Dynamic Tax Reforms

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Abstract

We develop a general method to study tax reforms in dynamic settings by proposing a sophisticated set of perturbations of any tax system. The first contribution of the paper is to derive a general formula for the welfare gains of dynamic tax reforms in a compact and easily interpretable way. The second contribution of our paper is to sequentially decompose the gains arising from each additional element of reform as the tax system becomes more sophisticated. We derive closed form expressions for the effects of age- and history-dependence, and of joint taxation of labor and capital incomes. This decomposition reveals several new elements determining the gains of tax reforms present in the dynamic but not in the static environment. Specifically, we show that the multivariate hazard rates of the labor and capital income distributions as well as several new elasticities are essential to the analysis of taxation. We then further decompose the gains of jointly taxing labor and capital incomes within and across periods at the tails of the distribution, by introducing methods based on copulas. Thirdly, we calibrate the model and show that the effects of tax reforms which introduce more sophisticated tax systems in the dynamic setting are quantitively significant. One advantage of our theoretical analysis is that it allows exceptionally simple, almost back of the envelope, calculations of the revenue gains of various elements of tax reform.

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1 Introduction

Tax reform is one of the most important and controversial policy issues. How should different groups of people (rich versus poor, wealthy versus non-wealthy, young versus old) be taxed? Which tax reforms of the current tax system would increase government revenue and social welfare?

Our paper proposes a general method to analyze tax reforms in dynamic settings. We study a dynamic model, in which individuals' characteristics evolve over their lifetime, either deterministically or stochastically. The government chooses a tax system and aims to maximize a weighted sum of individual utilities. The tax system consists of a sequence of tax functions which can be arbitrarily non-linear and joint in the entire history of labor and capital incomes. The generality of the tax functions allows to study the age-dependence and history dependence of taxes as well as the joint conditioning of taxes on labor and capital incomes.

We develop a sophisticated set of perturbations of the tax functions which we refer to as the tax reforms. Specifically, we propose perturbations of the gradient of the tax functions, which may depend jointly on labor and capital incomes over the individual's lifetime and within periods. We analyze the impact of such perturbations of the tax system on individual behavior, government revenue, and social welfare.

The first main contribution of the paper is to derive a general formula for the welfare effects of tax reforms in a compact and easily interpretable form despite the complexity of the dynamic problem and the corresponding general tax system. Importantly, these formulas are written only as a function of empirically observable and easily interpretable sufficient statistics. This approach to welfare analysis is advocated for instance by Chetty (2009).

The second main contribution of our paper is to sequentially decompose the gains coming from each additional element of reform as the tax function becomes more sophisticated. We show the effects of taking into account individuals' intertemporal optimization decisions, and allowing for more general, age- and history-dependent, tax reforms. This sequential decomposition of increasingly sophisticated tax systems shows the importance of the multivariate hazard rates of the labor and capital income distributions as well as several new elasticites in determining the effects of the reforms.

Specifically, we proceed as follows to sequentially increase the complexity of the tax systems.

Our initial tax system is a stylized version of the current US tax system, which is potentially sub-optimal. This baseline tax system is on purpose simple. Tax functions are separable between labor and capital incomes, and between incomes across time periods. Thus, the tax system does not feature age or history dependence nor joint taxation of labor and capital incomes. For clearer exposition, we also consider a simplified model in which the utility function has no income effect.

We start by considering a perturbation of the income tax that is identical to the static tax reform considered by Saez (2001). These results are well understood and we use them as a benchmark, considered, for example, in the Mirrlees Review (2010). We show that the welfare gains of changing the labor income taxes in a separable way (that is, independently of labor and capital incomes in other periods) is equal to the gain obtained in the static model plus an additional term. This additional term summarizes the income effects that a change of the labor income tax in a given period has on savings in all periods.
We show that this savings effect leads to the desirability of age-dependent taxes even when one considers perturbations that preserve the separability of the tax system. The optimal adjustment of savings following a perturbation of the labor income taxes therefore induces new effects that are absent from the analysis of tax reforms in the static model. We also consider perturbations of the capital income tax that preserves the separability of the tax function, and derive new expressions for the revenue gains of reforming the capital income tax schedule. We show that the parameters governing the effects of this tax reform are the elasticities of the responses of savings to taxes, and the hazard rate of the capital income distribution in a given period.

Second, we consider joint perturbations of incomes across different periods or within periods. That is, we analyze the effects of introducing history-dependence in the tax system, or joint taxation of labor and capital incomes. We show that the welfare gains of such perturbations can be expressed as the gains of the previous separable perturbations plus an additional term. We prove that this additional term is determined by the multivariate hazard rates of the joint distributions of labor incomes across time, or labor and capital incomes within a period. The intuition for this effect is as follows. A fundamental insight of the static model (see Diamond 1998 and Saez 2001) is that the univariate hazard rates of the labor income distribution are the key parameters determining the gains of static, or separable, perturbations of the labor income tax schedule. This is because the hazard rate measures the behavioral cost of the additional distortion (measure of individuals affected by the increase in the marginal tax rate) relative to the increase in revenue generated by the change in the tax rates (measure of individuals who pay the additional lump-sum tax). We show that in the dynamic setting the welfare gains of joint perturbations of the tax functions, e.g. perturbations which introduce history-dependent taxes, are determined by the multivariate hazard rates of the joint distribution of incomes (see, e.g., Johnson and Kotz 1975). In the dynamic setting, the ratios of the multivariate hazard rates to the univariate hazard rate measure the additional gains of the joint perturbations relative to the separable (i.e., not history-dependent) perturbation. Similarly, the multivariate hazard rates of the labor and capital income distributions determine the gains of introducing joint taxation of labor and capital income in a given period. The importance of the multivariate hazard rate is new to the analysis of taxation.

We then use methods based on copulas (see, e.g., Nelsen 1999) to gain further understanding of the determinants of the welfare gains of jointly perturbing the labor and capital income taxes across time and within periods. We derive a closed-form solution for the welfare gains at the tails of the joint distribution under the assumption of Pareto-distributed tails. There are several key parameters that determine the gains from joint perturbations of taxes: the thickness of the Pareto tails of the labor and capital income distributions in each of the periods, and various measures of the correlation between the marginal distributions of labor and capital incomes within and across periods. The copula-based methods, while used in the analysis of the lifecycle income behavior, are new to the literature on taxation.

Our third contribution is that the theoretical analysis that we develop allows exceptionally simple, almost back of the envelope, calculations of the revenue gains of various elements of tax reforms. We calibrate the model and show that the effects of reforms introducing more sophisticated tax systems in the dynamic setting are quantitatively significant. Our theoretical decomposition allows us to simply plug
the calibrated parameters into the formulas for the welfare gains without the need to solve a numerical optimization problem. Specifically, the quantitative results are as follows. The savings effect of separable perturbations implies optimal (separable) age-dependent labor income tax schedules that are up to 10 percentage points higher for older individuals than for younger individuals. We then utilize the copula-based methods to further decompose quantitatively the revenue gains of introducing joint taxation of labor and capital incomes within periods and across time, both for the tails and the bulk of the income distributions. The quantitative results show that the analysis of the tax reforms in the dynamic setting can yield a significant gains compared to the separable tax reforms.

The design of tax reform under heterogeneity has been studied in several strands of the literature. First, Piketty (1997), Diamond (1998), and Saez (2001) have expressed the optimal tax formulas first derived in Mirrlees (1971) in terms of easily interpretable and empirically estimable sufficient statistics: elasticities of labor supply and shape of the income distribution. They introduced the method based on local perturbations of an initial tax system that we generalize in our paper, and derived formulas characterizing the welfare gains of such perturbations. The focus of these models is primarily static, and does not deliver insights regarding optimal capital income tax rates, or the optimal evolution of distortions over the life-cycle. Piketty and Saez (2013) extend this analysis to dynamic models, with a focus on a limited set of tax instruments (linear tax rates); our paper is thus complementary to theirs. The dynamic version of the Mirrlees’ model (e.g., Golosov, Kocherlakota and Tsyvinski 2003, Golosov, Tsyvinski and Werning 2006, Kocherlakota 2010) characterizes the fully optimal tax system subject to informational frictions. Recent papers by Farhi and Werning (2012) and Golosov, Troshkin and Tsyvinski (2013), derive general formulas for the optimal labor and savings distortions in the dynamic setting as the solution to a dynamic mechanism design problem. Finally, Conesa and Krueger (2006), Conesa, Kitao and Krueger (2009), Blundell and Shephard (2013) compute numerically the optimal tax system imposing specific functional forms and a set of tax instruments. Our paper brings together these three strands by considering the effects of general tax reforms in dynamic settings, deriving easily interpretable and empirically estimable formulas, and allowing a simple quantitative analysis of the effects of reforming various elements of the tax system.

The paper is organized as follows. Sections 2 to 6 describe the deterministic version of the model. In Section 2, we describe the individual’s problem, define the various elasticity concepts, and set up the planner’s problem. In section 3, we describe the solution method, based on perturbations of the initial tax system. In Section 4, we derive the individuals’ behavioral responses to such perturbations, or tax reforms. In section 5, we obtain the general formula for the welfare gains of tax reforms. In Section 6, we study qualitatively and quantitatively a simple version of the general model, which allows us to obtain new theoretical and quantitative insights about the welfare gains of major tax reforms. In Section 7, we solve the stochastic version of the model, following the same steps as for the deterministic model.
2 Environment

2.1 The Model

There is a measure one of agents in the economy. An agent lives for \( T \leq \infty \) periods. At the beginning of period \( t = 1 \), there is a draw of an exogenous vector of characteristics \( \theta \) for each individual, consisting of an initial level of capital stock \( k_0 \), and a sequence of shocks. These shocks can be, for instance, the individual’s sequence of productivities, tastes, etc. over his lifetime. The environment is deterministic. Individuals know at the beginning of period \( t = 1 \) their entire vector of characteristics \( \theta \). We extend the analysis to the stochastic environment in Section 7. We denote by \( F(\theta) \) the c.d.f. of vectors \( \theta \in \Theta \) in the economy, and \( f(\theta) \) the corresponding density function.

Given the draw of vector \( \theta \), the individual chooses a sequence of consumption \( \{c_t\}_{t=1}^{T} \), labor incomes \( \{z_t\}_{t=1}^{T} \), and savings \( \{k_t\}_{t=1}^{T} \) for each period \( t \in \{1, \ldots, T\} \). The utility function \( U \) can be a general, not necessarily time separable, function of the vector of choices of consumption, labor income and capital. It is increasing in each period’s consumption (and capital if it enters explicitly the utility function), and decreasing in each period’s labor income. An example, which we focus on in Section 6, is the case where \( U(\theta) \equiv \max \left\{ c, z \right\} \) is increasing in each period’s consumption and capital if it enters explicitly the utility function, and decreasing in each period’s labor income. An example, which we focus on in Section 6, is the case where \( U = \sum_{t=1}^{T} \beta^{t-1} u(c_t, z_t/\theta_t) \). In this case, \( \theta_t \) is a shock to the productivity of labor supply in period \( t \).

In each period \( t \in \{1, \ldots, T\} \), the government levies a tax \( T_t \). The tax liability \( T_t(\cdot) \) in period \( t \) is an arbitrarily non-linear function of the individual’s entire history of labor incomes \( \{z_s\}_{s=1}^{T} \), capital incomes \( \{r_{s+1}k_s\}_{s=1}^{T-1} \), and last-period savings \( k_T \) if \( T < \infty \). The interest rate \( r_t \) in each period is exogenous. (For the ease of notation, we define \( r_{T+1} \equiv 1 \) so that we can write \( T_t \) as a function of \( \{r_{s+1}k_s\}_{s=1}^{T-1} \). The sequence of tax functions \( \{T_t(\cdot)\}_{t=1}^{T} \) is known to the individual at the beginning of period \( t = 1 \), and the government can commit to it. The tax function in period \( t \), \( T_t(\{z_s\}_{s=1}^{T}, \{r_{s+1}k_s\}_{s=1}^{T}) \), is assumed twice continuously differentiable in its \( 2T \) variables.

The individual maximization problem is given by:

\[
\mathcal{U}(\theta) = \max_{\{c_t, z_t, k_t\}_{t=1}^{T}} U\left(\{c_s\}_{s=1}^{T}, \{z_s\}_{s=1}^{T}, \{k_s\}_{s=1}^{T}, \theta\right)
\text{ s.t. } c_t + k_t = z_t + (1 + r_t) k_{t-1} - T_t(\{z_s\}_{s=1}^{T}, \{r_{s+1}k_s\}_{s=1}^{T}), \forall t = 1, \ldots, T.
\] (1)

We denote by \( \mathcal{U}(\theta) \) the indirect utility attained by an individual characterized by the vector \( \theta \). The budget constraint in period \( t \) imposes that the sum of his consumption \( c_t \) and savings \( k_t \) are no greater than the sum of his labor income \( z_t \) and capital income \( (1 + r_t) k_{t-1} \), net of the tax liability \( T_t \).

Note that the case \( T = 1 \) corresponds to the static model studied by Mirrlees (1971), Diamond (1998) and Saez (2001). In this case, the individual is indexed by a one-dimensional shock \( \theta \), and his maximization problem is:

\[
\mathcal{U}^S(\theta) = \max_{c,z} u(c, z, \theta) \text{ s.t. } c = z - T(z),
\] (2)

where the superscript \( S \) stands for “static”. The tax function \( T(\cdot) \) in this case is a function only of the

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Footnote: 

1In a given period \( t \), the planner can tax incomes that will be earned in the future periods \( s > t \) because the model in this section is deterministic.
labor income $z$.

For all $j \in \{1, \ldots, 2T\}$, let $x_j$ denote the $j^{th}$ variable of the tax function $T_t(\cdot)$. That is, $x_j = z_j$ and $x_{T+j} = r_{j+1}k_j$ for all $j \in \{1, \ldots, T\}$. We define the period-$t$ marginal tax rates as the partial derivatives of the tax function $T_t(\cdot)$ with respect to its $2T$ variables:

\[
\begin{align*}
\tau_{t,z_j} & = \frac{\partial T_t}{\partial z_j} \left( \{z_s\}_{s=1}^T, \{r_{s+1}k_s\}_{s=1}^T \right), \quad \forall j \in \{1, \ldots, T\}, \\
\tau_{t,k_j} & = \frac{\partial T_t}{\partial x_{T+j}} \left( \{z_s\}_{s=1}^T, \{r_{s+1}k_s\}_{s=1}^T \right), \quad \forall j \in \{1, \ldots, T\}.
\end{align*}
\]

We define the period-$t$ virtual income $R_t$, for all $t \in \{1, \ldots, T\}$, as

\[
R_t = \sum_{j=1}^{T} \tau_{t,z_j} z_j + \sum_{j=1}^{T} \tau_{t,k_j} r_{j+1}k_j - T_t \left( \{z_s\}_{s=1}^T, \{r_{s+1}k_s\}_{s=1}^T \right).
\]

To understand this notion, consider first the case $T = 1$, i.e., the static model in (2). We can rewrite the net income $z - T(z)$ of an individual who earns income $z$ as $(1 - \tau)z + R$, where $\tau = T'(z)$ is the marginal tax rate he faces, and $R = \tau z - T(z)$ is his virtual income. The function $H(x) = \tau x - R$ is the line that is tangent to the tax function $T(\cdot)$ at point $z$. The budget constraint faced by the individual at point $z$ is thus equal to $z - T(z) = z - H(z) = (1 - \tau)z + R$. The virtual income $R$ is the income the individual would earn if he faced this linearized budget constraint and earned no income ($z = 0$). Linearizing in this way the budget constraint allows us to express income $z$ and, later, the elasticities, as a function of the marginal tax rate $\tau$ at income level $z$, and of the virtual income $R$. In the dynamic model ($T \geq 2$), we define similarly the linearized budget constraint for the individual with labor and capital incomes $(z_1, \ldots, z_T, r_{2k1}, \ldots, r_Tk_{T-1}, k_T)$, by replacing the tax function $T_t(\cdot)$ by its tangent hyperplane at point $(z_1, \ldots, z_T, r_{2k1}, \ldots, r_Tk_{T-1}, k_T)$, which is defined by $H_t(x_1, \ldots, x_{2T}) = (-R_t) + \sum_{j=1}^{T} \tau_{t,z_j} x_j + \sum_{j=1}^{T} \tau_{t,k_j} x_{T+j}$, with $R_t = -T_t(0, \ldots, 0)$. Thus $R_t$ is the income the individual would get if he faced this linearized tax function and earned no labor or capital income $(z_1 = \ldots = z_T = 0, r_{2k1} = \ldots = r_Tk_{T-1} = 0, k_T = 0)$. The actual and the linearized tax functions coincide at the point $(z_1, \ldots, z_T, r_{2k1}, \ldots, r_Tk_{T-1}, k_T)$ for $R_t$ defined as in equation (4). The choices of incomes and, later, the elasticities, can then be expressed as functions of all the marginal tax rates at point $(z_1, \ldots, z_T, r_{2k1}, \ldots, r_Tk_{T-1}, k_T)$ and the virtual income $R_t$. We linearize the budget constraint in this way by replacing the tax function by its tangent hyperplane to obtain, for all $t \in \{1, \ldots, T\}$:

\[
\begin{align*}
\tau_{t,z_j} z_j & = (1 + \tau_t) z_t - T_t \left( \{z_s\}_{s=1}^T, \{r_{s+1}k_s\}_{s=1}^T \right)
\end{align*}
\]

\[
\begin{align*}
& = (1 - \tau_{t,z_{t-1}}) z_t + (1 + (1 - \tau_{t,k_{t-1}}) r_t) k_{t-1} - (1 + \tau_{t,k_t} r_{t+1}) k_t \\
& \quad - \sum_{j=1}^{T} \tau_{t,z_j} z_j + \sum_{j=1}^{T} \tau_{t,k_j} k_j + R_t.
\end{align*}
\]

\begin{footnote}{Whenever it simplifies the exposition, we will also use the notation $\tau_{t,r_{j+1}k_j}$ for the marginal tax rate $\tau_{t,k_j}$.}
\end{footnote}
We then rewrite the individual problem as follows:

$$\max_{\{c_s,z_s,k_s\}_{s=1}^T} U\left(\{c_s\}_{s=1}^T, \{z_s\}_{s=1}^T, \{k_s\}_{s=1}^T, \theta\right) \quad \text{s.t.} \quad (5).$$

(6)

The original and the linearized maximization problems, respectively (1) and (6), have the same solution. The first-order conditions, expressed in terms of the marginal tax rates and the virtual incomes, write

$$\begin{align*}
(1 - \tau_t, z_t) U_{c_t} - \sum_{s \neq t}^{T} \tau_{s,z_t} U_{c_s} &= -U_{z_t}, \quad \forall t \in \{1, \ldots, T\}, \\
(1 + (1 - \tau_{t+1,k_t}) r_{t+1}) U_{c_{t+1}} 1_{t<T} - (1 + \tau_{t,k_t} r_{t+1}) U_{c_t} - \sum_{s \neq t}^{T} \tau_{s,k_t} r_{t+1} U_{c_s} &= -U_{k_t}, \quad \forall t \in \{1, \ldots, T\}.
\end{align*}$$

(7)

If the utility function depends only on consumption and labor income, and the tax function in each period $t$ depends on labor and capital incomes earned in period $t$ only, i.e. $T_t \left(\{z_s\}_{s=1}^T, \{r_{s+1} k_s\}_{s=1}^T\right) = T_t(z_t, r_t k_{t-1})$, then we obtain the familiar first-order conditions. In this case, $\tau_{s,z_t} = 0$ for all $s \neq t$ and $\tau_{s,k_t} = 0$ for all $s \neq t + 1$. Hence, the first line of (7) states that the marginal rate of substitution between labor income and consumption, $-U_{z_t}/U_{c_t}$, is equal to the labor wedge, $1 - \tau_{t,z_t}$, and the second line of (7) is the Euler equation which equates the marginal rate of substitution between consumption in period $t$ and period $t + 1$, $U_{c,t}/U_{c_{t+1}}$, to the savings wedge, $(1 + (1 - \tau_{t+1,k_t}) r_{t+1})$. The other terms on the left-hand sides of the first-order conditions (7) appear when the tax function also depends on the past and future values of labor and capital incomes.

These first-order conditions define implicitly the Marshallian (uncompensated) labor and capital income functions, $z_t$ and $r_{t+1} k_t$ for $t \in \{1, \ldots, T\}$. These are functions of the marginal tax rates and virtual incomes defined in (3) and (4). Let

$$X(\theta) = (z_1, \ldots, z_T, r_2 k_1, \ldots, r_{T+1} k_T)'$$

(8)

be the vector of choice variables of the individual with the vector of characteristics $\theta$. All the choice variables that define the vector $X$, i.e. the labor incomes $z_1, \ldots, z_T$ and capital incomes $r_2 k_1, \ldots, r_{T+1} k_T$, depend on the $2T$ marginal tax rates in each period $t$ (i.e., the gradients of all the tax functions $T_t(\cdot)$) and the virtual incomes in each period $t$, at the point $X$ that the individual chooses. That is, $X$ is a function of $\{\tau_{t,z_t}, 1 - \tau_{t,z_t}\}$ for all $t \in \{1, \ldots, T\}$ and $j \in \{1, \ldots, T\} \setminus \{t\}$, of $\{\tau_{t,k_t}, 1 - \tau_{t,k_t}\}$ for all $t \in \{1, \ldots, T\}$ and $j \in \{1, \ldots, T\} \setminus \{t-1\}$, and of $\{R_t\}$ for all $t \in \{1, \ldots, T\}$.

The first-order conditions of the dual individual problem define similarly the Hicksian (compensated) labor and capital income functions, respectively $z_t^*$ and $r_{t+1} k_t^*$ for $t \in \{1, \ldots, T\}$. As the uncompensated demands, the compensated demands are functions of all the marginal tax rates (3) and virtual incomes (4) in all periods.

Finally, note that in the static model (2) (case $T = 1$), the choice vector $X$ is simply the scalar $z$.
(labor income), and it is a function of the net-of-tax rate $1 - \tau$ (where $\tau = T'(z)$) and the virtual income $R$. Hence the $z$ is the solution to $z = z(1 - \tau, R)$.

2.2 Elasticities

In this section, we define the key elasticity concepts. We expressed the choices of labor and capital incomes as functions of the marginal tax rates (3) and virtual incomes (4) and therefore can define the uncompensated and compensated income elasticities with respect to these marginal tax rates and virtual incomes. Some of the elasticities are standard in the literature. However, given the very general structure of the tax systems we consider (in particular, we allow for jointly taxing incomes within and across periods), there are several new elasticities. Specifically, for all $t \in \{1, \ldots, T\}$, we define the elasticities of labor and capital incomes with respect to every component of the gradient of the period-$t$ tax function $T_t(\cdot)$, i.e., the marginal tax rates in each of the $2T$ directions. We then gather these elasticities in matrices, which will prove useful to express the formulas of the next sections in a compact way.

In order to keep notations concise, we let, for all $x \in \{z_s, k_s\}_{s=1}^T$, $\tau_{t,y} = 1 - \tau_{t,z_t}$ if $y = z_t$, $\tau_{t,y} = \tau_{t,z_j}$ if $y = z_j, j \neq t$, $\tau_{t,y} = 1 - \tau_{t,k_{t-1}}$ if $y = k_{t-1}$, and $\tau_{t,y} = \tau_{t,k_j}$ if $y = k_j, j \neq t - 1$. These are the marginal and net-of-tax rates with respect to which the elasticities are defined.

We now proceed to formally define the uncompensated elasticities. Consider an individual with a vector of characteristics $\theta$ who chooses the vector of labor and capital incomes $X = X(\theta)$. Let $x \in \{z_s, r_{s+1}k_s\}_{s=1}^T$ be one of his choice variables. We define the uncompensated elasticities of the choice variable $x$ with respect to the marginal tax rates $\tau_{t,y}$ in period $t$ for this individual as:

$$\zeta_{x,\tau_{t,y}}^{u,(X)} = \frac{\tau_{t,y} x(\theta)}{x(\theta)} \frac{\partial x(\theta)}{\partial \tau_{t,y}}.$$  

The elasticity $\zeta_{x,\tau_{t,y}}^{u,(X)}$ gives the percentage change in the choice variable $x$ (labor or capital income in a given period $s$) of the individual $X$ if the marginal tax rate (or the net-of-tax rate) he faces in period $t$ and direction $y$, $\tau_{t,y}$, increases by one percent.\(^3\)

We define the income-effect parameter in period $t$ for this individual as:

$$\eta_{x,R_t}^{(X)} \equiv \tau_{t,x} \frac{\partial x(\theta)}{\partial R_t}.$$  

The parameter $\eta_{x,R_t}^{(X)}$ gives the change in the choice variable $x$ of the individual $X = X(\theta)$ if he receives an additional lump-sum income $dR_t$ in period $t$. Note that this parameter is weighted by the marginal tax rate he faces in period $t$ and direction $x$.

\(^3\)Note that in general, these elasticities depend on the value of the vector of shocks $\theta$. That is, if two individuals choose the same vector $X$ of labor and capital incomes but have different vectors of exogenous characteristics $\theta$, their responses to changes in the tax rates, i.e. their elasticities $\zeta_{x,\tau_{t,y}}^{u,(\theta)}$, are general different. For simplicity, we assume throughout the paper that for all $X \in \mathbb{R}_+^T \times \mathbb{R}_T^T$, there is at most one vector $\theta$ such that, given the initial tax system, $X(\theta) = X$. Thus $\zeta_{x,\tau_{t,y}}^{u,(\theta)}$ depends on $\theta$ only through $X(\theta)$, and we can write the elasticities defined above simply as $\zeta_{x,\tau_{t,y}}^{(X)}$. We discuss below how the formulas for the welfare gains of tax reforms write if this assumption is not satisfied.
Let $x^c \in \{z_i^c, r_{i+1}^c k_i^c\}_{i=1}^T$ denote the compensated demand functions. We define the compensated elasticities with respect to the marginal tax rates $\tau_{t,y}$ in period $t$ as

$$\zeta^c_{x, \tau_{t,y}}(X) \equiv \frac{\tau_{t,y} \partial x^c(\theta)}{x(\theta) \partial \tau_{t,y}}.$$  

(10)

The uncompensated elasticities, the compensated elasticities, and the income-effect parameters are related through the Slutsky equations, which are derived in the Appendix. Moreover, we also derive in the Appendix the closed-form expressions for all these elasticities and income effect parameters. These closed-form expressions are important as not all of these elasticities and income effect parameters have been estimated in the empirical literature.

For all $t \in \{1, \ldots, T\}$, we define the $2T \times 2T$-matrix of compensated elasticities with respect to the period-$t$ marginal tax rates for an individual with choice vector $X$ as follows. The $(i,j)$-element of the matrix is given by, for all $(i,j) \in \{1, \ldots, 2T\}^2$,

$$\left[\zeta^c_{X, \tau_{t}}(X)\right]_{i,j} = \frac{\partial x^c_i}{\partial \tau_{t,j}},$$

(11)

where $x_i$ is the $i$th element of the vector $X$, so that $x_i = z_i$ if $i \in \{1, \ldots, T\}$, and $x_i = r_{i+1} - T k_i - T$ if $i \in \{T+1, \ldots, 2T\}$. That is, the first $T$ rows of the matrix are the partial derivatives of compensated labor incomes with respect to the marginal tax rates, and the next $T$ rows of the matrix are the partial derivatives of compensated capital incomes with respect to the marginal tax rates. The first $T$ columns of the matrix are the partial derivatives of compensated labor and capital incomes with respect to the marginal tax rates on labor income, and the next $T$ columns of the matrix are the partial derivatives of compensated labor and capital incomes with respect to the marginal tax rates on capital income. The elements of this matrix (the partial derivatives) can be expressed in terms of the compensated elasticities defined in (10), by multiplying each element of the matrix by the corresponding marginal tax rate and dividing it by the corresponding choice variable.

Similarly, for all $t \in \{1, \ldots, T\}$, we define the $2T \times 1$-vector of income effect parameters for an individual with choice vector $X$ as

$$\eta^{(X)}_{X, R_t} = \frac{\partial X}{\partial R_t},$$

(12)

which again can be expressed in terms of the income effect parameters defined in (9), by multiplying each element of the vector by the corresponding marginal tax rate.

### 2.3 Social Welfare

The government chooses the tax system $T = \{T_t(\cdot)\}_{t=1}^T$ at the beginning of period $t = 1$. Given the joint c.d.f. $F(\theta)$ of the vectors $\theta \in \Theta$ in the economy, the planner’s objective is given by:

$$\int_{\Theta} G(Y(\theta)|\mathcal{T}) \ dF(\theta),$$

(13)
where \( \mathcal{U}(\theta)|_\mathcal{F} \) is the indirect utility attained by the individual \( \theta \) given the tax system \( \mathcal{F} \), and \( G: \mathbb{R} \to \mathbb{R} \) is the social welfare function. The function \( G(\cdot) \) is defined over lifetime utilities of the individuals, and is assumed increasing and concave. The tax system must satisfy the following government budget constraint:

\[
\int_{\Theta} \left[ \sum_{t=1}^{T} \delta^{t-1} T_t \left( X(\theta)|_\mathcal{F} \right) \right] dF(\theta) \geq Y, 
\]  

where \( \delta \) is the marginal rate of transformation of resources across periods for the government, and \( Y \geq 0 \) is an exogenous revenue requirement. Equation (14) states that the discounted sum of government tax revenue must be larger than an exogenous revenue requirement \( Y \), where the sum is over time \( t \in \{1, \ldots, T\} \) and over individuals \( \theta \in \Theta \). This budget constraint takes into account the optimizing behavior of individuals given the tax system. \( X(\theta)|_\mathcal{F} \) denotes the choice vector (defined in (8)) of individual \( \theta \), given the tax system \( \mathcal{F} \). We finally define social welfare as

\[
\mathcal{W} = \frac{1}{p} \int_{\Theta} G(\mathcal{U}(\theta)|_\mathcal{F}) dF(\theta) + \int_{\Theta} \left[ \sum_{t=1}^{T} \delta^{t-1} T_t \left( X(\theta)|_\mathcal{F} \right) \right] dF(\theta),
\]  

where \( p \) denote the shadow value of public funds. One can think of this measure (15) of social welfare as being expressed in monetary units.

The redistributive tastes of the government can be summarized by the marginal social welfare weights \( g_t(X) \). These weights are defined such that the government is indifferent between having \( g_t(X) \) more dollars of public funds in period \( t \) and giving one more dollar in period \( t \) to the taxpayers with choice vector \( X \). The smaller \( g_t(X) \), the less the government values marginal consumption of individuals \( X \).

We formally define the period-\( t \) marginal social welfare weight associated with an individual with the choice vector \( \bar{X} \) (and type \( \bar{\theta} \) such that \( X(\bar{\theta}) = \bar{X} \)) as follows,

\[
g_t(\bar{X}) = \delta^{t-1} \frac{G' \left( \mathcal{U}(\bar{\theta}) \right) U_{ct}(\bar{\theta})}{p}.
\]  

The marginal welfare weight \( g_t(\bar{X}) \) represents the value, as of period \( t \), in terms of public funds, of giving one additional dollar in period \( t \) uniformly to all individuals whose choice vector is \( \bar{X} \). This additional dollar increases the individual’s indirect utility by \( d\mathcal{U} = U_{ct} \), and using the envelope theorem, social welfare increases by \( d \left[ G(\mathcal{U}) \right] = G'(\mathcal{U}) d\mathcal{U} = G'(\mathcal{U}) U_{ct} \). We express this welfare gain in terms of the value of public funds by dividing this expression by the multiplier \( p \).

Note that in the static model (2) (case \( T = 1 \)), the social welfare and the marginal social welfare

\footnote{If the assumption that there is a unique draw \( \theta \), such that an individual with this particular draw chooses the vector \( X \), is not satisfied, then the welfare weight \( g_t(\bar{X}) \) should be defined as the average of the expression (16) over all individuals \( \bar{\theta} \) who choose the vector \( \bar{X} \).}
weights are simply given by:

\[
\psi^S = \frac{1}{p} \int_{\mathbb{R}_+} G \left( \psi^S (\theta) \right) dF (\theta) + \int_{\mathbb{R}_+} T \left( z (\theta) \right) dF (\theta),
\]

\[
g^S (\tilde{z}) = \frac{1}{p} G' (\psi^S (\theta)) u_c (\theta), \quad \text{where } \tilde{z} = z (\theta). \tag{17}
\]

3 Perturbations

In this section, we formally define the perturbations of the tax system that we study, i.e., our tax reforms. We start from an initial tax function in period \( t \), \( T_t (\cdot) \), which we then locally perturb to obtain a new tax function \( \tilde{T}_t (\cdot) \). We then analyze the individuals’ responses to such perturbations, and subsequently the change in social welfare (15) that they imply.

The perturbations we study consist of changing the marginal tax rates (the gradient of the tax function) around a given point \( X \) in the space \( \mathbb{R}_T^+ \times \mathbb{R}_T^T \). More general tax reforms, which simultaneously alter the marginal tax rates at several points in the space, have the same effects as linear combinations of the elementary perturbations we define here.

We start with a quick recap of the perturbation that Saez (2001) considers in the static model (2). The perturbation of the tax function is defined as follows. Consider an initial tax function \( T (z) \), defined on \( \mathbb{R}_+ \). Fix \( \bar{z} > 0 \). The new (perturbed) tax function is given by \( \tilde{T} (z) = T (z) + v_{\bar{z}} (z) \), where the function \( v_{\bar{z}} (\cdot) \) is defined on \( \mathbb{R}_+ \) as

\[
v_{\bar{z}} (z) = 0, \quad \forall z \in [0, \bar{z}]
\]

\[
v_{\bar{z}} (z) = (z - \bar{z}) d \tau, \quad \forall z \in [\bar{z}, \bar{z} + d\bar{z}]
\]

\[
v_{\bar{z}} (z) = d\bar{z} d \tau \equiv dR, \quad \forall z \in [\bar{z} + d\bar{z}, \infty). \tag{18}
\]

This perturbation increases the marginal tax rate of individuals with income \( z \in [\bar{z}, \bar{z} + d\bar{z}] \) by \( d \tau \). All the individuals with income \( z \geq \bar{z} + d\bar{z} \) face a lump-sum tax increase \( dR = d\bar{z} d \tau \) (and hence, a change in their virtual income equal to \( -dR < 0 \)). This is because the total tax liability of an individual with initial income \( z > \bar{z} + d\bar{z} \) increases by \( dR \) due to the increase in the marginal tax rates by \( d \tau \) on an interval of size \( d\bar{z} \) below their income level \( z \). The individuals with income \( z < \bar{z} \) are not affected by the perturbation. This perturbation is represented in Figure 1.
We now proceed to our dynamic case, $T \geq 2$. Let $t \in \{1, \ldots, T\}$. Consider an initial tax function in period $t$, $T_t \left( \{z_s\}_{s=1}^{T_t}, \{r_{s+1}k_s\}_{s=1}^{T_t} \right)$, defined on the space $\mathbb{R}_+^T \times \mathbb{R}^T$. We perturb this tax function along several directions of this space.

For instance, we can perturb the tax function along the first direction only (first period income $z_1$), at point $\bar{z}_1 \in \mathbb{R}_+$. That is, we perturb the marginal tax rate $\tau_{t,z_1} = \partial T_t / \partial z_1$ for all individuals with first-period income $z_1 \in [\bar{z}_1, \bar{z}_1 + d\bar{z}_1]$, irrespective of their other choice variables $z_2, \ldots, z_T, r_2k_1, \ldots, r_Tk_{T-1}, k_T$ in $\mathbb{R}_+^{T-1} \times \mathbb{R}^T$. We can also perturb the tax function jointly along the first and the second directions of the space (first and second period incomes $z_1, z_2$), at point $(\bar{z}_1, \bar{z}_2) \in \mathbb{R}_+^2$. That is, we perturb both marginal tax rates $\tau_{t,z_1}$ and $\tau_{t,z_2}$ for the individuals with incomes $(z_1, z_2) \in [\bar{z}_1, \bar{z}_1 + d\bar{z}_1] \times [\bar{z}_2, \bar{z}_2 + d\bar{z}_2]$, irrespective of their other choice variables $z_3, \ldots, z_T, r_2k_1, \ldots, r_Tk_{T-1}, k_T$ in $\mathbb{R}_+^{T-2} \times \mathbb{R}^T$.

More generally and more formally, we choose $d \in \{1, \ldots, 2T\}$ directions in the space $\mathbb{R}_+^T \times \mathbb{R}^T$, and a point $\bar{X}_d = (\bar{x}_1, \ldots, \bar{x}_d) \in \mathbb{R}_+^d$ in the subspace defined by these $d$ directions. In this section, every point $X$ in the space $\mathbb{R}_+^T \times \mathbb{R}^T$ will be denoted by $X = (X_d, X_{-d})$, where $X_d = (x_1, \ldots, x_d)$ are the coordinates of the vector $X$ in the $d$ chosen directions, and $X_{-d}$ are its $2T - d$ other coordinates.

We define a $d$-multilinear perturbation of the period-t initial tax function as follows. Let $d\bar{x} > 0$ and $d\tau \in \mathbb{R}$. The new (perturbed) tax function is given by $\bar{T}_t(X) = T_t(X) + v_{t,\bar{X}_d}(X)$, where the function $v_{t,\bar{X}_d}()$ is defined on $\mathbb{R}_+^T \times \mathbb{R}^T$ as

$$
\begin{align*}
&v_{t,\bar{X}_d}(X) = (x_i - \bar{x}_i) \cdot d\tau, \quad \text{if} \quad \left\{ \begin{array}{l}
\exists i \in \{1, \ldots, d\} \text{ s.t. } x_i \in [\bar{x}_i, \bar{x}_i + d\bar{x}] \\
\forall j \in \{1, \ldots, d\} \setminus \{i\}, \quad x_j \geq \bar{x}_j + d\bar{x}
\end{array} \right. \\
&v_{t,\bar{X}_d}(X) = d\tau d\bar{x} \equiv dR_t, \quad \text{if} \quad \forall j \in \{1, \ldots, d\}, \quad x_j \geq \bar{x}_j + d\bar{x} \\
&v_{t,\bar{X}_d}(X) = 0, \quad \text{if} \quad \exists j \in \{1, \ldots, d\} \text{ s.t. } x_j \leq \bar{x}_j.
\end{align*}
$$

We complete this definition on the remaining regions of the space (hypercubes of size $d\bar{x}$) by making $v_{t,\bar{X}_d}$ continuous and multilinear on each of these regions.\footnote{That is, fixing any $d - 1$ variables, the function is linear in the remaining $d^{th}$ variable. Since these hypercubes are defined by $2^d$ grid points, and a multilinear function has $2^d$ coefficients, $v_{t,\bar{X}_d}$ is uniquely defined by its values at each of the grid points. These values are given by the formulas above.

\footnote{The exact definition of the perturbation on these regions is not important, as we show that its effect in these regions}

Note that the function $v_{t,\bar{X}_d}(X)$ depends
only on the $d$ variables $(x_1, \ldots, x_d)$ (i.e., on $X_d$).

For all $i \in \{1, \ldots, d\}$, this perturbation increases by $d\tau$ the $i^{th}$ marginal tax rate of individuals if they choose $x_i \in [\bar{x}_i, \tilde{x}_i + d\bar{x}]$ and their other $d - 1$ variables are above their respective threshold; that is, $x_1 \geq \bar{x}_1, \ldots, x_{i-1} \geq \bar{x}_{i-1}, x_{i+1} \geq \bar{x}_{i+1}, \ldots, x_d \geq \bar{x}_d$. Moreover, all the individuals with $x_j \geq \tilde{x}_j + d\tilde{x}$ for all $j \in \{1, \ldots, d\}$ face a lump-sum tax increase by $dR_t$. Finally, the individuals with $x_j < \tilde{x}_j$ for at least one $j \in \{1, \ldots, d\}$ are not affected by the perturbation.

Figure 2 and Figure 3 illustrate some examples of possible perturbations. Figure 2 shows the perturbation (19) of the tax function along only one direction (here, first-period income $z_1$), hence for which $d = 1$. For simplicity, only two directions of the space $\mathbb{R}^T_+ \times \mathbb{R}^T$ are represented in the figures. We call such perturbations ($d = 1$) separable throughout the paper. This is because if the initial tax function is separable in income $x_i$ (the direction in which the tax function is perturbed), i.e. of the form $T_i(x_i) + T_{-i}(X_{-i})$, then the perturbed tax function $\tilde{T}(\cdot)$ is still separable, as the perturbation affects only the function $T_i(\cdot)$. The point represented by a star in Figure 2 is the point $(z_1 = \bar{z}_1, z_2 = \cdots = z_T = 0, r_2k_1 = \cdots = r_Tk_{T-1} = k_T = 0)$ at which we perturb the tax function. The marginal tax rate $\tau_{t,z_1}$ is increased by the amount $d\tau$ for the individuals who choose $z_1 \in [\bar{z}_1, \bar{z}_1 + d\bar{x}]$, irrespective of their choices of other labor and capital incomes; that is, for any $(z_2, \ldots, z_T, r_2k_1, \ldots, r_Tk_{T-1}, k_T) \in \mathbb{R}^{T-1}_+ \times \mathbb{R}^T$. This is the dark shaded band in Figure 2. (For simplicity, the only variable other than $z_1$ that we represent in Figure 2 is $z_2$.) The tax liability increases in a lump-sum way by the amount $dR > 0$ (i.e., the virtual income $R_t$ decreases by $-dR$) for all individuals with $z_1 \geq \bar{z}_1 + d\bar{x}$, irrespective of the values of their other choice variables. This is the light shaded region in Figure 2. We report the height of the function $v_{t,\tilde{z}_1}(z_1, X_{-1})$ along the (dark shaded) region where the marginal rates are perturbed; it is equal to $0$ for all $z_1 \leq \bar{z}_1$ and to $dR$ for all $z_1 \geq \bar{z}_1 + d\bar{x}$. The function $v_{t,\tilde{z}_1}(z_1, X_{-1})$ is linear in $z_1$ between $\bar{z}_1$ and $\bar{z}_1 + d\bar{x}$. Note finally that this perturbation parallels the one constructed in the static setting, described in (18) and Figure 1.

Figure 3 shows the perturbation (19) of the tax function along two directions (here, first- and second-period incomes $z_1, z_2$), hence for which $d = 2$. We call such perturbations ($d \geq 2$) joint throughout the paper. This is because even if the initial tax function is separable, the perturbed tax function $\tilde{T}(\cdot)$ is no longer so. The perturbation introduces a dependence, or non-separability, between the individual’s choice variables. For instance, here, the marginal tax rate $\partial \tilde{T}/\partial z_1$ now depends on the value of $z_2$, which would not be the case if $\tilde{T}$ were separable. The point represented by a star is the point $(z_1 = \bar{z}_1, z_2 = \tilde{z}_2, z_3 = \cdots = z_T = 0, r_2k_1 = \cdots = r_Tk_{T-1} = k_T = 0)$ at which we perturb the tax function. The marginal tax rate $\tau_{t,z_1}$ is increased by the amount $d\tau$ for the individuals who choose $z_1 \in [\bar{z}_1, \bar{z}_1 + d\bar{x}]$, only if their choice of second-period income $z_2$ is larger than $\bar{z}_2$ (and irrespective of their other choices of labor incomes and savings). This is the vertical dark shaded band in Figure 3. Similarly, the marginal tax rate $\tau_{t,z_2}$ is increased by the amount $d\tau$ for the individuals who choose $z_2 \in [\bar{z}_2, \bar{z}_2 + d\bar{x}]$, only if their choice of first-period income $z_1$ is larger than $\bar{z}_1$. This is the horizontal dark shaded band in Figure 3. Finally, the tax liability increases in a lump-sum way by the amount $dR > 0$ for all individuals with $z_1 \geq \bar{z}_1 + d\bar{x}$ and $z_2 \geq \bar{z}_2 + d\bar{x}$. This is the light shaded region in Figure 3. We report the height of the
Figure 2: Separable Perturbation in the Dynamic Model: Case $d = 1$

The function $v_t, (\bar{z}_1, \bar{z}_2, X - d)$ along the (dark shaded) regions where the marginal rates are perturbed; it is equal to 0 for all $(z_1, z_2) \notin [\bar{z}_1, \infty) \times [\bar{z}_2, \infty)$ and to $dR$ for all $(z_1, z_2) \in [\bar{z}_1 + d\bar{x}, \infty) \times [\bar{z}_2 + d\bar{x}, \infty)$. The function $v_t, (\bar{z}_1, \bar{z}_2, X - d)$ is linear in $z_1$ for $(z_1, z_2) \in [\bar{z}_1, \bar{z}_1 + d\bar{x}] \times [\bar{z}_2, \bar{z}_2 + d\bar{x}]$, and in $z_2$ for $(z_1, z_2) \in [\bar{z}_1 + d\bar{x}, \infty) \times [\bar{z}_2, \bar{z}_2 + d\bar{x}]$. (It is bilinear on the square $(z_1, z_2) \in [\bar{z}_1, \bar{z}_1 + d\bar{x}] \times [\bar{z}_2, \bar{z}_2 + d\bar{x}]$, i.e. of the form $a_0 + a_1 z_1 + a_2 z_2 + a_3 z_1 z_2$.)

Finally, Figure 4 shows the same perturbation as in Figure 3, but the height of the function $v_t, X_d$ is now represented on the vertical axis.

4 Behavioral Responses to Perturbations

We now analyze the individual responses to the perturbations we have defined in Section 3. These behavioral responses are the key elements determining the welfare gains of tax reforms.

Consider a period $t \in \{1, \ldots, T\}$ and a direction $x \in \{z_s, k_s\}_{s=1}^T$. Consider an increase in the marginal tax rate $\tau_{t,x}$ or the virtual income $R_t$ in the region $D \subset \mathbb{R}^T_+ \times \mathbb{R}^T$. These changes have an effect on the choice vector of an individual who chooses $X \in D$ if the initial tax system is maintained.\(^7\) All the components of the vector $X$, i.e., all the choices of labor and capital incomes that the individual makes over his lifetime, may change.

In this section, we derive the total change in every individual’s choice vector following any perturbation of the initial tax system.\(^8\) We show that despite the apparent complexity of the problem, we can derive,\(^7\) The behavior of individuals who chose $X \notin D$ under the initial tax system is unaltered by the perturbation, as the marginal tax rates and virtual incomes that they face are unchanged.

\(^8\)The perturbations that we consider in this section are more general than those individuals actually end up facing.
Figure 3: Joint Perturbation in the Dynamic Model: Case $d = 2$

Figure 4: Perturbation Function $v_{t,X_d}$
using the matrix notations introduced above, a compact and transparent formula (24) giving the change in individual’s behavior following the perturbation.\(^9\) We express these individual behavioral responses in terms of: (i) the elasticities defined in Section 2, and (ii) the local characteristics of the initial tax system, which we now define.

Consider an initial tax system \(\{T_s(\cdot) : 1 \leq s \leq T\} \). Let \(\nabla T_t(X)\) denote the gradient of the period-\(t\) tax function \(T_t(\cdot)\). The gradient is a vector of size \(2T\), defined, for all \(i \in \{1, \ldots, 2T\}\), as

\[
[\nabla T_t(X)]_i = \frac{\partial T_t(X)}{\partial x_i},
\]

where \(x_i = z_i\) if \(i \in \{1, \ldots, T\}\), and \(x_i = r_{i+1-T-k_{i-T}}\) if \(i \in \{T+1, \ldots, 2T\}\). In the static model (2), \(\nabla T_t(X)\) is a scalar, equal to the marginal tax rate \(T'(z)\).

Let \((D^2T_t)(X)\) denote the Hessian matrix of the tax function \(T_t(\cdot)\). It is defined, for all \(i, j \in \{1, \ldots, 2T\}\), as

\[
[(D^2T_t)(X)]_{ij} = \frac{\partial^2 T_t(X)}{\partial x_i \partial x_j}.
\]

In the static model (2), \((D^2T_t)(X)\) is a scalar, equal to the second derivative of the tax function \(T''(z)\).

Finally, we use the following notation,

\[
T'(X) \equiv \sum_{t=1}^{T} \delta^{t-1} (\nabla T_t(X))'.
\]

This row vector is the discounted (by the government’s discount factor \(\delta\)) sum of the gradients of the initial tax functions \(T_t(\cdot)\), for \(t = 1, \ldots, T\). For instance, the first coordinate of \(T'(X)\) is the discounted sum of all the marginal tax rates that an individual with choice vector \(X\) will face on his first-period income \(z_1\) over his lifetime.

### 4.1 The Static Case

We briefly recap the derivation of Saez (2001) of the behavioral responses to perturbations in the static model (2).

First consider an individual with income \(z\) which belongs to a region \(D \subset \mathbb{R}\) where the marginal tax rate is perturbed from \(T'(z)\) to \(T'(z) + d\tau\).\(^{10}\) The change \(d\tau\) for this individual has an elasticity effect which produces a small income change \(dz\).\(^{11}\) We consider only the first-order effects of the perturbation according to definition (19). In this section we derive the behavioral responses to any perturbation that simultaneously changes every component of the gradient (i.e., all the marginal tax rates) and the virtual income faced by the individual. In (19), on the other hand, each region of the space \(\mathbb{R}_+^T \times \mathbb{R}^T\) is affected by the change in only one of these components at a time. Deriving a single formula (24) for the behavioral responses to the most general perturbations allows us to economize on notation. From Section 4 onward, we apply the general formula we obtain here to each region of the space.

\(^9\)We assume that the initial tax system is such that a local perturbation does not induce some individuals to jump to a new choice vector \(X'\), i.e. change their behavior by a discrete amount. Thus, if a marginal tax rate (resp., virtual income) is perturbed by an infinitesimal amount \(d\tau\) (resp., \(d\tau\)), the change \(dX\) in the individual’s choice vector \(X\) is first-order in \(d\tau\) (resp., \(d\tau\)). This means that the individual’s budget constraint is everywhere strictly below its tangent at point \(X\), the linearized budget constraint defined in (5).

\(^{10}\)These are the individuals in the region \([\bar{z}, \bar{z} + dz]\) in Figure 1.

\(^{11}\)We assume that infinitesimal changes \(d\tau\) in the marginal tax rate or \(d\tau\) in the virtual income induce infinitesimal changes \(dz\) in labor income. That is, individuals’ incomes do not jump discontinuously following a perturbation. As
on labor income \( z \) as \( d\tau \to 0 \). This change \( d\tau \) leads to two effects. First, there is a direct elasticity effect due to the exogenous increase \( d\tau \). This effect is equal to \((\partial z^c/\partial \tau) d\tau = \zeta_{c,1-\tau} d\tau\), by definition of the compensated elasticity (10). Second, this shift by \( dz \) of the taxpayer along the non-linear tax function induces an additional indirect change in the marginal tax rate equal to \( d(T' (z)) = T'' (z) dz\), and hence an additional (indirect) elasticity effect on income. Therefore, we obtain that the individual changes his income \( z \) in response to the perturbation by an amount

\[
dz = \left[ \frac{-z}{1 - \tau} \zeta^c_{z,1-\tau} \right] \times d\tau + \left[ \frac{-z}{1 - \tau} \zeta^c_{z,1-\tau} \right] \times T'' (z) dz = \left[ \frac{-z}{1 - \tau} \zeta^c_{z,1-\tau} \right] d\tau \\
\equiv \left[ 1 - \zeta^c_{z,1-\tau} T'' (z) \right]^{-1} \left[ \zeta^c_{z,1-\tau} T'' (z) \right] d\tau.
\]

Now consider an individual with income \( z \) which belongs to a region \( D \subset \mathbb{R} \) where the virtual income is perturbed from \( R \) to \( R - dR \). That is, we consider the individuals’ response to a perturbation where the tax liability \( T (z) \), is replaced by \( T (z) + dR \).\(^{13}\) The change \( dR \) for the individual \( z \) induces a small change in income \( dz \). We consider only the first-order effects of the perturbation on labor income \( z \) as \( dR \to 0 \). This change in income is the consequence of two effects. First, there is a direct income effect due to the exogenous decrease \((-dR)\). This effect is equal to \((\partial z^c / \partial R)(-dR) = \eta^c_{z,R} (-dR)\), by definition of the income effect parameter. Second, this shift by \( dz \) of the taxpayer along the non-linear tax function induces an additional indirect change in the marginal tax rate equal to \( d(T' (z)) = T'' (z) dz\), and hence an additional (indirect) elasticity effect on income. Therefore, the individual changes his income \( z \) in response to the perturbation by an amount

\[
dz = \left[ \frac{1}{1 - \tau} \eta_{z,R} \right] (-dR) + \left[ \frac{-z}{1 - \tau} \zeta^c_{z,1-\tau} \right] \times T'' (z) dz = \left[ \frac{1}{1 - \tau} \eta_{z,R} \right] (-dR) \\
\equiv - \left[ 1 - \zeta^c_{z,1-\tau} T'' (z) \right]^{-1} \left[ \eta^c_{z,R} dR \right].
\]

If an individual is in a region where both the marginal tax rate and the virtual income are perturbed simultaneously (by \( d\tau \) and \((-dR)\), respectively), his behavioral response is equal to the sum of the responses to these two perturbations taken separately. Thus the response to this general perturbation is equal to

\[
dz = \left[ 1 - \zeta^c_{z,1-\tau} T'' (z) \right]^{-1} \left[ \zeta^c_{z,1-\tau} d\tau - \eta^c_{z,R} dR \right].
\]

\(^{12}\)Note that in the static case, we have \( \zeta^c_{z,1-\tau} = \frac{\partial z}{\partial \tau} \zeta^c_{z,1-\tau} = \partial z/\partial (1 - \tau) = \partial z/\partial \tau \).

\(^{13}\)The individuals affected by the perturbation are those in the region \([\bar{z}, \infty)\) in Figure 1, with \( dR = d\bar{d} \). More precisely, the individuals in the region \([\bar{z}, \bar{z} + d\bar{d}]\) face a change in their virtual income equal to \(-dR = -d\bar{d}\), while those in the region \([\bar{z} + d\bar{d}, \infty)\) face a change in their virtual income equal to \(-dR (z) = -(z - \bar{z}) d\tau\). However, the difference between \( dR (z) \) and \( dR \) on \([\bar{z}, \bar{z} + d\bar{d}]\) has only a second-order effect on government revenue, and can thus be ignored. Therefore, to a first order we can consider that all the individuals in \([\bar{z}, \infty)\) face a change in their virtual income from \( R \) to \( R - dR \equiv R - d\bar{d} \).
4.2 The Dynamic Case

We now derive the behavioral responses to perturbations in our dynamic model. We consider perturbations that affect both the gradient of the tax function (i.e., the marginal tax rates) and the virtual income faced by individuals in a given region $\mathcal{D} \subset \mathbb{R}_t^T \times \mathbb{R}^T$ of the space in period $t$. For example, in Figure 3, the gradient of the tax function is perturbed by the vector $d\tau_t = (d\tau_{t,z_1}, 0, \ldots, 0)'$ in the region $(z_1, z_2) \in [\bar{z}_1, \bar{z}_1 + d\bar{x}] \times [\bar{z}_2 + d\bar{x}, \infty)$ (vertical dark shaded band), and by the vector $d\tau_t = (0, d\tau_{t,z_2}, 0, \ldots, 0)'$ in the region $(z_1, z_2) \in [\bar{z}_1 + d\bar{x}, \infty) \times [\bar{z}_2, \bar{z}_2 + d\bar{x}]$ (horizontal dark shaded band); the virtual income is perturbed by $-dR_t = -drd\bar{x}$ in the light-shaded region $(z_1, z_2) \in [\bar{z}_1, \infty) \times [\bar{z}_2, \infty)$.

In this section we consider the behavioral responses induced by a general perturbations, in which all the coordinates of the gradient as well as the virtual income in period $t$ are simultaneously perturbed in the region $\mathcal{D}$. That is, for all $x \in \{z_s, r_{s+1}k_s\}_{s=1}^T$, the period-$t$ marginal tax rates $\partial T_t/\partial x(X)$ are replaced by $(\partial T_t/\partial x)(X) + d\tau_{t,x}$, and the period-$t$ virtual income $R_t$ is replaced by $R_t - dR_t$.

Consider an individual who draws a vector of characteristics $\theta$, and chooses a pre-perturbation choice vector $X(\theta) = X \in \mathbb{R}_t^T \times \mathbb{R}^T$. We fix $t \in \{1, \ldots, T\}$, and perturb the tax function $T_t(\cdot)$ as described in the previous paragraph. We analyze the effects of the perturbation on the choice vector $X$ which are first-order in $d\tau$ and $dR$. The proposition that follows describes the individual’s behavioral response to a general perturbation.

**Proposition 1.** Assume that, under the initial tax system $\mathcal{T} = \{T_s(\cdot) : 1 \leq s \leq T\}$, an individual chooses a vector $X$ which belongs to a region of the space $\mathcal{D} \subset \mathbb{R}_t^T \times \mathbb{R}^T$ where the gradient of the tax function $T_t(\cdot)$ is perturbed by the amount $d\tau_t \in \mathbb{R}^{2T}$, with

$$d\tau_t = \left( \begin{array}{cccc} d\tau_{t,z_1} & \ldots & d\tau_{t,z_T} & d\tau_{t,k_1} & \ldots & d\tau_{t,k_T} \end{array} \right)' ,$$

and the period-$t$ virtual income $R_t$ is perturbed by the amount $(-dR_t) \in \mathbb{R}$. This perturbation induces a first-order change $dX$ in the individual’s choice vector $X$ given by

$$dX = \left[ I_{2T} - \sum_{s=1}^{T} \mathcal{C}^{(X)}_{X,T_s}(D^2T_s)(X) \right]^{-1} \times \left[ \mathcal{C}^{(X)}_{X,T_t} d\tau_t - \eta^{(X)}_{X,R_t} dR_t \right] ,$$

where $I_{2T}$ is the $2T \times 2T$ identity matrix.$^{14}$

**Proof.** See Appendix.

The expression (24) generalizes to the dynamic case the formula (23) derived in the static model. We write it in a format that parallels the static formula (23). Specifically, the elasticities, marginal tax rate, and second-derivative of the tax function are replaced by the elasticity matrices, gradient, and Hessian of the tax function, respectively. We view the derivation of this formula and, most importantly, the compact representation that preserves the intuition of the static model as one of the main contributions

$^{14}$A perturbation that affects the tax functions in several periods simultaneously induces a behavioral response that is the sum of the responses to the perturbations of each tax function $T_t(\cdot)$ taken separately.
of the paper. The key difficulty in the dynamic model compared to the static model is to develop the
perturbation approach for the case of many variables.

We now sketch the main steps of the proof of Proposition 1. The individual’s behavior under the
initial tax system $\mathcal{T} = \{ T_s (\cdot) : 1 \leq s \leq T \}$ is described by the first-order conditions (7). Under the new
(perturbed) tax system $\tilde{\mathcal{T}} = \{ \tilde{T}_s (\cdot) : 1 \leq s \leq T \}$, the individual’s behavior is described by a similar set
of equations, in which the marginal tax rates in period $t$, $\tau_{t,x_j}$, are replaced by $\tau_{t,x_j} + d\tau_{t,x_j}$, and the
virtual income in period $t$, $R_t$, is replaced by $R_t - dR_t$. The perturbation generates a new solution to the
individual’s optimization problem: the entire vector $X$ (the solution to the initial system of first-order
conditions) is replaced by the new vector $\tilde{X} = X + dX$ (the solution to the new system). Note that in
general, a perturbation that affects only the period-$t$ tax function may have an effect on the entire choice
vector $X$; that is, it potentially affects the choices of labor and capital incomes in all the previous, current,
and future periods. First-order Taylor approximations of this new set of equations around the initial tax
system (i.e., around $d\tau_t = (0, \ldots, 0)'$ and $dR_t = 0$) yields a system of linear equations whose solution is
the first-order change in each of the choice variables $dz_1, \ldots, dz_T, d(\tau_2 k_1), \ldots, d(\tau_T k_{T-1}), dk_T$. These
are rather complicated equations which depend on the first and second partial derivatives of the utility
function $U$, and the first and second derivatives of the tax functions. The main difficulty in the proof
is to show that the solution to this system can be expressed in matrix form in terms of the elasticities
of labor incomes and savings, the gradients and the Hessians of the tax functions. Using the explicit
expressions that we derived for the elasticities, we prove that we can write the solution to this system
(the vector $dX$) as a function of the elasticity matrices that we defined in (11), and the vector of income
effect parameters (12). Thus, despite the complexity of the problem, driven by the fact that adjustments
in each of the variables affect simultaneously all the other ones, using matrix representations allows us
to express in a compact way all the dynamic effects of the general perturbations, and obtain the formula
(24).

We now provide the intuition underlying (24). As in the static case, the change $dX$ following the
perturbation $(d\tau_t, dR_t)$ is the consequence of two effects. First, there is a direct elasticity effect due to
the exogenous increase in the marginal tax rate $d\tau_t$, and a direct income effect due to the exogenous
decrease in the virtual income $(-dR_t)$. This effect is equal to $\zeta_{X,\tau_t}^{c,(X)} d\tau_t + \eta_{X,R_t}^{(X)} (-dR_t)$, by definition of
the compensated elasticity matrix $\zeta_{X,\tau_t}^{c,(X)}$ and the vector of income effect parameters $\eta_{X,R_t}^{(X)}$. Second, this
shift of the taxpayer $X$ along the non-linear tax function by $dX$ produces an additional change in the
marginal rates in all periods $s \in \{1, \ldots, T\}$ equal to $d(\nabla T_s) (X) = (D^2 T_s) (X) dX$. This induces indirect elasticity effects, leading to a further change in $X$. Note that since the vector $X$ is taxed in every period,
there is such an indirect elasticity effect $\zeta_{X,\tau_t}^{c,(X)} d(\nabla T_s) (X)$ in every period $s \in \{1, \ldots, T\}$. Therefore, we
obtain that the individual changes his choice vector $X$ in response to the perturbation by an amount

$$dX = \left[ \zeta_{X,\tau_t}^{c,(X)} d\tau_t + \eta_{X,R_t}^{(X)} (-dR_t) \right] + \sum_{s=1}^{T} \zeta_{X,\tau_s}^{(X)} (D^2 T_s) (X) \times dX$$

$$= I_{2T} - \sum_{s=1}^{T} \zeta_{X,\tau_s}^{c,(X)} (D^2 T_s) (X) \times \left[ \zeta_{X,\tau_t}^{c,(X)} d\tau_t - \eta_{X,R_t}^{(X)} dR_t \right],$$
which is exactly formula (24).\footnote{If the assumption that for all $X$, there is a unique draw $\theta$, such that an individual with this particular draw chooses the vector $X$, is not satisfied, then we define $dX$ as the average of the expression (24) over all individuals $\theta$ who choose the same vector $X$. If the initial tax system is locally linear around the point $X$, then this average behavioral response is simply given by $dX = \zeta_{X,\tau_1}^{C(X)} dt_1 - \eta_{X,\tau_1}^{C(X)} dR_1$, where $\zeta$ and $\eta$ are the average elasticities and income effect parameters among these individuals.}

5 Welfare Gains of Tax Reforms

Having described the perturbations and the effect that they induce on individual behavior, we now derive the revenue and welfare gains of tax reforms.

5.1 The Static Case

We first recap the static model (2) studied in Saez (2001). Let $h_z(\cdot)$ denote the density of incomes $z$ under the initial tax schedule $T(\cdot)$, and $H_z(\cdot)$ denote the corresponding c.d.f. Starting from this initial system, we perturb $T(\cdot)$ at point $\bar{z}$, as described in (18).

The first effect of this perturbation is the mechanical effect net of the welfare loss. If they did not change their labor supply, all the individuals with income $z \geq \bar{z}$ would pay an additional $dR$ of taxes.\footnote{More rigorously, only the individuals with income $z \geq \bar{z} + d\bar{z}$ pay that much additional taxes, but the difference is second-order.} This mechanically increases government revenue. However, this additional tax revenue is only valued $\{1 - g(z)\} dR$ by the government, by definition of the marginal welfare weights (16). Since there are $h_z(z) dz$ individuals with income $z$, the total mechanical effect of the perturbation is equal to

$$M = \left[ \int_{\bar{z}}^{\infty} \{1 - g(z)\} h_z(z) dz \right] dR. \quad (25)$$

The second effect of this perturbation is the elasticity effect. This is the change in government revenue due to the behavioral response of individuals with income in the interval $z \in [\bar{z}, \bar{z} + d\bar{z}]$, where the marginal tax rate $T'(z)$ is increased by $d\tau$. We saw in Section 4 (equation (26)) that these individuals change their income by the amount $dz|_{\bar{z}} < 0$ following the perturbation, with $\{ dz|_{\bar{z}} / d\tau \} \equiv \zeta_{z,1-\tau}(\bar{z}) / (1 - \zeta_{z,1-\tau}(\bar{z}) T''(\bar{z})).$ As a consequence, the government loses revenue $T'(\bar{z}) dz|_{\bar{z}}$, since the marginal tax rate on this income is $T'(\bar{z})$. There are $h_z(\bar{z}) d\bar{z}$ such individuals. Thus, the total elasticity effect of the perturbation is equal to

$$E = T'(\bar{z}) \{ dz|_{\bar{z}} \} h(\bar{z}) d\bar{z} = \left[ T'(\bar{z}) \left\{ \frac{dz|_{\bar{z}}}{d\tau} \right\} h_z(\bar{z}) \right] dR. \quad (26)$$

Finally, the third effect of this perturbation is the income effect. This is the change in government revenue due to the behavioral response of individuals with income in the interval $z \in [\bar{z} + d\bar{z}, \infty)$, where the virtual income $R$ is reduced by $(-dR) < 0$, with $dR = d\tau d\bar{z}$. We saw in Section 4 (equation (22)) that these individuals change their income by the amount $dz|_{\bar{z}}$ following the perturbation, where $\{ dz|_{\bar{z}} / (-dR) \} \equiv \eta_{z,R}(z) / (1 - \zeta_{z,1-\tau}(z) T''(z)).$ As a consequence, the government’s revenue increases
by \( T'(z) \, dz \mid_z \), since the marginal tax rate on this income is \( T'(z) \). There are \( h_z(z) \, dz \) such individuals. Thus, the total income effect of the perturbation is equal to

\[
I = \int_x^\infty T'(z) \{ dz \} \left[ \int_x^\infty T'(z) \left\{ \frac{dz}{-dR} \right\} h_z(z) \, dz \right] (-dR). \tag{27}
\]

Collecting these three effects (25), (26), and (27), we obtain the total welfare effect of the perturbation (18) expressed in revenue (dollar) units:

\[
\Gamma (\bar{z}) = \left[ \int_x^\infty (1 - g(z)) h_z(z) \, dz + T'(\bar{z}) \frac{\xi_{c,1-\eta}^c(z)}{1 - \xi_{c,1-\eta}^c(z)} h_z(z) \right. \\
\left. - \int_x^\infty T'(z) \frac{n_{x,R}(z)}{1 - \xi_{c,1-\eta}^c(z)} h_z(z) \, dz \right] dR. \tag{28}
\]

If \( \Gamma (\bar{z}) \) is positive, then the perturbation (18) raises welfare, and hence the government should implement this tax reform. Conversely, if it is negative, then the government should implement the opposite reform (reducing the marginal tax rate on \( \bar{z} + d\bar{z} \), and increasing the virtual income above \( \bar{z} + d\bar{z} \)).

The optimal, that is welfare maximizing, tax schedule must be such that no perturbation, or tax reform, yields positive welfare gains. Therefore, the optimal tax schedule satisfies

\[
\Gamma (\bar{z}) = 0, \quad \forall \bar{z} \in \mathbb{R}_+^+. \tag{29}
\]

Equation (29) thus gives a characterization of the optimal tax schedule on labor income in the static model.

**5.2 The Dynamic Case**

We now derive the welfare gains of the perturbations in the dynamic model. Consider \( d \) directions of the space \( \mathbb{R}_+^T \times \mathbb{R}^T \), and let \( \bar{X}_d = (\bar{x}_1, \ldots, \bar{x}_d) \) be a vector in the \( d \)-dimensional subspace of \( \mathbb{R}_+^T \times \mathbb{R}^T \) generated by these directions. In the example of Figure 3, \( d = 2 \) and \( \bar{X}_d \) is the point \( (\bar{z}_1, \bar{z}_2) \in \mathbb{R}_+^2 \). We assume that \( \bar{X}_d \) belongs to the interior of the set

\[
\mathcal{C}_d = \{ X_d \in \mathbb{R}_+^d : \exists \theta \in \Theta \, s.t. \, X_d(\theta) = X_d \}.
\]

This interiority assumption implies that the measure of individuals in the region \( \prod_{j=1}^d [\bar{x}_j, \bar{x}_j + d\bar{e}] \) (the square around \( \bar{X}_d = (\bar{z}_1, \bar{z}_2) \) in Figure 3) is second-order relative to the regions \( [\bar{x}_i, \bar{x}_i + d\bar{e}] \times \prod_{j=1,j\neq i}^d [\bar{x}_j, \infty) \) (the dark shaded bands). Thus the change in behavior of individuals in these regions has only a second-order effect on government revenue, and can thus be ignored. Let \( H_X(\cdot) \) denote the joint distribution function of \( X \) on \( \mathbb{R}_+^T \times \mathbb{R}^T \), and \( h_X(\cdot) \) denote the corresponding density function.\(^\text{17}\)

Consider the perturbation \( \nu_{t, \bar{X}_d} \) of the period-\( t \) tax function \( T_t \), defined in (19). In each of the regions of the space defined by this perturbation, either a marginal tax rate or a virtual income is perturbed. The

\(^{17}\)There are some technical difficulties involved in the definition of this density if the distribution of choice vectors \( X \) is degenerate along some directions. One way to resolve this issue is to assume individual-specific interest rates.
general formula (24) we derived in Section 4 describes the change in the choice vector $X$ in response to a simultaneous perturbation of the whole set of marginal tax rates and virtual incomes that the individual faces. To derive the welfare effects of the particular perturbation $v_{t,\bar{X}_d}$, we apply this general formula to each region of the space that it delimits. As in the static case, this tax reform has several effects.

The first effect of the perturbation is the mechanical effect net of the welfare loss. This effect captures the mechanical increase in government revenue due to the tax reform, assuming that individuals do not change their behavior in response to the perturbation. Consider a taxpayer with the choice vector $X$ such that $X_d = (x_1, \ldots, x_d)$ is in the region $\prod_{j=1}^d [\bar{x}_j, \infty] = [\bar{x}_1, \infty] \times \cdots \times [\bar{x}_d, \infty]$, i.e. $x_j \geq \bar{x}_j$ for all $j = 1, \ldots, d$. Thus $X \in \mathcal{D}_R$, where $\mathcal{D}_R = \prod_{j=1}^d [\bar{x}_j, \infty] \times \mathbb{R}^{2T-d}$.\footnote{The first $d$ components of the vector $X$ are restricted to be in the set $\prod_{j=1}^d [\bar{x}_j, \infty]$, and the last $2T - d$ components of $X$ are unrestricted, i.e. they are allowed to be anywhere in their domain of definition. For ease of notation, we denote by $\mathbb{R}$ the domain of definition of all of these $2T - d$ components, although more rigorously some of them (labor incomes) are in $\mathbb{R}_+$.} This taxpayer pays $dR_t = d\tau_t d\bar{x}$ additional taxes in period $t$.\footnote{As in the static case, this is only true for the individuals in the region $X_d \in \prod_{j=1}^d [\bar{x}_j + d\bar{x}, \infty)$, but considering that it is the case for all $X_d \in \prod_{j=1}^d [\bar{x}_j, \infty)$ instead yields formulas that differ only by second-order terms. Since we consider only the first-order effects of the perturbation on revenue and welfare, this difference does not matter.}

In the example of Figure 3, these individuals are in the light shaded region where $z_1 \geq \bar{z}_1$ and $z_2 \geq \bar{z}_2$. This additional tax revenue in period $t$ is valued $\delta^{t-1} \{1 - g_t(X)\} dR_t$ by the government in period one, by definition of the marginal social welfare weights. There are $dH_X(X)$ such individuals. Therefore, the overall mechanical effect net of the welfare loss is equal to this additional tax revenue summed over all individuals who pay the additional tax liability (i.e., $X \in \mathcal{D}_R$). The mechanical effect of the perturbation, denoted by $M(\bar{X}_d)$, can thus be written as

$$M(\bar{X}_d) = \delta^{t-1} \left[ \int_{\mathcal{D}_R} \{1 - g_t(X)\} dH_X(X) \right] dR_t. \quad (30)$$

The second effect of the perturbation is the elasticity effect. This effect captures the change in government revenue due to the behavioral response of individuals whose choice vector $X$ is in a region directly affected by a change in marginal tax rates. In Figure 3, these are the two dark shaded bands. Fix $i \in \{1, \ldots, d\}$. Each choice of $i$ corresponds to the choice of one of the bands where a marginal tax rate is perturbed. In Figure 3, $i = 1$ corresponds to the vertical dark shaded band, and $i = 2$ corresponds to the horizontal dark shaded band. Consider a taxpayer in the region $X \in [\bar{x}_i, \bar{x}_i + d\bar{x}] \times \mathcal{D}_{E_i,d}$, where $\mathcal{D}_{E_i,d} = \prod_{j=1,j\neq i}^d [\bar{x}_j, \infty] \times \mathbb{R}^{2T-d}$. That is, $x_i \in [\bar{x}_i, \bar{x}_i + d\bar{x}]$, while for all other $j \in \{1, \ldots, d\} \setminus \{i\}$, we have $x_j \geq \bar{x}_j$. The other coordinates of $X$, $x_j$ for $j \in \{d+1, \ldots, 2T\}$, are allowed to be anywhere in their domain of definition. Hence, $X \in \mathcal{D}_{E_i,d}$ means that $X$ is in the band where the marginal tax rate $\tau_{t,x_i}$ is perturbed. This taxpayer faces an increase $d\tau_t$ in the marginal tax rate $\tau_{t,x_i}$. This perturbation has an elasticity effect which produces a change $dX$ in the vector of incomes and savings, as described in Section 4.2. This change $dX$ induces a change in government’s revenue in period $s$ given by $d(T_s(X)) = (\nabla T_s(X)) dX$, so that the discounted sum of revenues due to this change $dX$ is equal to $\sum_{s=1}^T \delta^{s-1} (\nabla T_s(X)) dX = T'(X) dX$, where $T'(X)$ is defined in (20). Finally, there are $dH_X(X) = h_{x_i}(\bar{x}_i) d\bar{x} \times dH_{X-\bar{x}_i}(X_{-i} \mid \bar{x}_i)$ such individuals. Integrating over all individuals in the band where the marginal tax rate $\tau_{t,x_i}$ is perturbed ($X \in \mathcal{D}_{E_i,d}$), we obtain that the overall effect of this
behavioral change on tax receipts, denoted by $E_i(\bar{X}_d)$:

$$E_i(\bar{X}_d) = \int_{D_{E_i,d}} T'(\bar{x}_i, X_{-i}) \left( dX|_{(\bar{x}_i, X_{-i})} h_{x_i}(\bar{x}_i) dH_{X_{-i}}|_{x_i} (X_{-i}|\bar{x}_i) d\bar{x} \right)$$

$$= \left[ \int_{D_{E_i,d}} T'(\bar{x}_i, X_{-i}) \left( \frac{dX|_{(\bar{x}_i, X_{-i})}}{dT_t} h_{x_i}(\bar{x}_i) dH_{X_{-i}}|_{x_i} (X_{-i}|\bar{x}_i) \right) dR_{t} \right]$$

where $dX|_{(\bar{x}_i, X_{-i})}/dT_t$ is given in (24),

$$\frac{dX|_{(\bar{x}_i, X_{-i})}}{dT_t} = \left[ I_{dT} - \sum_{s=1}^{T} \zeta_{c,(\bar{x}_i, X_{-i})} (D^2T_s)(\bar{x}_i, X_{-i}) \right]^{-1} \times \left[ \zeta_{c,(\bar{x}_i, X_{-i})} \right],$$

where $\zeta_{X,\tau_{s_x}}$ is the $i^{th}$ column of the elasticity matrix $\zeta_{X,\tau_{s_x}}$. Note that there are $d$ such elasticity effects, $E_i(\bar{X}_d)$ for $i \in \{1, \ldots, d\}$, each of them corresponding to one of the bands where a marginal tax rate is perturbed (the vertical, resp. horizontal, dark shaded band in Figure 3 for $i = 1$, resp. $i = 2$).

The third effect of the perturbation is the *income effect*. It captures the change in government revenue due to the behavioral response of individuals whose choice vector $X$ is in the region where the virtual income is perturbed. In Figure 3, this corresponds to the light shaded region. Consider a taxpayer in the region $X \in D_{R}$ defined above. That is, for all $j \in \{1, \ldots, d\}$, $x_j \geq \bar{x}_j$. A taxpayer in this region faces a lump-sum increase $dR_t = d\bar{x}dT_t$ in his tax liability $T_t(X)$. This perturbation has an income effect which produces a change $dX_t$ in the vector of incomes and savings, as described in Section 4. This change $dX_t$ induces a discounted change in government’s revenue equal to $\sum_{s=1}^{T} \delta^{s-1} (\nabla T_s(X)) dX = T'(X) dX$. Finally, there are $dH(X)$ such individuals. Integrating over all individuals in the region where the virtual income $R_t$ is perturbed (i.e., $X \in D_{R}$), we obtain that the overall effect of this behavioral change on tax receipts, denoted by $I(\bar{X}_d)$:

$$I(\bar{X}_d) = \int_{D_R} T'(X) dX|_X dH(X)$$

$$= \left[ \int_{D_R} T'(X) \left( \frac{dX|_X}{-dR_t} \right) dH(X) \right] dR_t,$$

where $dX|_X / (-dR_t)$ is given by (24)

$$\frac{dX|_X}{-dR_t} = \left[ I_{dT} - \sum_{s=1}^{T} \zeta_{c,(X)} (D^2T_s)(X) \right]^{-1} \times \left[ \eta_{X,R_t} \right].$$

Summing the mechanical effect (net of welfare loss), the $d$ elasticity effects, and the income effect, we obtain the following result:

**Proposition 2.** Consider the perturbation $v_{t,\bar{X}_d}$ defined in (19) of the period-$t$ tax function $T_t$. The net
welfare gain of this perturbation is given by

\[ \Gamma_t (\bar{X}_d) = M (\bar{X}_d) + \sum_{i=1}^{d} E_i (\bar{X}_d) + I (\bar{X}_d), \]

where \( M (\bar{X}_d) \), \( E_i (\bar{X}_d) \), and \( I (\bar{X}_d) \) are respectively given by equations (30), (31), and (32).

Proof. See Appendix.

If there exists a vector \( \bar{X}_d \) such that \( \Gamma_t (\bar{X}_d) \) is positive, then the tax reform \( v_t, \bar{X}_d \) raises welfare. If \( \Gamma_t (\bar{X}_d) \) is negative, then the opposite tax reform raises welfare. The optimal tax system is such that no tax reform yields a positive welfare effect. Thus, the optimal tax system must satisfy,

\[ \Gamma_t (\bar{X}_d) = 0, \quad \forall t, \forall \bar{X}_d. \]

We end this section by comparing the formulas giving the welfare gains of tax reforms in the static case (equation (28)), and in the dynamic case (equation (33)). Despite the greater complexity of the dynamic model, the matrix representation allows us to write the dynamic welfare gains (33) in a remarkably compact way. Our formula generalizes, yet formally resembles, the static formula (28). There are two fundamental differences between them.

The first difference is that in the static case, an increase \( d\tau \) in the marginal tax rate creates an additional distortion at the income level \( \bar{z} \) which unambiguously reduces labor supply through a substitution effect, and hence reduces the revenue gain from increasing taxes. On the other hand, a lump-sum increase \( dR \) in the tax liability above the income level \( \bar{z} \) increases labor supply through the income effect, and hence reinforces the mechanical effect of the tax increase. In short, an increase in revenue obtained by increasing the tax liability of individuals with income \( z \) larger than \( \bar{z} \) comes at the cost of distorting the labor supply of individuals with income \( z \) equal to \( \bar{z} \). In the dynamic model, however, an increase in the marginal tax rate in a given direction, say \( \tau_{t, 1} \), has an effect not only on that variable (here, labor income \( z_1 \)), but on the entire vector \( \bar{X} \) of past, current, and future labor incomes and savings. Consequently, an increase in \( \tau_{t, 1} \) may decrease labor supply in the current period \( (z_1) \), which reduces tax revenue, but may at the same time increase or decrease labor supply in all (or some) other periods \( (z_2, \ldots, z_T) \), and increase or decrease savings in all (or some) periods \( (k_1, \ldots, k_T) \). These behavioral responses are new in the dynamic model, and have important effects on government revenue. It is thus crucial to consider the joint decision of all labor and capital incomes over individuals’ lifetimes, and recognize that all of these decisions are affected by a given change in the tax system, even if this change affects only the marginal tax rate on one of these variables (e.g., \( z_1 \)). These effects can either reduce or amplify the welfare gains predicted by the static formula (28). For example, if an increase in the marginal tax rate on labor income increases savings or labor supply in other periods, and the marginal tax rates on savings or labor supply in other periods are positive, then formula (28) underestimates the welfare gains of the perturbation. Therefore, the welfare effects of perturbing the tax system described by formula (33) are richer than in
the static model, which considered only the effect of tax reforms on current labor supply. We return to
this point in more detail in Section 6.

The second difference is that there is only one elasticity effect in the static model, because there is
only one region where the marginal tax rate is perturbed; namely, the interval \([\bar{z}, \bar{z} + d\bar{z}]\) (the dark shaded
region in Figure 2). This is also the case in the dynamic model if we consider a separable perturbation, i.e.
\(d = 1\). In this case, the only region where a marginal tax rate \((\tau_{t,x_1})\) is perturbed is \([\bar{x}_1, \bar{x}_1 + d\bar{x}] \times \mathbb{R}^{2T-d}\).
However, when we consider joint perturbations in the dynamic model, i.e. \(d \geq 2\), there are \(d \geq 2\) regions
where the marginal tax rates are perturbed, and perturbations in these different regions have different
effects on government revenue. For example, in Figure 3, \(\tau_{t,z_1}\) is perturbed on the vertical dark shaded
region, and \(\tau_{t,z_2}\) is perturbed on the horizontal dark shaded region. Correspondingly, there are thus \(d \geq 2\)
elasticity effects to take into account, which is why we have a sum of \(d\) elasticity effects, \(\sum_{i=1}^{d} E_i(\bar{X}_d)\), in
(28). In short, separable perturbations (or perturbations in the static model) induce only one elasticity
effect in a “large” region of the space (the entire band \([\bar{x}_1, \bar{x}_1 + d\bar{x}] \times \mathbb{R}^{2T-d}\)), whereas joint perturbations
in the dynamic model induce several elasticity effects in smaller regions of the space (the \(d\) shorter bands
\([\bar{x}_i, \bar{x}_i + d\bar{x}] \times \mathcal{D}_{E_i,d}\)). Similarly, the region where the lump sum tax is increased, \(D_R\), is smaller in the
case of a joint perturbation \((d \geq 2)\) than in the case of a separable perturbation \((d = 1)\) that has the
same value of \(dR_t\). This is clear when we compare the light shaded regions in Figures 2 and 3. This
implies that the mechanical and income effects concern a smaller set of individuals in the case of joint
perturbations than in the case of separable perturbations (or in the static model). We return to this
point in more detail in Section 6, where we show that these differences between the effects of separable
and joint perturbations regarding the number of individuals affected by increases in marginal tax rates
and lump-sum taxes are the key to understanding the welfare gains of introducing history-dependence,
or joint taxation of labor and capital, into the tax system.

6 Evaluating Elements of the Tax System

We now focus on a simpler version of the model to provide a decomposition of the key elements of the
general formula for the welfare gains (33). We discuss in the text the role of these assumptions and
which of them can be easily generalized. Importantly, from this simple dynamic model we obtain rich
qualitative and quantitative insights regarding the welfare gains of major potential tax reforms of the
current US tax code.

In Section 6.1, starting from any (potentially suboptimal) tax system, we perturb the labor income
tax rate and analyze theoretically how the welfare gains predicted by the dynamic formula (33) differ
from the formula (28) obtained in the static model. We show that taking into account the dynamic
nature of the model and the intertemporal optimization decisions of individuals significantly alters the
conclusions obtained in a static framework, inducing positive welfare effects of introducing age-dependent
taxes. Specifically, the adjustment of savings in response to perturbations of the labor income taxes plays
a key role. In Section 6.2, we discuss the welfare effects of perturbing the capital income tax rate,
yielding new insights on the important determinants of the capital taxes. In Section 6.3, we analyze the
welfare effects of jointly perturbing several dimensions of the initial tax system. We introduce history-dependence (joint taxation of past and current labor incomes), or income-dependence (joint taxation of current labor and capital incomes). We show that the gains from jointly taxing several types of incomes in a dynamic setting are given by the multivariate hazard rates of the joint distributions of labor and capital incomes. These are a new variable in the analysis of taxation. We use an approach based on copulas to estimate these multivariate hazard rates. Finally, in Section 6.4, we calibrate the model and show that the effects of the dynamic tax reforms are quantitatively significant. We further show that the key quantitative elements of the tax reform are the savings elasticities and the ratios of the multivariate hazard rates to the univariate hazard rates of the joint distributions of incomes.

We start by describing all the simplifying assumptions we make in this section. First, we assume that the utility function \(U\) is time-separable, and that the flow utility has no income effect:

\[
U_i = \sum_{t=1}^{T} \beta^{t-1} u\left( c_t - v \left( \frac{z_t}{\theta_t} \right) \right).
\]

The agent’s labor income \(z_t = \theta_t l_t\) in each period \(t\) is given by the product of his productivity in period \(t\), \(\theta_t\), and his choice of labor supply in period \(t\), \(l_t\). For some of the results, we also need to specify the functional form of the flow utility. When necessary, we assume that the utility of consumption has constant relative or absolute risk aversion, i.e. \(u(c) = e^{1-\sigma} / (1 - \sigma)\) or \(u(c) = -\exp(-\alpha c)\), and that the disutility of labor has constant labor supply elasticity, i.e. \(v(l) = l^{1+\varepsilon} / (1 + \varepsilon)\). The primary role of these assumptions is to make many of the elasticities of interest constant and independent of an individual’s labor or capital incomes, which simplifies the welfare gains formulas.

Second, as our starting point we consider a stylized tax system, not necessarily optimal, that captures the important elements of the current US tax code. First, the tax system is separable. That is, labor and capital income are not jointly taxed, either across time or within periods. Hence, individuals with identical values of period-\(t\) labor income \(z_t\) pay the same tax liability on their labor income in period \(t\), even if they have different levels of labor incomes in periods \(s \neq t\) and capital incomes in periods \(1 \leq s \leq T\). Second, the labor and capital income tax schedules are age-independent, i.e. a given labor or capital income \(z\) or \(r k\) is taxed identically in all periods. Finally, the tax schedule on capital is linear. We thus define formally the baseline tax system as, for all \(t = 1, \ldots, T\),

\[
T_t (\{z_s\}_{s=1}^{T}, \{r_{s+1}k_s\}_{s=1}^{T}) = T_z (z_t) + \tau_k (r_t k_{t-1}).
\]

The assumptions that the flow utility function has no income effect, i.e. that it is of the form \(u(c - v(l))\), and that the initial tax system is separable and age-independent, imply that perturbing the income tax rate on labor or capital income in period \(t\) does not affect labor income in periods \(s \neq t\). Labor income in a given period \(t\) is only a function of the marginal tax rate on labor income in period

---

\(^{20}\text{This functional form for the utility function simplifies the general formula (33) by making several of the dynamic effects of the perturbations equal to zero. For instance, labor income in period } t \text{ does not change in response to perturbations of the marginal tax rates on labor and capital incomes in other periods, or to lump-sum changes in taxes. This allows us to focus in greater detail on some of the key effects of tax reforms. Of course, formula (33) is valid for other functional forms of the utility function, e.g. with income effects and separable between consumption and labor.}\)
Third, we assume that the planner is Rawlsian. This assumption about the redistributive tastes of the government is equivalent to assuming that the social objective is to maximize tax revenue. Therefore, we can interpret the welfare gain formulas obtained in this section as characterizing the distance to the top of the Laffer curve (revenue-maximizing tax rate). For more general social welfare functions, there is an additional term in the formula that depends on the welfare weights of the relevant individuals.

In the three sections that follow, we sequentially add more sophisticated elements to the tax system: age-dependence, non-linear capital taxes, history dependence, joint taxation of labor and capital incomes. Our goal is to use the general perturbation method described above to show the effects that each additional element of the more sophisticated tax system has on government revenue.

6.1 Separable Perturbations of the Labor Income Tax Schedules

We first analyze the effects of reforming the baseline tax system (35) in a way that introduces age-dependent taxes but keeps the separability of the tax system. That is, we consider separable perturbations (see Figure 2) that affect the marginal tax rate on labor income in a given period \( t \) only. We thus perturb the period-\( t \) labor income tax schedule \( T_{z, t}(\cdot) \) on an interval \( [\bar{z}_t, \bar{z}_t + d\bar{x}] \), irrespective of the individual’s choices of \( z_s, s \neq t \) and \( k_s, 1 \leq s \leq T \).

We compare the revenue gains obtained from perturbing the period-\( t \) labor income in the static model (formula (28)) with the revenue gains in the dynamic model (formula (33)). When we consider the static framework, we specifically mean that we view the economy as the succession of static models described in (2). As we explained in Section 3, the separable perturbations to \( T_{z, t}(\cdot) \) (Figure 2) are equivalent to the perturbations defined in the static model (Figure 1). We show that even in the static framework, age-dependent taxes are desirable. When considering the dynamic framework, we show that taking into account the intertemporal optimization of individuals delivers qualitatively different predictions of revenue gains, implying the optimality of further age-dependence in the tax system.

In Subsection 6.1.1, we analyze the revenue gains of separable perturbations to the labor income tax schedules obtained in the static framework. Then, in Subsection 6.1.2, we contrast these gains with those obtained in the dynamic framework. Note that perturbing the tax schedules in only one period at a time introduces age-dependence in the baseline tax system (35). A given income \( z \) will then be taxed differently in different periods, because only one of the schedules \( T_{z, t}(\cdot) \) is perturbed.

6.1.1 Perturbations of the Labor Income Tax Schedule in the Static Framework

In this subsection we analyze the effects of separable perturbations of the labor income tax schedule in period \( t \) in the static framework. Mostly, the analysis here parallels that of Saez (2001).

Fix a period \( t \in \{1, \ldots, T\} \). We perturb the marginal labor income tax rate \( \tau_{t,z_t} \) by \( d\tau_{t,z_t} \) on the interval \( [\bar{z}_t, \bar{z}_t + d\bar{x}] \), and the virtual income \( R_t \) by \( -dR_t = -d\tau_{t,z_t} d\bar{x} \) on the interval \( [\bar{z}_t + d\bar{x}, \infty) \), as represented in Figures 1 and 2. Let \( h_{z_t}(z_t) \) denote the density of labor incomes in period \( t \), and let

\[ h_{z_t}(z_t) = \frac{1}{\bar{z}_t - \bar{z}_t - d\bar{x}} \]

Several microeconometric studies have found small income effects empirically (see, e.g., Gruber and Saez 2002, or the survey by Saez, Slemrod and Giertz 2012). This justifies making this assumption as an important benchmark.
$H_{z_t}(z_t)$ denote the corresponding c.d.f. Let $\Gamma^S_{t,z_t}(z_t)$ denote the revenue gain of this perturbation, given by formula (28). The revenue gain per capita (i.e., normalized by the measure of individuals affected by the perturbation) as of period 1, $\gamma^S_{t,z_t}(z_t)$, is given by the following Proposition, which is implied by the analysis of Saez (2001):

**Proposition 3.** The static revenue gains of separable perturbations, defined in (18), of the period-1 labor income tax schedule $T_{z_t}(\cdot)$ are given by

$$
\gamma^S_{t,z_t}(z_t) = \frac{\Gamma^S_{t,z_t}(z_t)}{1 - H_{z_t}(z_t)}
= \left[1 - \zeta^{c,(z_t)}_{z_t,1-\tau_{z_t}}\right] \frac{T'_{z_t}(z_t)}{1 - T'_{z_t}(z_t)} + \frac{z_t h_{z_t}(z_t)}{1 - H_{z_t}(z_t)} \delta^t dR_t.
$$

(36)

**Proof.** The utility function (34) implies that $\eta_{z_t,R_t} = 0$. Hence the result follows immediately from (28). □

The first term in formula (36), $1 \times \delta^t dR_t$, is the mechanical gain of the perturbation (25), normalized by the number of individuals earning income $z_t$ above $\bar{z}_t$, $1 - H_{z_t}(z_t)$: each individual above $\bar{z}_t$ pays $dR_t$ additional taxes, which brings $\delta^t dR_t$ dollars in revenue to the government. The second term of the formula is the elasticity effect of the perturbation (26), normalized by $1 - H_{z_t}(z_t)$. It is proportional to the labor income elasticity evaluated at point $\bar{z}_t$, $\zeta^{c,(z_t)}_{z_t,1-\tau_{z_t}}$, the marginal tax rate on labor income at point $\bar{z}_t$, $T'_{z_t}(z_t)$, and the hazard rate of the income distribution at point $\bar{z}_t$, defined by,

$$
\lambda_{z_t}(z_t) \equiv -\frac{d}{dz_t} \ln (1 - H_{z_t}(z_t)) \bigg|_{z_t} = \frac{h_{z_t}(z_t)}{1 - H_{z_t}(z_t)}.
$$

(37)

This term captures the cost of the perturbation. An increase in the marginal tax rate at point $\bar{z}$ raises revenue on all individuals with income $z \geq \bar{z}$ (through the mechanical effect). On the other hand, this increase in the marginal tax rate induces individuals with income $z = \bar{z}$ to reduce their labor income (by an amount proportional to $\zeta^{c,(z_t)}_{z_t,1-\tau_{z_t}}(z_t)$), which in turn reduces government revenue. Importantly, the hazard rate (37) is the key variable determining the revenue gains of such perturbations, and in turn the shape of the optimal tax rates. Intuitively, it represents the measure of individuals whose labor supply is distorted by the increase in the marginal tax rate, $h_{z_t}(z_t)$, relative to the measure of individuals who pay the additional lump-sum tax, $1 - H_{z_t}(z_t)$. The larger the hazard rate, the smaller the revenue gain of the perturbation.

Formula (36) can also be used to characterize the optimal labor income tax schedule $\tau^{*}_{t,z_t}(z_t)$ in this model, by equating $\gamma^S_{t,z_t}(z_t)$ to zero (see Saez 2001). The optimal tax rates are proportional to the inverse of the labor income elasticity, $1/\zeta^{c,(z_t)}_{z_t,1-\tau_{z_t}}$, and the inverse of the hazard rate of the labor income distribution, $[1 - H_{z_t}(z_t)]/z_t h_{z_t}(z_t)$. We discuss quantitatively the revenue gains of the separable perturbations implied by formula (36), as well as the optimal labor income tax rates, in Section 6.4.

---

22Note that the hazard rates and the elasticities are endogenous to the tax schedule.
Note that the economy in this subsection is viewed as the succession of $T$ static models. Nevertheless, equation (36) implies that, starting from the baseline age-independent tax system (35), the revenue gains obtained by increasing the marginal tax rate on labor income in period $t$ at a given point $\tilde{z}$, $\Gamma^S_{t,zt} (\tilde{z})$, are in general different from those obtained by the same perturbation in period $s \neq t$, $\Gamma^S_{s,zs} (\tilde{z})$. There are three differences. The measure $1 - H_{zt} (\tilde{z})$ of individuals with income above the point $\tilde{z}$ where the marginal tax rate is perturbed, the labor income elasticities $\zeta^c_{zt,1-\tau_{zt}}$ (see Kremer 1999), and the hazard rates (37) of the labor income distributions, may all vary with age. The fact that identical perturbations implemented in different periods induce different revenue gains, starting from an age-independent baseline tax system, implies that the optimal labor income tax system, even in the static model, should already feature some age-dependence.

6.1.2 Separable Perturbations of the Labor Income Tax Schedules in the Dynamic Model

While the previous subsection essentially paralleled the analysis in the static model, here we show that even simple separable perturbations have significantly different effects in the dynamic setting. Most importantly, the adjustments in savings in response to the perturbations play an essential role in the analysis of these tax reforms.

We start from the baseline tax system (35) and again consider the effect of separable perturbations defined in (19) with $d = 1$, represented in Figure 2. That is, for a given $t \in \{1, \ldots, T\}$, we perturb the marginal tax rates $\tau_{zt}$ by $d\tau$ on the interval $[\tilde{z}_t, \tilde{z}_t + d\tilde{x}]$, and perturb the virtual income $R_t$ by $-dR_t = -d\tau d\tilde{x}$ on the interval $[\tilde{z}_t + d\tilde{x}, \infty)$, irrespective of the individuals’ other choices of labor and capital incomes. By perturbing the labor income tax schedule in period $t$ only, these perturbations introduce age-dependence in the baseline tax system (35). Let $\Gamma_{t,zt}(\tilde{z}_t)$ denote the revenue gains of these perturbations, given by formula (33). Let $\gamma_{t,zt} (\tilde{z}_t)$ denote the revenue gains per capita, i.e. normalized by the measure of individuals affected by the perturbation. For any income level $z_t \geq \tilde{z}_t$, we let $\eta_{s,zt}^{(z_t)}$ denote the average income effect parameter (of period-$s$ savings w.r.t. period-$t$ lump-sum income) among individuals with income $z_t$ in period $t$, i.e.

$$
\eta_{s,zt}^{(z_t)} = \int_{\mathbb{R}_+}^{z_t} \eta_{s,R_t} (X_{-t} | z_t) \frac{dH_{X_{-t} | z_t} (X_{-t} | z_t)}{1 - \tau_{s,k_t}} dR_t.
$$

We show the following Proposition:

**Proposition 4.** The revenue gains of the separable perturbations of the period-$t$ labor income tax schedule $T_{zt} (\cdot)$, defined in (19) with $d = 1$, are given in the dynamic model by

$$
\gamma_{t,zt} (\tilde{z}_t) = \frac{\Gamma_{t,zt}(\tilde{z}_t)}{1 - H_{zt}(\tilde{z}_t)} = \gamma_{t,zt}^S (\tilde{z}_t) - \left[ \tau_k \int_{\tilde{z}_t}^{\infty} \left\{ \sum_{s=1}^{T} \delta^{s-1} \frac{\eta_{s,R_t}^{(z_t)}}{1 - \tau_{s,k_t}} \frac{h_{zt}(\tilde{z}_t)}{1 - H_{zt}(\tilde{z}_t)} d\tilde{z}_t \right\} dR_t, \right. \tag{38}
$$

where $\gamma_{t,zt}^S$ is the per capita revenue gain of the same perturbation obtained in the static framework, given by (36). If the utility function is CRRA or CARA, i.e. $u(c) = c^{1-\sigma}/(1-\sigma)$ or $u(c) = -\exp(-\alpha c)$,
then the income effect parameters $\eta_{k_s,R_t}$ are constant and formula (38) simply writes

$$\gamma_{t,z_t}(\bar{z}_t) = \gamma^s_{t,z_t}(\bar{z}_t) - \left[ \tau_k \sum_{s=1}^{T} \delta^{s-1} \frac{\eta_{k_s,R_t}}{1 - \tau_{s,k_t}} \right] dR_t.$$  
(39)

Proof. See the Appendix.

If the utility function is CRRA or CARA, the income effect parameters $\eta^{(X)}_{k_s,R_t}$ are constant, i.e. do not depend on the individual’s labor and capital incomes. The explicit expressions for these parameters, given in the Appendix, depend only on $\beta, \sigma, \tau_k, T$ and the periods $s, t$ considered. Thus the discounted sum of average income effect parameters in formula (38) can be taken out of the integral, and we obtain the simpler expression (39).

Equation (38) shows that the revenue gains of separable perturbations are the same as those obtained in the static model, plus an additional term. This new term captures the fact that a decrease in the period-$t$ virtual income $R_t$ (for individuals with income $z_t \geq \bar{z}_t$) induces a change in savings in every period $s$, given by $-\partial k_s/\partial R_t = -\eta_{k_s,R_t}/(1 - \tau_{s,k_t}).$ This in turn induces a change in government revenue in every period $s$ equal to, as of period 1, $-\tau_k \delta^{s-1} (\partial k_s/\partial R_t)$. Since a change in individual’s income in period $t$ induces a change in savings in all periods $s$, the total effect on government revenue is the discounted sum of these changes. These effects of the perturbations of labor income taxes on individuals’ savings cannot be captured in the static framework.

The closed-form expressions for the income effect parameters $\eta_{k_s,R_t}$ (in the Appendix) show that $-\eta_{k_s,R_t} > 0$ for all $s < t$, and $-\eta_{k_s,R_t} < 0$ for all $s \geq t$. Thus, the change in savings is positive in all the periods preceding the perturbation in period $t$, and negative in all the periods after $t$. Intuitively, this is because increasing labor income taxes (i.e., lowering income) in period $t$ induces individuals to save more when they are younger, i.e. in all periods $s < t$, as they anticipate that they will be poorer in the future. Since the capital income tax rate $\tau_k$ is positive in the baseline tax system, an increase in savings induces an additional gain in revenue for the government, relative to what the static formula predicts. The opposite reasoning holds for periods $s \geq t$: in every period after the tax increase, individuals are poorer and hence reduce their savings, which decreases government revenue if the capital income tax rate is positive. Therefore, an increase (resp., decrease) in the labor income tax rate late (resp., early) in life has an additional positive effect on government revenue. It is thus desirable for the government to implement tax reforms that reduce the tax rates on labor income for young individuals, and increase the tax rates on labor income for older individuals, relative to the optimal static tax schedule.

Let

$$\mathcal{S}_t \equiv -\tau_k \sum_{s=1}^{T} \delta^{s-1} \frac{\eta_{k_s,R_t}}{1 - \tau_{s,k_t}}$$  
(40)

denote the additional savings term in formula (39). The discussion above implies that $\mathcal{S}_t$ is increasing.

---

23 Our assumptions about the utility function (absence of income effects) and the baseline tax system (separability) imply that savings respond only to a change in the virtual income; the compensated elasticities of savings with respect to the marginal labor income tax rates, $\xi^{(X)}_{s,1-\tau_t,z_t}$, are equal to zero.
in \( t \), with \( \mathcal{J}_t < 0 \) and \( \mathcal{J}_T > 0 \). Thus there exists a period \( t^\ast \) such that \( \mathcal{J}_t < 0 \) for \( t \leq t^\ast \), and \( \mathcal{J}_t > 0 \) for \( t > t^\ast \). The welfare gain of the perturbation in period \( t \) is smaller than predicted by the static formula in all periods \( t \leq t^\ast \), and larger in all periods \( t > t^\ast \). We show in the Appendix the following Corollary:

**Corollary 5.** The optimal separable labor income tax schedule\(^{24}\) in period \( t \) in the dynamic framework, \( T^a_t(\cdot) \), is given by

\[
\frac{T^a_t(z_t)}{1 + T^a_t(z_t)} = (1 + \mathcal{J}_t) \frac{T^{a,S}_z(z_t)}{1 + T^{a,S}_z(z_t)},
\]

where \( T^{a,S}_z(\cdot) \) is the optimal tax schedule in period \( t \) in the static framework. The optimal separable labor income tax rates are increasing with age \( t \), and there exists a period \( 1 < t^\ast < T \) such that they are strictly smaller than in the static framework for all \( t \leq t^\ast \), and strictly larger for all \( t > t^\ast \).

*Proof.* See the Appendix. \( \square \)

We discuss quantitatively these optimal separable tax rates in Section 6.4.

This effect, which is the result of the intertemporal optimization of individuals, is present only in a dynamic model. This gives one important theoretical reason explaining the large welfare gains found numerically in the previous literature of moving the tax system from age-independence to age-dependence (see, e.g., Weinzierl 2011, Blundell and Shephard 2012). We discuss the quantitative importance of this effect in Section 6.4.

Note that in this subsection we are capturing only some of the gains from age-dependence. Indeed, a fully age-dependent tax system may also feature joint taxation of labor and capital incomes within each period. We will discuss this case in Section 6.3. Moreover, note that in the general version of the model analyzed in Sections 2 to 5, there are other effects of introducing age-dependence. Namely, perturbing the marginal tax rate in period \( t \) affects not only current labor income (as is the case when the utility function has no income effects), but also labor income in all periods \( s \neq t \). The important parameters in this case would be the elasticities of labor income in all periods \( s \) with respect to the marginal tax rate on labor income in period \( t \). The welfare gains would be given by the general formula (33).\(^{25}\)

### 6.2 Separable Perturbations of the Capital Income Tax Schedules

In this section we apply the general formula (33) to obtain the revenue gains of reforms of the capital income tax rates in the baseline tax system (35). The reforms we consider here are separable perturbations. Hence, the capital tax schedule in period \( t + 1 \) after the perturbation is still a function of current capital income \( r_{t+1} k_t \) only. That is, we do not introduce joint taxation of labor and capital incomes nor of capital incomes across periods. We consider two types of separable perturbations of the capital income tax rates. First, we derive the revenue gains of the reforms described in (19), which change the tax rate by \( d\tau \) in period \( t + 1 \) at a given point \( r_{t+1} k_t \) in the distribution. Such reforms introduce a non-linearity

---

\(^{24}\) That is, the labor income tax rates that are optimal if the capital income tax rate is kept unchanged, and if the tax system is kept separable between labor and capital incomes and between labor incomes across time (history-independent). The amount of age-dependence implied by (41) depends on the capital tax rate \( \tau_k \) in the baseline tax system.

\(^{25}\) In Section 6.3 we refer to the evidence that these latter elasticities are typically close to zero in the data, which supports our assumption of a utility function with no income effects as an important benchmark.
in the capital income tax schedule. Second, we analyze reforms that preserve the linearity of the baseline tax functions. That is, we increase the marginal tax rates by $d\tau$ for all $t, k \in \mathbb{R}$. We use such reforms to obtain as a particular case of our analysis the Chamley-Judd result about the suboptimality of linearly taxing capital in the long-run.

**Non-linear Tax Reforms.** Starting from the baseline tax system (35), we consider separable perturbations of the form (19) with $d = 1$, that affect the capital income tax schedule $T_k (\cdot)$. Formally, we perturb the marginal tax rate $\tau_{t+1,k_t}$ in period $t+1$ by $d\tau$ on the interval $[\hat{k}_t, \bar{k}_t + d\bar{x}]$, and perturb the virtual income $R_{t+1}$ by $-dR_{t+1} = -d\tau d\bar{x}$ on the interval $[\hat{k}_t + d\bar{x}, \infty)$. Note that this perturbation introduces a non-linearity at point $\hat{k}_t$ in the capital tax schedule, which was initially linear in the baseline tax system. Let $h_{k_t} (k_t)$ denote the density of savings in period $t+1$, and let $H_{k_t} (k_t)$ denote the corresponding c.d.f. Let $\zeta_{k_t,1-\tau_{t+1,k_t}}$ denote the average compensated elasticity (of period-t+1 capital w.r.t. the net of tax rate on capital income in period $t+1$) among individuals who have capital $k_t$ in period $t+1$, weighted by their relative level of capital $k_s/\hat{k}_t$ in period $s + 1$ versus period $t + 1$, i.e.

$$
\zeta_{k_t,1-\tau_{t+1,k_t}} = \int_{R_T^t \times R_{T-1}} \frac{k_t \zeta_{k_t,1-\tau_{t+1,k_t}} (X_{-t+1} | k_t)}{dH_{X-k_t}} (X_{-t+1} | k_t) \, dH_{X-k_t}.
$$

Similarly, let $\eta_{k_t,R_{t+1}}$ denote the average income effect parameter (of period-$s+1$ capital w.r.t. period-$t+1$ lump-sum income) among individuals who have capital $k_t$ in period $t + 1$, i.e.

$$
\eta_{k_t,R_{t+1}} = \int_{R_T^t \times R_{T-1}} \frac{\eta_{k_t,R_{t+1}} (X_{-t+1} | k_t)}{dH_{X-k_t}} (X_{-t+1} | k_t) \, dH_{X-k_t}.
$$

Applying formula (33), this perturbation yields per capita revenue gains given by the following Proposition:

**Proposition 6.** The revenue gains of separable perturbations of the capital income tax schedule $T_k (\cdot)$, defined in (19) with $d = 1$, are given by

$$
\gamma_{t,k_t} (\hat{k}_t) = \frac{\Gamma_{t,k_t} (\hat{k}_t)}{1 - H_{k_t} (\hat{k}_t)} = 1 - \frac{\tau_k}{1 - \tau_k} \left\{ \sum_{s=1}^{T} \delta^{s-t-1} \zeta_{k_t,1-\tau_{t+1,k_t}} \right\} \frac{\hat{k}_t h_{k_t} (\hat{k}_t)}{1 - H_{k_t} (\hat{k}_t)} - \tau_k \int_{\hat{k}_t}^{\infty} \left\{ \sum_{s=1}^{T} \delta^{s-t-1} \eta_{k_t,R_{t+1}} \right\} \frac{h_{k_t} (k_t)}{1 - H_{k_t} (k_t)} \, dk_t \right\} \delta^t dR_{t+1}.
$$

If the utility function is CRRA or CARA, i.e. $u(c) = c^{1-\sigma} / (1 - \sigma)$ or $u(c) = -\exp (-\alpha c)$, then the income effect parameters $\eta_{k_t,R_{t+1}}$ are constant, and formula (42) simply writes

$$
\gamma_{t,k_t} (\hat{k}_t) = \left[ 1 - \frac{\tau_k}{1 - \tau_k} \left\{ \sum_{s=1}^{T} \delta^{s-t-1} \zeta_{k_t,1-\tau_{t+1,k_t}} \right\} \frac{\hat{k}_t h_{k_t} (\hat{k}_t)}{1 - H_{k_t} (\hat{k}_t)} - \tau_k \sum_{s=2}^{T} \delta^{s-t-1} \eta_{k_t,R_{t+1}} \right\} \delta^t dR_{t+1}.
$$

**Proof.** See Appendix.
The first term of (42) describes the mechanical effect of the perturbation (30): \( 1 \times dR_{t+1} \) is the additional tax paid in period \( t+1 \) by individuals with capital \( k_t \geq \bar{k}_t \) in period \( t+1 \). The second term describes the elasticity effect (31) due to the additional distortion incurred by individuals with capital \( k_t = \bar{k}_t \). It is proportional to the discounted sum of savings elasticities in all periods \( s \), \( \zeta^c_{k_t,1-\tau_{t+1,k_t}} > 0 \), because the individual reduces his savings in all periods \( s \in \{1, \ldots, T\} \) following a change in the marginal tax rate on capital in period \( t+1 \).\(^{26}\) It is also a function of the marginal tax rate on capital \( \tau_k \), and the hazard rate of the capital income distribution,

\[
\lambda_{k_t}(k_t) \equiv -\frac{d}{dk_t} \ln (1 - H_{k_t}(k_t)) = \frac{h_{k_t}(\bar{k}_t)}{1 - H_{k_t}(\bar{k}_t)}. 
\]

(44)

The interpretation of the hazard rate of the capital distribution is the same as described for perturbations of the labor income distribution above. If this hazard rate is large, the measure of individuals whose capital accumulation decisions are distorted by the perturbation, \( h_{k_t}(\bar{k}_t) \), is large relative to the gain in tax revenue coming from the lump-sum tax increase faced by individuals above \( \bar{k}_t \), \([1 - H_{k_t}(\bar{k}_t)]\), and therefore the perturbation induces a large behavioral loss. Note that since the marginal tax rate on capital income is constant in the baseline tax system (35), there is no term of the form \( T''_{k_t}(r_{t+1}k_t) \) in expression (42). Finally, the third term in (42) describes the income effect of the perturbation (32); it is equal to the change in capital income (in all periods \( s \)) of the individuals who pay the additional tax (i.e., who save \( k_t \geq \bar{k}_t \) in period \( t \)), measured by the discounted sum of income effect parameters \( \eta_{c_s,R_{t+1}} \), times the marginal tax rate on capital \( \tau_k \).

The perturbation induces no other effects on individual behavior. This is because we have assumed that the utility function (34) has no income effects and the tax system is initially separable. This implies that an increase in the capital tax rate does not affect the choices of labor incomes \( z_t \), which depend only on the marginal tax rates on labor income. Formula (42) would include these other effects with more general preferences or tax systems, as the general formula (33) shows. The important parameters would be the elasticities of labor income in all periods with respect to the marginal tax rate on capital in period \( t+1 \).

**Linear Tax Reforms.** We now derive the revenue gains of tax reforms of the capital income tax schedule that leave the capital tax rate constant (and separable from the labor income taxes). The capital tax rate is now perturbed in a given period by \( d\tau \) for all levels of capital in the distribution. That is, the linear capital income tax rate \( \tau_k \) is replaced by the linear tax rate \( \tau_k + d\tau \).

Let \( \tilde{\zeta}^c_{k_t,1-\tau_{t+1,k_t}} \) denote the average uncompensated elasticity (of period-\( s + 1 \) capital w.r.t. the net of tax rate on capital in period \( t+1 \)) among all individuals in the population, weighted by their level of

\(^{26}\)Recall that separable perturbations of the labor income tax schedule in period \( t \) induce an adjustment in current labor income \( z_t \) only. The compensated elasticities were therefore not compounded in (39).
Let $\hat{k}_t$ denote the average capital stock in the economy in period $t+1$, i.e.

$$\hat{k}_t \equiv \frac{1}{R_t} \int_{R_t} k h_k (k_t) d k_t.$$  

Such perturbations yield revenue gains given by the following Proposition:

**Proposition 7.** The revenue gain of the perturbation that consists of increasing the linear capital income tax rate $\tau_{t+1,k_t}$ in period $t+1$ by $d\tau$ is given by

$$\Gamma_{t,\tau_k} = \left[ \hat{k}_t - \frac{\tau_k}{1 - \tau_k} \sum_{s=1}^{T} \delta^{s-t} \tilde{\gamma}^u_{k_s,1-\tau_{t+1,k_t}} \right] \delta^t d\tau.$$  

(45)

**Proof.** See Appendix.

Since this perturbation increases the linear tax rate on capital income from $\tau_k$ to $(\tau_k + d\tau)$ for all values of capital $k_t$ in period $t+1$, the elasticity that is relevant to compute the revenue gain is the uncompensated elasticity of savings, which includes both substitution and income effects. Note again that this perturbation induces individuals to change their savings in every period $s$, so that the change in government revenue is given by the discounted sum of these elasticities. Moreover, the perturbation has no effects on the labor income choices of individuals, since it does not affect the marginal tax rates on labor income and the utility function has no income effects. We show in the Appendix that the compensated elasticities $\xi_{k_s,1-\tau_{t+1,k_t}}^{(X)}$ are always positive, whereas the income effect parameters $\eta_{k_s,R_{t+1}}^{(X)}$ are negative in periods $s \leq t$ and positive in periods $s \geq t+1$. Therefore, from the Slutsky equations, the uncompensated elasticities $\xi_{k_s,1-\tau_{t+1,k_t}}^{(X)}$ will be unambiguously positive in all periods $s \geq t+1$ (the income and the substitution effects go in the same direction, inducing a decrease in savings following an increase in the marginal tax rate $\tau_k$ in period $t+1$), but can be positive or negative in periods $s \leq t$, depending on the choice vector $X$ the individual chooses. If the income effect dominates the substitution effect, increasing the marginal tax rate on capital may lead to revenue gains, as it induces individuals to increase their savings in response to an increase in the tax rate.

This leads us to discuss a particular case of our results: the Chamley (1986) and Judd (1985) theorem implying the suboptimality of taxing capital linearly in the steady-state of an infinite-horizon model. Assume that $T = \infty$ and fix a period $\tilde{t} < \infty$. Assume moreover that $\gamma_k = 0$ in all periods $t \geq \tilde{t}$ in the initial tax system, and that the labor income tax schedules $\tau_{t,z_t}$ are linear. We perturb the capital income tax rates $\tau_{t,k_t}$ by $d\tau$, as described in Proposition 7, in all periods $t \geq \tilde{t}$, and simultaneous perturb the labor income tax rates $\tau_{t,z_t}$, such that the government revenue remains constant in every period. We prove in the Appendix that the income effects associated with the smaller labor income tax rates exactly
compensate the income effects associated with the larger capital income tax rates.\textsuperscript{27} Individuals thus only face the additional distortion due to the increase in $\tau_k$, with no loss in revenue. The relevant elasticities for this perturbation are therefore the compensated savings elasticities, which are strictly positive: savings unambiguously decrease in every period following the increase in each of the $\tau_{t,k}$, because there is no income effect associated with the tax increase. We then show that summing these behavioral effects over all periods $t \geq \bar{t}$, implies that the compounded elasticity of savings $\sum_{t=\bar{t}}^{\infty} \sum_{s=1}^{\infty} \delta^{s-1} \zeta_{k_{s,t},1-\tau_{t+1,k_{s,t}}}$, which determines the behavioral effect of the perturbation, diverges to infinity as $\bar{t} \rightarrow \infty$. Introducing a small tax $d\tau$ on capital starting from a situation with no tax is therefore suboptimal in the long run.

6.3 Joint Taxation

6.3.1 Revenue Gains of Joint Perturbations

In this section, we start from the baseline tax system (35) and consider joint perturbations of the form (19), that is, with $d \geq 2$. We perturb the period-$t$ tax function jointly in two directions $x_1, x_2 \in \{z_s, r_{s+1,k_s}\}_{s=1}^{T}$, at point $(\bar{x}_1, \bar{x}_2)$. This perturbation is represented in Figure 3. Recall that the initial tax function $T_t(\cdot)$, defined in (35), is separable in all its variables. In particular, this means that the marginal tax rates of $T_t$ with respect to period-$s$ labor income $z_s$ or period-$t$ capital income $r_{t,k_{t-1}}$ do not depend on the value of period-$t$ labor income $z_t$, and vice versa. The joint perturbations we consider in this section introduce such dependence, and hence break the separability of the tax function that the individual faces in period $t$. If the directions $x_1, x_2$ of the perturbation are labor incomes in periods $s$ and $t$, $z_s$ and $z_t$, then this joint perturbation introduces history-dependence in the tax system, since the new (perturbed) tax function $T_t(\cdot)$ depends jointly on the values of $(z_s, z_t)$. If the directions $x_1, x_2$ are period-$t$ labor and capital incomes, $z_t$ and $r_{t,k_{t-1}}$, then this perturbation introduces joint taxation between labor and capital incomes in the period-$t$ tax function. It is straightforward to generalize the analysis of this section to joint perturbation in several directions $x_1, \ldots, x_d$ (see Appendix).

We start by defining important concepts for what follows.

**Definition 8.** Given a joint density function of $n$ variables $h_x(x_1, \ldots, x_n)$, we denote the survival c.d.f. (i.e., the probability that $x_j \geq \bar{x}_j$ for all $j \in \{1, \ldots, n\}$) by

$$\bar{H}_x(\bar{x}_1, \ldots, \bar{x}_n) = \int_{\bar{x}_1}^{\infty} \cdots \int_{\bar{x}_n}^{\infty} h_x(x_1, \ldots, x_n) \, dx_1 \cdots dx_n. \quad (46)$$

\textsuperscript{27}To be more precise, in general the decrease of the labor income tax rates does not exactly compensate the income loss of every individual due to the increase in the capital tax rate. For some individuals, the rebate is larger than their income loss, while for others it is smaller. However, we show that on average, the total income effect of the simultaneous perturbations is exactly zero. Thus the aggregate economy behaves as if there were only substitution effects (compensated perturbations), i.e. as if everyone were given back in a rebate exactly their loss of income due to the increase in $\tau_k$. 

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We define the multivariate hazard vector (see Johnson and Kotz (1975)) as follows:

\[
\lambda(x_1, \ldots, x_n) = -\nabla \left( \ln H(x_1, \ldots, x_n) \right) \\
= \left( -\frac{\partial}{\partial x_1} \ln H(x_1, \ldots, x_n), \ldots, -\frac{\partial}{\partial x_n} \ln H(x_1, \ldots, x_n) \right)^t
\]

\[
= \left( -\frac{\partial}{\partial x_1} \frac{\bar{H}(x_1, \ldots, x_n)}{H(x_1, \ldots, x_n)}, \ldots, -\frac{\partial}{\partial x_n} \frac{\bar{H}(x_1, \ldots, x_n)}{H(x_1, \ldots, x_n)} \right)^t
\]

\[
= \left( [\lambda_1(x_1, \ldots, x_n)]_1, \ldots, [\lambda_n(x_1, \ldots, x_n)]_n \right)^t.
\]

We denote the ratio between the \(j\)th component of the multivariate hazard rate (47) of the distribution \(h(x_1, \ldots, x_n)\), and the univariate hazard rate (37) of the marginal distribution \(h_x(x_j)\), as

\[
\alpha_j(x_1, \ldots, x_n) = \frac{[\lambda(x_1, \ldots, x_n)]_j}{\lambda_j(x_j)}, \quad \forall j \in \{1, \ldots, n\}.
\]

We return to the interpretation of the parameters \(\alpha_j(x_1, \ldots, x_n)\) below.

Consider the perturbation \(v_{t_1, t_2}(x_1, x_2)\), defined in (19) and represented in Figure 3, of the baseline tax system (35) in period \(t\). That is, the marginal rate \(\tau_{t_1, x_1}\) (resp., \(\tau_{t_2, x_2}\)) is perturbed by an amount \(d\tau\) for all individuals with incomes \((x_1, x_2) \in [x_1, x_1 + d\bar{x}] \times [x_2, \infty)\) (resp., \((x_1, x_2) \in [x_1, \infty) \times [x_2, x_2 + d\bar{x}]\)). The individuals with incomes \(x_1 \geq \bar{x}_1\) and \(x_2 \geq \bar{x}_2\) face a lump-sum tax change \(dT_t = -d\bar{x}d\tau\). Hence, the tax liability of an individual with income \(x_1 = \bar{x}_1\) is perturbed only if his income \(x_2\) is larger than the threshold \(\bar{x}_2\), and vice versa. Only a subset of individuals with income \(\bar{x}_1\) thus face a tax change. In contrast, in the case of the separable perturbation considered in Sections 6.1.2 and 6.2 and represented in Figure 2, the tax liability faced by an individual with income \(\bar{x}_1\) was perturbed irrespective of the values of his other choices of incomes.

Let \(h_{x_1, x_2}(x_1, x_2)\) denote the joint density of incomes \((x_1, x_2)\), and \(\bar{H}_{x_1, x_2}(\bar{x}_1, \bar{x}_2)\) denote the measure of individuals with labor incomes \(x_1 \geq \bar{x}_1\) and \(x_2 \geq \bar{x}_2\) (the shaded region in Figure 3), as defined in (46). Let \(\Gamma_{t_1, t_2}(\bar{x}_1, \bar{x}_2)\) denote the revenue gain induced by this joint perturbation, and \(\gamma_{t_1, t_2}(\bar{x}_1, \bar{x}_2)\) denote the revenue gain per capita (i.e., normalized by the measure of individuals affected by the perturbation), that is, \(\gamma_{t_1, t_2}(\bar{x}_1, \bar{x}_2) = \Gamma_{t_1, t_2}(\bar{x}_1, \bar{x}_2) / \bar{H}_{x_1, x_2}(\bar{x}_1, \bar{x}_2)\). It is natural to compare the revenue gain induced by this joint perturbation at point \((\bar{x}_1, \bar{x}_2)\), \(\gamma_{t_1, t_2}(\bar{x}_1, \bar{x}_2)\), to the revenue gains induced by the two corresponding separable perturbations, namely the separable perturbation of \(\tau_{t_1, x_1}\) at point \(\bar{x}_1\) (represented in Figure 2), and that of \(\tau_{t_2, x_2}\) at point \(\bar{x}_2\). Applying the formula (33), we obtain the following Proposition:

\[\text{Consider for simplicity the case where } x_1, x_2 \text{ are labor incomes in periods } s \text{ and } t > s. \text{ The first separable perturbation that we are considering here is to increase the marginal tax rate on } z_s \text{ on } [\bar{z}_s, \bar{z}_s + d\bar{x}] \text{ in period } t, \text{ i.e. } \partial T_t / \partial z_s. \text{ This marginal tax rate was originally equal to zero, because } z_s \text{ is taxed only in period } s \text{ in the baseline tax system. Recall that in Subsection 6.1.2 we considered a separable perturbation of the period-} s \text{ marginal tax rate } \partial T_{t_1} / \partial z_s \text{ and obtained the corresponding revenue gain } \gamma_s(x_s) \text{ in (38). However, we can show that with a tax-system that is initially history-independent, i.e. } \partial T_t / \partial z_s = 0, \text{ perturbing } \partial T_t / \partial z_s = \tau_{t, z_s} \text{ by } d\tau' = d\bar{\tau}'(1 - \tau_s)^{1-t} \text{, i.e. } \gamma_{t, z_s}(\bar{z}_s) \text{. In this subsection, we perturb } z_s \text{ and } z_t \text{ jointly in period } t, \text{ and express the resulting welfare gain } \gamma_{z_s, z_t}(\bar{z}_s, \bar{z}_t) \text{ as a function of the welfare gains of the two separable perturbations in period } t, \gamma_{t, z_s}(\bar{z}_s) \text{ and } \gamma_{t, z_t}(\bar{z}_t).} \]
Proposition 9. Assume that the utility function is CRRA or CARA, i.e. \( u(c) = c^{1-\sigma}/(1-\sigma) \) or \( u(c) = -\exp(-\alpha c) \). Then the revenue gain of the joint perturbation (19) at point \((\bar{x}_1, \bar{x}_2)\) is given by

\[
\gamma_{t,(x_1,x_2)}(\bar{x}_1, \bar{x}_2) = \alpha_1 (\bar{x}_1, \bar{x}_2) \gamma_{t,x_1}(\bar{x}_1) + \alpha_2 (\bar{x}_1, \bar{x}_2) \gamma_{t,x_2}(\bar{x}_2) \\
+ [1 - \alpha_1 (\bar{x}_1, \bar{x}_2) - \alpha_2 (\bar{x}_1, \bar{x}_2)] (1 + \mathcal{J}_t) dR_t,
\]

where \( \mathcal{J}_t \) is given by (40), \( \gamma_{t,x_1}(\bar{x}_1) \) and \( \gamma_{t,x_2}(\bar{x}_2) \) are given by (39) and (43), and

\[
\alpha_1 (\bar{x}_1, \bar{x}_2) = \frac{[\lambda_{x_1,x_2}(\bar{x}_1, \bar{x}_2)]_1}{\lambda_{x_1}(\bar{x}_1)} = \left( \frac{h_{x_1}(\bar{x}_1)}{1 - H_{x_1}(\bar{x}_1)} \right)^{-1} \left( \frac{\int_{\bar{x}_2}^{\infty} h_{x_1,x_2}(\bar{x}_1, x_2) dx_2}{H_{x_1,x_2}(\bar{x}_1, \bar{x}_2)} \right)
\]

is the ratio between the first component of the multivariate hazard rate of the joint distribution of incomes \((x_1, x_2)\), and the univariate hazard rate (37) of the distribution of income \(x_1\), as defined in (48). Similarly, \( \alpha_2 (\bar{x}_1, \bar{x}_2) = [\lambda_{x_1,x_2}(\bar{x}_1, \bar{x}_2)]_2 / \lambda_{x_2}(\bar{x}_2) \).

Proof. See the Appendix.

Proposition 9 shows that the revenue gain of the joint perturbation (19) at point \((\bar{x}_1, \bar{x}_2)\) is equal to a weighted sum of the revenue gains of the two corresponding separable perturbations, plus an additional term that captures the gains of the joint taxation. The magnitude of this additional term is determined by the value of

\[
g_{x_1,x_2}(\bar{x}_1, \bar{x}_2) \equiv 1 - \alpha_1 (\bar{x}_1, \bar{x}_2) - \alpha_2 (\bar{x}_1, \bar{x}_2)
\]

Hence, the key parameters determining the gains of introducing history-dependence or joint taxation of labor and capital incomes in the baseline tax system are the multivariate hazard rates of the joint distribution of incomes. More precisely, it is their ratios with the univariate hazard rates of the marginal distributions of incomes in each period. We derive theoretical and quantitative properties of \( g_{x_1,x_2}(\bar{x}_1, \bar{x}_2) \) in Sections 6.3.2 and 6.4.

Consider for concreteness the case \((x_1, x_2) = (z_s, z_t)\), i.e. the introduction of history-dependence between labor incomes in periods \(s\) and \(t\) in the period-\(t\) tax function. The discussion of the joint taxation of labor and capital incomes in period \(t\), i.e. \((x_1, x_2) = (x_s, r_t k_{t-1})\), is similar. Note that separable perturbations of the baseline tax system can only yield revenue gains that are a linear combination of \( \gamma_{t,z_s}(\bar{z}_s) \) and \( \gamma_{t,z_t}(\bar{z}_t) \), as we derived in Subsection 6.1.2. The optimal separable tax system is such that both of these are equal to zero, as in this situation the planner cannot increase revenue by perturbing the tax system in a way that keeps the separability of the tax function. The gain from introducing history-dependence is then proportional to \( g_{z_s,z_t}(\bar{z}_s, \bar{z}_t) \). Hence, as soon as the sum of the two relative hazard rates \( \alpha_1 (\bar{z}_s, \bar{z}_t) \) and \( \alpha_2 (\bar{z}_s, \bar{z}_t) \) is different from one, i.e. \( g_{z_s,z_t}(\bar{z}_s, \bar{z}_t) \neq 0 \), there is a gain from jointly perturbing the tax system, and hence, introducing history-dependence.

Recall that a fundamental insight of the static model (see Saez (2001) and Subsection 6.1.1) is that the \textit{univariate} hazard rates \( \lambda_{z_i}(z_i) \) of the labor income distribution in period \(t\) are the key parameters determining the welfare gains of the \textit{separable} perturbations of the period \(t\) tax function, as well as the
shape of the optimal separable tax schedules. Proposition 8 generalizes this result to the dynamic setting and shows that the welfare gains of joint perturbations, which introduce history-dependent taxes between incomes in periods $s$ and $t$, are determined by the multivariate hazard rates of the joint distribution of incomes $z_s$ and $z_t$. More precisely, the terms $\alpha_1(\bar{z}_s, \bar{z}_t)$ and $\alpha_2(\bar{z}_s, \bar{z}_t)$ appear in formula (49) for the following reason, similar to the intuitions we gave regarding the role of the hazard rates in Sections 6.1 and 6.2. The gain from jointly taxing $z_s$ and $z_t$ is given by the ratio between the measure of individuals distorted by the increase in the marginal tax rate (the dark shaded bands in Figure 3) and the measure of individuals who pay the additional lump-sum tax (the light shaded region in Figure 3) in the joint perturbation; these are the components of the multivariate hazard vector, $[\lambda_{z_s, z_t}(\bar{z}_s, \bar{z}_t)]_1$ and $[\lambda_{z_s, z_t}(\bar{z}_s, \bar{z}_t)]_2$. Similarly, the gains from the corresponding separable perturbations are given by the ratios between the number of individuals distorted by the increase in the marginal tax rates (e.g., the dark shaded band in Figure 2) and the measure of individuals who pay the larger lump-sum tax in the separable perturbations (e.g., the light shaded region in Figure 2); these are the univariate hazard rates, $\lambda_{z_s}(\bar{z}_s)$ and $\lambda_{z_t}(\bar{z}_t)$. Therefore, the revenue raised by joint perturbations, above what can be achieved by separable perturbations alone, depends on the relative magnitude of the components of the multivariate hazard rate and the corresponding univariate hazard rates; this gives $\alpha_1(\bar{z}_s, \bar{z}_t)$ and $\alpha_2(\bar{z}_s, \bar{z}_t)$.

6.3.2 Estimation of the Joint Distribution of Incomes Using Copulas

In this subsection, we develop a method to quantify the revenue effects of joint perturbations by using copulas. Importantly, this allows us to express these revenue gains as a function of a few parameters of the marginal and joint distributions of incomes, without necessarily having full knowledge of these joint distributions.

We first need to estimate the univariate distributions of labor or capital income for individuals in different periods $t$. These distributions determine the univariate hazard rates $\lambda_{z_t}(z_t)$ and $\lambda_{k_t}(k_t)$ defined in (37) and (44). An important feature of the labor income distribution as well as the wealth distribution is that their right tails are Pareto distributed (see, e.g., Nirei and Souma 2004). This implies that $z_t\lambda_{z_t}(z_t)$ and $k_t\lambda_{k_t}(k_t)$ converge to constants, which are equal to the respective Pareto coefficients of these distributions.\footnote{The Pareto coefficient is typically between 1.5 and 2.5 in the US for the labor income distribution.} Intuitively, a Pareto coefficient equal to $a$ implies that the mean of incomes $z$ above a given (large) threshold $\bar{z}$ is equal to $a/(a-1)$ times $\bar{z}$. Thus, the larger the Pareto coefficient $a$, the thinner the tail of the income distribution, and the less unequal the income distribution. We assume that the tails of the income distributions in each period (i.e., the distributions of young and older individuals’ labor incomes) are also Pareto distributed. By definition we then have, for all $t \in \{1, \ldots, T\}$,

$$
P(z_t \geq \bar{z}_t) = \left(\frac{\bar{z}_t}{c_t}\right)^{-a_t}, \quad \forall \bar{z}_t \geq c_t,
$$

where $c_t$ is a constant and $a_t$ is the Pareto parameter determining the thinness of the tail of the distribution in period $t$.

In order to quantify the revenue effects of joint perturbations, given by (49), we need to obtain the
of incomes subsection the case distributions based on the knowledge of the marginal distributions. For concreteness, we consider in this joint distribution. We thus use an approach based on copulas (see, e.g., Nelsen 1999) to estimate the joint labor and capital incomes in period \( \bar{H}_{z_s} \). We show in the Appendix that if \( \bar{H}_{z_s} \) is a joint survival distribution function with survival marginal distributions \( H_{z_s} \equiv 1 - H_{z_t} \) and \( H_{z_t} \equiv 1 - H_{z_s} \), then there exists a survival copula \( \hat{C} (\cdot, \cdot) \) such that for all \( (z_s, z_t) \in \mathbb{R}_+^2 \),

\[
\bar{H}_{z_s, z_t} (z_s, z_t) = \hat{C} (\bar{H}_{z_s} (z_s), \bar{H}_{z_t} (z_t)),
\]

and \( \hat{C} \) is unique if \( \bar{H}_{z_s} \) and \( \bar{H}_{z_t} \) are continuous. Conversely, if \( \hat{C} \) is a survival copula and \( \bar{H}_{z_s} \) and \( \bar{H}_{z_t} \) are distribution functions, then the function \( \hat{H}_{z_s, z_t} \) defined by formula (52) is a joint distribution function with marginals \( \bar{H}_{z_s} \) and \( \bar{H}_{z_t} \). Therefore, assuming that we know the relevant copula, we only need to estimate the marginal distributions in order to obtain the joint distribution function. Several papers (e.g., Bonhomme and Robin 2003 and Dearden, Fitzsimons, Goodman and Kaplan 2006) estimated the copulas that best fit the joint distribution of incomes over the lifetime. We can then compute the parameter \( g_{z_1, z_2} (\bar{z}_1, \bar{z}_2) \) defined in (50), and hence obtain the revenue gains from jointly perturbing the tax function.

We start with the tails of the joint distribution of labor incomes \( (z_s, z_t) \). The tails of the marginal distributions of incomes \( z_s \) and \( z_t \) are Pareto distributed. We bind these two marginal distributions together into a joint distribution using the generalized Clayton survival copula, defined as

\[
\hat{C} (u, v) = \left\{ \left[ \left( u^{-1/\rho} - 1 \right)^d + \left( v^{-1/\rho} - 1 \right)^d \right]^{1/d} + 1 \right\}^{-\rho}, \quad \text{for } \rho \in (0, \infty) \text{ and } d \geq 1,
\]

\[
\hat{C} (u, v) = uv, \quad \text{for } \rho = \infty.
\]

This copula leads to a generalization of the bivariate Pareto distribution for labor incomes in periods \( s \) and \( t \).

The parameters \( \rho \) and \( d \) govern the correlation between the two marginal distributions. When \( \rho \to \infty \), the generalized Clayton copula converges to the product copula \( \hat{C} (u, v) = uv \), so that the two marginal distributions are independent. The joint distribution is then the product of the two marginal distributions, i.e. \( h_{z_s, z_t} (z_s, z_t) = h_{z_s} (z_s) h_{z_t} (z_t) \). The parameter \( d \) pins down the coefficient of upper tail dependence between the two marginal distributions, defined as

\[
\lambda_u = \lim_{q \to 1} P \left( z_t > H_{z_t}^{-1} (q) \mid z_s > H_{z_s}^{-1} (q) \right)
\]

We show in the Appendix that \( \lambda_u = 2 - 2^{1/d} \). Given \( d \), the parameter \( \rho \) governs the rank correlation
between the two marginal distributions, as measured for instance by Kendall’s tau coefficient.\textsuperscript{30} The larger the $\rho$, the more correlated are labor incomes in periods $s$ and $t$. We show in the Appendix that Kendall’s tau is equal to $\rho_{s,t} = 1 - 2/\left(2d + \frac{d}{\rho}\right)$. As $\rho \to 0$ (Fréchet-Hoeffding upper bound), incomes in both periods are then perfectly correlated, and richer individuals in period $s$ are also richer in period $t$.\textsuperscript{31} We can therefore analyze the gains of history-dependence without having to estimate the whole joint distribution of labor incomes. We simply need to estimate a few important parameters which will determine the revenue gains of the joint perturbations: thickness of the Pareto tails and measures of correlation between labor incomes at different ages.

Applying Sklar’s theorem (52) using the generalized Clayton copula (53) and the fact that the marginal distributions of incomes are Pareto distributed, we obtain the following Proposition:

**Proposition 10.** Assume that the marginal distributions of incomes in periods $s$ and $t$ are Pareto distributed with coefficients $a_s$ and $a_t$, respectively. Assume that the joint distribution of period-$s$ and period-$t$ labor incomes is related to its marginal distributions by the generalized Clayton copula (53) with parameters $\rho \geq 0$ and $d \geq 1$. Then, the parameter $g_{s,t}(\tilde{z}_s, \tilde{z}_t)$ defined in (49), which governs the revenue gains of joint perturbations, is given by

$$g_{s,t}(\tilde{z}_s, \tilde{z}_t) = 1 - \frac{1}{d} \left[ A_s(\tilde{z}_s)^d + A_t(\tilde{z}_t)^d \right]^{1/d} \left[ A_s(\tilde{z}_s)^{d-1} + A_t(\tilde{z}_t)^{d-1} \right],$$

where $A_j(\tilde{z}_j) \equiv \left( \frac{\tilde{z}_j}{\tilde{z}_j} \right)^{a_j/\rho} - 1$ for $j \in \{s, t\}$. In particular, for all $\tilde{z}_s, \tilde{z}_t$, we have

$$-1 < g_{s,t}(\tilde{z}_s, \tilde{z}_t) < 0.$$

Moreover, $g_{s,t}(\tilde{z}_s, \tilde{z}_t)$ converges to $(-1)$ for all $(\tilde{z}_s, \tilde{z}_t)$ as $\rho \to \infty$, and $g_{s,t}(\tilde{z}_s, \tilde{z}_t)$ to $0$ for all $(\tilde{z}_s, \tilde{z}_t)$ as $\rho \to 0$.\hfill $\square$

Since $g_{s,t}(\tilde{z}_1, \tilde{z}_2)$ is always negative, there is always a gain (over the best possible combination of separable perturbations) from jointly decreasing the marginal income tax rates at point $(\tilde{z}_s, \tilde{z}_t)$ and decreasing in a lump-sum way the tax liability for individuals above $\tilde{z}_s$ and above $\tilde{z}_t$. Moreover, as

\textsuperscript{30}Kendall’s tau is defined as follows. Consider two random variables $\bar{z}_s, \bar{z}_t$, independent of $z_s, z_t$, but with the same joint distribution. Then $\rho_{s,t}(z_s, z_t) \equiv \mathbb{E}[\text{sign}((z_s - \bar{z}_s) \cdot (z_t - \bar{z}_t))$. There are other measures of correlation and tail-dependence that are commonly used, and that could pin down $\rho$ and $d$ independently; e.g., the coefficient of lower tail dependence, defined as $\lambda_l = \lim_{q \to 0} F \left( z_t \leq H_{x_t}^{-1}(q) \mid z_s \leq H_{x_s}^{-1}(q) \right)$, and Spearman’s rank correlation coefficient, defined as the correlation between the two uniform random variables $H_{x_s}(z_s)$ and $H_{x_t}(z_t)$, i.e. $\rho_S(z_s, z_t) \equiv \text{Corr}(H_{x_s}(z_s), H_{x_t}(z_t))$.

\textsuperscript{31}Suppose that the Pareto coefficients are equal in periods $s$ and $t$, i.e. the thinness of the tail of the income distribution is the same for young and older individuals ($a_1 = a_2$), that the upper tails of the marginal distributions are asymptotically independent, i.e. $d = 1$, and that the coefficient $\rho$ is equal to the Pareto coefficients, $a_1 = a_2 = \rho$. Then the joint distribution of $(z_s, z_t)$ is the bivariate Pareto distribution. We gain in generality by letting the values of $a_1, a_2$, and $\rho$ differ, and letting $d \neq 1$. This allows us to consider different levels of income inequality across ages ($a_1 \neq a_2$), as well as different levels of correlation between individuals’ labor income over their lifetime ($\rho \in (0, \infty)$, $d \geq 1$).
incomes in periods $s$ and $t$ become less correlated, i.e. $\rho \to \infty$, $g_{z_s,z_t}(\bar{z}_s,\bar{z}_t)$ converges to $(-1)$ for all $(\bar{z}_s,\bar{z}_t)$, and the gains from introducing history-dependence are the largest. As incomes become perfectly correlated, i.e. $\rho \to 0$, $g_{z_s,z_t}(\bar{z}_s,\bar{z}_t)$ converges to 0 for all $(\bar{z}_s,\bar{z}_t)$, and there are no gains from jointly perturbing the tax function, above what can already be achieved with separable and age-dependent taxes. We will run numerical exercises on a calibrated version of the model in Section 6.4.

For the bulk of the income distribution (away from the tails), both Bonhomme and Robin (2003) and Dearden, Fitzsimons, Goodman and Kaplan (2006) find that the Plackett copula best fits the data on lifetime labor incomes. The Plackett copula is defined as a function of the parameter $\rho \in [0,\infty)$ by

$$\hat{C}(u,v) = u + v - 1 + \frac{1}{2(\rho-1)} \left\{ \left[1 + (\rho-1)(2-u-v)\right] \right. - \sqrt{\left[1 + (\rho-1)(2-u-v)\right]^2 - 4(1-u)(1-v)(\rho(\rho-1))}, \quad \text{if } \rho \in [0,\infty) \setminus \{1\}$$

$$\hat{C}(u,v) = uv, \quad \text{if } \rho = 1.$$  (54)

The parameter $\rho$ measures again the dependence between the two marginal distributions. When $\rho = 1$, incomes in period one and two are independent (product copula). When $\rho \to \infty$, the copula reaches the Fréchet-Hoeffding upper bound, and incomes become perfectly correlated. Negative dependence between the two marginal distributions is obtained when $\rho < 1$. Spearman’s rho coefficient of rank correlation, defined as the correlation between $H_{z_1}(z_1)$ and $H_{z_2}(z_2)$, is given by $\rho_S = \left[2(\rho-1) + (\rho-1)^2 - 2\rho \ln \rho \right] / (\rho-1)^2$. Using this copula, we can again compute numerically the variable $g_{z_s,z_t}(\bar{z}_s,\bar{z}_t)$ as a function of labor incomes $\bar{z}_s$ and $\bar{z}_t$, given a value of the parameter $\rho$.

### 6.4 Numerical Exercise

In this Section we calibrate the model of Sections 6.1 to 6.3 and obtain quantitative magnitudes for the revenue gains of various elements of tax reforms.

We showed that the revenue gains of tax reforms are determined by three key objects which are all empirically observable. First, the revenue gains depend on the characteristics of the initial tax system that we perturb; in particular, on the gradient and the Hessian of the tax functions. The larger the initial marginal tax rate, the larger the adverse effect on government revenue of a decrease in labor supply due to a distortion. We use the baseline tax system (35) as our initial tax system, calibrated to the US tax code.

The second set of key parameters that we need to calibrate is the set of elasticities (10) and income effect parameters (9). Some of these elasticities have been estimated in the data. In particular, there is a large empirical literature that provides estimates of the labor supply elasticities, and we use these values for our calibration (see, e.g., Saez, Slemrod and Giertz 2012). Other elasticities such as the elasticities and income effect parameters of incomes and savings with respect to past and future tax rates have not been estimated, mainly because most of the empirical literature has focused on the static setting. We view as one of the contributions of the paper that we establish the importance of these elasticities for the analysis of tax reforms. In order to determine these elasticities, we use the explicit closed-form
expressions derived in the general model of Section 2.

The third set of key parameters that we need to calibrate is the set of multivariate hazard rates of the (joint) labor and capital income distributions. As described in Section 6.3, we use an approach based on copulas that allows to derive the joint income distributions based only on the knowledge of the marginal distributions. Determining the multivariate hazard rates and the joint distributions of labor and capital incomes as central elements of the revenue gains of the tax reforms, as well as using a copula methodology, are new to the literature on taxation.

6.4.1 Calibration of the Baseline Tax System

The initial tax system is the baseline system described in (35). We calibrate it to match some prominent features of the US tax code. We take a tax rate on interest income of 30%, and an interest rate of 3%. The labor income tax schedule, \( T_z (\cdot) \), consists of several brackets which we take from the US tax code. We need the values of these marginal tax rates only for the welfare gains in the static framework (36). The insights we obtained in the dynamic model depend only on the capital income tax rate and on the shapes of the labor and capital income distributions.

6.4.2 Calibration of the Elasticities

We now calibrate the elasticities and income effect parameters, (9) and (10). First, the assumptions that the utility function (34) has no income effects, and that the initial tax system (35) is separable, imply that several of the elasticities and income effect parameters are equal to zero. Since labor income \( z_s \) depends only on the marginal tax rate \( \tau_{s,z_s} \), the compensated elasticities \( \zeta^c_{s,z_s,t} \) for \( t \neq s \) and \( \zeta^c_{s,z_s,t,k_s} \), as well as the income effect parameters \( \eta_{z_s,R_t} \), are all equal to zero. Moreover, we show that the compensated elasticities \( \zeta^c_{k_s,t,z_s} \) are also equal to zero. Consistent with our assumption that the utility function has no income effects, several empirical studies found that the income effect parameters \( \eta_{z_s,R_t} \) are close to zero (see e.g. Saez, Slemrod and Giertz 2012 for a survey of the empirical literature).

To calibrate the other elasticities and income effect parameters, we assume that the utility function in (34) has the specific functional form

\[
    u (c - v (l)) = \ln \left( c - \frac{I^{1+1/\varepsilon}}{1 + 1/\varepsilon} \right).
\]

There are \( T = 40 \) periods (years), corresponding to individuals between 20 and 60 years old. This functional form for the utility implies constant labor supply elasticities, i.e. \( \zeta^c_{z_t,1-\tau_{t,z_t}} = \varepsilon \) for all \( t \in \{1, \ldots, T\} \). We choose a compensated income elasticity \( \varepsilon = 0.5 \), consistent with many empirical studies for the US, e.g. Gruber and Saez (2002). The income effect parameters of savings \( \eta_{k_s,R_t} \) implied by this specific functional form are constant, i.e. independent of an individual’s choice of labor and capital incomes; they depend only on the parameters \( \beta, \tau_k, T, s, t \). The compensated savings elasticities \( \zeta^c_{k_s,t,z_t,k_t} \) also have simple closed-form expressions, which we derive formally in the Appendix. We use these explicit expressions in our numerical simulations.
6.4.3 Empirical Joint Distribution of Incomes

We finally calibrate the hazard rates that enter formulas (36), (42) and (49). The univariate hazard rates, which allow to quantify the revenue gains from separable perturbations, can be estimated using the empirical income distribution in the US, fitting Pareto distributions at the tails. We compute the multivariate hazard rates using an approach based on copulas, as described in Section 6.3.

For the tails of the joint distribution of labor incomes in periods $s$ and $t > s$, we choose the following coefficients. The Pareto coefficients we choose are $a_s = 2.2$ in period $s$ and $a_t = 1.8$ in period $t$. Thus, the distribution of incomes is more unequal among older individuals. The values we choose for the parameters $\rho, d$ of the generalized Clayton copula (53) are $\rho = 1.5$ and $d = 3$; these imply a coefficient of upper tail dependence equal to $\lambda_u = 0.75$, a coefficient of lower tail dependence equal to $\lambda_l = 0.7$, and a coefficient of rank correlation between the two marginal distributions (Kendall’s tau) equal to $\rho_\tau = 0.75$.

For the bulk of the distribution (away from the tails), we use the Plackett copula (54), which is the copula that empirically best fits the data. We calibrate the parameter $\rho$ of the Plackett copula (54) by matching the value of Spearman’s rho correlation coefficient observed in the data with its closed-form expression, derived in the previous section, as a function of $\rho$.

6.4.4 Quantitative Results

We now quantify the revenue gains of the various dynamic tax reforms we studied in Sections 6.1 to 6.3. We start by quantifying the gains (36) obtained in the static framework. These give a measure of the distance between the current US tax schedule and the optimal static tax schedule of Saez (2001). We then compare these values to the revenue gains predicted by formula (38) in the (separable) dynamic setting.

We show that the additional dynamic effects in (38) due to the adjustments of savings are quantitatively significant, and we show the optimal separable age-dependent taxes implied by formula (41). Finally, we quantify the revenue gains, given by (49), of introducing history-dependence or joint taxation of labor and capital incomes in the tax system.

We consider the static framework first. Consider the revenue gain implied by formula (36) of a separable perturbation of the labor income tax rate in period $t$ at the point $\bar{z}_t$ in the Pareto tail of the distribution. With a marginal tax rate $T'(\bar{z}_t) = 40\%$, compensated elasticity $\varepsilon = 1/2$, and Pareto parameter $a_t = 2$ (implied by a hazard rate $\bar{z}_t h_{z_t}(\bar{z}_t) / (1 - H_{z_t}(\bar{z}_t)) = 2$), (36) yields $\gamma^S_{t,\bar{z}_t}(\bar{z}_t) = 0.33 \times \delta^{t-1} dR_t$. This implies that increasing the labor income tax by one dollar for all the individuals in the top of the labor income distribution (above the point $\bar{z}_t = $200K per year, which corresponds to the beginning of the Pareto tail) increases government revenue in period $t$ by $\varepsilon 33$ per individual above $\bar{z}_t$. The tax revenue that the government effectively raises is strictly lower than $1$ per individual, even though their tax liability has been increased by $1$, because this revenue gain calculation takes into account the behavioral response of individuals with income $z_t = \bar{z}_t$, who work less as a result of the larger marginal tax rate that they face. This shows that there are large revenue gains from increasing the labor income tax at the top of the distribution. In other words, the current US top tax rate is below the revenue maximizing tax rate, or the Rawlsian optimum. More generally, equating $\gamma^S_{t,\bar{z}_t}(\bar{z}_t)$ to zero at each point $\bar{z}_t$ gives a formula for the optimal tax rates on labor income. Diamond (1998) and Saez (2001) show that
the optimal tax schedule is U-shaped, reflecting the shape of the hazard rates of the income distribution (see Figure 5).

We saw, however, that in the dynamic framework the formula giving the revenue gains of separable perturbations is quite different from the formula obtained in the static setting. To illustrate quantitatively the importance of the additional savings term in formula (38), we show in Figure 5 the optimal labor income tax schedule implied by the static formula (36), and the optimal (age-dependent) tax system implied by the dynamic formula (41). In this Figure we assume that the characteristics of the populations of different ages are identical (same elasticities and income distributions). This implies that the optimal tax system in the static setting is age-independent. We thus focus purely on the additional effect of savings implied by the dynamic framework. Figure 5 shows that the revenue gains obtained by perturbing the optimal static tax schedule toward age dependence are significant. The optimal top tax rate on labor income (keeping the separability over time and with the capital tax schedule) should be between 45% for the youngest individuals (20 years old) and 55% for the oldest (60 years old), while the static formula implies an age-independent top tax rate equal to 50%. These results are driven by the fact that the later the additional tax is levied, the more capital the individual accumulates, since he saves more in all the periods before the perturbation; this effect increases government revenue above what the static model predicts. The reverse holds for the early periods, so that the optimal tax rates are then lower than in the static model.
We finally compute the revenue gains of joint perturbations of the tax system, given by formula (49). The revenue gains of introducing history-dependence are given by the function $g_{z_s, z_t}(z_s, z_t)$. We plot this function for the tails and the bulk of the income distribution in Figures 6 and 7.

7 The Stochastic Case

In this section we derive the general formulas for the revenue and welfare effects of tax reforms in the stochastic model. We only give an outline of the derivation here, and the details are collected in the Appendix. For the clarity of the exposition, we consider the case where the horizon is $T = 2$ periods, but our results generalize to the case $T \leq \infty$. We also assume that the shocks $\theta_1, \theta_2$ are productivity shocks, as in the model of Section 6.

In period one, an individual knows his first-period type, or productivity, $\theta_1 \in [0, \infty)$, and his initial capital stock $k_0 \in \mathbb{R}$. He then chooses his first-period consumption $c_1 \geq 0$, labor income $z_1 \geq 0$, and savings $k_1 \in \mathbb{R}$ to carry over to period two, subject to a budget constraint. In period two, he draws his second-period type $\theta_2 \in [0, \infty)$. For all $\theta_1 \in \mathbb{R}_+$, the second-period type $\theta_2$ is drawn from a distribution $F_{2|1}(\theta_2 | \theta_1)$ whose density $f_{2|1}(\theta_2 | \theta_1)$ is strictly positive on the interval $[\bar{\theta}_2, \bar{\theta}_2]$. For simplicity we assume that $[\bar{\theta}_2, \bar{\theta}_2] = \mathbb{R}_+$. (Our results generalize to intervals $[\bar{\theta}_2, \bar{\theta}_2] \subseteq \mathbb{R}_+$.). The individual then chooses his second-period consumption $c_2 \geq 0$ and labor income $z_2 \geq 0$, subject to a budget constraint. In each
period $t$, the individual receives flow utility from consumption and labor given by $u(c_t, l_t) = u(c_t, z_t/\theta_t)$. Given his initial draw $(k_0, \theta_1)$, he chooses $c_1(k_0, \theta_1)$, $k_1(k_0, \theta_1)$, and $\{c_2(k_0, \theta_1, \theta_2) : \theta_2 \in \mathbb{R}_+\}$ in order to maximize the expected discounted value of his utility. The components of the choice vector $X$ of an individual with initial capital and productivity $(k_0, \theta_1)$ are the choices of first-period labor income $z_1(k_0, \theta_1)$, savings $k_1(k_0, \theta_1)$, and all the values $\{z_2(k_0, \theta_1, \theta_2) : \theta_2 \in \mathbb{R}_+\}$ corresponding to the possible draws of $\theta_2$ in period two.

In each period $t = 1, 2$, the government levies a tax $T_t$. The first-period tax function $T_1$ is a function of the individual’s first-period labor income $z_1$ and capital income $r_2k_1$ only. The second-period tax function $T_2$ is a function of the individual’s entire history of labor incomes $\{z_1, z_2\}$ and capital income $r_2k_2$. The assumptions about the tax functions are identical to those we made in the deterministic model. The planner chooses the tax system $\{T_1(\cdot, \cdot), T_2(\cdot, \cdot, \cdot)\}$, subject to a budget constraint. Social welfare is then a weighted sum of individuals’ indirect utilities $\mathcal{W}(k_0, \theta_1)$.

It is important to note that there are many more marginal tax rates and virtual incomes that are relevant for the individual than there were in the deterministic model. Since $\theta_2$, and hence $z_2$ and $T_2(\cdot, \cdot, \cdot)$, are unknown when $z_1$ and $k_1$ are chosen, the two decision variables $(z_1, k_1)$ depend on the set of all possible marginal tax rates and virtual incomes that the individual may end up facing in period two, depending on his type $\theta_2$. Thus, $z_1$ and $k_1$ depend on the whole set $\{(\tau_2(z_1, x^2_2, k_1), R_2(z_1, x^2_2, k_1)) : x^2_2 \in \mathbb{R}_+\}$, parametrized by the possible values $x^2_2$ of second-period incomes that the individual may face in period two. Moreover, even though $z_2$ is chosen after a value of $\theta_2$ has been drawn (say $\theta^*_2$), $z_2(\theta^*_2)$ does not depend only on the marginal tax rate and virtual income that he ends up actually facing (i.e., $\tau_2(z_1, z_2(\theta^*_2), k_1)$), unless the utility function has no income effects. This is because $z_1$ and $k_1$, which have been chosen before the draw (taking into account the probabilities of all possible draws of $\theta_2$), are not in general the optimal values given this particular draw $\theta^*_2$, and this in turn affects the choice of $z_2(\theta^*_2)$. We thus obtain that for all $\theta^*_2 \in \mathbb{R}_+$, $z_2(\theta^*_2)$ depends on the entire set of marginal tax rates and virtual incomes $\{(\tau_2(z_1, x^2_2, k_1), R_2(z_1, x^2_2, k_1)) : x^2_2 \in \mathbb{R}_+\}$. In particular, when we perturb the tax function in the second period, $T_2(\cdot, \cdot, \cdot)$, at a given point $x^2 = (z_1, x^2_2, k_1)$, all the choice variables, $(z_1, \{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+, k_1\})$, adjust, even if the individual turns out not to be affected at all by the perturbation (i.e., even if $z_2(\theta^*_2) \neq x^2_2$). This is the main conceptual difficulty that needs to be addressed in the stochastic model.

We first define the elasticities of labor incomes $z_1$, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and savings $k_2$ with respect to the marginal tax rates on $z_1$ and $k_1$ that the individual actually faces in period one: $\gamma_{1,z_1}, \gamma_{1,k_1}$. We then define the elasticities of $z_1$, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and $k_1$ with respect to all the marginal tax rates $\{\tau_{2,z_1}(x^2), \tau_{2,z_2}(x^2), \tau_{2,k_1}(x^2) : x^2 = (z_1, x^2_2, k_1) \in \mathbb{R}_+^2 \times \mathbb{R}\}$ that the individual can possibly face in period two, depending on the possible values $x^2_2$ of second-period incomes that the individual may end up choosing in period two. Similarly we first define the income effect parameters of $z_1$, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and $k_1$ with respect to the individual’s virtual income in period one, $R_1$. We then define the income effect parameters of $z_1$, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and $k_1$ with respect to all the virtual incomes that the individual

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32 In the stochastic model, the government cannot tax second-period labor income $z_2$ in period one, as $z_2$ depends on the value of $\theta_2$ that the individual will draw in period two, and hence is not known with certainty in period one. On the other hand, if the interest rate $r_2$ is deterministic, and hence capital income $r_2k_1$ is already known with certainty in period one.
can possibly face in period two, \( \{ R_2 (x^2) : x^2 = (z_1, x^2_2, k_1) \in \mathbb{R}_+^2 \times \mathbb{R} \} \). We thus need to consider many more elasticities and income effect parameters than in the deterministic setting. These elasticities (e.g., of labor income \( z_2 \) with respect to the marginal tax rate at level \( z_2' \neq z_2 \)) are new to the literature on taxation, and have not been estimated empirically. Whether they are quantitatively significant or not is an empirical question, but we show that they matter theoretically. We derive explicit, closed-form expressions for all these elasticities, as we did in the deterministic setting.

We then go on to derive the behavioral responses to the multilinear perturbations we have defined in Section 3. The results are proved in the same way as in the deterministic setting, but the added degree of complexity we just described makes the derivations more involved both theoretically and conceptually. The formulas we obtain are accordingly more complex. Remarkably, however, we show that we can define the elasticity matrices (as well as the gradients and Hessians of the tax functions) in a way that allows to write the formula in a similar compact and empirically estimable form as (24) in the deterministic model. Moreover, we provide a heuristic derivation of this formula that, although conceptually more difficult than in the deterministic case, follows the same steps and the intuition of the static case. The general formula and heuristic derivation are given in the following Proposition, and the rigorous proof is in the Appendix:

**Proposition 11.** Consider an individual with type \((k_0, \theta_1)\) in the first period, which leads him to choose a vector \(X\) of incomes and savings. Let \(X_1 = (z_1, k_1)\) be his choice of first-period income and savings. Assume that \(X\) belongs to a region of the space where the gradient of the tax function \(T_1(\cdot, \cdot)\) is perturbed by the amount \(d\tau_1 = (d\tau_{1,z_1}, d\tau_{1,k_1})'\), the gradient of the tax function \(T_2(\cdot, \cdot, \cdot)\) is perturbed by the amount \(d\tau_2 = (d\tau_{2,z_1}, d\tau_{2,z_2}, d\tau_{2,k_1})'\) at the point \(x^2 = (z_1, x^2_2, k_1)\), the period-one virtual income \(R_1\) is perturbed by the amount \(dR_1\), and the period-two virtual income is perturbed by the amount \(dR_2\) at the point \(x^2 = (z_1, x^2_2, k_1)\). Let \(\theta^*_2\) be the second period type such that \(z_2(k_0, \theta_1, \theta^*_2) = x^2_2\). Define the vectors \(d\tau_1, d\tau_2(x^2)\) as the change in \(DT_1(z_1, k_1)\) and \(DT_2(x^2)\) due to the perturbation, i.e., \(d\tau_1 = \left(\begin{array}{cccc} d\tau_{1,z_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & d\tau_{1,k_1} \end{array}\right)\)' and \(d\tau_2(x^2) = \left(\begin{array}{cccc} d\tau_{2,z_1} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & d\tau_{2,k_1} \end{array}\right)\)', where the only interior non-zero component of \(d\tau_2(x^2)\) is the row indexed by \(\theta^*_2\). Then the individual changes his vector \(X\) in response to the perturbation by an amount

\[
dX = \left[ I - \zeta_t^1 (X) (D^2T_1) (X_1) - \int_0^\infty \zeta_t^2 (x^2_1) (X) (D^2T_2) \left((x^2)\right) d(x^2_2) \right]^{-1}
\times \left[ \zeta_t^1 (X) d\tau_1 + \zeta_t^2 (X) d\tau_2 (x^2) + \eta_{X,R_1} (X) dR_1 + \eta_{X,R_2(x^2)} (X) dR_2 (x^2) \right]
\]

*Proof.* See the Appendix.

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We then derive the general welfare gains of tax reforms as we did in Section 5 for the deterministic case. These welfare gains are given by the sum of (i) a mechanical gain in tax revenue (net of the welfare loss) due to the increase in the tax liability faced by individuals above the point of the perturbation (e.g., individuals who earn \(z_1 \geq \bar{z}_1\) and \(z_2 \geq \bar{z}_2\) if the second-period tax function is perturbed at point \((\bar{z}_1, \bar{z}_2)\), similar to what shown in Figure 3); (ii) the elasticity effects, due to the increases in the marginal tax
rates in the corresponding regions (similar to the dark shaded bands in Figure 3), and the behavioral responses that these changes induce on the choice vectors $X$ of individuals; and (iii) an income effect due to the lump-sum increase in the tax liability in the corresponding region (the light shaded band in Figure 3), and the behavioral responses that this change induces on the choice vectors $X$ of individuals.

8 Conclusion

We derive formulas giving the revenue and welfare effects of general tax reforms of potentially suboptimal tax systems in a dynamic economy. These formulas are written only as a function of empirically observable, and easily interpretable, sufficient statistics. We then sequentially decompose the gains coming from each additional element of tax reform as the tax function becomes more sophisticated. We highlight the importance of the labor and capital income elasticities, and the multivariate hazard rates of the income distributions. Finally, the theoretical analysis allows for an exceptionally simple, almost back of the envelope, quantification of the gains of the tax reforms.

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