{aligned}
&V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}}) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\uparrow, p_\psi^{\text{high}}) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi^{\text{high}}) \\
&\quad = -p_\varphi + p_\varphi^\uparrow - 0 \\
&\quad \geq -p_\varphi + p_\varphi^\downarrow - 0 \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi) + V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^\downarrow, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi) \\
&\quad = V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi, p_\psi) - V_i(p_{\Omega \smallsetminus \{\varphi,\psi\}}, p_\varphi^{\text{high}}, p_\psi),
\end{aligned}$$

   which is exactly (7).

Now, suppose that the preferences of i are not substitutable. We suppose moreover that the preferences of i fail the first condition of Defintion 2.[30] Hence, for some price vectors $p, p' \in \mathbb{R}^\Omega$ such that $|D_i(p)| = |D_i(p')| = 1$, $p_\omega = p'_\omega$ for all $\omega \in \Omega_{i\rightarrow}$, and $p_\omega \geq p'_\omega$ for all $\omega \in \Omega_{\rightarrow i}$, we have that for the unique $\Psi \in D_i(p)$ and $\Psi' \in D_i(p')$, either $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\rightarrow i}$ or $\Psi_{i\rightarrow} \not\subseteq \Psi'_{i\rightarrow}$. We suppose that $\{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\} \not\subseteq \Psi_{\rightarrow i}$; the latter case is analogous. Let $\varphi \in \Psi_{\rightarrow i} \smallsetminus \{\omega \in \Psi'_{\rightarrow i} : p_\omega = p'_\omega\}$. Let p_φ^{high} be a price for trade φ high enough such that φ is

[30] The case where the preferences of i fail the second condition of Defintion 2 is analogous.

not demanded at either $(p_\varphi^{\text{high}}, p_{\Omega \smallsetminus \{\varphi\}})$ or $(p_\varphi^{\text{high}}, p'_{\Omega \smallsetminus \{\varphi\}})$. Hence,

$$V_i(p_\varphi, p'_{\Omega \smallsetminus \{\varphi\}}) - V_i(p_\varphi^{\text{high}}, p'_{\Omega \smallsetminus \{\varphi\}}) = 0$$

while

$$V_i(p_\varphi, p_{\Omega \smallsetminus \{\varphi\}}) - V_i(p_\varphi^{\text{high}}, p_{\Omega \smallsetminus \{\varphi\}}) > 0.$$

Thus we see that V_i is not submodular.

Proof of Theorem 7

The proof is an adaptation of the proof of Theorem 1 of Sun and Yang (2009) to our setting. As our model is more general than that of Sun and Yang (2009)—we do not impose either monotonicity or boundedness on the valuation functions, and we do not require that the seller values each bundle at 0 and thus sells everything that he could sell—we have to carefully ensure that the Sun and Yang (2009) approach remains valid.

"If" Direction. We show first that (IDFS) and (IIFS) imply the single improvement property. Fix an arbitrary price vector $p \in \mathbb{R}^\Omega$ and a set of trades $\Psi \notin D_i(p)$ such that $u_i(\Psi) \neq -\infty$. Fix a set of trades $\Xi \in D_i(p)$. We focus exclusively on the trades in Ψ and Ξ by rendering all other trades that agent i is involved in irrelevant. To this end, we first define a very high price Π,

$$\Pi \equiv \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; p]) - U_i([\Omega_2; p)| + \max_{\omega \in \Omega_i} |p_\omega| + 1,$$

and then, starting from p, we construct a preliminary price vector p' as follows:

$$p'_\omega = \begin{cases} p_\omega & \omega \in \Psi \cup \Xi \text{ or } \omega \notin \Omega_i \\ p_\omega + \Pi & \omega \in \Omega_{\to i} \smallsetminus (\Psi \cup \Xi) \\ p_\omega - \Pi & \omega \in \Omega_{i \to} \smallsetminus (\Psi \cup \Xi). \end{cases}$$

Observe that $\Psi \notin D_i(p')$ and $\Xi \in D_i(p')$. As $\Psi \neq \Xi$, we have to consider two cases (each with several subcases), which taken together will show that there exists a set of trades $\Phi' \neq \Psi$ that satisfies conditions 2 and 3 of Definition 5 and $U_i[(\Phi'; p)] \geq U_i([\Psi; p])$.

Case 1: $\Xi \smallsetminus \Psi \neq \varnothing$**.** Select a trade $\xi_1 \in \Xi \smallsetminus \Psi$. Without loss of generality, assume that agent i is the buyer of ξ_1 (the case where i is the seller is completely analogous).

Starting from p', construct a modified price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}) \text{ or } \omega \notin \Omega_i \\ p'_\omega + \Pi & \omega \in (\Xi_{\to i} \smallsetminus (\Psi_{\to i} \cup \{\xi_1\})) \cup \Psi_{i\to}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\xi_1 \in \Xi$, and $p'_{\xi_1} = p''_{\xi_1}$, full substitutability (Definition A.5) implies that there exists $\Xi'' \in D_i(p'')$ such that $\xi_1 \in \Xi''$. Second, observe that following the price change from p' to p'', $(\Xi''_{\to i} \smallsetminus \Psi_{\to i}) \subseteq \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. Thus, $\Xi''_{\to i} \smallsetminus \Psi_{\to i} = \{\xi_1\}$ and $\Psi_{i\to} \subseteq \Xi''_{i\to}$. We consider three subcases.

Subcase (a): $\Xi''_{i\to} \smallsetminus \Psi_{i\to} \neq \varnothing$. Let $\xi_2 \in \Xi''_{i\to} \smallsetminus \Psi_{i\to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i\to} \smallsetminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \smallsetminus (\Psi_{i\to} \cup \{\xi_2\})) \cup \Psi_{\to i}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\xi_2 \in \Xi''$, and $p''_{\xi_2} = p'''_{\xi_2}$, full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\xi_2 \in \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \subseteq \Xi'''$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1, \xi_2\}$. Thus, $\Psi \smallsetminus \Xi''' = \varnothing$ and $\Xi''' \smallsetminus \Psi = \{\xi_1, \xi_2\}$ or $\{\xi_2\}$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from the perspective of agent i the only differences from Ψ to Ξ''' are making one new sale ξ_2, i.e., $e_{i,\xi_2}(\Psi) > e_{i,\xi_2}(\Xi''')$ with $\xi_2 \in \Omega_{i\to} \smallsetminus \Psi$, and (possibly) making one new purchase ξ_1, i.e. $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Subcase (b): $\Xi''_{i\to} \smallsetminus \Psi_{i\to} = \varnothing$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \neq \varnothing$. Let $\psi \in \Psi_{\to i} \smallsetminus \Xi''_{\to i}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus ((\Xi_{i\to} \smallsetminus \Psi_{i\to}) \cup (\Psi_{\to i} \smallsetminus \{\psi\})) \text{ or } \omega \notin \Omega_i \\ p''_\omega - \Pi & \omega \in (\Xi_{i\to} \smallsetminus \Psi_{i\to}) \cup (\Psi_{\to i} \smallsetminus \{\psi\}). \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi \notin \Xi''$, and $p''_\psi = p'''_\psi$, by full substitutability (Definition A.6) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Psi \smallsetminus \Xi''' \subseteq \{\psi\}$ and $\Xi''' \smallsetminus \Psi \subseteq \{\xi_1\}$. Thus, $\Psi \smallsetminus \Xi''' = \{\psi\}$ and $\Xi''' \smallsetminus \Psi = \{\xi_1\}$ or $\varnothing$.

Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one purchase ψ, i.e., $e_{i,\psi}(\Psi) > e_{i,\psi}(\Xi''')$ with $\psi \in \Psi_{\to i}$, and (possibly) making one new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Subcase (c): $\Xi'' = \Psi \cup \{\xi_1\}$. Let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only difference from Ψ to Ξ''' is making a new purchase ξ_1, i.e., $e_{i,\xi_1}(\Psi) < e_{i,\xi_1}(\Xi''')$ with $\xi_1 \in \Omega_{\to i} \smallsetminus \Psi$.

Case 2: $\Xi \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi \neq \varnothing$. Select a trade $\psi_1 \in \Psi \smallsetminus \Xi$. Without loss of generality, assume that agent i is a buyer in ψ_1 (the case where i is a seller is completely analogous).

Starting from p', construct price vector p'' as follows:

$$p''_\omega = \begin{cases} p'_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{\to i} \smallsetminus \{\psi_1\}) \text{ or } \omega \notin \Omega_i \\ p'_\omega - \Pi & \omega \in \Psi_{\to i} \smallsetminus \{\psi_1\}. \end{cases}$$

First, since $\Xi \in D_i(p')$, $\psi_1 \notin \Xi$, and $p'_{\psi_1} = p''_{\psi_1}$, full substitutability (Definition A.6) implies that there exists $\Xi'' \in D_i(p'')$ such that $\psi_1 \notin \Xi''$. Second, observe that following the price change from p' to p'', $\Xi'' \subseteq \Psi$ and $\Psi_{\to i} \smallsetminus \Xi''_{\to i} \subseteq \{\psi_1\}$. Thus, $\Psi_{\to i} \smallsetminus \Xi''_{\to i} = \{\psi_1\}$ and $\Xi'' \subseteq \Psi$. We consider two subcases.

Subcase (a): $\Psi_{i\to} \smallsetminus \Xi''_{i\to} \neq \varnothing$. Let $\psi_2 \in \Psi_{i\to} \smallsetminus \Xi''_{i\to}$. Starting from p'', construct price vector p''' as follows:

$$p'''_\omega = \begin{cases} p''_\omega & \omega \in \Omega_i \smallsetminus (\Psi_{i\to} \smallsetminus \{\psi_2\}) \text{ or } \omega \notin \Omega_i \\ p''_\omega + \Pi & \omega \in \Psi_{i\to} \smallsetminus \{\psi_2\}. \end{cases}$$

First, since $\Xi'' \in D_i(p'')$, $\psi_2 \notin \Xi''$, and $p''_{\psi_2} = p'''_{\psi_2}$, full substitutability (definition A.5) implies that there exists $\Xi''' \in D_i(p''')$ such that $\psi_2 \notin \Xi'''$. Second, observe that following the price change from p'' to p''', $\Xi''' \subseteq \Psi$ and $\Psi \smallsetminus \Xi''' \subseteq \{\psi_1, \psi_2\}$. Thus, $\Xi''' \smallsetminus \Psi = \varnothing$ and $\Psi \smallsetminus \Xi''' = \{\psi_1, \psi_2\}$ or $\{\psi_2\}$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from agent i's perspective the only differences from Ψ to Ξ''' are canceling one sale ψ_2, i.e., $e_{i,\psi_2}(\Psi) < e_{i,\psi_2}(\Xi''')$ with $\psi_1 \in \Omega_{i\to} \smallsetminus \Psi$, and (possibly) canceling one purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) > e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Psi_{\to i}$.

Subcase (b): $\Xi'' = \Psi \smallsetminus \{\psi_1\}$. In this subcase, let $p''' = p''$ and $\Xi''' = \Xi''$. Since $\Xi''' \in D_i(p''')$, we have $U_i([\Psi, p''']) \leq U_i([\Xi''', p'''])$. Furthermore, observe that from the perspective of agent i, the only difference from Ψ to Ξ''' is canceling purchase ψ_1, i.e., $e_{i,\psi_1}(\Psi) < e_{i,\psi_1}(\Xi''')$ with $\psi_1 \in \Omega_{\to i} \setminus \Psi$.

Taking together all the final statements from each subcase of Cases 1 and 2, if we take $\Phi' \equiv \Xi'''$, we obtain that we always have a price vector p''' and the sets Ψ and Φ' that satisfy conditions (2) and (3) of Definition 5. Moreover, since we always have $\Phi \in D_i(p''')$, $U_i([\Phi', p''']) \geq U_i([\Psi, p'''])$.

Next, we show that $U_i([\Phi', p''']) - U_i([\Psi, p''']) \geq 0$ implies $U_i([\Phi', p]) \geq U_i([\Psi, p])$. First, observe that when taking the difference the prices of all trades $\omega \in \Phi' \cap \Psi$ cancel each other out. Thus, replacing the prices p'''_ω with p_ω for all trades $\omega \in \Phi' \cap \Psi$ leaves the difference unchanged. Second, observe that in all previous subcases, the construction of p''' implies that for all $\omega \in ((\Psi \setminus \Phi') \cup (\Phi' \setminus \Psi))$, $p_\omega = p'''_\omega$. Combining the two observations above, $U_i([\Phi', p''']) - U_i([\Psi, p''']) = U_i([\Phi', p]) - U_i([\Psi, p])$, and therefore $U_i([\Phi', p]) \geq U_i([\Psi, p])$.

We now show that there exists a set of trades Φ that satisfies all conditions of Definition 5. Since $\Psi \notin D_i(p)$, $V_i(p) > U_i([\Psi; p])$. Since i's utility is continuous in prices, there exists $\varepsilon > 0$ such that $V_i(q) > U_i([\Psi; q])$ where q is defined as follows:

$$q_\omega = \begin{cases} p_\omega + \varepsilon & \omega \in (\Omega_{\to i} \setminus \Psi_{\to i}) \cup \Psi_{i\to} \\ p_\omega - \varepsilon & \omega \in (\Omega_{i\to} \setminus \Psi_{i\to}) \cup \Psi_{\to i}. \end{cases}$$

Our arguments above imply that there exists a set of trades $\Phi \neq \Psi$ such that $U_i([\Phi; q]) \geq U_i([\Psi; q])$. Using the construction of q, we obtain $U_i([\Phi; p]) - U_i([\Psi; p]) = U_i([\Phi; q]) - U_i([\Psi; q]) + \varepsilon|(\Psi \setminus \Phi) \cup (\Phi \setminus \Psi)| > U_i([\Phi; q]) - U_i([\Psi; q]) \geq 0$. Thus, $U_i([\Phi; p]) > U_i([\Psi; p])$. This completes the proof that (IDFS) and (IIFS) imply the single improvement property.

We now show that the single improvement property implies full substitutability (DCFS). More specifically, we will establish that single improvement implies the first condition of Definition A.4. The proof that the second condition of Definition A.4 is also satisfied uses a completely analogous argument.

Let $p \in \mathbb{R}^\Omega$ and $\Psi \in D_i(p)$ be arbitrary. It is sufficient to establish that for any $p' \in \mathbb{R}^\Omega$ such that $p'_\psi > p_\psi$ for some $\psi \in \Omega_{\to i}$ and $p'_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$, there exists a set of trades $\Psi' \in D_i(p')$ that satisfies the first condition of Definition A.4.

Fix one $p' \in \mathbb{R}^\Omega$ that satisfies the conditions mentioned in the previous paragraph and let $\psi \in \Omega_{\to i}$ be the one trade for which $p'_\psi > p_\psi$. Note that if either $\psi \notin \Psi$ or $\Psi \in D_i(p')$, there is nothing to show. From now on, assume that $\psi \in \Psi$ and $\Psi \notin D_i(p')$.

For any real number $\varepsilon > 0$ define a price vector $p^\varepsilon \in \mathbb{R}^\Omega$ by setting $p^\varepsilon_\psi = p_\psi + \varepsilon$ and $p^\varepsilon_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. Let $\Delta \equiv \max\{\varepsilon : \Psi \in D_i(p^\varepsilon)\}$. Note that Δ is well defined since i's utility function is continuous in prices. Furthermore, given that $\Psi \notin D_i(p')$, we must have $\Delta < p'_\psi - p_\psi$.

Next, for any integer n, define a price vector $p^n \in \mathbb{R}^\Omega$ by setting $p^n_\psi = p_\psi + \Delta + \frac{1}{n}$ and $p^n_\omega = p_\omega$ for all $\omega \in \Omega \setminus \{\psi\}$. By the definition of Δ we must have $\Psi \notin D_i(p^n)$ for all $n > 0$. By the single improvement property, this implies that for all $n > 0$, there exists a set of trades Φ^n such that the following conditions are satisfied:

1. $U_i([\Psi, p^n]) < U_i([\Phi^n, p^n])$,
2. there exists at most one trade ω such that $e_{i,\omega}(\Psi) < e_{i,\omega}(\Phi^n)$, and
3. there exists at most one trade ω such that $e_{i,\omega}(\Psi) > e_{i,\omega}(\Phi^n)$.

Note that we must have $\psi \notin \Phi^n$ for all $n \geq 1$. This follows since for any $n \geq 1$ and any set of trades Φ such that $\psi \in \Phi$, $U_i([\Phi; p^n]) = U_i([\Phi; p]) - \Delta - \frac{1}{n} \leq U_i([\Psi; p]) - \Delta - \frac{1}{n} = U_i([\Psi; p^n])$ given that $\Psi \in D_i(p)$.

Conditions 2 and 3 imply that for all $n > 0$, we must have $\{\omega \in \Psi_{\to i} : p'_\omega = p_\omega\} = \{\omega \in \Psi_{\to i} : p^n_\omega = p_\omega\} \subseteq \Phi^n_{\to i}$ and $\Phi^n_{i\to} \subseteq \Psi_{i\to}$.

Since the set of trades is finite, it is without loss of generality to assume that there is a set of trades $\Phi^* \in \Omega_i$ and an integer $\bar{n}$ such that $\Phi^n = \Phi^*$ for all $n \geq \bar{n}$. Since i's utility function is continuous with respect to prices and $p^n \to p^\Delta$, we must have $U_i([\Phi^*; p^\Delta]) \geq U_i([\Psi; p^\Delta])$. Since $\Psi \in D_i(p^\Delta)$, this implies $\Phi^* \in D_i(p^\Delta)$. Since $\Delta < p'_\psi - p_\psi$ and V_i is decreasing in the prices of trades for which i is a buyer, we must have $V_i(p^\Delta) \geq V_i(p')$. Since $\psi \notin \Phi^*$, we have that $U_i([\Phi^*; p']) = U_i([\Phi^*; p^\Delta]) = V_i(p^\Delta)$. Hence, $\Phi^* \in D_i(p')$ and setting $\Psi' \equiv \Phi^*$ yields a set that satisfies the first condition of Definition A.4.

Proof of Theorem 8

The proof is an adaptation of the proof of Theorem 1 of Gul and Stacchetti (1999). Since we impose neither monotonicity nor boundedness conditions on valuation functions, there are a number of details needed in order to check that Gul and Stacchetti (1999) proof strategy works in our setting.

Throughout the proof, for any price vector $p \in \mathbb{R}^\Omega$, we denote by $\tilde{D}_i(p)$ the sets of objects that correspond to the optimal sets of trades in $D_i(p)$.

We show first that the single improvement property in object-language implies the no complementarities condition. Let p be an arbitrary price vector and $\mathbf{\Phi}, \mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary. Let $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \smallsetminus \mathbf{\Phi}$ be arbitrary. Let $\mathbf{\Xi} \in \tilde{D}_i(p)$ be a set of objects such that $\mathbf{\Xi} \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$ and $\mathbf{\Psi} \smallsetminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}$, and such that there is no $\mathbf{\Xi}' \in \tilde{D}_i(p)$ for which $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$, $\mathbf{\Psi} \setminus \bar{\mathbf{\Psi}} \subseteq \mathbf{\Xi}'$, and $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$. If $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \varnothing$, we are done. If not, let Π be a very large number[31] and define $p(\varepsilon)$ by setting $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = \Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{\to i} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = -\Pi$ if $\boldsymbol{\omega} \in \mathbf{\Omega}_{i \to} \smallsetminus (\mathbf{\Phi} \cup \mathbf{\Psi})$, $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})}$ if $\boldsymbol{\omega} \in (\mathbf{\Phi} \cup \mathbf{\Psi}) \smallsetminus \bar{\mathbf{\Psi}}$, and $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})} + \varepsilon$ if $\boldsymbol{\omega} \in \bar{\mathbf{\Psi}}$. Note that for all $\varepsilon > 0$ we must have $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$ (since $\bar{\mathbf{\Psi}} \subseteq \mathbf{\Psi} \setminus \mathbf{\Phi}$) and $U_i([\mathbf{\Phi}; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Since $\mathbf{\Xi} \in \tilde{D}_i(p)$, we must have $u_i(\mathbf{\Xi}) \neq -\infty$. Hence, we can apply the single improvement property (in object-language) to infer that there must exist a set of objects $\mathbf{\Xi}'$ such that $|\mathbf{\Xi}' \setminus \mathbf{\Xi}| \leq 1$, $|\mathbf{\Xi} \setminus \mathbf{\Xi}'| \leq 1$, and $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$. Given the definition of $p(\varepsilon)$ and Π, we must have $\mathbf{\Xi}' \subseteq \mathbf{\Phi} \cup \mathbf{\Psi}$. Since $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ holds for arbitrarily small values of ε, we must have $\mathbf{\Xi}' \in \tilde{D}_i(p)$. But $U_i([\mathbf{\Xi}'; p(\varepsilon)]) > U_i([\mathbf{\Xi}; p(\varepsilon)])$ is equivalent to $U_i([\mathbf{\Xi}'; p]) - |\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}|\varepsilon > U_i([\mathbf{\Xi}; p]) - |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|\varepsilon$. Given that $\mathbf{\Xi}, \mathbf{\Xi}' \in \tilde{D}_i(p)$, the last inequality is equivalent to $|\mathbf{\Xi}' \cap \bar{\mathbf{\Psi}}| < |\mathbf{\Xi} \cap \bar{\mathbf{\Psi}}|$ and we thus obtain a contradiction to the definition of $\mathbf{\Xi}$. Hence, it has to be the case that $\mathbf{\Xi} \cap \bar{\mathbf{\Psi}} = \varnothing$ and this completes the proof that single improvement implies no complementarities.

Next, we show that the generalized no complementarities condition implies object-language full substitutability. Let p, p' be two price vectors such that $p \leq p'$. Let $\mathbf{\Psi} \in \tilde{D}_i(p)$ be arbitrary.[32] Let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega} \in \mathbf{\Omega}_i : p_{\mathsf{t}(\boldsymbol{\omega})} < p'_{\mathsf{t}(\boldsymbol{\omega})}\}$. The proof will proceed by induction on $|\tilde{\mathbf{\Omega}}_i|$. Consider first the case of $|\tilde{\mathbf{\Omega}}_i| = 1$ and let $\tilde{\mathbf{\Omega}}_i = \{\boldsymbol{\omega}\}$. Clearly, if $\boldsymbol{\omega} \notin \mathbf{\Psi}$ or $\mathbf{\Psi} \in \tilde{D}_i(p')$, there is nothing to show. So suppose that $\boldsymbol{\omega} \notin \mathbf{\Psi}$ and that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. For any $\varepsilon \geq 0$, define a price vector $p(\varepsilon)$ by setting $p_{\mathsf{t}(\boldsymbol{\varphi})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$, and $p_{\mathsf{t}(\boldsymbol{\omega})}(\varepsilon) = p_{\mathsf{t}(\boldsymbol{\omega})} + \varepsilon$. Let $\bar{\varepsilon} = \max\{\varepsilon : \mathbf{\Psi} \in \tilde{D}_i(p(\varepsilon))\}$ and note that $\bar{\varepsilon} < p'_{\mathsf{t}(\boldsymbol{\omega})} - p_{\mathsf{t}(\boldsymbol{\omega})}$ given that $\mathbf{\Psi} \notin \tilde{D}_i(p')$. Consider some $\varepsilon > \bar{\varepsilon}$ and fix a set of objects $\mathbf{\Phi} \in \tilde{D}_i(p(\varepsilon))$. It is easy to see that $\boldsymbol{\omega} \notin \mathbf{\Phi}$ and that $\mathbf{\Phi} \in \tilde{D}_i(p(\bar{\varepsilon}))$. By the generalized no complementarities condition, there must exist a set of objects $\mathbf{\Xi} \subseteq \mathbf{\Phi}$ such that $\mathbf{\Psi}' := \mathbf{\Psi} \setminus \{\boldsymbol{\omega}\} \cup \Xi \in \tilde{D}_i(p(\bar{\varepsilon}))$. Clearly, we must also have $\mathbf{\Psi}' \in \tilde{D}_i(p')$ and this completes the proof in case of $|\tilde{\mathbf{\Omega}}_i| = 1$. Now suppose that the statement has already been established for all pairs of price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| \leq K$ for some $K \geq 1$. Consider two price vectors p, p' such that $|\tilde{\mathbf{\Omega}}_i| = K + 1$. Fix a set of objects $\mathbf{\Psi} \in \tilde{D}_i(p)$. Let $\boldsymbol{\omega} \in \tilde{\mathbf{\Omega}}_i$ be arbitrary and consider a

[31]For instance, let

$$\Delta = \max_{\Omega_1 \subset \Omega_i, \Omega_2 \subset \Omega_i, u_i(\Omega_1) > -\infty, u_i(\Omega_2) > -\infty} |U_i([\Omega_1; p]) - U_i([\Omega_2; p])|,$$

and $\Pi = 1 + \Delta + \max_{\omega \in \Omega_i} |p_\omega|$.

[32]There is no need to rule out the possibility of several optimal bundles of objects in this proof.

price vector p'' such that $p''_{\mathfrak{t}(\boldsymbol{\omega})} = p_{\mathfrak{t}(\boldsymbol{\omega})}$ and $p''_{\mathfrak{t}(\boldsymbol{\varphi})} = p'_{\mathfrak{t}(\boldsymbol{\varphi})}$ for all $\boldsymbol{\varphi} \neq \boldsymbol{\omega}$. By the inductive assumption, there is a set $\boldsymbol{\Psi}'' \in \tilde{D}_i(p'')$ such that $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p''_{\mathfrak{t}(\boldsymbol{\varphi})} = p_{\mathfrak{t}(\boldsymbol{\varphi})}\} \subseteq \boldsymbol{\Psi}''$. Note that $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p'_{\mathfrak{t}(\boldsymbol{\varphi})} = p_{\mathfrak{t}(\boldsymbol{\varphi})}\} = \{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p''_{\mathfrak{t}(\boldsymbol{\varphi})} = p_{\mathfrak{t}(\boldsymbol{\varphi})}\} \setminus \{\boldsymbol{\omega}\}$. Applying the inductive assumption one more time, there has to be a set $\boldsymbol{\Psi}' \in \tilde{D}_i(p')$ such that $\boldsymbol{\Psi}'' \setminus \{\boldsymbol{\omega}\} \subseteq \boldsymbol{\Psi}'$. Combining this with the previous arguments, we obtain $\{\boldsymbol{\varphi} \in \boldsymbol{\Psi} : p'_{\mathfrak{t}(\boldsymbol{\varphi})} = p_{\mathfrak{t}(\boldsymbol{\varphi})}\} \subseteq \boldsymbol{\Psi}'$. This completes the proof.

Proof of Theorem 9

As Ω is finite and non-empty, for each agent i the domain of u_i is bounded and non-empty. Hence, by Part (b) of Theorem 7 of Murota and Tamura (2003), we see that u_i is $M^\natural$-concave over objects if and only if the preferences of i have the single-improvement property.[33] The result then follows from Theorem 7.

Proof of Theorem 10

We prove the Law of Aggregate Demand; the proof of the Law of Aggregate Supply is analogous.

Fix a fully substitutable valuation function u_i for agent i. Take two finite sets of contracts Y and Y' such that $|C_i(Y)| = |C_i(Y')| = 1$, $Y_{i\to} = Y'_{i\to}$, and $Y_{\to i} \subseteq Y'_{\to i}$. Assume that for any $\omega \in \Omega_{i\to}$, $(\omega, r) \in Y_{i\to}$ and $(\omega, r') \in Y_{i\to}$ implies $r = r'$ (this is without loss of generality, because for a given trade in $\Omega_{i\to}$, agent i, as a seller, can only choose a contract with the highest price available for that trade, and thus we can disregard all other contracts involving that trade). Let $W \in C_i(Y)$ and $W' \in C_i(Y')$. Define a modified valuation $\tilde{u}_i$ on $\tau(Y'_i)$ for agent i by setting, for each $\Psi \subseteq \tau(Y'_i)$,

$$\tilde{u}_i(\Psi) = u_i(\Psi_{\to i} \cup (\tau(Y') \smallsetminus \Psi)_{i\to}).$$

Let $\tilde{C}_i$ denote the choice correspondence associated to $\tilde{u}_i$. By construction,

$$\tilde{u}_i(\Psi) = u_i(\mathfrak{o}_i(\Psi)), \tag{8}$$

[33]Strictly speaking, Theorem 7(b) shows the equivalence of $M^\natural$-convexity and the (M$^\natural$-SI) property of a function f. It is, however, immediate that this result implies the equivalence of $M^\natural$-concavity and the single-improvement property for a function $g = -f$.

where here the object operator is defined with respect to underlying set of trades $\tau(Y')$:

$$\mathfrak{o}_i(\Psi) = \{\mathfrak{o}(\omega) : \omega \in \Psi_{\to i}\} \cup \{\mathfrak{o}(\omega) : \omega \in \tau(Y') \smallsetminus \Psi_{i\to}\}.$$

As the preferences of i are fully substitutable, the restriction of those preferences to $\tau(Y')$ is fully substitutable, as well. Object-language full substitutability of those preferences, as well as (8), together imply that $\tilde{u}_i$ satisfies the gross substitutability condition of Kelso and Crawford (1982).

Now, we must have $\tilde{C}_i(Y) = \{W_{\to i} \cup (Y' \smallsetminus W)_{i\to}\}$ and $\tilde{C}_i(Y') = \{W'_{\to i} \cup (Y' \smallsetminus W')_{i\to}\}$. As we assume quasilinearity, the Law of Aggregate Demand for two-sided markets applies to $\tilde{C}_i$ (by Theorem 7 of Hatfield and Milgrom (2005)). As $Y \subseteq Y'$, this implies that $|W'_{\to i} \cup (Y' \smallsetminus W')_{i\to}| \geq |W_{\to i} \cup (Y' \smallsetminus W)_{i\to}|$. The last inequality is equivalent to $|W'_{\to i}| - |W_{\to i}| \geq |W'_{i\to}| - |W_{i\to}|$, which is precisely the Law of Aggregate Demand.

The proof that the Law of Aggregate Demand for the case in which choice correspondences are single-valued implies the more general case in which they can be multi-valued is analogous to the proof of the implication (DFS)⇒(DEFS) of Theorem B.1.

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