Disagreement, Speculation, and Aggregate Investment

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Abstract

We study the effects of speculation caused by heterogeneous beliefs in a dynamic general equilibrium production economy. Speculation leads to speculative aggregate consumption risk since output shocks affect individual consumption shares and aggregate consumption dynamics. Speculative aggregate consumption risk leads to stochastic volatility in aggregate consumption growth, increasing the equity premium and Sharpe ratios and reducing interest rate volatility and price-dividend ratio volatility all relative to an endowment economy with heterogeneous beliefs. Relative to an endowment economy, optimistic investors’ portfolios are less tilted towards stocks and pessimistic investors’ portfolios are more tilted towards stocks because speculative aggregate consumption risk amplifies speculation risk for optimists and causes stock prices to hedge speculation risk for pessimists.
1 Introduction

Financial markets allow investors with diverse beliefs to speculate on those beliefs. As a consequence, equilibrium asset prices should reflect investors’ heterogeneous beliefs. Speculation can then also affect the allocation of physical capital, since firms’ investment decisions depend on asset prices. But existing equilibrium models of differences of opinion extend the standard class of endowment economies of Lucas [1978] and Breeden [1979], in which aggregate consumption and aggregate investment are entirely exogenous by assumption. In these models, asset prices bear all the consequences of beliefs heterogeneity amongst investors, since the dynamics of aggregate output is exogenous.

In a production economy, the allocation of output to aggregate consumption and aggregate investment should be affected by differences of opinion amongst investors in the financial markets. We develop a parsimonious and analytically tractable model to study the impact of differences of opinion and the resulting speculative trade of financial assets on the allocation of aggregate output between aggregate consumption and aggregate investment. We also study equilibrium asset prices, portfolios and financial trade in a production economy with speculation.

We study a dynamic general equilibrium economy with two types of investors having heterogeneous beliefs. Each investor observes aggregate capital and the investment-capital ratio and both investors know that the investment-capital ratio affects the drift of the aggregate capital stock, but they disagree on how effectively investment transforms into installed capital. Accordingly, each investor in isolation would choose a different aggregate investment and aggregate consumption allocation. In the competitive equilibrium with both types of investors, asset prices reflect the beliefs of both types of investors and their respective wealth levels—the firm’s cost of capital therefore depends on the heterogeneous beliefs and investors’s relative wealths. As a consequence, the value maximizing level of investment depends on the investors’ beliefs and the investors’ relative wealths.

When positive shocks hit the economy, investors’ speculation leads to a higher share
of aggregate consumption for optimistic investors and lower aggregate investment so that there is higher level of overall aggregate consumption to be shared among the two investors. The increased share of consumption for optimistic investors occurs in endowment economies is often called sentiment risk (Dumas et al. [2009]). The effect of sentiment on aggregate investment and the dynamics of aggregate consumption is a new channel and it appears exclusively in production economies in which investment is chosen optimally. We call the effects of sentiment on consumption dynamics speculative aggregate consumption risk.

We use a calibrated economy to demonstrate that speculative aggregate consumption risk has important effects on aggregate quantities, asset prices, investors’ portfolios and trade in financial markets. Speculative aggregate consumption risk leads to endogenous stochastic volatility in aggregate consumption. This stochastic volatility increases the equity premium and the Sharpe ratio relative to and endowment economy compared to an equivalent endowment economy. The interest rate, Tobin’s q and the price-dividend ratio become less volatile than in endowment economies with heterogeneous beliefs because aggregate investment and aggregate consumption react to speculation. This is in contrast to the results in Jermann [1998] who shows that interest rates become more volatile and the risk premium lower in production economies with habits than in endowment economies with habits.

We also study the investors’ optimal portfolios. Optimistic investors’ portfolios are less tilted towards stocks and pessimistic investors’ portfolios are more tilted towards stocks than in an equivalent endowment economy because speculative aggregate consumption risk amplifies the overall risk of speculation for the optimists and provides a hedge for the pessimists. The portfolios are less tilted because the investors’ speculation causes positive output shocks to increase in aggregate consumption and to increase in the individual consumption share of the optimists and decrease the consumption share of the pessimists. As a consequence of the impact on portfolios, trade in the stock market and the market-leverage ratio are both lower in the production than in endowment economies.
The model can be consistent with the size, book-to-market and investment-to-assets anomalies. Firms characteristics may appear to be priced because they carry some dimension of the risk introduced by differences of opinion and, in particular, speculative aggregate consumption risk.

Our work is related to the larger literature that studies the asset pricing implications of heterogeneous beliefs in endowments economies. Basak [2005] shows how to characterize asset prices using the martingale approach in an endowment economy; Gallmeyer and Hollifield [2008] study the impact of a short-sales constraint with heterogeneous beliefs in an endowment economy; Kogan et al. [2006] show that heterogeneous beliefs can have a significant impact on asset prices even when irrational investors are a small fraction of the investors in the economy; David [2008] studies an endowment economy with heterogeneous beliefs showing that heterogeneous beliefs can significantly increase the equity premium relative to a homogenous beliefs economy; Dumas et al. [2011] show that heterogeneous beliefs can help explain several puzzles in international finance; and Dumas et al. [2009] shows that sentiment risk can have significant impacts on investors’ optimal portfolios and equilibrium asset prices. All these papers study endowment economies. Detemple and Murthy [1994] study the equilibrium in an economy with heterogeneous beliefs and endogenous output in which all investors have logarithmic utility, which gives fixed equity price to capital ratios. As a consequence of the logarithmic utility, heterogeneous beliefs mainly affect interest rates. We allow for non-logarithmic investors and capital adjustment costs, which allows for time variation in Tobin’s q and price-dividend ratios.

Altı and Tetlock [2013] estimate a structural model of an individual firm’s optimal investment decisions in the presence of heterogeneous beliefs, providing empirical evidence that investors have over-confident and trend following beliefs in a partial equilibrium model, in which there is no feedback from the heterogeneous beliefs to the dynamics of aggregate consumption or aggregate asset prices, nor to investors’ optimal portfolio strategies. We focus the feedback from the heterogeneous beliefs to individual and aggregate consumption,
to aggregate asset prices, and to investors’ optimal portfolio strategies.

2 The model

We study a one sector continuous-time production economy in which we allow for differences of opinion among investors. The economy is a one-sector version of Cox et al. [1985] with capital adjustment costs as in the Eberly and Wang [2011] and modified to include investors with heterogeneous beliefs.

The model is set in continuous time with an infinite horizon. Let $K_t$ denote the representative firm’s capital stock, $I_t$ the aggregate investment rate, and $Y_t$ the aggregate output rate. The representative firm has a constant-returns-to-scale production technology:

$$Y_t = AK_t,$$

with constant coefficient $A > 0$. Capital accumulation has the dynamics:

$$dK_t = \Phi_z(I_t, K_t) dt + \sigma K_t dW^z_t; \quad K_0 > 0,$$

where $\sigma > 0$ is the volatility of capital growth and $W^z_t$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^z)$ with the objective probability measure $\mathcal{P}^z$ governing empirical realizations of the process. The function $\Phi_z(I_t, K_t)$ measures the effectiveness of converting investment goods into installed capital under the objective probability measure.

As in the neoclassical investment literature (i.e. Hayashi [1982]), the firm’s adjustment cost is homogeneous of degree one in $I_t$ and $K_t$:

$$\Phi_z(I_t, K_t) = K_t \phi_z(i_t),$$

with constant coefficient $A > 0$. Capital accumulation has the dynamics:
where \( i_t \equiv \frac{I_t}{K_t} \) is the firm’s investment capital ratio and \( \phi_z(i_t) \) is an increasing and concave function. We use a quadratic adjustment cost function

\[
\phi_z(i_t) = i_t - \frac{1}{2} \theta i_t^2 - \delta_z,
\]

(4)

where \( \theta > 0 \) is the adjustment cost parameter. When \( \theta = 0 \), there are no adjustment costs, the expected growth rate of capital is \( \phi(i_t) = i_t - \delta_z \) and the model is a one-sector Cox et al. [1985] economy.

We denote by \( c_t \equiv \frac{C_t}{K_t} \) the aggregate consumption-capital ratio. The aggregate resource constraint is

\[
c_t + i_t = A.
\]

(5)

The \( AK \) production technology has three useful properties. First, capital growth equals output growth: \( \frac{dK_t}{K_t} = \frac{dY_t}{Y_t} \). Accordingly, we refer to moments of capital growth or moments of output growth interchangeably. Second, investment-capital ratio is proportional to the investment-output ratio: \( \frac{I_t}{Y_t} = \frac{1}{A} \frac{K_t}{K_t} = \frac{1}{A} i_t \). Third, the aggregate consumption-capital ratio is proportional to the aggregate consumption-output ratio: \( \frac{C_t}{Y_t} = \frac{1}{A} \frac{C_t}{K_t} = \frac{1}{A} c_t \).

Investors have different beliefs about the effectiveness of converting investment into installed capital. Upon observation of \( K_t \) and \( i_t \) it is not possible for investors to distinguish whether shifts in capital are driven by Brownian shocks or by the impact of the investment-capital ratio \( i_t \) in the drift of capital growth. In order to capture different beliefs about the technology, investors can disagree about the value of \( \delta_z \).

There are two types of Investors, \( a \) and \( b \). Type \( a \) investors are optimistic because they believe that every unit invested transforms into installed capital at a rate higher than the rate believed by the pessimistic type \( b \) investors. Type \( a \) investors believes the parameter is \( \delta_a \) and type \( b \) investors believes the parameters is \( \delta_b > \delta_a \).\(^1\) Since both types of investors have common information and not asymmetric information, they are aware of each others’

\(^1\)We could allow investors to learn about the parameter \( \delta \) by observing realizations of the capital process. We use dogmatic beliefs for simplicity.
different perception about $\delta_z$, and each investor thinks investors of the other type are wrong. Hence, investors agree to disagree. The true value of the parameter $\delta_z$ lies between both investors’ beliefs:

$$\delta_a < \delta_z < \delta_b. \quad (6)$$

We rewrite the dynamics of the capital stock in terms of investor specific Brownian motions. Define the Brownian motion process $W^a_t$ on the investor $a$ specific filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^a)$ and define the Brownian motion process $W^b_t$ on investor type $b$ specific filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}, \mathcal{P}^b)$. The dynamics of capital under each investors’ beliefs is

$$dK_t = \phi_j(i_t) K_t dt + \sigma K_t dW^j_t; \quad K_0 > 0, \quad (7)$$

where the subjective drift of capital growth is

$$\phi_j(i_t) = i_t - \frac{1}{2} \theta i_t^2 - \delta_j,$$

for $j = \{a, b, z\}$. The relation between the investor specific Brownian motions is:

$$dW^b_t = \bar{\mu} dt + dW^a_t, \quad (8)$$

where

$$\bar{\mu} = \frac{\delta_b - \delta_a}{\sigma} > 0, \quad (9)$$

We call $\bar{\mu}$ disagreement.

Without loss of generality, we use Investor $b$’s probability measure as the reference measure for our analysis. From Equation (8) and Girsanov’s theorem, the change from $b$’s

\footnote{All the equilibrium quantities are the same if we use instead $a$’s probability measure as the reference measure.}
measure to $a$'s measure is given by the exponential martingale $\eta_t$ with dynamics$^3$:

$$d\eta_t = \bar{\mu}\eta_t dW_t^b; \quad \eta_0 = 1. \quad (10)$$

We compute expectations using the process $\eta_t$. For any $T > t$ measurable random variable

$$E_t^a[X_T] = E_t^b\left[\frac{\eta_T}{\eta_t} X_T\right], \quad (11)$$

where $E_t^j$ denotes investor $j$’s conditional expectations.

We call $\eta_t$ sentiment following Dumas et al. [2009]. The role of $\eta_t$ is to show how Investor $a$ over-estimates or under-estimates the probability of a state relative to Investor $b$. Because Investor $b$ is pessimistic, Investor $a$ views positive innovations as more probable than Investor $b$ does. Since $\bar{\mu} > 0$, $K_t$ and $\eta_t$ are positively correlated to innovations in the Brownian motion. Equations (7) and (10) completely characterize the evolution of the economy in the eyes of Investor $b$.

### 3 Equilibrium with agreement

Before presenting the results with differences of opinion, we summarize the model solution when all investors are of the same type $j$ and agree on the value of the parameter $\delta_j$. The $AK$ production technology and the quadratic adjustment cost imply that economies investment opportunities are constant so that the aggregate investment-capital ratio and Tobin’s q are constant over time. We use such a simple benchmark to highlight the dynamic effects of disagreement on the economy.

All investors have power utility with the same risk aversion coefficient $1 - \alpha > 0$ and subjective discount rate $0 < \rho < 1$. We let $\alpha < 0$ so that investors are more risk averse than the case of logarithmic preferences. Assuming complete markets, a competitive equilibrium

$^3$With Gaussian priors and Bayesian learning rather than dogmatic priors, the disagreement process would be deterministic and $\eta_t$ would have a deterministic time-varying diffusion coefficient.
allocation is the solution to the planner’s problem:

\[
\sup_{c_t} \mathbb{E}_0^j \left[ \int_0^\infty e^{-\rho t} \frac{1}{\alpha} (K_t c_t)^\alpha \, dt \right],
\]

subject to the capital accumulation rule in Equation (7) and to the aggregate resource constraint in Equation (5).

The associated social planner value function is:

\[
V(K_t) = \Lambda \frac{1}{\alpha} K_t^\alpha.
\]

The Appendix reports the constant \( \Lambda \); the solution for the investment-capital ratio \( i_j^* \) is

\[
i_j^* = A + \frac{1-\alpha}{\vartheta} - \frac{\sqrt{(A - \frac{1-\alpha}{\vartheta})^2 + 2\frac{2-\alpha}{\vartheta} \left( \rho - \alpha \left( \frac{A}{2-\alpha} - (\delta_j + \frac{1}{2} (1-\alpha) \sigma^2) \right) \right)}}{2-\alpha}.
\]

and, from the aggregate resource constraint in Equation (5), the aggregate consumption-capital ratio is \( c_j^* = A - i_j^* \).

Because \( i_j^* \) and \( c_j^* \) are constant, the equilibrium capital stock follows a geometric Brownian motion. Since aggregate output, aggregate consumption, and aggregate investment are all proportional to the capital stock, these aggregate quantities also follow geometric Brownian motions:

\[
\frac{dY_t}{Y_t} = \frac{dK_t}{K_t} = \frac{dC_t}{C_t} = \frac{dI_t}{I_t} = \phi_j (i_j^*) \, dt + \sigma dW_t^j; \quad C_0 = (A - i_j^*) K_0 > 0.
\]

The dynamics of all aggregate quantities depend on the underlying technology and preferences through the investors’ optimal capital allocation decision in Equations (14) and (15).

The equilibrium interest rate, market price of risk and Tobin’s q are all constant.

\[4\]Given power utility and the linearly homogenous capital accumulation process, the value function is homogenous of degree \( \alpha \) in capital.
4 Equilibrium with disagreement

Following Basak [2005], we compute the competitive equilibrium from the solution to a planner’s problem. The Appendix shows that the planner’s problem is a weighted average of the expected utility of each investor:

\[
\sup_{c_{a,t} + c_{b,t} = K_t} \left\{ \mathbb{E}_0^b \left[ \int_0^\infty e^{-\rho t} \frac{1}{\alpha} c_{a,t} \alpha \right] + \frac{\lambda}{\alpha} \right\},
\]

(16)

where \( \lambda \) is the initial weight of the planner on Investor \( a \), and subject to the capital accumulation rule in Equation (7) and to the aggregate resource constraint in Equation (5). Using the change of measure process \( \eta_t \), the planner’s problem under \( b \)'s probability measure is

\[
\sup_{c_{a,t} + c_{b,t} = K_t} \left\{ \mathbb{E}_0^b \left[ \int_0^\infty e^{-\rho t} \frac{1}{\alpha} c_{b,t} \alpha \right] + \frac{\lambda}{\alpha} \right\}.
\]

(17)

The objective function is maximized subject to the aggregate resource constraint in Equation (5), the capital accumulation rule in Equation (7), and the sentiment dynamics in Equation (10). There are two dimensions of this optimization problem: The optimal individual consumption allocation rule among the two investors, and the optimal capital allocation between investment and aggregate consumption. We consider first the individual optimal consumption allocation, given a level of aggregate consumption-capital ratio \( c(\eta_t) \).

The Appendix shows that the optimal consumption sharing rule for investors in the two groups, for any level of aggregate consumption-capital ratio \( c(\eta_t) \), is:

\[
c_{a,t} = \omega(\eta_t) c(\eta_t) K_t; \quad c_{b,t} = [1 - \omega(\eta_t)] c(\eta_t) K_t,
\]

(18)

where \( c(\eta_t) K_t \) is aggregate consumption at \( t \) and \( \omega_t \) is the consumption share of the optimist

\(^5\)This initial weight depends on the relative initial endowments of the two investors.
Investor $a$, equal to
\[
\omega_t = \omega(\eta_t) \equiv \left( \frac{\lambda t}{\mu_t} \right)^{\frac{1}{1-\alpha}} \frac{1}{1 + \left( \frac{\lambda t}{\mu_t} \right)^{\frac{1}{1-\alpha}}}.
\] (19)

From equation (19) the consumption share of Investor $a$, $\omega$ is monotonically related to the change of measure $\eta$. We therefore can express the equilibrium in terms of $\omega$ because of its convenient domain, $\omega_t \in (0, 1)$. Applying Ito’s Lemma and using the investor specific Brownian motions, for $j = \{a, b, z\}$, $\omega_t$ evolves as
\[
d\omega_t = \mu^j_\omega(\omega_t) \, dt + \sigma_\omega(\omega_t) \, dW^j_t,
\] (20)
where
\[
\mu^b_\omega(\omega_t) = -\frac{1}{2} \left( \frac{1}{1-\alpha} \bar{\mu} \right)^2 \omega_t (1 - \omega_t) (2\omega_t - \alpha),
\] (21)
\[
\mu^a_\omega(\omega_t) = \mu^b_\omega(\omega_t) + \bar{\mu} \sigma_\omega(\omega_t), \quad \mu^z_\omega(\omega_t) = \mu^b_{\omega,t} + \frac{1}{2} \bar{\mu} \sigma_\omega(\omega_t),
\] (22)
and
\[
\sigma_\omega(\omega_t) = \frac{1}{1-\alpha} \omega_t (1 - \omega_t) \bar{\mu}.
\] (23)
The drift and volatility of $\omega_t$ are analyzed in Section 6.

We now turn to the solution of the optimal capital allocation problem. In other words, we now obtain the optimal investment-capital ratio $i_t^*$ and aggregate consumption-capital ratio $c_t^*$ as a function of $\omega_t$. In order to characterize the capital allocation, we plug the optimal individual consumption allocation in Equation (18) into the planner’s problem spelled out in Equation (17), subject to the aggregate resource constraint in Equation (5), the capital accumulation rule in Equation (7), and the sentiment dynamics in Equation (10). We conjecture and the value function is given by:
\[
V(K_t, \omega_t) = \frac{1}{(1 - \omega_t)^{1-\alpha}} \frac{1}{\alpha} H(\omega_t) \frac{1}{\alpha} K_t^\alpha.
\] (24)
The solution is a set of functions $i(\omega_t)$, $c(\omega_t)$ and $H(\omega_t)$ satisfying the first order condition, the Hamilton-Jacobi-Bellman equation, the aggregate resource constraint and the boundary conditions. We report the first order condition and Hamilton-Jacobi-Bellman equation in the Appendix.

The boundary conditions are:

$$
\lim_{\omega_t \to 0} H(\omega_t) = \frac{1}{(A - i_b^*)^{1-\alpha} (1 - \theta i_b^*)}; \quad \lim_{\omega_t \to 1} H(\omega_t) = \frac{1}{(A - i_a^*)^{1-\alpha} (1 - \theta i_a^*)},
$$

(25)

where $i_a^*$ and $i_b^*$ are the constant investment-capital ratios that are consistent with homogeneous beliefs economies populated only by Investor $a$ or only by Investor $b$. The boundary conditions are such that when the consumption share of Investor $a$ tends to zero or one, we converge to the homogeneous beliefs solution for each type of investor. We solve numerically for the functions $i(\omega_t)$, $c(\omega_t)$ and $H(\omega_t)$ consistent with optimality.

Using the solution for $c(\omega_t)$, we apply Itô’s Lemma to characterize the dynamics of aggregate consumption $C_t = c(\omega_t)K_t$. The process under type $j$ Investor’s beliefs is

$$
dC_t = \mu^j_C(\omega_t)C_t dt + \left( \sigma + \frac{c'(\omega_t)}{c(\omega_t)} \sigma_\omega(\omega_t) \right) C_t dW^j,
$$

(26)

where $c'(\omega_t)$ is the derivative of the consumption-capital ratio with respect to $\omega$.

From equation (26), aggregate consumption growth volatility is stochastic, being driven not only by the constant diffusion of capital growth $\sigma$, but also by the aggregate consumption-capital ratio policy and the diffusion of the consumption share, $\frac{c'(\omega_t)}{c(\omega_t)} \sigma_\omega(\omega_t)$. Positive Brownian shocks increase not only capital but also change the allocation of that capital to aggregate consumption $c(\omega_t)$. The endogenous investment/consumption channel is the impact of speculation on aggregate consumption risk.

From Equation (18), the optimistic investor’s consumption reacts to capital shocks through two channels. The first channel is that $\omega_t$ is positively correlated to the Brownian shocks: investor $a$’s share of aggregate consumption is positively correlated to the Brownian shocks.
The first channel occurs in endowment economies with heterogenous beliefs. The second channel is that aggregate consumption depends on the consumption share itself. In the numerical example in Section 6, the aggregate consumption-capital ratio is an increasing function of the consumption share $\omega_t$. From Equation (26), this implies that aggregate consumption has a positive diffusion.

The second channel does not occur in endowment economies with heterogeneous beliefs and is a consequence of the impact of sentiment on capital allocation in our production economy. We call the second channel speculative aggregate consumption risk. In production economies the impact of sentiment on aggregate consumption adds a new dimension of risk to speculation: By speculating on their beliefs, investors place bets not only on their shares of aggregate consumption, but also on the aggregate consumption that is being shared.

The state price density under the reference measure of Investor $b$, $\xi^b_t$, is obtained from the marginal period-utility of the planner with respect to aggregate consumption, and is

$$\xi^b_t = e^{-\rho t} \{ (1 - \omega_t) c(\omega_t) K_t \}^{\alpha - 1}. \quad (27)$$

The state price density captures the fundamental risk embedded in $K_t$ and it also incorporates a sentiment risk factor. As $\eta_t$ fluctuates, the consumption share $\omega_t$ fluctuates, affecting the pricing of financial securities. As it was the case for individual consumption, the state price density $\xi^b_t$ is affected by sentiment through two channels: individual consumption shares $(1 - \omega_t)$, and the aggregate consumption-capital ratio $c(\omega_t)$. Accordingly, the pricing of all financial assets are affected by sentiment through the optimal capital allocation embedded in the associated aggregate consumption-capital ratio policy $c(\omega_t)$.

Because there is only one Brownian motion under any investors’ subjective probability measure, two linearly independent securities are required for a complete market to implement the Pareto optimal allocation. We assume that there is a locally riskless bond with endogenous rate of interest $r_t$ and also a stock with endogenous value $P_t$ that pays a dividend
stream equal to aggregate consumption $C_t$. That makes two securities, one of them being instantly risky meaning that we can implement the competitive equilibrium with these securities.\footnote{We require that the stock has non-zero diffusion in order to implement the equilibrium. We verify that stock volatility is non-zero in our numerical implementation.}

The equilibrium price of equity, or the total wealth of the economy is the discounted sum of all future dividends:

$$P_t = \mathbb{E}_t^b \left[ \int_t^\infty \frac{b}{\xi_t^b} C_u du \right].$$

(28)

Tobin’s $q$ is defined as ratio of the market value of the firm $P_t$ and the book value of the firm $K_t$ and therefore, its inverse is the book-to-market ratio. Tobin’s $q$ satisfies the optimality condition:\footnote{We emphasize that $\phi'$ is not indexed by $j$ because $\phi'$ is independent of $\delta_j$.}

$$q (\omega_t) = \frac{1}{\phi' (i (\omega_t))}. \quad (29)$$

The capital stock increases by $\phi' (i)$ per marginal unit of investment, and each unit of capital is valued at $q$. The firm optimally chooses investment to equate $\phi' (i) q (i)$ to unity—the marginal cost of the investment. Because $\phi$ is concave in $i$ as a consequence of adjustment costs, $q$ is increasing in $i$. Using the quadratic specification for $\phi$,

$$q (\omega_t) = \frac{1}{1 - \theta i (\omega_t)}, \quad (30)$$

so that the price dividend ratio is:

$$\frac{P_t}{C_t} = q (\omega_t) \times \left( \frac{C_t}{K_t} \right)^{-1} = \frac{1}{[A - i (\omega_t)] [1 - \theta i (\omega_t)]}. \quad (31)$$

Starting from the equilibrium pricing measure in Equation (27) and the aggregate resource constraint in Equation (5), we use Ito’s lemma to obtain the interest rate and market
prices of risk. The interest rate \( r_t \) is

\[
\begin{align*}
\dot{r}(\omega_t) &= \rho + (1 - \alpha) \phi_b (A - c(\omega_t)) - \frac{1}{2} (1 - \alpha) (2 - \alpha) \sigma^2 \\
&\quad + (1 - \alpha) \omega_t \bar{\mu} \sigma - \frac{1}{2} \frac{\alpha}{1 - \alpha} \omega_t (1 - \omega_t) \bar{\mu}^2 \\
&\quad - (1 - \omega_t) \frac{c'(\omega_t)}{c(\omega_t)} \left\{ (1 - \alpha) \omega_t \bar{\mu} \sigma - \frac{1}{2} \frac{\alpha}{1 - \alpha} \omega_t (1 - \omega_t) \bar{\mu}^2 \\
&\quad \times \left[ \frac{1}{\alpha} \omega_t \left( (2 - \alpha) \frac{c'(\omega_t)}{c(\omega_t)} - \frac{c''(\omega_t)}{c'(\omega_t)} \right) - \frac{1 - 2 \omega_t}{1 - \omega_t} \right] \right\} \quad (32)
\end{align*}
\]

The interest rate has the familiar structure from equilibrium models of disagreement in endowment economies adjusted for our endogenous consumption process. On the first line, the first term captures time preference, the second term is the standard wealth effect, and the third term is the standard precautionary saving effect. The second line includes terms driven by disagreement and speculation that are standard in endowment economies. The third and fourth lines incorporate additional terms that appear as a consequence of speculative aggregate consumption risk in our production economy. This risk generates endogenous variation in the aggregate consumption-capital ratio \( c(\omega_t) \), as well as its first and second derivatives.

The market price of risk for investor \( b \) is

\[
\kappa_{b,t}(\omega_t) = (1 - \alpha) \left( \sigma + \frac{c'(\omega_t)}{c(\omega_t)} \sigma, (\omega_t) \right) - \omega_t \bar{\mu}, \quad (33)
\]

and the market price of risk for investor \( a \) is

\[
\kappa_{a,t}(\omega_t) = (1 - \alpha) \left( \sigma + \frac{c'(\omega_t)}{c(\omega_t)} \sigma, (\omega_t) \right) + (1 - \omega_t) \bar{\mu}. \quad (34)
\]

This form of the market price of risk is consistent with that in endowment economies with disagreement. As it is the case with endowment economies, the first term represents the market price of aggregate consumption risk, and the second term represents the market price.
of sentiment risk. The main difference is that the market price of aggregate consumption risk is now driven by sentiment itself, due to the endogenous stochastic volatility of aggregate consumption. Importantly, this additional source of risk increases the market price of risk relative to the endowment economy from the perspective of both investors by same magnitude.

We compute the investors’ portfolios in two steps. In the first step, we use the solution to the central planner’s problem to obtain the process for each individual investor’s optimal wealth process. In the second step, we compute the self-financing portfolio strategy that replicates the wealth processes. Details are given in the Appendix.

We solve for the optimal individual wealth of Investor $b$ and obtain the optimal individual wealth of Investor $a$ from market clearing. We interpret Investor $b$’s wealth, $X_{b,t}$, as a security that pays a dividend at a rate equal to his individual consumption, $c_{b,t}$. The investors’ wealths are

$$X_{b,t} = D_b(\omega_t) K_t \quad \text{and} \quad X_{a,t} = (q(\omega_t) - D_b(\omega_t)) K_t,$$

where $D_b(\omega_t)$ is the solution to an ordinary differential equation reported in the Appendix, and investor $a$ wealth follows from market clearing. The boundary conditions for the differential equation are that when the consumption share of Investor $a$ tends to zero, Investor $b$’s individual wealth converges to the value of equity in a homogeneous beliefs economy dominated by type $b$ investors. When the consumption share of Investor $a$ tends to one, Investor $b$’s wealth converges to zero. We solve numerically for the process $D_{b,t}$.

Following Cox and Huang [1989], the desired holding of equity for each investor can be calculated from the ratio of each investor’s individual wealth diffusion and the stock price diffusion. Investor $b$’s holdings of equity shares is

$$\zeta_{P_t}^b = \frac{D_b(\omega_t) \sigma + D'_b(\omega_t) \sigma_\omega(\omega_t)}{q(\omega_t) \sigma + q'(\omega_t) \sigma_\omega(\omega_t)},$$

(36)
and Investor \( b \)'s holdings of the locally riskless bond follow from the budget constraint:

\[
\zeta^b_t = X_{b,t} - \zeta^b_t P_t. 
\]

5 An equivalent endowment economy

In order to highlight how speculation driven by differences of opinions affects the equilibrium when the capital allocation is endogenous, we compare the outcomes to a similar endowment economy with disagreement. Such an economy has been extensively used to analyze the effects of heterogeneous beliefs on equilibrium assets prices and trading, for example by Basak [2005] or Dumas et al. [2009]. With this aim, we define an equivalent endowment economy in which the investment-capital ratio is fixed. While any fixed level for the investment-capital ratio would be compatible with an endowment economy, we select the level of the investment-capital ratio that corresponds to the optimal level in an homogeneous beliefs economy populated by investors who know the true value of \( \delta_z \). That way, our benchmark economy is not tilted towards the beliefs of any of the two investors. Following the results in Equation (14), our choice for the fixed-investment capital ratio in an equivalent endowment economy is

\[
i^*_z = \frac{A + \frac{1-\alpha}{\theta} - \sqrt{(A - \frac{1-\alpha}{\theta})^2 + 2\frac{2-\alpha}{\theta} (\rho - \alpha \left[ \frac{A}{2-\alpha} - \left( \frac{\delta_z}{2} + \frac{1}{2} (1 - \alpha) \sigma^2 \right) \right]}}{2 - \alpha}, \tag{37}
\]

from the aggregate resource constraint in Equation (5), the aggregate consumption-capital ratio is \( c^*_z = A - i^*_z \). Given the fixed investment-capital ratio \( i^*_z \) and aggregate consumption-capital ratio \( c^*_z \), exogenous aggregate consumption follows a Geometric Brownian motion:

\[
dC_t = \phi (i^*_z) C_t dt + \sigma C_t dW^z_t; \quad C_0 = (A - i^*_z) K_0 > 0. \tag{38}
\]

Here, we let investors disagree in exactly the same way as before but, because the
investment-capital ratio $i^*_z$ and the aggregate consumption-capital ratio $c^*_z$ are both now fixed, such disagreement does not affect the dynamics of aggregate consumption so that the model is equivalent to one in which aggregate consumption is exogenous and investors disagree about the expected growth rate of aggregate consumption. The solution to this restricted model is standard in the literature.\textsuperscript{8}

\section{Quantitative analysis}

To illustrate the effect of disagreement on investment, consumption allocations, asset prices and portfolios, we present a numerical example. Even though our goal is not to match the magnitude of particular moments in the data, we would like to work with reasonable parameter values. We specify parameter values based on the calibration in Eberly and Wang [2011] for an economy similar to ours but with agreement. Our parameter values are given in Table 1.

As a first exercise, we compare economies populated by investors with different benchmark beliefs. Table 2 reports some basic statistics for the benchmark economies under agreement and according to objective ($z$), pessimist ($b$) or optimist ($a$) benchmark beliefs. In each economy, the objective beliefs are fixed, and we change the beliefs of the investors populating the economy in each column. In the first column, all investors have objective beliefs, in the second column all investors have pessimistic beliefs and in the third column, all investors have optimistic beliefs. The first row reports the endogenous investment-capital ratio in each economy; the second row reports the expected output growth under the objective measure with the endogenous capital-investment ratio; the third row reports the expected output under the investors’ subjective beliefs; the fourth row reports the Tobin’s q in each economy; the fifth row reports the interest rate; and the final row reports the equity premium under the investor’s subjective beliefs.

More optimistic investors believe investment transforms into capital at a higher rate

\textsuperscript{8}See Basak [2005] and Dumas et al. [2009].
than the true one. Accordingly, more optimistic investors choose to allocate less capital to investment and more to consumption, as shown in the first row in Table 2. Considering only this effect, an economy populated only by optimistic investors should be one in which there is lower capital, consumption and output growth compared to the benchmark case of objective beliefs, as shown in the second row of Table 2. However, these expectations are affected when we consider investors’ subjective beliefs about the rate at which investment transforms into capital. For instance, the third row in Table 2 shows that when subjective expectations are considered, the more optimistic the investor, the higher the expected output growth is under their subjective beliefs.

As explained in Section 4, Tobin’s $q$ is increasing in the investment-capital ratio $i$. As a result, economies populated only by optimists are characterized by lower investment-capital ratios and lower Tobin’s $q$. The fourth row of Table 2 shows that this is indeed the case.

In economies of full agreement, investors’ perceptions affect the interest rate through the standard wealth effect. Accordingly, in economies populated by optimists the subjective expected growth rate of consumption is higher, so that investors are less willing to save for the future, leading to a higher interest rate. Finally, the equity premium is just the product of risk aversion and the variance of consumption growth, which is the same in all cases considered.

6.1 The effects of heterogeneous beliefs

In our equilibrium model of disagreement, the consumption share of optimist Investor $a$, $\omega_t$ driven by sentiment $\eta_t$ is the single state variable that characterizes the equilibrium because all variables are proportional to the capital stock. Accordingly, we start our analysis by examining the dynamics of the consumption share. The dynamics of all the equilibrium prices, quantities and portfolios are related to the dynamics of $\omega_t$.

The top panel of Figure 1 plots the drift of the consumption share $\mu^j_\omega$ using each of the
subjective probability measures and the bottom panel plots the volatility $\sigma_\omega$.\textsuperscript{9} From the Figure, every investor believes his own consumption share will increase.\textsuperscript{10} Each investor believes his consumption share will increase because because the investors agree to disagree: each investor believes himself to be correct and that the other investor is wrong. When an investor has a small consumption share, he expects his consumption share to increase steeply when his current consumption share is less than 0.5 and expects his consumption share to increase but at a slower rate if the current consumption share is greater than 0.5. Under the objective measure, the drift in the consumption share generates a tendency for $\omega_t$ to revert to 0.5: the drift of $\omega_t$ is positive when $\omega_t < 0.5$ and negative when $\omega_t > 0.5$. Because we have chosen the objective measure to be centered between the two subjective measures, the drift is such that neither investor type is eliminated in the short run.

The consumption share volatility $\sigma_\omega$ is a quadratic function of $\omega_t$, attaining its highest value at $\omega_t = 0.5$ and tends to zero when $\omega_t$ tends to zero or one. When both investors have a similar consumption share, shocks generate bigger swings in the consumption shares. As is generally the case in equilibrium models of disagreement with expected utility, the consumption share $\omega_t$ does not have a stationary distribution.\textsuperscript{11}

Our goal is to understand how sentiment affects the optimal capital allocation between investment and aggregate consumption. Sentiment drives the optimal capital allocation by affecting both the investment-capital ratio $i(\omega_t)$ and the aggregate consumption-capital ratio $c(\omega_t) = A - i(\omega_t)$. Recall that the AK production technology implies that the aggregate consumption-capital ratio is proportional to the investment-output ratio: $\frac{C_t}{Y_t} = \frac{1}{A} \frac{C_t}{K_t} = \frac{1}{A} c_t$. Hence, the implications for the impact of sentiment on $c_t$ also apply to the impact of sentiment on $\frac{C_t}{Y_t}$.

Figure 2 plots the investment-capital ratio $i(\omega_t)$ and the aggregate consumption-capital

\textsuperscript{9}The expressions for $\mu_\omega$ and $\sigma_\omega$ are given in Equations (21) - (23).
\textsuperscript{10}Investor $b$ believes the drift of $\omega$, $\mu_\omega$, is negative means that he believes the drift of his own consumption share $1 - \omega$ is positive.
\textsuperscript{11}Borovička [2013] shows that a stationary distribution is possible with recursive preferences.
ratio $c(\omega_t)$, as well as some of their moments. There are two lines in all the plots in the figure. The solid line corresponds to our production economy, and the dotted line corresponds to an equivalent endowment economy.

In endowment economies, sentiment has no impact on the investment-capital ratio $i_t$ and the aggregate consumption-capital ratio $c_t$, since they are fixed by assumption. But in the production economy, sentiment has important implications for the dynamics of the investment-capital ratio $i(\omega_t)$ and the aggregate consumption-capital ratio $c(\omega_t)$. The general intuition is similar to that in endowment economies: the optimist provides insurance to the pessimist against adverse capital shocks. Accordingly, the individual consumption of the optimist increases when there are positive shocks to capital. The key difference in the production economy relative to the endowment economy is that individual consumptions of the two investors are affected by sentiment not only through swings in their consumption shares of aggregate consumption, but also through swings on aggregate consumption itself. The top panel of Figure 2 makes this point clear: the aggregate consumption-capital ratio $c(\omega_t)$ is increasing in the consumption share of optimists, $\omega_t$, and the investment-capital ratio $i(\omega_t)$ is decreasing in $\omega_t$. Because positive shocks to aggregate uncertainty increase simultaneously both $K_t$ and $c(\omega_t) = \frac{C_t}{K_t}$, it follows that because of sentiment, aggregate consumption $C_t$ becomes procyclical. Speculation among investors is now equivalent not only to bets on the shares of consumption, but also to bets on individual consumption, including changes on the aggregate consumption to be shared. It is this latter new dimension of aggregate risk implied by sentiment in production economies that provides new implications for asset prices, portfolios, consumption and investment.

We now turn to study the impact of sentiment on the first and second moments of the investment-capital ratio $i(\omega_t)$, the aggregate consumption-capital ratio $c(\omega_t)$, and aggregate consumption. The first and second moments are computed using Ito’s lemma, and the formulas are reported in the Appendix.

\footnote{The drifts and diffusions of functions of $\omega_t$ are computed using Ito’s lemma.}
Figure 2 plots the drift of the aggregate consumption-capital ratio $\mu_{c,t}$ using each of the subjective probability measures in the center panel, as well as the drift under the objective measure and the volatility $\sigma_{c,t}$ in the bottom panel. Note the similarity of these drifts and volatility to the drifts and volatility of the consumption share $\omega_t$ in Figure 1. In fact, because the aggregate consumption-capital ratio $c(\omega_t)$ is almost linearly increasing in $\omega_t$ as shown in the top panel of Figure 2, the intuition for the drift and volatility of the aggregate consumption-capital ratio is very similar to that of the drift and volatility of the consumption share $\omega_t$. As explained above, the dynamics of the drift of $\omega_t$ follow from investors’ agreement to disagree. This same intuition drives each investor’s perceived drift of $c(\omega_t)$. Because each investor believes his consumption share $\omega_t$ will increase over time and knowing that the aggregate consumption-capital ratio $c(\omega_t)$ is increasing in $\omega_t$, optimists expect $c(\omega_t)$ to increase and pessimists expect $c(\omega_t)$ to decrease.

The bottom panel of Figure 2 shows that the volatility of the aggregate consumption-capital ratio $c(\omega_t)$ is similar to the volatility of $\omega_t$ in Figure 1, because the consumption-capital ratio is almost linearly increasing in $\omega_t$. Even though our model for differences of opinion is very simple, it generates stochastic volatility in the aggregate consumption-capital ratio $c(\omega_t)$, despite the constant volatility of capital. This stochastic volatility is driven by the speculative aggregate consumption risk induced by sentiment, as described above. The $AK$ aggregate production function implies that the volatility of the investment-capital ratio $i(\omega_t)$ must reduce in order to keep the output-capital ratio constant at $A$.

The dynamics of investment and aggregate consumption depend on the dynamics of capital $K_t$, as well as on the dynamics of the aggregate consumption-capital ratio $c(\omega_t)$ and the investment-capital ratio $i(\omega_t)$. We report formulas for their dynamics in the Appendix.

In the top panel of Figure (3) the solid line plots the drift of capital growth $\phi_j(i(\omega_t))$, the dotted-dashed line is the drift of aggregate consumption growth $\mu^j_{C,t}$ and the dashed

\[\text{21}\]

\begin{footnote}{From Ito’s lemma, the instantaneous volatility of any smooth function $f(\omega_t)$ is given by $f'(\omega_t)\sigma_{\omega,t}$. If $\sigma_{\omega,t}$ has an inverted U-shape in terms of $\omega_t$, the volatility of $f(\omega_t)$ will tend to retain a U-shape when $f'(\omega_t) > 0$ and will have an inverted U-shape when $f'(\omega_t) < 0$, assuming the function $f(\omega_t)$ is close to linear.}

line is the drift of investment growth $\mu_{I,t}^j$, under the measure of every investor $j$. It is clear that both the drift of aggregate consumption growth $\mu_{C,t}^j$ as well as drift of investment growth $\mu_{I,t}^j$ are driven almost entirely by the impact of the drift of capital growth $\phi_j(i(\omega_t))$. This is reasonable, considering that the drift of capital growth $\phi_j(i(\omega_t))$ is affected by the investment policy $i(\omega_t)$, which is itself driven by sentiment $\omega_t$. Comparing the views of the pessimist Investor $b$ in the left plot and the optimist Investor $a$ in the right plot, we see that the optimist not only believes that output growth will be higher, but he also believes that aggregate consumption growth will be even higher, while investment growth will be relatively lower. The intuition is again driven by the fact that investors agree to disagree.

For instance, the optimist expects that positive shocks will generate higher aggregate consumption and relatively lower investment, for every unit of capital. By contrast, the pessimist expects a lower output growth and an even lower aggregate consumption growth: he expects negative shocks will generate higher investment and relatively lower aggregate consumption for every unit of capital. In the left plot on the bottom panel we note that under the objective measure the intuition above still goes through. We get an intermediate level of expected output growth and, given our chosen parameters, the expected aggregate consumption growth is relatively higher than expected investment growth under the objective measure.

In the right plot of the bottom panel of Figure (3) we show aggregate consumption growth volatility $\sigma_{C,t}$. The solid line corresponds to our production economy and the dashed line to an equivalent endowment economy. In the production economy aggregate consumption volatility behaves similarly to aggregate consumption-capital ratio volatility in the bottom panel of Figure 2, implying that capital growth volatility has a relatively lower impact than the impact on the drifts. The reason is that the capital growth volatility $\sigma$ is constant. Hence, the stochastic aggregate consumption growth volatility in our model is entirely driven by the impact of sentiment on the aggregate consumption-capital ratio $c(\omega_t)$. Similarly, investment growth volatility is mainly driven by the investment-capital ratio volatility and therefore
is close to $-\sigma_{C,t}$. In bad times when $\omega_t$ is low, aggregate consumption growth volatility is procyclical and investment growth volatility is countercyclical. In good times, when $\omega_t$ is high aggregate consumption growth volatility is countercyclical and investment growth volatility is procyclical.

The top panel of Figure 4 plots the interest rate as well as its volatility. The solid line corresponds to our production economy while the dotted line plots the same objects in the case of an equivalent endowment economy. The endogenous variation in the aggregate consumption-capital ratio $c(\omega_t)$ leads to an interest rate that is less volatile than in equivalent endowment economies. For instance, given our calibration, an equivalent endowment economy would imply that the interest rate lies in the range of $[-1.4\%, 8.6\%]$, while in our production economy we obtain that the interest rate fluctuates within the more realistic range of $[1.6\%, 5.6\%]$ as indicated in Table 1. The interest rate volatility adopts the inverted U-shape in terms of $\omega_t$ from the volatility of $\omega_t$, because the interest rate is almost linearly increasing in $\omega_t$.

The market prices of risk in our production economy have the same structure as in endowment economies. The key difference is that aggregate consumption risk $\sigma_{C,t}(\omega_t)$ now incorporates speculative aggregate consumption risk driven by sentiment. The bottom panel of Figure 4 plots the market prices of risk, and it illustrates how risk is transferred from the pessimist to the optimist as is standard in endowment economies with heterogeneous beliefs. The plots also highlight how both investors face an additional speculative aggregate consumption risk driven by the impact of sentiment on the consumption-capital ratio described above.

To further illustrate this point, we derive from the market prices of risk a CAPM for every investor. Instead of the standard consumption CAPM, we disentangle aggregate consumption risk into fundamental capital growth risk and sentiment risk, including speculative aggregate consumption risk driven by sentiment. For any asset $S$, the following CAPM holds for investor $b$. 

23
\[ \mu_b^b - r = (1 - \alpha) \left\{ \text{Cov} \left( \frac{dS_t}{S_t}, \frac{dK_t}{K_t} \right) + \left[ -\frac{\omega_t}{1 - \omega_t} + \frac{c'(\omega_t)}{c(\omega_t)} \right] \text{Cov} \left( \frac{dS_t}{S_t}, \frac{d\omega_t}{\omega_t} \right) \right\}. \] (39)

A similar pricing formula for Investor \( a \).

The first term is the standard aggregate risk factor \( \frac{dK_t}{K_t} = \frac{dY_t}{Y_t} \), any risky security’s risk premium is positively related to the covariance of its return with the change in the fundamental risk factor equal to aggregate output growth. In endowment economies this term is aggregate consumption risk \( \frac{dK_t}{K_t} = \frac{dC_t}{C_t} \), as a consequence of the fixed aggregate consumption-capital ratio. We focus on the expression in square brackets multiplying the sentiment risk factor.

We interpret the first term in the squared bracket as follows. For the pessimistic Investor \( b \), an increase in \( \omega_t \) is unfavorable because his consumption share decreases as a consequence of losses from speculation. Accordingly, he would be willing to pay an insurance premium, or a negative price of risk for assets positively correlated with \( \frac{d\omega_t}{\omega_t} \). For optimist Investor \( a \), an increase in \( \omega_t \) is favorable because his consumption share increases as a consequence of gains from speculation. Hence, he would require a risk premium, or positive price of risk, for assets positively correlated with \( \frac{d\omega_t}{\omega_t} \).

The last term in the squared bracket does not occur in to the endowment economy, and stems from the pricing of sentiment driven by speculative aggregate consumption risk. In our model both investors require an additional risk premium for any asset positively correlated with \( \frac{d\omega_t}{\omega_t} \) because in states in which \( \omega_t \) increases, aggregate consumption increases for every unit of capital. As long as the risk of capital is already priced in the first component in the CAPM \( \left( \frac{dK_t}{K_t} = \frac{dY_t}{Y_t} \right) \), the last term incorporates the risk of changes in aggregate consumption for a given level of capital that are driven by sentiment only.

Our model of disagreement in a production economy identifies additional terms in the standard CAPM that account for the risk of disagreement and speculation of beliefs through
the impact on the aggregate consumption-investment decision. At the same time, our model generates less volatile interest rates and a higher equity premium. Consequently, we believe models of disagreement and sentiment risk in which capital allocation is endogenous may help to better reconcile the data with the well-known equity premium and risk-free rate puzzles.

6.2 Asset prices

We now examine how disagreement affects the behavior of asset prices. The top panel of Figure 5 plots Tobin’s q and the price-dividend ratio. As in previous plots, the solid line corresponds to our production economy while the dotted line plots the same objects in the case of an equivalent endowment economy. As we have argued before, the optimality of investment and the concavity of adjustment costs \( \phi_j (i_t) \) implies that Tobin’s q must be positively related to the investment-capital ratio. For that reason, q is decreasing in \( \omega_t \). As shown in Equations (30) and (31), the price-dividend ratio equals q divided by the aggregate consumption-capital ratio \( c (\omega_t) \). Because the consumption-capital ratio is increasing in \( \omega_t \), the price-dividend ratio must be decreasing in \( \omega_t \). Accordingly, in our production economy disagreement affects both Tobin’s q and the price-dividend ratio through \( c (\omega_t) \) and \( i (\omega_t) \).

In endowment economies, the impact of disagreement is rather different. For instance, in endowment economies, the price-dividend ratio is obtained directly from discounting future cash flows incorporating the expected future dynamics of the consumption shares \( \omega_t \) knowing that the investment-capital ratio \( i (\omega_t) \) and aggregate consumption-capital ratio \( c (\omega_t) \) will remain fixed along every path. Precisely because these remain fixed, the price reaction to current and future shifts in \( \omega_t \) must be larger, as is apparent from the top panel in Figure 5.

The center panel in Figure 5 plots the volatility of the growth in Tobin’s q as well as the stock return volatility. Given that q is (almost linearly) decreasing in \( \omega_t \), the volatility of the growth in Tobin’s q has an inverted U-shape when plotted against \( \omega_t \), similar to \( \sigma_{\omega_t} (\omega_t) \). Because the stock price \( P_t \) equals \( q \times K_t \) and capital growth volatility is constant, stock
return volatility has the inverted U-shape when plotted against $\omega_t$. The top panel shows that the slope of Tobin’s $q$, as well the slope of the price-dividend ratio with respect to $\omega_t$ are much higher in the endowment economy than in the production economy, because of the high reaction of prices as a consequence of the fixed aggregate consumption-capital ratio, as discussed above. These higher slopes lead to higher signed volatility effects. For instance, a given positive Brownian shock increases the consumption share, which decreases Tobin’s $q$ and the price-dividend ratio by much more in an endowment economy than in a production economy.

The bottom panel on Figure 5 plots the equity premium under the beliefs of every investor. The equity premium is obtained by multiplying the investor’s market price of risk times the stock return volatility. As explained above, production economies have a higher market price of risk, and a higher stock return volatility than endowment economies. Therefore, the equity premium is also higher in production economies.

### 6.3 Portfolios

Investors speculate on their beliefs by trading in the financial market. Pessimist Investor $b$ lends to optimist Investor $a$ by purchasing the locally riskless bond. As a result, pessimist Investor $b$ switches his stock holdings into locally riskless debt, thereby decreasing the risk in his wealth. The optimist Investor $a$, on the other hand, takes the funds obtained from selling the locally riskless bond to the pessimist Investor $b$ and uses them to finance his own levered purchase of additional stock shares.

The top panel of Figure 6 plots the stock holdings $\zeta^a_{P_t}$ and bond holdings $\zeta^a_{B_t}$ of the optimist Investor $a$. As in previous plots, the solid line corresponds to the production economy and the dotted line corresponds to an equivalent endowment economy. The bond holdings of the optimist Investor $a$ are negative in all states, which means that he is borrowing from the pessimist Investor $b$. Moreover, the top panel of Figure 6 shows that borrowing is maximized around a middle point of $\omega_t$. Because there is only one share of the stock,
the holdings of the stock by the optimist Investor $a$ would reach one only if $\omega_t \to 1$ and he became the only investor in the economy, being then forced to hold the only unit of the stock. In all other cases, the optimist Investor $a$ chooses lower holdings of the stock, while the remaining is held by the pessimist Investor $b$.

To obtain a better understanding of investors’ portfolios we consider, in addition to the stock and bond holdings of the optimist Investor $a$, we compute each investors’ weight of stock in their portfolio. These are given by $\frac{\zeta_a P_q}{D_a(\omega_t)}$ and $\frac{\zeta_b P_q}{D_b(\omega_t)}$. The center panel of Figure 6 plots the portfolio weight on the stock for each investor. The optimist Investor $a$ holds a portfolio weight on the stock that is always larger than one, which means he uses leverage to purchase the stock. By contrast, the portfolio weight of the pessimist Investor $b$ is always between zero and one, because he always maintains a part of his portfolio in the riskless bond.

Interestingly, we note that the portfolio weight of the optimist Investor $a$ is less tilted towards stocks in the production economy than in the endowment economy. By way of contrast the portfolio weight of the pessimist Investor $b$ is more tilted towards stocks in the production economy than in the endowment economy. The intuition for this results follows from the asymmetric impact of sentiment on the aggregate consumption-capital ratio $c(\omega_t)$ for optimists and pessimists. It amplifies sentiment risk for the optimist Investor $a$, because in good times he gets simultaneously a higher consumption share of aggregate consumption and a higher aggregate consumption; it hedges sentiment risk for the pessimist Investor $b$, because in good times he gets simultaneously a lower consumption share of aggregate consumption and a higher aggregate consumption.

Accordingly, we expect portfolios to be less extreme in our production economy than in an endowment economy. This generates a reduction in trade in financial assets. In order to quantify these effects, we compute a proxy for trade in stocks. In a continuous-time setting like ours, with the diffusive nature of the information flow, trading volume in the conventional sense would be infinite. We follow Xiong and Yan [2010] and Longstaff and
Wang [2012] by using the absolute volatility of the stock holding of any investor $\zeta^j_t$ to gauge stock trading activity. The left plot in the bottom panel panel of Figure 6 confirms that the maximum level of stock trading activity, reached when $\omega_t$ lies close to the center at 0.5, is lower in a production economy than in an endowment economy.

In order to assess the importance of the credit market, we follow Longstaff and Wang [2012] and define the market-leverage ratio as the ratio of aggregate credit in the market to the total value of assets held by investors $|\zeta^j_t| \overline{P}_t$. The right plot in the bottom panel panel of Figure 6 confirms that the maximum market-leverage ratio, reached when $\omega_t$ lies close to the center at 0.5, is lower in a production economy than in an endowment economy.

The lower market-leverage in production economies is consistent with the lower volatility of interest rates in production economies discussed above. Despite a lower stock trading activity in production economies, its riskier aggregate dividend due to the impact of sentiment on the aggregate consumption-capital ratio, induces a higher equity premium and stock return volatility in production economies.

7 Anomalies

The previous section has explained in detail the implications of disagreement for investment, aggregate consumption, the market price of risk, the interest rate, asset prices and portfolios; in our production economy framework. To our knowledge we are the first to study those effects in a general equilibrium production economy.

In this section, we evaluate whether our model could help explaining some well-known asset pricing anomalies. We focus on three particular firm’s characteristics that have been shown to explain expected returns. Small firms earn higher average returns than big firms (e.g., Banz [1981]); high book-to-market stocks earn higher average returns than low book-to-market stocks (e.g., Fama and French [1993] and Lakonishok et al. [1994]); and high investment-to-assets firms earn lower average returns than low investment-to-assets firms.
These anomalies have been studied empirically in the cross-section of returns, based on sorted portfolios. Rather than attempting to replicate those experiments, which would be unfeasible in our simple model of a single-firm with two investors, we ask the more fundamental question of whether some dimension on the apparent ability of these firms’ characteristic to explain expected returns may be related to the underlying risk of sentiment. The CAPM, derived in Equation (39), suggests firm’s characteristics driven by sentiment may seem to explain returns, to the extent that they capture the underlying sentiment risk that is priced in the CAPM.

Figure 7 plots expected stock returns against $\omega_t$ and each of the firm characteristics described above. In each plot there are three lines. The solid line plots the expected stock return under the objective measure $z$, the dotted-dashed line is the expected stock return under the measure of the optimist Investor $a$, and the dashed line is the expected stock return under the measure of the pessimist Investor $b$. In all cases, the expected stock return is highest under the measure of the optimistic Investor $a$, followed by the expected stock return under the objective measure $z$, and the lowest is the expected stock return under the pessimistic Investor $b$. This is intuitive, and we include expected returns under all these three measures to highlight that the sign of the slope between expected returns and the given firm characteristic is the same under all measures.

The left plot in the top panel of Figure 7 shows that the expected stock return is increasing in the proportion of optimists in the economy. We include this plot as a reference, in order to convey intuition about the slope in the other three plots. The right plot in the top panel of Figure 7 illustrates the negative relation between expected stock return and firm’s size, consistent with the empirical evidence. In our model the firm’s size $P_t (\omega_t) = q (\omega_t) K_t$ is scaled by capital $K_t$, which we fix to 1 in the plot, in order to keep the analysis in two dimensions. One justification for this is that the main source of co-movement in our model is driven by $\omega_t$ and, while the realization of $q (\omega_t)$ is directly impacted by $\omega_t$, the
dynamics of capital $K_t$ are impacted by $\omega_t$ only through its drift. Accordingly, we expect the effect of Tobin’s q to be strongest. The intuition for the size anomaly in our setup is the following: when the proportion of optimists increases we obtain simultaneously an increase in the expected stock return and a decrease in Tobin’s q and firm’s size. As explained above, Tobin’s q co-moves with the investment-capital ratio $i(\omega_t)$, which is decreasing in $\omega_t$.

The left plot in the bottom panel of Figure 7 illustrates the negative relation between expected stock return and book-to-market, consistent with the empirical evidence. In our model book-to-market is the inverse of Tobin’s q. Therefore, the intuition is strongly related to that of the size anomaly: the expected stock return is positively related to Tobin’s q and, therefore, the expected stock return is negatively related to the inverse of Tobin’s q.

The right plot in the bottom panel of Figure 7 illustrates the negative relation between expected stock return and investment-to-assets, consistent with the empirical evidence. In our model, we proxy investment-to-assets with the investment-capital ratio $i(\omega_t)$. That is, ours is a measure of the ratio of investments to the book value of assets, which is also the denominator of Tobin’s q. Because in our model the investment-capital ratio $i(\omega_t)$ and Tobin’s q co-move, the intuition for the investment-to-assets anomaly is strongly related to the size anomaly: the expected stock return is positively related to Tobin’s q and, therefore, the expected stock return is also positively related to the investment-capital ratio $i(\omega_t)$.

We conclude that, despite its simplicity, our model relates firm’s characteristics to expected stock return. In our setup, these three well-known anomalies are just a consequence of these firm’s characteristics capturing some dimension of sentiment risk, including the speculative aggregate consumption risk driven by sentiment in a production economy setting, introduced for the first time in this paper.
8 Conclusion

We provide a tractable continuous-time production framework to study the implications of disagreement on the allocation of capital to aggregate investment and aggregate consumption, as well as on equilibrium asset prices, portfolios and financial trade. We show that, in production economies in which investment is chosen optimally, a new dimension of risk driven by disagreement and speculation emerges. We call it speculative aggregate consumption risk. The main intuition for this additional risk dimension is that, in a setup with optimal capital allocation, disagreement affects not only the shares of aggregate consumption among investors with different views, but also the aggregate consumption to be shared among those investors.

We provide a full characterization of the impact of speculative aggregate consumption risk on equilibrium quantities, asset prices, portfolios and financial trade. Moreover, we show that our model can be consistent with the size, book-to-market and investment-to-assets anomalies. These firms’ characteristics may appear to be priced because they carry some dimension of the risk introduced by differences of opinion and, in particular, speculative aggregate consumption risk.

Our model studies the simplest possible type of sentiment risk: all the investors in our model are dogmatic and have a fixed symmetric bias in their beliefs. As a consequence, disagreement risk has very simple dynamics. Even in such a simple setting, our model produces endogenous stochastic volatility of aggregate consumption and several interesting implications on equilibrium quantities, asset prices and portfolios. Yet, it may be interesting to study how the magnitude of these effects changes in an environment in which investors can have time-varying and stochastic beliefs. AWe leave those avenues for future work.
References


A Appendix

A.1 Solution with agreement

Given the conjectured value function in Equation (13), the solution is a pair of constants $(i, \Lambda)$ that satisfy the first order condition

$$0 = A - i - [(1 - \theta i) \Lambda]^{\frac{1}{\alpha-1}},$$

and the Hamilton-Jacobi-Bellman equation

$$0 = (1 - \theta i)\Lambda^{\frac{\alpha}{\alpha-1}} + \alpha\phi(i) - \frac{1}{2} \alpha (1 - \alpha) \sigma^2 - \rho. \quad (2)$$

The solution for $\Lambda$ is

$$\Lambda^*_j = \frac{1}{(A - i^*_j)^{1-\alpha}(1 - \theta i^*_j)} , \quad (3)$$

and Equation (14) is the optimal investment rate.

A.2 Social planner’s value function with disagreement

Let $V(\cdot, \cdot)$ be the value function of the planner in an economy with heterogeneous beliefs, where the first argument is capital stock, and the second argument captures type $a$ investors’ Pareto-weight. Also, let $V_a(K_0)$ be the expected utility of a type $a$ investors when the current aggregate capital stock is $K_0$ with $V_b(K_0)$ defined similarly. The planner’s value function at time zero, with initial capital stock $K_0$ and weight $\lambda$ on type $a$ investors, satisfies the recursion:

$$V(K_0, \lambda) = V_b(K_0) + \lambda V_a(K_0) \quad (4)$$

where the final line follows from the law of iterated expectations and the definition of the value function.

It then follows that the value function evaluated at time $t$ with weight $\lambda t_{b,0}$ on the type $a$
investors satisfies

\[
V \left( K_t, \lambda \eta \frac{\eta_t}{\eta_0} \right) = \sup_{c_{a,t} + c_{b,t} = K_t c_t} \mathbb{E}_t^b \left[ \int_t^\infty e^{-\rho(s-t)} \left( \frac{1}{\alpha} c_{b,s} + \lambda \eta \frac{\eta_s}{\eta_0} \frac{1}{\alpha} c_{a,s} \right) ds \right]
\]

\[
= \sup_{c_{a,t} + c_{b,t} = K_t c_t} \left\{ \mathbb{E}_0^b \left[ \int_t^\infty e^{-\rho(s-t)} \frac{1}{\alpha} c_{b,t}^\alpha dt \right] + \lambda \eta \frac{\eta_t}{\eta_0} \mathbb{E}_t^a \left[ \int_t^\infty e^{-\rho(s-t)} \frac{1}{\alpha} c_{a,s}^\alpha ds \right] \right\}
\]

\[
= V_b(K_t) + \lambda \eta \frac{\eta_t}{\eta_0} V_a(K_t).
\]  

(5)

Accordingly, we obtain the boundary conditions for the value function

\[
\lim_{\eta_t \to 0} V \left( K_t, \lambda \eta \frac{\eta_t}{\eta_0} \right) = V_b(K_t); \quad \lim_{\eta_t \to \infty} V \left( K_t, \lambda \eta \frac{\eta_t}{\eta_0} \right) = V_a(K_t).
\]  

(6)

Using the definition of the consumption share in Equation (19) and the form of the value function in Equation (24), we obtain the boundary conditions in Equation (25).

Using market clearing in consumption: \(c_{a,t} + c_{b,t} = K_t c_t\), it follows that

\[
V \left( K_t, \lambda \eta \frac{\eta_t}{\eta_0} \right) = V_b(K_t) + \lambda \eta \frac{\eta_t}{\eta_0} V_a(K_t)
\]  

(7)

\[
= \sup_{c_t} \mathbb{E}_0^b \left[ \int_0^\infty e^{-\rho t} \frac{1}{\alpha} K_t^\alpha c_t^\alpha \left( 1 + \left( \frac{\eta t}{\eta_0} \right)^{1-\alpha} \right)^{1-\alpha} \right] dt,
\]  

(8)

and the planner’s equilibrium can be decentralized using the consumption share in Equations (18) and (19).

### A.3 Solution of the social planner’s problem with disagreement

The Hamilton-Jacobi-Bellman (HJB) equation for the planner’s problem is

\[
\rho V = \sup_{i_t} \left\{ \frac{1}{\alpha} K_t^\alpha (A - i_t)^\alpha \left( 1 + \left( \frac{\eta t}{\eta_0} \right)^{1-\alpha} \right)^{1-\alpha} + V_K \phi_b (i_t) K_t + \frac{1}{2} V_{KK} \sigma^2 K_t^2 
\] + \frac{1}{2} V_{\eta \eta t} \eta_t^2 \right\}.
\]  

(9)

Using these dynamics of \(\omega_t\) in Equation (20) and our conjectured value function in Equation (24) we obtain the first order conditions in Equation (10):

\[
0 = A - i (\omega_t) - (H (\omega_t) [1 - \theta i (\omega_t)])^\frac{1}{\alpha - 1}.
\]  

(10)
Plugging the first order conditions back into the HJB we obtain Equation (11):

\[
0 = \left[ 1 - \theta i (\omega_t) \right]^{\frac{\alpha}{1 - \alpha}} H (\omega_t)^{\frac{1}{1 - \alpha}} + \alpha \phi_b (i (\omega_t)) - \frac{1}{2} \alpha (1 - \alpha) \sigma^2 \\
+ \frac{1}{2} \mu^2 \left( \frac{1}{1 - \alpha} \right)^2 \omega_t (1 - \omega_t) \left[ \alpha (1 - 2\omega_t) \frac{H' (\omega_t)}{H (\omega_t)} + \omega_t (1 - \omega_t) \frac{H'' (\omega_t)}{H (\omega_t)} + \alpha \right] \\
+ \alpha \sigma \left[ \frac{1}{1 - \alpha} \omega_t (1 - \omega_t) \frac{H' (\omega_t)}{H (\omega_t)} + \omega_t \right] - \rho. 
\]  

(A.4) Portfolios

We first show how to obtain equilibrium wealth and we then proceed to obtain portfolios.

Let \( X_{b,t} = D_b (\omega_t) K_t \) be the wealth of Group \( b \). Because \( X_{b,t} \) is equivalent to a security that pays out a dividend equal to the optimal consumption of Group \( b \), \( c_{b,t} \), no arbitrage implies that \( X_{b,t} \) satisfies

\[
\mathbb{E}_t^b \left[ d \left( \xi^b_t X_{b,t} \right) \right] + \xi^b_t c_{b,t} dt = 0. 
\]  

(12)

Ito's lemma implies that the dynamics of \( X_{b,t} \) are

\[
dX_{b,t} = m (\omega_t) X_{b,t} dt + n (\omega_t) X_{b,t} dW_t, 
\]  

(13)

where

\[
m (\omega_t) = \phi_b (i (\omega_t)) + \frac{D'_b (\omega_t)}{D_b (\omega_t)} \sigma_\omega (\omega_t) \left[ \frac{1}{2} \cdot \frac{1}{1 - \alpha} (2\omega_t - \alpha) \mu + \sigma \right] + \frac{1}{2} \frac{D''_b (\omega_t)}{D_b (\omega_t)} \sigma_\omega^2 (\omega_t), 
\]

\[
n (\omega_t) = \sigma + \frac{D'_b (\omega_t)}{D_b (\omega_t)} \sigma_\omega (\omega_t), 
\]  

(14)

and we have used the dynamics of \( K_t \) in Equation (7) and the dynamics of \( \omega_t \) in Equation (20). In addition, we know that the state price density dynamics are given by:

\[
d\xi^b_t = -r_t \xi^b_t dt - \kappa_{b,t} \xi^b_t dW_t. 
\]  

(15)

Using Ito's lemma one more time we obtain the dynamics of \( \xi^b_t X_{b,t} \), in terms of the dynamics of \( \xi^b_t \) and \( X_{b,t} \) given in Equations (15) and (13), respectively. In particular, we get

\[
d \left( \xi^b_t X_{b,t} \right) = \left( \xi^b_t X_{b,t} \right) \left( \frac{d\xi^b_t}{\xi^b_t} + \frac{dX_{b,t}}{X_{b,t}} + \frac{d\xi^b_t}{\xi^b_t} \frac{dX_{b,t}}{X_{b,t}} \right). 
\]  

(16)

Plugging these dynamics into our no arbitrage restriction in Equation (12) and simplifying
we get the differential equation for $D_b(\omega_t)$:

$$0 = \left[ -r(\omega_t) + \phi_b(i(\omega_t)) - \kappa_b(\omega_t) \sigma \right] D_b(\omega_t)$$

$$+ \frac{1}{1 - \alpha} \omega_t (1 - \omega_t) \mu \left[ -\frac{1}{2} \frac{1}{1 - \alpha} (2\omega_t - \alpha) \mu + \sigma - \kappa_b(\omega_t) \right] D_b'(\omega_t)$$

$$+ \frac{1}{2} \left[ \frac{1}{1 - \alpha} \omega_t (1 - \omega_t) \mu \right]^2 D_b''(\omega_t) + (1 - \omega_t) [A - i(\omega_t)],$$

with boundary conditions

$$\lim_{\omega_t \to 0} D_b(\omega_t) = \frac{1}{(c_b^*)^{1-\alpha} (1 - \theta_i^*)}; \quad \lim_{\omega_t \to 1} D_b(\omega_t) = 0. \quad (18)$$

The boundary conditions are intuitive and explained in are that the wealths of the investors converge to the single-agent wealths at the boundaries.

The diffusion coefficients are obtained by applications of Ito’s lemma to the expressions for the wealth of Group $b$ and the Equity price, $P_t$, respectively. The portfolio construction in terms of diffusions follows from Cox and Huang (1989).

Under the subjective measure $j = \{a, b, z\}$ their dynamics are given by

$$di = \mu^i(\omega_t) dt + \sigma^i(\omega_t) dW^j_t,$$

$$dc = \mu^c(\omega_t) dt + \sigma^c(\omega_t) dW^j_t,$$

where

$$\mu^i(\omega_t) = i'(\omega_t) \mu^i(\omega_t) + \frac{1}{2} i''(\omega_t) \sigma^2(\omega_t), \quad \sigma^i(\omega_t) = i'(\omega_t) \sigma_{\omega}(\omega_t),$$

$$\mu^c(\omega_t) = -\mu^i(\omega_t), \quad \sigma_{c,t} = -\sigma_i(\omega_t). \quad (21)$$

$$C_t = K_t c(\omega_t); \quad I_t = K_t i(\omega_t). \quad (23)$$

Accordingly, under the measure of $j = \{a, b, z\}$, the dynamics of investment and aggregate consumption growth are given by

$$\frac{dI_t}{I_t} = \mu^i(\omega_t) dt + \sigma^i(\omega_t) dW^j_t,$$

$$\frac{dC_t}{C_t} = \mu^i(\omega_t) dt + \sigma^c(\omega_t) dW^j_t,$$

$$\frac{dC_t}{C_t} = \mu^i(\omega_t) dt + \sigma^c(\omega_t) dW^j_t,$$
where

\[ \mu_j^I(\omega_t) = \phi_j(i(\omega_t)) + \frac{1}{i(\omega_t)} \mu_j^I(\omega_t) + \frac{i'(\omega_t)}{i(\omega_t)} \sigma(\omega_t), \quad (26) \]

\[ \mu_j^C(\omega_t) = \phi_j(i(\omega_t)) + \frac{1}{c(\omega_t)} \mu_j^I(\omega_t) + \frac{c'(\omega_t)}{c(\omega_t)} \sigma(\omega_t), \quad (27) \]

\[ \sigma_I(\omega_t) = \sigma + \frac{i'(\omega_t)}{i(\omega_t)} \sigma(\omega_t), \quad (28) \]

\[ \sigma_C(\omega_t) = \sigma + \frac{c'(\omega_t)}{c(\omega_t)} \sigma(\omega_t). \quad (29) \]

The first element of the drift and volatility of investment and aggregate consumption growth are, respectively, the drift \( \phi(i(\omega_t)) \) and volatility \( \sigma \) of capital growth. The second term of the drift and volatility of investment and aggregate consumption growth are related to the drift and volatility of the investment-capital ratio \( i(\omega_t) \) and the aggregate consumption-capital ratio \( c(\omega_t) \). The remaining terms in the drifts are covariance terms.

A.5 Implementing the planner’s investment decisions

In equilibrium both investors agree on the firm’s value, which is equal to the observed stock price. Therefore, the investment allocation can be implemented through a representative firm that chooses the investment plan to maximize firm value.

Lemma 1 Given their subjective beliefs and taking prices as given, both investors agree on the investment plan that maximizes firm value.

Proof. The optimal value maximizing plan for the firm from the perspective of Investor \( j = \{a, b\} \) solves

\[
\sup_{\xi_j^t} \mathbb{E}^j_t \left[ \int_t^\infty \frac{\xi_j^s}{\xi_j^t} K_s (A - i_s) ds \right],
\]

s.t. \( dK_t = \phi_j(i(\omega_t)) K_t dt + \sigma K_t dW_t^j \),

\[ d\xi_j^t = -\xi_j^t r_t dt - \xi_j^t \kappa_t dW_t^j, \quad (31) \]

where \( \xi_j^t \) is Investor \( j \)'s state price density.

The maximand in the first equation is the observed stock price \( P_t \). Because investors must agree on observed prices, the maximand is identical for any investor \( j \).

The relation in Equation (8) implies that investors observe capital \( K_t \), and leads to the consistency on the dynamics of capital \( K_t \) under the measure of any investor \( j \) in Equation (31).

Investor-specific state price densities \( \xi_j^t \) and sentiment \( \eta_t \) are related by:

\[ \eta_t = \lambda \frac{\xi_b^t}{\xi_a^t}. \quad (33) \]
Using this relation, and the dynamics of sentiment in Equation (10), leads to consistency on the dynamics of the state price density $\xi^j_t$ under the measure of any investor $j$ in Equation (32).

Because the optimization problem is the same under the eyes of each investor $j$, both investors must agree on the optimal investment policy that maximizes firm value, given their subjective beliefs and assuming they take equilibrium prices as given. ■
Table 1: **Parameter values used in all the numerical examples**

<table>
<thead>
<tr>
<th>Object</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant in the drift of output growth</td>
<td>$\delta$</td>
<td>0</td>
</tr>
<tr>
<td>Volatility of output growth</td>
<td>$\sigma$</td>
<td>0.1</td>
</tr>
<tr>
<td>Productivity</td>
<td>$A$</td>
<td>0.1</td>
</tr>
<tr>
<td>Time Preference</td>
<td>$\rho$</td>
<td>0.04</td>
</tr>
<tr>
<td>Adjustment cost</td>
<td>$\theta$</td>
<td>10</td>
</tr>
<tr>
<td>Relative risk aversion</td>
<td>$1 - \alpha$</td>
<td>4</td>
</tr>
<tr>
<td>Disagreement</td>
<td>$\bar{\mu}$</td>
<td>0.25</td>
</tr>
<tr>
<td>Optimist Beliefs</td>
<td>$\delta_a$</td>
<td>-0.0125</td>
</tr>
<tr>
<td>Pessimist Beliefs</td>
<td>$\delta_b$</td>
<td>0.0125</td>
</tr>
</tbody>
</table>
Table 2: **Summary statistics under agreement**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Objective (δ)</th>
<th>Pessimist (δ_b)</th>
<th>Optimist (δ_a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment-capital ratio</td>
<td>0.028</td>
<td>0.039</td>
<td>0.018</td>
</tr>
<tr>
<td>Expected output growth (objective measure)</td>
<td>2.40%</td>
<td>3.15%</td>
<td>1.65%</td>
</tr>
<tr>
<td>Expected output growth (subjective measures)</td>
<td>2.40%</td>
<td>1.90%</td>
<td>2.90%</td>
</tr>
<tr>
<td>Tobin’s q</td>
<td>1.387</td>
<td>1.644</td>
<td>1.222</td>
</tr>
<tr>
<td>Interest rate</td>
<td>3.60%</td>
<td>1.60%</td>
<td>5.60%</td>
</tr>
<tr>
<td>Equity premium (subjective measures)</td>
<td>4.00%</td>
<td>4.00%</td>
<td>4.00%</td>
</tr>
</tbody>
</table>
Figure 1: **Dynamics of Investor $a$’s consumption share, $\omega$:** The Figure provides plots of the drift and volatility of the type $a$ Investor’s share of aggregate consumption $\omega_t$ against the current value of $\omega_t$. The drifts are shown under the type $b$ Investor’s beliefs in the top left plot, under the type $a$ Investor’s beliefs in the top right plot and under the objective measure in the bottom left plot. The bottom right plot is the diffusion coefficient for $\omega$ under all beliefs. The parameters used are reported in Table 1.
Figure 2: **Aggregate investment and consumption**: The figure provides plots of investment and consumption against $\omega$, Investor type $a$’s share of aggregate consumption. In all plots, the solid line shows results for a production economy and the dotted line shows results in the baseline endowment economy. The top left plot is the investment capital ratio, and the top right plot is the consumption capital ratio. The middle left plot reports the drift of the consumption-capital ratio under Investor type $b$’s beliefs; the middle right plot the drift of the consumption-capital ration under Investor type $b$’s beliefs; and the bottom left plot the drift of the consumption-capital ration under the Objective beliefs. The bottom right plot is the diffusion coefficient for the investment-consumption ratio under all beliefs. The parameters used are reported in Table 1.
Figure 3: **Growth of capital, consumption, and investment:** The figure provides plots of the drifts capital growth, investment growth and consumption growth and the diffusion of consumption growth under all beliefs. The top panels show the drifts of capital growth with a solid line, consumption growth with a dash-dotted magenta line and and investment growth with a dashed green line. The top left plot reports the drifts under type $b$ Investor’s beliefs; the top right plot reports the drifts under type $a$ Investor’s beliefs; and bottom left plot reports the drifts under the Objective beliefs. The bottom right plot is the diffusion coefficient for the consumption growth under all beliefs and the dashed red line in the plot is the diffusion in the reference endowment economy, which is a constant. The parameters used are reported in Table 1.
Figure 4: **Interest rate and the market price of risk** The figure provides plots showing the level of the interest rate, interest rate volatility and the market prices of risk for both investor types plotted against $\omega_t$. The solid lines correspond to the production economy, the dotted lines to the endowment economy. The parameters used are reported in Table 1.
Figure 5: Equity statistics The figure provides plots of several equity statistics relative to the optimistic Investor $a$’s consumption share, $\omega$. The solid lines correspond to the production economy, the dotted lines to the endowment economy. The top left plot is Tobin’s $q$ and the top right plot is the price-dividend ratio. The left middle plot is the diffusion coefficient for the percentage change in Tobin’s $q$ and the middle left plot is stock return volatility. The bottom left plots is the equity premium under each type $b$’s beliefs and the right plot is the equity premium under type $a$’s beliefs. The parameters used are reported in Table 1.
Figure 6: **Portfolios:** The plots characterize portfolios relative to the optimistic Investor a’s consumption share, \( \omega \). The solid lines correspond to the production economy, the dotted lines to the endowment economy. In the top right plot, we see that Investor a generally borrows more from b in the endowment economy than in the production economy. This supports more extreme shifts in investor’s portfolio weights on the stock, as shown in the center two plots. In turn this implies generally more volatile stock holdings and credit markets, as shown in the bottom plots. In both production and endowment economies, the credit market is largest when \( \omega \approx 0.5 \), which also corresponds to periods of heavy trading.
Figure 7: **Anomalies**: The plots characterize the relation between the expected stock market return and different firm’s characteristics. The solid lines corresponds to the expected stock market return under the objective measure, the dotted-dashed line under the measure of (optimist) Investor a, and the dashed line under the measure of (pessimist) Investor b. In each plot, the expected stock return is plotted against the object indicated in the legend on the x-axis. The relation between the expected stock return and each firm’s characteristic are in line with the empirical evidence.