Differences of opinion and foreign exchange markets\textsuperscript{1}

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Abstract

This article develops a two-country, two-good international financial market model in which domestic and foreign residents differ in their beliefs about the future output growth rates in each country. Endogenous differences of opinion generate an equilibrium international sentiment risk world factor, which is common to the state price densities in units of goods of the two countries, while output risk is the standard country-specific risk factor. We show that our model can help explain a number of foreign exchange related anomalies: (i) rejection of the uncovered interest rate parity hypothesis (the so-called forward premium puzzle); (ii) the low correlation between exchange rates and relative cross-country consumption (the so-called exchange rate disconnect puzzle pointed out by Backus and Smith (1993)); and (iii) apparent high risk sharing implied by state price densities coexisting with the apparent poor risk sharing implied by consumption (Brandt et.al. (2006)).
1 Introduction

Foreign exchange markets are characterized by several empirical regularities which are hard to reconcile with standard equilibrium models. Perhaps the most studied is the rejection of the uncovered interest rate parity (hereafter UIP) hypothesis, or so-called “forward premium puzzle”. According to the UIP hypothesis, the slope coefficient from the regression of exchange rate changes on the interest rate differential should be one. Empirical tests typically find that the slope coefficient is significantly different from one, and often negative.\(^1\) Negative slope coefficients imply that currencies with high interest rates tend to appreciate. This means that an investor who borrows in the country of lower interest rates and invests the proceeds in the country with higher interest rates could potentially make profits: not only does he obtain a higher interest rate abroad, but he is also paid in units of the foreign currency which becomes more valuable. Rejection of UIP implies that the return on this long-short strategy, also called carry trade, is positive and predictable.

Another empirical regularity in foreign exchange markets, pointed out by Backus and Smith (1993), is the low correlation between exchange rates changes and the difference between the two countries consumption growth.\(^2\) Finally, a related empirical regularity pointed out by Brandt et al. (2006) is that the actual real exchange rate is much smoother than the theoretical one implied by standard asset pricing models.

In this paper, we propose a model based upon international differences in perceptions of the data generating process of country outputs, which can be used to address the foreign exchange regularities mentioned above. In our model, two representative investors live in two countries, and each is endowed with an output-paying process in units of a country-specific good. These investors do not observe the time-varying expected value of the growth rate of either output, but they try to infer it from the observations of outputs and a public signal. We assume that the country investors agree to disagree about the true dynamics of the public signal. In equilibrium, this disagreement introduces a new risk factor which we call international sentiment.

Assuming that international investors consume both goods but with a stronger preference toward the Home good, we obtain that in equilibrium the country-specific state price densities contain

\(^1\)For example, Froot and Thaler (1990) report that the average slope coefficient, across 75 studies, is -0.88.
\(^2\)This correlation is equal to one in standard equilibrium models with CRRA preferences.
a common international sentiment world-risk factor, as well as a standard country-specific output risk factor. In our complete markets setup, the real exchange rate in units of local goods per foreign goods is the ratio of the foreign to domestic state price densities, which in equilibrium is increasing in the local to foreign output ratio and decreasing in relative international sentiment risk. The exchange rate is increasing in the local to foreign output ratio due to supply effect: when the local output increases, all else equal, the relative value of the local currency in terms of the foreign currency decreases as it has become less scarce, and as a result the exchange rate (defined as the value of a unit of local good per unit of foreign good) increases. Relative sentiment drives exchange rates due to a demand effect: differences of opinion induce investors to engage in intertemporal risk sharing and their marginal utility increases upon realization of the states that the agent considers relatively more likely. While the marginal utility of investors in both countries are affected by relative sentiment, home bias in consumption ensures that there is an asymmetric effect which generates an impact in exchange rates.

Why is this model able to reproduce rejection of the UIP hypothesis? As explained above, the exchange rate is linked to the ratio of local to foreign output and hence it is affected by domestic and foreign output shocks. The exposure to these shocks drives the return on the carry trade strategy described above. For example, positive shocks in the local output increase the exchange rate (depreciating the local currency, which becomes less scarce due to the positive shock) and therefore raises the value in local currency of the proceeds from having invested in the foreign bond market, thereby increasing the return of the carry trade strategy performed by home investors. Hence, implementing the carry trade strategy involves output risk, which in turn generates a positive risk premium from the point of view of the local investor. This intuition was first suggested by Backus et al. (2001), showing that the expected currency risk premium (the expected return on the carry trade strategy) is proportional to the difference in the conditional variances of the domestic and foreign state price densities. Importantly, when the domestic state price density has a relatively high conditional variance, domestic investors performing the carry trade strategy expect a positive risk premium.

In our model, implementing the carry trade involves an exposure not only to output risk but also to international sentiment risk. When foreign investors are relatively pessimistic about the
local to foreign output ratio two effects take place simultaneously: (i) local investors offer them insurance against negative local shocks (or positive foreign shocks) in exchange for an additional currency risk premium from the perspective of the local investor, and (ii) due to wealth effects driven by pessimism about the local to foreign output ratio, foreign investors are more willing to save in local bonds (which pays in units of local output), leading to a reduction of the interest rate at home, relative to the one abroad. Because international sentiment risk produces this endogenous negative correlation between the expected currency risk premium and the interest rate differential, UIP is naturally rejected in our setup.

The model is also able to reproduce the exchange rate disconnect puzzle. While standard theoretical models require a perfect correlation between the exchange rate changes and relative consumption growth, Backus and Smith (1993) showed that this correlation is low in the data. In our setup this occurs because the exchange rate is driven not only by the ratio of marginal utilities of consumption, but also by relative sentiment. The stochastic wedge between the exchange rate and marginal utility of consumptions induced by relative sentiment decreases the correlation between the exchange rate changes and relative consumption growth, bringing it closer to the magnitude observed in the data.

A related empirical regularity is associated with international risk sharing. Brandt et al. (2006) point out an apparent disconnect between the relatively low empirical cross-country consumption growth correlations and the extremely high degree of international risk sharing implied by the relatively low real exchange rate volatility observed in the data. In our setup, this occurs because international sentiment simultaneously: (i) appears as a common world risk factor in the state price densities of the two countries, increasing the cross-country correlation of state price densities changes and hence decreasing the exchange rate volatility; and (ii) induces international investors to engage in intertemporal risk sharing such that they get a higher consumption share when the state that the agent considers relatively more likely occurs. But this must imply a lower consumption share for investors in the other country, which take the other side of the bet and hence obtain a lower consumption share. Therefore, irrespective of which investor gets a higher or lower consumption share, the speculation induced by international sentiment decrease the cross-country consumption growth correlation.
The fact that the international sentiment generates a common world risk factor and a country-specific risk factor in the state price densities of both countries is consistent with the structure of the state price densities assumed in recent papers by Lustig et al. (2010) and Verdelhan (2011). While they impose this structure exogenously to build intuition of their empirical findings on foreign exchange markets, in this paper we show that international sentiment can produce those world and country-specific risk factors in the country state price densities endogenously.

To show that the model can quantitatively reproduce jointly all these empirical regularities we conduct a simulation exercise. We derive closed-form expressions for the conditional interest rate differential, currency risk premium, and instantaneous conditional UIP slope coefficient when output is correlated across countries. A calibration exercise is then made to match the unconditional first and second moments of output growth rates, as well as the volatility of the differential of interest rates and the volatility of the state price densities, assuming the US is the home country and the UK is the foreign country. The model reproduces a negative and significant unconditional UIP slope coefficient in every path, and averaging across 50,000 paths the model matches closely the slope, t-statistic and $R^2$ of the UIP regression with their empirical counterparts. In addition, the model reproduces a low correlation between the exchange rate changes and relative consumption growth, consistent with the evidence reported by Backus and Smith (1993). Related to this point, the model is consistent with highly correlated state price densities and low volatility of the exchange rate (so that prices imply high risk sharing) and poorly correlated cross-country consumption growth rates (so that quantities indicate poor risk sharing), consistent with the finding of Brandt et. al. (2006).

However, the simulation also highlights some weaknesses of the model. Although we are able to match the volatility of the differential of interest rates across countries, the volatility of the interest rate in each country is too high. This is a common challenge faced by models of heterogeneous beliefs. As discussed by David (2008), the only way to obtain sufficiently low interest rate volatility in models of heterogeneous beliefs and constant relative risk aversion is by setting risk aversion to very low levels. Similarly, Xiong and Yan (2010) are able to match interest rate volatility, at the expense of not matching the volatility of equity return. In our setup with homogeneous separable constant relative risk aversion preferences, the speculative term driven by international sentiment risk appearing in the equilibrium interest rate is very similar across the two countries, so it has a
minor impact in the volatility of the interest rate differential. When computing individual interest rates, by contrast, this term adds significant volatility. Similarly, although we match the cross-country consumption correlation, the volatility of each country consumption is too high due to the highly volatile consumption shares driven by international sentiment risk. This is also a common feature of standard equilibrium models with heterogeneous beliefs and CRRA preferences.  

This paper adds to a large literature studying the UIP condition. Hodrick (1989), Backus et al. (1993), Bansal et al. (1995) and Bekaert (1996) study the ability of consumption and money-based general equilibrium models to explain the forward premium puzzle. Hollifield and Uppal (1997) explore the effect of segmentation of international commodity markets in the relationship between expected exchange rates and differential of interest rates. Verdelhan (2008) and Heyerdahl-Larsen (2011) study how habits can reproduce rejection of UIP. Colacito and Croce (2010), Bansal and Shaliastovich (2010) and Gavazzoni (2007) argue that long-run risk, together with Epstein-Zin preferences, can explain the forward premium puzzle. Backus et al. (2011) study the impact of monetary policy on rejection of UIP. Stathopoulos (2011) studies how habits can help explain the exchange rate disconnect puzzle reported by Backus and Smith (1993) as well as the apparent contradiction on international risk sharing implied by prices and quantities as shown by Brandt et. al. (2006). To our knowledge, this is the first paper that is able to address jointly (i) rejection of UIP; (ii) the low correlation between exchange rate changes and countries’ relative consumption growth reported by Backus and Smith (1993); and (iii) the fact that the actual real exchange rate is much smoother than the theoretical one implied by standard asset pricing models, as reported by Brandt et. al. (2006).

Our model is close to the one in Dumas et al. (2011), who study international equity markets and portfolios in an international economy that also exhibits differences of opinions across country investors. By contrast, we construct a two-country, two-good international economy in which the exchange rate plays an important role in determining the dynamics of international trade. Our model is also related to the one by Pavlova and Rigobon (2007), who develop a tractable two-country, two-good international economy. Our model is different in focus. They study the joint dynamics of stocks, bonds and foreign exchange markets. By contrast, our focus is on understanding

\textsuperscript{3}Recursive preferences of the Epstein-Zin type may allow to match moments of the interest rate and equity returns.
the impact of international sentiment risk in the currency risk premium and interest rates, and assessing if it could help explain the foreign exchange market empirical regularities.

In related work, Yu (2011) studies the impact of sentiment on rejection of UIP. His model is in the long-run risk class. As in Colacito and Croce (2010), Bansal and Shaliastovich (2010) and Gavazzoni (2007), it is the combination of long-run risk and Epstein-Zin preferences that drives the result. His main contribution is empirical: he shows that sentiment data possesses the same empirical characteristics as the long-run risk factor so that, combined with recursive preferences, produces rejection of UIP. Finally, his model is silent concerning international risk sharing and the disconnect between exchange rates and relative consumption.

In addition to reproducing endogenous rejection of UIP, the model developed in this paper is in line with several empirical regularities related to the impact of differences of opinion in the expected exchange rates. Frankel and Froot (1987) use foreign exchange rate forecasts and find that expectational errors play a significant role in explaining the forward premium puzzle. In addition, Bacchetta, Mertens and Van Wincoop (2006) find a strong relationship between forecast errors and currency excess returns, which implies there is a link between foreign exchange forecasts and underlying currency market behavior. More recently, Beber et.al. (2010) find that an empirical proxy for dispersion of beliefs has significant predictive power for subsequent carry trade returns beyond the forward premium. This result holds for all currencies and at almost all horizons in their sample. The current paper provides a structural model which is consistent with all of these empirical results and offers a theoretical framework that justifies them while being consistent with several empirical regularities in foreign exchange markets.

The next section presents the main setup of the international model with heterogeneity of beliefs. Section 3 characterizes the equilibrium consumption and instantaneous pricing of risk. Section 4 analyzes the impact of differences of opinion on the foreign exchange market. Section 5 contains a quantitative exercise which shows that differences of opinions can generate jointly the empirical regularities described above. Section 6 contains the conclusions.
2 An international model of heterogeneous beliefs

We consider a continuous-time pure-exchange international economy along the lines of Lucas (1982). The international economy has an infinite horizon and it includes two countries: home and foreign, indexed by \( i = \{ H, F \} \). Each country produces its own perishable good via strictly positive output processes modeled as Lucas trees. We assume that the financial markets are perfectly integrated, and there are no frictions such as transaction costs or capital flow restrictions. There are two types of investors in the international economy called Investor \( H \) and Investor \( F \), according to their country of residence. These investors are heterogeneous in endowments and beliefs, which provides incentives for trade in the financial markets.

2.1 Exogenous outputs

Both investors commonly observe the country-specific outputs, but have incomplete (yet symmetric) information on its dynamics. The country-specific output processes \( \delta_{i,t} \) follow:

\[
\frac{d\delta_{H,t}}{\delta_{H,t}} = f_{H,t} dt + \sigma_{\delta_H} dZ_{\delta_H,t} \tag{1}
\]

\[
\frac{d\delta_{F,t}}{\delta_{F,t}} = f_{F,t} dt + \sigma_{\delta_F} dZ_{\delta_F,t} \tag{2}
\]

where \( Z_{\delta,t} \) are Brownian motions under the objective probability measure, which governs empirical realizations of the processes. We allow there to be a cross-country output correlation by letting these Brownians be correlated in the sense that:\(^4\)

\[
dZ_{\delta,t} = \begin{pmatrix} dZ_{\delta_H,t} \\ dZ_{\delta_F,t} \end{pmatrix} \sim \mathcal{N} \left[ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right] dt
\]

Without loss of generality the Home output \( \delta_{H,t} \) is assumed to be the numeraire and therefore its price is normalized to one. The (relative) price of the foreign output \( \delta_{F,t} \) is denoted by \( p_t \).\(^5\)

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\(^4\)We define a set of standard uncorrelated Brownians \( d\tilde{Z}_{\delta,t} = \left( d\tilde{Z}_{\delta_H,t}, d\tilde{Z}_{\delta_F,t} \right) \) such that they are related by \( dZ_{\delta,t} = d\tilde{Z}_{\delta,t} \Omega \) where \( \Omega = \begin{pmatrix} \sqrt{1 - \rho^2} & \rho \\ 0 & 1 \end{pmatrix} \). This relation holds true under any measure so, in the rest of the paper, we use \( \sim \) to refer to standard uncorrelated output Brownians.

\(^5\)The relative price of two international goods is sometimes also called terms of trade.
The output growth rate fundamentals $f_{i,t}$ are also stochastic, with dynamics given by:

$$df_{H,t} = \zeta (\bar{f}_H - f_{H,t}) \, dt + \sigma_f dZ_{f_{H,t}}$$  \hspace{1cm} (3)$$

$$df_{F,t} = \zeta (\bar{f}_F - f_{F,t}) \, dt + \sigma_f dZ_{f_{F,t}}$$

where $\zeta > 0$ and $Z_{f_{i,t}}$ are independent Brownian motions under the objective probability measure.\(^6\)

2.2 Public information and heterogeneous beliefs

The output growth rate fundamentals $f_{i,t}$ are not observed by any investor. All investors must estimate, or filter out, the current value of $f_{i,t}$ and its future behavior. They estimate these values by observing current outputs and a public signal $s_t$. The true dynamics of $s_t$ are given by:

$$ds = dZ_{s,t}$$

where $Z_{s,t}$ is also a Brownian motion under the objective probability measure, independent of all the other Brownian motions. That is, $s_t$ is pure noise. However, we assume that investors in each country $i$ believe that $s_t$ is partly correlated with the innovations to the conditional output growth rate of their own country $f_{i,t}$. In their mind, the signal dynamics are given by:

$$ds = \phi_s dZ_{f_{i,t}} + \sqrt{1 - \phi_s^2} dZ_{s,t}$$

where $\phi_s < 1$ measures the perceived correlation between the signal and the output growth rate fundamental innovation $dZ_{f_{i,t}}$. Following Xiong and Yan (2010) we also assume that investors from Country $i$ misperceive the volatility of the unobserved own output growth rate fundamental by a factor $k_i$, and $k_i = 1$ corresponds to the case in which investors correctly perceive the volatility of the unobserved own output growth rate fundamental.\(^7\) For instance, the “model” that home investors have in mind is

$$df_{H,t} = \zeta (\bar{f}_H - f_{H,t}) \, dt + k_H \sigma_f dZ_{f_{H,t}}$$  \hspace{1cm} (4)$$

\(^6\)To keep tractability, we let all the cross-country output correlation be captured by $\rho$.

\(^7\)We show in the next subsection how the assumption of $k_i \neq 1$ allows us to isolate of the impact of beliefs’ heterogeneity. This specification follows closely Xiong and Yan (2010).
\[ df_{F,t} = \zeta (\bar{f}_F - f_{F,t}) \, dt + \sigma_f dZ_{f_{F,t}} \]

\[ ds = \phi_H dZ_{f_{H,t}} + \sqrt{1 - \phi_H^2} dZ_{s,t} \]

Symmetrically, the “model” that foreign investors have in mind is:

\[ df_{H,t} = \zeta (\bar{f}_H - f_{H,t}) \, dt + \sigma_f dZ_{f_{H,t}} \quad (5) \]

\[ df_{F,t} = \zeta (\bar{f}_F - f_{F,t}) \, dt + k_F \sigma_f dZ_{f_{F,t}} \]

\[ ds = \phi_F dZ_{f_{F,t}} + \sqrt{1 - \phi_F^2} dZ_{s,t} \]

### 2.3 The econometrician’s viewpoint

Following Xiong and Yan (2010) and Dumas et al. (2012), we will evaluate the impact of heterogeneous beliefs from the viewpoint of an “econometrician” who understands the true structure of the economy. Hence, we first derive the solution of the filtering problem of the econometrician. To that end, we define \((\hat{f}_{H,t}^E, \hat{f}_{F,t}^E)^\top\) as the countries’ output growth rate fundamentals beliefs by investor \(j = \{H, F, E\}\), where \(E\) stands for “econometrician”. From standard filtering theory, the dynamics of the econometrician beliefs are given by

\[ d\hat{f}_{H,t}^E = \zeta (\bar{f}_H - \hat{f}_{H,t}^E) \, dt + \frac{1}{1 - \rho^2} \left[ \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_H} \right) d\bar{W}_{\delta_H,t}^E + \left( \frac{\gamma_{HF}}{\sigma_F} - \rho \frac{\gamma_H}{\sigma_H} \right) d\bar{W}_{\delta_F,t}^E \right] \quad (6) \]

\[ d\hat{f}_{F,t}^E = \zeta (\bar{f}_F - \hat{f}_{F,t}^E) \, dt + \frac{1}{1 - \rho^2} \left[ \left( \frac{\gamma_{HF}}{\sigma_H} - \rho \frac{\gamma_F}{\sigma_F} \right) d\bar{W}_{\delta_H,t}^E + \left( \frac{\gamma_F}{\sigma_F} - \rho \frac{\gamma_{HF}}{\sigma_H} \right) d\bar{W}_{\delta_F,t}^E \right] \quad (7) \]

where we have introduced the subjective uncorrelated Brownian motions perceived by the econometrician \((d\bar{W}_{\delta_H,t}^E, d\bar{W}_{\delta_F,t}^E)^\top\): \(\bar{W}_{\delta_{H,t}}^E = \frac{1}{\sigma_H} \left( \frac{d\delta_{H,t}}{\delta_{H,t}} - \hat{f}_{H,t}^E dt \right), \quad \bar{W}_{\delta_{F,t}}^E = \frac{1}{\sigma_F} \left( \frac{d\delta_{F,t}}{\delta_{F,t}} - \hat{f}_{F,t}^E dt \right)\)

and \(\gamma_H, \gamma_F, \gamma_{HF}\) are the elements of the variance-covariance matrix of the posterior distribution of the econometrician about \((f_{H,t}, f_{F,t})^\top\).\(^8\) The elements of this variance-covariance matrix would

\(^8\)In particular, \(\gamma_H\) is the posterior variance of \(f_{H,t}\), \(\gamma_F\) is the posterior variance of \(f_{F,t}\) and \(\gamma_{HF}\) is the posterior covariance of \(f_{H,t}\) and \(f_{F,t}\).
normally be deterministic functions of time. But for simplicity we assume, as did Scheinkman and Xiong (2003), Dumas et al. (2009, 2011) and Xiong and Yang (2010), that there has been a sufficiently long period of learning to achieve convergence to the steady state level of the variance-covariance matrix, independently of priors. As a result $\gamma_H, \gamma_F$ and $\gamma_{HF}$ are all constants, given in the appendix.

It is worth noting that, due to cross-country output correlation $\rho$, perceived innovations to the output growth rates of both country outputs $\left( d\tilde{W}_H^E, d\tilde{W}_F^E \right)^\top$ are useful to filter out each output growth rate fundamental $\hat{f}_{i,t}^E$.

2.4 The investors’ viewpoint

We now solve the filtering problem of each country investor. To that end, we consider first the perception of home investors, spelled out in equations (4). From standard filtering theory, the dynamics of the home investor’s beliefs are given by:

\[
\begin{align*}
    \frac{d\hat{f}_{H,t}^H}{d\hat{f}_{H,t}^H} &= \zeta \left( f_H - \hat{f}_{H,t}^H \right) dt + \frac{1}{1-\rho^2} \left[ \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^H + \left( \frac{\gamma_{HF}}{\sigma_H} - \rho \frac{\gamma_H}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^F \right] + k_H \phi_H \sigma_f ds \\
    \frac{d\hat{f}_{F,t}^H}{d\hat{f}_{F,t}^H} &= \zeta \left( f_F - \hat{f}_{F,t}^H \right) dt + \frac{1}{1-\rho^2} \left[ \left( \frac{\gamma_{HF}}{\sigma_H} - \rho \frac{\gamma_F}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^H + \left( \frac{\gamma_F}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^F \right]
\end{align*}
\]

(8) (9)

where we have introduced the subjective uncorrelated Brownian motions perceived by the home investor $\frac{d\tilde{W}_{\delta_H,t}^H}{d\tilde{W}_{\delta_H,t}^H}, \frac{d\tilde{W}_{\delta_H,t}^F}{d\tilde{W}_{\delta_H,t}^F}$:

\[
\begin{align*}
    d\tilde{W}_{\delta_H,t}^H &= \frac{1}{\sigma_H} (\frac{d\delta_{H,t}}{\delta_{H,t}} - \frac{d\hat{f}_{H,t}^H}{\hat{f}_{H,t}^H} dt) \\
    d\tilde{W}_{\delta_H,t}^F &= \frac{1}{\sigma_H} (\frac{d\delta_{F,t}}{\delta_{F,t}} - \frac{d\hat{f}_{F,t}^H}{\hat{f}_{F,t}^H} dt)
\end{align*}
\]

Because home investors believe that the signal is partly correlated with the unobserved home output growth rate fundamental innovation, they use it in order to filter out $\hat{f}_{H,t}^H$.

Similarly, the perception of foreign investors is spelled out in equations (4). From standard filtering theory, the dynamics of the foreign investor’s beliefs are given by

\[
\begin{align*}
    \frac{d\hat{f}_{H,t}^F}{d\hat{f}_{H,t}^F} &= \zeta \left( f_H - \hat{f}_{H,t}^F \right) dt + \frac{1}{1-\rho^2} \left[ \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^F + \left( \frac{\gamma_{HF}}{\sigma_H} - \rho \frac{\gamma_H}{\sigma_H} \right) d\tilde{W}_{\delta_H,t}^H \right]
\end{align*}
\]

(10)
where we have introduced the subjective Brownian motions perceived by the Foreign investor

\[
d\tilde{f}_{H,t} = \zeta \left( f_F - \tilde{f}_{F,t} \right) dt + \frac{1}{1 - \rho^2} \left[ \left( \frac{\gamma_H}{\delta_H} - \rho \frac{\gamma_F}{\delta_F} \right) d\tilde{W}_{\delta_H,t} + \left( \frac{\gamma_F}{\delta_F} - \rho \frac{\gamma_H}{\delta_H} \right) d\tilde{W}_{\delta_F,t} \right] + k_F \phi_F \sigma_F ds
\] (11)

Because foreign investors believe that the signal is partly correlated with the unobserved foreign output growth rate fundamental innovation, they use it in order to filter out \( \tilde{f}_{F,t} \).

Beliefs’ heterogeneity affects the economy through several channels. First, disagreement induces speculative trading between the two country investors and trading leads to endogenous relative wealth fluctuation, which in turn affects the equilibrium asset prices. Second, the average belief of the two country investors about the total output growth rate, \( f_{H,t} + f_{F,t} \), could differ from the econometrician’s belief, and the erroneous average belief would affect equilibrium asset price dynamics. Third, the country investors’ posterior belief variance-covariance matrix may differ from that of the econometrician. On the one hand, investors in Country \( i \) feel they have extracted useful information about their own output growth rate fundamental \( f_{i,t} \) from \( s_t \), which reduces their posterior variance; on the other hand, their subjective perception about the volatility of \( f_{i,t} \) may affect the elements of their posterior variance-covariance matrix when \( k_i \) is different from 1.

While all these channels are important in their own, we are primarily interested in the first one. Hence, in order to isolate the first channel, we follow Xiong and Yan (2010) and choose the value of \( \phi_i \) and \( k_i \) that effectively shuts down the second and third channels. In particular, we impose the following conditions

\[
-\phi_H = \phi_F = \phi > 0 \quad (12)
\]

\[
k_H = k_F = \frac{1}{\sqrt{1 - \phi^2}} > 1 \quad (13)
\]

The first condition implies that investors react to \( s_t \) in opposite directions but with exactly in the same magnitude. As a result, their beliefs about their own country output growth rate fundamental diverge in response to \( s_t \). In addition, it can be shown that the beliefs structure is
such that the variance-covariance matrix of the posterior belief of each country investor is different to that of the econometrician only if \( k_i (1 - \phi^2) \neq 1 \). Therefore, the second condition implies that the variance-covariance matrix of the posterior belief of each country investor is the same as that of the econometrician.\(^9\) In summary, these two conditions imply that if the average of the two country investors beliefs about the world output growth rate at time zero coincide with the econometrician’s, then the average belief will always keep track of the econometrician’s belief in the future. These properties are stated formally in the next proposition.

**Proposition 1.** Let the differences of opinion between each investors and the econometrician be given by

\[
\hat{g}_{H,t} = \hat{f}_{H,t}^E - \hat{f}_{H,t}^F; \quad \hat{g}_{F,t} = \hat{f}_{F,t}^E - \hat{f}_{F,t}^F
\]

Given conditions (12) and (13), their dynamics are given by

\[
d\hat{g}_{H,t} = - \left[ \zeta + \left( \frac{1}{1 - \rho^2} \right)^{\frac{3}{2}} \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \right) \right] \hat{g}_{H,t} \, dt + \frac{\phi}{\sqrt{1 - \phi^2}} \, ds
\]

\[
d\hat{g}_{F,t} = - \left[ \zeta + \left( \frac{1}{1 - \rho^2} \right)^{\frac{3}{2}} \left( \frac{\gamma_F}{\sigma_F} \right) \left( \rho^2 + \sqrt{1 - \rho^2} \right) - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \left( 1 + \sqrt{1 - \rho^2} \right) \right] \hat{g}_{F,t} \, dt - \frac{\phi}{\sqrt{1 - \phi^2}} \, ds
\]

Moreover, if \( \frac{1}{2} \left( \hat{f}_{H,0}^E + \hat{f}_{F,0}^E \right) + \frac{1}{2} \left( \hat{f}_{H,0}^H + \hat{f}_{F,0}^F \right) = \hat{f}_{H,0}^E + \hat{f}_{F,0}^E \), then the average of the two country investors beliefs about the total output growth rate always tracks the econometrician’s belief:

\[
\frac{1}{2} \left( \hat{f}_{H,t}^E + \hat{f}_{H,t}^F \right) + \frac{1}{2} \left( \hat{f}_{F,t}^E + \hat{f}_{F,t}^F \right) = \hat{f}_{H,t}^E + \hat{f}_{F,t}^E
\]

The changes from the probability measure of the econometrician to those of investors in each Country \( i = H, F \) capturing the differences of probability beliefs between each of these country investors and the econometrician, are given by:

\[
d\eta_{H,t} = \frac{1}{\sigma_{H\delta_H} \sqrt{1 - \rho^2}} \hat{g}_{H,t} d\tilde{W}_{H,t}^E
\]

\[
d\eta_{F,t} = \frac{1}{\sigma_{F\delta_F} \sqrt{1 - \rho^2}} \hat{g}_{F,t} d\tilde{W}_{F,t}^E
\]

\( ^9 \)This naturally implies that: \( \nu_{\delta_H}^H = \nu_{\delta_H}^F = \frac{1}{1 - \rho^2} \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \right) \); \( \nu_{\delta_F}^H = \nu_{\delta_F}^F = \frac{1}{1 - \rho^2} \left( \frac{\gamma_F}{\sigma_F} - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \right) \); \( \epsilon_{\delta_H}^H = \epsilon_{\delta_H}^F = \frac{1}{1 - \rho^2} \left( \frac{\gamma_{HF}}{\sigma_{H\delta_F}} - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \right) \); \( \epsilon_{\delta_F}^H = \epsilon_{\delta_F}^F = \frac{1}{1 - \rho^2} \left( \frac{\gamma_{HF}}{\sigma_{H\delta_F}} - \rho \frac{\gamma_{HF}}{\sigma_{H\delta_F}} \right) \).
\[
\frac{d\eta_{F,t}}{\eta_{F,t}} = \frac{\rho}{\sqrt{1 - \rho^2}} \frac{1}{\sigma_\delta_F} \hat{g}_{F,t}d\hat{W}_{\delta_H,t} - \frac{1}{\sigma_\delta_F} \hat{g}_{F,t}d\hat{W}_{\delta_F,t}
\]

Proof. See the appendix

Proposition 1 shows that, concerning the beliefs about the world output growth rate, we can isolate belief-dispersion effects from other learning-related effects, such as those caused by erroneous average world output growth rate belief and underestimation of risk, by choosing a particular set of parameter values in investors’ learning processes.\(^{10}\) This particular parametrization that isolates beliefs dispersion effects was first proposed by Xiong and Yang (2010).

In principle, the econometrician should have two pairs of disagreements with each country’s investor. However, the econometrician and each country investor agree about the estimate of the conditional growth rate of the output of the other country, since they both ignore the signals of the other country. Therefore, \(\hat{f}_{H,t} - \hat{f}_{H,t} = \hat{f}_{F,t} - \hat{f}_{F,t} = 0\), so that only the disagreements between the econometrician and the investor’s forecast of his own output growth rate, move over time.

3 Equilibrium consumption and pricing

In this section we first describe the optimization problem faced by each investor and then, assuming a complete markets setting, we compute the equilibrium in the world economy by clearing the market of each good.\(^{11}\) This provides the equilibrium interest rates as well as the corresponding market prices of risk and the exchange rates.

3.1 Individual optimization problem

We let \(c^{i}_{j,t}\) be the consumption by Investor \(i\) of the good produced in Country \(j\). Both representative investors derive utility from home and foreign goods. Investor \(i\) faces the optimization problem:

\[
\sup_{\{c^{i}_{H,t},c^{i}_{F,t}\}} \mathbb{E}_{0}^{E} \int_{0}^{\infty} \eta_{i,t}e^{-\beta t} \frac{1}{\alpha} C^{\alpha}_{i,t} dt; \quad \alpha < 0
\]

\(^{10}\)We could have imposed a more restrictive parametrization to isolate the belief-dispersion effect for the fundamentals of each country. However, for the purpose of understanding currency risk premium this more general parametrization is enough.

\(^{11}\)The results in this paper are independent of the menu of assets and only require the financial market to be complete.
subject to the lifetime budget constraint:
\[
\mathbb{E}_0^E \int_0^\infty m_{H,t}^E \left( c_{H,t}^i + p_t c_{F,t}^i \right) dt \leq \mathbb{E}_0^E \int_0^\infty m_{i,t}^E \delta_{i,t} dt
\]
(15)

and given the consumption basket:\footnote{Our choice of preferences and consumption basket can be interpreted as one in which investors exhibit CRRA preferences over a CES type of consumption basket, where we have restricted the constant elasticity of substitution to be the inverse of relative risk aversion \(1 - \alpha\), for tractability. If we take the limit when \(\alpha \to 0\) our preference specification converges to the log-linear setup, used by Pavlova and Rigobon (2007), Cole and Obstfeld (1991), Zapatero (1995) and many others in international finance models.}

\[
\bar{C}_{i,t} = \left[ \tau_i \left( c_{H,t}^i \right)^{\alpha} + \left( 1 - \tau_i \right) \left( c_{F,t}^i \right)^{\alpha} \right]^{\frac{1}{\alpha}}
\]
(16)

where \(\tau_i > 0\) is the weight on the home good in the preferences of Investor \(i\), \(\beta\) is the subjective discount rate, and \(m_{i,t}^E\) is the state price density required to discount output in units of goods produced in Country \(i\) (which we call in the rest of the paper "output \(i\) state price density") under the measure of the econometrician, satisfying

\[
m_{F,t}^E = m_{H,t}^E \times p_t
\]

The above equation says that the output \(F\) state price density \(m_{F,t}^E\) can be expressed as the product of two components. The first, \(m_{H,t}^E\), is the Arrow-Debreu state price of one unit of the numeraire \(\delta_{H,t}\). The second, \(p_t\), is the relative price which converts units of the foreign good \(\delta_{F,t}\) into units of the numeraire \(\delta_{H,t}\).

Note that we write the optimization problem in equation (14) using the expectation of the econometrician at initial time 0. We indicate this expectation with the superscript \(E\) in the expectation operator, \(\mathbb{E}_0^E\). We multiply the period utility of \(H\) at time \(t\) by the change of measure variable, \(\eta_{H,t}\), to get back to the expectation under the measure of \(H\). The next proposition provides the equilibrium optimal good consumption, relative price, basket consumption, price index and exchange rate.
Proposition 2. Investor $i$'s optimal consumption of each good are:

$$c_{H,i} = \left( \lambda_{i} \frac{e^{\beta t}}{\tau_{i} \eta_{i,t}} m_{H,i}^{E} \right)^{-\frac{1}{\alpha}}, \quad c_{F,i} = \left( \lambda_{i} \frac{e^{\beta t}}{(1 - \tau_{i}) \eta_{i,t}} p_{t} m_{F,i}^{E} \right)^{-\frac{1}{1-\alpha}} \quad (17)$$

where $\lambda_{H}$ is the constant Lagrangian multiplier of the static budget constraint.

Optimality of consumption ensures that each country investor equates his ratio of marginal utility of consumption of the two goods to the relative price of the two goods:

$$p_{t} = \frac{1 - \tau_{H}}{\tau_{H}} \left( \frac{c_{F,t}^{H}}{c_{H,t}^{F}} \right)^{\alpha-1} = \frac{1 - \tau_{F}}{\tau_{F}} \left( \frac{c_{F,t}^{F}}{c_{H,t}^{F}} \right)^{\alpha-1} \quad (18)$$

Country $i$'s consumption basket is:

$$\overline{C}_{i,t} = \left( \lambda_{i} \frac{e^{\beta t}}{\eta_{i,t}} m_{H,i}^{E} P_{i,t} \right)^{-\frac{1}{\alpha}} \quad (19)$$

where $P_{i,t}$ is the price index of country $i$, defined as the price of one unit of country $i$ consumption basket $\overline{C}_{i,t}$, related to relative prices $p_{t}$ by:

$$P_{i,t} = \left( \frac{1 - \tau_{H}}{\tau_{i}} + (1 - \tau_{i}) \frac{1}{\tau_{i}} p_{t} \right)^{\frac{1-\alpha}{\alpha}} \quad (20)$$

The exchange rate is the ratio of price indexes, and is therefore given by

$$e_{t} = \frac{P_{F,t}}{P_{H,t}} = \left( \frac{\tau_{F}^{1-\alpha} + (1 - \tau_{F})^{\frac{1}{1-\alpha}} p_{t}^{\frac{\alpha}{1-\alpha}}}{\tau_{H}^{1-\alpha} + (1 - \tau_{H})^{\frac{1}{1-\alpha}} p_{t}^{\frac{\alpha}{1-\alpha}}} \right)^{-\frac{1-\alpha}{\alpha}} \quad (21)$$

Proof. Optimal consumption follows from optimization of the objective function in Equation (14), subject to the corresponding constraints. The price-index follows from minimizing the expenditure function.

The proposition above illustrates important properties of the equilibrium. In particular, It is important to highlight that while the relative price of the two goods $p_{t}$ is defined as the number of units of the Home good $\delta_{H,t}$ needed to get one unit of the Foreign good $\delta_{F,t}$, the exchange rate $e_{t}$ is the number of units of the Home consumption basket $\overline{C}_{H,t}$ needed to get one unit of the Foreign
consumption basket $\mathcal{C}_{F,t}$. It now becomes clear that home bias in consumption is required to obtain non-trivial dynamics in exchange rates. For instance, even with rich dynamics in the relative prices of the two goods, heterogeneity in the consumption baskets of the two countries ($\tau_H \neq \tau_F$) is necessary to prevent the real exchange rate to be constant and equal to one (purchasing power parity). Home bias in consumption is introduced by letting $1 > \tau_H > \tau_F > 0$.

### 3.2 Equilibrium consumption and pricing measures

The aggregate resource constraint for the good produced in Country $i$ implies that in equilibrium:

$$c_{i,t}^H + c_{i,t}^F = \delta_{i,t}$$  \hspace{1cm} (22)

Plugging optimal consumption from Proposition 2 into Equation (22) we can solve for the equilibrium output $i$ state price densities:

$$m^E_{H,t} = e^{-\beta t} \left[ \left( \frac{\eta_{H,t}}{\lambda_H} \right)^{\frac{1}{1-\alpha}} + \left( \frac{\eta_{F,t}}{\lambda_F} \right)^{\frac{1}{1-\alpha}} \right] \delta_{H,t}^{-\alpha-1}$$  \hspace{1cm} (23)

$$m^E_{F,t} = m^E_{H,t} \times p_t = e^{-\beta t} \left[ \left( 1 - \tau_H \right) \frac{\eta_{H,t}}{\lambda_H} \right]^{\frac{1}{1-\alpha}} + \left( 1 - \tau_F \right) \frac{\eta_{F,t}}{\lambda_F} \right]^{\frac{1}{1-\alpha}} \delta_{F,t}^{-\alpha-1}$$  \hspace{1cm} (24)

Equations (23) and (24) show that while output is a country-specific risk factor world average sentiment resembles a common world risk factor appearing in both good state price densities. In particular, the world sentiment risk factor appearing in each state price density is in both cases a harmonic average of the same country sentiment risk factors, but the weights in that harmonic average are country-specific, due to consumption home bias. For instance we can write the Good $i$ specific state price densities as:

$$m^E_{i,t} = e^{-\beta t} \times m_{W,t} \times m_{C_i,t}$$  \hspace{1cm} (25)
where

\[ m_{W,t} = \left[ \theta_H \left( \frac{\eta_{H,t}}{\lambda_H} \right)^{\frac{1}{1-\alpha}} + \theta_F \left( \frac{\eta_{F,t}}{\lambda_F} \right)^{\frac{1}{1-\alpha}} \right]^{1-\alpha} \] (26)
\[ m_{C_{i,t}} = \delta_{i,t}^{\alpha-1} \] (27)

are the world (international sentiment) and Country i-specific (output) risk factors, and where the constant weights \( \theta = (\theta_H, \theta_F) \) are country-specific.\(^{13}\) This structure of output state price densities is in line with the setup used in recent papers studying the dynamics of exchange rates. For instance, this structure of state price densities is used in Lustig et al. (2010) to argue that counter-cyclical dollar risk premium reflects time-varying compensation to U.S. investors for taking on U.S. specific risk by shorting the dollar. Similarly, Verdelhan (2011) uses this structure and finds that bilateral exchange rates (his dollar factor) depend on differences in country-specific and world shocks, while difference in baskets of exchange rates (his carry factor) accounts for world shocks. While these papers exogenously assume this risk structure in their state price densities in order to give economic intuition to their empirical findings, this paper shows that international sentiment can reproduce such structure endogenously. In particular, our model implies that the world risk factor is driven by international sentiment risk and the country risk factor is driven by (country-specific) output risk. This endogenous structure for the state price densities will have important implications for equilibrium relative price of goods and exchange rate.

In equilibrium, each country’s consumption share of the output \( \delta_{i,t} \) is given by a monotonic transformation of the ratio of the two changes of probability measures \( \omega_i \):

\[ c_{i,t}^H = \omega_i \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{i,t}; \quad c_{i,t}^F = \left[ 1 - \omega_i \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \right] \delta_{i,t} \] (28)

where

\[ \omega_H \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) = \frac{\left( \frac{\tau_H \lambda_F}{\tau_F \lambda_H} \frac{\eta_{H,t}}{\eta_{F,t}} \right)^{\frac{1}{1-\alpha}}}{1 + \left( \frac{\tau_H \lambda_F}{\tau_F \lambda_H} \frac{\eta_{H,t}}{\eta_{F,t}} \right)^{\frac{1}{1-\alpha}}} \; \quad \omega_F \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) = \frac{\left( \frac{1-\tau_H \lambda_F}{1-\tau_F \lambda_H} \frac{\eta_{H,t}}{\eta_{F,t}} \right)^{\frac{1}{1-\alpha}}}{1 + \left( \frac{1-\tau_H \lambda_F}{1-\tau_F \lambda_H} \frac{\eta_{H,t}}{\eta_{F,t}} \right)^{\frac{1}{1-\alpha}}} \] (29)

are the consumption shares of each output for the Home investor. Since \( \omega_H \) is increasing in \( \frac{\tau_H}{\tau_F} \) and \( \omega_F \) is increasing in \( \frac{1-\tau_H}{1-\tau_F} \), we obtain that home bias in preferences generates home bias in

\(^{13}\)In particular, \( \theta = (\tau_H, \tau_F) \) for Country H and \( \theta = (1-\tau_H, 1-\tau_F) \) for Country F.
equilibrium consumption.

The corresponding optimal consumption basket for residents in each country are given, respectively, by

\[ C_{H,t} = \left( \tau_H \left( \omega_H \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{H,t} \right) \right)^\alpha + \left( 1 - \tau_H \right) \left( \omega_F \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{F,t} \right)^\alpha \]  

\[ C_{F,t} = \left( \tau_F \left( \left[ 1 - \omega_H \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{H,t} \right) \right)^\alpha + \left( 1 - \tau_F \right) \left( \left[ 1 - \omega_F \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{F,t} \right) \right)^\alpha \]  (30) (31)

The equations above show the impact of outputs \( \delta_{i,t} \) and relative sentiment \( \frac{\eta_{H,t}}{\eta_{F,t}} \) on the consumption basket in each country. Since both country investors consume both goods, the consumption in each country is increasing in the output of both goods. The way in which output shocks are distributed across countries’ consumption baskets is entirely driven by relative sentiment \( \frac{\eta_{H,t}}{\eta_{F,t}} \). When investors in Country \( H \) assign a higher probability to positive shocks, they have bet on those states occurring through \( \frac{\eta_{H,t}}{\eta_{F,t}} \). If such states do occur, investors in Country \( H \) get to consume more than investors in Country \( F \). Alternatively, this can be understood as the ratio of marginal utility of home to foreign investors being driven by relative international sentiment \( \frac{\eta_{H,t}}{\eta_{F,t}} \). Investors which have bet on positive shocks have a higher marginal utility in those states, because it provides them with a higher consumption share in those states due to side bets placed through \( \frac{\eta_{H,t}}{\eta_{F,t}} \).

### 3.3 Equilibrium relative price and exchange rate

Given the equilibrium output state price densities we can obtain, from Proposition 2, the equilibrium relative good price:

\[ P_t = \frac{1 - \tau_H}{\tau_H} \left[ \frac{\omega_H \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{H,t}}{\omega_F \left( \frac{\eta_{H,t}}{\eta_{F,t}} \right) \delta_{F,t}} \right]^{1-\alpha} \]  

Relative prices depend on the output ratio, \( \frac{\delta_{H,t}}{\delta_{F,t}} \), and on relative sentiment, \( \frac{\eta_{H,t}}{\eta_{F,t}} \), through the home investor’s ratio of home and foreign output consumption shares. The dependence on the output ratio is standard in international finance models: when the output of one country is hit by a positive supply shock it becomes less scarce and its relative price decreases. In this model, the relative price is also driven by relative sentiment \( \frac{\eta_{H,t}}{\eta_{F,t}} \). When investors in Country \( H \) assign a higher probability to positive shocks, they have bet on those states occurring through \( \frac{\eta_{H,t}}{\eta_{F,t}} \). Therefore, their marginal utility of those states increases but, given consumption home bias, if they win these bets they get
a higher consumption share of foreign output than that of home output. As a result, increased relative demand for foreign output decrease its relative price.

The exchange rate is the number of units of the Home consumption basket $C_{H,t}$ needed to get one unit of the Foreign consumption basket $C_{F,t}$. Accordingly, it will prove convenient to characterize the corresponding consumption state price densities. Denote by $\xi_{i,t}$ the state price density required to discount the country $i$ consumption basket $C_{i,t}$ (which we call in the rest of the paper "Consumption $i$ state price density"). It can be shown that these state price densities are weighted averages of the Home and Foreign output state price densities, $m^E_{H,t}$ and $m^E_{F,t}$. In particular, we have

$$
\xi^E_{H,t} = m^E_{H,t} \times P_{H,t} = \left[ \frac{\tau_F}{\tau_H} \left( m^E_{H,t} \right)^{-\frac{1}{1-\alpha}} + \left( 1 - \frac{\tau_F}{\tau_H} \right) \left( m^E_{F,t} \right)^{-\frac{1}{1-\alpha}} \right]^{-\frac{1}{1-\alpha}} = e^{-\beta t} \eta_{H,t} \xi^\alpha_{H,t} (32)
$$

$$
\xi^E_{F,t} = m^E_{F,t} \times P_{F,t} = \left[ \frac{\tau_H}{\tau_F} \left( m^E_{H,t} \right)^{-\frac{1}{1-\alpha}} + \left( 1 - \frac{\tau_H}{\tau_F} \right) \left( m^E_{F,t} \right)^{-\frac{1}{1-\alpha}} \right]^{-\frac{1}{1-\alpha}} = e^{-\beta t} \eta_{F,t} \xi^\alpha_{F,t} (33)
$$

The Consumption $i$ state price density $\xi_{i,t}$ can be expressed as the product of two components. The first, $m_{H,t}$, is the Arrow-Debreu state price of one unit of the numeraire $\delta_{H,t}$. The second, $P_{i,t}$, is the price index which converts units of the $i$ consumption basket $C_{i,t}$ into units of the numeraire $\delta_{H,t}$. Using equilibrium consumption allocations and previous results it can be shown that the equilibrium exchange rate is given by:

$$
e_t = \frac{\xi^E_{F,t}}{\xi^E_{H,t}} = \frac{P_{F,t}}{P_{H,t}} = \left( \frac{m^E_{F,t}}{m^E_{H,t}} \right)^{-\frac{1}{1-\alpha}} \left( \frac{\tau^\alpha_{F,t} + (1 - \tau^\alpha_{F,t}) \frac{1}{1-\alpha} \left( \tau^\alpha_{H,t} + (1 - \tau^\alpha_{H,t}) \frac{1}{1-\alpha} \right)}{\tau^\alpha_{F,t} + (1 - \tau^\alpha_{F,t}) \frac{1}{1-\alpha} \left( \tau^\alpha_{H,t} + (1 - \tau^\alpha_{H,t}) \frac{1}{1-\alpha} \right)} \right)^{\frac{1}{1-\alpha}} = \frac{\eta_{F,t}}{\eta_{H,t}} \left( \frac{C_{F,t}}{C_{H,t}} \right)^{\alpha-1} (34)
$$

The relation between the exchange rate and relative price follows from Proposition 2. Given that the exchange rate is entirely driven by the relative price the intuition on how relative output and relative sentiment affect it are very similar to the intuition conveyed for the relative price.\(^{14}\)

\(^{14}\)It can be shown that, given consumption home bias, $\frac{\partial \alpha}{\partial p_H} > 0$. 

The relation between exchange rate and consumption baskets implies that, despite optimality of consumption baskets that are themselves driven by relative sentiment, the ratio of marginal utility of the consumption baskets of the two countries do not equal the exchange rate. The wedge between the two is driven by relative sentiment $\frac{\eta_{F,t}}{\eta_{H,t}}$. 

\(^{14}\)
3.4 Interest rates and consumption risk

Consider a Country $i$ bond in zero net supply which pays an instantaneous interest rate $r_{i,t}$ and is locally riskless in units of the consumption basket in Country $i$: $\bar{C}_{i,t}$. The dynamics of the consumption state price densities are given by

$$d\xi_{i,t}^E = -\xi_{i,t}^E \left[ r_{i,t} dt + \kappa_{i,t}^E d\tilde{W}_t^E \right]$$  \hspace{1cm} (35)

where $\kappa_{i,t}^E$ is the Country $i$ market price of consumption risk from the perspective of the econometrician. For the analysis of currency risk premium, it will be useful to define the Country $i$ consumption risk premium $rp_{i,t}^E$ as one-half of the conditional variance of the log consumption state price density

$$rp_{i,t}^E = \frac{1}{2} (\kappa_{i,t}^E)^\top (\kappa_{i,t}^E)$$

The next proposition shows the Country $i$ interest rate $r_{i,t}$ and its consumption risk premium $rp_{i,t}$

**Proposition 3.** The equilibrium interest rate $r_{i,t}$ of Country $i = \{H, F\}$ are given by:

$$r_{i,t} = \beta + (1 - \alpha) \left( s_{i,t} \tilde{f}_{H,t}^E + (1 - s_{i,t}) \tilde{f}_{F,t}^E \right) - (s_{i,t} (\pi_{i,t} - \omega_{H,t}) \tilde{g}_{H,t} + (1 - s_{i,t}) (1 - \pi_{i,t} - \alpha (1 - \omega_{F,t})) \tilde{g}_{F,t})$$

$$-\frac{1}{2} \left( 1 - \alpha \right) \left\{ s_{i,t}^2 \sigma_{\delta_H}^2 + (1 - s_{i,t})^2 \sigma_{\delta_F}^2 + 2 s_{i,t} (1 - s_{i,t}) \rho \sigma_{\delta_H} \sigma_{\delta_F} + (1 - \alpha) \left( s_{i,t} \sigma_{\delta_H}^2 + (1 - s_{i,t}) \sigma_{\delta_F}^2 \right) \right\}$$

$$+ \frac{1}{2} \frac{1}{1 - \alpha} \left\{ \pi_{i,t} (1 - \pi_{i,t}) - (1 + \alpha) \left[ s_{i,t} \omega_{H,t} + (1 - s_{i,t}) \omega_{F,t} \right] \right\}$$

$$\times \frac{1}{1 - \rho^2} \left[ \left( \tilde{g}_{H,t} / \sigma_{\delta_H} \right)^2 + \left( \tilde{g}_{F,t} / \sigma_{\delta_F} \right)^2 + 2 \rho \tilde{g}_{H,t} \tilde{g}_{F,t} / \sigma_{\delta_H} \sigma_{\delta_F} \right]$$

and the corresponding Country $i$ consumption risk premium under the measure of the econometrician $rp_{i,t}$ follow:

$$rp_{i,t}^E = (1 - \alpha) (s_{i,t} \pi_{i,t} \tilde{g}_{H,t} + (1 - s_{i,t}) (1 - \pi_{i,t}) \tilde{g}_{F,t})$$

$$+ \frac{1}{2} (1 - \alpha)^2 \left( s_{i,t}^2 \sigma_{\delta_H}^2 + (1 - s_{i,t})^2 \sigma_{\delta_F}^2 + 2 s_{i,t} (1 - s_{i,t}) \rho \sigma_{\delta_H} \sigma_{\delta_F} \right)$$

$$+ \frac{1}{2} \frac{1}{1 - \rho^2} \left\{ s_{i,t}^2 \left( \tilde{g}_{H,t} / \sigma_{\delta_H} \right)^2 + (1 - \pi_{i,t})^2 \left( \tilde{g}_{F,t} / \sigma_{\delta_F} \right)^2 - 2 \rho \pi_{i,t} (1 - \pi_{i,t}) \tilde{g}_{H,t} \tilde{g}_{F,t} / \sigma_{\delta_H} \sigma_{\delta_F} \right\}$$
In the above expressions $s_{i,t}$ represent the shares $s_{i,t}$ of the Country $i$ consumption basket driven by consumption of the good produced in Country $H$, defined by

$$s_{H,t} = \frac{\tau_H (\omega_{H,t} \delta_{H,t})^\alpha}{\tau_H (\omega_{H,t} \delta_{H,t})^\alpha + (1 - \tau_H) (\omega_{F,t} \delta_{F,t})^\alpha}; \quad s_{F,t} = \frac{\tau_F [(1 - \omega_{H,t}) \delta_{H,t}]^\alpha}{\tau_F [(1 - \omega_{H,t}) \delta_{H,t}]^\alpha + (1 - \tau_F) [(1 - \omega_{F,t}) \delta_{F,t}]^\alpha}$$

In addition, we have introduced the function $\pi_{i,t}$ related to the basket consumption share of the home investor, defined by

$$\pi_{i,t} = s_{i,t} \omega_{H,t} + (1 - s_{i,t}) \omega_{F,t}$$

**Proof:** The results follow from applying Ito’s lemma to equations (32) - (33); and then identifying terms with equation (35).

We first note that the interest rate and consumption risk premium differ across countries only on the shares $s_{i,t}$. This is reasonable, since the only difference between the state price densities of the two countries is that they discount consumption baskets that have different preference weights on the two goods (consumption home bias).

The first term in the interest rate $r_{i,t}$ reflects investors’ time preference. The second term represents a wealth effect: when the expected growth rate of output increases, investors have a lower incentive to save in bonds which are locally riskless in units of the consumption basket of that country $C_{i,t}$, and this generates a higher interest rate. The expected growth rates are weighted by the shares $s_{i,t}$. The third term represents an additional wealth effect generated by differences of opinion. In periods in which investors are relatively optimistic about the expected output growth rate compared to the econometrician ($\tilde{g}_{i,t} < 0$), investors have a lower incentive to save in Country $i$ bonds and the interest rate increases. Conversely, in periods when the average world investor is relatively pessimistic about the expected output growth rate of Country $i$ compared to the econometrician ($\tilde{g}_{i,t} > 0$) there is a negative wealth effect and the interest rate decreases. Differences of opinion ($\tilde{g}_{i,t}$) also interact with the shares $s_{i,t}, \omega_{i,t}$ and $\pi_{i,t}$. The fourth term represents precautionary savings driven by the consumption-basket-related output growth rate volatility, which is naturally weighted by a quadratic form of the shares $s_{i,t}$. The last term is driven by the impact of speculative demands on savings. Its relative size depends on two components: (i) the product of shares $(\pi_{i,t} (1 - \pi_{i,t}), \omega_{H,t} (1 - \omega_{H,t}))$ and (ii) the dispersion of beliefs across investors concerning
the country output growth rates \( (\hat{g}_{i,t}^2) \). In contrast to the added wealth effect (third term), whose sign depends on the average world investor being optimistic or pessimistic, this speculative term always increases the interest rate.

The consumption risk premiums \( r^E_{i,t} \) contain three terms. The first term is driven by the world average difference of opinion on the fundamentals of each country \( (\hat{g}_{i,t}) \). The main intuition of this component is that in our setup risk is transferred from the more pessimistic investor to the more optimistic. The transfer of risk is proportional to the extent of difference of opinion \( \hat{g}_{i,t} \). For example, when the average world investor is relatively pessimistic about expected output growth rate \( (\hat{g}_{i,t} > 0) \) compared to the econometrician there is a positive expected reward from the perspective of the econometrician, in exchange for providing insurance to pessimist investors against bad states realizations. Differences of opinion \( (\hat{g}_{i,t}) \) interact with the shares \( s_{i,t}, \omega_{i,t} \) and \( \pi_{i,t} \). The second term is the standard reward for consumption-basket-related output risk, naturally weighted by a quadratic form of the shares \( s_{i,t} \). The last term is the reward for the risk of beliefs’ dispersion \( (\hat{g}_{i,t}^2) \), which drive sentiments risk irrespective of investors being optimistic or pessimistic. This last term also interacts with the product of shares \( \pi_{i,t} \).

The analysis above reveals that beliefs’ heterogeneity may affect the correlation between the interest rate and the consumption risk premium through different channels. First, pessimism about the expected output growth rate decreases the interest rate through a wealth effect, and simultaneously increases the consumption risk premium from the perspective of the econometrician, as compensation for providing insurance to pessimists against the realization of bad states. Second, output risk related to the consumption basket decreases the interest rate through a precautionary savings motive, but naturally increases the consumption risk premium. Note that even though the output growth rate volatility is constant, the output risk that is relevant for the pricing of consumption is stochastic because it is weighted by the shares \( s_{i,t} \), that are driven by output and sentiment. Finally, sentiment risk may generally increase both the interest rate and the consumption risk premium. However, the product of shares components associated with sentiment risk driving the interest rate and consumption risk premium are different. Therefore, while the first two channels clearly induce a negative correlation, the last one may induce a positive correlation. The total effect of beliefs’ heterogeneity on the correlation between the interest rate and the consump-
tion risk premium is ultimately an empirical question, which we address in the next sections of the paper.

4 Foreign exchange markets

In this section we first obtain the equilibrium relative price, real exchange rates and currency risk premium, and then analyze the impact of differences of opinion in these quantities.

4.1 Exchange rate risk premium

We now define the exchange rate risk premium.

Definition 1. Consider the following consumption carry trade strategy: the home investor borrows at the domestic interest rate \( r_{H,t} \) which is locally riskless in units of the Home consumption basket \( C_{H,t} \), and invests the proceeds per unit of domestic consumption basket in the foreign interest rate \( r_{F,t} \) which is locally riskless in units of the Foreign consumption basket \( C_{F,t} \). We denote the instantaneous excess return from this strategy the exchange rate risk premium \( d\ln r_{e,t} \), given by:

\[
d\ln r_{e,t} = d\ln e_t + (r_{F,t} - r_{H,t}) \, dt
\]

The domestic investor earns the foreign interest rate \( r_{F,t} \) but has to repay the domestic interest rate \( r_{H,t} \). If the exchange rate increases he makes a profit from converting the interest rate gained abroad to local consumption basket units.

Backus et al. (2001) have shown that in affine settings, like the one in this paper, the expected exchange rate risk premium is equal to one-half of the difference between the conditional variance of the local and the foreign log consumption state price density. In our setting, this leads to the following relation:

\[
E_t^E [d\ln r_{e,t}] = (r_{pH,t}^E - r_{pF,t}^E) \, dt
\]

This relation contains important insights provided by Backus et al. (2001). Predictability of currency excess returns is only possible when the difference in the country risk premiums is predictable. Proposition 3 provides insights on how output and sentiment risk drive jointly interest rates \( r_{i,t} \) and consumption risk premiums \( r_{pE_{i,t}} \). Because these differ across countries on the shares
that are themselves driven by output and sentiment risk, it is clear that the differential of interest rates and the differential of consumption risk premiums (the expected exchange rate risk premium) will be also jointly driven by output and sentiment risks. This relates to the explanation of UIP, as shown in the next proposition.

4.2 Uncovered interest rate parity

Proposition 4. The relation between the log of the real exchange rate and the domestic and foreign real interest rates is

\[ d \ln e_t = (1 + \Psi_t) (r_{H,t} - r_{F,t}) dt + (\kappa^E_{H,t} - \kappa^E_{F,t}) d\widetilde{W}^E_t \]

\[ = (r_{H,t} - r_{F,t}) dt + \mathbb{E}_t^E [d \ln r_{e,t}] + (\kappa^E_{H,t} - \kappa^E_{F,t}) d\widetilde{W}^E_t \]

where

\[ \Psi_t = \frac{\mathbb{E}_t^E [d \ln r_{e,t}]}{(r_{H,t} - r_{F,t}) dt} = \frac{\tau p^E_{H,t} - \tau p^E_{F,t}}{r_{H,t} - r_{F,t}} \]

Proof: See the appendix.

Taking expectations from the above proposition we obtain

\[ \mathbb{E}_t^E [d \ln e_t] = (1 + \Psi_t) (r_{H,t} - r_{F,t}) dt \quad (36) \]

On the other hand, the standard test for UIP is of the form

\[ \mathbb{E}_t^E [d \ln e_t] = a + b (r_{H,t} - r_{F,t}) dt \quad (37) \]

where the null hypothesis is \( a = 0; \ b = 1 \). Comparing equations (36) and (37), we see that the effect of differences of opinion on the relation between real exchange rate changes and the real interest rate differential is captured by the term \( \Psi_t \). In particular, the model resembles the UIP failure in the data when \( \Psi_{tt} < 0 \). In light of Proposition 3 (and the corresponding analysis), this will hold true when differences of opinion generate a negative correlation between the interest rate and consumption risk premium. The quantitative section addresses this issue in more detail.

Proposition 4 shows that the expected exchange rate risk premium, \( \mathbb{E}_t^E [d \ln r_{e,t}] \), generates the
wedge between the change in the (log) real exchange rate and the differential of interest rates. In the setup of this paper the expected exchange rate risk premium is just compensation for bearing output risk, for providing insurance to pessimistic investors, and for bearing sentiment risk.

4.3 Disconnect between exchange rates and relative consumption

As shown in Equation (34), the equilibrium exchange rate is not equal to the ratio of marginal utilities of the consumption baskets in each country. In our model there is a wedge driven by international sentiment. This is the essence on the disconnection between exchange rate and relative consumption in our setup. For instance, taking logs to Equation (34) we obtain that

\[ d \ln e_t = d \ln \xi_{F,t} - d \ln \xi_{H,t} = d \ln \frac{\eta_{F,t}}{\eta_{H,t}} + (\alpha - 1) d \ln \frac{C_{F,t}}{C_{H,t}} \]  

(38)

which implies that there need not be a perfect correlation between \(d \ln e_t\) and \(d \ln \frac{C_{F,t}}{C_{H,t}}\) anymore. Accordingly, international sentiment may help explain the low correlation between the exchange rate and relative country consumption, as documented by Backus and Smith (1993).

4.4 International risk sharing

It is clear from the previous analysis that this model incorporates a new dimension of international "sentiment" risk sharing. Country investors share risk by placing bets with each other on their beliefs about fundamentals. This new dimension may potentially help reconciling a high degree of risk sharing implied by prices and a relatively lower degree of risk sharing implied by consumption, as documented by Brandt et al. (2006). To see this, we note that taking variances to Equation (38) we obtain

\[
\begin{align*}
\text{Var} (d \ln e_t) & = \text{Var} (d \ln \xi_{F,t}) + \text{Var} (d \ln \xi_{H,t}) - 2 \sqrt{\text{Var} (d \ln \xi_{F,t}) \text{Var} (d \ln \xi_{H,t}) \text{Corr} (d \ln \xi_{F,t}, d \ln \xi_{H,t})} \\
& = (1 - \alpha)^2 \left[ \text{Var} (d \ln C_{F,t}) + \text{Var} (d \ln C_{H,t}) - 2 \sqrt{\text{Var} (d \ln C_{F,t}) \text{Var} (d \ln C_{H,t})} \right] \\
& \times \text{Corr} (d \ln C_{F,t}, d \ln C_{H,t}) + \text{Var} \left( d \ln \frac{\eta_{F,t}}{\eta_{H,t}} \right) - 2 (1 - \alpha) \sqrt{\text{Var} \left( d \ln \frac{\eta_{F,t}}{\eta_{H,t}} \right) \text{Var} \left( d \ln \frac{C_{F,t}}{C_{H,t}} \right)} \\
& \times \text{Corr} \left( d \ln \frac{\eta_{F,t}}{\eta_{H,t}}, d \ln \frac{C_{F,t}}{C_{H,t}} \right)
\end{align*}
\]  

(39)
The puzzle pointed out by Brandt et al. (2006) is the following. Given data on the second moments of foreign exchange rates and on each country’s Sharpe ratios, the equality in the first line of Equation (34) implies a very high correlation of state price densities, higher than 0.9: "international risk sharing is better than you think". On the other hand, in the standard setting with CRRA utility and without sentiment (up to the second line on the above equation), the correlation of state price densities equals the cross-country consumption correlation, which in the data is at most 0.3. Hence, if we were to use this data from consumption, it would imply an excessively high variance for the volatility of exchange rate compared to what we observe in the data: "exchange rates are too smooth". Because of beliefs’ heterogeneity, the third line of Equation (34), representing the extent of sentiment risk sharing, appears. This new dimension of risk sharing is missing in Brandt et al. (2006).

In the current model investors engage in output risk sharing and international sentiment risk sharing. While output risk may increase cross-country consumption correlation, international sentiment generates an opposing effect. Because investors place bets with each other, sentiment risk decreases the cross-country consumption correlation. Hence international sentiment may help explain a simultaneous low volatility of exchange rates and associated high correlation in country state price densities (implying a high degree of risk sharing), and a low cross-country consumption correlation (implying a poor degree of risk sharing), as found by Brandt et al. (2006).

5 Quantitative Exercise

In this section, we evaluate the ability of the model to replicate the rejection of the uncovered interest rate parity observed in empirical studies. To this end, we choose the parameters of the model in order to match a set of unconditional moments on real GDP, the interest rates and the state price densities.\footnote{Note that calibrating the output processes $\delta_{i,t}$ to consumption data would be inappropriate in our setting, since total consumption of one country need not be equal to its total output in our fully integrated economy.} In particular, we assume the United States is the domestic country and the United Kingdom is the foreign country. We calibrate the model to match the unconditional output growth rate means and variances as well as their correlation, using a sample of quarterly data from 1960 until 2011 obtained from Datastream. In addition, we target the variance of state
price densities and the interest rate differential. Since we are trying to match empirical data, all the simulations of the model are done under the objective probability measure, which governs empirical realizations of the processes. Table 1 presents the parameters chosen.

Table 1: Parameter values used in all the numerical examples

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters for investors’ output processes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation between output growth rate in the two countries</td>
<td>$\rho$</td>
<td>18.00%</td>
</tr>
<tr>
<td>Volatility of the growth rate of output in Country $H$</td>
<td>$\sigma_{\delta_H}$</td>
<td>1.25%</td>
</tr>
<tr>
<td>Volatility of the growth rate of output in Country $F$</td>
<td>$\sigma_{\delta_F}$</td>
<td>1.55%</td>
</tr>
<tr>
<td>Mean reversion of output growth rate</td>
<td>$\zeta$</td>
<td>2.00%</td>
</tr>
<tr>
<td>Long-term value of output growth rate</td>
<td>$f$</td>
<td>2.70%</td>
</tr>
<tr>
<td>Volatility of output growth rate</td>
<td>$\sigma_f$</td>
<td>1.00%</td>
</tr>
<tr>
<td>Parameters for investors’ preferences and model uncertainty</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation between signal and mean growth rate</td>
<td>$\phi$</td>
<td>0.75%</td>
</tr>
<tr>
<td>Subjective discount rate</td>
<td>$\beta$</td>
<td>10.00%</td>
</tr>
<tr>
<td>Relative risk aversion</td>
<td>$1 - \alpha$</td>
<td>7.5</td>
</tr>
<tr>
<td>Weight on the home good by home investor</td>
<td>$\tau_H$</td>
<td>0.8</td>
</tr>
<tr>
<td>Weight on the home good by foreign investor</td>
<td>$\tau_F$</td>
<td>0.4</td>
</tr>
<tr>
<td>Ratio of shadow prices for the static budget constraints</td>
<td>$\lambda_H / \lambda_F$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2 shows the unconditional moments for the data, as well as the average moments obtained from simulating the model across 50,000 paths of 205 quarterly observations each. We do a reasonable job in matching the unconditional moments: volatility and correlations of the countries output growth rates, volatility of changes in (log) exchange rates, volatility of the differential of interest rates and volatility of the changes in (log) state price densities. It is worth noting that, even though we were able to obtain a low volatility for the differential of interest rates, the volatility of the interest rate in each country is too high. As discussed by David (2008), the only way to obtain sufficiently low interest rate volatility in models of heterogeneous beliefs and constant relative risk aversion is by setting risk aversion to very low levels. The reason is that the last term in the interest rate expression in Proposition 3, driven by speculation, is highly volatile. Because that term is highly correlated across countries, it has a smaller impact in the volatility of the interest rate differential. When computing individual interest rates, by contrast, this term adds significant volatility.

In Table 3 we present the mean of the estimated intercept and slope of the UIP regression over
Table 2: Moments on outputs, exchange rate, and financial variables

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean of home output growth rate</td>
<td>3.04%</td>
<td>2.69%</td>
</tr>
<tr>
<td>Mean of foreign output growth rate</td>
<td>2.24%</td>
<td>2.69%</td>
</tr>
<tr>
<td>Volatility of home output growth rate</td>
<td>1.72%</td>
<td>1.73%</td>
</tr>
<tr>
<td>Volatility of foreign output growth rate</td>
<td>1.95%</td>
<td>1.96%</td>
</tr>
<tr>
<td>Correlation of home and foreign output growth rates</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>Volatility of the exchange rate</td>
<td>10.5%</td>
<td>16.17%</td>
</tr>
<tr>
<td>Volatility of the differential of interest rates</td>
<td>3.00%</td>
<td>3.28%</td>
</tr>
<tr>
<td>Volatility of the state price density in the US</td>
<td>0.69</td>
<td>0.64</td>
</tr>
<tr>
<td>Volatility of the state price density in the UK</td>
<td>0.69</td>
<td>0.65</td>
</tr>
</tbody>
</table>

50,000 simulated paths of the model in 205 quarterly observations each. In addition, we present the mean of the t-statistic of the slope coefficient and the mean $R^2$ of the regression. It is evident that the model is able to replicate rejection of the uncovered interest rate parity. In fact the slope coefficient, which is statistically significant, is negative and close to its empirical counterpart. In addition, the mean $R^2$ is higher than in the data. This implies that the current model allows for predictability of exchanges rates.

Table 3: UIP regression results

<table>
<thead>
<tr>
<th>UIP regression</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept estimate</td>
<td>-0.0015</td>
<td>0.0035</td>
</tr>
<tr>
<td>Intercept t-statistic</td>
<td>-0.2806</td>
<td>0.5087</td>
</tr>
<tr>
<td>Slope estimate</td>
<td>-1.7835</td>
<td>-1.2605</td>
</tr>
<tr>
<td>Slope t-statistic</td>
<td>-1.8989</td>
<td>-3.6341</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0392</td>
<td>0.0623</td>
</tr>
<tr>
<td>$\text{Corr} \left( d \ln e_t, -d \ln R_{e,t} \right)$</td>
<td>-0.9988</td>
<td></td>
</tr>
<tr>
<td>$\frac{\text{Var}(d \ln R_{e,t})}{\text{Var}(d \ln e_t)}$</td>
<td>1.0279</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1 presents histograms on the variables of interest in the UIP regression across the 50,000 simulated paths. It is noteworthy that, as shown by the histogram, the slope coefficient is negative and significant in virtually every path. The intercept, by contrast, was insignificant most of the times (histogram not reported), in line with the empirical evidence. The relatively high $R^2$ suggests that the model is also consistent with predictability.

Table 4 shows the correlation between state price densities, country consumptions, and the
Figure 1: Histograms of the coefficients, t-statistics and $R^2$ in the UIP regression
correlation between the exchange rates and relative consumption both in the data as well as the corresponding average correlations across 50,000 simulated paths of our model in 205 quarterly observations each. The model replicates a much higher correlation for state price densities than cross-country consumption correlation, in line with the empirical evidence provided by Brandt et al. (2006). In addition, the model is able to reproduce the disconnection between exchange rates and relative consumption, reported by Backus and Smith (1993). In fact, we obtain a correlation of zero.

Table 4: International risk sharing and the disconnect between exchange rates and relative consumption

<table>
<thead>
<tr>
<th>International risk sharing</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cross country correlation of state price densities</td>
<td>0.99</td>
<td>0.93</td>
</tr>
<tr>
<td>Cross country consumption correlation</td>
<td>0.33</td>
<td>0.32</td>
</tr>
<tr>
<td>Correlation between exchange rates and relative consumption</td>
<td>0.05</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Figure 2 presents histograms on the variables of interest concerning international risk sharing across the 50,000 simulated paths. As it was the case with UIP results, these histograms show that our results hold in most paths. For instance, we note that the empirical distribution of the correlation of state price densities (in the range from 0.6 to 1) is such that it is almost always higher than the empirical distribution of consumption growth cross-country correlation (in the range from -0.6 to 0.8). In addition, the correlation between exchange rates and relative consumption is centered in zero but may take several values in the range from -0.3 to 0.3, according to the extent of disagreement across country investors.
Figure 2: Histograms of the International Risk Sharing relevant correlations
6 Conclusions

In this paper we show that economies with heterogeneous beliefs, which have recently proved successful in explaining salient features of financial asset pricing and portfolio choice, can also help explain important features of foreign exchange markets.

Rejection of the UIP hypothesis implies that currencies with high interest rates tend to appreciate. In our model, when the foreign investor is relatively pessimistic about the home output growth rate two simultaneous effects occur: (i) because the pessimistic investor transfers risk to the local optimistic one, the optimistic investor expects a positive risk premium from implementing the carry trade strategy; and (ii) negative wealth effects induce these pessimistic investors to increase their savings in local bonds at home, which reduces the local interest rate compared to the one abroad. Thus optimistic local investors expect a positive risk premium from carry trade when interest rates are lower at home than abroad. This is the channel through which the UIP hypothesis is endogenously rejected due to differences of opinion.

The model is also able to reproduce the exchange rate disconnect puzzle. While standard theoretical models require a perfect correlation between the exchange rate changes and relative consumption growth, Backus and Smith (1993) showed that this correlation is low in the data. In our setup this occurs because the exchange rate is driven not only by the ratio of marginal utilities of consumption, but also by relative sentiment. The stochastic wedge between the exchange rate and marginal utility of consumptions induced by relative sentiment decreases the correlation between the exchange rate changes and relative consumption growth, bringing it closer to the magnitude observed in the data.

A related empirical regularity is associated with international risk sharing. Brandt et. al. (2006) point out an apparent disconnect between the relatively low empirical cross-country consumption growth correlations and the extremely high degree of international risk sharing implied by the relatively low real exchange rate volatility observed in the data. In our setup, this occurs because international sentiment simultaneously: (i) appears as a common world risk factor in the state price densities of the two countries, increasing the cross-country correlation of state price densities changes and hence decreasing the exchange rate volatility; and (ii) induces international investors to engage in intertemporal risk sharing such that they get a higher consumption share when the state
that the agent considers relatively more likely occurs. But this must imply a lower consumption share for investors in the other country, which take the other side of the bet and hence obtain a lower consumption share. Therefore, irrespective of which investor gets a higher or lower consumption share, the speculation induced by international sentiment decrease the cross-country consumption growth correlation.

A quantitative exercise confirms the intuition of these empirical regularities. For tractability, we chose a very simple model to illustrate the main impact of differences of opinion on the foreign exchange markets. While more general modeling could potentially help explaining other empirical regularities, we believe the contribution of this paper is to show, using a simple model, how differences of opinion may help explain jointly these three empirical regularities in foreign exchange markets.
Econometrician filtering. From Lipster and Shiryaev (2001, Theorem 12.7, page 36), the econometrician’s posterior output growth rate fundamentals’ distribution is given by

\[
\begin{pmatrix}
    f_{H,t} \\
    f_{F,t}
\end{pmatrix}
\left( \delta_{H,t}, \delta_{F,t} \right)_{t=0} \sim \mathcal{N}
\begin{pmatrix}
    \left( \hat{f}_{H,t} \right) \\
    \left( \hat{f}_{F,t} \right)
\end{pmatrix}
\begin{pmatrix}
    \gamma_H & \gamma_{HF} \\
    \gamma_{HF} & \gamma_F
\end{pmatrix}
\]

The steady state elements of the variance-covariance matrix are

\[
\begin{align*}
\gamma_H &= \sigma^2_{\delta H} \left( \sqrt{(1 - \rho^2) \left( \zeta^2 (1 - \rho^2) + \left( \frac{\sigma_f}{\sigma_{\delta H}} \right)^2 - \left( \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}} \right)^2 - 2 \zeta \rho \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}} \right) - \zeta (1 - \rho^2) + \rho \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}}} \right) \\
\gamma_F &= \sigma^2_{\delta F} \left( \sqrt{(1 - \rho^2) \left[ \zeta^2 (1 - \rho^2) + \left( \frac{\sigma_f}{\sigma_{\delta F}} \right)^2 - \left( \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}} \right)^2 - 2 \zeta \rho \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}} \right] - \zeta (1 - \rho^2) + \rho \frac{\gamma_{HF}}{\sigma_{\delta H} \sigma_{\delta F}}} \right) \\
\gamma_{HF} &= \rho \sigma_{\delta H} \sigma_{\delta F} \left( \sqrt{A + B + C - 2 \left( \frac{1}{\sigma_{\delta H} \sigma_{\delta F}} \right)^2 \left[ \frac{(1 - \rho^2) \sigma^6_{\delta H} N}{1 + \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^2 D} - \zeta \right]} \right)
\end{align*}
\]

where the constants \( A, B, C, N \) and \( D \) are:

\[
\begin{align*}
A &= \frac{\zeta^2}{1 - 2 (1 - 2 \rho^2) \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^2 + \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^4} \\
B &= \frac{\sigma_f^2 + \zeta^2 \sigma^2_{\delta H}}{\sigma^2_{\delta H} - 2 (1 - 2 \rho^2) \sigma^2_{\delta F} + \sigma^2_{\delta H}} \\
C &= \frac{\sigma_f^2 + 2 \zeta^2 \rho^2 \sigma^2_{\delta H}}{\sigma^2_{\delta F} - 2 (1 - 2 \rho^2) \sigma^2_{\delta H} + \sigma^2_{\delta F}} \\
N &= \left( \sigma_f^2 + \sigma^2_{\delta H} \zeta^2 \right) \left( \frac{\sigma_f}{\sigma_{\delta H}} \right)^2 + \zeta^2 \right) - \sigma^2_{\delta H} \zeta^4 \rho^2 \\
D &= \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^6 - 4 (1 - 2 \rho^2) \left[ 1 + \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^4 \right] + 2 \left[ 3 - 8 \rho^2 (1 - \rho^2) \right] \left( \frac{\sigma_{\delta H}}{\sigma_{\delta F}} \right)^2
\end{align*}
\]
Proof of Proposition 1. From filtering theory, the investor’s forecasts are

\[ d\tilde{f}^H_{t,t} = \left[ \zeta \left( \bar{f}_H - \tilde{f}^H_{H,t} \right) + \left( \frac{1}{1 - \rho^2} \right)^\frac{3}{2} \left( \frac{\gamma_H}{\sigma^2_H} - \rho \frac{\gamma_H \sigma_F}{\sigma_H \sigma_F} \right) \tilde{g}^H_{H,t} \right] dt \]

\[ + \frac{1}{1 - \rho^2} \left[ \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_H \sigma_F}{\sigma_F} \right) dW^E_{\delta H,t} + \left( \frac{\gamma_H}{\sigma_F} - \rho \frac{\gamma_H \sigma_H}{\sigma_F} \right) dW^E_{\delta F,t} \right] - \frac{\phi}{\sqrt{1 - \phi^2}} ds \]

\[ d\tilde{f}^F_{t,t} = \left[ \zeta \left( \bar{f}_F - \tilde{f}^F_{H,t} \right) + \left( \frac{1}{1 - \rho^2} \right)^\frac{3}{2} \left[ \left( \frac{\gamma_F}{\sigma^2_H} \right) \left( \rho^2 + \sqrt{1 - \rho^2} \right) - \rho \frac{\gamma_H \sigma_F}{\sigma_H \sigma_F} \left( 1 + \sqrt{1 - \rho^2} \right) \right] \tilde{g}^F_{F,t} \right] dt \]

\[ + \frac{1}{1 - \rho^2} \left[ \left( \frac{\gamma_H}{\sigma_H} - \rho \frac{\gamma_H \sigma_F}{\sigma_F} \right) dW^E_{\delta H,t} + \left( \frac{\gamma_F}{\sigma_F} - \rho \frac{\gamma_H \sigma_H}{\sigma_F} \right) dW^E_{\delta F,t} \right] + \frac{\phi}{\sqrt{1 - \phi^2}} ds \]

The dynamics of disagreements then follow from their definition in the proposition, given the dynamics of the forecasts from the perspective of the econometrician. The changes of probability measure follow directly from an application of Girsanov theorem.

Proof of Proposition 4. We know that

\[ d\ln e_t = d\ln \xi^E_{F,t} - d\ln \xi^E_{H,t} \]  \hspace{1cm} (40)

Applying Ito’s lemma to the log of equation (35) we get

\[ d\ln \xi^E_{F,t} = (-r_{F,t} - r_{F,t}) dt - \kappa_{F,t} d\tilde{W}^E_t \]  \hspace{1cm} (41)

\[ d\ln \xi^E_{H,t} = (-r_{H,t} - r_{H,t}) dt - \kappa_{H,t} d\tilde{W}^E_t \]  \hspace{1cm} (42)

Plugging these expressions into Equation (40) we obtain

\[ d\ln e_t = [r_{H,t} - r_{F,t} + r_{H,t} - r_{F,t}] dt + (\kappa_{H,t} - \kappa_{F,t}) d\tilde{W}^E_t \]  \hspace{1cm} (43)

Plugging back this last expression into Definition 1 we get:

\[ d\ln r^E_{t,t} = (r_{H,t} - r_{F,t}) dt + (\kappa_{H,t} - \kappa_{F,t}) d\tilde{W}^E_t \]  \hspace{1cm} (44)
By taking expectations we obtain the result:

$$\mathbb{E}_t^E [d \ln r_{e,t}] = (r_{pH,t} - r_{pF,t}) \, dt$$  \hspace{1cm} (45)
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