A Model of Secular Stagnation[*]

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Abstract

In this paper we propose a simple overlapping generations New Keynesian model in which a permanent (or very persistent) slump is possible without any self-correcting force to full employment. The trigger for the slump is a deleveraging shock which can create an oversupply of savings. Other forces that work in the same direction and can both create or exacerbate the problem include a drop in population growth, an increase in income inequality and a fall in the relative price of investment. High savings, in turn, may require a permanently negative real interest rate. In contrast to earlier work on deleveraging, our model does not feature a strong self-correcting force back to full employment in the long-run, absent policy actions. Successful policy actions include, among others, an increase in the inflation target and an increase in government spending. We also establish conditions under which an income redistribution can increase demand. Policies such as committing to keep nominal interest rates low or temporary government spending, however, are less powerful than in models with temporary slumps. Our model sheds light on the long persistence of the Japanese crisis, the Great Depression, and the slow recovery out of the Great Recession.

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1 Introduction

During the closing phase of the Great Depression in 1938, the President of the American Economic Association, Alvin Hansen, delivered a disturbing message in his Presidential Address to the Association (see Hansen (1939)). He suggested that the Great Depression might just be the start of a new era of ongoing unemployment and economic stagnation without any natural force towards full employment. This idea was termed the ”secular stagnation” hypothesis. One of the main driving forces of secular stagnation, according to Hansen, was a decline in the population birth rate and an oversupply of savings that was suppressing aggregate demand. Soon after Hansen's address, the Second World War led to a massive increase in government spending effectively ending any concern of insufficient demand. Moreover, the baby boom following WWII drastically changed the population dynamics in the US, thus effectively erasing the problem of excess savings of an aging population that was of principal importance in his secular stagnation hypothesis.

Recently, Hansen's secular stagnation hypothesis has gained renewed attention. One obvious motivation is the Japanese malaise that has by now lasted two decades and has many of the same symptoms as the U.S. Great Depression - namely decreasing population growth, a nominal interest rate at zero, and subpar GDP growth. Another reason for renewed interest is that, even if the financial panic of 2008 was contained, growth remains weak in the United States and unemployment high. Most prominently, Lawrence Summers raised the prospect that the crisis of 2008 may have ushered in the beginning of secular stagnation in the United States in much the same way as suggested by Alvin Hansen in 1938. Summers suggests that this episode of low demand may have begun well before 2008 but was masked by the housing bubble before the onset of the crisis of 2008. In Summers' words, we may have found ourselves in a situation in which the natural rate of interest - the short-term real interest rate consistent with full employment - is permanently negative (see Summers (2013)). And this, according to Summers, has profound implications for the conduct of monetary, fiscal, and financial stability policy today.

Despite the prominence of Summers' discussion of the secular stagnation hypothesis and a flurry of commentary that followed it (see e.g. Krugman (2013), Taylor (2014), Delong (2014) for a few examples), there has not, to the best of our knowledge, been any attempt to formally model this idea - to write down an explicit model in which unemployment is high for an indefinite amount of time due to a permanent drop in the natural rate of interest. The goal of this paper is to fill this gap.

It may seem somewhat surprising that the idea of secular stagnation has not already been studied in detail in the recent literature on the liquidity trap. This literature already invites the possibility that the zero bound on the nominal interest rate is binding for some period of time due to a drop in the natural rate of interest. The reason for this omission, we suspect, is that secular stagnation does not emerge naturally from the current vintage of models in use in the literature.

This, however - and perhaps unfortunately - has less to do with economic reality than with the limitations of these models. Most analyses of the current crisis takes place within representative agent models (see e.g. Krugman (1998), Eggertsson and Woodford (2003), Christiano, Eichenbaum and Rebelo (2011) and Werning (2012) for a few well known examples). In these models, the long run real interest rate is directly determined by the discount factor of the representative agent which is fixed. The natural rate of interest can then only temporarily deviate from this fixed state of affairs due to preference shocks. And changing the discount rate permanently (or assuming a permanent preference shock) is of no help either since this leads the intertemporal budget constraint of the representative household to "blow up" and the maximization problem of the household is no longer well defined.[1]

Meanwhile, in models in which there is some heterogeneity in borrowing and lending (such as Curdia and Woodford (2010), Eggertsson and Krugman (2012) and Mehrotra (2012)), it remains the case that there is a representative saver whose discount factor pins down a positive steady state interest rate. But, as we shall see, moving away from a representative savers framework, to one in which households transition from borrowing to saving over their lifecycle will have a major effect on the steady state interest rate and can open up the possibility of a secular stagnation.[2]

In what follows, we begin by outlining a simple endowment economy (in the spirit of Samuelson (1958)) with overlapping generations (OLG) where households go through three stages of life: young, middle aged and old. The endowment income is concentrated within the middle generation. This gives rise to demand for borrowing by the young, and gives the middle aged an incentive to save part of their endowment for old age by lending it to the young. This lending, however, is constrained by a debt limit faced by young agents. In this environment, we will see that the steady state real interest rate is no longer determined solely by households' discount factor. Instead, it depends on the relative supply of savings and demand for loans, and the equilibrium real interest rate may easily be permanently negative. Forces that work in this direction include a slowdown in population growth, which increases the relative supply of savings, along with a tighter debt limit, which directly reduces demand for loans. Under some conditions, income inequality, either across generations or within generations, may also generate a negative real interest rate. Interestingly enough, all three factors - increases in inequality, a slowdown in population growth, and a tightening of borrowing limits - have been at work in several economies that have experienced low interest rates and subpar growth in recent years.

One interesting result emerges when we consider a debt deleveraging shock of the kind com-

[1] One alternative that is consistent with representative agents models and we do not consider here is an increase in uncertainty, which may also lower the natural rate of interest. Caballero and Farhi (2014) contains some examples of how uncertainty may affect the natural rate of interest and thus trigger a liquidity trap.

[2] Two closely related papers, Guerrieri and Lorenzoni (2011) and Philippon and Midrigin (2011) feature richer heterogeneity than the papers cited above, although not lifecycle dynamics which are critical to our results. These papers too, however, only deliver a temporary liquidity trap in response to a negative deleveraging shock.

mon in the literature (see e.g. Eggertsson and Krugman (2012), Guerrieri and Lorenzoni (2011), Philippon and Midrigin (2011), Hall (2011), Mian and Sufi (2011) and Mian and Sufi (2012) for both theoretical and empirical analysis). In that work, typically, this shock leads to a temporary reduction in the real interest rate as debtors pay down their debt and savers need to compensate for the drop in overall spending by increasing their spending. Once the deleveraging process is completed (debt is back to a new debt limit), the economy returns to its steady state with a positive interest rate. In our framework, however, no such return to normal occurs. Instead, a period of deleveraging puts even more downward pressure on the real interest rate so that it becomes permanently negative. The key here is that households shift from borrowing to saving over their lifecycle. If a borrower takes on less debt today (due to the deleveraging shock) then when tomorrow comes, he has greater savings capacity since he has less debt to repay. This implies that deleveraging - rather than facilitating the transition to a new steady state with a positive interest rate - will instead reduce the real rate even further by increasing the supply of savings in the future. The point here is not to predict that the natural rate of interest will necessarily remain negative indefinitely. Instead, in contrast to existing models, our model accommodates a negative natural rate of interest of arbitrary persistence that can, therefore, explain long-lasting slumps. This is particularly relevant when considering the Great Depression in the US (where the short term interest rate started to drop in 1929 only to finally start rising again in 1947) or current day Japan (where interest rates started falling in 1994 and remain at zero).

We then augment our simple endowment economy with a nominal price level. A key result that emerges is that, under flexible prices, there exists a lower bound on steady state inflation, which can be no lower than the negative of the natural rate of interest. Thus, for example, if the natural rate of interest is -4%, then there is no equilibrium that is consistent with inflation below 4% in steady state. The secular stagnation hypothesis implies that long-run price stability is impossible when prices are flexible. This has profound implications for an economy with realistic pricing frictions. If a central bank can force inflation below this "natural" lower bound, it does so at the expense of generating a permanent recession.

We next turn to a model in which wages are downwardly rigid as originally suggested by Keynes (1936) as an explanation for persistent unemployment in the Great Depression. In this economy, we show that if the central bank is unwilling to tolerate high enough inflation, there is a permanent slump in output. In line with the literature that emphasizes deleveraging shocks that have short-term effects, we find that, in this economy, a long slump is one in which usual economic rules are stood on their head. The old Keynesian paradox of thrift is in full force, as well as the more recent "paradox of toil" (Eggertsson (2010)), where an increase in potential output decreases actual output as well as the proposition that increasing wage flexibility only worsens the shortfall in output (Eggertsson and Krugman (2012)).

Finally, we consider the role of monetary and fiscal policy. We find that a high enough inflation target can always do away with the slump altogether as it accommodates a negative natural interest rate. Importantly, however, an inflation target which is below what is required has no effect in this context. This result formalizes what Krugman (2014) has referred to as the ”timidity trap” - an inflation target that is too low will have no effect in an economy experiencing a secular stagnation. We show this trap explicitly in the context of our model which only arises if the shock is permanent. Similarly, we illustrate that, in a secular stagnation environment, there are strong limitations of forward guidance with nominal interest rates. Forward guidance relies on manipulating expectations after the zero lower bound shock has subsided; as the shock in our model is permanent, manipulating these types of expectations is of much more limited value. Moving to fiscal policy, we show that either an increase in government spending or a redistribution of income from savers to borrowers can eliminate the output gap, although this latter result depends on the details of the distribution of income (we provide examples in which income redistribution is ineffective).

The paper closes by extending the main model to consider investment and capital accumulation. While introducing capital does not change any of the results already derived in the baseline model, it allows us to illustrate a new mechanism for secular stagnation emphasized both by Keynes (1936) and Summers (2013). According to these authors a decline in the relative price of investment may also put downward pressures on the natural rate of interest and thus serve as an additional trigger for secular stagnation - an insight that we confirm in our model.

We have already noted that our work is related to the relatively large literature on the zero bound in which the natural rate of interest is temporarily negative. Our paper also relates to a different strand of the literature on the zero bound which argues that the zero bound may be binding due to self-fulfilling expectations (see e.g. Benhabib, Schmitt-Grohé and Uribe (2001) and Schmitt-Grohé and Uribe (2013)). As in that literature, our model describes an economy that may be in a permanent liquidity trap. A key difference is that the trigger for the crisis in our model is a real shock rather than self-fulfilling expectations. This implies different policy options, but also that, in our model, the liquidity trap is the unique determinate equilibria facilitating policy experiments. Moreover, it also ties our paper to the hypothesis of secular stagnation. Closer to our study is Kocherlakota (2013) which also features a permanent liquidity trap. One key difference is the focus in that paper on falling land prices as a trigger for the crisis, while our focus is on the forces usually associated with secular stagnation: population dynamics, income inequality, a decline in the relative price of investment, and a debt deleveraging shock.[3]

[3]In the context of the Japanese crisis, Krugman (1998) suggested that population dynamics might be driving some of the decline in the natural rate of interest although he did not explore this possibility explicitly. Carvalho and Ferrero (2014) quantify this force in a medium scale DSGE model in the Japanese context and argue that demographic pressures were a significant contributor.

2 Endowment Economy

Imagine a simple overlapping generations economy. Households live for three periods. They are born in period 1 (young), they become middle age in period 2 (middle age), and retire in period 3 (old). Consider the case in which no aggregate savings is possible (i.e. there is no capital), but that generations can borrow and lend to one another. Moreover, imagine that only the middle and old generation receive any income in the form of an endowment Y_t^m and Y_t^o. In this case, the young will borrow from the middle aged households, which in turn will save for retirement when old when they spend all their remaining income and assets. We assume, however, that there is a limit on the amount of debt the young can borrow. Generally, we would like to think of this as reflecting some sort of incentive constraint, but for the purposes of this paper, it will simply take the form of an exogenous constant D_t (as the "debtors" in Eggertsson and Krugman (2012)).

More concretely, consider a representative household of a generation born at time t. It has the following utility function:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o} \mathbb{E}_t \left\{ \log\left(C_t^y\right) + \beta \log\left(C_{t+1}^m\right) + \beta^2 \log\left(C_{t+2}^o\right) \right\}$$

where C_t^y is the consumption of the household when young, C_{t+1}^m its consumption when middle aged, and C_{t+2}^o its consumption while old. We assume that borrowing and lending take place via a one period risk-free bond denoted B_t^i where $i = y, m, o$ at an interest rate r_t. Given this structure, we can write the budget constraints facing households of the generation born at time t at in each period as:

$$C_t^y = B_t^y \tag{1}$$

$$C_{t+1}^m = Y_{t+1}^m - (1 + r_t) B_t^y + B_{t+1}^m \tag{2}$$

$$C_{t+2}^o = Y_{t+2}^o - (1 + r_{t+1}) B_{t+1}^m \tag{3}$$

$$(1 + r_t) B_t^i \leq D_t \tag{4}$$

where equation (1) corresponds to the budget constraint for the young where consumption is constrained by what the household can borrow. Equation (2) corresponds to the budget constraint of the middle aged household that receives the endowment Y_t^m and repays what was borrowed when young as well as accumulating some additional savings B_{t+1}^m for retirement. Finally, equation (3) corresponds to the budget constraint when the household is old and consumes its savings and endowment received in the last period.

The inequality (4) corresponds to the exogenous borrowing limit (as in Eggertsson and Krugman (2012)), which we assume will bind for the young so that:

$$C_t^y = B_t^y = \frac{D_t}{1 + r_t} \tag{5}$$

where the amount of debt that the young can borrow depends on their ability to repay when middle aged and, therefore, includes interest payments (hence a drop in the real interest rate increases borrowing by the young).

The old at any time t will consume all their income so that:

$$C_t^o = Y_t^o - (1 + r_{t-1}) B_{t-1}^m \tag{6}$$

The middle aged, however, are at an interior solution and satisfy a consumption Euler equation given by:

$$\frac{1}{C_t^m} = \beta \mathbb{E}_t \frac{1 + r_t}{C_{t+1}^o} \tag{7}$$

We assume that the size of each generation is given by N_t. Let us define the growth rate of births by $1 + g_t = \frac{N_t}{N_{t-1}}$. Equilibrium in the bond market requires that borrowing of the young equals the savings of the middle aged so that $N_t B_t^y = -N_{t-1} B_t^m$ or:

$$(1 + g_t) B_t^y = -B_t^m \tag{8}$$

An equilibrium is now defined as set of stochastic processes $\{C_t^y, C_{t,}^o, C_t^m, r_t, B_t^y, B_t^m\}$ that solve $(1), (2), (5), (6), (7)$, and (8) given an exogenous process for $\{D_t, g_t\}$.

To analyze equilibrium determination, let us focus on equilibrium in the market for savings and loans given by equation (8) using the notation L_t^s and L_t^d; the left hand side of (8) denotes the demand for loans, L_t^d, and the right hand side its supply, L_t^s. Hence the demand for loans (using (5)) can be written as:

$$L_t^d = \frac{1 + g_t}{1 + r_t} D_t \tag{9}$$

while the supply for savings - assuming perfect foresight for now - can be solved for by substituting out for C_t^m in (2), using (3) and (7), and for B_t^y by using (5). Then solving for B_t^m, we obtain the supply of loans given by:

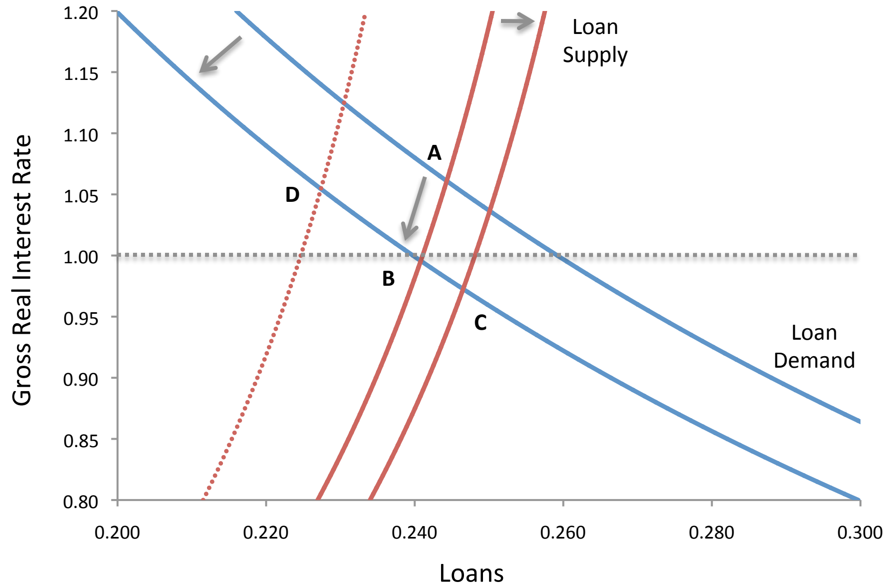
$$L_t^s = \frac{\beta}{1 + \beta} (Y_t^m - D_{t-1}) - \frac{1}{1 + \beta} \frac{Y_{t+1}^o}{1 + r_t} \tag{10}$$

An equilibrium, depicted in Figure 1, is then determined by the intersection of the demand (L_t^d) and supply (L_t^s) for loans at the equilibrium level of real interest rates given by:

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} + \frac{1}{\beta} \frac{Y_{t+1}^o}{Y_t^m - D_{t-1}} \tag{11}$$

Observe that the real interest rate will in general depend on the relative income between the middle aged and the young as well as on the debt limit, population growth, and the discount factor.

Figure 1: Equilibrium in the asset market



2.1 Deleveraging and Slowdown in Population Growth

The objective of this paper is not to develop a theory of the debt limit D_t and we will simply follow the existing literature in assuming that there is a sudden and permanent reduction in its value from some high level, D^H, to a new low level D^L. Eggertsson and Krugman (2012), for example, suggest that a shock of this kind can be thought of as a ”Minsky moment”; a sudden realization that households' collateral constraints were too lax and their focus is on the adjustment of the economy to this new state of affairs. This shock has been described as a deleveraging shock.

A key contribution of this paper, however, is that rather than resulting in only a temporary reduction in the real interest rate as the economy adjusts, the deleveraging shock can instead lead to a permanent reduction in the real interest rate. As we shall see, this has fundamental implications for the nature of the slump in the model once it is extended to include other realistic frictions.

Point B in Figure 1 shows the equilibrium level of the real interest rate when there is a deleveraging shock in our model in the first period of the shock. As we can see, the shock leads directly to a reduction in the demand for loans since the demand curve shifts from $\frac{1+g_t}{1+r_t}D^H$ to $\frac{1+g_t}{1+r_t}D^L$. For now there is no change in the supply of loanable funds as we see in (10) because the debt repayment of the middle aged agents depends upon D_{t-1}.

Let us compare the equilibrium in point B to point A. Relative to the previous equilibrium, the young are now spending less at a given interest rate, while the middle aged and old are spending the same. Since all the endowment has to be consumed in our economy, this fall in spending by the young then needs to be made up by inducing some agents to spend more. This adjustment takes place via reduction in the real interest rate.

The drop in the real interest stimulates spending via two channels. As equation (11) shows, a fall in the real interest rate makes consumption today more attractive to the middle aged, thus increasing their spending.[4] For the credit-constrained young generation, a reduction in the real interest rate relaxes their borrowing constraint. A lower interest rate allows them to take on more debt, B_t^y at any given D_t because their borrowing is limited by their ability to repay in the next period and that payment depends on the interest rate. Observe that the spending of the old is unaffected by the real interest rate at time t; these households will simply spend all their existing savings and their endowment irrespective of the interest rate.

So far, the mechanism described in our model is exactly the same as in Eggertsson and Krugman (2012) and the literature on deleveraging. There is a deleveraging shock that triggers a drop in spending by borrowers at the existing rate of interest. The real interest rate then needs to drop for the level of aggregate spending to remain the same. It is only in what happens next, however, that the path followed by these two models diverge.

In Eggertsson and Krugman (2012), the economy reaches a new steady state next period in which, once again, the real interest rate is determined by the discount factor of the representative savers in the economy. In that setting, the loan supply curve shifts back so that the real interest rate is exactly the same as before (as seen in point D). Loan supply shifts back in Eggertsson and Krugman (2012) because borrower deleveraging reduces interest income accruing to savers which implies that their supply of savings falls in equilibrium.

In contrast, in our model there is no representative saver; instead, households are both borrowers and savers at different stages in their lives. The fall in the borrowing of the young households in period t then implies that in the next period - when that agent becomes a saver household - the middle-aged agent has more resources to save (since he has less debt to pay back due to the reduction in D_t). Therefore, at time $t+1$ the supply of savings L_t^s shifts outwards as shown in Figure 1. In sharp contrast to Eggertsson and Krugman (2012) where the economy settles back on the old steady state after a brief transition with a negative interest rate during the deleveraging phase, the economy settles down at a new steady state with a permanently lower real rate of interest. Moreover, this interest rate can easily be negative depending on the size of the shock. This, as we shall see, is a condition for a permanent liquidity trap. Moreover, in a lifecycle model with more periods, this process would serve as a powerful and persistent propagation mechanism for the original deleveraging shock. More generally, even if the drop in D_t is not permanent, the

[4] Observe that the L_t^s curve is upward sloping in $1+r_t$. A drop in the real interest rate reduces desired saving as the present value of future income rises. If preferences exhibit very weak substitution effects and if the endowment is received only by the middle-aged generation, it is possible to have a downward sloping L_t^s curve. Saving that is increasing in the interest rate (and conversely demand that increases when interest rates go down) is in our view the more empirically relevant case.

natural rate of interest will inherit the dynamics of the drop in D_t which may be of arbitrary duration. Other factors may matter too and, as we shall see, changes in these variables may also be arbitrarily persistent.

We see that D_t is not the only factor that can shift the demand for loanable funds. As revealed in equation (9), a reduction in the birth rate of the young will also reduce the demand for loans. This has the consequence of shifting back the demand for loans, and thus triggering a reduction in the real interest rate in just the same way as a reduction in D_t. The intuition here is straightforward: if population growth is slower (a lower g_t), then there are fewer young households. Since the young are the source of borrowing in this economy, the reduction in population growth diminishes demand for loans. The supply for loanable funds, however, remains unchanged, since we have normalized both L_t^d and L_t^s by the size of the middle generation. The result is a drop in the real interest rate.

There are other forces, however, that may influence the supply of loanable funds - one of which may be inequality. This is what we turn to next.

2.2 Income Inequality

There is no general result about how an increase in inequality affects the real rate of interest. In general, this relationship will depend on how changes in income affect the relative supply and demand for loans. As we shall see, however, there are relatively plausible conditions under which higher inequality will in fact reduce the natural rate of interest.

Consider one measure of inequality that measures the degree of income inequality across generations. As we have already seen in (11), the relative endowment of the old versus the middle generation affects the real interest rate; moving resources from the old to the middle generation will increase savings and thereby put downward pressure on the real interest rate. The conclusion here is that redistribution that raises savings increases downward pressure on the real rate. Consider now the alternative: if all generations receive the same endowment $Y_t^y = Y_t^m = Y_t^o$, then it is easy to see that there is no incentive to borrow or lend, and, accordingly, the real interest rate is equal to the inverse of the discount factor $1 + r_t = \beta^{-1}$. It is thus inequality of income across generations that is responsible for our results and what triggers possibly negative real interest rates.

Generational inequality, however, is typically not what people have in mind when considering inequality; instead commentators often focus on unequal income of individuals within the working-age population. Inequality of that kind can also have a negative effect on the real interest rate. Before getting there, however, let us point out that this need not be true in all cases. Consider, for example an endowment distribution $Y_t(z)$ where z denotes the type of an individual that is known once he is born. Suppose further that once again $Y_t^y(z) = Y_t^m(z) = Y_t^o(z)$ for all z. Here once again, income is perfectly smoothed across ages and the real interest rate is given by β^{-1}. The point is that we can choose any distribution of income $Y(z)$ to support that equilibrium so that, in this case, income distribution is irrelevant for the type of preferences we have assumed.[5]

Let us now consider the case when inequality in a given cohort can in fact generate negative pressure on real interest rates. When authors attribute demand slumps to a rise in inequality they typically have in mind - in the language of old Keynesian models - that income gets redistributed from those with a high propensity for consumption to those that instead wish to save their income. We have already seen how this mechanism works in the case of inequality across generations. But we can also imagine that a similar mechanism applies if income gets redistributed within a cohort as long as some of the agents in that cohort are credit constrained.

Again let us assume that only middle aged and the old generation receive an income endowment. Now, however, suppose that some fraction of households receive a larger endowment in their middle generation (i.e. high-income households) while the remaining households (i.e. low-income households) receive a very small endowment in the middle period of life. For simplicity, all households receive the same income endowment in old age (this could be thought of as some sort of state-provided pension like Social Security). For sufficiently low levels of the middle-period endowment and a sufficiently tight credit constraint, low-income households will remain credit constrained in the middle period of life. These households will rollover their debt in the middle generation and only repay their debts in old age, consuming any remaining endowment. In this situation, only the high-income households will save in the middle period and will, therefore, supply savings to both credit-constrained middle-generation households and the youngest generation.

As before, we can derive an explicit expression for real interest rate in this richer setting with multiple types of households. Under the conditions described above, the only operative Euler equation is for the high-income households who supply loans in equilibrium. The demand for loans is obtained by adding together the demand from young households and from the credit constrained low-income households. The expression we obtain is a generalization of the case obtained in equation (11):

$$1 + r_t = \frac{1+\beta}{\beta} \frac{(1 + g_t + \eta_s)\, D_t}{(1-\eta_s)\left(Y_t^{m,h} - D_{t-1}\right)} + \frac{1}{\beta} \frac{Y_{t+1}^{o}}{\left(Y_t^{m,h} - D_{t-1}\right)} \tag{12}$$

where η_s is the fraction of low-income households, $Y_t^{m,h}$ is the income of the high-income middle-generation and Y_{t+1}^{o} is the income of these households in the next period (i.e. the pension income received by all households). If $\eta_s = 0$ we recover the expression for the real interest rate derived in (11).

[5]More general preference specifications, however, can easily make a difference.

Total income for the middle generation is a weighted average of high and low income workers: $Y_t^m = \eta_s Y_t^{m,l} + (1-\eta_s) Y_t^{m,h}$. Let us then define an increase in inequality as a redistribution of middle-generation income from low to high-income workers, without any change in Y_t^m. While this redistribution keeps total income for the middle generation constant by definition, it must necessarily lower the real interest rate by increasing the supply of savings which is only determined by the income of the wealthy. This can be seen in equation (12) where the real interest rate is decreasing in $Y_t^{m,h}$ without any offsetting effect via $Y_t^{m,l}$.[6]

As this extension of our model suggests, the secular rise in wage inequality in recent decades in the US and other developed nations may have been one factor in exerting downward pressure on the real interest rate. Labor market polarization - the steady elimination of blue-collar occupations and the consequent downward pressure on wages for a large segment of the labor force - could show up as increase in income inequality among the working-age population, lowering the real interest rate in the manner described here (for evidence on labor market polarization, see e.g. Autor and Dorn (2013) and Goos, Manning and Salomons (2009)). Several other theories have been suggested for the rise in inequality. To the extent that they imply an increase in savings, they could fit into our story as well.

3 Price Level Determination

To this point, we have not described the behavior of the price level or inflation in this economy. We now introduce nominal price determination and first consider the case in which prices are perfectly flexible. In this case, we will see that with negative real rates the economy "needs" permanent inflation, and there is no equilibrium that is consistent with price stability. This will turn out to be critical for the analysis when we introduce realistic pricing frictions.

As is by now standard in the literature, we introduce a nominal price level by assuming that there is traded one period nominal debt denominated in money, and that the government controls the rate of return of this asset (the nominal interest rate).[7] The saver in our previous economy (middle generation household) now has access to risk-free nominal debt which is indexed in dollars in addition to one period risk-free real debt.[8] This assumption gives rise to the consumption

[6] As emphasized earlier, not all forms of income inequality should be expected to have a negative effect on the real interest rate. If middle-generation income is drawn from a continuous distribution, a mean-preserving spread that raises the standard deviation of income could be expected to have effects on both the intensive and extensive margin. That is the average income among savers would rise, but this effect would be somewhat offset by an increase in the fraction of credit-constrained households (i.e an increase in η_s). The extensive margin boosts the demand for loans and would tend to increase the real interest rate. Whether inequality raises or lowers rates would depend on the relative strength of these effects.

[7] There are various approaches to microfound this such as money in the utility function or cash-in-advance constraints.

[8] For simplicity, this asset trades in zero net supply, so that in equilibrium the budget constraints already analyzed

Euler equation which is the nominal analog of equation (7):

$$\frac{1}{C_t^m} = \beta E_t \frac{1}{C_{t+1}^o}(1+i_t)\frac{P_t}{P_{t+1}} \tag{13}$$

where i_t is the nominal rate and P_t is the price level. We impose a nonnegativity constraint on nominal rates. Implicitly, we assume that the existence of money precludes the possibility of a negative nominal rate. At all times:

$$i_t \geq 0 \tag{14}$$

Equation (7) and (13) imply (assuming perfect foresight) the standard Fisher equation:

$$1 + r_t = (1+i_t)\frac{P_t}{P_{t+1}} \tag{15}$$

where again recall that r_t is exogenously determined as before by equation (11). The Fisher equation simply states that the real interest rate should be equal to the nominal rate deflated by the growth rate of the price level (inflation).[9]

From (14) and (15), it should be clear that if the real rate of interest is permanently negative, there is no equilibrium consistent with stable prices. To see this, assume there is such an equilibrium so that $P_{t+1} = P_t = P^*$. Then the Fisher equation implies that $i_t = r_t < 0$ which violates the zero bound. Hence a constant price level – price stability – is inconsistent with our model when r_t is permanently negative. The next question we ask, then, is what constant growth rates of the price level are consistent with equilibrium?

Let us denote the growth rate of the price level (inflation) by $\Pi_t = \frac{P_{t+1}}{P_t} = \bar{\Pi}$. The zero bound and the Fisher equation then implies that for an equilibrium with constant inflation to exist, there is a bound on the inflation rate given by $\Pi(1+r) = 1 + i \geq 1$ or:

$$\bar{\Pi} \geq \frac{1}{1+r} \tag{16}$$

which implies that steady state inflation is bounded from below by the real interest rate due to the zero bound.

Observe that at a positive real interest rate this bound may seem of little relevance. If, as is common in the literature using representative agent economies, the real interest rate in steady state is equal to inverse of the discount factor, then this bound says that $\Pi \geq \beta$. In a typical calibration this implies a bound on steady state inflation rate of about -2% to -4% - well below the inflation target of most central banks meaning that the bound is of little empirical relevance.

are unchanged.

[9]Again, we can define an equilibrium as a collection of stochastic processes $\{C_t^y, C_{t,}^o, C_t^m, r_t, i_t, B_t^y, B_t^m, P_t\}$ that solve (1), (2), (5), (6), (7) and (8) and now in addition (13) and (14) given the exogenous process for $\{D_t, g_t\}$ and some policy reaction function for the government like an interest rate rule.

With a negative real rate, however, this bound takes on a greater practical significance. If the real interest rate is permanently negative, it implies that, under flexible prices, steady state inflation needs to be permanently above zero and possibly well above zero depending on the value of the steady state real interest rate. This insight will be critical once we move away from the assumption that prices are perfectly flexible. In that case, if the economy calls for a positive inflation rate – and cannot reach that level due to policy (e.g. a central bank committed to low inflation) – the consequence will be a permanent drop in output instead.

4 Endogenous Output

We now extend our model to allow output to be endogenously determined. We simplify our exposition by assuming that only the middle aged agents obtain income. Now, however, they supply labor to generate this income. This assumption simplifies the exposition slightly and allows us to prove some results analytically. We show the full model in the Appendix.

The budget constraint of the agents is again given by equations (1) and (4), but now we replace the budget constraint of the middle aged (2) and old generation (3) with:

$$C^m_{t+1} = \frac{W_{t+1}}{P_{t+1}} L_{t+1} + \frac{Z_{t+1}}{P_{t+1}} - (1+r_t)B^y_t + B^m_{t+1} \tag{17}$$

$$C^o_{t+2} = -(1+r_{t+1})B^m_{t+1} \tag{18}$$

where W_{t+1} is the nominal wage rate, P_{t+1} the aggregate price level, L_{t+1} the labor supply of the middle generation, and Z_{t+1} the profits of the firms. For simplicity, we assume that the middle generation will supply its labor inelastically at $\bar{L}$. Note that if the firms do not hire all available labor supplied, then L_t may be lower than $\bar{L}$ as we further outline below. Under this assumption, each of the generations' optimization conditions remain exactly the same as before.

On the firm side, we assume that firms are perfectly competitive and take prices as given. They hire labor to maximize period-by-period profits. Their problem is given by:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t \tag{19}$$

$$\text{s.t.} \quad Y_t = L_t^\alpha \tag{20}$$

The firms' labor demand condition is then given by:

$$\frac{W_t}{P_t} = \alpha L_t^{\alpha-1} \tag{21}$$

So far we have described a perfectly frictionless production side and if this were the end of the story, our model would be analogous to what we have already studied in the endowment economy. Output would now be given by $Y_t = L_t^\alpha = \bar{L}^\alpha$ and equation (21) would determine the real wage (and $\frac{W_t}{P_t}L_t + \frac{Z_t}{P_t} = Y_t$ so the middle generation's budget constraint would remain the same as in the endowment economy).

We will, however, now deviate from this frictionless world. The friction with longest history in Keynesian thought is that of downward nominal wage rigidities. Keynes put this friction at the center of his hypothesis of the slump of the Great Depression, and we follow his example here. More recently these type of frictions have been gaining increasing attention due to the weak recovery of employment from the Great Recession (see e.g. Shimer (2012) and Schmitt-Grohé and Uribe (2013) - the latter has a similar specification for wage-setting as we adopt here).[10] A key implication of downwardly rigid wages is that output can fall permanently below the full-employment level. As we will see, if a central bank is unwilling to let inflation be what it "needs to be" (given by the inequality (16)), it will have to tolerate some level of unemployment.

To formalize this, consider a world in which households will never accept working for wages that fall below their wage in the previous period so that nominal wages at time t cannot be lower than what they were time $t-1$. Or to be slightly more general, imagine that the household would never accept lower wages than a wage norm given by $\tilde{W}_t = \gamma W_{t-1} + (1-\gamma)\alpha\bar{L}^{\alpha-1}$. If $\gamma = 1$ this corresponds to a perfectly downwardly rigid wage, but, if $\gamma = 0$, we obtain the flexible wage case considered earlier in which our model simplifies to the endowment economy. Formally, if we take this assumption as given, it implies that we replace $L_t = \bar{L}$ with:

$$W_t = \max\left\{\tilde{W}_t, P_t\alpha\bar{L}^{\alpha-1}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma)P_t\alpha\bar{L}^{\alpha-1} \tag{22}$$

We see that nominal wages can never fall below the wage norm $\tilde{W}_t$ - wages are downwardly rigid with the degree of rigidity parameterized by γ. If labor market clearing requires higher nominal wages than the past nominal wage rate, this specification implies that demand equals supply and the real wage is given by (21) evaluated at $L_t = \bar{L}$.

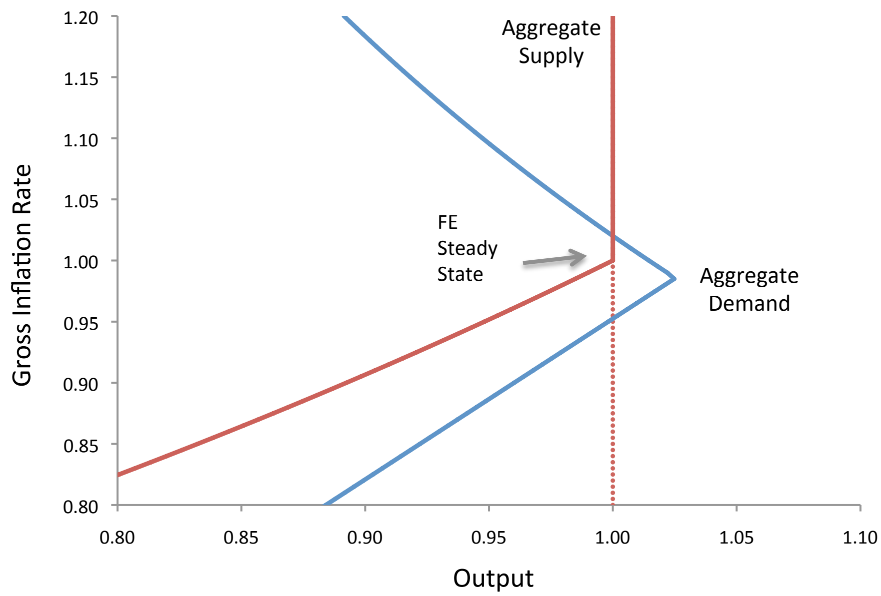
Moving to monetary policy, now suppose that the central bank sets the nominal rate according to a standard Taylor rule:

$$1 + i_t = \max\left(1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right) \tag{23}$$

where $\phi_\pi > 1$ and Π^* and i^* are parameters of the policy rule that we hold constant. This rule states that the central bank tries to stabilize inflation around an inflation target Π^* (determined by the steady state nominal interest rate i^* and the inflation target Π^*) unless it is constrained by the zero bound. We define an equilibrium as a collection of quantities $\{C_t^y, C_{t,}^o, C_t^m, B_t^y, B_t^m, L_t, Y_t, Z_t\}$ and prices $\{P_t, W_t, r_t, i_t\}$ that solve (1), (5), (6), (7), (8), (13), (14), (17), (19), (20), (21), (22) and (23) given an exogenous process for $\{D_t, g_t\}$.

[10] The presence of substantial nominal wage rigidity has been established empirically recently in US administrative data by Fallick, Lettau and Wascher (2011), in worker surveys by Barattieri, Basu and Gottschalk (2010), and in cross-country data by Schmitt-Grohé and Uribe (2011).

Figure 2: Steady state aggregate demand and aggregate supply curves



We will now analyze the steady state of this economy and show that it may very well imply a permanent contraction at zero interest rates. We will defer to later sections discussion of transition dynamics in response to exogenous shocks and an analysis of the determinacy of the steady state.

It is fairly easy to analyze the steady state of the model. This can be done in two steps: first by tracing out the aggregate supply of the economy and then aggregate demand of the economy. Consider first aggregate supply.

The aggregate supply specification of the model consists of two regimes: one in which real wages are equal to the market clearing real wage (if $\Pi \geq 1$), and the other when the bound on nominal wages is binding ($\Pi < 1$). Intuitively, positive inflation in steady state means that wages behave as if they are flexible since nominal wages must rise to keep real wages constant.[11] If $\Pi \geq 1$, then labor demand equals the exogenous level of labor supply $\bar{L}$ defining the full employment level of output Y^f:

$$Y = \bar{L}^{\alpha} = Y^f \text{ for } \Pi \geq 1 \tag{24}$$

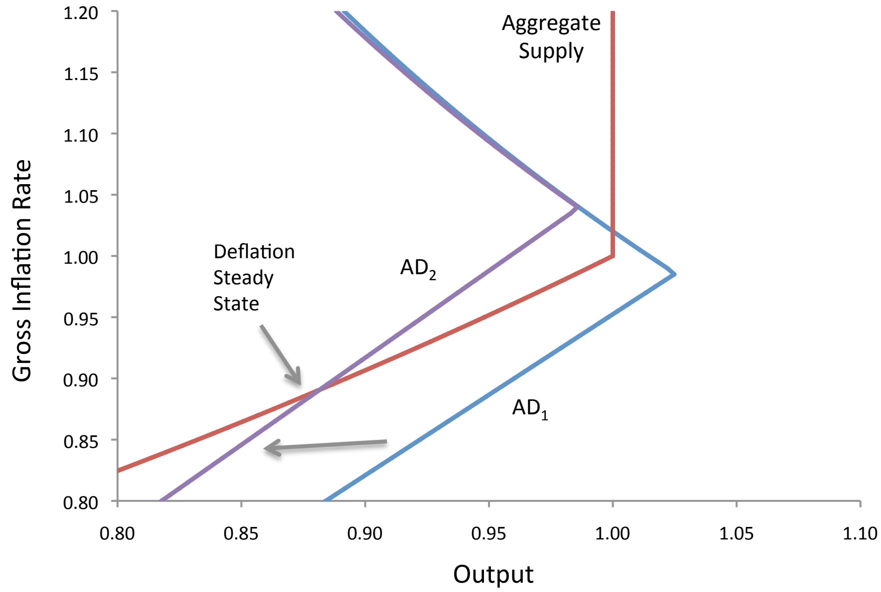
This is shown as a solid vertical segment in the AS curve in Figure 2.[12]

Let us now turn to the case in which $W = \tilde{W}$ which obtains when $\Pi < 1$. The real wage is then given by $w = \frac{(1-\gamma)\alpha\bar{L}^{\alpha-1}}{1-\gamma\Pi^{-1}}$ and substituting this into equation (21) and using the production

[11] First observe that for W to exceed $\tilde{W}$ in steady state, it must be the case that the real wage - denoted $w = \frac{W}{P}$) - is $w \geq \gamma w \Pi^{-1} + (1-\gamma)\,\alpha\bar{L}^{\alpha-1}$. This is satisfied as long as $\Pi \geq 1$.

[12] We normalize $Y^f = 1$.

Figure 3: Steady state aggregate demand and aggregate supply curves



function (20) to express the result in terms of output, we obtain:

$$\frac{\gamma}{\Pi} = 1 - (1-\gamma)\left(\frac{Y}{Y^f}\right)^{\frac{1-\alpha}{\alpha}} \quad \text{for } \Pi < 1 \tag{25}$$

which shows that output increases with inflation. Equation (25) is simply a nonlinear Phillips curve. The intuition is straightforward: as inflation increases, real wages decrease (since wages are rigid) and hence firms hire more labor. As γ approaches unity, the Phillips curve gets flatter, and as γ approaches zero, the Phillips curve becomes vertical. Importantly, this Phillips curve relationship is not a short-run relationship; instead, it describes the behavior of steady state inflation and output. This is because we assume that wages are downwardly rigid - even in the long run, a point that will be critical to ensure existence of equilibrium in our model as we will see. The aggregate supply curve is shown in Figure 2, with the vertical segment given by (24) and the upward-sloping segment given by (25) with the kink at $\Pi = 1$.

Let us now turn to the aggregate demand side. As in the case of aggregate supply, the demand side consists of two regimes: one when the zero bound is not binding, the other when it is binding. Let us start with deriving aggregate demand at positive nominal rates. Demand is obtained by summing up the consumption of the three generations and substituting out for the nominal interest rate with the policy reaction function (23). Combining equations (11) , (15) , (23) and assuming $i > 0$, we get:

$$Y = D + \frac{(1+\beta)(1+g)D\Gamma^*}{\beta}\frac{1}{\Pi^{\phi_\pi - 1}} \quad \text{for } i > 0 \tag{26}$$

where $\Gamma^* \equiv (1+i^*)^{-1} (\Pi^*)^{\phi_\pi}$ is the policy parameter given in the policy reaction function. This relationship is the top portion of the AD curve in Figure 2 and shows that as inflation increases output falls. The logic for this relationship is familiar: if inflation increases, then the central bank raises the nominal interest rate by more than one for one (since $\phi_\pi > 1$), which in turn increases the real interest rate and reduces demand.

Consider now the situation in which $i = 0$. In this case, again combining equations (11), (15), (23) and assuming $i = 0$, we get:

$$Y = D + \frac{(1+\beta)(1+g) D}{\beta} \Pi \text{ for } i = 0 \tag{27}$$

which now leads to an upward sloping relationship between inflation and output. The logic should again be relatively straightforward for those familiar with the literature on the short-run liquidity trap: as inflation increases, the nominal interest rate remains constant, thus reducing the real interest rate. This change in the real rate raises consumption demand as shown by the upward sloping segment of the AD curve in Figure 2.

The kink in the aggregate demand curve occurs at the inflation rate at which monetary policy is constrained by the zero lower bound. That is, the AD curve depicted in Figure 2 will become upward sloping when the inflation rate is sufficiently low that the implied nominal rate the central bank would like to set is below zero. Mathematically, we can derive an expression for this kink point by solving for the inflation rate that equalizes the two arguments in the max operator of equation (23):

$$\Pi_{kink} = \left(\frac{1}{1+i^*} \right)^{\frac{1}{\phi_\pi}} \Pi^* \tag{28}$$

The location of the kink in the AD curve depends on both parameters of the policy rule: the inflation target Π^* and the targeted real interest rate i^*.

In what follows, it will be useful to define the natural rate of interest - the interest rate at which output is at its full employment level. The natural rate can be obtained by evaluating equation (11) at $Y_t = Y^f$:

$$1 + r_t^f = \frac{1+\beta}{\beta} \frac{(1+g_t) D_t}{Y^f - D_{t-1}} \tag{29}$$

4.1 Normal Equilibrium and A Long Slump

Equilibrium output and inflation is determined at the intersection of the aggregate demand and aggregate supply curves. We first consider a normal equilibrium when the natural rate of interest is positive followed by a secular stagnation equilibrium when the natural rate of interest is negative. Here, we assume that the centralbank aims for a positive inflation target (that is, $\Pi^* > 1$) and assume that $1 + i^* = (1 + r^f)\Pi^*$ (consistent with a Taylor rule).

Table 1: Parameter values used in AD and AS figures

Description	Parameter	Value
Population growth	g	0.2
Collateral constraint	D	0.28
Discount rate	β	0.77
Labor share	α	0.7
Wage adjustment	γ	0.3
Taylor coefficient	ϕ_π	2
Gross inflation target	Π^*	1.01
Labor supply	L	1

First, consider the case when the natural rate of interest is positive. If $r_f > 0$, then the aggregate demand curve intersects the aggregate supply curve in the vertical segment of the AS curve. The exact intersection is determined by Π^*.[13] This full employment equilibrium is displayed in Figure 2. This figure displays the nonlinear AD and AS curves where the parameters chosen are given in Table 1. An important point to observe is that if there are shocks to D and g that reduce the natural rate of interest, the intersection is unchanged as long as $r_f > 0$. This is because the central bank will fully offset these shocks via cuts in the nominal interest rate. Under our assumed policy rule, the equilibrium depicted in Figure 2 is unique for a small enough inflation target and high enough γ.[14]

The making of a secular stagnation equilibrium is shown in Figure 3. Here, we illustrate the effect of a tightening of the collateral constraint D_t (or a fall in population growth). We assume the shock is large enough so that the natural rate of interest is negative, $\left(r^f < 0\right)$. In particular, we display the effect of a 7.5% tightening of the collateral constraint, which shifts the AD curve from AD_1 to AD_2. As can be seen from equation (26), a reduction in D reduces output for any given inflation rate. This fall in output stems from the decline in consumption of the younger households. Young households cannot borrow as much as before to finance their spending in the early stages of their lives. In the normal equilibrium, this drop in spending would be made up drop in the real interest rate which restores spending back to where it was prior to the shock. The zero bound, however, prevents this adjustment. Hence, the shock moves the economy off the full employment segment of the AS curve to a deflationary equilibrium where the nominal interest rate is zero. Here, steady state deflation raises steady state real wages, thus depressing demand for labor and contracting output. We show in the Appendix (Proposition 1) that the secular stagnation

[13] In case $\Pi^* = 1$ the intersection is at the kink of the AS curve.

[14] Uniqueness is guaranteed if $\gamma = 1$. For large enough values of γ this remains the case. As γ approaches zero, more equilibria are possible. We discuss these additionally equilibria when they appear in a more general setting in Section 6

equilibrium is unique for any $\gamma > 0$. Hence, we have seen that a sufficiently large shock to the collateral constraint will land us into the strange land of the liquidity trap, which we now consider further.

5 Keynesian Paradoxes

As with standard New Keynesian analyses of the zero lower bound, our OLG environment delivers many of the same Keynesian paradoxes discussed in Eggertsson (2010) and Eggertsson and Krugman (2012). These paradoxes can be illustrated graphically using the aggregate demand and aggregate supply framework developed in the previous section.

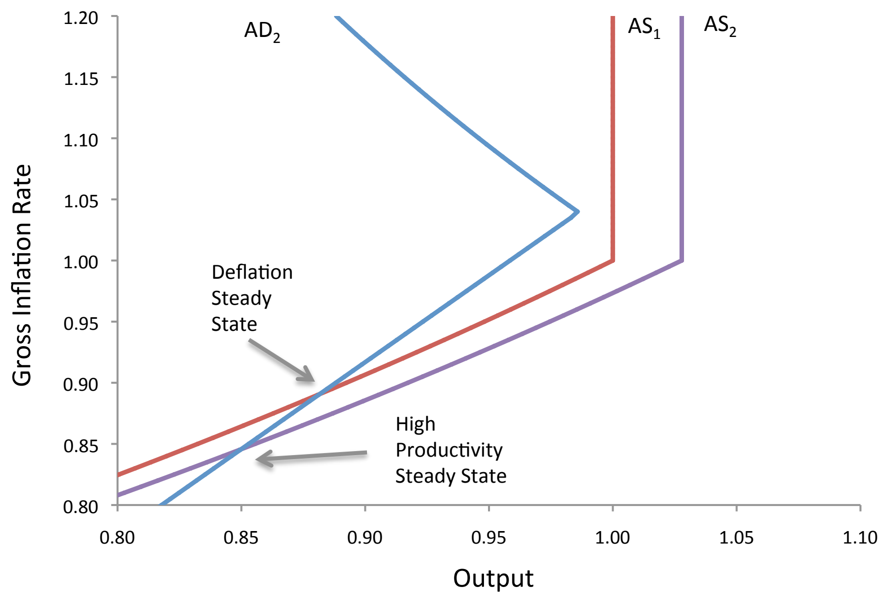
The paradox of thrift in the old Keynesian literature is the proposition that if all households try to save more, aggregate demand falls and, accordingly, households have less income from which to save. Aggregate savings then falls as a consequence of all households trying to save more. For now, we do not have any aggregate saving in this economy as there is no investment. However, precisely the same mechanism is operative in the sense that an increase in the desire to save reduces demand and thereby income. Once we extend the model to incorporate investment in Section 9, we will demonstrate this paradox explicitly.

Two other more recently discovered paradoxes also appear in the model much in the same way as in the literature on temporary liquidity traps. First, there is the paradox of toil first illustrated in Eggertsson (2010). This paradox states that if all households try to work more, there will be less work in equilibrium. More generally any force that increases the overall productive capacities of the economy - for example, a positive technology shock - can have contractionary effects. A shift out in aggregate supply triggers deflationary pressures, raising the real interest rate and further suppressing aggregate demand. The key behind this result is that aggregate demand is increasing in inflation due to the zero bound (see Figure 4) - a drop in inflation cannot be offset by a cut in the nominal rate leading to higher real interest rates.

Figure 4 displays the effect of a labor supply shock that raises labor supply by 4%. This shock shifts the AS curve from AS_1 to AS_2 while leaving the AD curve unchanged and results in an output contraction. A positive productivity shock would have similar effects, shifting down the AS curve and shifting out the full employment line.

Lastly, there is the paradox of flexibility as in, for example, Eggertsson and Krugman (2012). This paradox states that as prices become more flexible, output contracts. This is paradoxical since if all prices and wages were flexible, then there would be no contraction at all. The paradox emerges because an increase in price/wage flexibility triggers a drop in expected inflation, thus increasing the real interest rate. This cannot be offset by interest rate cuts due to the zero bound. As wages become more flexible (i.e. a decrease in the parameter γ), the slope of the AS curve steepens as shown in Figure 5. For any given deflation steady state, a decrease in γ will shift the

Figure 4: Paradox of toil



steady state along the AD curve, increasing the rate of deflation, raising real wages, and therefore increasing the shortfall in output.

Figure 5 illustrates the effect of increasing wage flexibility and the resulting paradox of flexibility. We consider a decrease in γ from 0.3 to 0.27 - a ten percent decrease in wage rigidity. This experiment is illustrated in the figure as a change in the slope of the AS curve depicted by the movement from AS_1 to AS_2. Importantly, the full employment line is left unaffected by any change in γ due to the assumption that wages are only downwardly rigid.

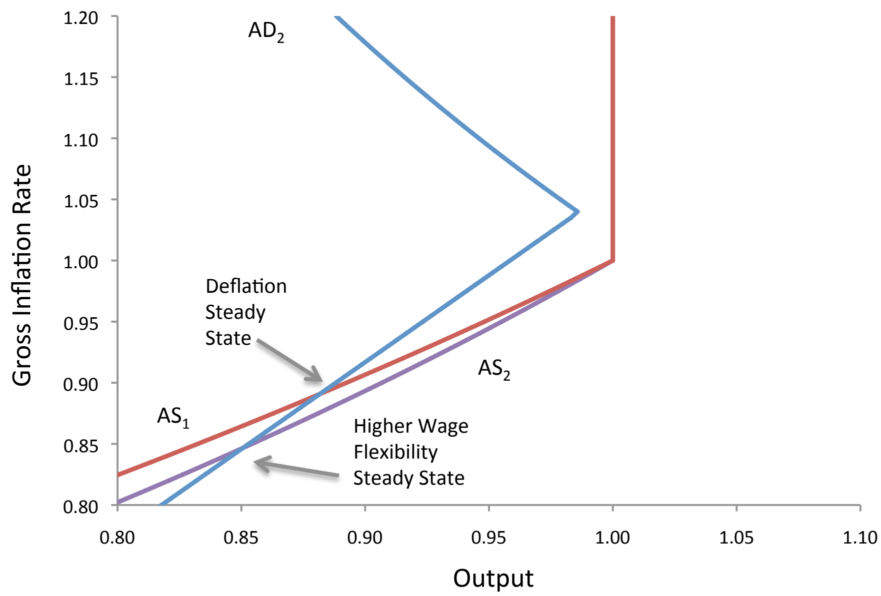
It is worth pointing out that the paradoxes we illustrate in this section rely on the fact that the AD curve is steeper than the AS curve around the equilibrium point. In the Appendix, in Proposition 1, we establish that this is always the case in the model.

6 Monetary Policy

Can monetary policy prevent a secular stagnation? The classic solution to a demand slump in the modern literature on the liquidity trap is a credible commitment to future inflation (see e.g. Krugman (1998)). This commitment reduces the real interest rate and thus stimulates spending. Here we will see that this remedy can also move the economy out of a secular stagnation.

Consider an increase in the inflation target, Π^*, given by our policy rule. This change has no effect on the AS curve but instead shifts the AD curve. Specifically, it shifts out the kink in the AD curve as shown in Figure 6. In Figure 6, the initial inflation target is set at zero percent. As the inflation target increases, the point at which the aggregate demand curve kinks moves

Figure 5: Paradox of flexibility



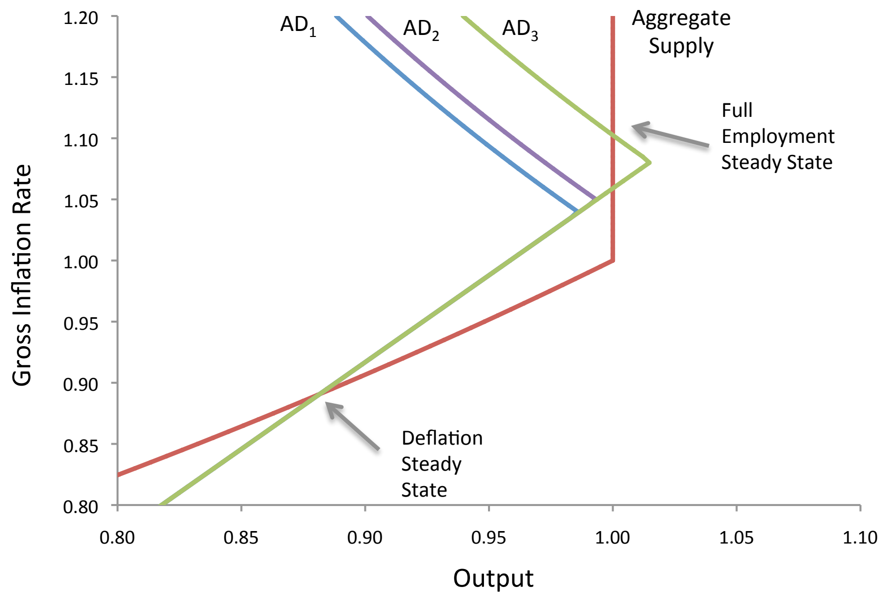
upward effectively shifting out the downward sloping portion of the AD curve. AD_1 shows the original aggregate demand curve with a unique deflationary steady state. AD_2 shows the effect of a modest increase in the inflation target, while AD_3 shows the effect of a large increase in the inflation target. Notice that AD_3 now intersects the aggregate supply curve at three points.

AD_2 illustrates the perils of too small an increase in the inflation target. For a sufficiently negative natural rate of interest, a small increase in the inflation target will not shift the AD curve enough to intersect the full employment line. Under AD_2, the economy remains stuck at the secular stagnation steady state. The contention that small increases in an inflation target will be ineffective has been labeled by Krugman variously as the "timidity trap" or, in reference to Japan in the late 1990s, as the "law of the excluded middle." Our framework captures this idea. Formally, the inflation target needs to be high enough so that $\left(1+r^f\right)\Pi^* \geq 1$; otherwise, one can show that the kink in the AD curve occurs to the left of the full employment line.

For a sufficiently large increase in the inflation target as in the case of AD_3, our model features three equilibria. The AD curve intersects the full employment line twice while the secular stagnation steady state remains unchanged. The AD curve intersects the full employment line twice because there are two positive rates of inflation that ensure the real interest rate is equal to the natural rate of interest.

The equilibrium at the top intersection of the two curves is the "normal" equilibrium at which point inflation is equal to the inflation target of the central bank Π^*. At this point the nominal interest rate is positive because the inflation target is large enough to accommodate the negative shock, i.e. $(1+r^f)\Pi^* > 1$. However, there is another equilibrium at full employment which is

Figure 6: Effect of raising the inflation target



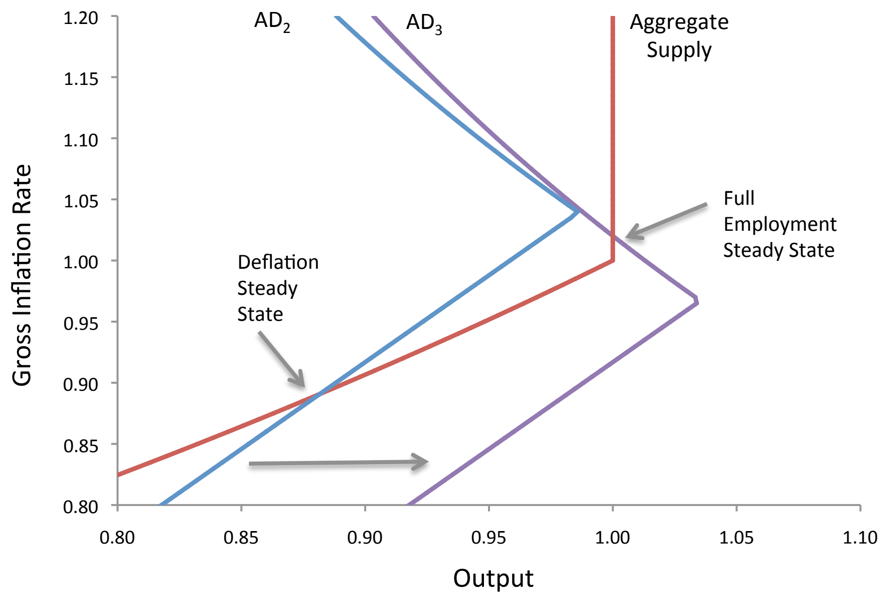
consistent with the policy rule. This is the second intersection of the two curves where $i = 0$ and $\Pi < \Pi^{*}$. This equilibrium, however, is locally indeterminate as we will see in Section 8.

Finally observe that even if an increase in the inflation target made possible two new equilibria – both of which are consistent with full employment – the increase in the inflation target does not eliminate the original secular stagnation equilibrium. As seen in Figure 6, this equilibrium is still present and is fully consistent with a policy rule with a higher inflation target.

In this respect, monetary policy is less effective in our environment than in models that feature temporary liquidity traps such as Krugman (1998) or Eggertsson and Woodford (2003). In those models, a permanent increase in the inflation target will always have an effect because, by assumption, one can always reach it at some point in the future once the shock that gives rise to the liquidity trap is over. Working backwards, a commitment of this sort will always have expansionary effects during the liquidity trap and, provided the inflation target is high enough, it may even eliminate the demand slump altogether. Since the trap is permanent in our model, however, this backward induction breaks down, i.e., there is no future date at which one can be certain that the higher inflation target is reached (even if the policy regime is fully credible in the sense that people do not expect the government to deviate from the policy rule). For the same reason, commitments to keep nominal rates low for a long period of time in a secular stagnation is of limited use, and does not by itself guarantee a recovery. Indeed, interest rate commitments of the variety currently pursued by the Federal Reserve would be irrelevant in shifting the economy out of the deflationary equilibrium since households are expecting rates to stay at zero forever.

Our model is silent on how the government could coordinate expectations on the ”good” full-

Figure 7: Effect of an increase in government spending



employment equilibria via monetary policy. This, then, motivates the consideration of other policy options which we turn to now.

7 Fiscal Policy

While the implications for monetary policy at the zero lower bound differ somewhat in our OLG model from the existing literature, the implications for fiscal policy are closer to the lessons from the representative agent New Keynesian model. In particular, government purchases can be a useful tool for reducing the excess of savings and raising real rates above zero. Importantly, this policy not only creates the possibility of a full employment equilibrium as in the case of monetary policy but also eliminates the possibility of the secular stagnation equilibrium altogether.

If middle-generation households are taxed to finance some level of government expenditure and the government runs a balanced budget, then an increase in government expenditure reduces the supply of loans and raises the real interest rate. The resulting expression for the real rate in the endowment economy in Section 2 is given below:

$$1 + r_t = \frac{1+\beta}{\beta} \frac{(1+g_t)D_t}{(Y_t - G_t - D_{t-1})} \tag{30}$$

where G_t is aggregate government purchases. The expression for the real interest rate in equation (30) is a generalization of the expression for the real interest rate obtained in equation (11), with the real interest rate strictly increasing in the level of government purchases.

Moving to the model with endogenous production, Figure 7 shows the effect of increasing government spending in that environment. In contrast to the case of expansionary monetary policy, higher government spending shifts the whole AD curve outward. Thus, an increase in government spending does not only allow for the possibility of a full employment steady state, it also eliminates the secular stagnation equilibrium.

It is straightforward to consider what the multiplier of government spending is in a secular stagnation for small deviations of government spending from steady state. To a first order approximation, this multiplier is given by:

$$\frac{dY}{dG} = \frac{1}{1 - \psi\kappa} > 1$$

which necessarily is greater than unity.[15] The reason why the multiplier must be greater than one is the positive effect spending has on inflation. This effect is governed by κ which implies that the real interest rate falls with higher levels of spending. If wages are completely fixed, then $\kappa = 0$ and the multiplier is exactly 1. Recall that this version of the model is one in which the middle income household receives all income. If the young or the old also receive some income, then this multiplier can be higher as these households will spend every extra dollar of income on consumption. The multiplier then starts looking similar to that in Eggertsson and Krugman (2012).

The effect of transfers is relatively straightforward in this model. The key determinant of the transfer multiplier is the extent to which transfers redistribute income from the unconstrained agents (the saver) to those that are constrained - namely the young and the old (and possibly credit-constrained middle aged households). In this case, transfers will be expansionary, and many redistribution schemes may satisfy this condition.

When we extend the model to include government debt, we have found that the primary channel through which the public debt affects the real interest rate is through its effect on the distribution of income across generations.[16] Overall the model paints a pretty positive picture for both government spending and redistribution as policies for restoring full employment.

[15]When the model is linearized around a zero inflation steady state, the parameters κ and ψ are as follows:

$$\kappa = \frac{1-\alpha}{\alpha}\frac{1-\gamma}{\gamma}$$
$$\psi = \frac{1+\beta}{\beta}\left(1+g\right)D$$

[16]One can show, for example, that if the government cuts taxes on the middle generation by issuing debt while paying off that debt the next period via taxes on the old, then Ricardian equivalence holds and debt is irrelevant. More generally, however, debt issuance of this kind is likely to imply redistribution across generations which may increase the real interest rate.

8 Determinacy

So far, our analysis of secular stagnation has focused exclusively on the permanent state of affairs (the steady state) for output, inflation, real interest rates and real wages. Left unanswered is whether the transition to this steady state is well behaved once the economy is pushed out of a "normal" equilibrium, due to a deleveraging shock or changing demographics. We now consider this question by linearizing the model around the steady states considered and finding conditions for determinacy of the equilibrium.

Consider a secular stagnation steady state with $\Pi_{ss} < 1$ and $Y_{ss} < Y^f$. Our model can be distilled into a linearized aggregate demand and a linearized aggregate supply curve. In this case, the dynamic AD curve can be obtained by combining (11), (13), and (23) and imposing a binding zero lower bound. The dynamic AS curve can be obtained by combining relations (20), (21) and (22) with $W_t = \tilde{W}_t$. The log-linearized AD and AS curves are given below:

$$s_y y_t = E_t \pi_{t+1} + d_t + s_d d_{t-1} + g_t \tag{31}$$

$$\pi_t = \frac{1}{\gamma_w} \frac{1-\alpha}{\alpha} \left(y_t - \gamma_w y_{t-1} \right) \tag{32}$$

where $\gamma_w = \frac{\gamma}{\Pi_{ss}}$, $s_y = \frac{Y_{ss}}{Y_{ss}-D}$ and $s_d = \frac{D}{Y_{ss}-D}$ and lower case denotes that the variables are expressed in terms of log deviations from steady state.[17] When inflation is stable and output is at its full employment level, the coefficient s_y is equal to ψ^{-1} defined in the previous section. The coefficient s_y must be greater than 1.

We can substitute equation (32) into (31) to eliminate π_{t+1} yielding a first-order stochastic difference equation in terms of y_t:

$$y_t = \phi_y E_t y_{t+1} + \phi_d \left(d_t + s_d d_{t-1} + g_t \right) \tag{33}$$

$$\phi_y = \frac{1}{\gamma_w \left(1 + s_y \frac{\alpha}{1-\alpha} \right)} \tag{34}$$

$$\phi_d = \frac{1}{s_y + \frac{1-\alpha}{\alpha}} \tag{35}$$

A locally unique bounded solution now exists for this system as long as $\phi_y < 1$. It turns out that this condition requires that the slope of the AS curve is flatter (in inflation-output space) than the AD curve. The existence of a unique secular stagnation equilibrium when the natural rate is negative implies that this condition will always be satisfied as we show in Proposition 1 in the Appendix. Thus, the secular stagnation equilibrium always exists for any $\gamma > 0$ and will always be determinate. The determinacy of the deflation steady state in our environment stands in contrast to the deflation steady state analyzed in Schmitt-Grohé and Uribe (2013).[18] Observe

[17] Note that g_t is an exogenous shock to population growth, not a shock to government spending.

[18] For further discussion on the stability properties in Benhabib, Schmitt-Grohé and Uribe (2001), see Bullard (2010).

that equation (33) is also informative about the transition dynamics of this economy in response to deleveraging shocks. For permanent deleveraging shocks, d_t, the transition to steady state occurs in two periods.

We can also analyze the stability properties of the other steady states present when the natural rate is negative but the inflation target is sufficiently high to ensure the existence of full employment steady states. As shown in Figure 6, the full employment steady state involves positive steady state inflation rates and, therefore, real wages adjust to keep output constant at all times. For these steady states, the local dynamics of the system are governed by a single linearized AD curve since $y_t = 0$ at all times:

$$\pi_t = \frac{1}{\phi_\pi} \left(E_t \pi_{t+1} + g_t + d_t + s_d d_{t-1} \right)$$

where d_t is the exogenous collateral shock and g_t is the exogenous population growth shock. This equation is a forward-looking equation in inflation.

The Taylor principle is necessary and sufficient for ensuring existence and determinacy of current inflation. In the full employment steady state where the AD curve is downward sloping, inflation is locally determinate since the nominal rate adjusts more than one for one for a change in inflation. In the second full employment steady state, $\phi_\pi = 0$ and current inflation is indeterminate. This second steady state conforms to the typical deflation steady state studied in Benhabib, Schmitt-Grohé and Uribe (2001), and demonstrates that the deflation steady state discussed in our framework is conceptually different from the deflation steady state in representative-agent New Keynesian models. Our deflation steady state remains locally determinate even when a sufficiently high inflation target ensures the existence of a second locally determinate full-employment steady state.

9 Introducing Capital

So far we have abstracted from capital accumulation. In this section, we show that incorporating capital does not qualitatively change the results we have derived. It does, however, introduce some new mechanisms that can generate a secular stagnation. To introduce capital, we assume that middle-generation households can now not only make loans to young households but also invest in capital that is rented to firms. The firms now produce output using both labor and capital given by a Cobb-Douglas production function:

$$Y_t = K_t^{1-\alpha} L_t^{\alpha}$$

where K_t denotes the capital stock. The firms' labor demand equation (21) remains unchanged, but now firms also rent capital so that the marginal product of capital is equated to the rental rate of capital r_t^k:

$$r_t^k = (1-\alpha)\frac{Y_t}{K_t} \tag{36}$$

For now, let us assume wages are flexible implying that $L_t = \bar{L}$. Thus, we only consider how the introduction of capital affects the natural rate of interest.

The household's objective function is unchanged, and the household's budget constraints are now adjusted for the middle and old generation taking into account capital accumulation as follows:

$$C_{t+1}^m + p_{t+1}^k K_{t+1} + (1+r_t) B_t^y = w_{t+1} L_{t+1} + r_{t+1}^k K_{t+1} + B_{t+1}^m \tag{37}$$

$$C_{t+2}^o + (1+r_{t+1}) B_{t+1}^m = p_{t+2}^k (1-\delta) K_{t+1} \tag{38}$$

where the relative price of capital in terms of the consumption good, p_t^k, is an exogenous variable. We assume that capital is rented out in the same period as when investment takes place, and that the rental income is received by the middle generation household. The household sells the capital it buys in the next period (net of capital depreciation) at a price p_{t+2}^k.

Observe that if consumption could be transformed into capital without any cost, then $p_t^k = 1$ at all times. Instead, we allow for the possibility that the transformation of the consumption good into the capital good may be costly, implying a relative price p_t^k of investment goods. More generally, as shown by Greenwood, Hercowitz and Huffman (1988), this specification emerges from a two sector model with an "investment good" and a "consumption good" that are produced using two different technologies.

Optimal choices for consumption by the middle generation household again result in consumption Euler equation like (7) supplemented with a new equation for the optimal choice of capital:

$$\left(p_t^k - r_t^k\right) C_{t+1}^o = \beta p_{t+1}^k (1-\delta) C_t^m \tag{39}$$

We can combine (7) and (39) to obtain a no arbitrage condition linking the capital rental rate to the real interest rate:

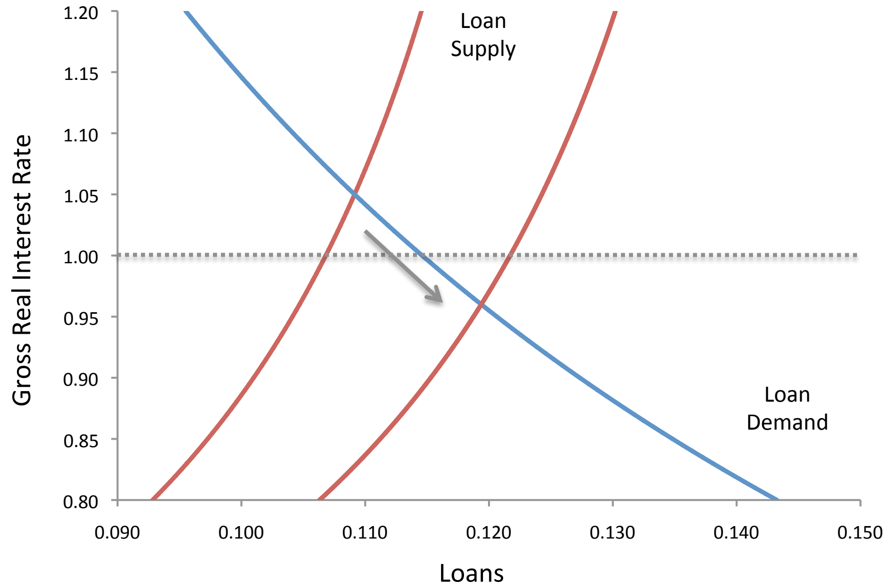
$$r_t^k = p_t^k - p_{t+1}^k \frac{1-\delta}{1+r_t} \geq 0$$

where the inequality follows from the fact that the marginal product of capital given by (36) cannot be negative. If we assume that the relative price of capital is constant in steady state, $p_{ss}^k = \bar{p}$, then the fact that marginal product of capital cannot be negative imposes a lower bound on the steady state real interest rate given by[19]:

$$r_{ss} \geq -\delta$$

[19]The steady state condition is only slightly more complicated if p_t^k features a trend rate of growth, $\frac{p_{t+1}^k}{p_t^k} = \Delta p^k$. Then the expression becomes $r_{ss} \geq \Delta p^k (1-\delta) - 1$. See Karabarbounis and Neiman (2014) for evidence on the long-term decline in the relative price of capital goods.

Figure 8: Effect of decrease in relative price of capital goods



Let us now turn to the equilibrium real interest rate. We can once again equate the demand and supply of loans in the bond market as in (8). The demand for loans L_t^d is still given by equation (9) and the supply for loans is once again given by $L_t^s = -B_t^m$. The key difference is that now to find B_t^m, we must take into account capital accumulation in equation (37). Going through the same steps as in Section 2, we derive the supply of bonds:

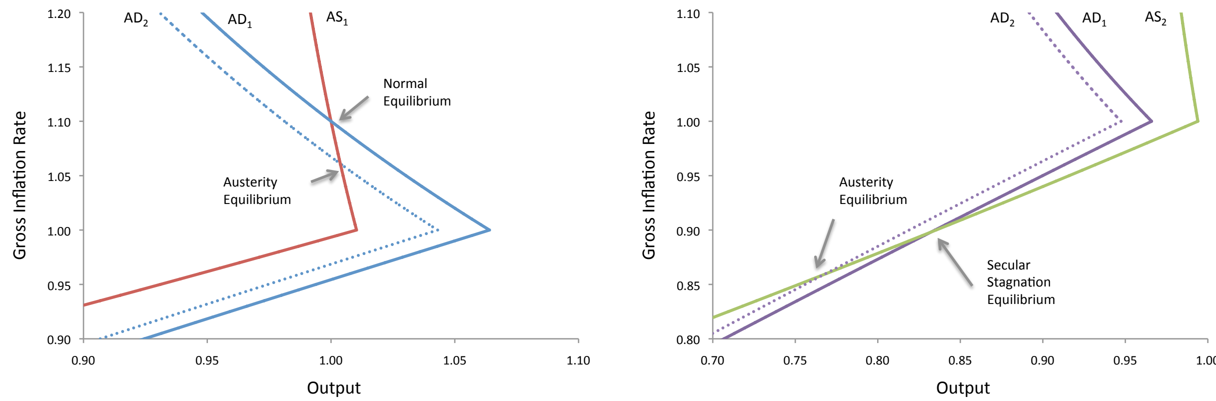
$$L_t^s = \frac{\beta}{1+\beta}(Y_t - D_{t-1}) - \frac{\beta}{1+\beta}\left(p_t^k + \frac{p_{t+1}^k(1-\delta)}{\beta(1+r_t)}\right)K_t \tag{40}$$

Relative to our earlier derivation of loan supply in (10), we see that the last term on the right hand side is new - the presence of capital will reduce the supply of savings available in the bond market. However, as before there is no reason for the two curves, L_t^s and L_t^d, to intersect at a positive real interest rate. Loan supply and demand can intersect at positive or negative real interest rate depending on the value of D_t and g_t. Relative to our earlier exposition, however, there are now interesting new forces at play: the relative price of investment and depreciation.

In outlining his secular stagnation hypothesis, Summers (2013) suggests that ”declines in the cost of durable goods, especially those associated with information technology, mean that the same level of saving purchases more capital every year.” In our model, this idea can be captured as a decline in the relative price of investment goods p_t^k, which results in a decline in the natural rate of interest.

We see that this idea is borne out in the supply for loanable funds in equation (40). A decline in the relative cost of investment leaves more funds left for saving in the bonds market for any given level of the capital stock, thus increasing the supply of funds in the bond market. Meanwhile

Figure 9: Paradox of thrift



the demand for loanable funds in the bond market is unchanged. This is shown graphically in Figure 8 for the parameter values considered in Table 1. A shock that decreases the relative price of investment goods p^k shifts out the loan supply schedule, thereby reducing the natural rate as suggested by Summers.

Keynes (1937), in a speech that was a precursor to the Hansen (1939) address, suggested that "modern inventions are directed towards finding ways of reducing the amount of capital investment necessary to produce a given result" echoing Summers' comments. He also argued that "a result of our experience as to the rapidity of change in tastes and technique, our preference is decidedly directed towards those type of capital goods which are not too durable" arguing that a preference shift towards less durable capital goods would also put downward pressure on the real interest rate. In the context of our model, a natural interpretation of this idea is to explore the effect of an increase in the depreciation rate of the capital stock. The result of this experiment is much the same as the decline in p^k. A decline in the depreciation rate shifts out the loan supply schedule as in Figure 8, thus reducing the real interest rate.

In steady state, we can also compute the investment to GDP ratio to examine the sensitivity of investment to shocks that cause the zero lower bound to bind. The ratio of investment to GDP in steady state can be expressed as follows:

$$\frac{I}{Y} = \delta\frac{K}{Y} = \frac{\delta}{A}\left(\frac{\alpha A}{r_k}\right)^{1-\alpha}$$

The investment to output ratio will fall if the rental rate on capital rises. A shock that leads to the secular stagnation equilibrium will raise the rental rate of capital if the degree of deflation is high enough. At the ZLB, high rates of deflation raise the real interest rate and, therefore, the rental rate on capital suggesting that endogenous capital accumulation may be an important amplification mechanism to generate a large drop in aggregate demand.

In the Appendix, we confirm our previously derived results in the model with capital. While we leave most of the illustrations of these extensions to the Appendix, it is worth highlighting one feature of the model that is new: it provides an explicit illustration of the old Keynesian paradox of thrift that we mentioned in Section 5. In Figure 9, we show the AS and AD schedule in the model with capital and consider the effect of an increase in the discount factor β; that is, we consider a shock that increases household preferences for saving (see the Appendix for further discussion of the derivation of these curves and the underlying assumptions).

Consider first the effect of higher saving at positive interest rates. This is shown on the left hand side of Figure 9. We see that the AS and AD curves intersect on the downward sloping part of the AD curve. The dashed line then shows the effect of an increase in β which shifts back the AD curve. This shift back in the AD curve reduces the real interest rate, which in turn increases output because it increases investment. This latter effect is captured by the fact that the AS curve is downward sloping at positive inflation rates. The downward slope is due to capital accumulation - real interest rates (and hence rental rates) are increasing with inflation over this range. In normal circumstances, if households try to save more, the real interest rate drops, the capital stock increases, and aggregate output rises as shown in Figure 9.

The paradox of thrift states that if everybody tries to save more, there will be less savings in the aggregate. The paradox of thrift is illustrated in the right-hand side of Figure 9 where the AD and AS curve intersect at the zero interest rate in the secular stagnation equilibrium. The figure shows what happens if there is an increase in the desire to save. As the figure shows, this leads to a backward shift in demand, which in turn leads to both lower output and inflation. In normal times, an increase in the desire to save lowers the real interest rate increasing investment and output. At the zero lower bound, however, this adjustment cannot take place - a higher desire to save lowers effective demand and worsens deflation thereby raising the real rate and the rental rate, further depressing investment and output. Thus, our model conclusively demonstrates an important feature of classical Keynesian theory.

10 Conclusion

In this paper, we formalize the secular stagnation hypothesis in an overlapping generations model with nominal wage rigidity. We show that, in this setting, any combination of a permanent collateral (deleveraging) shock, slowdown in population growth, or an increase in inequality can lead to a permanent output shortfall by lowering the natural rate of interest below zero on a sustained basis. Absent a higher inflation target, the zero lower bound on nominal rates will bind, real wages will exceed their market clearing rate, and, output will fall below the full employment level.

We also demonstrate that the Keynesian paradoxes of thrift, toil and flexibility continue to hold in our OLG setting. Like the representative agent model, fiscal policy remains an effective tool for shifting back the AD curve towards full employment. However, commitments by a central bank to keep nominal rates low are ineffective if nominal rates are expected to remain low indefinitely. Sufficiently raising the central bank's inflation target will ensure a full employment steady state when the natural rate is negative, but such a policy will not eliminate the secular stagnation equilibrium.

The main takeaway from our analysis is not a prediction that the world as we see it today will remain mired in a recession forever. Instead, the purpose is to establish the conditions under which a permanent recession can take hold, or more to the point, provide a formalization of the secular stagnation hypothesis. Perhaps an important conclusion from our analysis is not just that a permanent recession is possible, but instead that a liquidity trap can be of arbitrary duration and last as long as the particular shocks that give rise to it (such as a deleveraging shock and/or rise in inequality and/or population growth slowdown). In a richer setting, changes in household formation and worsening income inequality may emerge endogenously from the initial collateral shock prolonging a zero lower bound episode. This would suggest that a passive attitude to a recession of this kind is inappropriate.

We anticipate that the mechanisms illustrated in this environment would persist in a richer, quantitative lifecycle model that could be calibrated to match moments of saving and income. This framework with negative real interest rates may have further implications for asset prices as well, and also, we hope, provide sharper predictions for the future rather than simply illuminating and suggesting a depressing possibility.

References

Autor, David, and David Dorn. 2013. "The Growth of Low-Skill Service Jobs and the Polarization of the US Labor Market." *American Economic Review*, 103(5): 1553–1597.

Barattieri, Alessandro, Susanto Basu, and Peter Gottschalk. 2010. "Some Evidence on the Importance of Sticky Wages." National Bureau of Economic Research.

Benhabib, Jess, Stephanie Schmitt-Grohé, and Martın Uribe. 2001. "The Perils of Taylor Rules." *Journal of Economic Theory*, 96(1): 40–69.

Bullard, James. 2010. "Seven Faces of the Peril." *Federal Reserve Bank of St. Louis Review*, 92(5): 339–52.

Caballero, Ricardo, and Emmanuel Farhi. 2014. "The Safety Trap." National Bureau of Economic Research.

Carvalho, Carlos, and Andrea Ferrero. 2014. "What Explains Japan's Persistent Deflation." Mimeo, PUC-Rio.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo. 2011. "When Is the Government Spending Multiplier Large?" *Journal of Political Economy*, 119(1): 78–121.

Curdia, Vasco, and Michael Woodford. 2010. "Credit Spreads and Fiscal Policy." *Journal of Money, Credit and Banking*, 42(6): 3–35.

Delong, Brad. 2014. "What Market Failures Underlie our Fears of Secular Stagnation?" Blog post.

Eggertsson, Gauti B. 2010. "The Paradox of Toil." Staff Report, Federal Reserve Bank of New York.

Eggertsson, Gauti B., and Michael Woodford. 2003. "The Zero Bound on Interest Rates and Optimal Monetary Policy." *Brookings Papers on Economic Activity*, , (1): 139–234.

Eggertsson, Gauti B, and Paul Krugman. 2012. "Debt, Deleveraging, and the Liquidity Trap: A Fisher-Minsky-Koo Approach." *The Quarterly Journal of Economics*, 127(3): 1469–1513.

Fallick, Bruce, Michael Lettau, and William Wascher. 2011. "Downward Nominal Wage Rigidity in the United States During the Great Recession." Mimeo, Federal Reserve Board.

Goos, Maarten, Alan Manning, and Anna Salomons. 2009. "Job Polarization in Europe." *American Economic Review*, 99(2): 58–63.

Greenwood, Jeremy, Zvi Hercowitz, and Gregory W. Huffman. 1988. "Investment, Capacity Utilization, and the Real Business Cycle." *American Economic Review*, 78(3): 402–417.

Guerrieri, Veronica, and Guido Lorenzoni. 2011. “Credit Crises, Precautionary Savings, and the Liquidity Trap.” National Bureau of Economic Research.

Hall, Robert E. 2011. “The Long Slump.” *American Economic Review*, 101(2): 431–469.

Hansen, Alvin. 1939. “Economic Progress and Declining Population Growth.” *American Economic Review*, 29(1): 1–15.

Karabarbounis, Loukas, and Brent Neiman. 2014. “The Global Decline of the Labor Share.” *The Quarterly Journal of Economics*, 129(1): 61–103.

Keynes, John Maynard. 1936. *General Theory of Employment, Interest and Money.* Atlantic Publishers & Dist.

Keynes, John Maynard. 1937. “Some Economic Consequences of a Declining Population.” *Eugenics Review*, 29(1): 13–17.

Kocherlakota, Naryana. 2013. “Impact of a Land Price Fall when Labor Markets are Incomplete.” Mimeo, Federal Reserve Bank of Minneapolis.

Krugman, Paul R. 1998. “It’s Baaack: Japan’s Slump and the Return of the Liquidity Trap.” *Brookings Papers on Economic Activity*, 2: 137–205.

Krugman, Paul R. 2013. “Secular Stagnation, Coalmines, Bubbles, and Larry Summers.” Blog post.

Krugman, Paul R. 2014. “The Timidity Trap.” The New York Times.

Mehrotra, Neil. 2012. “Fiscal Policy Stabilization: Purchases or Transfers?” Mimeo, Columbia University.

Mian, Atif, and Amir Sufi. 2011. “House Prices, Home Equity-Based Borrowing, and the US Household Leverage Crisis.” *American Economic Review*, 101(5): 2132–56.

Mian, Atif, and Amir Sufi. 2012. “What Explains High Unemployment: The Aggregate Demand Channel.” National Bureau of Economic Research.

Philippon, Thomas, and Virgiliu Midrigin. 2011. “Household Leverage and the Recession.” National Bureau of Economic Research.

Samuelson, Paul A. 1958. “An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money.” *Journal of Political Economy*, 66(6): 467–482.

Schmitt-Grohé, Stephanie, and Martin Uribe. 2011. “Pegs and Pain.” National Bureau of Economic Research.

Schmitt-Grohé, Stephanie, and Martın Uribe. 2013. "The Making of Great Contraction with a Liquidity Trap and a Jobless Recovery." Mimeo, Columbia University.

Shimer, Robert. 2012. "Wage Rigidities and Jobless Recoveries." *Journal of Monetary Economics*, 59(S): S65–S77.

Summers, Lawrence. 2013. "Why Stagnation Might Prove to be the New Normal." The Financial Times.

Taylor, John B. 2014. "The Economic Hokum of 'Secular Stagnation'." The Wall Street Journal.

Werning, Ivan. 2012. "Managing a Liquidity Trap: Monetary and Fiscal Policy." Mimeo, Massachusetts Institute of Technology.

A Derivation of Model

A.1 Households' Problem

In this section, we specify and solve the household's problem in the general case of income received in all periods and taxes paid in all periods. For household i, their objective function and budget constraints are given below:

$$\max_{C_t(i),C_{t+1}(i),C_{t+2}(i)} \mathbb{E}_t \left\{ \log\left(C_t\left(i\right)\right) + \beta \log\left(C_{t+1}\left(i\right)\right) + \beta^2 \log\left(C_{t+2}\left(i\right)\right) \right\} \tag{A.1}$$

$$\text{s.t.} \quad C_t\left(i\right) = w_t L_t\left(i\right) - T_t + B_t\left(i\right) \tag{A.2}$$

$$C_{t+1}\left(i\right) = Z_t + w_{t+1} L_{t+1}\left(i\right) - T_{t+1} + B_{t+1}\left(i\right) - \frac{\left(1+i_t\right)}{\Pi_{t+1}} B_t\left(i\right) \tag{A.3}$$

$$C_{t+2}\left(i\right) = w_{t+2} L_{t+2}\left(i\right) - T_{t+2} - \frac{\left(1+i_{t+1}\right)}{\Pi_{t+2}} B_{t+1}\left(i\right) \tag{A.4}$$

$$B_{t+j}\left(i\right) \leq \mathbb{E}_{t+j}\left(1+r_{t+j+1}\right) D_{t+j} \quad \text{for } j = 0, 1 \tag{A.5}$$

where the household i has exogenous labor supply endowments in each period of life, D_{t+j} is an exogenous collateral constraint, and T_{t+j} are lump sum taxes imposed by the government. For simplicity, taxes do not differ across household types, but taxes may change over time.

We restrict ourselves to cases in which the collateral constraint is binding in the first period of life and possibly binding in the second period of life. In particular, we will assume two types of households - a household that has sufficiently low labor endowment in its middle period of life and remains credit constrained, and a household that has sufficiently high labor endowment in its middle period of life and is unconstrained. For the former, borrowing in the young and middle generations is determined by the binding collateral constraints. For the latter, borrowing is determined by the collateral constraint only while young; in the middle-generation, an Euler equation determines the optimal level of saving:

$$\frac{1}{C_t^{m,h}} = \beta \mathbb{E}_t \frac{1+i_t}{\Pi_{t+1} C_{t+1}^{o,h}} \tag{A.6}$$

Let L^y be the labor endowment for young generation, $L^{m,l}$ be the labor endowment for the poor middle-generation household, $L^{m,h}$ the labor endowment for the wealthy middle-generation household, and L^o the labor endowment in the last period. We adopt the normalization that $L^y + \eta_s L^{m,l} + \left(1-\eta_s\right) L^{m,h} + L^o = 1$. The budget constraints for each type of household alive at any point in time is given below:

$$C_t^y = \alpha Y_t \frac{L^y}{L_t^{flex}} - T_t + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{A.7}$$

$$C_t^{m,l} = \alpha Y_t \frac{L^{m,l}}{L_t^{flex}} + (1-\alpha) Y_t - T_t - D_{t-1} + \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{A.8}$$

$$C_t^{m,h} = \alpha Y_t \frac{L^{m,h}}{L_t^{flex}} + (1-\alpha) Y_t - T_t - D_{t-1} - B_t^{m,h} \tag{A.9}$$

$$C_t^{o,l} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t - D_{t-1} \tag{A.10}$$

$$C_t^{o,h} = \alpha Y_t \frac{L^o}{L_t^{flex}} - T_t + B_{t-1}^{m,h} \frac{1+i_{t-1}}{\Pi_t} \tag{A.11}$$

where T_t are lump sum taxes per capita and Y_t is output per middle-generation household.[20] Aggregate consumption in this economy is given by the following expression:

$$C_t = N_t C_t^y + N_{t-1} \left(\eta_s C_t^{m,l} + (1-\eta_s) C_t^{m,h} \right) + N_{t-2} \left(\eta_s C_t^{o,l} + (1-\eta_s) C_t^{o,h} \right)$$

A.2 Firms' Problem, Labor Supply and Wage Determination

In this section, we specify the firm's problem in the baseline case with no capital accumulation. Firms choose labor to maximize profits subject to a standard decreasing returns to scale production function, taking wages as given:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t^d \tag{A.12}$$

$$\text{s.t.} \quad Y_t = A_t \left(L_t^d \right)^\alpha \tag{A.13}$$

where L_t^d is firm's labor demand. Firms' labor demand is determined by equating the real wage to the marginal product of labor:

$$\frac{W_t}{P_t} = \alpha A_t \left(L_t^d \right)^{\alpha-1} \tag{A.14}$$

Each middle-generation household operates a firm and collects profits from its operation. The total measure of firms in the economy is N_{t-1}, and therefore grows with the total population. All firms are identical sharing the same labor share parameter α.

Labor supply is exogenous and fixed over a household's lifetime. When population is constant ($g=0$), then labor supply is constant and can be normalized to unity. In the absence of downward nominal wage rigidity, the real wage equalizes labor supply to labor demand:

$$\left(N_t L^y + N_{t-1} \left(\eta_s L^{m,l} + (1-\eta_s) L^{m,h} \right) + N_{t-2} L^o \right) = N_{t-1} L_t^{flex} \tag{A.15}$$

[20] Output is not expressed in per capita terms to avoid a proliferation of population growth rate terms. In this economy, aggregate output is $N_{t-1} Y_t$ while the total population is $N_t + N_{t-1} + N_{t-2}$.

where w_t^{flex} defines the market-clearing real wage.

In the presence of downward nominal wage rigidity, the real wage may exceed the market clearing real wage. In this case, labor is rationed with a proportional reduction in labor employed across all households (i.e. if total labor demand is 10% below the full-employment level, then labor falls 10% for all cohorts).

We assume that nominal wages are downwardly rigid implying that real wages exceed the market-clearing level in the presence of deflation. The process determining the real wage is given below:

$$W_t = \max\left\{\tilde{W}_t, P_t w_t^{flex}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) P_t w_t^{flex} \tag{A.16}$$

A.3 Monetary and Fiscal Policy

Monetary and fiscal policy are straightforward. We assume a monetary policy rule of the following form and a balanced budget where per capita taxes always equal per capita government spending:

$$1 + i_t = \max\left(1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi}\right) \tag{A.17}$$

$$G_t = T_t \tag{A.18}$$

where i^* is a the targeted natural rate and Π^* is the central bank's gross inflation target. If the central bank has the correct natural rate target i^*, then inflation is stabilized at $\Pi = 1$ in steady state.

A.4 Market Clearing and Equilibrium

Asset market clearing requires that total lending from savers equals total borrowing from credit constrained young households and poor middle-generation households. This condition is given below:

$$(1-\eta_s) N_{t-1} B_t^{m,h} = N_t \frac{D_t}{1+r_t} + \eta_s N_{t-1} \frac{D_t}{1+r_t} \tag{A.19}$$

$$(1-\eta_s) B_t^{m,h} = (1 + g_t + \eta_s) \frac{D_t}{1+r_t} \tag{A.20}$$

It can be verified that asset market clearing implies that aggregate consumption equals aggregate output less aggregate government purchases:

$$C_t = N_{t-1} Y_t - (N_t + N_{t-1} + N_{t-2}) G_t \tag{A.21}$$

A competitive equilibrium is a set of aggregate allocations $\left\{Y_t, C_t^{m,h}, C_t^{o,h}, B_t^{m,h}, L_t^{flex}, T_t\right\}_{t=0}^{\infty}$, price processes $\left\{i_t, \Pi_t, w_t, w_t^{flex}\right\}_{t=0}^{\infty}$, exogenous processes $\{G_t, g_t, D_t\}_{t=0}^{\infty}$ and initial values of household saving, nominal interest rate, and real wage $\left\{B_{-1}^{m,h}, i_{-1}, w_{-1}\right\}$ that jointly satisfy:

1. Household Euler equation ($A.6$)
2. Household budget constraints ($A.9$) and ($A.11$)
3. Asset market clearing ($A.20$)
4. Fiscal policy rule ($A.18$)
5. Monetary policy rule ($A.17$)
6. Full-employment labor supply ($A.15$)
7. Full-employment wage rate: $w_t^{flex} = \alpha A_t \left(L_t^{flex}\right)^{\alpha-1}$
8. Labor demand condition: $w_t = \alpha A_t \left(\frac{Y_t}{A_t}\right)^{\frac{\alpha-1}{\alpha}}$
9. Wage process: $w_t = \max\left\{\tilde{w}_t, w_t^{flex}\right\}$ where $\tilde{w}_t = \gamma\frac{w_{t-1}}{\Pi_t} + (1-\gamma)w_t^{flex}$

B Derivation of Model with Capital

In this section, we derive the equilibrium conditions characterizing the model with capital presented in Section 9. For simplicity, we assume that only the middle generation supplies labor - the labor endowment for the young and the old generations is zero.

B.1 Households' Problem

In this section, we specify and solve the household's problem in the case of capital income received in the middle period. Capital now provides a new asset by which households can transfer wealth across generations. For household i, their objective function and budget constraints are given below:

$$\max_{C_t(i),C_{t+1}(i),C_{t+2}(i)} \mathbb{E}_t\left\{\log\left(C_t\left(i\right)\right) + \beta\log\left(C_{t+1}\left(i\right)\right) + \beta^2\log\left(C_{t+2}\left(i\right)\right)\right\} \tag{B.1}$$

$$\text{s.t.} \quad C_t\left(i\right) = B_t\left(i\right) \tag{B.2}$$

$$C_{t+1}\left(i\right) = w_{t+1}L_{t+1} + r_{t+1}^k K_{t+1}\left(i\right) + B_{t+1}\left(i\right) - p_{t+1}^k K_{t+1}\left(i\right) - \frac{(1+i_t)}{\Pi_{t+1}}B_t\left(i\right) \tag{B.3}$$

$$C_{t+2}\left(i\right) = p_{t+2}^k K_{t+1}\left(i\right)(1-\delta) - \frac{(1+i_{t+1})}{\Pi_{t+2}}B_{t+1}\left(i\right) \tag{B.4}$$

$$B_{t+j}\left(i\right) \leq \mathbb{E}_{t+j}\left(1+r_{t+j+1}\right)D_{t+j} \quad \text{for } j = 0, 1 \tag{B.5}$$

where p_t^k is an exogenous relative price of capital goods and $K_t\left(i\right)$ are households purchases of the physical capital good.

As before, the middle generation's optimal choice for supplying loans results in a standard Euler equation depending on the expected real interest rate. However, the middle generation now chooses their optimal level of capital accumulation as well. A second Euler equation gives optimal capital supply and determines the relationship between the rental rate of capital and the expected real interest rate.

$$\frac{1}{C_t^m} = \beta \mathbb{E}_t \frac{1+i_t}{\Pi_{t+1} C_{t+1}^o} \tag{B.6}$$

$$\frac{p_t^k - r_t^k}{C_t^m} = \beta \mathbb{E}_t \frac{p_{t+1}^k (1-\delta)}{C_{t+1}^o} \tag{B.7}$$

The budget constraints for each type of household at any point in time is given below:

$$C_t^y = \mathbb{E}_t \Pi_{t+1} \frac{D_t}{1+i_t} \tag{B.8}$$

$$C_t^m = Y_t - D_{t-1} - p_t^k K_t - B_t^m \tag{B.9}$$

$$C_t^o = p_t^k K_{t-1} (1-\delta) + B_{t-1}^m \frac{1+i_{t-1}}{\Pi_t} \tag{B.10}$$

B.2 Firms' Problem, Labor Supply and Wage Determination

Firms now choose both labor and capital to maximize profits subject to a standard Cobb-Douglas production function. Firms are competitive, taking wages and rental rates as given:

$$Z_t = \max_{L_t, K_t} P_t Y_t - W_t L_t - P_t r_t^k K_t \tag{B.11}$$

$$\text{s.t.} \quad Y_t = A_t K_t^{1-\alpha} L_t^{\alpha} \tag{B.12}$$

where r_t^k is the real rental rate on capital. Firms' labor and capital demand equates marginal products with the marginal costs of each factor:

$$\frac{W_t}{P_t} = w_t = \alpha \frac{Y_t}{L_t} \tag{B.13}$$

$$r_t^k = (1-\alpha) \frac{Y_t}{K_t} \tag{B.14}$$

Since production is now constant returns to scale, we need not specify who operates firms in the economy and the number of firms in the economy. As before, we assume that nominal wages are downwardly rigid implying that real wages exceed the market-clearing level in the presence of deflation. The process determining the real wage is given below:

$$W_t = \max\left\{\tilde{W}_t, P_t w_t^{flex}\right\} \text{ where } \tilde{W}_t = \gamma W_{t-1} + (1-\gamma) P_t w_t^{flex} \tag{B.15}$$

where w_t^{flex} is the real wage that would obtain if nominal wages were fully flexible. Even though labor supply is exogenous and constant, this flexible price wage depends on the level of capital accumulation which, in turn, depends on the real interest rate:

$$w_t^{flex} = \alpha A_t K_t^{1-\alpha} \bar{L}^{\alpha-1} \tag{B.16}$$

B.3 Market Clearing and Equilibrium

Asset market clearing requires that total lending from savers equals total borrowing from credit constrained young households. This condition is given below:

$$B_t^m = (1+g_t)\frac{D_t}{1+r_t} \tag{B.17}$$

A competitive equilibrium in this economy is a set of aggregate allocations $\{Y_t, L_t, C_t^m, C_t^o, B_t^m, K_t\}_{t=0}^{\infty}$, price processes $\left\{i_t, \Pi_t, r_t^k, w_t, w_t^{flex}\right\}_{t=0}^{\infty}$, exogenous processes $\{G_t, g_t, D_t\}_{t=0}^{\infty}$ and initial values of household saving, capital, nominal interest rate, and nominal wage $\left\{B_{-1}^m, K_{-1}, i_{-1}, w_{-1}\right\}$ that jointly satisfy:

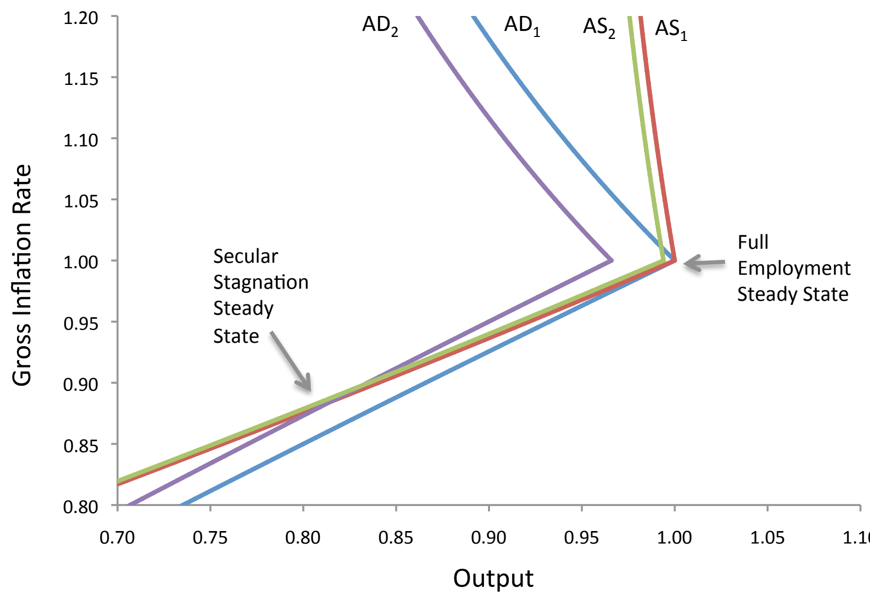
1. Household Euler equations ($B.6$) and ($B.7$)
2. Household budget constraints ($B.9$) and ($B.10$)
3. Asset market clearing ($B.17$)
4. Monetary policy rule ($A.17$)
5. Full-employment wage rate ($B.16$)
6. Production function ($B.12$)
7. Labor demand condition ($B.13$)
8. Capital demand condition ($B.14$)
9. Wage process: $w_t = \max\left\{\tilde{w}_t, w_t^{flex}\right\}$ where $\tilde{w}_t = \gamma\frac{w_{t-1}}{\Pi_t} + (1-\gamma)w_t^{flex}$

B.4 Derivation of Steady State AD and AS Curves

To derive the steady state AD and AS curves, we consider the equilibrium conditions in steady state listed in the previous subsection. For simplicity, we assume an inflation target $\Pi^* = 1$ and an initial natural rate of interest of zero so that $i^* = 0$. We choose this level for the natural rate so that the AD and AS curve kink at the same point. In general, if these curves do not kink at the same point, then both the AD and AS curve will kink at multiple points. These multiple kink points occur because the level of capital effects both the AS and AD curve. Setting $i^* = 0$ simplifies the presentation of the AD and AS curves without materially changing the effect of various shocks.

When inflation is above unity, under the assumption made, the zero lower bound is not binding and the bound on nominal wages is slack so that labor is fully employed: $L = \bar{L}$. In this case,

Figure 10: Effect of a collateral shock with capital



the model with capital in steady state can be summarized by the following conditions:

$$r^k = p^k \left(1 - \frac{(1-\delta)}{1+r}\right) \tag{B.18}$$

$$r^k = (1-\alpha)\frac{Y}{K} \tag{B.19}$$

$$1 + r = \Pi^{\phi_\pi - 1} \tag{B.20}$$

$$Y = AK^{1-\alpha}\bar{L}^\alpha \tag{B.21}$$

$$Y = D + G + \frac{1+\beta}{\beta}\frac{(1+g)}{1+r}D + p^k K \left(1 + \frac{1}{\beta}\frac{(1-\delta)}{1+r}\right) \tag{B.22}$$

The first three equations can be used to eliminate endogenous variables K, r^k, and r to obtain the AS and AD curves respectively.

When inflation is below unity, under the assumptions made, the zero lower bound is binding and the bound on nominal wages is binding so that labor is rationed. In this case, the model with capital in steady state can be summarized by the following conditions:

$$L = \left(\frac{1 - \frac{\gamma}{\Pi}}{1-\gamma}\right)^{\frac{1}{1-\alpha}} \bar{L} \tag{B.23}$$

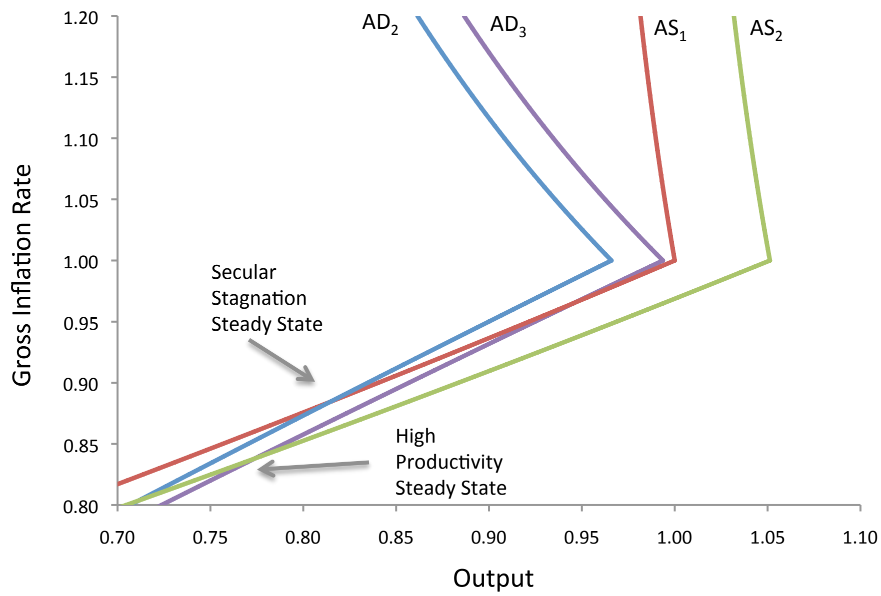
$$r^k = p^k \left(1 - \Pi(1-\delta)\right) \tag{B.24}$$

$$r^k = (1-\alpha)\frac{Y}{K} \tag{B.25}$$

$$Y = AK^{1-\alpha}L^\alpha \tag{B.26}$$

$$Y = D + G + \frac{1+\beta}{\beta}\Pi(1+g)D + p^k K \left(1 + \frac{\Pi}{\beta}(1-\delta)\right) \tag{B.27}$$

Figure 11: Effect of a productivity shock with capital



The first three equations can be used to eliminate endogenous variables L, K, and r^k and obtain an AS curve and an AD curve relating output and inflation.

B.5 Properties of the Model with Capital

In this subsection, we display steady state aggregate supply and demand diagrams and show that a collateral shock still results in a unique secular stagnation equilibrium. We also displays diagrams that illustrate the paradox of toil and the paradox of flexibility.

Figure 10 displays both the normal equilibrium and the effect of a collateral shock. The initial level of the collateral shock is chosen such that the natural rate of interest is zero for reasons discussed earlier. The relative price of capital and productivity are both normalized to unity, the depreciation rate is set at 7.5% per year (implying $\delta = 0.79$ over a 20-year lifetime). All other parameters are the same as considered in Table 1 in the main text. The collateral shock is chosen to deliver a natural rate of -5%. Notice that a collateral shock has effects on both the AS and AD curve shifting both curves backward. The effect on the AS curve is relatively minimal so that the shift in the AD curve dominates.

Figure 11 displays the effect of a 4% positive shock to total factor productivity A. As can be seen in the figure, an increase in productivity shifts out both the AS schedule and the AD schedule from AD_2 to AD_3. An increase in productivity worsens the effect of nominal wage rigidities, but has a slightly offsetting effect of raising demand for capital which shifts the AD curve inward. Under this calibration, the supply effect dominates and a rise in total factor productivity lowers total output and worsens deflation.

Figure 12: Effect of increased wage flexibility with capital

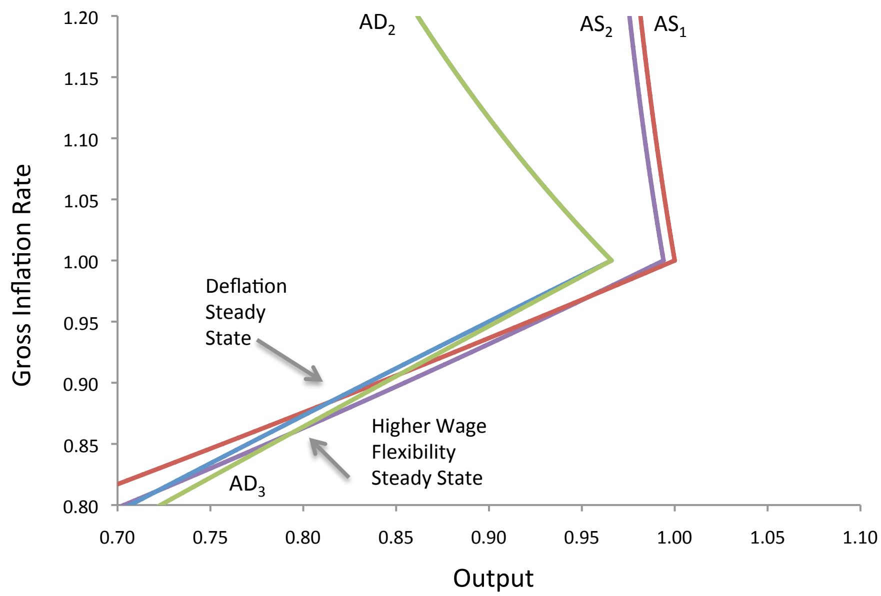


Figure 12 displays the effect of an increase in wage flexibility of the same magnitude as considered in Section 5. As can be seen in the figure, both the AS curve and the AD curve shift. The AS curve shifts downward in the region where nominal wages are rigid. Higher degrees of wage flexibility worsen the effect of deflation and the increase in real wages. Higher wage flexibility also slightly shifts downward the AD curve somewhat offsetting the shift in the AS curve. However, output falls and deflation worsens illustrating the paradox of flexibility.

C Linearization and Solution

In this section, we detail the linearization and general solution to the model without capital but with income received in all periods. For simplicity, we do not consider the effect of population growth shocks which greatly complicate the linearization and the computation of analytical solutions.

The generalized model with income received in all three periods and credit constrained middle-generation households can be summarized by the following linearized AD curve and linearized AS curve.

$$i_t = E_t\pi_{t+1} - s_y\left(y_t - g_t\right) + \left(1 - s_w\right) E_t\left(y_{t+1} - g_{t+1}\right) + s_w d_t + s_d d_{t-1} \tag{C.1}$$

$$y_t = \gamma_w y_{t-1} + \gamma_w \frac{\alpha}{1-\alpha}\pi_t \tag{C.2}$$

where various coefficients are given in terms of their steady state values.

$$\gamma_w = \frac{\gamma}{\bar{\pi}}$$
$$s_y = \frac{\bar{Y}_{m,h}}{\bar{Y}_{m,h} - \bar{D}}$$
$$s_d = \frac{\bar{D}}{\bar{Y}_{m,h} - \bar{D}}$$
$$s_w = \frac{1+\beta}{\beta} \frac{(1+\bar{g}+\eta_s)\,\bar{D}}{\bar{i}/\bar{\pi}\left(\bar{Y}_{m,h} - \bar{D}\right)}$$

The exogenous shocks are the collateral shock d_t and the government spending shock g_t, which means that a solution to this linear system takes the form:

$$y_t = \beta_y y_{t-1} + \beta_g g_t + \beta_d d_t + \beta_{d,l} d_{t-1} \tag{C.3}$$
$$\pi_t = \alpha_y y_{t-1} + \alpha_g g_t + \alpha_d d_t + \alpha_{d,l} d_{t-1} \tag{C.4}$$

Solving by the method of undetermined coefficients, we obtain the following expressions for the coefficients that determine equilibrium output and inflation in response to collateral and government spending shocks.

$$\beta_y = 0 \tag{C.5}$$
$$\alpha_y = -\frac{1-\alpha}{\alpha} \tag{C.6}$$
$$\beta_{d,l} = \frac{s_d}{s_y + \frac{1-\alpha}{\alpha}} \tag{C.7}$$
$$\alpha_{d,l} = \frac{1-\alpha}{\gamma_w \alpha}\beta_{d,l} \tag{C.8}$$
$$\beta_d = \frac{s_w + \beta_{d,l}\left(\frac{1-\alpha}{\gamma_w \alpha} + (1-s_w)\right)}{s_y + (1-s_w)\,\rho_d + \frac{1-\alpha}{\alpha}\,(1 - 1/\gamma_w \rho_d)} \tag{C.9}$$
$$\alpha_d = \frac{1-\alpha}{\gamma_w \alpha}\beta_d \tag{C.10}$$
$$\beta_g = \frac{s_y + (1-s_w)\,\rho_g}{s_y + (1-s_w)\,\rho_g + \frac{1-\alpha}{\alpha}\,(1 - 1/\gamma_w \rho_g)} \tag{C.11}$$
$$\alpha_g = \frac{1-\alpha}{\gamma_w \alpha}\beta_g \tag{C.12}$$

By substituting ($C.2$) into ($C.1$), we can obtain a first-order difference equation in output. This forward-looking difference equation implies that inflation and output will be determinate if and only if the following condition obtains:

$$s_y - (1-s_w) > \frac{1-\alpha}{\alpha}\frac{1-\gamma_w}{\gamma_w}$$

When $s_w = 0$, this condition is the same determinacy condition as discussed in the main text. When the above condition holds, there is a unique rational expectations equilibrium in the deflation steady state. The left hand side is always positive, so in the case of perfect price rigidity (i.e. $\gamma_w = 1$), this condition is satisfied and the deflation steady state is locally unique.

D Properties of Secular Stagnation Equilibrium

Here we provide a formal proof for various properties of the secular stagnation equilibrium described in the body of the text.

Proposition 1. *If $\gamma > 0$, $\Pi^* = 1$, and $i^* = r^f < 0$, then there exists a unique determinate secular stagnation equilbrium.*

Proof. Under the assumptions of the proposition, the inflation rate at which the zero lower bound binds given in equation (28) is strictly greater than unity. Let Y_{AD} denote the level of output implied by the aggregate demand relation and Y_{AS} denote the level of output implied by the aggregate supply relation. For gross inflation rates less than unity, Y_{AD} and Y_{AS} are given by:

$$Y_{AD} = D + \psi\Pi \tag{D.1}$$

$$Y_{AS} = \left(\frac{1-\frac{\gamma}{\Pi}}{1-\gamma}\right)^{\frac{\alpha}{1-\alpha}} Y^f \tag{D.2}$$

where $\psi = \frac{1+\beta}{\beta}(1+g)\,D > 0$. The AD curve is upward sloping because $\Pi < 1 < \Pi_{kink}$ under our assumptions and, therefore, the zero lower bound binds.

When $\Pi = \gamma$, $Y_{AD} > Y_{AS} = 0$. When $\Pi = 1$, the real interest rate equals $\Pi^{-1} = 1 > r^f$. Thus, when $\Pi = 1$, $Y_{AD} < Y^f$. Furthermore, from the equations above, when $\Pi = 1$, $Y_{AS} = Y^f$. Therefore, it must be the case that $Y_{AD} < Y_{AS}$ when $\Pi = 1$. Since the AS and AD curve are both continuous functions of inflation, it must be the case that there exists a Π_{ss} at which $Y_{AD} = Y_{AS}$.

To establish uniqueness, we first assume that their exist multiple distinct values of Π_{ss} at which $Y_{AD} = Y_{AS}$. In inflation-output space (output on the x-axis), the AS curve lies above the AD curve when inflation equals γ and the AS curve lies below the AD curve for inflation at unity - see equation (25). Thus, if multiple steady states exist, given that AS is a continuous function, there must exist at least three distinct points at which the AS and AD curve intersect.

At the first intersection point, the slope of AS curve crosses the AD curve from above and, therefore, at the second intersection the AS curve crosses the AD curve from below. Therefore, the AS curve as a function of output is locally convex in this region. Similarly, between the second and third intersection, the AS curve is locally concave. A third segment drawn from the third intersection point to the point $\Pi = 1$, $Y = Y^f$ lies above the AS curve and is therefore locally convex. Thus, over the region $Y = 0$ to $Y = Y^f$, the second derivative of the AS curve must change signs at least twice.

We can take the second derivative of inflation with respect output of the AS curve and derive the following expression:

$$\frac{d^2\Pi}{dY^2} = G(Y)\left((1+\phi)(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi} + (\phi - 1)\right) \tag{D.3}$$

$$G(Y) = \frac{\phi\gamma(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi}}{Y^2\left(1-(1-\gamma)\left(\frac{Y}{Y^f}\right)^{\phi}\right)} \tag{D.4}$$

$$\phi = \frac{1-\alpha}{\alpha} \tag{D.5}$$

As can be seen, over the region considered, the function $G(Y)$ is positive and, therefore, the convexity of the AS curve is determined by the second term. This term may be negative if $\phi < 1$, but the second derivative can only change signs over this region at most once. Thus, we have derived a contradiction by assuming multiple steady states. Therefore, there exists a unique intersection point.

As established before, it must be the case that the AS curve has a lower slope than the AD curve at the point of intersection. The slope of the AS curve is:

$$\frac{d\Pi}{dY} = \frac{1-\alpha}{\alpha}\frac{1}{\gamma}\frac{\Pi}{Y}(\Pi - \gamma) \tag{D.6}$$

If the slope of the AS curve is less than the slope of the AD curve at the intersection point, then it must be the case that:

$$\begin{aligned}
\frac{1-\alpha}{\alpha}\frac{\Pi}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< \psi^{-1} \\
\frac{1-\alpha}{\alpha}\frac{\psi\Pi}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< 1 \\
\frac{1-\alpha}{\alpha}\frac{Y-D}{Y}\left(\frac{\Pi}{\gamma}-1\right) &< 1 \\
s_y\frac{\alpha}{1-\alpha}+1 &> \frac{\Pi}{\gamma} \\
\frac{\gamma}{\Pi}\left(s_y\frac{\alpha}{1-\alpha}+1\right) &> 1
\end{aligned}$$

The last inequality here is precisely the condition for determinacy discussed in Section 8. Thus, the unique secular stagnation steady state is always determinate as required. □