Credit Ratings and Security Design*

Jens Josephson‡ and Joel Shapiro‡
Stockholm University and University of Oxford

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Abstract

The poor performance of credit ratings on structured finance products leading up to the financial crisis has prompted investigation into the role of Credit Rating Agencies (CRAs) in designing and marketing these products. We analyze a two-period model where a credit rating agency with reputational concerns both designs and rates structured products that are sold to different clienteles: unconstrained investors and investors constrained by minimum quality requirements. The motivation for pooling assets derives from the attempt to tailor products for constrained investors and reputational incentives. When quality requirements for constrained investors are higher it becomes more difficult to sell securities, making rating inflation increase as the benefits of maintaining reputation for the future decrease. Ratings inflation decreases if the quality of the asset pool is higher, as the incentive to pass off more valuable assets reduces opportunism.

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‡Department of Economics, Stockholm University, SE-106 91 Stockholm Sweden Contact: jens.josephson@ne.su.se

‡Saïd Business School, University of Oxford, Park End Street, Oxford OX1 1HP. Contact: Joel.Shapiro@sbs.ox.ac.uk
1 Introduction

The recent financial crisis has prompted much investigation into the role of credit-rating agencies (CRAs). With the dramatic increase in the use of structured finance products, the agencies quickly expanded their business and earned outsized profits (Moody’s, for example, tripled its profits between 2002 and 2006). Ratings quality seems to have suffered, as structured finance products were increasingly given top ratings shortly before the financial markets collapsed. In this paper, we ask how the design of such products is influenced by CRAs, and how security design changes with market incentives.

The design of structured finance products is marked by close collaboration between issuers and rating agencies. Issuers depend on rating agencies to certify quality and to be able to sell to regulated investors. Beyond directly paying CRAs for ratings (the “issuer pays” system), Griffin and Tang (2012) write that “The CRA and underwriter may engage in discussion and iteration over assumptions made in the valuation process.” Agencies also provide their models to issuers even before the negotiations take place (Benmelech and Dlugosz, 2009). These products are characterized by careful selection of the underlying asset pool and private information about asset quality.

We present a reputation-based two period model of rating structured products. Each period an issuer has a set of good and bad assets that it can pool, tranche, and issue securities against. The CRA is long-lived and may be of two types, truthful or opportunistic. Reputation for the CRA consists of the probability that investors perceive it to be truthful. This perception can change according to inferences from ratings and security performance. There are two types of rational investors1, constrained and unconstrained. Constrained investors need the quality of securities to be above a certain level, while unconstrained investors can purchase any type of security.

We present several findings on the drivers of ratings inflation. When quality requirements for constrained investors are higher, it becomes more difficult to sell securities. This leads to an increase in rating inflation, as the benefits of maintaining reputation for the future decrease. This implies that tighter regulation of constrained investors may have negative equilibrium effects on the quality of assets sold. If the quality of the asset pool is higher, ratings inflation decreases, as the incentive to pass off more valuable assets reduces opportunism. This provides a link between fundamental asset values and ratings inflation, which is important when thinking about how ratings vary over time.

1By rational, we mean that they make inferences based on available information using Bayes’ rule when possible and maximize their payoff given their constraints.
We provide two new motivations for the pooling of assets. First, in our model structuring motives derive from the need to tailor products for constrained investors. Second, a CRA can balance the informational advantage over investors with the need to maintain its reputation by choosing the right mix of good and bad assets to include.

Reputational incentives affect the equilibrium configuration and design of tranches. If the CRA is considered more likely to be truthful, it can sell two tranches of securities, with rating inflation only for the top tranche. As its reputation decreases, it may only sell one tranche that has inflated ratings (to unconstrained investors) or not be hired by the issuer.

The key building blocks of our model are as follows:

- **Security design**: The assets can be pooled, tranched, and/or retained by the issuer. The design is constrained by the incentive to pass off bad securities as good ones (a lemons problem) and the demands of investors, but buffeted by the possibility of milking reputation.

- **Reputation concerns for CRAs**: As rating agencies executives often argue, CRAs are concerned about maintaining their reputation for providing timely and accurate assessments of default risk.

- **Clientele effects**: A principal motivation for securitization is to apportion risk to investor groups with heterogeneous preferences for risk. The obvious example of this was the increased demand for safe investments in the 2000s by regulated entities (e.g. banks, pension funds, insurance companies).

There is substantial evidence of asymmetric information and strategic asset pool selection for structured finance products. Downing, Jaffee, and Wallace (2009) compare the performance of pools of mortgages that are pass-through MBS with no tranching with securitized REMICs (Real Estate Mortgage Investment Conduits) with tranching. The extra layer of securitization and anonymity in sales allows for a selection of worse performing pools due to private information. This is shown to be true with ex-post performance data. Moreover, there is a “lemons spread” due to rational discounting of these securities. An, Deng, and Gabriel (2011) show that portfolio lenders use private information to pass off lower quality loans to commercial mortgage backed securities (CMBS). Conduit lenders, who originate loans for direct sale into securitization markets do not select loans and hence have higher quality loans conditioning on the observables. The analysis shows a lemons discount for portfolio loans. This lemons discount is lower for multifamily loans, which have lower levels of uncertainty and lender private information than retail,
office, and industrial loans. Elul (2011) demonstrates that securitized mortgages perform worse than portfolio loans, with the largest differences in prime mortgages in private (non-GSE) securitizations, consistent with the presence of adverse selection. Ashcraft, Goldsmith-Pinkham, and Vickery (2011) find that the MBS deals that were most likely to underperform were the ones with more interest-only loans (because of limited performance history) and lower documentation, that is, loans that were more opaque or difficult to evaluate.

We find that rating inflation occurs for the highest quality tranche levels. In the data, Gorton and Metrick (2012) show that AAA-rated asset backed securities have significantly higher cumulative default rates compared to AAA-rated corporate bonds. This is also true for lower rating categories, but the differences lessen as ratings worsen. Cornaggia, Cornaggia, and Hund (2013), also find that structured products are overrated compared to corporate issues, while municipal and sovereign bonds are underrated, over the sample period 1980-2010. Ashcraft, Goldsmith-Pinkham, and Vickery (2011) find that as MBS issuance volume shot up between 2005 and mid-2007, ratings quality declined. Specifically, subordination levels\(^2\) for subprime and Alt-A MBS deals decreased over this period when conditioning on the overall risk of the deal. Subsequent ratings downgrades for the 2005 to mid-2007 cohorts were dramatically larger than for previous cohorts. Griffin and Tang (2012) show that CRA adjustments to their models’ predictions of credit risk in the CDO market were positively related to future downgrades. These adjustments were overwhelmingly positive and the amount adjusted (the width of the AAA tranche) increased sharply from 2003 to 2007 (from 6% to 18.2%). He, Qian, and Strahan (2012) find that top rated MBS tranches sold by larger issuers\(^3\) performed significantly worse (prices drop more) and have higher initial yields than those sold by small issuers during the boom period of 2004 to 2006. Stanton and Wallace (2012) demonstrate that the spread between CMBS and corporate bond yields for ratings AA and AAA fell significantly after 2002 (and did not fall for bonds with worse ratings), when risk-based capital requirements for top rated CMBS were lowered significantly. Also, CMBS rated below AA were upgraded to AA or AAA significantly more than the rate observed in a comparable sample of RMBS leading up to the crisis.

In the following subsection, we review related theoretical work. In Section 2, we examine the problem of the issuer when there is no rating agency. In Section 3, we add a CRA and analyze the second period. In Section 4, we

\(^2\)The subordination level they use is the fraction of the deal that is junior to the AAA tranche. A smaller fraction means that the AAA tranche is less “protected” from defaults, and therefore less costly from the issuer’s point of view.

\(^3\)They define larger by market share in terms of deals. As a robustness check, they also look at market share in terms of dollars and find similar results.
look at the first period of the game. Section 5 concludes. All proofs are in the Appendix.

1.1 Theoretical Literature

This paper is related to two strands of the finance literature, one on credit rating agencies and the other on security design.

The link between ratings quality and reputation is key for our results. Mathis, McAndrews, and Rochet (2009) examines how a CRA’s concern for its reputation affects its ratings quality. They present a dynamic model of reputation in which a monopolist CRA may mix between lying and truthtelling to build up/exploit its reputation. The authors focus on whether an equilibrium in which the CRA tells the truth in every period exists, and they demonstrate that truthtelling incentives are weaker when the CRA has more business from rating complex products. Strausz (2005) is similar in structure to Mathis et al. (2009), but examines information intermediaries in general. Bar-Isaac and Shapiro (2013) incorporate economic shocks and show that CRA accuracy may be countercyclical, which is also consistent with our results. Our model of reputation is similar to those above, but the ability of the CRA to strategically structure what type of securities are sold while at the same time rating those securities is new and links our work directly to the phenomenon of structured finance.

In addition to Mathis et al. (2009), there are several other recent theoretical papers on CRAs. Fulghieri, Strobl and Xia (2012) focus on the effect of unsolicited ratings on CRA and issuer incentives. Bolton, Freixas, and Shapiro (2012) demonstrate that competition among CRAs may reduce welfare due to shopping by issuers. Conflicts of interest for CRAs may be higher when exogenous reputation costs are lower and there are more naïve investors. Skreta and Veldkamp (2009) and Sangiorgi, Sokobin, and Spatt (2009) assume that CRAs relay their information truthfully, and they demonstrate how noisier information creates more opportunity for issuers to take advantage of a naive clientele through shopping. Opp, Opp, and Harris (2013) examine how ratings-contingent regulation affects the informativeness of ratings. In Pagano and Volpin (2012), CRAs also have no conflicts of interest, but can choose ratings to be more or less opaque depending on what the issuer asks for. They show that opacity can enhance liquidity in the primary market but may cause a market freeze in the secondary market.

The research on security design which is closest to this paper are work by DeMarzo and Duffie (1999), DeMarzo (2005), and Hartman-Glaser (2012). In DeMarzo and Duffie (1999), an issuer can first apportion some of the cash flows of a project into a security before it has private information about the
project (designing the security) and then after learning private information about the project can retain a fraction of the security and sell off the rest. The second stage is a signaling problem where the issuer faces the tradeoff between wanting to sell its entire position and use the cash for other projects, or retaining some fraction, which signals higher value. For some security designs the optimal design is debt. This differs from our paper in that here the issuer (via the CRA) has another tool beyond retention to sell, which is reputation. Even in the one-period problem, the presence of an honest CRA makes sale possible. CRAs make reputation possible as most issuers do not have a long observable issuance performance history. We also have a clientele effect. DeMarzo (2005) uses a variant of the model of DeMarzo and Duffie (1999) to demonstrate two effects of pooling assets. The information destruction effect occurs because the issuer loses his private information advantage over each asset if they are pooled. The risk diversification effect allows the issuer to create low risk securities. In our model pooling occurs to cater to constrained investors, and the lemons effect is weaker due to reputational forces. Hartman-Glaser (2012) has a long lived issuer who can be truthful or opportunistic. The issuer signals through amount retained, an explicitly costly signal and a type of “security design” (although restrictive). In our paper, we focus on the ability of the issuer to select assets and pooling and tranching can occur due to the clientele effect.

In addition to their empirical results, An, Deng, and Gabriel (2011) have a theoretical model where a portfolio lender can only pass off some loans because of the lemons problem and must sell at a discount. Their results suggest that the magnitude of the lemons discount associated with portfolio loan sales varies positively with the dispersion of loan quality in the pool and inversely with the seller’s cost of holding the loans in its portfolio.

There is a literature on security design and information acquisition by investors initiated by Myers and Majluf (1984) that looks at how securities can be split between information sensitive and information insensitive parts (Boot and Thakor (1993), Fulghieri and Lukin (2001), Dang, Gorton, and Holmstrom (2011), and Hanson and Sundaram (2012)). We do not allow for differential information among investors, but include heterogeneity in demand as well as a rating agency whose structuring incentives are reputational.

In the economics literature, Albano and Lizzeri (2001) extends the framework of Lizzeri (1999) and has a producer that can choose a quality of a good that is unobservable to consumers but observable to a certification intermediary. The intermediary commits to a fee schedule and a disclosure rule. The optimal allocation involves underprovision of quality. Our paper differs in several ways. Rather than commit to a disclosure rule, the rating agency in our model uses reputation as a disciplining device. We also have
2 The Model without a Rating Agency

We begin with two types of agents: an issuer and investors. All agents are risk neutral. We will analyze the issuers’ problem first without any rating agency, and then look at the effect of introducing a ratings agency.

The issuer has assets of measure $M \geq 2$. A mass $\mu$ of the assets are good and are worth $G$ to investors. A mass $M - \mu$ are bad and are worth $B$ to investors. The issuer’s valuations of the assets are lower than the investors’ values for the assets: a good asset is worth $g$ and a bad asset is worth $b$ to the issuer. This can occur for several reasons: the issuer has valuable alternative investment opportunities, has regulatory capital requirements for holding the assets, and/or the need to transfer risk off of its balance sheet.

We assume the following ordering:

$$b < B < g < G$$

The inequality of $g > B$ indicates that issuers prefer to keep good assets rather than sell them off if investors perceive them as $B$.

There is a measure 1 of atomistic investors each willing to buy 1 security. Investors can be one of two types: constrained (measure $\alpha$) and unconstrained (measure $1 - \alpha$). We will assume that $1 - \alpha > \mu$ for ease of exposition. Constrained investors will only purchase securities that they believe are high quality and have value of at least $\bar{V}$. We assume that $g < \bar{V} < G$, which implies that constrained investors would not purchase a security that has a payoff of $B$.

Constrained investors may be constrained because of regulations (for example banks, pension funds, and insurance companies are often restricted in the types of assets they may hold) or because of their portfolio hedging requirements. In practice, regulations currently require these types of institutions to hold investment products that have specific ratings. We relax this requirement for two reasons. First, regulations are being changed to weaken the dependence on ratings, and are tending toward using institutional risk models. Second, we do not want ratings to be driven by regulation, which has been discussed amply in the literature (see Opp, Opp, and Harris (2013) and White (2010)).

An important argument for securitization is the clientele effect, which is what we are directly modeling here.

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4 The rationale for this inequality will be explained below.

5 In a different version of this paper, we look at a model where constrained investors need certain ratings. This model is more complex, but has very similar qualitative properties.
The unconstrained investors are willing to purchase any security. They may be hedge funds or other institutional investors. We assume both types of investor are rational in the sense that they update given available information and maximize their payoff.

The issuer can put together portfolios of good and bad assets through securitization. We define securitization as selling securities based on the payoffs of the portfolio. We restrict the space of securities by defining the payoff of a security as the average payoff of an underlying pool of assets. The average payoff for securities based on portfolio $i$ will be defined by the fraction of assets in the portfolio that are good $\gamma_i$. The actual average payoff, or valuation, will be represented by $V_i$.

In this model, the maximum number of different types of securities it could create are two: one for unconstrained investors ($U$), one for constrained investors ($C$). We will call a block of securities designated for a specific type of investor a tranche. The assets retained by the issuer ($I$) may be considered the equity tranche. The quantity of securities in a specific tranche is denoted by $q_i$, where $i \in \{U, C, I\}$. Securitization changes the quality profile, but does not change the overall quality of the assets. The constraints on securitization are:

$$q_U \gamma_U + q_C \gamma_C + q_I \gamma_I = \mu, \quad (1)$$
$$q_U (1 - \gamma_U) + q_C (1 - \gamma_C) + q_I (1 - \gamma_I) = M - \mu. \quad (2)$$

The first equation says that the sum of the claims on good assets equals the amount of good assets. The second equation is analogous for bad assets. These equations imply that $q_I + q_U + q_C = M$. This, of course, is an extremely stylized model of how securitization works, in practice things are much more complex (see Coval, Jurek, and Stafford (2009) for a detailed description of the process).

We will assume that the demand by all constrained investors cannot be met, as there is a scarcity of good assets.

$$\alpha V > \mu G + (\alpha - \mu)B$$ \quad (A1)

(although the interpretation of those properties will vary given the interpretation of rating-based regulation versus quality constraints).

There has been much discussion about the naivete of investors in the RMBS market, e.g. see Bolton, Freixas, and Shapiro (2012). However, not all structured finance markets are necessarily characterized in such a way, as Stanton and Wallace (2012) point out:

“All agents in the CMBS market can reasonably be viewed as sophisticated, informed investors.”
The right-hand side of this equation describes the maximum value of a tranche sold to all constrained investors. This value is attained by allocating all good assets to the constrained tranche. It follows that $\alpha > \mu$.

We also assume that the issuer can’t observe investor types. This will not matter, as the issuer can use simple incentive contracts (giving an epsilon more of surplus to unconstrained investors) to perfectly screen them.

Issuers make take it or leave it offers to investors. The reservation utility of all investors is normalized to zero.

2.1 Full Information

Suppose that there is full information about the securities’ profile. The issuer has two choices. It can sell just to unconstrained types, in which case its payoff net of the initial value of the portfolio, $(M - \mu) b + \mu g$, is (we will use the convention of reporting net payoffs in the rest of the paper):

$$q_U (\gamma_U (G - g) + (1 - \gamma_U) (B - b)).$$

(3)

The second possibility for the issuer is to sell to both types of investors. In this case, the net payoff is

$$q_U (\gamma_U (G - g) + (1 - \gamma_U) (B - b)) + q_C (\gamma_C (G - g) + (1 - \gamma_C) (B - b)).$$

(4)

Here, the issuer tries to sell as much as it can to the constrained investors, and passes the rest off to the unconstrained investors. Notice that just selling to constrained investors (and not to unconstrained investors) is dominated by this option, as unconstrained investors place a higher value on any remaining tranche than the issuer does. We now solve for the issuer’s optimal allocation.

**Lemma 1** The optimal allocation has:

1. A constrained tranche containing all of the good assets and an amount of bad assets such that the average value in the pool equals $\bar{V}$,

2. An unconstrained tranche of size $1 - \alpha$ with only bad assets,

3. The remaining bad assets retained by the issuer.

The issuer’s optimal strategy is to sell to as many constrained investors as possible. Since demand from such investors exceeds the potential supply, they are rationed. The remaining securities are set to the lowest quality and split among the unconstrained investors and the issuer.
The issuer therefore engages in securitization when there is full information. It will also retain part of the worst securities if necessary to maximize its sales. Notice that the issuer is maximizing total welfare. This will not be the case when there is asymmetric information with a rating agency.

2.2 Asymmetric Information

When the issuer’s type is private information, the issuer faces the problem of persuading investors that the securities are of a certain quality. We will demonstrate that this directly leads to a lemons problem. This is similar to the adverse selection problem found in the empirical work of Downing, Jaffee, and Wallace (2009) and An, Deng, and Gabriel (2011), who document a lemons spread and worse ex-post performance when issuers have more scope for selecting the loans that are securitized.

We assume the issuer will offer a range of securities to investors with labels of their quality. Investors will observe the quantity of securities $q_i$ and the reported value of the securities $R_i$, where $i \in \{U,C\}$. We employ the equilibrium concept of Perfect Bayesian Equilibrium. In the following lemma, we describe the equilibrium allocation.

**Lemma 2** In equilibrium, the issuer will sell $1 - \alpha$ securities of value $B$ to unconstrained investors and retain the rest.

This represents a breakdown of the market typical for adverse selection problems. The issuer can’t include any good assets in equilibrium. If it did, and investors believed the good assets were included and raised their valuations, the issuer would then replace the assets with bad ones to capture the extra rents. This temptation leads to only bad assets being sold.

The welfare loss from asymmetric information is equal to $(G - g)\mu + (B - b)\mu (G - \bar{V}) / (\bar{V} - B)$, the loss from being unable to sell a tranche backed by a mass $\mu$ of good assets and a mass $\mu (G - \bar{V}) / (\bar{V} - B)$ of bad assets to the constrained investors.

3 The Model with a Rating Agency

In this section, we examine whether a rating agency can reduce or eliminate the asymmetric information problem. We also study how ratings interact with the structuring of the investments. We focus on a monopoly rating agency.
The CRA reduces the lemons problem through the reputation it acquires over time. We model two types of rating agency: truthful ($T$) and opportunistic ($O$). This follows the approach of Fulghieri, Strobl, and Xia (2012) and Mathis, McAndrews, and Rochet (2009) (who in turn follow the classic approach of modeling reputation of Kreps and Wilson (1984) and Milgrom and Roberts (1984)). The opportunistic CRA will announce the value for each security, but will choose its announcement and the security design on the basis of its incentives. The truthful CRA is a behavioral player and announces truthfully the value of the securities it rates. Despite being behavioral, the truthful CRA may be strategic in our model. It choose the design of the security sold, while being restricted to report the actual value of its design. This is a departure from the literature, which reduces the opportunistic player to a nonstrategic player.\footnote{The only exception we are aware of is Hartman-Glaser (2012) where the truthful issuer can decide how much to retain of a security.}

Our model will have two periods. The CRA will be the same for both periods and each period there will be a different issuer. For ease of exposition, we will begin by describing a one-period version of this model. The probability of facing a truthful CRA at the beginning of the period is given by the prior, $\theta$, which, together with the structure of the game and payoffs, is common knowledge. We also assume the issuer knows the type of the rating agency.\footnote{As the issuer knows the quality of its securities, this is the most natural assumption; otherwise, both types of rating agency would be involved in a two-sided signaling game as in Bouvard and Levy (2010), Frenkel (2012), and Bar-Isaac and Deb (2012). Other papers on CRAs do not need to make an assumption about this as the issuer has no choice variable.}

The CRA observes perfectly the quality of the issuer’s assets and makes a take-it-or-leave-it offer to the issuer. As part of its services, the CRA designs and rates the securities offered by the issuer for a fee $f \geq 0$. This fee is unobservable to investors. While in practice, the issuer will initially design the securities and get feedback from the rating agencies about modifications necessary to achieve certain ratings\footnote{See details in Griffin and Tang (2012). Rating agencies also provide their basic model to issuers to communicate further. For example, Benmelech and Dlugosz (2009) write, “The CDO Evaluator software [from S&P, publicly available] enabled issuers to structure their CDOs to achieve the highest possible credit rating at the lowest possible cost... the model provided a sensitivity analysis feature that made it easy for issuers to target the highest possible credit rating at the lowest cost.”}, we incorporate this back and forth into one step for simplicity. If the issuer does not use a rating agency it may issue securities nevertheless. Therefore the issuer can get at least its asymmetric information net payoff of $(1 - \alpha)(B - b)$ by not purchasing ratings. We will
assume the CRA incurs a positive, but arbitrarily small cost of issuing a rating. Hence, in any equilibrium the CRA is hired if and only if it can create additional surplus.

Denote a message that can be sent by a CRA of type \( d \in \{T, O\} \) by 
\[
\begin{align*}
m^d = (R^d_C, R^d_U, q^d_C, q^d_U)
\end{align*}
\]
and the set of such messages by \( M \), where \( R^d_i \) is the reported value (rating) of a security intended for an investor of type \( i \in \{C, U\} \). The quantity \( q^d_i \) represents the amount of securities available for investors of type \( i \in \{C, U\} \) from an issuer paired with a CRA of type \( d \in \{T, O\} \). This quantity is observable to investors. If the issuer does not issue a security intended for group \( i \), we set \( q^d_i = 0 \) and \( R^d_i = 0 \). Denote the true valuation of the securities issued by \( v^d = (V^d_C, V^d_U) \). A strategy for a CRA of type \( d \) is thus 
\[
\begin{align*}
\begin{pmatrix} m^d \\ v^d \end{pmatrix} \in S_d,
\end{align*}
\]
where \( S_d \) is the strategy space of type \( d \).

Let \( \beta : M \rightarrow \Delta \) be the belief function of the investors, assigning a probability distribution over the set of CRA types upon observing \( m \), so that \( \beta(d|m) \) is the conditional belief that a CRA is of type \( d \in \{T, O\} \) given a message \( m \). Let \( v^\beta_i(m) \) be the expected valuation of investors conditional on message \( m \) under the beliefs \( \beta \) if \( q_i > 0 \), and zero otherwise. Each unconstrained investor makes a bid of \( p_U(m) = v^\beta_U(m) \). Each constrained investor makes a bid of \( p_C(m) \), where \( p_C(m) = v^\beta_C(m) \) if \( v^\beta_C(m) \leq V \) and \( p_C(m) = 0 \) otherwise.

While we allow ratings to be continuous, in reality, CRAs use discrete ratings. In principle, ratings correspond to ranges of default probabilities - although CRAs do not publish the ranges corresponding to the ratings. Allowing for ratings from a continuous range in the model has several benefits. First, it does not make us impose an arbitrary scaling and allows us to be general. Second, it allows us to abstract from “rating at the edge”, i.e. setting securities to the lowest value of a prescribed range. While this may have been an important phenomenon, rational investors should anticipate this and adjust accordingly, thus undoing its effect. Third, even legislation such as the Dodd-Frank bill has recognized that structured finance ratings are different from corporate bond ratings, meaning that in the model we are effectively allowing the rating agency to set its standards.\(^{10}\)

Note that our assumption that \( M \geq 2 \) guarantees that the opportunistic CRA has sufficiently many bad assets to create tranches of size equal to the truthful CRA’s, containing only bad assets.

To summarize, the timing of the game with one issuer is as follows:

\(^{10}\)For a basic metric, Gorton (2012) shows that asset backed securities have significantly higher cumulative default rates compared to equivalently rated corporate bonds.
0. Nature selects the type $d$ of the CRA.

1. The CRA offers the issuer a contract for fee $f^d$.

2. If the issuer accepts, then the CRA designs the securities, selects tranche sizes, and reports values. Otherwise, the issuer designs the securities, selects tranche sizes and sells the securities himself, without any rating.

3. Investors observe the sizes of tranches, any reported values (and who is reporting them) and buy securities at their conditional expected value.

We suppose steps 1-3 are repeated in a second period, and that the issuer is different in each period. If the different types of CRAs separate in the first period, then second-period investors update their priors about the type of the CRA accordingly. If the different types of CRAs pool in the first period, investors are still able to update their priors. The reason is that in this case, we will assume that investors discover the type of the opportunistic CRA between periods with a positive probability. This probability depends on the amount of rating inflation the opportunistic CRA chooses. We will define this probability and the dynamics explicitly in Section 4. Now, we focus on the second period choices.

### 3.1 The Second Period

In this section, we will analyze the second period, when the type of the CRA has not been revealed in the first period and the posterior that the CRA is truthful is $\theta_2$. Since this is the last period, the opportunistic CRA has no reputation concerns. An alternative interpretation of this section is that it analyzes a one-period version of the model.

Our first result concerns the securities offered by the issuer at the opportunistic CRA.

**Lemma 3** In any equilibrium of the second period, any security rated by the opportunistic CRA will have a value of $B$.

Without reputation concerns, the opportunistic CRA has no incentive to include good assets in the pool of assets to sell since the actual composition is not observable to investors.

We say that an equilibrium is pooling if it has the property that both types of CRAs report the same values of all securities and the sizes of all tranches are the same (we will also include any equilibrium where both CRAs are not hired in this category). We call any equilibrium which is not pooling and where at least one CRA is hired, a separating equilibrium.
Lemma 4 In the second period, there is no separating equilibrium.

This is an important result in the characterization of the equilibrium. If there were a separating equilibrium, the opportunistic CRA would be recognized and the best it could do is sell bad assets to unconstrained investors at fair value. As the issuer could do this without the CRA, the opportunistic CRA would not be hired given the small fixed cost of operating.

Given this result, if a CRA is hired, only pooling equilibria are possible. The possible pooling equilibria where CRAs are active could have securities sold in one tranche only to unconstrained investors, securities sold in one tranche only to constrained investors, or securities sold in two tranches, one meant for each type of investor. All of these possible pooling equilibria exist. However, after we refine the set of equilibria, there will no longer be one where securities are sold in one tranche only to constrained investors.

Given the numerous equilibria that can be supported by a variety of off-the-equilibrium path beliefs, we use the refinement concept of Unbeaten Equilibrium, introduced by Mailath, Okuno-Fujiwara, and Postlewaite (1993). Placing restrictions on off-the-equilibrium path beliefs using a concept such as the Intuitive Criterion (Cho and Kreps, 1987) has little bite in this environment, whereas the Unbeaten Equilibrium concept selects a unique equilibrium outcome for a given set of parameters. We give a brief intuitive discussion of the concept here, and define it formally in the Appendix.

The undefeated equilibrium concept is used to select among different Pure Strategy Perfect Bayesian Equilibria (PBEs). In our setting, these are equilibria such that (1) each type of CRA is using a pure strategy and maximizing profits given the investors' bids and the other CRA's strategy, (2) each investor bids his expected value conditional upon observed tranche sizes and reported values, and (3) beliefs are calculated using Bayes' rule for tranche sizes and reported values used with positive probability.

A PBE, \(E\), is said to defeat another PBE, \(E'\), if: (1) there is a message \(m\) sent only in \(E\), (2) the set of types \(K\) who send this message are all better off in \(E\) than in \(E'\) and at least one of them is strictly better off, and (3) under \(E'\), the beliefs about some such a type are not a posterior off-the-equilibrium path assuming only types in \(K\) send \(m\). A PBE is said to be undefeated if the game has no other PBE that defeats it.

The undefeated concept essentially works by checking that no types in one equilibrium are better off in another equilibrium where they choose a different action/message.\(^{11}\)

\(^{11}\)While this works by comparing equilibrium payoffs, Mailath, Okuno-Fujiwara, and
We now write two conditions which will help define the parameter space for the unique undefeated equilibrium outcome.

\begin{align*}
\theta_2(G - B) &> g - b \quad \text{(C1)} \\
\theta_2 G + (1 - \theta_2) B &\geq \bar{V} \quad \text{(C2)}
\end{align*}

The first condition says that the truthful CRA strictly prefers to add one more good asset rather than a bad asset to the asset pool being sold. The second condition states that if the truthful CRA placed only good assets into the tranche for the constrained investors and the opportunistic CRA put only bad assets into that tranche, the constrained investors would be willing to invest.

We now proceed to find the undefeated equilibrium.

**Proposition 1** If and only if C2 holds, the unique outcome of any undefeated equilibrium, \( E^* \), has two tranches of the following form:

1. The tranche designed by the truthful CRA for constrained investors contains all of the good assets and an amount of bad assets such that, given the opportunistic CRA only uses bad assets, the average value in the pool equals \( \bar{V} \).

2. The tranche designed by the truthful CRA for unconstrained investors contains a measure \( 1 - \alpha \) of bad assets.

3. Profits for the truthful CRA are equal to:

\[
(\bar{V} - b) \mu \theta_2 (G - B) / (\bar{V} - B) - \mu (g - b)
\]

4. Profits for the opportunistic CRA are equal to:

\[
(\bar{V} - b) \mu \theta_2 (G - B) / (\bar{V} - B)
\]

5. The report for the constrained tranche is \( R_C = (\bar{V} - (1 - \theta_2)B) / \theta_2 \) and the quantity sold for that tranche is \( q_C = \mu \theta_2 (G - B) / (\bar{V} - B) \).

---

Postlewaite (1993) suggest this places more realistic restrictions on off-the-equilibrium path beliefs than other concepts by using beliefs from an actual equilibrium. In the examples they examine, this selects the most reasonable equilibria. This concept is also used in several other papers, including Taylor (1999), Gomes (2000), and Fishman and Hagerty (2003).
In the proposition, the unique undefeated equilibrium outcome has two tranches: it sells to both constrained investors and unconstrained investors. The issuer at the truthful CRA puts all of its good assets as well as some bad assets in the tranche for the constrained investors, while the issuer at the opportunistic CRA puts in only bad assets. Both put in only bad assets for the unconstrained tranche. Both tranches are priced according to the rational expectations of investors, meaning the prices are also dependent on the investors’ perception that the CRA is truthful. Note that the issuer at the truthful CRA generates strictly lower revenues than in full information and sells off strictly fewer assets. The opportunistic CRA does strictly better than the truthful CRA as it receives the same price and sells off more bad assets (and retains more good assets). The issuer with an opportunistic CRA offloads more bad assets than if there were asymmetric information with no CRA.

For our next set of parameters, we find a unique one-tranche undefeated equilibrium outcome.

**Proposition 2** If and only if $C_2$ does not hold and $C_1$ holds, the unique outcome of any undefeated equilibrium, $E_*$, has one tranche of the following form:

1. The truthful CRA issues one tranche of size $1 - \alpha$ for the unconstrained investors backed by a measure $\mu$ of good assets and $1 - \alpha - \mu$ of bad assets.

2. The profits for a truthful CRA are:

   $$\mu (\theta_2(G - B) + b - g)$$

3. The profits for an opportunistic CRA are:

   $$\mu \theta_2(G - B)$$

In this proposition, the unique undefeated equilibrium outcome has one tranche with securities sold to all of the unconstrained investors. The truthful CRA places all of its good assets in the tranche, and as many bad assets as it can to satisfy the demand of the unconstrained investors. The price of the securities reflects the value and the perceived probability that the CRA is truthful. Because of the presence of the good assets, the issuer at both types of CRA generates strictly larger revenues than in the case of asymmetric information with no CRA. Once again, the opportunistic CRA does better than the truthful CRA.

For the last set of parameters, no CRA is hired:
**Corollary 1** If C2 and C1 do not hold, any equilibrium, $E_\emptyset$, has neither of the CRAs being hired.

This follows directly from the proofs of Proposition 1 and Proposition 2. In this equilibrium the CRA can’t generate value for the issuer, so the issuer does not hire the CRA and issues securities of value $B$, which are purchased by unconstrained investors.

From the above, it follows immediately that any undefeated equilibrium where the CRAs are hired has rating inflation. For the equilibrium with two tranches, the top tranche’s rating is equal to the value of what the truthful CRA is offering, but this is above the expected value by investors since the opportunistic CRA sells only bad assets. For the equilibrium with one tranche, there is a similar type of inflation. Despite the potential for a large amount of rating inflation, it is clear that securitization improves welfare in the second period compared to the benchmark of no CRA, as otherwise the issuers would not hire the CRA.

Given Proposition 1, Proposition 2, and Corollary 1, we can now look at the equilibrium configuration, i.e. the parameter space for which each equilibrium exists.

**Corollary 2** The equilibrium configuration has the following features:

1. If $\bar{V} - g > B - b$: for $\theta_2 \geq \frac{\bar{V} - B}{\bar{C} - B}$, the equilibrium is $E_{**}$, for $\frac{\bar{V} - B}{\bar{C} - B} > \theta_2 > \frac{g - b}{\bar{G} - \bar{B}}$, the equilibrium is $E_*$, and for $\frac{g - b}{\bar{G} - \bar{B}} \geq \theta_2$, the equilibrium is $E_\emptyset$.

2. If $B - b \geq \bar{V} - g$: for $\theta_2 \geq \frac{\bar{V} - B}{\bar{G} - \bar{B}}$, the equilibrium is $E_{**}$, for $\frac{\bar{V} - B}{\bar{G} - \bar{B}} > \theta_2$, the equilibrium is $E_\emptyset$.

We do not prove the corollary, as it follows directly from the above propositions and the assumption that $\bar{V} - g > 0$. We illustrate the equilibrium configuration in Figure 1.
The corollary provides several insights. First, a one-tranche equilibrium only exists if $V - g > B - b$. This reflects the fact that the quality requirement of constrained investors is high relative to the benefit of retaining $G$ assets and a tranche dedicated to constrained investors will not always be sustainable. It also means that the benefit to pushing $B$ assets onto investors is not that large, which makes it desirable to sell off $G$ assets to the unconstrained investors. Second, the two tranche equilibrium exists when $\theta_2$ is large. This means that it takes a substantial amount of reputation for honesty to be able to sell to constrained investors. Third, the larger the quality requirement of constrained investors, the less likely it is that there will be a two tranche equilibrium.\footnote{This can be found directly from the corollary by shifting $V$.} 

In the next section, we go to the first period and examine how the payoffs of the second period create reputation effects for the opportunistic CRA and whether they can eliminate conflicts of interest.

\section{The First Period}

In this section, we will analyze equilibrium behavior in the first period. We begin by defining a reputation mechanism to link periods 1 and 2. Define $p$ as the probability that the type of the opportunistic CRA is discovered after period 1 ends and before period 2 begins. We posit that the type of the opportunistic CRA will be more likely to be discovered the more inaccurate
its ratings are.\textsuperscript{13} Therefore we define $p$ such that $p = 1$ if the CRAs separate in the first period, and otherwise $p = h(z)$, where the argument $z$ measures how inflated (or inaccurate) the ratings are. We assume a functional form for $z$:

$$z = q_U(R_U - V_U^O) + q_C(R_C - V_C^O)$$  \hspace{1cm} (5)

This represents the aggregate difference between reported and actual values for all securities issued.\textsuperscript{14} Therefore the likelihood of being caught depends both on the magnitude of the divergence between the ratings and the actual quality and on the quantity of securities that had inflated ratings. The function $h$ is assumed to be strictly convex and continuously differentiable such that $h(z) = 0$ for $z \leq 0$, $h'(0) = 0$, and $h'(\mu(G - B)) = \infty$. This functional form implies that aggregate rating deflation has no benefit. It also rules out corner solutions where the opportunistic CRA only includes bad assets ($V_U^O = V_C^O = B$) whenever the truthful CRA includes all of issuer’s good assets.

Each CRA wants to maximize its expected discounted profits. For the opportunistic CRA, expected discounted profits are given by:

$$\Pi^O = \pi_1^O + \delta(1 - p)\pi_2^O.$$  

Here, $\pi_1^O$ is first-period profits, $\pi_2^O$ is second-period profits, $p$ is a probability that the type of the CRA is discovered and $\delta$ is the discount factor. If there is no rating inflation at all, the opportunistic CRA is secure and will earn its full period two profits. If there is rating inflation and the opportunistic CRA is discovered, it is not hired in period 2. If the CRA’s type is not revealed in period one, then the equilibrium posterior that it is truthful is $\theta_2 = \theta_1 / (\theta_1 + (1 - p)(1 - \theta_1))$ in the beginning of period two, where $\theta_1$ denotes the prior at the beginning of period one. It follows immediately from this formula that in this case $\theta_1 \leq \theta_2$. Given that an opportunistic CRA was not revealed in the first period, it is more likely that the CRA is truthful.

We have already shown that there are no separating equilibria in the second period. The following Lemma extends this result to the first period.

\textsuperscript{13}Note that in the CRA literature (e.g. Fulghieri, Strobl, and Xia (2012), Mathis, McAndrews, and Rochet (2009), and Bar-Isaac and Shapiro (2013)) the reputation mechanism is much simpler, as those papers have an investment that is binary, and only defaults in the bad state. Therefore something rated good that defaults leads directly to learning. Because of the generality of our setup, we define this mechanism as ex-post learning from the divergence between the rating and the realized performance.

\textsuperscript{14}Note that the reports and quantities are not indexed by the type of CRA. We will show below that all equilibria are pooling, and in a pooling equilibrium, both types must have the same report and quantity.
Lemma 5 There is no equilibrium where the CRAs separate in the first period.

Separating in the first period implies that the opportunistic CRA won’t have business in either the first or second period and therefore will have a profitable deviation. We can thus restrict ourselves to looking only at pooling equilibria. In any pooling equilibrium, the opportunistic CRA’s choice of assets to include in the tranches, \((V^{O^*}_U, V^{O^*}_C)\), must be optimal given the first-period reports \(R^*_U = R^*_C\) and the quantities \(q^*_U = q^*_C\). More specifically, if the equilibrium has two tranches in the first period, then \((V^{O^*}_U, V^{O^*}_C)\) must be a solution to the following maximization problem (an analogous program applies if the equilibrium has only one tranche):

\[
\max_{V^*_U \in [B,G], V^*_C \in [B,G]} \{q^*_U (\theta_1 R^*_U + (1 - \theta_1)V^{O^*}_U - (g - b)(V^{O^*}_U - B)/(G - B) - b) + q^*_C (\theta_1 R^*_C + (1 - \theta_1)V^{O^*}_C - (g - b)(V^{O^*}_C - B)/(G - B) - b) + (1 - h(q^*_U (R^*_U - V^*_U) + q^*_C (R^*_C - V^*_C)))\delta \pi^O}
\]

The first line represents the opportunistic CRA’s profits from selling to unconstrained investors. Notice that the price paid depends on the beliefs of investors, which must be held fixed while finding the equilibrium choice. The cost to the CRA is opportunity cost of not holding the assets. The fraction of good assets is given by \(\gamma^O_U = (V^*_U - B)/(G - B)\), which is used to derive this cost. The second line represents the opportunistic CRA’s profits from selling to constrained investors. This is analogous to the first line. The third line represents the probability the opportunistic CRA will have a business in the second period times the discounted equilibrium profits in the second period (given equilibrium inflation in period 1). Note that the probability depends on the opportunistic CRA’s choice, as more distortion from the reported value will lower the likelihood of survival, but the equilibrium second period profits do not.

In any pooling equilibrium, the first order conditions with respect to the securities issued in period 1 are given by

\[
-(g - b)/(G - B) + h'(z)\delta \pi^O \leq 0, \quad (6)
\]

where the inequality can be replaced by an equality when \(V^{O^*}_U > B\) or \(V^{O^*}_C > B\).

The assumption that \(h'(\mu(G - B)) = \infty\) guarantees that (6) holds with equality when all good assets are included.\(^{15}\) Hence, in this case we can use

\(^{15}\)See Lemma 8 in the Appendix.
the first-order condition to solve for $z^*$, the total amount of rating inflation chosen by the opportunistic CRA. Notice that after substituting for $\pi_2^O$ in equation (6), which depends on $z$ through $\theta_2$, $z^*$ can be expressed only in terms of exogenous parameters.

We will now provide sufficient conditions for the existence of a two-tranche equilibrium. The following condition is analogous to condition C2, except that the prior, $\theta_2$, has been replaced by $\theta_1$.

$$\theta_1 G + (1 - \theta_1) B \geq \bar{V} \tag{C2'}$$

Condition C2' guarantees that in the first period, it is feasible to sell to constrained investors. Note that as $\theta_1 \leq \theta_2$ if the type of the CRA is not revealed in the first period, the above conditions then implies the corresponding second-period condition.

The two-period game analyzed in this paper has multiple equilibria and in order to select among them we would ideally like to apply something similar to the undefeated equilibrium concept that was employed to the second-period game in the previous section. However, the undefeated equilibrium concept is formally defined for one-stage signalling games and therefore has to be amended to fit our framework.\textsuperscript{16}

Let the second-period game given prior $\theta_2$ be the one-period game described in Section 3 where the prior is given by $\theta_2$ and CRA payoffs are defined by corresponding one-period profits. Let the first-period game be the one-period game described in Section 3 where the prior is given by $\theta_1$ and CRA payoffs are defined by the first-period profits plus the discounted expected second-period profits in an undefeated equilibrium of the second-period game given prior $\theta_2$, where $\theta_2$ is the posterior conditional upon the action by the CRA and whether the type was revealed between periods.

**Definition 1** We say that $\mathcal{E}$ is an **undefeated equilibrium of the full game** if:

1. For every prior $\theta_2 \in (0, 1)$, the restriction of $\mathcal{E}$ to the second period is an undefeated equilibrium of the second-period game given prior $\theta_2$.

2. The restriction of $\mathcal{E}$ to the first period is an undefeated equilibrium of the first-period game.

Using this definition, we can prove the following proposition. We will use the following notation to denote the posterior in period two if there is no

\textsuperscript{16}Mailath et al (1993) briefly discuss the possibility of extending their concept to general games with more stages and multiple players.
discovery, \( \theta_2(z) := \theta_1 / (\theta_1 + (1 - h(z))(1 - \theta_1)) \). Note that this is an increasing function of \( z \).

**Proposition 3** If \( C_2' \) holds, the unique outcome of any undefeated equilibrium of the full game, \( \mathcal{E}_{**} \), has two tranches of the following form in the first period:

1. Inflation, \( z^* \), by the opportunistic CRA occurs in tranche for constrained investors and is implicitly defined by:
   \[
   \frac{g - b}{G - B} = \frac{h'(z^*)(G - B)\delta (\bar{V} - b) \mu \theta_2(z^*)}{(\bar{V} - B)}.
   \]

2. The tranche for constrained investors has a size of
   \[
   q_C = \frac{\mu (G - B) - (1 - \theta_1)z^*}{\bar{V} - B}.
   \]

3. For the constrained investors, the truthful CRA issues securities backed by a measure \( \mu \) of good assets and the rest bad assets, and the opportunistic CRA issues securities backed by a measure \( \mu - \frac{z^*}{G - B} \) of good assets and the rest bad assets, such that the expected value in the constrained tranche equals \( \bar{V} \).

4. The tranche for unconstrained investors contains a measure \( 1 - \alpha \) of bad assets.

5. First-period profits for the truthful CRA are equal to:
   \[
   q_C(\bar{V} - b) - \mu(g - b).
   \]

6. First-period profits for the opportunistic CRA are equal to:
   \[
   q_C(\bar{V} - b) + \frac{g - b}{G - B}z^* - \mu(g - b).
   \]

In \( \mathcal{E}_{**} \), a two-tranche equilibrium similar to \( E_{**} \) is played in the first period. More specifically, in the first period the tranche for unconstrained investors will contain securities of value \( R_U = V_U^O = V_U^T = B \) and the quantity of securities will equal the quantity of unconstrained investors. The tranche for constrained investors will contain as many securities as possible with an expected value of \( \bar{V} \). The truthful CRA will place all of its good assets and some bad assets in this tranche, whereas the opportunistic will
place a fraction of the good assets in to maintain reputation concerns, and fill the rest with bad assets.

In $E_{++}$, the second period equilibrium outcome is $E_{++}$, which also has two tranches, as $C_2'$ implies $C_2$ (given that $\theta_2 > \theta_1$).

In the Appendix, we prove Proposition 3 by showing that given the prescribed sufficient condition, the above equilibrium outcome maximize both the truthful and the opportunistic CRAs’ payoffs over the set of potential pooling equilibria. Hence, the equilibrium outcome is not just Pareto efficient in the sense that no type could be made better off without another being made worse off. It goes beyond this to say that these are the equilibria both types would select.

We will now proceed with comparative statics for equilibria of type $E_{++}$. The following proposition describes how rating inflation changes with the parameters in equilibrium.

**Proposition 4** If $C_2'$ holds the rating inflation by the opportunistic CRA in period 1 in any undefeated equilibrium of the full game is:

1. Increasing in $g$ and $V$
2. Decreasing in $\delta, \mu, G, b,$ and $\theta_1$
3. Increasing in $B$ if $2\bar{V} > G + B$ and decreasing in $B$ if $2\bar{V} > G + B$.

Rating inflation thus increases if there is a larger payoff to retaining good assets as the incentive to pass off bad assets is larger. For the same reason, it decreases if the premium for good assets is larger. Holding those fixed, inflation decreases with the fraction of good assets. This set of results is quite interesting; if the quality of the asset pool improves, then there will be less ratings inflation.

Rating inflation increases if constrained investors demand higher quality assets. This occurs because second-period profits are decreasing in the quality requirement of constrained investors, as it is more difficult to push securities onto them. As second-period profits decline, the cost of inflating ratings dissipated.

Inflation goes down if the prior that the CRA is truthful in period 1 is larger. The insight on the prior comes from the fact that the more likely the period 1 CRA is truthful, the more there is to gain for the opportunistic CRA in period 2, implying it will choose less rating inflation in period 1 to increase the chance of survival. Interestingly, one might posit that there should be a tradeoff, as if the prior is larger, the opportunistic has higher gains from inflation in period 1. This effect is not there, however, since when
the opportunistic CRA chooses which assets to include in period 1 (and hence inflation) it must take the price of a security and the beliefs of investors as fixed, meaning that there is no marginal impact of the prior on its inflation choice.

Lastly, rating inflation decreases if reputation is more important, which is proxied for by the discount factor.

Finally, we will look into the welfare properties of \( \mathcal{E}_{**} \). Welfare is given by the weighted sum of CRA payoffs for the two-period game plus the surplus of the issuer. This is:

\[
W_{**} = \theta_1 \pi_1^T + (1 - \theta_1) \pi_1^O + \delta \{ \theta_1 \pi_2^T + (1 - \theta_1)(1 - p) \pi_2^O \} \\
\quad + (1 + \delta)(1 - \alpha)(B - b).
\]

Substituting and simplifying, we find that

\[
W_{**} = \left( \frac{\tilde{V} - b}{V - B} - \frac{g - b}{G - B} \right) \left( (1 + \delta \theta_1) \mu (G - B) - z^*(1 - \theta_1) \right) \\
\quad + (1 + \delta)(1 - \alpha)(B - b).
\]

Using this, the following comparative statics are straightforward to compute.

**Proposition 5** If \( C2' \) holds, the ex ante welfare in any undefeated equilibrium of the full game \( \mathcal{E}_{**} \) is increasing in \( \theta_1 \) and \( \mu \), and decreasing in \( \tilde{V} \) and \( z^* \).\(^{17}\)

Welfare decreases with the minimum quality requirement of constrained investors, \( \tilde{V} \), as that reduces the possibility of selling assets. This suggests that the benefits of regulation that constrain investors must be traded off with the reduced efficiency of capital allocation. Welfare also increases in the probability of a CRA being truthful, \( \theta_1 \), and in the measure of good assets, \( \mu \), as both allow the total amount of assets sold to increase. Lastly, an increase in the probability of catching rating inflation will increase welfare - transparency and provision of historical data is beneficial in this environment.

5 Conclusion

In this paper we examine the interaction between security design, credit rating agencies, and investor clienteles. This is particularly important in the wake of the poor performance of ratings for structured products.

\(^{17}\)The change in \( z^* \) is assumed to be the result of an exogenous change in the \( h \) function, which will thus not affect the other primitives of the model.
We model rating agencies as long lived players with incentives driven by reputation. They structure products with issuers for constrained and unconstrained investors. The presence of constrained investors provides a new motivation for the pooling and tranching of assets; catering to a specific clientele. We find that when quality requirements for constrained investors are higher, rating inflation increases in response, as lower future profits create more incentives to take advantage of current investors.

We also demonstrate that reputational incentives affect security design in the form of the number of tranches. Rating inflation occurs in the top tranches and will depend negatively on the quality of the asset pool, as the incentive to pass off a bad asset as a good one is lower.

There are several future avenues of research to explore. It would be of interest to add risk (and risk aversion) to the model, to relate our results to others in the literature. Furthermore, we would also like to examine the role of competition and shopping in this environment.

References


Appendix

Proof of Lemma 1

Consider first selling only to unconstrained investors. The net payoff in (3) is increasing in $q_U$, so the issuer will set $q_U = 1 - \alpha$ and sell to all unconstrained investors. Furthermore, the expression is increasing in $\gamma_U$ if $G - g > B - b$ and decreasing in $\gamma_U$ if $G - g < B - b$. Therefore if $G - g > B - b$, the issuer will set $\gamma_U = \frac{\mu}{1 - \alpha}$. This yields a net payoff of:

$$\mu(G - g - B + b)) + (1 - \alpha)(B - b). \quad (8)$$

If $G - g < B - b$, the issuer will set $\gamma_U = 0$, as the issuer can sell all $B$ assets to the unconstrained investors, given that $M \geq 2$. This yields a net payoff of

$$(1 - \alpha)(B - b). \quad (9)$$

The same net payoff entails if $G - g = B - b$.

Now consider the case of selling to both types of investor. If securities worth $V > V$ were sold to constrained investors, the issuer could always
add more bad assets to the constrained portfolio and increase its profits. Therefore, securities sold to constrained investors will be worth \( V \) to them,

\[
\gamma'_c G + (1 - \gamma'_c) B = V. \tag{10}
\]

For the same reason, the issuer will set \( q_U = 1 - \alpha \). Finally, it is easy to see that the net payoff is maximized if all good assets are allocated to the constrained tranche, so that \( q'_C \gamma'_C = \mu \) given A1. This implies that \( q'_C = \mu(G - B)/(V - B) \) and that \( \gamma'_U = 0 \), \( q'_U = 1 - \alpha \), and \( \gamma'_I = 0 \). The maximum net payoff selling to both types is thus:

\[
(1 - \alpha) (B - b) + (G - g) \mu + (B - b) \mu (G - V)/(V - B). \tag{11}
\]

As the issuer is able to sell all of the good assets at their maximum value and more than \( 1 - \alpha \) of the bad assets at their maximum value, the payoff from selling to both types of investor is strictly larger than the payoff of selling to just unconstrained investors.

**Proof of Lemma 2**

First, consider the issuer announcing the full information equilibrium allocation, i.e. \( \mu(G - B)/(V - B) \) securities worth \( V \) and \( 1 - \alpha \) securities worth \( B \), where it retains the rest. This can’t be an equilibrium, as the issuer has the incentive to substitute its retained assets for the assets in the portfolio meant for constrained investors. It will be unable to sell to constrained investors at all in fact, because if at some point constrained investors believe they are getting a high quality security, the issuer will move more bad assets into the portfolio. The issuer is also unable to sell securities with a value greater than \( B \) to unconstrained investors, as the issuer can’t commit not to substitute something more \( B \) assets into the portfolio. Therefore, the only possible allocation is selling securities of value \( B \) to the \( 1 - \alpha \) unconstrained investors. The payoff for the issuer is \( \mu g + (1 - \alpha) B + (M - \mu - (1 - \alpha)) b \).

**Proof of Lemma 3**

Suppose there is an equilibrium where the CRA is hired where this is not true and the price of a security is \( p \geq B \). Then, since \( B < g \), the opportunistic CRA could gain by retaining the good assets backing the tranches and replacing them with bad assets (recall that there are always enough bad assets to do this since \( M \geq 2 \)) without changing its reported values.
Proof of Lemma 4

In a separating equilibrium, the type of each CRA would be revealed perfectly. Hence, the opportunistic CRA would only be able to issue securities worth $B$, by Lemma 3, and it would thus only be able to sell to unconstrained investors. Since it could not create any value, it would not be hired by the issuer.

The truthful CRA could not be issuing securities resulting in a positive surplus, or the opportunistic CRA would have a profitable deviation by mimicking the sizes and ratings of its issues. Furthermore, if it were issuing securities worth $B$, it would not be hired.

Undefeated Equilibria: Definition and Application

In this subsection, we define the concept of Undefeated Equilibria, as put forth by Mailath, Okuno-Fujiwara, and Postlewaite (1993). We begin with the definition of a Pure Strategy Perfect Bayesian Equilibrium (PBE).

In addition to the notation in section 3.1, we add the following. Denote an arbitrary CRA type by $d$ and the set of such types by $D = \{T, O\}$. Let $p = (p_U, p_C)$. The profits to the CRA of type $d$ are denoted by $u(s, p, d)$. Finally, define the probability function $\Theta(d)$ such that $\Theta(T) = \theta$ and $\Theta(O) = 1 - \theta$.

**Definition 2** $E^* = (s^*, p^*, \beta^*)$ is a Pure Strategy Perfect Bayesian Equilibrium (PBE) if and only if:

1. $\forall d \in D : s^*(d) \in \arg \max_{s \in S^d} u(s, p, d)$,

2. $\forall m \in M : p_U(m) = v_U^\beta(m)$, and $p_C(m) = v_C^\beta(m)$ if $v_U^\beta(m) \geq \bar{V}$ and $p_C(m) = 0$ otherwise,

3. $\forall d \in D$ and $\forall m \in M : \beta^*(d|m) = \Theta(d)1_{m(d) = m}/\sum_{d' \in D} \Theta(d')1_{m(d') = m}$ if the denominator is positive, where $1_{m(d) = m}$ is an indicator function that takes the value 1 if $m(d) = m$ and 0 otherwise.

In words, a strategy profile and a belief function constitute a Pure Strategy Perfect Bayesian Equilibrium if: 1. each type of CRA is using pure strategy maximizing profits given the investors’ bids and the other CRA’s strategy, 2. each investor bids his expected value conditional upon observed tranche sizes and reported values, 3. beliefs are calculated using Bayes’ rule for tranche sizes and reported values used with positive probability.
Definition 3 A PBE, $E = (s, p, \beta)$, **defeats** another PBE, $E' = (s', p', \beta')$, if and only if:

1. $\forall d \in D : m'(d) \neq m$ and $K = \{d \in D : m(d) = m\} \neq \emptyset$,
2. $\forall d \in K : u(s, p, d) \geq u(s', p', d)$ and $\exists d \in K : u(s, p, d) > u(s', p', d)$,
3. $\exists d \in K : \beta'(d | m) \neq \Theta(d) \pi(d) / \sum_{d' \in D} \Theta(d') \pi(d')$ for some $\pi : D \rightarrow [0, 1]$ satisfying:
   - $d' \in K$ and $u(s', p', d') < u(s, p, d') \Rightarrow \pi(d') = 1$, and
   - $d' \notin K \Rightarrow \pi(d') = 0$.

In words, an equilibrium $E$ defeats another equilibrium $E'$ if: 1. there is a message $m$ sent only in $E$, 2. the set of types $K$ who send this message are all better off in $E$ than in $E'$ and at least one of them is strictly better off, and 3. under $E'$, the beliefs about some such a type are not a posterior off-the-equilibrium path assuming only types in $K$ send $m$ and that they do so with probability one if they are strictly worse off than under $E$.

A PBE is said to be **undefeated** if the game has no other PBE that defeats it.

In order to apply the undefeated concept, we define a **strictly Pareto dominant equilibrium** as a PBE that has strictly higher payoff for both types of CRAs than any PBE with a different strategy profile.

**Lemma 6** If a strictly Pareto dominant equilibrium exists, then it defeats any PBE with a different strategy profile.

**Proof.** If the game has a unique PBE the proof is trivial. Suppose therefore that the game has a strictly Pareto dominant equilibrium, $E$, and a PBE with a different strategy profile, $E'$. First note that by Lemma 4, both must be pooling (although the CRAs may not be hired in one of them). Second, since 1. the truthful CRA is restricted to honest reports and 2. the truthful must use different strategies in $E$ and $E'$, the messages sent in the two equilibria must be different, $m \neq m'(if the CRAs are not hired in one of the equilibria, the corresponding message is empty). Third, both CRAs are strictly better off under $E$ than under $E'$. Finally, beliefs in $E'$ given the message $m$ cannot be a posterior assuming the truthful CRA sends this message with probability one, or there would be profitable unilateral deviation for both types of CRAs. Hence, $E$ defeats $E'$.

Therefore, it suffices to find a strictly Pareto dominant equilibrium.
Proof of Proposition 1

We begin by finding the equilibrium with two tranches that maximizes the profits of the truthful CRA.

It is clear that a necessary condition for the equilibrium is that $\theta_2 G + (1 - \theta_2)B \geq V$, since otherwise the constrained investors cannot be served.

To make the notation slightly easier, let $\mu_U$ and $\mu_C$ be the measure of good securities backing the unconstrained and constrained tranches of the truthful CRA, and let $\nu_U$ and $\nu_C$ be the measure of bad securities backing the unconstrained and constrained tranches of the truthful CRA. Therefore mapping this to our notation in the text, $\mu_j = q_j \gamma_j^T$ and $\nu_j = q_j (1 - \gamma_j^T)$, where $j \in \{U, C\}$.

Using this, and Lemma 3, we can write the net profits in any possible pooling equilibrium as:

$$\theta_2 (\mu_U G + \nu_U B) + (1 - \theta_2) B (\mu_U + \nu_U) - \mu_U g - \nu_U b + \theta_2 (\mu_C G + \nu_C B) + (1 - \theta_2) B (\mu_C + \nu_C) - \mu_C g - \nu_C b - (1 - \alpha)(B - b)$$

The first line is the payoff from selling to unconstrained investors rather than retaining the assets. The second line is the payoff from selling to constrained investors rather than retaining the assets, minus the payoff an issuer could achieve without the CRA. We simplify this expression:

$$\mu_U (\theta_2 G + (1 - \theta_2) B - g) + \nu_U (B - b) + \mu_C (\theta_2 G + (1 - \theta_2) B - g) + \nu_C (B - b) - (1 - \alpha)(B - b).$$

This should be maximized with respect to $\mu_U, \mu_C, \nu_U, \nu_C$ given the restrictions:

$$0 \leq \mu_U + \nu_U \leq 1 - \alpha,$$
$$0 \leq \mu_C + \nu_C \leq \alpha,$$
$$0 \leq \mu_U + \mu_C \leq \mu,$$
$$\mu_C (\theta_2 G + (1 - \theta_2) B - \tilde{V}) / (\tilde{V} - B) \geq \nu_C.$$

The first two inequalities say that the amount of securities sold to each type of investor can’t exceed the amount of each type of investor. The third inequality states that the amount of good assets sold can’t surpass the amount of good assets available. The last inequality represents the fact that constrained investors demand securities with expected value of at least $\tilde{V}$.

It is clear that any solution has $\mu_U + \nu_U = 1 - \alpha$, since otherwise profits can always be increased by increasing $\nu_U$ without affecting any of the other constraints.
Any solution must have the second constraint not binding, as assumption A1 tells us it is impossible given the scarcity of good assets to sell to all constrained investors. Without the second constraint, the fourth constraint will bind since the amount of bad assets for the constrained, $C$, will be increased until the minimum quality requirement of the constrained investors is binding. This yields:

$$\mu_C(G\theta_2 + (1 - \theta_2)B - \bar{V})/(\bar{V} - B) = \nu_C.$$  

Substituting the binding constraints into the expression for net profits gives:

$$\mu_U(\theta_2G + (1 - \theta_2)B - g) + (1 - \alpha - \mu_U)(B - b) + \mu_C(\theta_2G + (1 - \theta_2)B - g) +$$

$$\mu_C(B - b)(\theta_2G + (1 - \theta_2)B - \bar{V})/(\bar{V} - B) - (1 - \alpha)(B - b)$$

It is clear that $\mu_U$ has a strictly lower partial derivative than $\mu_C$.

Hence, if the partial derivative with respect to $\mu_C$ is positive, the solution has $\mu_U = 0$, $\mu_C = \mu$, $\nu_U = 1 - \alpha$, and $\nu_C = \mu(\theta_2G + (1 - \theta_2)B - \bar{V})/(\bar{V} - B)$. This clearly implies a positive profit, and therefore the CRA would be hired.

If the partial with respect to $\mu_C$ is negative, then the partial with respect $\mu_U$ is also negative, and hence $\mu_U = 0$, $\mu_C = 0$, $\nu_U = 1 - \alpha$, and $\nu_C = 0$. Since this implies zero profits, excluding rating costs, the CRA would not be hired.

If the partial with respect to $\mu_C$ is zero, then the partial with respect $\mu_U$ is negative, and hence (c) $\mu_U = 0$, $\mu_C = \mu$, $\nu_U = 1 - \alpha$, and $\nu_C = \mu_C(\theta_2G + (1 - \theta_2)B - \bar{V})/(\bar{V} - B)$. Since the profits are zero excluding rating costs, the CRA would not be hired.

It is easy to see that the solutions can be implemented as equilibria. Just assume beliefs are equal to the prior for any out-of-equilibrium message. The truthful CRA then has no incentive to deviate, since the equilibrium is maximizing net profits. Moreover, the opportunistic CRA is also weakly better off than for any other message.

Finally, we will show that the above equilibrium maximizes the profits of the truthful CRA. If we denote the truthful CRA’s profits by $\pi_T$, then the profits for the opportunistic CRA can be written $\pi_T + (\mu_U + \mu_C)(g - b)$. Since the partial of profits with respect to $\mu_C$ is even higher than for the truthful CRA, if the equilibrium is payoff maximizing for the truthful CRA, then so is it for the opportunistic CRA. Therefore it is a Pareto Dominant equilibrium, which by Lemma 6 means that it is undefeated.

Lastly, we can solve for the report and quantity issued for the constrained tranche (going back to our notation in the text) using the following two equations:
\[ \theta_2 R_C + (1 - \theta_2) B = \bar{V} \]
\[ q_C R_C = \mu G + (q_C - \mu) B, \]

The first expression says that the expected value of the constrained tranche will be equal to \( \bar{V} \). The second expression defines \( q_C \) from the choice of the truthful CRA: the total value of the truthful CRA’s pool of assets for the constrained tranche is equal to the total amount of \( G \) assets it has and the number of \( B \) assets it includes.

There are two unknowns in the two equations above, \( q_C \) and \( R_C \). Solving, we obtain the following:

\[ R_C = \frac{(\bar{V} - (1 - \theta_2) B)}{\theta_2} \]
\[ q_C = \frac{(\theta_2 \mu (G - B))}{(\bar{V} - B)}. \]

**Proof of Proposition 2**

As with the previous proposition, we begin by finding the equilibrium that maximizes the profits of the truthful CRA.

If \( \theta_2 G + (1 - \theta_2) B < \bar{V} \), the truthful CRA cannot sell to the constrained investors. If the restriction \( \mu_C = 0 \) is added to the optimization problem in the proof of Proposition 1, the solution has \( \mu_U = \mu \), \( \mu_C = 0 \), \( \nu_U = 1 - \alpha - \mu_U \), and \( \nu_C = 0 \) if the partial with respect to \( \mu_U \), \( \theta_2 (G - B) - (g - b) \), is positive. For sufficiently small rating cost, this is clearly payoff superior to issuing \( 1 - \alpha \) securities worth \( B \) to unconstrained investors, and hence the CRA will be hired.

If the partial is negative, then the solution has \( \mu_U = 0 \), \( \mu_C = 0 \), \( \nu_U = 1 - \alpha \), and \( \nu_C = 0 \). However, in this case, due to the cost of rating, the CRA would not be hired.

If the partial is zero, then the solution has \( \mu_U \leq \mu \), \( \mu_C = 0 \), \( \nu_U = 1 - \alpha - \mu_U \), and \( \nu_C = 0 \). However, since the payoff is the same as in the solution with negative partial, the CRA would not be hired.

It is easy to see that the first solution can be implemented as an equilibrium. Just assume beliefs are equal to the prior for any out-of-equilibrium message. The truthful CRA then has no incentive to deviate, since the equilibrium is maximizing net profits. Moreover, the opportunistic CRA is also weakly better off than for any other message.

Finally, the above equilibrium maximizes the profits of the truthful CRA. If we denote the truthful CRA’s profits by \( \pi_T^2 \), then the profits for the opportunistic CRA are \( \pi_T^2 + \mu (g - b) \). Since the partial of profits with respect
to $\mu_C$ is even higher than for the truthful CRA, if the equilibrium is payoff maximizing for the truthful CRA, then so is it for the opportunistic CRA. Therefore it is a Pareto Dominant equilibrium, which by Lemma 6 means that it is undefeated.

**Proof of Lemma 5**

In a separating equilibrium, the type of each CRA would be revealed perfectly. Hence, by Lemma 3 the opportunistic CRA would only be able to issue securities worth $B$ in period two, and it would thus not be hired then. This implies that it has no reputation concerns and would never issue a security worth more than $B$ in period one either, and as it is separating in the first period, it would not be hired in the first period either.

The truthful CRA could not issue securities in period one resulting in a positive surplus on its own, or the opportunistic CRA would have a profitable deviation by mimicking the sizes and ratings of its issues (with actual values equal to or lower than the reported). Furthermore, if it were issuing securities worth $B$, it would not be hired.

**Proof of Proposition 3**

We will prove the proposition by showing that the equilibrium outcome under $E_{ss}$ yields each type of CRA a higher payoff than any other equilibrium of the first-period game, thus demonstrating that it is strictly Pareto dominant, and thereafter invoke Lemma 6 to show it is undefeated. We start with a number of useful Lemmata.

**Lemma 7** If $C2'$ holds, then $E_{ss}$ can be sustained as an equilibrium of the full game.

**Proof.** Assume out-of-equilibrium path beliefs assigning probability one to the CRA being opportunistic. We know from the above that $\theta_1 \leq \theta_2$ if the type of the CRA is not revealed in period 1. Hence, $C2'$ implies C2. This guarantees that $E_{ss}$ is a unique undefeated equilibrium outcome of the game in the second period and that both types of CRAs earn positive profits then if no type is revealed in the first period.

In the first period, the issuer at the opportunistic CRA issues securities worth $V_C^O \geq B$ to constrained investors. Hence, by C2' such investors can be served. Moreover, the fact that $V_C^O \geq B$ also implies that the first-period profit of the truthful CRA will be larger than $(V - b) \mu \theta_1 (G - B)/(V - B) - \mu (g - b)$
(which would be the profits if the opportunistic CRA only included bad assets in the constrained tranche), and hence positive by C2’.

Consider a deviation by the truthful CRA in the first period. Such a deviation would, according to the beliefs assumed, imply that the CRA is not hired in any of the periods and hence earns a profit of zero.

The opportunistic CRA would not deviate by changing its value $V_O^U$ or $V_O^C$ as these are given by the incentive compatibility constraint (6). It will also not deviate by choosing a different message given the off-the-equilibrium-path beliefs.

**Lemma 8** If the truthful CRA sells all of its good assets, then equilibrium inflation in the first period is given by the interior solution to the opportunistic CRA’s incentive compatibility constraint (equation 6).

**Proof.** This follows since, by assumption, $h'(\mu(G-B)) = \infty$ and at any corner solution where the opportunistic CRA sells no good assets, first-period inflation is given by $\mu(G-B)$.

**Lemma 9** If C2’ holds, $E^{**}$ strictly Pareto dominates any equilibrium where the truthful CRA includes zero good assets.

**Proof.** First, we know from Lemma 7 that under C2’, $E^{**}$ exists and profits in the strictly Pareto dominant second-period equilibrium (assuming the type was not revealed in period one) are increasing in $\theta_2$ for both types of CRAs.

Second, in any potential equilibrium where no good assets are sold in the first period, the CRA will not be hired then and subsequently $\theta_2 = \theta_1$.

Third, in $E^{**}$ the truthful CRA sells all of the good assets in the first period, earns positive first-period profits, and has a posterior strictly larger than the prior, $\theta_2 > \theta_1$.

Fourth, in $E^{**}$ the opportunistic CRA also earns a strictly higher payoff than in any equilibrium when no good assets are sold. It this were not true, it would have a profitable deviation by issuing securities that are identical to the ones issued by the truthful CRA in the first period. Such a deviation would give positive first-period profits (identical to those of the truthful CRA), zero revelation probability, and second-period profits identical to the maximum that could be obtained if no good assets were sold in the first period.

**Lemma 10** If C2’ holds, the outcome of $E^{**}$ maximizes payoffs to the truthful CRA in the first-period game.
Proof: We know from Lemma 5, that any equilibrium must be pooling. Consider the objective function of the truthful CRA:

\[
\theta_1(\mu_U G + \nu_U B) + (1 - \theta_1) (\mu_U^0 G + (\mu_U + \nu_U - \mu_U^0) B) - \mu_U g - \nu_U b + \\
\theta_1(\mu_C G + \nu_C B) + (1 - \theta_1) (\mu_C^0 G + (\mu_C + \nu_C - \mu_C^0) B) - \mu_C g - \nu_C b \\
-(1 - \alpha)(B - b) + \delta \pi_2^T(\theta_2),
\]

where \(\mu_U^0\) and \(\mu_C^0\) are the measures of good assets sold by the opportunistic CRA for the unconstrained and constrained tranche respectively, and \(\pi_2^U(\theta_2)\) are the second-period profits to the truthful CRA in the unique undefeated equilibrium outcome of the corresponding second-period game. Note that \(\pi_2^U(\theta_2)\) depends positively on inflation, \(z\), through \(\pi_2^U(\theta_2)\). We are looking for the strictly Pareto dominant equilibrium, which implies this expression should be maximized with respect to all of the choice variables \(\mu_U, \mu_C, \nu_U, \nu_C, \mu_U^0,\) and \(\mu_C^0\) given non-negativity constraints and the restrictions:

1. \(1 - \alpha - \nu_U - \mu_U \geq 0\), \(\text{(A)}\)
2. \(\mu - \mu_U - \mu_C \geq 0\), \(\text{(C)}\)
3. \(\theta_1(\mu_C G + \nu_C B) + (1 - \theta_1) (\mu_C^0 G + (\mu_C + \nu_C - \mu_C^0) B)
   - \bar{V}(\mu_C + \nu_C) \geq 0\), \(\text{(D)}\)
4. \((\mu_U - \mu_U^0)(G - B) + (\mu_C - \mu_C^0)(G - B) - \zeta' = 0\), \(\text{(E)}\)

where
\[
\zeta' = \min \{z^*, (\mu_U + \mu_C)(G - B)\}
\]
and \(z^*\) is the inflation when the opportunistic CRA’s incentive compatibility constraint holds with equality (equation 6).

The last constraint says that equilibrium inflation is given by the first-order condition unless this is ruled out by the non-negativity constraints, \(\mu_U^0 \geq 0\) and \(\mu_C^0 \geq 0\), in which case inflation is given by \((\mu_U + \mu_C)(G - B)\).

Note that due to assumption A1, the constraint \(\alpha - \mu_C - \nu_C \geq 0\) is redundant.

We set up the Lagrangian \(L\) with multipliers named after each constraint \((A, C, D, \text{and } E)\) and obtain the following (simplified) first-order conditions.\(^\text{18}\) Each holds with equality if the relevant variable is greater than zero.

\(^{18}\)We here abstract from the complication due to the fact that if \(C = 0\), so that both \(E_*\) and \(E_{**}\) are possible, \(\pi_2(\theta_2)\) may not be differentiable with respect to \(\theta_2\) at the point where the second period equilibrium switches from \(E_*\) to \(E_{**}\). We later show that \(C > 0\).
\[
\frac{\partial L}{\partial \mu_U^C} = 1 - \theta_1 - E \leq 0
\] (12)

\[
\frac{\partial L}{\partial \mu_C} = 1 - \theta_1 + D (1 - \theta_1) - E \leq 0
\] (13)

\[
\frac{\partial L}{\partial \nu_U} = B - b - A \leq 0
\] (14)

\[
\frac{\partial L}{\partial \mu_U} = \theta_1 G + (1 - \theta_1) B - g + \delta \pi_2^{Tr}(\theta_2) \frac{\partial \theta_2}{\partial \mu_U} \frac{\partial z'}{\partial \mu_U}
\] (15)

\[
-A - C + E (G - B) - E \frac{\partial z'}{\partial \mu_U} \leq 0
\]

\[
\frac{\partial L}{\partial \nu_C} = B - b + D (B - \bar{V}) \leq 0
\] (16)

\[
\frac{\partial L}{\partial \mu_C} = \theta_1 G + (1 - \theta_1) B - g + \delta \pi_2^{Tr}(\theta_2) \frac{\partial \theta_2}{\partial \mu_C} \frac{\partial z'}{\partial \mu_C}
\] (17)

\[-C + D \left( \theta_1 G + (1 - \theta_1) B - \bar{V} \right) + E (G - B) - E \frac{\partial z'}{\partial \mu_C} \leq 0.
\]

Note that either we have a corner solution where \( \frac{\partial z'}{\partial \mu_C} = \frac{\partial z'}{\partial \mu_U} = G - B \), or an interior solution where \( \frac{\partial z'}{\partial \mu_C} = \frac{\partial z'}{\partial \mu_U} = 0 \).

We can find the unique solution in 5 steps.

1. Condition (14) implies that \( A > 0 \), and hence, by the assumption that \( 1 - \alpha > \mu, \nu_U > 0 \) and \( A = B - b \).

2. Condition (16) implies \( D \geq \frac{B - b}{V - \bar{V}} > 0 \).

3. Together with (12) and (13), this implies \( E > 1 - \theta_1 \) and \( \mu_U^C = 0 \).

4. Since \( \mu_C + \mu_U > 0 \), by Lemma (9), and \( \frac{\partial L}{\partial \nu_U} < \frac{\partial L}{\partial \nu_C} \), due to \( D > 0 \) and \( A > 0 \), we must have \( \mu_C > 0 \), and \( \mu_U = 0 \).

5. From (16) and (C2'), it follows that

\[
\theta_1 G + (1 - \theta_1) B - g + D \left( \theta_1 G + (1 - \theta_1) B - \bar{V} \right) > 0,
\]

which, together with (17), imply \( C > 0 \). Thus, by Lemma (8), \( z' = z^* \). \( \blacksquare \)

It remains to show that this equilibrium is also payoff maximizing for the opportunistic CRA under the specified conditions.
Lemma 11 If $C_2'$ holds, $\mathcal{E}_{**}$ maximizes the payoffs to the opportunistic CRA in the first-period game.

Consider the objective function of the opportunistic CRA:

$$
\theta_1(\mu_U G + \nu U B) + (1 - \theta_1) \left( \mu^O_U G + (\mu_U + \nu_U - \mu^O_U) B \right) - \mu^O_G g - (\mu_U + \nu_U - \mu^O_U) b + \\
\theta_1(\mu_C G + \nu C B) + (1 - \theta_1) \left( \mu^O_C G + (\mu_C + \nu_C - \mu^O_C) B \right) - \mu^O_C g - (\mu_C + \nu_C - \mu^O_C) b \\
-(1 - \alpha)(B - b) + (1 - h'(z')) \delta \pi^O_2(\theta_2),
$$

where $\pi^O_2(\theta_2)$ are the second-period profits of the opportunistic CRA in the unique undefeated equilibrium outcome of the corresponding second-period game.

We set up the Lagrangian $L$ using each of the same constraints from the truthful CRA’s optimization problem, with the same notation for the multipliers ($A, C, D, \text{ and } E$). This expression should be maximized with respect to $\mu_U, \nu_U, \nu_C, \mu^O_U, \mu^O_C$. We obtain the following (simplified) first-order conditions. Each holds with equality if the relevant variable is greater than zero.

$$
\frac{\partial L}{\partial \mu_U} = 1 - \theta_1 - E - \left( \frac{g - b}{G - B} \right) \leq 0
$$

$$
\frac{\partial L}{\partial \mu_C} = 1 - \theta_1 + D (1 - \theta_1) - E - \left( \frac{g - b}{G - B} \right) \leq 0
$$

$$
\frac{\partial L}{\partial \nu_U} = B - b - A \leq 0
$$

$$
\frac{\partial L}{\partial \nu_C} = B - b + (1 - h'(z')) \delta \pi^O_2(\theta_2) \frac{\partial \theta_2}{\partial \nu_U} \frac{\partial z'}{\partial \nu_U} - A - C + E (G - B) - E \frac{\partial z'}{\partial \nu_U} \leq 0
$$

$$
\frac{\partial L}{\partial \nu_C} = B - b + (1 - h'(z')) \delta \pi^O_2(\theta_2) \frac{\partial \theta_2}{\partial \nu_C} \frac{\partial z'}{\partial \nu_C} - A - C + D (\theta_1 G + (1 - \theta_1) B - V) \leq 0
$$

We now find the unique solution in 5 steps.
1. Condition (20) implies that $A > 0$, and hence, by the assumption $1 - \alpha > \mu, \nu_U > 0$ and $A = B - b$.

2. Condition (22) implies $D \geq \frac{B - b}{V - B} > 0$.

3. Together with (18) and (19), this implies $E > 1 - \theta_1 - \frac{g - b}{G - B}$ and $\mu_U^O = 0$.

4. Since $\mu_C + \mu_U > 0$, by Lemma (9), and $\frac{\partial L}{\partial \mu_U} < \frac{\partial L}{\partial \mu_C}$, due to $D > 0$ and $A > 0$, we must have $\mu_C > 0$, and $\mu_U = 0$.

5. We know from (22) and (C2') that

$$\theta_1 G + (1 - \theta_1) B - b + D (\theta_1 G + (1 - \theta_1) B - V) > 0.$$ 

Moreover, in any equilibrium the following incentive compatibility constraint must hold (equation 6), or the opportunistic CRA would have a profitable deviation:

$$h'(z') \delta \pi^O_2 (\theta_2) \leq (g - b) / (G - B).$$

The latter implies,

$$g - b - h'(z') \delta \pi^O_2 (\theta_2) \frac{\partial z'}{\partial \mu_C} \geq g - b - (g - b) / (G - B) \frac{\partial z'}{\partial \mu_C} \geq 0.$$

Hence, unless $C > 0$, $\frac{\partial L}{\partial \mu_C}$ would be positive. ■

We are now in a position to complete the proof of Proposition 3. Propositions 1, 2, and Corollary 1 characterize the unique undefeated equilibrium outcome of the second-period game for any prior $\theta_2$. Lemmata 10 and 11 demonstrate that the restriction of $E_*$ to the first period is a strictly Pareto dominant equilibrium of the first-period game and therefore, by Lemma 6, also undefeated.

**Proof of Proposition 4**

Define the function $k_{\theta_1} ()$ implicitly by

$$h'(z)(\theta_1 / ((1 - h(z)) (1 - \theta_1))) = k_{\theta_1}^{-1}(z).$$

The left hand side of this equation is increasing in both $\theta_1$ and $z$. This implies the function $k_{\theta_1}(x)$ is increasing in $x$ and decreasing in $\theta_1$. Using this
function and the incentive compatibility constraint given that $E_{ss}$ is played in the last period, we can solve for $z$:

$$z^* = k\theta_1 \left( \frac{(g-b)(\bar{V}-B)}{\delta \mu (G-B)^2 (V-b)} \right)$$

(25)

It follows that $z^*$ is increasing in $g$ and $\bar{V}$ and decreasing in $\delta$, $\mu$, $G$, and $\theta_1$. It is increasing in $B$ if $2\bar{V} > G + B$ and decreasing in $B$ if $2\bar{V} > G + B$. It is increasing in $b$ if $g > \bar{V}$ and decreasing in $b$ if $g < \bar{V}$.

**Proof of Proposition 5**

Differentiating the welfare expression (7) gives:

$$\frac{dW}{d\theta_1} = \left( \frac{\bar{V} - b}{\bar{V} - B} - \frac{g-b}{G-B} \right) \left( \delta \mu (G-B) + z^* \frac{dz^*}{d\theta_1} (1 - \theta_1) \right)$$

$$\frac{dW}{dV} = - \left( \frac{\bar{V} - b}{\bar{V} - B} - \frac{g-b}{G-B} \right) \frac{dz^*}{d\theta_1} (1 - \theta_1)$$

$$\frac{dW}{d\mu} = \left( \frac{\bar{V} - b}{\bar{V} - B} - \frac{g-b}{G-B} \right) \left( (1 + \delta \theta_1) (G-B) - \frac{dz^*}{d\mu} (1 - \theta_1) \right)$$

$$\frac{dW}{dz^*} = - \left( \frac{\bar{V} - b}{\bar{V} - B} - \frac{g-b}{G-B} \right) (1 - \theta_1).$$

The signs of the derivatives follow since, by C2',

$$\frac{\bar{V} - b}{\bar{V} - B} - \frac{g-b}{G-B} > 0,$$

and, by $q_C > 0$,

$$\mu (G-B) - z^*(1 - \theta_1) > 0.$$