The Demise of Walk Zones in Boston: Priorities vs. Precedence in School Choice*

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Abstract

School choice plans in many cities grant students higher priority for some (but not all) seats at their neighborhood schools. This paper demonstrates how the precedence order, i.e. the order in which different types of seats are filled by applicants, has quantitative effects on distributional objectives comparable to priorities in the deferred acceptance algorithm. While Boston's school choice plan gives priority to neighborhood applicants for half of each school's seats, the intended effect of this policy is lost because of the precedence order. Despite widely held impressions about the importance of neighborhood priority, the outcome of Boston's implementation of a 50-50 school split is nearly identical to a system without neighborhood priority. We formally establish that either increasing the number of neighborhood priority seats or lowering the precedence order positions of neighborhood seats at a school have the same effect: an increase in the number of neighborhood students assigned to the school. We then show that in Boston a reversal of precedence with no change in priorities covers almost three-quarters of the range between 0% and 100% neighborhood priority. Therefore, decisions about precedence are inseparable from decisions about priorities. Transparency about these issues—in particular, how precedence unintentionally undermined neighborhood priority—led to the abandonment of neighborhood priority in Boston in 2013.

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1 Introduction

School choice programs aspire to weaken the link between the housing market and access to good schools. In purely residence-based public school systems, families can purchase access by moving to neighborhoods with desirable schools. Children from families less able to move do not have the same opportunity. With school choice, children are allowed to attend schools outside their neighborhoods without having to move homes. This may generate a more equitable distribution of school access.

Enabling choice requires specifying how school seats will be rationed among students. During the 1970s, court rulings on the appropriate balance of neighborhood and non-neighborhood assignment, often drawn on racial and ethnic lines, influenced the shape of urban America (Baum-Snow and Lutz 2011, Boustan 2012). The debate about the appropriate balance continues today throughout the design of school choice programs.

The initial literature on school choice mechanism design took students' property rights over school seats as primitives and focused on how these property rights are interpreted by the assignment mechanism (Balinski and Sönmez 1999, Abdulkadiroğlu and Sönmez 2003). Efforts in the field also avoided taking positions on how to endow agents with claims to schools, and instead advocated for strategy-proof mechanisms, which simplify applicant ranking decisions.[1] Now, with the growing use of mechanisms based on the student-proposing deferred acceptance algorithm (DA), it is possible to more explicitly consider the role of students' property rights.[2] By setting school **priorities**, such as rules giving higher claims to neighborhood applicants, districts using DA can precisely define applicants' property rights in a way that is independent of demand for school seats.

Specifying priorities is only one part of determining students' access to schools. Another component involves determining the fraction of seats at which given priorities apply. For the last thirteen years, Boston Public Schools (BPS) split schools' priority structures into two equally-sized pieces, with one half of the seats at each school giving students from that school's neighborhood priority over all other students, and half of the seats not giving neighborhood students priority.[3] When students in Boston rank a school in their preference list, they are considered for both types of seats.[4] The order in which the slots are processed, the slots' **precedence** order, determines how the seats are filled by applicants.

BPS's current 50-50 seat split emerged out of a city-wide discussion following the end of racial and ethnic criteria for school placement in 1999. Many advocated abandoning choice and returning

[1] For example, a December 2003 community engagement process in Boston considered six different proposals for alternative neighborhood zone definitions. However, the only recommendation adopted by the school committee was to switch the assignment algorithm (Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005).

[2] Other mechanisms often lack this complete separation. For instance, in the pre-2005 Boston mechanism, applicants' preference rankings first determined whose claims were justified; priorities were only used to adjudicate claims among equal-ranking applicants.

[3] The 50-50 school seat split was not altered when Boston changed their assignment mechanism in 2005 to one based on the student-proposing deferred acceptance algorithm (Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2006, Pathak and Sönmez 2008).

[4] Throughout this paper, we use the terms "slot" and "seat" interchangeably.

to neighborhood schools at that point, but the school committee chose only to reduce the fraction of seats where neighborhood, i.e. "walk-zone" priority, applies from 100% to 50% of seats within each school. The official policy document states (BPS 1999):

> Fifty percent walk zone preference means that half of the seats at a given school are subject to walk zone preference. The remaining seats are open to students outside of the walk zone.
>
> RATIONALE: One hundred percent walk zone preference in a controlled choice plan without racial guidelines could result in all available seats being assigned to students within the walk zone. The result would limit choice and access for all students, including those who have no walk zone school or live in walk zones where there are insufficient seats to serve the students residing in the walk zone.
>
> Patterns of parent choice clearly establish that many choose schools outside of their walk zone for many educational and other reasons. [...] One hundred percent walk zone preference would limit choice and access for too many families to the schools they want their children to attend. On the other hand, the policy also should and does recognize the interests of families who want to choose a walk zone school.
>
> Thus, I believe fifty percent walk zone preference provides a fair balance.

The 50-50 slot split was seen as "striking an uneasy compromise between neighborhood school advocates and those who want choice," while the Superintendent hoped that the "plan would satisfy both factions, those who want to send children to schools close by and those who want choice" (Daley 1999).

In this paper, we illustrate how both priorities and precedence are important for achieving distributional objectives in matching problems, including those described in BPS's policy statement. Although there are multiple possible interpretations, it is clear that the goal of Boston's school committee was not to completely eliminate the role of walk-zone status in student assignment. Indeed, if that were the intention, there would be no need to give any students walk-zone priority. Using data on students' choices and assignments in BPS, however, we show that BPS's policy, as implemented in practice, led to an outcome almost identical to the outcome that would arise if walk-zone priority were not used at all. We compare the BPS outcome to two extreme alternatives: one where none of the seats have walk-zone priority, and one where all seats have walk-zone priority. Table 1 shows that the outcome of the BPS mechanism is far from being a "compromise" be at the midpoint of these two extremal policies. Instead, it is nearly the same as if none of the seats had walk-zone priority.

Despite the perception that walk-zone applicants had been advantaged in the BPS system since 1999, they appear to have had little advantage in practice. Only 3% of Grade K1 (a main elementary school entry point) applicants obtained a different assignment under Boston's implementation than they would under open competition without walk-zone priority, as indicated in the column labeled 0% Walk. The difference is as low as 1% for Grade 6. Furthermore, this pattern is not simply a feature of student demand. Under the alternative in which all seats have walk-zone priority (labeled

100% Walk), the number of students assigned to schools in their walk zones increases to 19% and 17% for Grades K1 and K2, respectively. Although motivated as a compromise between the two factions, BPS's 50-50 school seat split is significantly closer to open competition than is at first apparent.

Why does Boston's assignment mechanism result in an assignment so close to one without any neighborhood priority, even though half of each school's seats give priority to neighborhood students? This paper provides an answer. We develop a theoretical framework for school choice mechanism design in which both priority and precedence play key roles. We show that the division of schools into walk-zone and open priority seats reveals little about the proximity of the outcome to a compromise between extremes without specifying the precedence order, which determines what happens when a student has claims for *both* walk-zone and open seats. Building on the work of Kominers and Sönmez (2012), we establish two new comparative statics:

1. Given a fixed slot precedence order, replacing an open slot at a school with a walk-zone slot weakly increases the number of walk-zone students assigned to that school.

2. Given a fixed split of slots into walk-zone and open slots, switching the precedence order position of a walk-zone slot with that of a subsequent open slot weakly increases the number of walk-zone students assigned to that school.

While the first of these results is intuitive, the second one is more subtle. Moreover, neither result follows from earlier comparative static approches used in simpler models (e.g., Balinski and Sönmez (1999)) because they involve simultaneous priority improvements for a large number of students. As a result, they involve developing a novel formal approaches, which may be useful in other environments which, like our model, have slot-specific priority structure. In a specialization of our model to the case of two-schools, our comparative statics can be sharpened to show that the types of priority and precedence order changes described above in fact increase neighborhood assignment at all schools. The impact in this case is entirely distributional—both instruments leave the aggregate number of students obtaining their top-choice schools unchanged.

We empirically examine the extent to which the comparative statics from our simplified model are relevant under the richer priority structure in Boston's school choice program. After demonstrating that BPS's current implementation of the 50-50 split is far from balancing the concerns of the neighborhood schooling and school choice proponents, we show that an alternative precedence order in which open slots are depleted before walk-zone slots results in 8.2% more students attending walk-zone schools in Grade K1. This represents nearly three-quarters of the maximal achievable difference between completely eliminating walk-zone priority and having walk-zone priority apply at all school seats.

We also examine alternative precedence orders that implement policies between the 0% Walk and 100% Walk extremes. Once a preliminary version of this paper was circulated, it entered the policy discussion in Boston. When our work clarified the role of the precedence order in undermining the intended effects of the 50-50 seat split, BPS completely eliminated neighborhood priority.

This paper contributes to a broader agenda, examined in a number of recent papers, that introduces concerns for diversity into the literature on school choice mechanism design (see, e.g., Budish, Che, Kojima, and Milgrom (2013), Echenique and Yenmez (2012), Erdil and Kumano (2012), Hafalir, Yenmez, and Yildirim (2012), Kojima (2012), and Kominers and Sönmez (2012)). When an applicant ranks a school with many seats, it is similar to expressing indifference among the school's seats. Therefore, our work parallels recent papers examining the implications of indifferences in school choice problems (Erdil and Ergin 2008, Abdulkadiroğlu, Pathak, and Roth 2009, Pathak and Sethuraman 2011). However, the question of school-side indifferences, the focus of prior work, is entirely distinct from the issue of indifferences in student preferences. Tools used to resolve indifferences in schools's priorities (e.g., random lotteries) do not immediately apply to the case of student-side indifferences. Our work is related to that of Roth (1985), which shows how to interpret a (Gale and Shapley 1962) college admissions (many-to-one) matching model as a marriage (one-to-one) matching model by splitting colleges into individual seats and assuming that students rank those seats in a given order. Our results show that implementation of this seat-split approach without attention to precedence can undermine priority policies. Finally, this paper builds on the theoretical literature on matching with contracts (Crawford and Knoer 1981, Kelso and Crawford 1982, Hatfield and Milgrom 2005, Ostrovsky 2008, Hatfield and Kojima 2010, Echenique 2012) and the applied motivation shares much with recent work on matching in the military (Sönmez and Switzer 2013, Sönmez 2013).

Our paper proceeds as follows. Section 2 introduces the model and illustrates the roles of precedence and priority. Section 3 reports on our empirical investigation of these issues in the context of Boston's school choice plan. Section 4 briefly reviews the debate in Boston and describes how the present paper affected the debate. Section 5 concludes. All proofs are relegated to the Appendix.

2 Model

There is a finite set I of students and a finite set A of schools. Each school a has a finite set of slots S^a. We use the notation a_0 to denote a "null school" representing the possibility of being unmatched; we assume that this option is always available to all students. Let $S \equiv \bigcup_{a \in A} S^a$ denote the set of all slots (excluding those at the null school). We assume that $|S| \geq |I|$, so that there are enough (real) slots for all students. Each student i has a strict preference relation P^i over $A \cup \{a_0\}$. Throughout the paper we fix the set of students I, the set of schools A, the set of schools' slots S, and the students' preferences $(P^i)_{i \in I}$.

For a school $a \in A$, each slot $s \in S^a$ has a linear priority order π^s over students in I. This linear priority order captures the "property rights" of the students for this slot in the sense that the higher a student is ranked under π^s, the stronger claims he has for the slot s of school a. Following the BPS practice, we allow slot priorities to be heterogeneous across slots of a given school. A subtle consequence of this within-school heterogeneity is that we must determine how slots are assigned when a student is "qualified" for multiple slots with different priorities at a school. The last primitive of the model regulates this selection: For each school $a \in A$, the slots in S^a are ordered according to a (linear) **order of precedence** $\rhd^a$. Given a school $a \in A$ and two of its slots $s, s' \in S^a$, the expression $s \rhd^a s'$ means that slot s is to be filled before slot s' at school a whenever possible.

A **matching** $\mu : I \to A$ is a function which assigns a school to each student such that no school is assigned to more students than its total number of slots. Let μ_i denote the assignment of student i, and let μ_a denote the set of students assigned to school a.

Our model generalizes the school choice model of Abdulkadiroğlu and Sönmez (2003) in that it allows for heterogenous priorities across the slots of a given school. Nevertheless, a mechanism based on the celebrated *student-proposing deferred acceptance algorithm* (Gale and Shapley 1962) easily extends to this model once the *choice function* of each school is constructed from the slot priorities and order of precedence.

Given a school $a \in A$ with a set of slots S^a, a list of slot priorities $(\pi^s)_{s \in S^a}$, an order of precedence $\rhd^a$ with

$$s_a^1 \rhd^a s_a^2 \rhd^a \cdots \rhd^a s_a^{|S^a|},$$

and a set of students $J \subseteq I$, the **choice of school *a* from the set of students *J***, denoted by $C^a(J)$, and is obtained as follows: Slots at school a are filled one at a time following the order of precedence $\rhd^a$. The highest priority student in J under $\pi^{s_a^1}$, say student j_1, is chosen for slot s_a^1 of school a; the highest priority student in $J \setminus \{j_1\}$ under $\pi^{s_a^2}$ is chosen for slot s_a^2 of school a, and so on.

For a given list of slot priorities $(\pi^s)_{s \in S}$ and an order of precedence $\rhd^a$ at each school $a \in A$, the outcome of the **student-proposing deferred acceptance mechanism (DA)** can be obtained as follows:

***Step* 1:** Each student i applies to her top choice school. Each school a with a set of Step 1 applicants J_1^a tentatively holds the applicants in $C^a(J_1^a)$, and rejects the rest.

***Step* ℓ:** Each student rejected in Step $\ell - 1$ applies to her most-preferred school (if any) that has not yet rejected her. Each school a considers the set J_ℓ^a consisting of the new applicants to a and the students held by a at the end of Step $\ell - 1$, tentatively holds the applicants in $C^a(J_\ell^a)$, and rejects the rest.

The algorithm terminates after the first step in which no students are rejected, assigning students to the schools holding their applications.

2.1 A Mix of Neighborhood-Based and Open Priority Structures

In this paper we are particularly interested in the slot priority structure used at Boston Public Schools. There is a master priority order π^o that is uniform across all schools. This master priority order is obtained via an even lottery often referred to as the **random tiebreaker**. At each school in Boston, slot priorities depend on students' walk-zone and sibling statuses and the random tiebreaker

π^o. For our theoretical analysis, we consider a simplified priority structure which only depends on walk-zone status and the random tiebreaker. Using data from BPS in Section 3, we show that this is a good approximation for BPS.

For any school $a \in A$, there is a subset $I_a \subset I$ of **walk-zone students**. There are two types of slots:

1. **Walk-zone** slots: For each walk-zone slot at a school a, any walk-zone student $i \in I_a$ has priority over any non-walk-zone student $j \in I \setminus I_a$, and the priority order within these two groups is determined according to the random tiebreaker π^o.

2. **Open** slots: $\pi^s = \pi^o$ for each open slot s.

For any school $a \in A$, define S^a_w to be the set of walk-zone slots and S^a_o to be the set of open slots. BPS currently uses a priority structure in which half of the slots at each school are walk-zone slots, while the remaining half are open slots. As described in the introduction, this structure has been historically interpreted as a compromise between the proponents of neighborhood assignment and the proponents of school choice.

An important comparative statics exercise concerns the impact of replacing an open slot with a walk-zone slot under DA for a given order of precedence. One might naturally expect such a change to weakly increase the number of students who are assigned to their walk-zone schools. Surprisingly, this intuition is not correct in general, as we show in the next example.

Example 1. There are four schools $A = \{k, l, m, n\}$. Each school has two available slots. There are eight students $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$. There are two walk-zone students at each school: $I_k = \{i_1, i_2\}$, $I_l = \{i_3, i_4\}$, $I_m = \{i_5, i_6\}$ and $I_n = \{i_7, i_8\}$. The random tiebreaker π^o orders the students as:

$$\pi^o : i_1 \succ \ i_8 \succ i_3 \succ i_4 \succ i_5 \succ i_6 \succ i_7 \succ i_2.$$

The preference profile is:

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}	P^{i_5}	P^{i_6}	P^{i_7}	P^{i_8}
k	k	l	l	m	m	n	k
l	l	k	k	k	k	k	l
m	m	m	m	l	l	l	m
n	n	n	n	n	n	m	n

First consider the case where each school has one walk-zone slot and one open slot. Also assume that the walk-zone slot has higher precedence than the open slot at each school.

The outcome of DA for this case is:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ k & n & l & l & m & m & n & k \end{pmatrix}. \tag{1}$$

Observe that six students (i.e. students $i_1, i_3, i_4, i_5, i_6, i_7$) are assigned to their walk-zone schools in this scenario.

Next we replace the open slot at school k with a walk-zone slot, so that both slots at school k are walk-zone slots. Each remaining school has one walk-zone slot and one open slot, with the walk-zone slot having higher precedence than the open slot.

The outcome of DA for the second case is:

$$\mu' = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ k & k & l & m & m & n & n & l \end{pmatrix}. \tag{2}$$

Observe that five students (i.e. students i_1, i_2, i_3, i_5, i_7) are assigned to their walk-zone schools in the second case—the total number of walk-zone assignments decreases when the open slot at school k is replaced with a walk-zone slot. □

A small modification of Example 1 illustrates that the difficulty persists even if open slots are replaced by walk-zone slots at all schools. To see this, suppose school k has one walk-zone slot and one open slot, with the walk-zone slot's precedence higher than the open slot's, and that schools l, m, and n each have two open slots. The outcome is the same as in (1). If we replace the open slot at school k, with a walk-zone slot, and replace the higher-precedence open slots at schools l, m, and n with walk-zone slots, then the outcome is the same as (2).

Despite these negative findings, the following proposition shows that the replacement of an open slot of school a with a walk-zone slot weakly increases the number of walk-zone students assigned to school a (even though it may decrease the total number of walk-zone assignments).

Proposition 1. *For any given order of precedence of slots, replacing an open slot of school a with a walk-zone slot weakly increases the number of walk-zone students who are assigned to school a under DA.*

When a school district increases the fraction of walk-zone slots, one objective behind this change is to increase the fraction of students assigned to walk-zone schools. As Proposition 1 shows, replacing an open slot with a walk-zone slot serves this goal through its "first-order effect" in the school directly affected by the change, although the overall effect across all schools might in theory be in the opposite direction.[5] Nevertheless, our empirical analysis in Section 3 suggests that the first-order effect dominates—the overall effect of an increase in the number of walk-zone slots is in the expected direction.

While the role of the number of walk-zone slots as a policy tool is quite clear, the role of the order of precedence is much more subtle. Indeed, the choice of the order of precedence is often considered a minor technical detail—and, to our knowledge, precedence has never entered policy discussions until the present work. Nevertheless, precedence significantly affects outcomes.

[5]A natural question is whether there is a relationship between the set of students who are assigned to school a when an open slot is changed to a walk-zone slot. Example C1 in the appendix shows that with this change, the set of students are disjoint. Another question is whether all walk-zone students at school a are weakly better off when an open slot is changed to a walk-zone slot. Example C2 shows that a walk-zone student can actually be worse off in this situation.

Qualitatively, the effect of decreasing the precedence order position of a walk-zone slot is similar to the effect of replacing an open slot with a walk-zone slot. While this may appear counter-intuitive at first, the reason is simple: By decreasing the precedence of a walk-zone slot, one increases the odds that a walk-zone student who has a lottery number high enough to make her eligible for both open and walk-zone slots is assigned to an open slot. This in turn increases the competition for the open slots and decreases the competition for walk-zone slots. Our next result formalizes this observation.

Proposition 2. *Fix the set of walk-zone slots and the set of open slots at each school. Then, switching the precedence order position of a walk-zone slot of school a with the position of a lower-precedence open slot weakly increases the number of walk-zone students assigned to school a under DA.*

Given Example 1, it is not surprising the aggregate effect of lowering walk-zone slot precedence may go against the "first order" effect. We now present a modified version of Example 1 that makes this point.

Example 2. To illustrate the conceptual relation between priority swaps and changes in the order of precedence, we closely follow Example 1. The only difference is a small modification in the second case.

There are four schools $A = \{k, l, m, n\}$. Each school has two available slots. There are eight students $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$. There are two walk-zone students at each school: $I_k = \{i_1, i_2\}$, $I_l = \{i_3, i_4\}$, $I_m = \{i_5, i_6\}$ and $I_n = \{i_7, i_8\}$. The random tiebreaker π^o orders the students as:

$$\pi^o : i_1 \succ i_8 \succ i_3 \succ i_4 \succ i_5 \succ i_6 \succ i_7 \succ i_2.$$

The preference profile is:

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}	P^{i_5}	P^{i_6}	P^{i_7}	P^{i_8}
k	k	l	l	m	m	n	k
l	l	k	k	k	k	k	l
m	m	m	m	l	l	l	m
n	n	n	n	n	n	m	n

First consider the case where each school has one walk-zone slot and one open slot. Also assume that, at each school, the walk-zone slot has higher precedence than the open slot.

The outcome of DA for this case is:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ k & n & l & l & m & m & n & k \end{pmatrix}.$$

Observe that six students (i.e. students $i_1, i_3, i_4, i_5, i_6, i_7$) are assigned to their walk-zone schools in this scenario.

Next, we change the order of precedence at school k so that the open slot has higher precedence than the walk-zone slot. Each remaining school maintains the original order of precedence with the walk-zone slot higher precedence than the open slot.

The outcome of DA for the second case is:

$$\mu' = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ k & k & l & m & m & n & n & l \end{pmatrix}.$$

Observe that five students (i.e. students i_1, i_2, i_3, i_5, i_7) are assigned to their walk-zone schools in the second case. Thus, we see that the total number of walk-zone assignments decreases following reduction in the precedence of the walk-zone slot of school k. $\square$

2.2 Additional Results for the Case of Two Schools

In this section, we obtain sharper theoretical results for the case of two schools ($|A| = 2$). This case is motivated in part by the commonly discussed policy objective of giving students from poorer neighborhoods access to desirable schools in richer neighborhoods. We assume that each student belongs to exactly one walk zone and that students rank both schools.

Proposition 3. *Suppose that there are two schools, that each student belongs to exactly one walk zone, and that students rank both schools. Then, replacing an open slot of either school with a walk-zone slot weakly increases the total number of students assigned to their walk-zone schools under DA.*

An immediate implication of Proposition 3 is the following intuitive result justifying why the school choice and neighborhood schooling lobbies respectively prefer 0% and 100% walk-zone priority.

Corollary 1. *Suppose that there are two schools, that each student belongs to exactly one walk zone, and that students rank both schools. Under DA (holding fixed the number of slots at each school):*

- *The minimum number of walk-zone assignments across all priority and precedence policies is obtained when all slots have open slot priority, and*
- *the maximum number of walk-zone assignments across all priority and precedence policies is obtained when all slots have walk-zone priority.*

Proposition 4. *Suppose that there are two schools, that each student belongs to exactly one walk zone, and that students rank both schools. Fix the set of walk-zone slots and the set of open slots at each school. Then, switching the order of precedence position of a walk-zone slot at either school with that of a subsequent open slot at that school weakly increases the total number of students assigned to their walk-zone schools under DA.*

Our empirical analysis in Section 3 shows that the fraction of students who receive their first choices, second choices, and so forth show virtually no response to changes in the fraction of walk-zone slots or the order of precedence. Our final theoretical result provides a basis for this empirical observation.

Proposition 5. *Suppose that there are two schools, that each student belongs to exactly one walk zone, and that students rank both schools. Then, the number of students assigned to their top-choice schools is independent of both the number of walk-zone slots and the choice of precedence order.*

An important policy implication of our last result is that the division of slots between walk-zone priority and open priority as well the order of precedence selection has little bearing on the aggregate number of students who receive their top choices; thus, the impact of these DA calibrations on student welfare is predominantly distributional.

3 Precedence and Priority in Boston Public Schools

3.1 50-50 vs. No Slot Split

To examine whether our theoretical comparative static predictions capture the main features of school choice with richer priority structures, we use data on submitted preferences from Boston Public Schools. Relative to our two-priority-type model, Boston has three additional priority groups:

1. **guaranteed applicants**, who are typically continuing on at their current schools,
2. **sibling-walk applicants**, who have siblings currently attending a school and live in the walk zone, and
3. **sibling applicants**, who have siblings attending a school and live outside the walk zone.

Under BPS's slot priorities, applicants are ordered as follows:

Walk-Zone Slots	Open Slots
Guaranteed	Guaranteed
Sibling-Walk Sibling	Sibling-Walk, Sibling
Walk No Priority	Walk, No Priority

A single random lottery number is used to order students within priority groups, and this number is the same for both types of slots.

We use data covering four years from 2009–2012, when BPS employed a mechanism based on the student-proposing deferred acceptance algorithm. Students interested in enrolling in or switching schools are asked to list schools each January for the first round. Students entering kindergarten can either apply for elementary school at Grade K1 or Grade K2 depending on whether they are four or five years old. Since the mechanism is based on the student-proposing deferred acceptance algorithm and there is no restriction on the number of schools that can be ranked, the assignment mechanism is strategy-proof.[6] BPS informs families of this property on the application form, advising:

> List your school choice in your true order of preference. If you list a popular school first, you won't hurt your chances of getting your second choice school if you don't get your first choice (BPS 2012).

Since the BPS mechanism is strategy-proof, we can isolate the effects of changes in priorities and precedence by holding submitted preferences fixed.[7]

The puzzle that motivated this paper is shown in Table 1, which reports a comparison of the assignment produced by BPS, under a 50-50 split of slots, to two extreme alternatives representing the ideal positions of the school choice and neighborhood school factions: (1) a priority structure without walk-zone priorities at any slot and (2) a priority structure where walk-zone priority applies at all slots. We refer to these two extremes as **0% Walk** and **100% Walk**, and we compute their outcomes using the same lottery numbers as BPS. Table 1 shows that the BPS assignment is nearly identical to the former of these two alternatives; it differs for only 3% of Grade K1 students. One might suspect that this phenomenon is driven by a strong preferences for neighborhood schools among applicants, which would bring the outcomes of these two assignment policies close together. However, comparing the 100% Walk outcome to BPS outcome, 19% of Grade K1 students obtain a different assignment. Therefore, the remarkable proximity of the current BPS outcome to the ideal of school choice proponents does not suggest (or reflect) negligible stakes in school choice.

For Grades K2 and 6, the BPS assignment is similarly close to the 0% Walk outcome. Just as with Grade K1, this fact is not entirely driven by applicants' preferences for neighborhood schools. The fraction of students who obtain a different assignment under the 100% Walk alternative are 17% and 10%, respectively. The differences are smaller at higher grades because there are more continuing students who obtain guaranteed priority. On average, 4.5% of Grade K2 applicants have guaranteed priority at their first choice compared to 13% of Grade 6 students. Hence, despite the adoption of a seemingly neutral 50-50 split, Table 1 shows that the BPS outcome is closely similar to the 0% Walk outcome across different grades and years.

[6] For analysis of the effects of restricting the number of choices which can be submitted, see the work of Haeringer and Klijn (2009), Calsamiglia, Haeringer, and Kljin (2010), and Pathak and Sönmez (2013).

[7] As a check on our understanding of the data, we verify that we can re-create the assignments produced by BPS. Across four years and three applicant grades, we can match 98% of the assignments. Based on discussions with BPS, we learned that the reason why we do not exactly re-create the BPS assignment is that we do not have access to BPS's exact capacity file, and instead must construct it ex-post from the final assignment. There are small differences between this measure of capacity and the capacity input to the algorithm due to the handling of unassigned students who are administratively assigned. In this paper, to hold this feature fixed in our counterfactuals, we take our re-creation as representing the BPS assignment.

3.2 The Impact of Precedence

To understand the source of this puzzle, in Table 2 we report the fraction of students who obtain a slot in a school in their walk zone under different priority and precedence policies. Each year, when a student is unassigned in BPS, they are administratively assigned via an informal process conducted by the central enrollment office. These students are reported as unassigned in Table 2, even though many are likely to be eventually assigned in this administrative aftermarket. Unassigned students are also the reason why the fraction of students who obtain a walk zone school is less than 50% even though most applicants at Grades K1 and K2 rank walk-zone schools as their first choices. Among those who are assigned, the BPS mechanism assigns 62.6% of Grade K1 and 58.2% of Grade K2 students to schools in their walk zones.

Before turning to variations in precedence, we compare the fraction of students assigned to a walk zone school under the two priority extremes: 0% Walk and 100% Walk. Recall that our Proposition 3 states that the number of students assigned to a walk zone school increases when the number of walk-zone slots increases. Table 2 shows this prediction borne out in Grades K1, K2, and 6, even though BPS's priority structure is more complicated than that studied in our model. Corollary 1 suggests that the fraction of students who obtain walk-zone assignments under the 0% Walk and 100% Walk policies provides a benchmark for what can be obtained under variations of priority or precedence given student demand. For Grade K1, this range spans from 46.2% to 57.4% walk-zone assignment; the 11.2% interval represents the maximum difference in allocation attainable through changes in either priorities or precedence for Grade K1.

The first alternative precedence we consider is **Walk-Open**, under which all walk-zone slots precede the open slots. (The actual BPS implementation is a slight variation on Walk-Open in which applicants with sibling priority and outside the walk zone apply to the open slots before applying to the walk-zone slots.) Focusing on Walk-Open provides useful intuition because as Table 2 shows the BPS system produces an outcome very close to Walk-Open. Under Walk-Open, the pool of walk-zone applicants will be depleted by the time the open slots start being filled. For instance, suppose a school has 100 slots, and there are 100 walk-zone applicants and 100 non-walk-zone applicants. With the Walk-Open precedence order, 50 of the 100 walk-zone applicants will fill the 50 walk-zone slots. The remaining competition for open slots is between 50 walk-zone applicants and 100 non-walk-zone applicants. Since there are twice as many non-walk-zone applicants as walk-zone applicants in this residual pool, the non-walk-zone students stand to get more of the open slots. This *processing bias* partly explains why having walk-zone applicants first apply to walk-zone slots ends up disadvantaging them.

Next we consider the **Open-Walk** precedence order in which all applicants fill open slots before filling walk-zone slots. This represents the other end of precedence policy spectrum. Proposition 4 states that in our model, switching the order of precedence positions of walk-zone slots with those of subsequent open slots weakly increases the total number of walk-zone assignments. For each grade, Table 2 shows that this effect appears in the data. 54.8% of Grade K1 students are assigned to their walk-zone schools under Open-Walk, relative to 46.6% with Walk-Open. The 8.2% range, all holding fixed the 50-50 split, represents 73% of the range attainable from going from 0% Walk to 100% Walk. For Grade K2, the two extreme precedence policies cover 74% of the 9.3% range between the priority policy extremes. For Grade 6, the two extreme precedence policies cover 67% of the 5.4% range between the extremes. Thus, we see that decisions about precedence order have impacts on the assignment of magnitude comparable to decisions about priorities.

The difference between Walk-Open and Open-Walk represents the range of walk zone assignments that can arise from alternative precedence orders all within the 50-50 split. The comparison of these two policies illustrates that precedence cannot be ignored for achieving distributional objectives.

We next turn to intermediate precedence policies still holding the 50-50 split fixed. We investigate several alternatives, each of which may represent a compromise that the Boston School committee intended in their 1999 statement. Comparisons of these alternatives also provides intuitions about how the biases associated with different policies.

The first alternative we examine attempts to mitigate the bias caused processing all of the slots of a particular type at once. The **Rotating** precedence order alternates between walk-zone and open slots. Under Rotating, the fraction of students who are assigned to schools in their walk zones increases by 2.5% relative to the Walk-Open precedence policy for Grade K1, but is still closer to Walk-Open than Open-Walk. The reason that Rotating is closer to Walk-Open is that alternating slots only partly undoes the processing bias, as we describe next.

3.3 The Impact of Lottery Numbers

The other feature that accounts for the effect of precedence policies are the lottery numbers of applicants. Recall the earlier example where a school has 100 slots, and there are 100 walk-zone applicants and 100 non-walk-zone applicants. Under the Walk-Open precedence order, the walk-zone applicants who compete for open slots are out-numbered by non-walk-zone students. They also come from a pool of applicants with adversely selected lottery numbers. Under Walk-Open, the walk-zone slots are filled by the 50 walk-zone applicants with the highest lottery numbers among walk-zone applicants. The competition for open slots is among the 50 walk-zone students *with lowest lottery numbers* and the 100 non-walk-zone students. When walk-zone applicants are considered for slots at the open slots, their adversely selected lottery numbers systematically place them behind applicants without walk-zone priority, leaving them unlikely to obtain any of the open slots. This *random number bias*—created by the precedence order—renders the outcomes under Walk-Open precedence very similar to the assignment that arises when all slots are open. In our example with 100 slots, with no walk zone priority, on average, 50 slots would be assigned to walk-zone applicants and 50 would be assigned to non-walk-zone applicants. Hence, in Boston's implementation of the 50-50 split, there is no difference between a system without walk zone priority and the Walk-Open precedence. Even though the official BPS policy (following the School Committee's 1999 policy declaration) states that there should be open competition for the open slots, walk-zone applicants are systematically disadvantaged in the competition for those slots.

Random number bias is also the reason why the Rotating precedence order is much closer to Walk-Open than Open-Walk in Table 2. With Rotating precedence, the pool of walk-zone applicants with favorable lottery numbers is depleted after the first few slots are allocated, so the bias in lottery numbers among the pool of walk-zone applicants re-emerges after a few rounds of rotation. While Rotating tackles the processing bias, if only one lottery number is used, then it retains the random number bias.

As we describe in detail in the next section, BPS was hesitant to have a system with more than one lottery number. Even with such a constraint, it may be possible to combat the random number bias via the precedence order. The **Compromise** precedence order first fills half of the walk-zone slots, then fills all the open slots, and then the fills the second half of the walk-zone slots. It attempts to even out the treatment of walk-zone applicants through changes in the order of slots. Initially, when the first few open slots are processed, the walk-zone applicant pool has adversely selected lottery numbers, but this bias becomes less important by the time the last open slots are processed. As a result, the fraction of applicants who attend a school for which they have walk-zone priority is close the midpoint between Walk-Open and Open-Walk. The results of this policy are shown in column (5) of Table 2. At Grade K1, Compromise assigns 50.7% to a walk-zone school—exactly the mid-point between the Walk-Open (46.6%) and Open-Walk (54.8%) assignments. Compromise is therefore a viable solution to achieve the school committee's policy aim of obtaining a balance between the pro-neighborhood and pro-choice factions.

When more than one lottery number can be used, there are additional possibilities. Table 3 reports on alternative policies that use two lottery numbers in ordering applicants, one for the walk-zone slots and one for the open slots. We report on these simulations because they allow for a quantitative comparison of the processing and random number biases in isolation. Column (2) reports on the Walk-Open precedence order with two lottery numbers. This deals with the random number bias, but retains the processing bias, as the pool of applicants from the walk zone is still depleted by the time the open slots are filled. Walk-Open with two lottery numbers is still close to the current BPS outcome. It assigns 48.1% of students to walk-zone schools at Grade K1 and is quite close to the 46.6% assigned when Walk-Open is used with only one lottery number. Walk-Open with two lottery numbers is still closer to 0% Walk than 100% Walk; this suggests that random number bias accounts for only part of the reason the 50-50 allocation is not midway between the two extremes.

To deal with both the processing and random number bias, we report on a Rotating treatment which uses two independent lottery numbers, one for walk-zone slots and the other for open slots. For Grade K1, 51.7% of students are assigned to walk-zone schools under this treatment; this point is near the 51.8% midpoint between 0% Walk and 100% Walk. Moreover, the difference between columns (2) and (3) shows the magnitude of the processing bias. This difference is 3.6%, while the difference due to random number bias (comparing column (2) in Table 2 and column (3) in Table 3) is 1.5%. Thus, we see that both biases are substantial. The patterns are similar for Grades K2 and 6.

The remedy of using two lottery numbers, however, has an important drawback. It is well-known that using multiple lottery numbers across schools with deferred acceptance may generate efficiency losses (Abdulkadiroğlu, Pathak, and Roth 2009). Even though the two lottery numbers are within schools (and not across schools), the same efficiency consequence arises here. Indeed, if we compare the Unassigned row in Table 2 to Table 3, there are slightly more unassigned students when two lottery numbers are used. For Grade K1, 25.0% of students are unassigned when using Rotating two lottery numbers, and this fraction is between 0.2-0.4% higher than any precedence policy reported in Table 2. The same pattern holds for Grade K2 and Grade 6. Although these are small numbers, they do suggest that using two lottery numbers could produce real efficiency losses.

If the efficiency cost of multiple lotteries is prohibitive or explaining a system with two lottery numbers is too challenging, then the Open-Walk precedence order with a single lottery number is a viable alternative. By removing the statistical bias in the processing for walk-zone applicants, Open-Walk respects the school committee's goal of keeping the competition for non-walk-zone seats open to all applicants. One implication of the Open-Walk treatment, as shown in Table 2, is that it leads to the highest fraction of students with a walk zone assignment, given the 50-50 split.

The Open-Walk policy makes it easy to understand what policy BPS has been implementing from 2009-2012. In Table 4, we compare the BPS policy to the two alternatives we've proposed: Open-Walk with one lottery number and Rotating with two lottery numbers. Table 4 reports how the actual BPS implementation compares to the Open-Walk treatment with a smaller set of walk zone seats.

From 2009–2012, BPS's implementation corresponds to Open-Walk with a 5-10% walk-zone priority depending on the grade. This fact stands in sharp contrast to the advertised 50-50 compromise. For Grade K1, the actual BPS implementation gives 47.2% of students walk-zone assignments; this is just above the Open-Walk treatment with 5% walk share, but below the Open-Walk treatment with 10% walk share. For Grade K2, the actual BPS implementation has 48.5% walk-zone assignment compared to 48.4% with the Open-Walk treatment with a 10% walk share. For Grade 6, the actual BPS implementation is bracketed by Open-Walk with 5% and 10% walk share. BPS's implementation also corresponds to Rotating with two lottery numbers where the fraction of walk zone seats is just above 10%.

Finally, the last issue we examine empirically is whether priority or precedence changes are mostly distributional as suggested by Proposition 5. Table 5 reports on how the overall distribution of choices received varies with precedence order. This table shows that there is almost no difference in the aggregate distribution of received choice rank across BPS, rotating with two lottery numbers, and Open-Walk with one lottery number. Consistent with Proposition 5, changes in precedence are a tool to achieve distributional objectives, having little overall impact on the total number of students who obtain their top choices.

3.4 Relation to BPS's Stated Policy Goals

Our simulations have established that both the processing and random number bias have substantial impact on school choice systems' ability to achieve distributional objectives. Here, we attempt to determine which policies implement the school committee's objective. This is a difficult question because the school committee's policy statement may be subject to multiple interpretations, and policy choices can be dictated in part by simplicity of implementation or explanation.

The deliberations leading up to the 50-50 policy indicate a goal of achieving some sort of compromise between pro-choice and pro-neighborhood constituents. By contrast, as implemented, BPS's 50-50 split conferred little benefit to walk zone applicants. Our analysis illustrates how such a surprising finding is possible, with the choice of Walk-Open precedence undermining BPS's priority policies.

Our empirical analysis highlights two leading alternatives that can achieve a compromise, between neighborhood assignment and full choice. With two independent lottery numbers, the Rotating treatment is near the midpoint between a system with 100% walk zone priority (which according to BPS's stated policy, "limits choice and access for students") and a system with 0% walk-zone priority (which according to BPS's stated policy, would not "recognize the interests of families who want to choose a walk zone school.") Our simulations indicate that the efficiency costs of multiple lotteries, in Boston, appear to be relatively small. Thus, Rotating precedence with two independent lotteries could in principle be an effective policy option for BPS.

However, BPS officials expressed concerns that using two separate lottery numbers could make the system difficult to explain to participants. With the constraint of having only one lottery number, the Open-Walk precedence order removes the statistical bias in the processing for walk-zone applicants, respecting the Boston school committee's goal of keeping the competition for non-walk seats open to all applicants. Using Open-Walk would lead to more students being assigned to their walk zone schools. However, the Open-Walk policy makes it possible to calibrate, in a transparent way, the fraction of seats reserved for walk-zone students, without adversely impacting walk-zone applicants' chances at the open seats.

4 The 2012-2013 Boston School Choice Debate

In Boston Mayor Thomas Menino's 2012 State of the City Address, Menino (2012) articulated support for increased neighborhood assignment in school choice:[8]

> "Something stands in the way of taking our [public school] system to the next level: a student assignment process that ships our kids to schools across our city. Pick any street. A dozen children probably attend a dozen different schools. Parents might not know each other; children might not play together. They can't carpool, or study for the same tests.
>
> [. . .]
>
> Boston will have a radically different school assignment process—one that puts priority on children attending schools closer to their homes."

[8] For more on this debate, see the materials available at http://bostonschoolchoice.org and press accounts by Goldstein (2012) and Handy (2012).

Menino's remarks initiated a policy debate; a preliminary version of our results played a role in the deliberations and subsequent re-design of Boston's school choice program. Here, we recount the details because they (1) show that the role of precedence was not appreciated until our work and (2) provide a case study of some political economy issues in school choice. (Readers not interested in how our analysis interacted with the public policy debate can skip the rest of this section.)

For elementary and middle school admissions, Boston was divided into three zones: North, West, and East. To respond to Menino's charge, a natural proposal was to try to increase walk-zone assignment by increasing the proportion of walk-zone slots at each school. But no one knew the impact of precedence at the time of Menino's speech—Boston's policymakers were unaware that their system was unintentionally disadvantaging walk-zone students in the competition for open slots.

In Fall 2012, BPS proposed five different plans which all restricted participant choice by reducing the number of schools students could rank.[9] The idea behind each of these plans was to reduce the fraction of non-neighborhood applicants in competition for seats at each school. These plans and other proposals from the community became the center of a year-long, city-wide discussion on school choice. Critics expressed concerns that with smaller choice menus, families from disadvantaged neighborhoods would be shut out of good schools if the neighborhood component of assignment were given more weight. This point was summarized by a community activist (Seelye 2012):

> "A plan that limits choice and that is strictly neighborhood-based gets us to a system that is more segregated than [BPS] is now."

Underlying the discussion was puzzlement as to how Menino's concerns could be correct given that walk-zone applicants were reserved 50% of each school's seats, while also having a shot at the open seats. A few proponents of neighborhood assignment were convinced that there had not been enough neighborhood assignment in recent years, but they could not determine why. After a preliminary version of our research became available, Pathak and Sönmez interacted with BPS's staff. Parts of our research were presented to the Mayor's twenty-seven-member Executive Advisory Committee (EAC), explaining that the BPS walk-zone priority was not having its intended impact because of the chosen precedence order. The EAC meeting minutes summarized the discussion (EAC 2013):

> "A committee member stated that the walk-zone priority in its current application does not have a significant impact on student assignment. The committee member noted that this finding was consistent with anecdotal evidence that the committee had heard from parents."

Following the presentation, BPS immediately recommended that the system switch to the Compromise precedence order for the Fall 2013 admissions cycle. The meeting minutes state:

[9] The initial plans suggested dividing the city into 6, 9, 11, or 23 zones, or doing away with school choice entirely and reverting assignment based purely on neighborhood.

> "BPS's recommendation is to utilize the [C]ompromise method in order to ensure that the walk-zone priority is not causing an unintended consequence that is not in stated policy."

Part of the reason for recommending the Compromise method is the anticipated difficulty of describing a system employing two lottery numbers. The switch to the Compromise treatment, as Table 2 shows, would lead to an increase in the number of students assigned to their walk-zone schools. This change, together with the proposals to shrink zones or adopt a plan with smaller choice menus, raised concerns that the equity of access would decrease.

Our discovery about the role of precedence proved so significant that it became part of the fight between those favoring neighborhood assignment and those favoring increased choice. Proponents of neighborhood assignment interpreted our findings as showing that (unintentionally) improper implementation of the 50-50 school split caused hundreds of students to be shut out of their neighborhood schools. They argued that a change in the precedence order would be the only policy consistent with the School Committee's 1999 policy goals. Switching to either Rotating with two lottery numbers or Open-Walk would also coincide with the Mayor's push towards moving children closer to home.

School choice proponents seized on our findings for multiple reasons. Some groups, such as the activist Metropolitan Area Planning Council, fought fiercely to keep the 50-50 seat split with the current precedence order (MAPC 2013):

> "The assignment priority given to walk-zone students has profound impacts on the outcomes of any new plan. The possible changes that have been proposed or discussed include increasing the set-aside, decreasing the set-aside, changing the processing order, or even reducing the allowable distance for walk zone priority to less than a mile. Actions that provide additional advantage to walk-zone students are likely to have a disproportionate negative impact on Black and Hispanic students, who are more reliant on out-of-walk-zone options for the quality schools in their basket."

The symbolism of the 50-50 split, combined with BPS's precedence order, resounded with sophisticated choice proponents because it created the impression that they were giving something away to neighborhood proponents even though they really were not.

Confirming the counterintuitive nature of our results, other parties expressed skepticism on how walk-zone priority as implemented did not have large implications for student assignment. For instance, the City Councillor in charge of education publicly testified (Connolly 2013):

> "MIT tells us that so many children in the walk zones of high demand schools 'flood the pool' of applicants, and that children in these walk zones get in in higher numbers, so walk zone priority doesn't really matter."
>
> "Maybe, that is true. But if removing the walk zone priority doesn't change anything, why change it all?"

In response to this and similar questions, we argued that moving away from the BPS priority/precedence structure would improve transparency and thus make it easier for BPS to target adequate implementation of its policy goals.

Choice proponents also interpreted our findings as an argument for removing walk-zone priority entirely. Indeed, given that walk-zone priority plays a relatively small role (as currently implemented by BPS relative to 0% Walk), simply eliminating it might increase transparency about how the system works. Getting rid of walk zone priority altogether avoids the (false) impression that applicants from the walk zone are receiving a boost under the mechanism. This argument eventually convinced Boston Superintendent Carol Johnson to eliminate walk zone priority all together. On March 13, 2013, Superintendent Johnson stated (Johnson 2013):

> "After viewing the final MIT and BC presentations on the way the walk zone priority actually works, it seems to me that it would be unwise to add a second priority to the Home-Based model by allowing the walk zone priority be carried over."
>
> ...
>
> "Leaving the walk zone priority to continue as it currently operates is not a good option. We know from research that it does not make a significant difference the way it is applied today: although people may have thought that it did, the walk zone priority does not in fact actually help students attend schools closer to home. The External Advisory Committee suggested taking this important issue up in two years, but I believe we are ready to take this step now. We must ensure the Home-Based system works in an honest and transparent way from the very beginning."

In March 2013, the Boston school committee voted to totally eliminate walk-zone priority, so as to avoid giving the false impression of advantaging walk-zone students. The district also voted for a Home-Based system which defines the choice menu for applicants based on their address, and substantially reduces the number of schools applicants can rank. This change affected applicants for elementary and middle school starting Fall 2013.

5 Conclusion

Those articulating a pro-neighborhood position in the school choice debate often lament how choice has "destroyed the concept of neighborhood schools" by scattering children across the city by assigning them to schools far from home (Ravitch 2011). In Boston, Mayor Menino's claim that the current system does not put priority on children attending schools closer to their homes seemed to be at odds with the fact that half of each school's seats prioritized applicants from the walk-zone. This paper explains this apparent puzzle by showing the important role played by the precedence order.

In addition to our new comparative static results, the we have shown how the precedence order effectively undermined the policy aim of the 50-50 slot split in Boston. Moreover, our empirical results show that in Boston, the precedence order (1) is an important lever for achieving distributional objectives, and (2) has quantitative impacts almost as large as changes in neighborhood priority. The role precedence played was so central in Boston that once its unintended consequences were made transparent, policymakers decided to abandon walk-zone priority altogether.

Even though explicit discussions of precedence have not been part of prior school choice policy debates (with the exception of the recent one at BPS), it is clear that they should accompany debates about priorities. It also seems likely that precedence could play an important role outside of student assignment, in other priority-based assignment problems where priorities depend on particular slots. Finally, it is worth noting that our paper uses market design techniques and analysis to show how to achieve given policy objectives. We have not considered normative questions like whether there should be walk-zone priority at all, or how to compute the optimal walk-zone set-aside. These important questions seem worth future investigation.

Table 1. Difference between the Current Boston Mechanism and Alternative Walk Zone Splits

	Grade K1			Grade K2			Grade 6		
		Difference relative to current BPS			Difference relative to current BPS			Difference relative to current BPS	
	# students	0% Walk	100% Walk	# students	0% Walk	100% Walk	# students	0% Walk	100% Walk
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
2009	1770	46	336	1715	28	343	2348	54	205
		3%	19%		2%	20%		2%	9%
2010	1977	68	392	1902	62	269	2308	41	171
		3%	20%		3%	14%		2%	7%
2011	2071	50	387	1821	90	293	2073	4	225
		2%	19%		5%	16%		0%	11%
2012	2515	88	504	2301	101	403	2057	24	247
		3%	20%		4%	18%		1%	12%
All	8333	252	1619	7739	281	1308	8786	123	848
		3%	19%		4%	17%		1%	10%

Notes. Table reports fraction of applicants whose assignments differ between the mechanism currently employed in Boston and two alternative mechanisms: one with a priority structure without walk-zone priorities at any seats (0% Walk), and the other with a priority structure with walk-zone priorities at all seats (100% Walk).

Table 2. Number of Students Assigned to School in Walk Zone (2009-2012), Single Random Number

	Priorities = 0% Walk	Priorities = 50% Walk, Changing Precedence						Priorities = 100% Walk
		Walk-Open	Actual BPS	Rotating	Compromise (O25-W50-O25)	Compromise (W25-O50-W25)	Open-Walk	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
				I. Grade K1				
Walk Zone	3849	3879	3930	4080	4051	4227	4570	4787
	46.2%	46.6%	47.2%	49.0%	48.6%	50.7%	54.8%	57.4%
Outside Walk Zone	2430	2399	2353	2187	2221	2044	1695	1468
	29.2%	28.8%	28.2%	26.2%	26.7%	24.5%	20.3%	17.6%
Unassigned	2054	2055	2050	2066	2061	2062	2068	2078
	24.6%	24.7%	24.6%	24.8%	24.7%	24.7%	24.8%	24.9%
				II. Grade K2				
Walk Zone	3651	3685	3753	3842	3853	3900	4214	4374
	47.2%	47.6%	48.5%	49.6%	49.8%	50.4%	54.5%	56.5%
Outside Walk Zone	2799	2764	2694	2601	2600	2538	2214	2036
	36.2%	35.7%	34.8%	33.6%	33.6%	32.8%	28.6%	26.3%
Unassigned	1289	1290	1292	1296	1286	1301	1311	1329
	16.7%	16.7%	16.7%	16.7%	16.6%	16.8%	16.9%	17.2%
				III. Grade 06				
Walk Zone	3439	3476	3484	3542	3555	3657	3797	3907
	39.1%	39.6%	39.7%	40.3%	40.5%	41.6%	43.2%	44.5%
Outside Walk Zone	4782	4750	4743	4686	4664	4561	4419	4309
	54.4%	54.1%	54.0%	53.3%	53.1%	51.9%	50.3%	49.0%
Unassigned	565	560	559	558	567	568	570	570
	6.4%	6.4%	6.4%	6.4%	6.5%	6.5%	6.5%	6.5%

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk zone priority; 100% implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (2)-(7) hold the 50/50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of open slots. Actual BPS implements the current BPS system. Rotating implements the precedence ordering alternating between walk-zone and open slots. Compromise implements the precedence order in which exactly half of the walk-zone slots come before all open slots, which are in turn followed by the half of the walk-zone slots. Balanced implements Rotating, but uses two random numbers for each student, one for walk-zone slots and the other for open slots. Open-Walk implements the precedence order in which all open slots are ahead of walk-zone slots.

Table 3. Number of Students Assigned to School in Walk Zone (2009-2012), Two Random Numbers

	Priorities = 0% Walk	Priorities = 10%	Priorities = 25% Walk	Priorities = 50% Walk Changing Precedence					Priorities = 100% Walk
		Rotating: Two Random (W-O-O-O-O-O-O-O-O-O)	Rotating: Two Random (W-O-O-O)	Walk-Open: Two Random	Compromise: Two Random (W25-O50-W25)	Compromise: Two Random (O25-W50-O25)	Rotating: Two Random (O-W-O-W-...)	Open-Walk: Two Random	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
						I. Grade K1			
Walk Zone	3849	3939	4133	4008	4272	4308	4305	4551	4787
	46.2%	47.3%	49.6%	48.1%	51.3%	51.7%	51.7%	54.6%	57.4%
Outside Walk Zone	2430	2339	2140	2245	1976	1945	1941	1721	1468
	29.2%	28.1%	25.7%	26.9%	23.7%	23.3%	23.3%	20.7%	17.6%
Unassigned	2054	2055	2060	2080	2085	2080	2087	2061	2078
	24.6%	24.7%	24.7%	25.0%	25.0%	25.0%	25.0%	24.7%	24.9%
						II. Grade K2			
Walk Zone	3651	3711	3872	3831	3982	4023	4037	4202	4374
	47.2%	48.0%	50.0%	49.5%	51.5%	52.0%	52.2%	54.3%	56.5%
Outside Walk Zone	2799	2736	2562	2579	2427	2391	2383	2211	2036
	36.2%	35.4%	33.1%	33.3%	31.4%	30.9%	30.8%	28.6%	26.3%
Unassigned	1289	1292	1305	1329	1330	1325	1319	1326	1329
	16.7%	16.7%	16.9%	17.2%	17.2%	17.1%	17.0%	17.1%	17.2%
						III. Grade 06			
Walk Zone	3439	3481	3568	3572	3676	3648	3691	3808	3907
	39.1%	39.6%	40.6%	40.7%	41.8%	41.5%	42.0%	43.3%	44.5%
Outside Walk Zone	4782	4726	4631	4608	4504	4535	4507	4397	4309
	54.4%	53.8%	52.7%	52.4%	51.3%	51.6%	51.3%	50.0%	49.0%
Unassigned	565	579	587	606	606	603	588	581	570
	6.4%	6.6%	6.7%	6.9%	6.9%	6.9%	6.7%	6.6%	6.5%

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk zone priority; 100% implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (4)-(8) hold the 50/50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of open slots, but uses two different random numbers for walk and open seats. Rotating implements the precedence ordering alternating between walk-zone and open slots. Compromise implements the precedence order in which exactly half of the walk-zone slots come before all open slots, which are in turn followed by the half of the walk-zone slots. Open-Walk implements the precedence order in which all open slots are ahead of walk-zone slots, but uses two different random numbers for walk and open seats.

Table 4. Choices Received by Students (2009-2012)

Choice Received	Priorities = 0% Walk	Priorities = 50% Walk Changing Precedence		Priorities = 100% Walk
		BPS	Rotating: Two Random (O-W-O-W-...)	
	(1)	(2)	(3)	(4)
		I. Grade K1		
1	44.2%	44.3%	44.0%	43.0%
2	13.2%	13.2%	13.1%	13.3%
3	8.4%	8.4%	8.3%	8.7%
4	4.6%	4.6%	4.3%	4.6%
5	2.3%	2.3%	2.5%	2.5%
6	1.0%	1.0%	1.1%	1.2%
7	0.7%	0.7%	0.7%	0.7%
8	0.4%	0.5%	0.5%	0.6%
9	0.3%	0.3%	0.3%	0.3%
10	0.1%	0.2%	0.2%	0.2%
Unassigned or Admin. Assigned	24.6%	24.6%	25.0%	24.9%
		II. Grade K2		
1	49.1%	49.2%	48.8%	49.1%
2	16.1%	16.0%	15.8%	15.8%
3	10.2%	9.9%	10.0%	9.7%
4	4.1%	4.2%	4.3%	4.3%
5	2.2%	2.1%	2.2%	2.0%
6	0.7%	0.8%	0.7%	0.8%
7	0.5%	0.5%	0.5%	0.4%
8	0.3%	0.3%	0.3%	0.4%
9	0.1%	0.1%	0.2%	0.2%
10	0.1%	0.1%	0.1%	0.2%
Unassigned or Admin. Assigned	16.7%	16.7%	17.0%	17.2%
		III. Grade 6		
1	65.8%	65.8%	65.0%	64.8%
2	17.7%	17.7%	17.5%	18.1%
3	7.1%	7.3%	7.6%	7.5%
4	1.9%	1.9%	2.3%	2.2%
5	0.8%	0.8%	0.7%	0.8%
6	0.1%	0.1%	0.1%	0.1%
7	0.0%	0.0%	0.0%	0.0%
8	0.1%	0.1%	0.0%	0.0%
9	0.0%	0.0%	0.0%	0.0%
10	0.0%	0.0%	0.0%	0.0%
Unassigned or Admin. Assigned	6.4%	6.4%	6.7%	6.5%

Notes. Table reports the distribution of choice ranks arising under different priority and precedence policies. Unassigned or Adminstrative Assignment means student is not assigned to any of the listed choices; some students will be adminstratively assigned after Round 1.

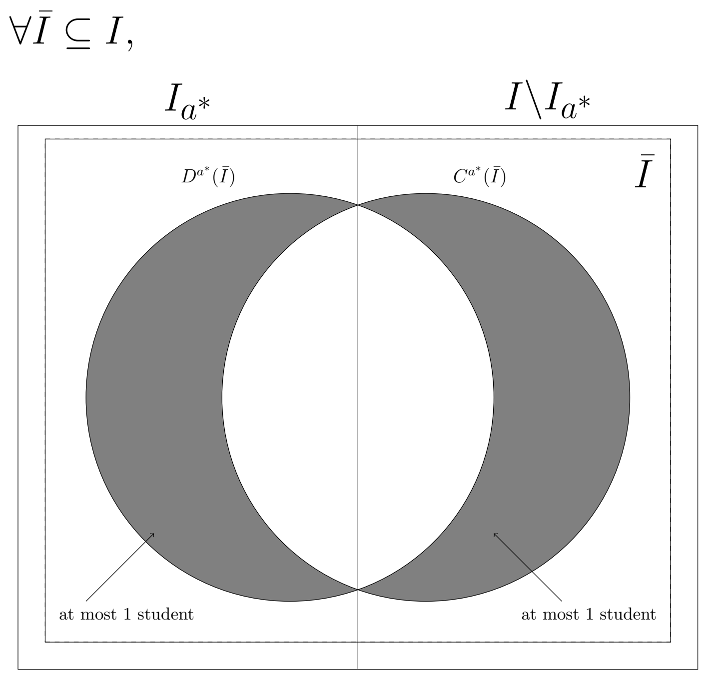

Figure 1: Comparison of $C^{a^*}(\bar{I})$ and $D^{a^*}(\bar{I})$, as described formally in the lemma.

A Appendix

A.1 Preliminaries for Proposition 1

For a school a^* and a slot $s^* \in S^{a^*}$ of school a^*, suppose that s^* is an open slot under priority structure π, and is a walk-zone slot under priority structure $\tilde{\pi}$. Suppose furthermore that $\pi^s = \tilde{\pi}^s$ for all slots $s \in [S^{a^*} \setminus \{s^*\}]$. Let C^{a^*} and D^{a^*} respectively be the choice functions for a^* induced by the priorities π and $\tilde{\pi}$, under (fixed) precedence order $\triangleright^{a^*}$.

Lemma 1. *For any set of students $\bar{I} \subseteq I$, as pictured in Figure 1:*

1. *All students in the walk-zone of a^* that are chosen from $\bar{I}$ under choice function C^{a^*} are chosen under choice function D^{a^*} (i.e. $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$). Moreover, $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$.*

2. *All students not in the walk-zone of a^* that are chosen from $\bar{I}$ under choice function D^{a^*} are chosen under choice function C^{a^*} (i.e. $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$). Moreover, $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$.*

Proof. We proceed by induction on the number q_{a^*} of slots at a^*. The base case $q_{a^*} = 1$ is immediate, as then $S^{a^*} = \{s^*\}$ and $C^{a^*}(\bar{I}) \neq D^{a^*}(\bar{I})$ if and only if a walk-zone student of a^* is assigned to s^* under D, but a non-walk-zone student is assigned to s^* under C, that is, if $D^{a^*}(\bar{I}) \subseteq$

I_{a^*} while $C^{a^*}(\bar{I}) \subseteq I \setminus I_{a^*}$. It follows immediately from this observation that $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$, $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$, $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$, and $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$.

Now, given the result for the base case $q_{a^*} = 1$, we suppose that the result holds for all $q_{a^*} < \ell$ for some $\ell \geq 1$; we show that this implies the result for $q_{a^*} = \ell$. We suppose that $q_{a^*} = \ell$, and let $\bar{s} \in S^{a^*}$ be the slot which is minimal (i.e., processed/filled last) under the precedence order $\triangleright^{a^*}$. A student eligible for one type of slot is also eligible for the other, and hence $\bar{s}$ is either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case if no student is assigned to $\bar{s}$ (under either priority structure); hence, we assume that

$$|C^{a^*}(\bar{I})| = |D^{a^*}(\bar{I})| = q_{a^*} = \ell. \tag{3}$$

If $\bar{s} = s^*$, then the result follows just as in the base case: It is clear from the algorithms defining C^{a^*} and D^{a^*} that in the computations of $C^{a^*}(\bar{I})$ and $D^{a^*}(\bar{I})$, the same students are assigned to slots s with higher precedence than $s^* = \bar{s}$ (i.e., slots s with $s \triangleright^{a^*} s^* = \bar{s}$), as those slots' priorities and relative precedence ordering are fixed. Thus, as in the base case, $C^{a^*}(\bar{I}) \neq D^{a^*}(\bar{I})$ if and only if a walk-zone student of a^* is assigned to s^* under D, but a non-walk-zone student is assigned to s^* under C.

If $\bar{s} \neq s^*$, then $s^* \triangleright^{a^*} \bar{s}$. We let $J \subseteq \bar{I}$ be the set of students assigned to slots in $S^{a^*} \setminus \{\bar{s}\}$ in the computation of $C^{a^*}(\bar{I})$, and let $K \subseteq \bar{I}$ be the set of students assigned to slots in $S^{a^*} \setminus \{\bar{s}\}$ in the computation of $D^{a^*}(\bar{I})$.

The first $q_{a^*} - 1 = \ell - 1$ slots of a^* can be treated as a school with slot-set $S^{a^*} \setminus \{\bar{s}\}$ (under the precedence order induced by $\triangleright^{a^*}$). Thus, the inductive hypothesis (in the case $\ell - 1$) implies:

$$[J \cap I_{a^*}] \subseteq [K \cap I_{a^*}]; \tag{4}$$

$$|[J \cap I_{a^*}] \setminus [K \cap I_{a^*}]| \leq 1; \tag{5}$$

$$[K \cap (I \setminus I_{a^*})] \subseteq [J \cap (I \setminus I_{a^*})]; \tag{6}$$

$$|[K \cap (I \setminus I_{a^*})] \setminus [J \cap (I \setminus I_{a^*})]| \leq 1. \tag{7}$$

If we have equality in (4) and (6),[10] then the set of students available to be assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$ is the same as in the computation of $D^{a^*}(\bar{I})$. Thus, as $\pi^{\bar{s}} = \tilde{\pi}^{\bar{s}}$ by assumption, we have $C^{a^*}(\bar{I}) = D^{a^*}(\bar{I})$; hence, the desired result follows immediately.[11]

If instead the inclusions in (4) and (6) are strict, then by (5) and (7), respectively, we have a unique student $k \in [K \cap I_{a^*}] \setminus [J \cap I_{a^*}]$ and a unique student $j \in [J \cap (I \setminus I_{a^*})] \setminus [K \cap (I \setminus I_{a^*})]$. Here, k is in the walk-zone of a^* and is assigned to a slot s with higher precedence than $\bar{s}$ (i.e. a slot s with $s \triangleright^{a^*} \bar{s}$) in the computation of $D^{a^*}(\bar{I})$, but is not assigned to such a slot in the computation of $C^{a^*}(\bar{I})$. Likewise, j is not in the walk-zone of a^*, is assigned to a slot s with $s \triangleright^{a^*} \bar{s}$ in the computation of $C^{a^*}(\bar{I})$, and is not assigned to such a slot in the computation of $D^{a^*}(\bar{I})$. By construction, k must

[10] As $|J| = |K|$ by (3), equality holds in one of (4) and (6) if and only if it holds for *both* inclusions (4) and (6).

[11] Note that as $C^{a^*}(\bar{I}) = D^{a^*}(\bar{I})$, we have $|[(C^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(D^{a^*}(\bar{I})) \cap I_{a^*}]| = 0 \leq 1$ and $|[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| = 0 \leq 1$.

be the π^o-maximal student in $[\bar{I} \setminus J] \cap I_{a^*}$ and j must be the π^o-maximal student in $[\bar{I} \setminus K] \cap (I \setminus I_{a^*})$ (indeed, j is π^o-maximal in $\bar{I} \setminus K$).

Now:

- If $\bar{s}$ is a walk-zone slot, then k is assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$; hence, $C^{a^*}(\bar{I}) = J \cup \{k\}$. Thus, as $k \in [K \cap I_{a^*}]$, we have $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$ by (4), and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$ by (5). In the computation of $D^{a^*}(\bar{I})$, meanwhile, if a walk-zone student is not assigned to $\bar{s}$, then j must be assigned to $\bar{s}$, as j is π^o-maximal among students in $[\bar{I} \setminus K] \cap (I \setminus I_{a^*})$. It follows that $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$ (by (6)), and $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$ (by (7)).

- If $\bar{s}$ is an open slot, then j is assigned to $\bar{s}$ in the computation of $D^{a^*}(\bar{I})$; hence, $D^{a^*}(\bar{I}) = K \cup \{j\}$. Thus, as $j \in [J \cap (I \setminus I_{a^*})]$, we have $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$ by (6), and $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$ by (7). Meanwhile, if a walk-zone student is assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$, then it must be k, as k is π^o-maximal among students in $[\bar{I} \setminus J] \cap I_{a^*}$. It follows that $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$ (by (4)), and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$ (by (5)).

These observations complete the induction. □

A.2 Preliminaries for Proposition 2

For a school a^*, suppose that $s^*_w \in S^{a^*}$ is a walk-zone slot that immediately precedes an open slot $s^*_o \in S^{a^*}$ under the precedence order $\triangleright^{a^*}$. That is, $s^*_w \triangleright^{a^*} s^*_o$, and there are no slots $s \notin \{s^*_w, s^*_o\}$ such that $s^*_w \triangleright^{a^*} s \triangleright^{a^*} s^*_o$. Suppose that precedence order $\tilde{\triangleright}$ is obtained from $\triangleright$ by swapping the positions of s^*_w and s^*_o and leaving the positions of all other slots unchanged.[12] Let C^{a^*} and D^{a^*} respectively be the choice functions for a^* induced by the precedence orders $\triangleright$ and $\tilde{\triangleright}$, under (fixed) slot priorities π^s.

Lemma 2. *For any set of students $\bar{I} \subseteq I$, as pictured in Figure 1:*

1. *All students in the walk-zone of a^* that are chosen from $\bar{I}$ under choice function C^{a^*} are chosen under choice function D^{a^*} (i.e. $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$). Moreover, $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$.*

2. *All students not in the walk-zone of a^* that are from $\bar{I}$ chosen under choice function D^{a^*} are chosen under choice function C^{a^*} (i.e. $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$). Moreover, $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$.*

Proof. We proceed by induction on the number q_{a^*} of slots at a^*.

First, we prove the base case $q_{a^*} = 2$.[13] We denote by $i_{s^*_w}$ and $i_{s^*_o}$ (resp. $j_{s^*_w}$ and $j_{s^*_o}$) the students respectively assigned to slots s^*_w and s^*_o in the computation of $C^{a^*}(\bar{I})$ (resp. $D^{a^*}(\bar{I})$).

[12] We prove Proposition 2 for the case of adjacent-slot swaps here; the full result follows upon observing that any precedence-swap can be implemented as a sequence of adjacent-slot swaps.

[13] Note that the setup requires at least two distinct slots of a^*, so $q_{a^*} = 2$ *a priori*.

Now:

- If $\{i_{s_w^*}, i_{s_o^*}\} \subset I_{a^*}$, then the ordering under π^o must rank $i_{s_w^*}$ highest among all students $i \in \bar{I}$, and rank $i_{s_o^*}$ second-highest among all students $i \in \bar{I}$, as otherwise some student $i \in \bar{I}$ with $i \neq i_{s_w^*}$ would have higher rank than $i_{s_o^*}$ under π^o, and would thus have higher claim than $i_{s_o^*}$ for (open) slot s_o^* under precedence order $\triangleright^{a^*}$. But then, $i_{s_w^*}$ is the π^o-maximal student in $\bar{I}$ and $i_{s_o^*}$ is the π^o-maximal walk-zone student in $\bar{I} \setminus \{i_{s_w^*}\}$; hence, we must have $j_{s_o^*} = i_{s_w^*}$ and $j_{s_w^*} = i_{s_o^*}$, so that $D^{a^*}(\bar{I}) = C^{a^*}(\bar{I})$. In this case, $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| = 0 \leq 1$ and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| = 0 \leq 1$.

- If $\{i_{s_w^*}, i_{s_o^*}\} \subset (I \setminus I_{a^*})$, then $\bar{I}$ contains no students in the walk-zone of a^* (i.e. $\bar{I} \cap I_{a^*} = \emptyset$) and $i_{s_w^*}$ and $i_{s_o^*}$ are then just the π^o-maximal non-walk-zone students in $\bar{I}$. In this case, we find that $j_{s_o^*} = i_{s_w^*}$ and $j_{s_w^*} = i_{s_o^*}$; hence, $D^{a^*}(\bar{I}) = C^{a^*}(\bar{I})$. Once again, we have $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| = 0 \leq 1$ and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| = 0 \leq 1$.

- If $i_{s_w^*} \in I_{a^*}$ and $i_{s_o^*} \in (I \setminus I_{a^*})$, then $i_{s_w^*}$ is the π^o-maximal walk-zone student of a^* in $\bar{I}$. If $i_{s_w^*}$ is also π^o-maximal among *all* students in $\bar{I}$, then we have $j_{s_o^*} = i_{s_w^*}$. Moreover, in this case either

  1. $j_{s_w^*} \in I_{a^*}$, or
  2. $i_{s_w^*}$ is the only walk-zone student of a^* in $\bar{I}$, so that $j_{s_w^*} = i_{s_o^*}$.

  Alternatively, if $i_{s_w^*}$ is not π^o-maximal among all students in $\bar{I}$, then $i_{s_o^*}$ must be π^o-maximal among all students in I_{a^*}, so that $j_{s_o^*} = i_{s_o^*}$ and $j_{s_w^*} = i_{s_w^*}$.

  We therefore find that

  $$[(C^{a^*}(\bar{I})) \cap I_{a^*}] = \{i_{s_w^*}\} \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}];$$

  hence, $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$. Additionally, we have

  $$[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq \{i_{s_o^*}\} = [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})],$$

  so that $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$.

- We cannot have $i_{s_w^*} \in (I \setminus I_{a^*})$ and $i_{s_o^*} \in I_{a^*}$, as s_w^* is a walk-zone slot (and thus gives all students in I_{a^*} higher priority than students in $I \setminus I_{a^*}$) and $s_w^* \triangleright^{a^*} s_o^*$.

The preceding four cases are exhaustive and the desired result holds in each; thus, we have the base case for our induction.

Now, given the result for the base case $q_{a^*} = 2$, we suppose that the result holds for all $q_{a^*} < \ell$ for some $\ell \geq 2$; we show that this implies the result for $q_{a^*} = \ell$. Thus, we suppose that $q_{a^*} = \ell$. We let $\bar{s} \in S^{a^*}$ be the slot which is minimal under the precedence order $\triangleright^{a^*}$. A student eligible for one type of slot is also eligible for the other, and hence $\bar{s}$ is either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case if no student is assigned to $\bar{s}$ (under either priority structure); hence, we assume that

$$|C^{a^*}(\bar{I})| = |D^{a^*}(\bar{I})| = q_{a^*} = \ell. \tag{8}$$

If $\bar{s} = s^*_o$, then the result follows just as in the base case, as it is clear from the algorithms defining C^{a^*} and D^{a^*} that the same students are assigned to slots s with higher precedence than s^*_w (i.e. slots s with $s \rhd^{a^*} s^*_w \rhd^{a^*} s^*_o = \bar{s}$) in the computations of $C^{a^*}(\bar{I})$ and $D^{a^*}(\bar{I})$.

If $\bar{s} \neq s^*_o$, then $s^*_w \rhd^{a^*} s^*_o \rhd^{a^*} \bar{s}$. We let $J \subseteq \bar{I}$ be the set of students assigned to slots in $S^{a^*} \setminus \{\bar{s}\}$ in the computation of $C^{a^*}(\bar{I})$, and let $K \subseteq \bar{I}$ be the set of students assigned to slots in $S^{a^*} \setminus \{\bar{s}\}$ in the computation of $D^{a^*}(\bar{I})$. The first $\ell - 1$ slots of a^* can be treated as a school with slot-set $S^{a^*} \setminus \{\bar{s}\}$ (under the precedence order induced by $\rhd^{a^*}$). Thus, the inductive hypothesis (in the case $q_{a^*} = \ell - 1$), implies:

$$[J \cap I_{a^*}] \subseteq [K \cap I_{a^*}]; \tag{9}$$

$$|[J \cap I_{a^*}] \setminus [K \cap I_{a^*}]| \leq 1; \tag{10}$$

$$[K \cap (I \setminus I_{a^*})] \subseteq [J \cap (I \setminus I_{a^*})]; \tag{11}$$

$$|[K \cap (I \setminus I_{a^*})] \setminus [J \cap (I \setminus I_{a^*})]| \leq 1. \tag{12}$$

If we have equality in (9) and (11),[14] then the set of students available to be assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$ is the same as in the computation of $D^{a^*}(\bar{I})$. Thus, as $\pi^{\bar{s}} = \tilde{\pi}^{\bar{s}}$ by assumption, we have $C^{a^*}(\bar{I}) = D^{a^*}(\bar{I})$; hence, the desired result follows immediately.[15]

If instead the inclusions in (9) and (11) are strict, then by (10) and (12), respectively, we have a unique student $k \in [K \cap I_{a^*}] \setminus [J \cap I_{a^*}]$ and a unique student $j \in [J \cap (I \setminus I_{a^*})] \setminus [K \cap (I \setminus I_{a^*})]$. Here, k is in the walk-zone of a^* and is assigned to a slot s with higher precedence than $\bar{s}$ (i.e. a slot s with $s \rhd^{a^*} \bar{s}$) in the computation of $D^{a^*}(\bar{I})$ but is not assigned to such a slot in the computation of $C^{a^*}(\bar{I})$. Likewise, j is not in the walk-zone of a^*, is assigned to a slot s with higher precedence than $\bar{s}$ (i.e. a slot s with $s \rhd^{a^*} \bar{s}$) in the computation of $C^{a^*}(\bar{I})$, and is not assigned to such a slot in the computation of $D^{a^*}(\bar{I})$. By construction, k must be the π^o-maximal student in $[\bar{I} \setminus J] \cap I_{a^*}$ and j must be the π^o-maximal student in $[\bar{I} \setminus K] \cap (I \setminus I_{a^*})$ (indeed, j is π^o-maximal in $\bar{I} \setminus K$).

Now:

- If $\bar{s}$ is a walk-zone slot, then k is assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$; hence, $C^{a^*}(\bar{I}) = J \cup \{k\}$. Thus, as $k \in [K \cap I_{a^*}]$, we have $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$ by (9), and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$ by (10). In the computation of $D^{a^*}(\bar{I})$, meanwhile, if a walk-zone student is not assigned to $\bar{s}$, then j must be assigned to $\bar{s}$, as j is π^o-maximal among students in $[\bar{I} \setminus K] \cap (I \setminus I_{a^*})$. It follows that $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$ (by (11)), and $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$ (by (12)).

[14] As $|J| = |K|$ by (8), equality holds in one of (9) or (11) if and only if it holds for *both* (9) and (11).

[15] Note that as $C^{a^*}(\bar{I}) = D^{a^*}(\bar{I})$, we have $|[J \cap I_{a^*}] \setminus [K \cap I_{a^*}]| = 0 \leq 1$ and $|[K \cap (I \setminus I_{a^*})] \setminus [J \cap (I \setminus I_{a^*})]| = 0 \leq 1$.

- If $\bar{s}$ is an open slot, then j is assigned to $\bar{s}$ in the computation of $D^{a^*}(\bar{I})$; hence, $D^{a^*}(\bar{I}) = K \cup \{j\}$. Thus, as $j \in [J \cap (I \setminus I_{a^*})]$, we have $[(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \subseteq [(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]$ by (11), and $|[(C^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})] \setminus [(D^{a^*}(\bar{I})) \cap (I \setminus I_{a^*})]| \leq 1$ by (12). Meanwhile, if a walk-zone student is assigned to $\bar{s}$ in the computation of $C^{a^*}(\bar{I})$, then it must be k, as k is π^o-maximal among students in $[\bar{I} \setminus J] \cap I_{a^*}$. It follows that $[(C^{a^*}(\bar{I})) \cap I_{a^*}] \subseteq [(D^{a^*}(\bar{I})) \cap I_{a^*}]$ (by (9)), and $|[(D^{a^*}(\bar{I})) \cap I_{a^*}] \setminus [(C^{a^*}(\bar{I})) \cap I_{a^*}]| \leq 1$ (by (10)).

These observations complete the induction. □

A.3 Proof of Propositions 1 and 2

In this section, we prove Propositions 1 and 2 using a completely parallel argument for the two results. Our proof makes use two technical machinery components. The first, which is well-known in the matching literature, is the *cumulative offer process* (Kelso and Crawford 1982, Hatfield and Milgrom 2005, Hatfield and Kojima 2010), a stable matching algorithm which is outcome-equivalent to DA but easier to analyze in our framework. The second, which to the best of our knowledge is completely novel, is construction of a *copy economy*, a setting in which each student i who are rejected by school a^* are replaced by two "copies"—a top copy i^{t} who takes the role of i in applying to schools i weakly prefers to a^*, and a bottom copy i^{b} who takes the role of i in applying to schools i likes less than a^*.

Because bottom copies i^{b} can act independently of top copies i^{t} and the cumulative offer process is independent of proposal order, constructing copies enables us to track how the market responds to rejection of students i by a^* *before i (or rather, i^{t}) applies to a^**.

A.3.1 The Cumulative Offer Process

Definition. In the *cumulative offer process* under choice functions $\bar{C}$, students propose to schools in a sequence of steps $\ell = 1, 2, \ldots$:

Step 1. Some student $i^1 \in I$ proposes to his favorite school a^1. Set $\bar{A}^2_{a^1} = \{i^1\}$, and set $\bar{A}^2_a = \emptyset$ for each $a \neq a^1$; these are the sets of students *available* to schools at the beginning of Step 2. Each school $a \in A$ *holds* $\bar{C}^a(\bar{A}^2_a)$ and *rejects* all other students in $\bar{A}^2_a$.

Step ℓ. Some student $i^\ell \in I$ not currently held by any school proposes to his most-preferred school that has not yet rejected him, a^ℓ. Set $\bar{A}^{\ell+1}_{a^\ell} = \bar{A}^\ell_{a^\ell} \cup \{i^\ell\}$, and set $\bar{A}^{\ell+1}_a = \bar{A}^\ell_a$ for each $a \neq a^\ell$. Each school $a \in A$ holds $\bar{C}^a(\bar{A}^{\ell+1}_a)$ and rejects all other students in $\bar{A}^{\ell+1}_a$.

If at any Step $\ell + 1$ no student is able to propose—that is, if all students not on hold have proposed to all schools they find acceptable—then the process terminates. The *outcome of the cumulative offer process* is the match $\bar{\mu}$ that assigns each school $a \in A$ the students it holds at the end of the last step before termination: $\bar{\mu}_a = \bar{C}^a(\bar{A}^{\ell+1}_a)$.

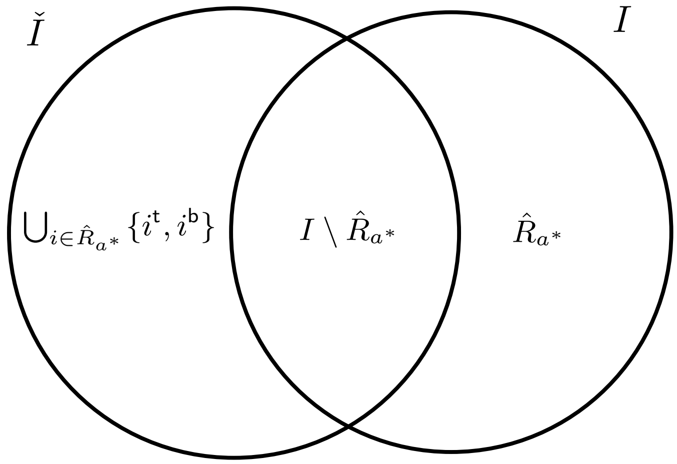

Figure 2: Relationship between the set $\check{I}$ of students in the copy economy and the set I of students in the original economy.

In our context, the cumulative offer process outcome is independent of the proposal order and is equal to the outcome of the student-optimal stable mechanism (see Hatfield and Kojima (2010) and Kominers and Sönmez (2012)).[16]

A.3.2 Copy Economies

We denote by $\bar{\mu}$ the outcome of cumulative offer process under choice functions $\bar{C}$ (associated to priorities $\bar{\pi}$). For a fixed school $a^* \in A$, we let $R_{a^*} \subseteq I$ be the set of students rejected by a^* during the cumulative offer process under choice functions $\bar{C}^a$.[17] Formally, we have $R_{a^*} = \{i \in I : a^* P^i \bar{\mu}_i\}$. We fix some $\hat{R}_{a^*} \subseteq R_{a^*}$ and construct a *copy economy* with set of schools A, set of slots S, and precedence order profile $\triangleright$. The set of students in the copy economy, denoted $\check{I}$, is obtained by replacing each student $i \in \hat{R}_{a^*}$ with a *top copy* i^{t} and a *bottom copy* i^{b}:

$$\check{I} = (I \setminus \hat{R}_{a^*}) \bigcup \left(\cup_{i\in\hat{R}_{a^*}} \{i^{\mathsf{t}}, i^{\mathsf{b}}\} \right).$$

The relationship between the set $\check{I}$ of students in the copy economy and the set I of students in the original economy is pictured in Figure 2. Note that we suppress the dependence of $\check{I}$ on $\hat{R}_{a^*}$, as the set $\hat{R}_{a^*}$ under consideration is clearly identified whenever we undertake a copy economy construction. For a copy i^{c} of agent i (where $\mathsf{c} \in \{\mathsf{t}, \mathsf{b}\}$), we say that i is the agent *underlying* i^{c}.

Copies' preferences correspond to specific (post- and pre-)truncations of their underlying agents' preferences, as pictured in Figure 3. The preference relation $P^{i^{\mathsf{t}}}$ of the top copy of i is defined so that

[16] This observation follows from the facts—proven respectively in Proposition 3 and Lemma D.1 of (Kominers and Sönmez 2012)—that the choice functions in our setting satisfy both the *substitutability* condition of (Hatfield and Milgrom 2005) and the *irrelevance of rejected contracts* condition of (Aygün and Sönmez 2013).

[17] Here, a^* need not be as defined in Sections A.1 and A.2, although it will be in Section A.3.3.

$$P^i : \underbrace{a^1 \succ \cdots \succ a^\ell \succ a^*}_{P^{i^{\mathsf{t}}}} \succ \underbrace{a^{\ell+1} \succ \cdots}_{P^{i^{\mathsf{b}}}}$$

Figure 3: Construction of copies' preference relations: $P^{i^{\mathsf{t}}}$ is the "top part" of P^i that ranks all the schools which i (weakly) prefers to a^*, while $P^{i^{\mathsf{b}}}$ is the "bottom part" of P^i that ranks all the schools which i likes (strictly) less than a^*. (Note that we must insert a_0 at the end of $P^{i^{\mathsf{t}}}$.)

$$\cdots \pi^s \underbrace{i}_{i^{\mathsf{t}}\breve{\pi}^s i^{\mathsf{b}}} \pi^s \underbrace{\bar{i}}_{\bar{i}^{\mathsf{t}}\breve{\pi}^s \bar{i}^{\mathsf{b}}} \pi^s \cdots$$

Figure 4: Construction of priorities in the copy economy: $\breve{\pi}^s$ is constructed so that each instance of an agent $i \in \hat{R}_{a^*}$ in priority order π^s is replaced with the "subrelation" $i^{\mathsf{t}}\breve{\pi}^s i^{\mathsf{b}}$.

- $aP^{i^{\mathsf{t}}}a' \iff aP^i a'$ for all $a, a' \in A \setminus \{a_0\}$, and
- $a_0 P^{i^{\mathsf{t}}} a \iff a^* P^i a$ for all $a \in A$.

That is, $P^{i^{\mathsf{t}}}$ is the "top part" of P^i that ranks all the schools which i (weakly) prefers to a^*.[18] The preference relation $P^{i^{\mathsf{b}}}$ of the bottom copy of i is defined similarly, with

- $aR^i a^* \implies a_0 P^{i^{\mathsf{b}}} a$ for all $a \in A$.
- $a^* P^i a P^i a' \implies a P^{i^{\mathsf{b}}} a'$ for all $a, a' \in A$.

That is, $P^{i^{\mathsf{b}}}$ is the "bottom part" of P^i that ranks all the schools which i likes (strictly) less than a^*.[19]

The priorities $\breve{\pi}^s$ in the copy economy (for each $s \in S$) are constructed by replacing agents in $\hat{R}_{a^*}$ with their copies as follows:

- $i^{\mathsf{t}}\breve{\pi}^s i^{\mathsf{b}}$ for all $i \in \hat{R}_{a^*}$;
- $i\pi^s\bar{i} \implies i\breve{\pi}^s\bar{i}$ for all $i, \bar{i} \in I \setminus \hat{R}_{a^*}$;
- $i\pi^s\bar{i} \implies i\breve{\pi}^s\bar{i}^{\mathsf{t}}\breve{\pi}^s\bar{i}^{\mathsf{b}}$ for all $i \in I \setminus \hat{R}_{a^*}$ and $\bar{i} \in \hat{R}_{a^*}$;
- $\bar{i}\pi^s i \implies \bar{i}^{\mathsf{t}}\breve{\pi}^s\bar{i}^{\mathsf{b}}\breve{\pi}^s i$ for all $i \in I \setminus \hat{R}_{a^*}$ and $\bar{i} \in \hat{R}_{a^*}$; and
- $i\pi^s\bar{i} \implies i^{\mathsf{t}}\breve{\pi}^s i^{\mathsf{b}}\breve{\pi}^s\bar{i}^{\mathsf{t}}\breve{\pi}^s\bar{i}^{\mathsf{b}}$ for all $i, \bar{i} \in \hat{R}_{a^*}$.

This construction is illustrated in Figure 4. We write $\check{C}$ for the choice functions induced by these priorities.

[18]Note that $P^{i^{\mathsf{t}}}$ could be empty, in the case that a^* is the first choice of student i.

[19]Note that $P^{i^{\mathsf{t}}}$ could be empty, in the case that a^* is least-preferred acceptable school of i.

We say that a set of students $\bar{I} \subseteq I$ is *equivalent up to copies* to a set of students $\check{\bar{I}} \subseteq \check{I}$ if $\bar{I}$ and $\check{\bar{I}}$ share the same set of students in $I \setminus \hat{R}_{a^*}$, and the set of students underlying the set of copy students in $\check{\bar{I}}$ exactly equals $\bar{I} \cap \hat{R}_{a^*}$. That is, $\bar{I} \subseteq I$ is *equivalent to* $\check{\bar{I}} \subseteq \check{I}$ *up to copies* if

- $[\bar{I} \cap (I \setminus \hat{R}_{a^*})] = [\check{\bar{I}} \cap (I \setminus \hat{R}_{a^*})]$ and
- $[\bar{I} \cap \hat{R}_{a^*}] = \left\{ i \in \hat{R}_{a^*} \subseteq I : i^{\mathsf{c}} \in \check{\bar{I}} \text{ for some } \mathsf{c} \in \{\mathsf{t}, \mathsf{b}\} \right\}$.

In this case, we write $\bar{I} \overset{\text{cp}}{=} \check{\bar{I}}$.

Because the priorities $\check{\pi}^s$ are constructed so that each instance of an agent $i \in \hat{R}_{a^*}$ in priority order π^s is replaced with the "subrelation" $i^{\mathsf{t}} \check{\pi}^s i^{\mathsf{b}}$, the following lemma is immediate.

Lemma 3. *Suppose that $\bar{I} \overset{\text{cp}}{=} \check{\bar{I}}$, and suppose moreover that for each $i \in \hat{R}_{a^*}$, there is at most one copy of i in $\check{\bar{I}}$. Then, for each school $a \in A$, we have $\bar{C}^a(\bar{I}) \overset{\text{cp}}{=} \check{C}^a(\check{\bar{I}})$.*

We let $\check{\mu}$ be the outcome of the cumulative offer process in the copy economy. We now show that $\check{\mu}$ in a certain sense corresponds to $\bar{\mu}$ under copy equivalence. Specifically, we show that under $\check{\mu}$ (in the copy economy):

- students in $i \in I \setminus \hat{R}_{a^*}$ have the same assignments under $\bar{\mu}$;
- all the top copies of students in $\hat{R}_{a^*}$ are assigned to a_0; and
- all the bottom copies of students $i \in \hat{R}_{a^*}$ receive the assignment received by i under $\bar{\mu}$.

Lemma 4. *We have:*

- $\check{\mu}_i = \bar{\mu}_i$ *for any* $i \in (I \setminus \hat{R}_{a^*})$;
- $\check{\mu}_{i^{\mathsf{t}}} = a_0$ *for any* $i \in \hat{R}_{a^*}$; *and*
- $\check{\mu}_{i^{\mathsf{b}}} = \bar{\mu}_i$ *for any* $i \in \hat{R}_{a^*}$.

Consequently, for each $a \in A$, we have $\bar{\mu}_a \overset{\text{cp}}{=} \check{\mu}_a$.

Proof. We let $\bar{\Sigma} = \left\langle (i^1 \to a^1), (i^2 \to a^2), \ldots, (i^L \to a^L) \right\rangle$ be a (full) proposal sequence that can arise in the cumulative offer process under choice functions $\bar{C}$ (i.e. in the original economy).[20] Now, for each ℓ for which $i^\ell \in \hat{R}_{a^*}$, we let

$$\check{i}^\ell = \begin{cases} i^{\ell t} & a^\ell R^{i^\ell} a^* \\ i^{\ell b} & a^* P^{i^\ell} a^\ell; \end{cases}$$

that is, $\check{i}^\ell$ is the copy of i^ℓ who finds school a^ℓ acceptable. For each ℓ for which $i^\ell \in (I \setminus \hat{R}_{a^*})$, we let $\check{i}^\ell = i^\ell$.

[20]Since the cumulative offer process outcome is independent of the proposal order, we can analyze an arbitrary proposal sequence for the original economy.

Claim. *The proposal sequence*

$$\check{\Sigma} = \left\langle (\check{i}^1 \to a^1), (\check{i}^2 \to a^2), \ldots, (\check{i}^\ell \to a^\ell), \ldots, (\check{i}^L \to a^L) \right\rangle \oplus \left\langle (i^{\mathrm{t}} \to a_0) : i \in \hat{R}_{a^*} \right\rangle \tag{13}$$

is a valid sequence of proposals for the cumulative offer process under choice functions $\check{C}$ (in the copy economy).[21]

Proof. For each ℓ and $a \in A$, we let $\bar{A}^\ell_a$ denote the sets of available students arising in the cumulative offer process (in the original economy) under proposal order $\bar{\Sigma}$, and let $\check{A}^\ell_a$ denote the sets of available students arising in the cumulative offer process (in the copy economy) under proposal order $\check{\Sigma}$.[22] We proceed by induction, showing that

1. $(\check{i}^\ell \to a^\ell)$ is a valid proposal in step ℓ of the cumulative offer process (in the copy economy), and
2. for each $\ell \leq L$ and $a \in (A \setminus \{a_0\})$, we have
$$\bar{A}^{\ell+1}_a \overset{\mathrm{cp}}{=} \check{A}^{\ell+1}_a. \tag{14}$$

Both hypotheses are clearly true in the base case $\ell = 1$, so we assume that the hold up to ℓ, and show that this implies them in the case $\ell + 1$.

The proposal $(i \to a)$ occurs in the sequence $\bar{\Sigma}$ at most once, as no student ever proposes to the same school twice in the cumulative offer process. Thus, by our construction of $\check{\Sigma}$, we see that there is no student i for whom two distinct copies propose to some (nonnull) school $a \in [A \setminus \{a_0\}]$. It follows that for each $i \in \hat{R}_{a^*}$, there is at most one copy of i in $\check{A}^{\ell+1}_a$. Thus, the conclusion of Lemma 3 applies:

$$\bar{C}^a(\bar{A}^{\ell+1}_a) \overset{\mathrm{cp}}{=} \check{C}^a(\check{A}^{\ell+1}_a) \tag{15}$$

for each school $a \in (A \setminus \{a_0\})$.

Now, if $i^{\ell+1} \in (I \setminus \hat{R}_{a^*})$, then $i^{\ell+1}$ is not held by any school $a \in A$ at the end of step ℓ of the cumulative offer process in the original economy, i.e. $i^{\ell+1} \notin (\cup_{a \in (A \setminus \{a_0\})} \bar{C}^a(\bar{A}^{\ell+1}_a))$, and $i^{\ell+1}$ has not proposed to school $a^{\ell+1}$ by the end of step ℓ of that process. We then see immediately from (15) that $(\check{i}^{\ell+1} \to a^{\ell+1}) = (i^{\ell+1} \to a^{\ell+1})$ is a valid proposal at step $\ell + 1$ of the cumulative offer process in the copy economy. Moreover, it follows from (14) that

$$\bar{A}^{\ell+2}_a = (\bar{A}^{\ell+1}_a \cup \{i^{\ell+1}\}) \overset{\mathrm{cp}}{=} (\check{A}^{\ell+1}_a \cup \{i^{\ell+1}\}) \overset{\mathrm{cp}}{=} (\check{A}^{\ell+1}_a \cup \{\check{i}^{\ell+1}\}) = \check{A}^{\ell+2}_a$$

for each $a \in (A \setminus \{a_0\})$, as desired.

If instead $i^{\ell+1} \in \hat{R}_{a^*}$, then $i^{\ell+1}$ is not held by any school $a \in A$ at the end of step ℓ of the cumulative offer process in the original economy, i.e. $i^{\ell+1} \notin (\cup_{a \in (A \setminus \{a_0\})} \bar{C}^a(\bar{A}^{\ell+1}_a))$, and $i^{\ell+1}$ has

[21] Here, $\oplus$ denotes the concatenation of sequences. Note that the ordering of the proposals in the appended subsequence $\left\langle (i^{\mathrm{t}} \to a_0) : i \in \hat{R}_{a^*} \right\rangle$ can be arbitrary.

[22] Formally, the sets $\bar{A}^\ell_a$ are given, and we construct the sets $\check{A}^\ell_a$ inductively, as we show by induction that $\check{\Sigma}$ is a valid cumulative offer process proposal order.

not proposed to school $a^{\ell+1}$ by the end of step ℓ of that process. From (15), then, we see that no copy of $i^{\ell+1}$ is held by any school $a \in (A \setminus \{a_0\})$ at the end of step ℓ of the cumulative offer process in the copy economy. Noting that no top copy i^{t} proposes to a_0 until after proposal $(\check{i}^L \to a^L)$, we see that both the top and bottom copies of $i^{\ell+1}$ are available to propose at the beginning of step $\ell+1$ of the cumulative offer process in the copy economy; hence, $\check{i}^{\ell+1}$ is available to propose at step $\ell+1$ irrespective of which copy of $i^{\ell+1}$ he or she is. Combining the preceding observations, we see that $(\check{i}^{\ell+1} \to a^{\ell+1})$ is a valid proposal at step $\ell+1$ of the cumulative offer process in the copy economy. As in the prior case, we then have from (14) that

$$\bar{A}_a^{\ell+2} = (\bar{A}_a^{\ell+1} \cup \{i^{\ell+1}\}) \overset{\text{cp}}{=} (\check{A}_a^{\ell+1} \cup \{i^{\ell+1}\}) = (\check{A}_a^{\ell+1} \cup \{\check{i}^{\ell+1}\}) = \check{A}_a^{\ell+2}.$$

The preceding observations show that

$$\left\langle (\check{i}^1 \to a^1), (\check{i}^2 \to a^2), \ldots, (\check{i}^\ell \to a^\ell), \ldots, (\check{i}^L \to a^L) \right\rangle \tag{16}$$

is a valid sequence of proposals for the cumulative offer process in the copy economy. Now, we note that following these proposals, all students in $(I \setminus \hat{R}_{a^*}) \cup \{i^{\mathsf{b}} : i \in \hat{R}_{a^*}\}$ are held by (possibly null) schools, and all students in $\{i^{\mathsf{t}} : i \in \hat{R}_{a^*}\}$ are available to propose again.

The final proposal of each top copy $\bar{i}^{\mathsf{t}} \in \{i^{\mathsf{t}} : i \in \hat{R}_{a^*}\}$ is $(\bar{i}^{\mathsf{t}} \to a^*)$, by construction of (16). As only the top copies $\bar{i}^{\mathsf{t}} \in \{i^{\mathsf{t}} : i \in \hat{R}_{a^*}\}$ are available to propose in step $L+1$ and those students' preference relations terminate after ranking a^*, the cumulative offer process in the copy economy is completed by running the sequence of proposals $\left\langle (i^{\mathsf{t}} \to a_0) : i \in \hat{R}_{a^*} \right\rangle$ (in any order). $\square$

Now, we observe that under proposal sequence $\check{\Sigma}$ as defined by (13):

O1. The last school each student $i \in (I \setminus \hat{R}_{a^*})$ proposes to is the school that i proposes to last in the cumulative offer process in the original economy (under proposal sequence $\bar{\Sigma}$).

O2. For each $i \in \hat{R}_{a^*}$, the last school i^{t} proposes to is a_0.

O3. For each $i \in \hat{R}_{a^*}$, the last school i^{b} proposes to is the school that i proposes to last in the cumulative offer process in the original economy (under proposal sequence $\bar{\Sigma}$).

Now, *any* valid cumulative offer process proposal sequence in the copy economy yields the outcome $\check{\mu}$. Thus, in particular, we see that $\check{\mu}$ is the outcome of the cumulative offer process in the copy economy under proposal sequence $\check{\Sigma}$. The desired result then follows from observations O1–O3. $\square$

A.3.3 Main Argument

In the sequel, we assume the setup of either Section A.1 or Section A.2, let $C^a = D^a$ for all schools $a \neq a^*$, and let μ and ν respectively denote the cumulative offer process outcomes under the choice functions C and D. We make use of an *Adjustment Lemma*, which is Lemma 1 for the case of Proposition 1 and Lemma 2 for the case of Proposition 2. We denote by $\mathfrak{n}_a(\bar{\mu}) \equiv |\bar{\mu}_a \cap I_a|$ the number of walk-zone students (of a) assigned to a in matching $\bar{\mu}$.

First, we note the following immediate corollary of the Adjustment Lemma.

Lemma 5. *For $\bar{I} \subseteq I$, if $|\bar{I}| > q_{a^*}$, then $|[\bar{I} \setminus (C^{a^*}(\bar{I}))] \cap [\bar{I} \setminus (D^{a^*}(\bar{I}))]| \geq |\bar{I}| - q_{a^*} - 1$.*[23]

Now, we let $\hat{R}_{a^*} \subseteq I$ be the set of students who are rejected from a^* in *both* the cumulative offer process under choice functions C *and* the cumulative offer process under choice functions D. As the students in $\hat{R}_{a^*}$ are rejected in the cumulative offer process under *both* choice functions C and D, we may consider the copy economy associated to the original economy by the construction introduced in Section A.3.2, for both choice function profiles. We denote by $\check{I}$ the set of students in this economy, and denote by $\check{C}$ and $\check{D}$ the copy economy choice functions associated to C and D, respectively.

We denote by μ and ν the cumulative offer process outcomes under choice functions C and D, respectively. Analogously, we denote by $\check{\mu}$ and $\check{\nu}$ the cumulative offer process outcomes under choice functions $\check{C}$ and $\check{D}$. By Lemma 4, we have

- $\check{\mu}_i = \mu_i$ and $\check{\nu}_i = \nu_i$ for any $i \in (I \setminus \hat{R}_{a^*})$;
- $\check{\mu}_{i^{\mathfrak{t}}} = a_0$ and $\check{\nu}_{i^{\mathfrak{t}}} = a_0$ for any $i \in \hat{R}_{a^*}$; and
- $\check{\mu}_i = \mu_{i^{\mathfrak{b}}}$ and $\check{\nu}_i = \nu_{i^{\mathfrak{b}}}$ for any $i \in \hat{R}_{a^*}$.

As every student in $\hat{R}_{a^*}$ is rejected from a^* in the cumulative offer process under choice functions C, we have

$$[\mu_{a^*} \cap I_{a^*}] = [(\mu_{a^*} \cap I_{a^*}) \setminus \hat{R}_{a^*}] = [\mu_{a^*} \cap (I_{a^*} \setminus \hat{R}_{a^*})] = [\check{\mu}_{a^*} \cap (I_{a^*} \setminus \hat{R}_{a^*})] = [\check{\mu}_{a^*} \cap I_{a^*}], \quad (17)$$

where the second-to-last equality follows from the fact that $\check{\mu}_i = \mu_i$ for each $i \in (I_{a^*} \setminus \hat{R}_{a^*})$, and the last equality follows because $[\check{\mu}_{a^*} \cap \hat{R}_{a^*}] = \emptyset$. It follows that

$$\mathfrak{n}_{a^*}(\mu) = |\mu_{a^*} \cap I_{a^*}| = |\check{\mu}_{a^*} \cap I_{a^*}|. \quad (18)$$

Analogously, we find that

$$\mathfrak{n}_{a^*}(\nu) = |\check{\nu}_{a^*} \cap I_{a^*}|. \quad (19)$$

Thus, to show our proposition it suffices to prove that weakly more walk-zone students of a^* are assigned to a^* under $\check{\nu}$ than under $\check{\mu}$. To show this, we recall that cumulative offer processes are always independent of proposal order, and consider particular orders for the cumulative offer processes under choice functions $\check{C}$ and $\check{D}$.

Under each process, we first execute as many proposals $(\check{i} \to a)$ as possible with $\check{i} \in \check{I}$ and $aP^{\check{i}}a^*$. Since $\check{C}$ and $\check{D}$ differ only with respect to $\check{C}^{a^*}$ and $\check{D}^{a^*}$, we can use the exact same order of proposals in each cumulative offer processes, in the initial sequence of proposals. Once such proposals are completed, each student in $\check{I}$ either

- is on hold at some (possibly null) school $a \neq a^*$, or

[23] Formally, this is also true in the case that $|\bar{I}| < q_{a^*} + 1$, as then $|[\bar{I} \setminus (C^{a^*}(\bar{I}))] \cap [\bar{I} \setminus (D^{a^*}(\bar{I}))]| < q_{a^*} + 1$, so that $|\bar{I}| - q_{a^*} - 1 < 0$.

- has proposed to all schools he prefers to a^* and is available to propose to a^*;

we let $\check{J}$ be the set of students in the latter of these two categories. By construction, at this stage, the sets of students on hold at schools $a \neq a^*$ are the same in both processes. Also, $\check{J}$ contains no bottom copies of any agent $i \in \hat{R}_{a^*}$, as bottom copies do not find a^* acceptable.

We continue the cumulative offer processes by having the students in $\check{J}$ propose to a^* in uninterrupted sequence. Following these proposals, the set of students available to a^* is exactly $\check{J}$.

If $|\check{J}| \leq q_{a^*}$, then all students in $\check{J}$ are held by a^*, and both processes terminate after all the students in $\check{J}$ have proposed to a^*. In this case, we have $\check{\mu}_{a^*} = \check{\nu}_{a^*}$; hence (18) and (19) together show that $\mathfrak{n}_{a^*}(\mu) = \mathfrak{n}_{a^*}(\nu)$.

If instead $|\check{J}| > q_{a^*}$, then we examine the set

$$\check{K} \equiv [(\check{J} \setminus \check{C}^{a^*}(\check{J})) \cap (\check{J} \setminus \check{D}^{a^*}(\check{J}))]$$

of students rejected under both $\check{C}^{a^*}$ and $\check{D}^{a^*}$ when (exactly) the set of students $\check{J}$ is available. By construction, $\check{K}$ is copy-equivalent to a subset of $\hat{R}_{a^*}$. Thus, we see that $\check{K}$ must consist entirely of top copies of students in $\hat{R}_{a^*}$, as represented in the exterior box of Figure 5. All such copies rank a_0 directly below a^*. Hence, we may continue the cumulative offer processes by having all of these students propose to a_0; we execute all such proposals.

Recall that up to this point, we have executed the cumulative offer processes under choice functions $\check{C}$ and $\check{D}$ in complete, step-by-step parallel. The sets of students held by at each school $a \neq a^*$ (including a_0) are exactly the same; meanwhile, a^* holds $\check{C}^{a^*}(\check{J})$ in the process under choice functions $\check{C}$ and holds $\check{D}^{a^*}(\check{J})$ in the process under choice functions $\check{D}$.

Using the fact that a^* fills all its slots when possible and Lemma 5, we can bound the size of $\check{K}$: we have

$$|\check{J}| - q_{a^*} \geq |\check{K}| \geq |\check{J}| - q_{a^*} - 1,$$

as pictured in the interior of Figure 5.

- If $|\check{J}| - q_{a^*} = |\check{K}|$, then (again because a^* fills all its slots when possible) we must have $\check{C}^{a^*}(\check{J}) = \check{D}^{a^*}(\check{J})$, as pictured in Figure 6. In this case, the cumulative offer process terminates after the final proposals of students in $\check{K}$ (which can be processed in the same order under choice functions $\check{C}$ and $\check{D}$; we then have $\check{\mu}_{a^*} = \check{\nu}_{a^*}$, which again shows that $\mathfrak{n}_{a^*}(\mu) = \mathfrak{n}_{a^*}(\nu)$.

- If instead $|\check{K}| = |\check{J}| - q_{a^*} - 1$, then $\check{C}^{a^*}(\check{J}) \neq \check{D}^{a^*}(\check{J})$. By the Adjustment Lemma, we see that

$$|(\check{D}^{a^*}(\check{J})) \cap I_{a^*}]| > |(\check{C}^{a^*}(\check{J})) \cap I_{a^*}]|. \tag{20}$$

  Moreover, the Adjustment Lemma shows that there is a unique student $\check{i} \in [[(\check{C}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})] \setminus [(\check{D}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})]]$ and a unique student $\check{j} \in [[(\check{D}^{a^*}(\check{J})) \cap I_{a^*}]] \setminus [(\check{C}^{a^*}(\check{J})) \cap I_{a^*}]]$, as pictured in Figure 7.

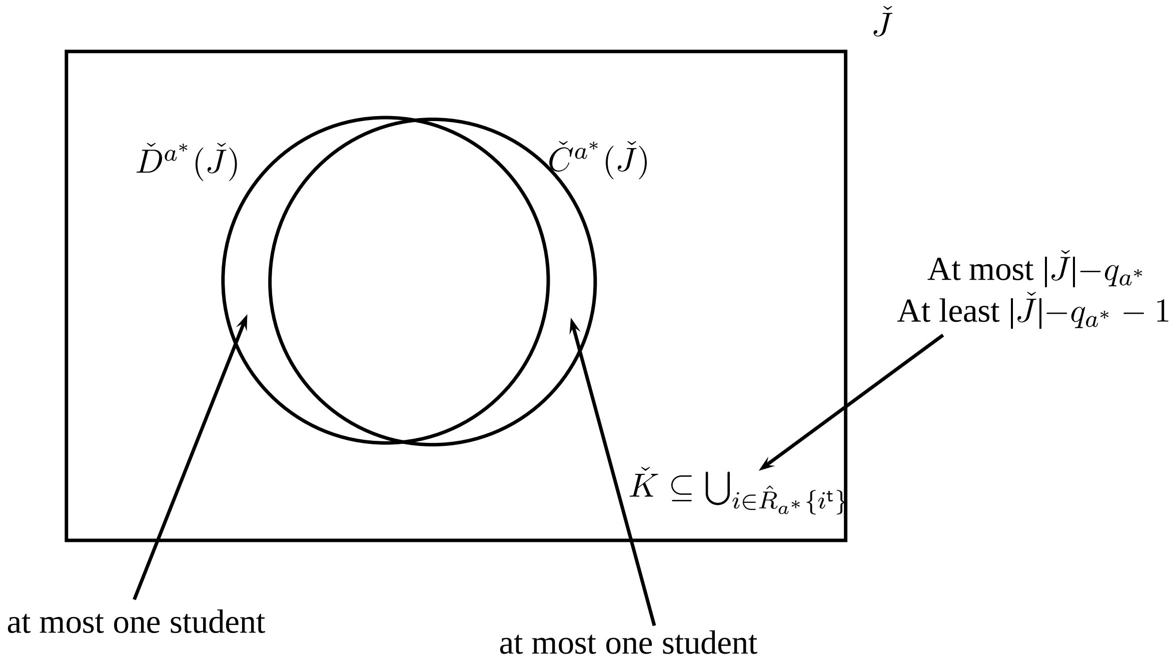

Figure 5: The structure of the choices of a^* from $\check{J}$ under choice functions $\check{D}^{a^*}$ and $\check{C}^{a^*}$, as implied by the Adjustment Lemma.

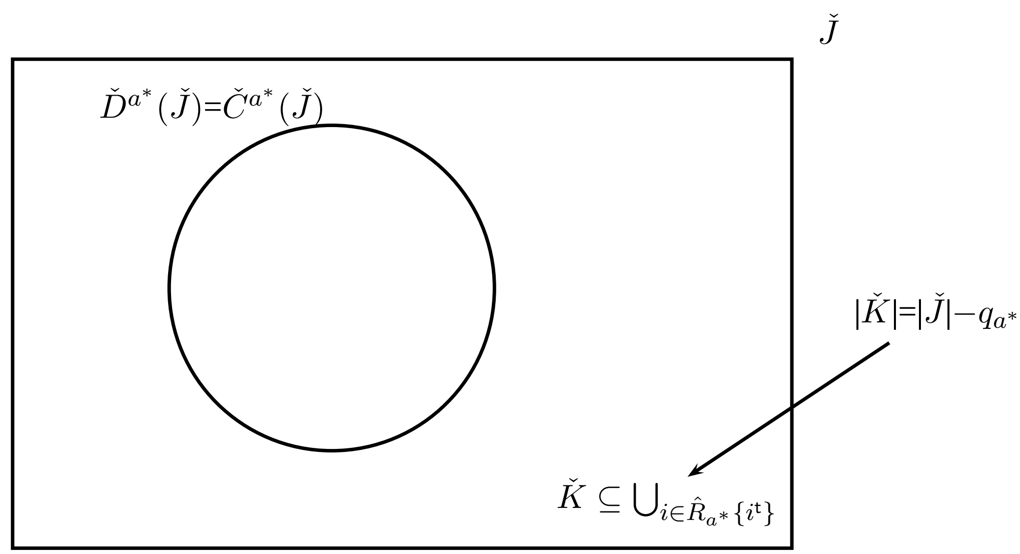

Figure 6: Case 1: $|\check{K}| = |\check{J}| - q_{a^*}$ and $\check{C}^{a^*}(\check{J}) = \check{D}^{a^*}(\check{J})$.

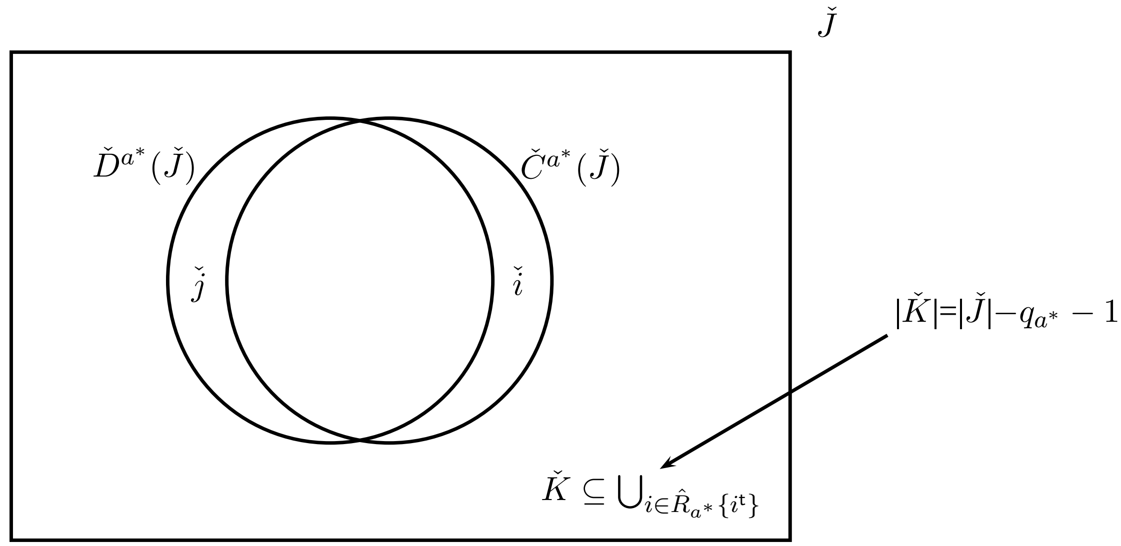

Figure 7: Case 2: $|\check{K}| = |\check{J}| - q_{a^*} - 1$ and $\check{C}^{a^*}(\check{J}) \neq \check{D}^{a^*}(\check{J})$.

– First, we suppose that $\check{J} \setminus \check{K}$ contains no copies of students in $\hat{R}_{a^*}$. Then, in particular, neither $\check{i}$ nor $\check{j}$ is a copy of a student in $\hat{R}_{a^*}$. In this case, we know that $\check{i}, \check{j} \in (\check{I} \setminus \hat{R}_{a^*}) = (I \setminus \hat{R}_{a^*})$.

  As $\check{i}$ is rejected by a^* in the cumulative offer process under choice functions $\check{D}$, we know that $a^* P^{\check{i}} \check{\nu}_{\check{i}} = \nu_{\check{i}}$; it follows that $\check{i}$ is rejected in the cumulative offer process under choice functions D. But $\check{i} \notin \hat{R}_{a^*}$, so we know that $\check{i}$ is *not* rejected in the cumulative offer process under choice functions C. Thus, $\mu_{\check{i}} R^{\check{i}} a^*$. As $\check{\mu}_{\check{i}} = \mu_{\check{i}}$ and $\check{i}$ proposes to a^* in the cumulative offer process under choice functions $\check{C}$, we find that we must have $\check{\mu}_{\check{i}} = a^*$, that is, $\check{i}$ is not rejected by a^* in the remainder of the cumulative offer process under choice functions $\check{C}$. By our choice of $\check{i}$, however, we know that $\check{i} \in (I \setminus I_{a^*})$ and that $\check{i}$ has the lowest rank under π^o among all non-walk-zone students in $\check{C}^{a^*}(\check{J})$. It follows that the number of non-walk-zone students assigned to a^* weakly increases throughout the remainder of the cumulative offer process under choice functions $\check{C}$, that is,

$$|\check{\mu}_{a^*} \cap (I \setminus I_{a^*})| \geq |(\check{C}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})|.$$

  This implies that

$$|\check{\mu}_{a^*} \cap I_{a^*}| \leq |(\check{C}^{a^*}(\check{J})) \cap I_{a^*}|. \tag{21}$$

  Analogously, we find that $\check{\nu}_{\check{j}} = a^*$, which implies that

$$|\check{\nu}_{a^*} \cap I_{a^*}| \geq |(\check{D}^{a^*}(\check{J})) \cap I_{a^*}|. \tag{22}$$

Now, we find that

$$
\begin{aligned}
\mathfrak{n}_{a^*}(\nu) &= |\check{\nu}_{a^*} \cap I_{a^*}| & (23)\\
&\geq |(\check{D}^{a^*}(\check{J})) \cap I_{a^*}| & (24)\\
&> |(\check{C}^{a^*}(\check{J})) \cap I_{a^*}| & (25)\\
&\geq |\check{\mu}_{a^*} \cap I_{a^*}| & (26)\\
&= \mathfrak{n}_{a^*}(\mu), & (27)
\end{aligned}
$$

where (23) follows from (19), (24) follows from (22), (25) follows from (21), (26) follows from (20), and (27) follows from (18).

– Finally, we consider the case in which $\check{J} \setminus \check{K}$ contains at least one copy of a student in $\hat{R}_{a^*}$. By construction, any such copy must be a top copy.

Claim. *If some copy $i^{\mathsf{t}} \notin \{\check{i}, \check{j}\}$ is in $\check{J} \setminus \check{K}$ for some $i \in \hat{R}_{a^*}$, then at least one of $\check{i}, \check{j}$ must be a copy, as well.*

Proof. We suppose first that the agent i underlying i^{t} is in the walk-zone of a^*, i.e. $i \in I_{a^*}$, but that $\check{j}$ is not a copy. Then, since $i^{\mathsf{t}} \in [(\check{J} \setminus \check{K}) \setminus \{\check{i}, \check{j}\}]$, we know that $i^{\mathsf{t}} \in (\check{C}^{a^*}(\check{J}))$; in particular, i has higher rank under π^o than $\check{j}$. As $i \in \hat{R}_{a^*}$, we know that i is rejected in the cumulative offer processes under both choice functions C and D. It follows that the lower-π^o-ranked student $\check{j}$ must also be rejected in the cumulative offer processes under both choice functions C and D, as he proposes to a^* in each of those processes; hence, we must have $\check{j} \in \hat{R}_{a^*}$. This is impossible, since we assumed that $\check{j}$ is not a copy, and all students in $\hat{R}_{a^*}$ are represented by copies in the copy economy. An analogous argument shows that if the agent i underlying i^{t} is not in the walk-zone of a^*, then $\check{i}$ must be a (top) copy. □

Claim. *There is at most one (top) copy in $\check{J} \setminus \check{K}$.*

Proof. We suppose there are at least two (top) copies in $\check{J} \setminus \check{K}$. By the preceding claim, either $\check{i}$ of $\check{j}$ is a (top) copy. We assume the former case ($\check{i}$ is a copy); the argument in the latter case is analogous. We let i^{t} be a (top) copy in $\check{J} \setminus \check{K}$ with $i^{\mathsf{t}} \neq \check{i}$.

As $\check{i} \notin \check{D}^{a^*}(\check{J})$, we may continue the cumulative offer process under choice functions $\check{D}$ (after having all students in $\check{K}$ propose to a_0) by having $\check{i}$ apply to his next-most-preferred school after a^*—since $\check{i}$ is a top copy, this school is a_0. At this point, $\check{i}$ is held by a_0. At the end of this cumulative offer process step, a^* must hold all the students in $(\check{J} \setminus \check{K}) \setminus \{\check{i}\}$, or else a^* holds fewer than q_{a^*} students. But this means that the process terminates, as all students are held by schools. We therefore have $\check{\nu}_{i^{\mathsf{t}}} = a^*$. This contradicts the fact that we must have $\check{\nu}_{i^{\mathsf{t}}} = a_0$ (by Lemma 4), as i^{t} is a top copy. □

The preceding claims show that there is exactly one (top) copy in $\check{J} \setminus \check{K}$, and that it is either $\check{i}$ or $\check{j}$. We assume the former case ($\check{i}$ is a copy); the argument in the latter case is analogous. We may continue the cumulative offer process under choice functions $\check{D}$ (after having all students in $\check{K}$ propose to a_0) by having $\check{i}$ propose to his next-most-preferred school after a^*—since $\check{i}$ is a top copy, this school is a_0, and the process terminates after the $(\check{i} \to a_0)$ proposal. Then, we have $\check{\nu}_{a^*} = \check{D}^{a^*}(\check{J})$, so

$$|\check{\nu}_{a^*} \cap (I \setminus I_{a^*})| = |(\check{D}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})| = |(\check{C}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})| - 1. \tag{28}$$

Meanwhile, the student underlying $\check{i}$ has lower rank under π^o than any non-walk-zone student in $\check{J} \setminus \check{K}$. It follows that $\check{i}$ will be the *first* student in $(\check{J} \setminus \check{K}) \setminus I_{a^*}$ rejected from a^* in the remainder of the cumulative offer process under choice functions $\check{C}$. After such a rejection occurs, we may have $\check{i}$ propose to his next-most-preferred school after a^*—as above, since $\check{i}$ is a top copy, this school is a_0 and the process terminates after the $(\check{i} \to a_0)$ proposal. Thus, we see that

$$|\check{\mu}_{a^*} \cap (I \setminus I_{a^*})| \geq |[(\check{C}^{a^*}(\check{J})) \setminus \{\check{i}\}] \cap (I \setminus I_{a^*})| = |(\check{C}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})| - 1. \tag{29}$$

Combining (28) and (29), we see that

$$|\check{\nu}_{a^*} \cap (I \setminus I_{a^*})| = |(\check{C}^{a^*}(\check{J})) \cap (I \setminus I_{a^*})| - 1 \leq |\check{\mu}_{a^*} \cap (I \setminus I_{a^*})|;$$

it follows that

$$|\check{\nu}_{a^*} \cap I_{a^*}| \geq |\check{\mu}_{a^*} \cap I_{a^*}|. \tag{30}$$

Combining (30) with (18) and (19), we find that $\mathfrak{n}_{a^*}(\nu) \geq \mathfrak{n}_{a^*}(\mu)$.

B The Two-School Model

B.1 Preliminaries

Matchings μ and ν are obtained as in Appendix A.3: Either one of the open slots is replaced with a walk-zone slot, or the precedence position of a walk-zone slot is switched with that of a subsequent open slot to obtain D from C, and ν and μ are, respectively, the associated cumulative offer process outcomes.

Lemma 6. *We have $|\nu_a| = |\mu_a|$ and $|\nu_b| = |\mu_b|$. That is, the number of slots filled at each school is the same under μ as under ν.*

Proof. If *both* of the schools a and b have an empty slot under either matching, stability implies that all students get their first choices under each matching; hence $\nu = \mu$ and the result holds immediately. Likewise, if neither school has an empty slot under either matching, the result holds immediately since then $|\nu_a| = |\mu_a| = |S^a|$ and $|\nu_b| = |\mu_b| = |S^b|$. Hence the only non-trivial case is when, under one of the matchings, one school is full but the other is not.

Without loss of generality, we suppose that under matching μ, school a has an empty slot whereas school b has all its slots full. Then not only does each student who is assigned a slot at school b under matching μ prefer school b to school a, but also there are at least as many students with a first choice of school b as the number of slots at school b. Thus by stability school b must fill all its slots under matching ν as well; hence, $|\nu_b| = |\mu_b| = |S^b|$. By assumption,

- there are at least as many slots as students, and
- all students find both schools acceptable;

therefore, we see that

$$|\nu_a| = |I| - |\nu_b| = |I| - |\mu_b| = |\mu_a|.$$

This observation completes the proof. $\square$

Proof of Propositions 3 and 4

We prove Propositions 3 and 4 using a completely parallel argument for the two results. We make use of an *Adjustment Proposition*, which is is Proposition 1 for the case of Proposition 3, and Proposition 2 for the case of Proposition 4.

Proposition 6. *There is weakly more neighborhood assignment under ν than under μ, that is,*

$$\mathfrak{n}_a(\nu) + \mathfrak{n}_b(\nu) \geq \mathfrak{n}_a(\mu) + \mathfrak{n}_b(\mu).$$

Proof. Without loss of generality, we assume the priority structure of school a has changed (i.e. that $a = a^*$ in the setup of Appendix A.3).

If $\nu_a = \mu_a$, then we have

$$\nu_b = I \setminus \nu_a = I \setminus \mu_a = \mu_b,$$

as by assumption

- there are at least as many slots as students, and
- all students find both schools acceptable.

Thus, in this case the result is immediate.

If $\nu_a \neq \mu_a$,

$$|\nu_a \cap I_a| = \mathfrak{n}_a(\nu) > \mathfrak{n}_a(\mu) = |\mu_a \cap I_a|$$

by the Adjustment Proposition. Therefore Lemma 6 implies that

$$|\nu_a \cap (I \setminus I_a)| = |\nu_a| - |\nu_a \cap I_a| < |\mu_a| - |\mu_a \cap I_a| = |\mu_a \cap (I \setminus I_a)|,$$

which in turn implies that

$$|\nu_a \cap I_b| < |\mu_a \cap I_b|$$

as $I \setminus I_a = I_b$ by assumption. Thus, we see that

$$\mathfrak{n}_b(\nu) = |\nu_b \cap I_b| = |I_b| - |\nu_a \cap I_b| > |I_b| - |\mu_a \cap I_b| = |\mu_b \cap I_b| = \mathfrak{n}_b(\mu)$$

as all students (and in particular all students in I_b) are matched under both μ and ν. Hence in this case

$$\mathfrak{n}_a(\nu) + \mathfrak{n}_b(\nu) > \mathfrak{n}_a(\mu) + \mathfrak{n}_b(\mu);$$

this completes the proof. $\square$

Proof of Proposition 5

Let r_a^1 denote the number of students who rank school a as first choice, and let r_b^1 denote the number of students who rank school b as first choice.

We can obtain the outcome of the SOSM by either the *student proposing deferred acceptance algorithm* or the *cumulative offer process*. We utilize the former in this proof.

By assumption, $|S^a| + |S^b| \geq |I|$. Thus, as each student has a first choice,

$$|S^a| + |S^b| \geq r_a^1 + r_b^1.$$

Hence, either:

1. $|S^a| \geq r_a^1$ and $|S^b| \geq r_b^1$, or
2. $|S^a| > r_a^1$ and $|S^b| < r_b^1$, or
3. $|S^a| < r_a^1$ and $|S^b| > r_b^1$.

In the first case, the student proposing deferred acceptance algorithm terminates in one step and all students receive their first choices under both μ and ν. Thus, the result is immediate in this case.

The analyses of the second and third cases are analogous, so it suffices to consider the case that $|S^a| > r_a^1$ and $|S^b| < r_b^1$.

Claim. *For this case, under both μ and ν,*

- *the number of students receiving their first choices is equal to $|S^b| + r_a^1$, and*
- *the number of students receiving their second choices is equal to $r_b^1 - |S^b|$.*

Proof. We consider the construction of either μ or ν through the student proposing deferred acceptance algorithm, and observe that school b receives $r_b^1 > |S^b|$ offers in Step 1, holding $|S^b|$ of these while rejecting $r_b^1 - |S^b|$. School a, meanwhile, receives $r_a^1 < |S^a|$ offers and holds all of them. In Step 2, all students rejected by school b apply to school a, bringing the total number of applicants at school a to $r_a^1 + (r_b^1 - |S^b|)$. As $r_a^1 + (r_b^1 - |S^b|) \leq |S^a|$ by assumption, no student is rejected by school a, and the algorithm terminates in Step 2. Hence, under both μ and ν,

- $|S^b|$ students are assigned to school b as their first choice,
- r_a^1 students are assigned to school a as their first choice, and

- $r_b^1 - |S^b|$ students are assigned to school a as their second choice.

These observations show the claim.

The preceding claim shows the result for the second case; since an analogous argument shows the result for the third case, this completes the proof. □

C Examples

The first example shows that there is no general relationship between the set of walk-zone students assigned to a school when an open slot is changed to a walk-zone slot. The second example shows that some walk-zone students at a school can be worse off when an open slot is changed to a walk-zone slot.

Example C1. There are three schools $A = \{k, l, m\}$. Each school has one available slot. There are 4 students $I = \{i_1, i_2, i_3, i_4\}$. There are two walk-zone students at k and one walk-zone student at l and m: $I_k = \{i_1, i_2\}$, $I_l = \{i_3\}$ and $I_m = \{i_4\}$. The random tiebreaker π^o orders the students as:

$$\pi^o : i_1 \succ i_4 \succ i_2 \succ i_3.$$

The preference profile is:

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}
l	k	m	k
k	m	l	a_0
m	l	k	l
a_0	a_0	a_0	m

First consider the case where k and m's slots are open slot and l's slot is walk-zone. The outcome of DA for this case is:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ k & m & l & a_0 \end{pmatrix}.$$

Next we replace the open slot at school k with a walk-zone slot.

The outcome of DA for the second case is:

$$\mu' = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ l & k & m & a_0 \end{pmatrix}.$$

Example C2. There are three schools $A = \{k, l, m\}$. Each school has one available slot. There are 3 students $I = \{i_1, i_2, i_3\}$. There are two walk-zone students at k and one walk-zone student at l: $I_k = \{i_1, i_2\}$ and $I_l = \{i_3\}$. The random tiebreaker π^o orders the students as:

$$\pi^o : i_3 \succ i_2 \succ i_1.$$

The preference profile is:

P^{i_1}	P^{i_2}	P^{i_3}
k	l	k
m	k	l
l	m	m
a_0	a_0	a_0

First consider the case where all slots are open slots. The outcome of DA for this case is:

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ m & l & k \end{pmatrix}.$$

Next we replace the open slot at school k with a walk-zone slot.
The outcome of DA for the second case is:

$$\mu' = \begin{pmatrix} i_1 & i_2 & i_3 \\ m & k & l \end{pmatrix}.$$

Here, i_2 is worse off.

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