SPECULATION, EXPECTATIONS AND RISK PREMIA:
AN AFFINE GAUSSIAN FRAMEWORK
PRELIMINARY

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ABSTRACT. This paper generalizes the Affine Gaussian asset pricing framework to settings in which rational traders have private information relevant for predicting future prices. The framework features a continuous cross-sectional distribution of expectations about future bond yields and we demonstrate how this makes it possible to exploit “raw” survey data in likelihood based estimation of model parameters. We use the framework to analyze the term structure of interest rates and show how to decompose bond yields into three components: (i) average expectations about short rates (ii) common risk premia and (iii) a speculative component due to traders taking advantage of private information about future risk free rates and risk premia. The empirical results suggest that the speculative component is quantitatively important, and more so for medium- and long-maturity bonds. The model nests the standard affine Gaussian model as a special case.

1. INTRODUCTION

Both casual observation and survey evidence suggest that different people have different views about what the return of a given asset will be. This paper presents a model that can fit this fact and analyzes its implications for the dynamics of the term structure of interest rates in a setting where all traders are rational. In the model presented below, heterogeneity in return expectations is caused by traders observing different signals. Relaxing the assumption that all traders have access to the same information is well-known to have theoretically interesting implications for asset prices but so far little has been done to estimate the quantitative importance of private information in asset markets. This paper helps to fill this gap by presenting a flexible affine framework in which rational traders have private information relevant for predicting future bond prices. The framework can be used to quantify the effects of private information on the dynamics of the term structure of interest rates. We find that speculative dynamics caused by traders exploiting what they perceive to be inaccurate market expectations can be quantitatively important and more so for long maturity bonds.

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1Some early examples of papers studying the theoretical implications of private information on asset prices are Hellwig (1980), Grossman and Stiglitz (1980), Admati (1985), Singleton (1987).
In markets where assets are traded among agents who may not want to hold on to the asset until it is liquidated, expectations about the resell value of an asset will matter for its current price. It is well-known that when traders have access to different information, the price of the asset may then deviate systematically from the “consensus value”, i.e. the hypothetical price that would reflect the average opinion of the (appropriately discounted) liquidation value of the asset (e.g. Allen, Morris and Shin 2006). The reason for this is that individual traders can then predict how the average expectation differs from their own best prediction. In addition, individual traders can also predict how average expectations will be revised in the future. That is, even though the expected liquidation value of the asset follows a martingale from the perspective of the individual trader, higher order expectations do not. To see why, consider an individual trader who believes that current average opinion about the liquidation value of an asset is inaccurate. If more information will be observed by other traders between the current period and the period of liquidation, it is rational for the individual trader to believe that other (rational) traders will adjust their expectations towards what the individual trader thinks is a more accurate prediction of the liquidation value of the asset. For example, if conditional on the individual traders information set the current average expectation appears too low, he can make a profit (in expectations) by taking a long position in the asset and reselling it in the next period when other traders have revised their expectations about the liquidation value upward. Conversely, if the current market expectation appears to high, taking a short position will result in a positive expected excess return.

The value at maturity of a zero coupon default-free bond is known with certainty, yet Nimark (2012) shows that similar speculative dynamics as those described above are at work in a term structure model when traders have private information relevant for predicting future short rates. There, an individual trader will take a long position in long maturity bonds if conditional on his own information, the average expectation about future short rates appears to be too high. As more information will be observed over time, it is rational for the individual trader to expect other rational traders to revise their expectations about future risk free short rates downwards resulting in an increase in bond prices.

In this paper we derive a flexible affine framework for quantifying the importance of the type of speculative behavior described above for the term structure of interest rates. The framework differs from most of the previous theoretical literature on asset pricing and private information in that we do not specify utility functions, nor do we model the portfolio decision of traders explicitly. Instead, we make an attempt to stay as close as possible to the large empirical literature that use affine models to study asset prices. In the standard full information affine no-arbitrage framework, variation across time in expected excess returns are explained by variation across time in either the amount of risk or in the required compensation for a given amount of risk. Gaussian models as in the $A_0(N)$ models of Dai and Singleton (2000) or in Joslin, Singleton and Zhu (2011) focus on the latter and identify the price of risk as an affine function of a small number of factors that also determine the dynamics of the risk free short rate. Similar to this approach, we specify a model in which variation in expected excess returns across individual traders must be accompanied by variation in the compensation for risk required by individual traders. That is, we identify how the required compensation for risk for an individual trader varies across time by assuming that
it is a function of the same trader specific state that also determines an individual trader’s expectations about future bond prices. The framework is flexible and it nests a standard affine Gaussian term structure model in the special case in which the private signals are perfectly informative about the state. This facilitates comparison of our results to the large existing literature on affine term structure models.

In the model presented below, dispersion of expectations about future interest rates arises from rational traders having private signals relevant for predicting future bond returns. Alternative approaches to model disagreement, or differences in beliefs, include modeling agents as having different degrees of overconfidence in the precision of a commonly observed signal as in Scheinkman and Xiong (2003), or by having agents learn from a common signal but starting from different priors as in Buraschi and Jiltsov (2006).

There exist papers that, like the present one, analyzes the implications of differences in beliefs (or expectations) on the term structure of interest rates. Xiong and Yan (2009) present a model in which two groups of traders with log-utility misinterpret an uninformative signal as being helpful in predicting future inflation. Since the two groups of agents have different subjective beliefs about the correlation between the uninformative signal and the inflation process, they have different subjective beliefs about future inflation and the real return on bonds. The model can generate speculative dynamics that to an econometrician appear to be time varying risk premia and the “excess” volatility of long bond yields documented by Gurkaynack, Sack, Swanson (2005). Similarly, Chen, Joslin and Tran (2010, 2012) develop a flexible affine term structure framework for modeling differences in beliefs in which at least one group of traders do not form model consistent expectations.

We think that one appealing feature of a fully rational framework as the one proposed here is that agents use the information contained in prices efficiently. In the model of Xiong and Yan (2009), an econometrician outside the model can predict excess returns conditioning only on publicly available bond prices, though the traders in the model can not (or perhaps choose not to). We find it unappealing to assume that we as econometricians could make larger trading profits than the traders inside the model simply by conditioning on publicly available prices. For these reasons, we think that a rational framework in which differences in expectations are caused by differences in information, and where the private signals remain unobservable to us as econometricians, is more suitable for empirical work.

The fact that traders form model consistent expectations also introduces additional restrictions on the dynamics of the model that are absent in models where differences in expectations are driven by exogenously specified differences in beliefs and where agents “agree to disagree”. In the model presented below, imposing model consistent expectations restricts the joint dynamics of bond prices and the cross-sectional dispersion of bond price expectations. In equilibrium, bond prices cannot reveal too much information since too informative bond prices would imply a counterfactually degenerate cross-sectional distribution of expectations. Put differently, the fact that agents learn from endogenous prices imposes more restrictions than models in which bond prices by assumption are not allowed to convey any information useful to individual traders.

2A game theorist might say that a trader’s active “type” must determine both his expectations about future returns as well as the compensation for risk that he requires.
Another difference-in-beliefs based alternative to model expectation heterogeneity is to let traders learn rationally from prices but starting from heterogenous priors as in Buraschi and Jiltsov (2006). However, rational learning from common signals implies that the beliefs of different traders will converge over time. This approach is thus not suitable for modeling and estimating phenomena that do not subside over time. Based on these considerations, we think modeling heterogenous expectations as arising from individual traders observing different signals is the most suitable approach for empirical work.

One of the aims of this paper is to quantify the importance of speculative dynamics driven by heterogenous expectations. To do so, we propose a new framework that combines the flexible specification for risk premia of the affine term structure literature with the speculative dynamics introduced by privately informed traders. This allows us to perform a three-way decomposition of the term structure of interest rates, which is the main empirical contribution of the paper. Two of the components correspond to the decomposition in Cochrane and Piazzesi (2008), i.e. they are the average expectation about future short rates and a risk premia that is common to all agents. The third term is the speculative component which is the result of traders taking advantage of private information to predict future short rates and risk premia. We find that this term is quantitatively important, and particularly so for long maturity bonds.

The model is estimated using quarterly bond yields data from 1971.Q4 to 2011.Q4. Since the model implied distribution of expectations across traders is continuous we use raw survey data, i.e. data on individual survey responses, when estimating the parameters of the model. To our knowledge, this is the first paper to use the full cross section of yield survey data to estimate a term structure model. This allows us to discipline the degree of cross sectional heterogeneity of expectations among traders. Since the model implies strong joint predictions about the cross-sectional dispersion of expectations and bond price dynamics, survey data sharpens our inference about the role private information plays in determining bond price dynamics.

The rest of paper is structured as follows. The next section presents an affine framework for modeling the term structure when traders have access to private information relevant for predicting future bond returns. Section 3 shows how the framework can be used to decompose the term structure into the standard components, i.e. risk premia and expectations about future short rates, as well as speculative components driven by traders exploiting what they perceive to be inaccurate average expectations about future bond prices. Section 4 describes how the model can be estimated and presents the empirical results. Section 5 concludes and the Appendix contains derivations and details left out of the main text.

2. An affine term structure model with private information

In this section we describe how bond prices can be determined in an arbitrage free framework when traders have private information relevant for predicting future bond returns. The basic set up follows a large part of the affine term structure literature (see Duffie and Kan

\[ \text{D'Amico Kim and Wei (2008), Lee Chun (2010) and Piazzesi and Schneider (2011) for example use survey data to better identify term structure dynamics. However, these studies use the median forecast as opposed to the entire cross-section.} \]
1996 and Dai and Singleton 2000) and posits that the short rate $r_t$ is an affine function of a vector of state variables. In the standard approach, a model is completed by specifying how traders price risk by assuming a functional form for the stochastic discount factor (SDF). This SDF can then be used to price contingent claims.

We will take a similar approach here, except that the SDFs must now be trader specific to accommodate the fact that each trader has its own private information. In the standard common information model, the price $P^n_t$ of a bond with $n$ periods to maturity is given by

$$P^n_t = E \left[ M_{t+1}^{n-1} P_{t+1}^{n-1} | \Omega_t \right]$$

(2.1)

where $P^{n-1}_{t+1}$ is the price of a $n - 1$ periods to maturity bond in period $t + 1$, $\Omega_t$ is the common information set in period $t$ and $M_{t+1}$ is the stochastic discount factor. In the absence of arbitrage, this relationship has to hold for all maturities $n$. In a model with private information a similar relation holds, expect that the SDF is now trader specific so that for all traders $j \in (0, 1)$ and all maturities $n$ the relationship

$$P^n_t = E \left[ M^j_{t+1} P^{n-1}_{t+1} | \Omega^j_t \right]$$

(2.2)

must hold. All traders pay the same price for bonds so the left hand side of (2.2) is common to all traders. However, private information relevant for predicting future bond prices introduces heterogeneity in expectations of $P^{n-1}_{t+1}$. For (2.2) to continue to hold when expectations differ across traders, $M^j_{t+1}$ must then be trader specific.

Allowing for privately informed traders introduces a few complications in terms of specifying and solving the model, relative to the standard common information set up. The first is related to the need for agents to “forecast the forecasts of others”, or what Allen, Morris and Shin (2006) has labeled the “beauty contest” aspect of asset markets with privately informed traders. When long maturity bonds are traded in secondary markets, traders need to predict what other traders will be willing to pay for a bond in the future. That is, when traders are price takers, the expectation of $P^{n-1}_{t+1}$ in the equilibrium condition (2.2) will depend on how much other traders will be willing to pay for an $n - 1$ periods to maturity bond in period $t + 1$. With private information, this may be different from what an individual trader would be willing to pay if he were to hold on to the bond until it matures. The reason for this is, as explained in the Introduction, that the law of iterated expectations does not hold for higher order expectations.

In practical terms, the beauty contest aspect introduced by private information implies that the factors determining the short rate are no longer a complete description of the state of the world. Instead, higher order expectations of the factors, i.e. expectations about other traders’ expectations about the factors, enter as endogenous state variables. The law of motion for the higher order expectations of the factors then has to be determined jointly with bond prices since traders use the information contained in bond prices to form expectations about the unobservable factors. Private information thus introduces an additional step in deriving a process for bond prices that is not present in the full information set up with only exogenous state variables.

The second difference relative to the common information set up is that we need to specify a functional form for the individual traders’ SDFs that allows for heterogeneity in expected returns. Below we propose a form that is analogous to the standard formulation under
common information and indeed, nests the standard formulation under the special case when there are no idiosyncratic shocks to the trader specific states.

Solving the model implies finding a fixed point on the mapping from traders’ expectations to bond prices and from bond prices to traders’ expectations. The approach we will take is the following. First, we will specify a functional form for the law of motion for higher order expectations of the exogenous factors. The exogenous factors as well as the average higher order expectations of these factors are the state of the model and bond prices are conjectured (and later verified) to be an affine function of this extended state. For a given law of motion of the state and functional form for the individual traders’ stochastic discount factor we can derive how bond prices depend on the state. Given bond prices as a function of the state, we can solve the traders’ filtering problem which in turn determines the law of motion for the (endogenous) state. This section presents the model framework but many of the details of how to solve the model are relegated to the Appendix. In the sections following this one, we will demonstrate how the model can be used to decompose the term structure into components driven by average expectations about future short rates, common risk premia and a speculative component driven by traders exploiting private information.

2.1. The conjectured processes for bond prices and the state. Following the affine term structure literature the one period risk-free rate $r_t$ is an affine function of a vector of state variables $x_t$

$$r_t = \delta_0 + \delta'_x x_t \quad (2.3)$$

and the vector $x_t$ follows a first order vector auto regression

$$x_{t+1} = \mu P + F P x_t + C \varepsilon_{t+1}. \quad (2.4)$$

In a full information setting, we would normally proceed by specifying a functional form for the stochastic discount factor that would allow us to derive the price of a bond of any maturity as an affine function of the factors $x_t$. In our private information set up the factors determining the short rate are not directly observable by the traders. Instead, traders receive noisy private signals about $x_t$. If long bonds are traded frequently and individual traders are price takers, individual traders expectations about future bond prices will depend on their expectations about other traders’ expectations about risk free rates and risk premia. These higher order expectations of future risk free rates and risk premia can be reduced to higher order expectations about the current latent factors $x_t$. The relevant state of the model is then the hierarchy of higher order expectations $X_t$ defined as

$$X_t \equiv \begin{bmatrix} x_t \\ x_t^{(1)} \\ \vdots \\ x_t^{(k)} \end{bmatrix} \quad (2.5)$$

with the average $k$ order expectations $x_t^{(k)}$ defined recursively as

$$x_t^{(k)} = \int E \left[ x_t^{(k-1)} \mid \Omega_t^j \right] \, dj$$
The information set of trader $j$ is denoted $\Omega^j_t$ and $\bar{k}$ is the maximum order of expectations considered. Nimark (2011) demonstrates that a finite $\bar{k}$ is sufficient to approximate the true equilibrium dynamics to an arbitrary accuracy.

We will conjecture (and later verify) that the state $X_t$ follows a first order vector auto regression process

$$X_{t+1} = \mu + MX_t + Nu_{t+1}$$

and that the price of a bond with maturity $n$ is an affine function of the state $X_t$ plus a maturity specific disturbance $v^n_t$

$$p^n_t = A_n + B'_n X_t + v^n_t.$$ (2.7)

That is, bond prices depend on the exogenous factors $x_t$ as well as on the average higher order expectations of these factors and without the maturity specific disturbances $v^n_t$ prices would reveal the expectations of others perfectly. The maturity specific disturbances thus plays a similar role as the noise traders in Admati (1985) and can be motivated as in Duffee (2011) as arising from “preferred habitats, special repurchase rates, or variations in liquidity”.

It is perhaps worth pointing out here that even though the state vector is high dimensional, this by itself does not increase our degrees of freedom in terms of fitting bond yields. The fact that the endogenous state variables $x_t^{(k)}$ are rational expectations of the lower order expectations in $x_t^{(k-1)}$ disciplines the law of motion (2.6) and the matrices $M$ and $N$ are completely pinned down by the parameters of the process governing the true exogenous factors $x_t$ and how precise traders’ signals about $x_t$ are.

2.2. The stochastic discount factor of trader $j$. When traders have access to private information, different traders may have different expectations about future returns. In the absence of arbitrage opportunities, any expected return in excess of the risk free rate must be compensation for risk. The price of risk must thus be determined partly by the same trader specific factors that determine return expectations. Here we show how this can be achieved by letting trader $j$’s SDF be a function of the same trader specific state that determines his expectations about future returns.

The stochastic discount factor of trader $j$ is denoted $M^j_{t+1}$ and in the absence of arbitrage, the relationship

$$p^n_{t+1} = \log E \left[ M^j_{t+1} p^n_{t+1} | \Omega^j_t \right]$$ (2.8)

must be satisfied for each trader $j$ and maturity $n$. With private information, expectations about future prices and discount factors may differ across traders. Following the full information affine term structure literature, we will specify the logarithm of trader $j$’s SDF to follow

$$m^j_{t+1} = -r_t - \frac{1}{2} \Lambda^j_t \Sigma_x \Sigma'_x \Lambda^j_t - \Lambda^j_t a^j_{t+1}.$$ (2.9)

The vector $a^j_{t+1}$ is the period $t + 1$ innovation to trader $j$’s state vector $X^j_t$, conditional on information available to trader $j$ up to period $t$ defined as

$$a^j_{t+1} = X^j_{t+1} - E(X^j_{t+1} | \Omega(j))$$ (2.10)
with conditional covariance matrix \( \Sigma \Sigma' \). The state \( X^j_t \) is given by:

\[
X^j_t = \left[ \begin{array}{c} x^j_t \\ E [X_t | \Omega^j_t] \end{array} \right]
\]

(2.11)

where

\[
x^j_t = x_t + Q \eta^j_t : \eta^j_t \sim N (0, I)
\]

(2.12)

is the vector of trader \( j \) specific exogenous factors. The vector \( x^j_t \) is also the source of trader \( j \)'s private information about the unobservable exogenous state \( x_t \). The precision of the signals \( x^j_t \) are common across traders and determined by the matrix \( Q \). The state \( X^j_t \) determines how much trader \( j \) requires to be compensated for risk. The individual vector of risk prices \( \Lambda^j_t \) in the stochastic discount factor (2.9) is an affine function of agent \( j \)'s state

\[
\Lambda^j_t = \Lambda_0 + \Lambda X^j_t
\]

(2.13)

Trader \( j \) observes \( x^j_t \) but cannot by direct observation disentangle the common factors \( x_t \) from the idiosyncratic component \( Q \eta^j_t \). Even though the idiosyncratic shocks \( \eta^j_t \) are purely transitory they will in general have persistent effects on trader \( j \)'s (higher order) expectations so that in general, the difference between trader \( j \)'s state \( X^j_t \) and the average state \( X_t \) can be expressed as

\[
X^j_t - X_t = Q(L) \eta^j_t
\]

(2.14)

where \( Q(L) \) is an infinite order lag polynomial. If \( Q = 0 \) so that \( x^j_t = x_t \), the vector \( a^j_{t+1} \) would simply be the innovation to the exogenous common factors implying that \( a^j_{t+1} = C \varepsilon_{t+1} \) and \( \Sigma \Sigma' = CC' \) in (2.4). The model then collapses to the full information affine model since the trader specific state \( x^j_t \) then perfectly reveals the true (common) latent factors \( x_t \) and all traders require the same compensation for risk. The history of realizations of the idiosyncratic components \( Q \eta^j_t \) also determines how trader \( j \)'s expectations about future returns differ from the average expectations, since the only source of private information is the observation of \( x^j_t \).

2.3. Traders’ filtering problem. The SDF introduced above prices conditional uncertainty about future bond prices. This uncertainty depends both on the uncertainty inherent in the unknowable nature of the future as well as on the uncertainty arising from the fact that the current state of the world is not directly observable. In order to form optimal expectations about future bond prices, traders thus need to form an optimal estimate of the current aggregate state \( X_t \). Since the model is linear with Gaussian shocks, the Kalman filter delivers optimal state estimates.

Traders know the law of motion as well as how the state maps into the vector of observable variables. They use this knowledge to form model consistent expectations of the aggregate state. In each period traders observe the short rate \( r_t \), a vector of current bond prices \( p_t \) as well as the trader specific factors \( x^j_t \). In period \( t \), the variables observable by trader \( j \) can be collected in the vector \( z^j_t \) defined as

\[
z^j_t \equiv \left[ \begin{array}{c} r_t \\ p_t \\ x^j_t \end{array} \right]
\]

(2.15)
Traders thus use equilibrium prices to learn about the unobservable state of the economy. This contrasts with difference-in-beliefs models where agents “agree to disagree”. When traders agree to disagree, the beliefs of all agents are common knowledge and from the traders’ perspective, there is no additional information contained in endogenous prices.

Traders do not forget and the information set of trader \( j \) is the filtration defined by

\[
\Omega_t^j \equiv \{ z_t^j, \Omega_{t-1}^j \} \tag{2.16}
\]

The law of motion of the state (2.6) and the definition of the observables then let us describe the filtering problem of trader \( j \) as a standard state space system

\[
X_{t+1} = \mu + MX_t + Nu_{t+1} \tag{2.17}
\]

\[
z_t^j = \mu_z + DX_t + \begin{bmatrix} R \\ R \end{bmatrix} \begin{bmatrix} u_t \\ \eta_t^j \end{bmatrix} \tag{2.18}
\]

where \( u_{t+1} \) is a vector containing the aggregate shocks \( \varepsilon_{t+1} \) and \( v_{t+1} \). Trader \( j \)’s state estimate evolves according to the Kalman filter updating equation

\[
E \left[ X_t \mid \Omega_t^j \right] = (I - KD) ME \left[ X_{t-1} \mid \Omega_{t-1}^j \right] + K z_t^j \tag{2.19}
\]

where \( K \) is the Kalman gain. Since bond prices are part of the observation vector \( z_t^j \), the selector matrix \( D \) in the measurement equation (2.18) is partly a function of the \( B_t' \) matrices in the bond price equation (2.7). This implies that we have to solve simultaneously for the filtering problem and the pricing equation (2.7).

2.4. The \( H \) operator. For future reference, it is useful to define the average expectations operator \( H : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1} \) that takes a hierarchy of expectations and moves all expectations one step up in orders of expectations by annihilating the first element in a hierarchy of expectations and sets all expectations of order \( k > \tilde{k} \) equal to zero so that

\[
\begin{bmatrix}
  x_t^{(1)} \\
  \vdots \\
  x_t^{(k)} \\
  0
\end{bmatrix}
= H
\begin{bmatrix}
  x_t \\
  x_t^{(1)} \\
  \vdots \\
  x_t^{(k)}
\end{bmatrix} \tag{2.20}
\]

The matrix \( H \) will be useful for deriving expressions that depend on expectations about the aggregate state \( X_t \) and expectations about other traders’ expectations.

2.5. The bond price recursions. The no-arbitrage condition (2.8) has to hold for all traders at all times. This fact, together with the functional form of trader \( j \)’s stochastic discount factor (2.9) implies that we could choose any trader \( j \)’s state \( X_t^j \) as being the state variable that bond prices are a function of. However, the most convenient choice from a modeling perspective is to let the “average” trader’s SDF price bonds. That is, we will let bonds be priced by the SDF of the fictional trader who’s state \( X_t^j \) coincides with the cross-sectional average state, defined as

\[
\int X_t^j \, dj \equiv X_t \tag{2.21}
\]
The identity of the average trader will change over time as idiosyncratic shocks cause individual traders’ states to move around within the cross-sectional distribution. That is, a trader who in a given period happens to be located in the middle of the distribution will only continue to be the average trader with probability zero. However, since (2.8) holds for all $j$ at all times, the identity of the average trader is of no consequence. The advantage of letting the average trader’s SDF price bonds is that it allows us to write bond prices in the conjectured form (2.7), i.e. as a function of the aggregate state $X_t$. In the Appendix we show that the recursions for the matrices $A_{n+1}$ and $B_{n+1}$ in the conjectured bond price equation are then given by

$$A_{n+1} = -\delta_0 + A_n + B_n'\mu^P + \frac{1}{2} \left[ B_n'\Sigma_{t+1|t} B_n' + V_n V_n' \right]$$

$$-B_n'\Sigma_{t+1|t} \Lambda_0 + B_n'N V_n' - V_n N' \Lambda_0$$

and

$$B_{n+1} = -\delta_X' X_0 + B_n'M H - (B_n'\Sigma_{t+1|t} + V_n N') \Lambda_X$$

where the matrix $\Sigma_{t+1|t}$ is the conditional variance of the errors of traders’ one step ahead expectations of the state defined as

$$\Sigma_{t+1|t} \equiv E \left( X_{t+1} - E \left[ X_{t+1} \mid \Omega^j_t \right] \right) \left( X_{t+1} - E \left[ X_{t+1} \mid \Omega^j_t \right] \right)' .$$

As in a full information set up, the recursions (2.22) and (2.23) can be started from

$$A_1 = -\delta_0$$

$$B_1 = \left[ \begin{array}{c} -\delta_X' \\ 0 \end{array} \right]$$

since $p^1_t = -r_t$.

The recursive expressions above are similar to the corresponding expressions under full information. Where there are differences, those arise because of the fact that under imperfect information, the conditional variance of the state $\Sigma_{t+1|t}$ now depends not only on the state innovations in the next period, i.e. the vector $\mathbf{N}\mathbf{u}_t$ in the law of motion (2.6), but also on the uncertainty about the current state $\Sigma_{t|t}$. In addition, the maturity specific disturbances $v^n_t$ affect the state $X_t$ directly when traders are imperfectly informed. The reason is that the shocks $v^n_t$ affect bond yields and traders use the information in bond yields to form an estimate of the state, and the state $X_t$ is partly made up of traders’ estimates of the state. Under full information, the maturity specific shocks do not affect traders’ estimates of the state since these are by construction equal to the exogenous state $x_t$. The term $V_n N'$ then equals zero, even though the maturity specific shocks still contribute to the overall variance of bond yields and traders still gets compensated for this risk through the term $V_n V_n'$ in equation (2.22) of the bond price recursions.

2.6. Expected returns and trader $j$’s price of risk. We now have all the components in place to show how the filtering problem of the agents together with the conjectured price process and the stochastic discount factor relate the expected return conditional on trader $j$’s information set to the compensation of risk as a function of trader $j$’s state $X^j_t$. Since $r_t$
and $p_{n+1}$ are observable we can express the expected excess return conditional on trader $j$’s information as
\[
E[p_{n+1}^n \mid \Omega_t] - p_{n+1} - r_t = A_n - A_{n+1} - \delta_0 + (B_n'\Sigma_{t+1} + V_n'N')\Lambda X^j_t
\]
where we used that
\[
E[p_{n+1}^n \mid \Omega_t] = A_n + B_n'MHX^j_t
\]
and the expression for $B_{n+1}$ from the recursion (2.23). As in the standard model, a positive expected excess return is compensation for risk. Some of this compensation for risk is constant over time which is given by $A_n - A_{n+1} - \delta_0$ and is a function of the constant price of risk $\Lambda_0$ as in the standard model. The difference here is that the time varying component of expected excess returns are partly trader specific. In the absence of arbitrage, it must be the case that a trader who expects a higher excess return than other traders must also require more compensation for risk. The trader specific price of risk must therefore be a function of the same trader specific state that determines a trader’s expectations. In equation (2.27) trader $j$’s compensation for risk is the last term on the right hand side, which is a function of the conditional variance $(B_n'\Sigma_{t+1} + V_n'N')$ and the hierarchy $X^j_t$. The matrix $\Lambda_X$ determines how they are combined into the time varying component of risk premia.

This ends the presentation of the model. We now show how the model can be used to decompose bond prices.

3. Decomposing the yield curve

This section demonstrates how the term structure can be decomposed into the standard components consisting of expected future short rates and risk premia as well as speculative components driven by traders exploiting private information. Harrison and Kreps (1978) defined speculative behavior as when the possibility of reselling an asset before it matures changes its equilibrium price. In the model presented here, we implicitly assume that long maturity bonds are traded in every period and that traders therefore need to predict what other traders will be willing to pay for a bond at the next trading opportunity. Since the price other traders will be willing to pay for a bond in the future is determined by future risk adjusted expectations of bond prices further into the future, individual traders will need to form higher order expectations about all the determinants of future bond prices. That is, traders need to form higher order expectations about both future risk premia as well as future short rates. Thus, relative to the model of Nimark (2012) there is now an additional speculative component to the term structure driven by traders exploiting what they rationally perceive to be inaccurate market expectations about future risk premia.

3.1. Defining speculative behavior. It is well known that higher order expectations are distinct from first order expectations when traders have access to private information (e.g. Allen, Morris and Shin 2006, Bacchetta van Wincoop 2006 and Nimark 2012). When individual traders believe that average expectations of the current state are inaccurate, that is, when first and higher order estimates of the current state do not coincide, individual traders can predict the direction that other traders will adjust their expectations in the future. When they take advantage of this predictability, speculative dynamics arise. However, if by chance an individual trader thinks that the average expectation is the same as his own best
prediction, no such predictability of others forecasts revisions are possible. The forecast of others’ forecasts then coincide with the individual’s own forecast, which is a martingale from the individual’s own perspective. This reasoning suggest that we should define the speculative component in an n period bond price as the difference between the actual price of the bond, and the counterfactual price that would prevail if average first and higher order expectations coincided. This counterfactual price is what Allen, Morris and Shin (2006) refers to as the “consensus value” of an asset and the difference between the “consensus value” and the actual price is what Bacchetta and van Wincoop (2006) refers to as the “higher order wedge”.

In our framework, the consensus value of an n period bond denoted $\overline{p}_t^n$ is defined as the price that would prevail if by chance

$$x_t^{(1)} = x_t^{(k)} : k = 1, 2, 3, ...$$

and given by

$$\overline{p}_t^n = A_n + B'_n H X_t$$

where the matrix $H$ is defined so that

$$\begin{bmatrix}
  x_t \\
  x_t^{(1)} \\
  \vdots \\
  x_t^{(k)}
\end{bmatrix} = H \begin{bmatrix}
  x_t \\
  x_t^{(1)} \\
  \vdots \\
  x_t^{(k)}
\end{bmatrix}$$

That is, $H$ is a matrix that takes a hierarchy of expectations and equates average higher order expectations with the average first order expectation. We can use $H$ to decompose the current n period bond price into a component that depends on the average first order expectations and a speculative component that is the difference between the actual price and the counterfactual consensus value $\overline{p}_t^n$ as follows

$$\overline{p}_t^n = A_n + B'_n H X_t + \left( B_n - B'_n H \right) X_t + v_t^n$$

The speculative term is thus the difference between the actual price and the answer you would get if you asked the “average” trader what he thinks the price would be if all traders, by chance, had the same state estimate as he did (while holding conditional uncertainty constant).

3.2. Higher order prediction errors. The speculative term (3.4) can be further decomposed into an (even!) more interesting form in order to separate speculation related to expected future short rates from speculation about future risk premia. First note that by repeated substitution in the bond price recursions (2.23), the vector $B_n$ can be expressed as

$$B'_n = -\sum_{s=0}^{n-1} \delta_X [MH - \Sigma_{t+1|t} \Lambda_X]^s + \sum_{s=1}^{n-1} V_s N \Lambda_X [MH - \Sigma_{t+1|t} \Lambda_X]^{n-1-s}$$

When expanded, the expression for $B'_n$ contains the term $-\sum_{s=0}^{n-1} \delta_X [MH]^s$. That is, the current price of an n period bond partly depends on the cumulative sum of higher order short
rate expectations between period $t$ and $t+n-1$. We can thus decompose $B'_n$ into a short rate (higher order) expectations term and a term $W_n$ capturing (higher order) expectations about future risk premia.

$$B'_n = -\delta_X \sum_{s=0}^{n-1} [MH]^s + W_n$$ (3.6)

That is, $W_n$ contains the sum of all terms in (3.5) that involves $\Lambda_X$. Using $\overline{H}$ and $W_n$ we can write the actual bond price as

$$p^n_t = A_n + W_n \overline{H} X_t - n (\delta_0 + \delta_X \mu_X) - \delta_X \sum_{s=0}^{n-1} [MH]^s \mu_X$$ (3.7)

$A_n$ and the cumulative sum of the average first order expectation of future risk premia $W_n \overline{H} X_t$. The second line is the cumulative sum of the average expectation of future short rates. The third line contains the higher order prediction error of future short rates, i.e. the cumulative difference between the average first order expectation and average higher order expectations of future short rates. Similarly, the fourth line contains the difference between average first order expectations and the higher order expectations about future risk premia. These two terms are thus capturing the speculative part of a bond price due to traders taking advantage of what they perceive to be mistakes in other traders’ expectations about future short rates and risk premia. As demonstrated in Nimark (2011) this type of speculative component must be orthogonal to public information in real time since individual traders cannot predict the errors other traders are making based on information available to all traders. This has implications for how these speculative terms can be estimated. In the next section we discuss the empirical specification, how the model can be estimated and the speculative terms quantified.
4. The estimated model

This section presents the main empirical results of the paper, but first we need to be more specific about some of the choices that need to be made in order to make the model presented above operational. In particular, we will explain how the factor processes are normalized and briefly discuss related identification issues as well as how the prices of risk can be parameterized when higher order expectations enter as state variables. We will also demonstrate how the the cross-sectional dimension of the Survey of Professional Forecasters can be exploited to make sharper inference about the precision of of information available to traders. Once the model is completely specified and estimated, we can use it to quantify the importance of speculative dynamics.

4.1. Exogenous factor dynamics. The first choice we need to make is to decide how many factors to include in the exogenous vector $x_t$. In the benchmark specification, $x_t$ is a three dimensional vector so that in the special case with perfectly informed traders, the model collapses to a standard three factor affine Gaussian no-arbitrage model. The factors $x_t$ follow a VAR(1) process under the physical measure

$$x_{t+1} = \mu_P + F_P x_t + C \epsilon_{t+1}$$

(4.1)

where the superscript $P$ is for “physical”. Since the factors are latent we need to impose restrictions on their laws of motion and on how those factors relate to the short rate. We follow Joslin, Singleton and Zhu (2011) and restrict the dynamics of the factors under the risk neutral measure which also follow VAR(1) dynamics

$$x_{t+1} = \mu_Q + F_Q x_t + C \epsilon_{Q,t+1}$$

(4.2)

The restrictions that we impose are that $\mu_Q = 0$ and that the matrix $F_Q$ is diagonal with the factors ordered in descending degree of persistence under the risk neutral dynamics. Furthermore, $C$ is restricted to be lower triangular. Finally, the vector $\delta_X$ of the short rate equation (2.3) is restricted to have its first three elements equal to one and all other elements equal to zero. This ensures that all parameters are identified in the special case of perfectly informed traders.

4.2. Trader’s information sets. Traders observe both exogenous and endogenous signals. Trader $j$ observes the factors $x_j$ as defined in (2.12) which is the source of trader $j$’s private information about the common factors $x_t$. Each trader also observes the risk free short rate $r_t$. In addition to these exogenous signals, all traders can observe all bond yields up to maturity $N$, where $N$ is the largest maturity used in the estimation of the model. Here, the longest maturity yield that we will use in estimation is a 10 year bond implying that $N = 40$ with quarterly data. That traders observe bond yields of all maturities is of no particular importance and there is little information added by observing additional bond yields as long as the yields of at least three different maturities (more or less evenly spread out over the yield curve) are observed.

4.3. Prices of risk parametrization. The traders in the model require compensation for the risk associated with the conditional variance of future bond prices. As explained in Section 2 above, expected excess returns are partly trader specific but must in the absence
of arbitrage be compensation for risk. The required compensation for risk must therefore also partly be trader specific and trader \(j\)'s required compensation for risk a function of the same variables that the expected excess returns are conditioned on. Following the full information affine literature as closely as possible, we specify the vector of risk prices of trader \(j\) as an affine function of trader \(j\)'s state \(X_t^j\)

\[
\Lambda_t^j = \Lambda_0 + \Lambda_X X_t^j \quad (4.3)
\]

The state vector \(X_t^j\) is high dimensional and as a consequence, leaving \(\Lambda_0\) and \(\Lambda_X\) completely unrestricted would imply a very large number of free parameters. In order to avoid an over-parameterized model we therefore restrict \(\Lambda_0\) and \(\Lambda_X\) as follows

\[
\Lambda_t^j = \begin{bmatrix}
\lambda_0 \\ 0_{(3k-3)\times3} \\
\lambda_1 \\ 0_{(3k-3)\times3} \\
\vdots \\ 0_{3\times3} \\
\alpha^2 \lambda_2 I_3 \\ 0_{3\times3} \\
\vdots \\
0_{3\times3} \\
\end{bmatrix} X_t^j
\]

\[
(I - H) X_t^j
\]

The sum of the two matrices multiplying \(X_t^j\) in (4.4) then equals the matrix \(\Lambda_X\) in (4.3). The matrix \((I - H)\) takes the difference between the expectations of the state and the true state, i.e.

\[
(I - H) X_t^j =
\]

and the term \((I - H) X_t^j\) thus equals zero in the special case when traders are perfectly informed. The three dimensional vector \(\lambda_0\) and the \(3 \times 3\) matrix \(\lambda_1\) in the first row of (4.4) thus completely specifies the price of risk in the perfect information case, just as it would in the standard full information affine three factor model. It then follows that the price of risk associated with traders taking advantage private information is governed by the scalars \(\lambda_2\) and \(\alpha\) which are the only two parameters added in the specification of the price of risk relative to the full information set up.

The parameter \(\lambda_2\) controls how the price of risk depends on differences in higher order expectations. To understand why the differences between adjacent orders of expectations in (4.5) should affect the required compensation for risk, recall that when traders have access to private information they can predict the average expectation error of other traders. When say, a trader’s first and second order expectations differ, that must imply that the individual trader thinks that other traders’ expectations are inaccurate since he thinks his own first order expectation is optimal. The excess return that can be earned by the individual trader from exploiting this difference must in the absence of arbitrage be compensation for risk. The parameter \(\lambda_2\) determines just by how much this compensation varies with the difference
between different orders of expectations. The parameter $\alpha$ in turn regulates how differences in beliefs between different orders of expectations depends on which orders are involved. For example, if $\alpha = 1$ then the difference between the first and the second order expectation price risk the same way that the difference between second and third order, and so on. If $\alpha < 1$ then differences between higher order expectations have a smaller impact on the price of risk than differences between lower order expectations.

The high dimensional matrix that prices risk as a function of differences in higher order expectations is thus parameterized by just two free parameters. We experimented with more flexible specifications, but found that restricting $\Lambda_X$ in this way did not appear to restrict the dynamics of the speculative component in bond prices due to the fact that the vector of higher order prediction errors $\left(I - H\right)X_t^j$ in practice appears to be a one factor process.\footnote{The virtual one factor structure of $\left(I - H\right)X_t^j$ can be verified numerically on a case-by-case basis by inspecting the scree plot of the matrix $\left(I - H\right)\Sigma_X^j \left(I - H\right)'$ where $\Sigma_X^j \equiv E \left[X_t^j X_t^j\right]$.}

### 4.4. Physical and full information risk neutral dynamics.

With the market prices of risk at hand, the dynamics of the true state vector $X_t$ and factors under the physical measure is related to what we may call the full information risk neutral measure as follows

$$M^{FIQ} = MH - \left(\Sigma_{t+1|t} + V_n N'\right) \Lambda_X$$  \hspace{1cm} (4.6)

That is, while the historical dynamics of the state are given by (2.17), bonds are priced as if traders were risk neutral and the state vector $X_t$ could be observed perfectly and followed the VAR process

$$X_t = M^{FIQ} X_{t-1} + Nu_t$$  \hspace{1cm} (4.7)

Of course, traders are neither risk neutral nor can they observe $X_t$ perfectly.

### 4.5. Estimating the model using bond yields and survey data.

The parameters of the model can be estimated using likelihood based methods. We use quarterly data on bond yields of maturity 1,2,3,4,5,6,7,8,9 and 10 years with the sample spanning the period 1971:Q4 to 2011:Q4. The yields data is taken from the CRSP data set. In addition to bond yields we also use one quarter ahead forecasts of the T-Bill rate and the 4 quarters ahead forecasts of the 10 year bond rate from the Survey of Professional Forecasters (SPF). In the model, the cross-sectional distribution of traders’ forecasts of the short rate is Gaussian with a mean and variance given by

$$E \left[r_{t+1} \mid \Omega_{t}^j\right] \sim N\left(A_1 + B_1 MH X_t, B_1 M \Sigma_j M' B_1'\right)$$  \hspace{1cm} (4.8)

and

$$E \left[y_{t+4}^{40} \mid \Omega_{t}^j\right] \sim N\left(A_{40} + B_{40} M^{40} H X_t, B_{40} M^{40} \Sigma_j (B_{40} M^{40})'\right)$$  \hspace{1cm} (4.9)

where $\Sigma_j$ is the cross-sectional covariance of expectations about the current state, i.e.

$$\Sigma_j \equiv EH \left(X_t^j - X_t\right) \left(X_t^j - X_t\right)' H'$$  \hspace{1cm} (4.10)

As econometricians, we can thus treat the individual survey responses of T-Bill rate forecasts as noisy measures of the average expectation of the short rate $r_t$ where the variance of the “noise” is determined by the cross-sectional model implied variance of short rate expectations.
The deviations of individual traders’ forecasts from the average forecasts are caused by the idiosyncratic shocks that are symmetric but independent across traders. The covariance of the measurement errors can thus be specified as the scalars $B_1M\Sigma_jM'B_1'$ and $B_40M^4\Sigma_j(B_40M^4)'$ times an identity matrix. To evaluate the log likelihood function we thus compute the innovations representation for the following state space system

$$X_t = \mu_X + MX_{t-1} + Nu_t$$

$$Z_t = d_t + \bar{D}_tX_t + \bar{R}_t\varepsilon_t : \varepsilon_t \sim N(0, I)$$

where

$$d_t = \begin{bmatrix} -\frac{1}{4}A_4 \\ -\frac{1}{8}A_8 \\ \vdots \\ -\frac{1}{40}A_{40} \\ 1_{(m\times1)} \times A_1 \end{bmatrix}, \quad \bar{D}_t = \begin{bmatrix} -\frac{1}{4}B_4 \\ -\frac{1}{8}B_8 \\ \vdots \\ -\frac{1}{40}B_{40} \\ 1_{(m\times1)} \times B_1MH \\ 1_{(m\times1)} \times B_{40}MH \end{bmatrix}$$

$$\bar{R}_t\bar{R}_t' = \begin{bmatrix} RR' & 0 & 0 \\ 0 & I_m \times B_1M\Sigma_jM'B_1' & 0 \\ 0 & 0 & I_m \times B_{40}M^4\Sigma_jM'B_{40}' \end{bmatrix}$$

and $m$ is the number of survey responses available in period $t$. The Appendix contains details of how to compute the cross sectional variance $\Sigma_j$ in practice. The number of survey responses vary over time and surveys are not available at all for the period before 1981:Q3. The dimensions of $d_t$, $\bar{D}_t$ and $\bar{R}_t$ are thus varying across time. This presents no particular problem, but may influence the precision of our estimates of the state, i.e. we will have more precise estimate of the latent state $X_t$ when there is a large number of survey responses available. Using “raw” survey data and likelihood based methods also naturally incorporates that we have more precise information about the expectations of traders when there are 50 responses (the sample maximum) compared to when there are only 9 responses (the sample minimum). This information is lost when using measures of central tendency like a mean or median forecast.

Using raw survey data also allow us to exploit the second moment of surveys, i.e. the cross sectional variance of surveys to make inference about the precision of traders information. The cross-sectional variance of expectations is in general a non-monotonic function of the precision of the private signals. When the private signals are very precise, the cross sectional dispersion is close to zero and the dynamics of bond yields will be close to those of the full information model. When the precision of the private signals decreases, the cross-sectional dispersion initially increases even though the weight traders attach to the private signal decreases. At some point, the private signals become so noisy that the reduced weight traders attach to them is dominating the increase in the dispersion of the signals. In the limit, the cross-sectional dispersion again tends to zero as traders attach no weight to an infinitely noisy private signal. However, the dynamics of bond yields no longer coincide with the full information case since traders expectations about the common factors do not
coincide with the actual factors. The model thus makes joint predictions about the dynamics of bond prices and the cross-sectional variance of expectations.

### Table 1 (Preliminary)

<table>
<thead>
<tr>
<th>Posterior Parameter Estimates 1971:Q2-2011:Q1</th>
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<tbody>
<tr>
<td>Factor processes</td>
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<tr>
<td>$F_{1,1}^Q$</td>
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<tr>
<td>$F_{2,2}^Q$</td>
</tr>
<tr>
<td>$F_{3,3}^Q$</td>
</tr>
<tr>
<td>$C_1$</td>
</tr>
<tr>
<td>$C_2$</td>
</tr>
<tr>
<td>$C_3$</td>
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<table>
<thead>
<tr>
<th>Maturity specific disturbances</th>
<th>Short rate constant $r_t$</th>
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<tbody>
<tr>
<td>$\sigma_v$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Risk Premia Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{X1}$</td>
</tr>
<tr>
<td>$\mu_{X2}$</td>
</tr>
<tr>
<td>$\mu_{X3}$</td>
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<tr>
<td>$\Lambda_{X,12}$</td>
</tr>
<tr>
<td>$\Lambda_{X,13}$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
</tr>
</tbody>
</table>

Log likelihood at $\theta$: 28239

An alternative strategy to use the survey data would be to treat individual responses as noisy measures of a common expectation held by all traders (see Kim and Ophanides (2005) and Chernov and Mueller (2012)). Other have used surveys by taken a measure of central tendency like a mean or median and then adding noise to them (see Piazzesi and Schneider (2011) and Kim, D’amico, Kim and Wei (2010)). The difference between this approach and ours is that the latter strategies do not impose any joint restrictions on the dynamics of bond yields and the cross-sectional dispersion of survey responses.

4.6. **Estimation and results.** The model is parameterized by the matrices $A^Q$ and $C$ which governs the processes of the common latent factors $x_t$, the matrix $Q$ which specifies the standard deviation of the trader specific factors in $x_j$, the scalar $\delta$ which is the unconditional mean of the short risk free rate $r_t$, $\sigma_v$ the standard deviation of the maturity specific disturbances $v_t$ and the vector $\lambda_0$ matrix $\lambda_1$ and scalars $\alpha$ and $\lambda_2$ which parameterizes the price of risk. In total, there are 28 parameters and the maximum likelihood estimates are reported in Table 1.

4.7. **Historical decompositions.** The Kalman smoother (see for instance Durbin and Koopman 2002) can be used to back out an estimate of the state $X_t$ conditional on the entire history of observables. Since the speculative components of the term structure are linear functions of the state, the backed out history $E \left[ X^T \mid Z^T \right]$ can be used to construct estimates of the historical contribution to the bond yields by the individual components of the decompositions (3.4) and (3.7).
In the top left panel of Figure 1, we have plotted the history of the 10 year yield together with a decomposition, splitting the yield into the term based on average expectations about future short rates, common risk premia and a speculative term. The individual terms are plotted with 95% probability intervals against the 10-year yield in the remaining three panels. It is clear form the figure that most of the variation in yields are driven by variation in average first order short rate expectations. The common risk premia term is the second most important term, but the speculative term speculative dynamics are also quantitatively important. At the posterior median, the speculative term account for up to a full percentage point of the yield at the peaks.

Speculative dynamics ar present at all maturities beyond 2 periods, but are quantitatively more important at medium to long maturities. This is illustrated in Figure 2 where the estimated speculative component in the 1- 5- and 10-year yields are plotted. The speculative component is positively correlated across maturities and most volatile for the 5-year bond. In comparison, the volatility of the speculative term in the 1-year bond is substantially less volatile. The speculative component in the 10 year bond is also less volatile than the term in the 5-year bond, but only marginally so.

The speculative component is positive when higher order expectations about future bond prices are above average first order expectations of future bond yields. Since future bond yields have both a expected short rate component and as well as a risk premia component, we can decompose the speculative component further into speculation about future short rates and speculation about future risk premia, compute the third and fourth term in (3.7) separately. The result is shown in Figure 3 where the total speculative component (black line) is plotted together with the term due to speculation about future short rates (blue

**Figure 1.** Individual terms in decomposition of the 10 year yield with 95 percent prob intervals.
Figure 2. Individual terms in decomposition of the 10 year yield with 95 percent prob intervals.

(Speculation about future risk premia is thus the difference between the black and the blue line.)

Figure 3. Decomposition of the 10 year yield speculative component.
4.8. **What drives speculative dynamics?** In order to address the question of what drives speculative dynamics we can decompose the variance of the speculative terms into the three sources: The three innovations to the exogenous factors in $x_t$ and the maturity specific disturbances $v_t$. The results of this exercise is displayed in Table 2 where the variance decompositions of the 1-, 5- and 10 years yields and the first three principal components are also presented. The variance decomposition reveals that while the shocks to the third factor explains most of the variance of yields of all maturities, the first shock explains the majority of the variance of the speculative component. It is also the first shock that explains most of the variance of the second principal component, i.e. the so called “slope” factor.

**Table 2** Variance decomposition

<table>
<thead>
<tr>
<th></th>
<th>$y^4_t$</th>
<th>$y^{20}_t$</th>
<th>$y^{10}_t$</th>
<th>$pc^4_t$</th>
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<tr>
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<td>0.11</td>
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<td>0.00</td>
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</tr>
<tr>
<td>$\varepsilon^3_t$</td>
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<tr>
<td>$v_t$</td>
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<td>0.05</td>
<td>0.12</td>
<td>0.12</td>
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</tr>
</tbody>
</table>

**Figure 4.** Compare to FI case

4.9. **Comparison to full information model.** Gaussian affine term structure models have been used by for instance Cochrane and Piazzesi (2005) and Joslin, Priebsch and Singleton (2011) to decompose the the term structure into risk premia and expected future short rates. It is interesting to look at how allowing for speculative dynamics changes the historical estimates of risk premia. In Figure 4 we have plotted the posterior estimate of the risk premia in the 10-year bond extracted using the model with privately informed traders together with the risk premia extracted using the a full information model. One may have suspected that allowing for speculative dynamics would reduce the role played by risk premia.
in the explaining the variance of bond yields. Interestingly, the opposite turns out to be the case, at least if the data is viewed through the lens of our model. Figure 4 shows that the risk premia term in the speculative model is actually estimated to be more volatile than the risk premia component estimated using the standard full information affine three factor model. Abstracting from speculative dynamics may thus lead a researcher to underestimate the importance of risk premia.

4.10. **How useful is the private information?** We can also use the estimated model to quantify how useful the private signals are in terms of helping agents to forecast future bond prices. Figure 5 plots the reduction in the conditional yield variance due to the private signals, as compared to the yield variance when conditioning only on publicly available bond yields. That is, Figure 4 plots the expression

\[
\frac{E \left( y_{t+1}^n - E \left[ y_{t+1}^n \mid \Omega_j^t \right] \right)^2}{E \left( y_{t+1}^n - E \left[ y_{t+1}^n \mid y^t \right] \right)^2}
\]

for \( n = 1, 2, ..., 40 \) and where \( y_t^n \equiv n^{-1} p_t^n \) and \( y^t \) is the history of bond yields. For short maturities, the conditional variance is reduced by about 6 per cent and for longer maturities the reduction reaches about 14 per cent for one quarter ahead forecasts of the 10-year yield.

5. **Conclusions**

In this paper we have presented an affine Gaussian framework for analyzing the term structure of interest rates when rational traders have access to private information. We showed that private information introduces a speculative term in bond yields due to average higher order prediction errors, i.e. differences between first and higher order expectations. Interpreting the history of US bond yields through the lens of the estimated model we find
that speculative dynamics are quantitatively important for medium to long maturity bonds. In the period of low interest rates in the first decade of the 2000s, the speculative component reaches a full percentage point at a time when the total yield on a ten year bond was below 5 per cent. We also find that allowing for speculative dynamics changes the estimates of historical risk premia. Interestingly, the posterior estimates of risk premia in the model with speculative dynamics are more volatile than in a standard three factor affine model with perfectly informed traders. This suggests that it may it may be important to control for private information and speculative dynamics when trying to establish the causes of time varying price of risk.

The paper also makes a methodological contribution by demonstrating how “raw” survey data can be incorporated into the estimation of a term structure model using likelihood based methods. As far as we know, this is the first paper to do so and may be of independent interest to some readers.

REFERENCES

[6] Buraschi and Whelan (2010), Term Structure Models and Differences in Beliefs, mimeo, Imperial College.


Appendix A. An Affine Framework for Term Structure Models with Private Information.

In this appendix we derive the recursions for bond prices, we show how we solve the model following the algorithm in Nimark (2012) and provide additional details on the computation of the likelihood.

A.1. Derivations under the Physical Measure. We first derive recursions for bond prices. For a one-period bond (n=1):

\[ P_t^1 = E_t[M_{t+1}^1 \Omega(j)] = \exp(-r_t) \]  \hspace{1cm} (A.1)

\[ = \exp(-\delta_0 - \delta_X X_t) \]  \hspace{1cm} (A.2)

Matching coefficients leads to \( A_1 = -\delta_0 \) and \( B_1 = -\delta_X \).

Now we show the recursion for any maturity \( n \). Bond prices are observed with measurement errors which we motivate by the existence of small misspecifications. They also serve the purpose of making equilibrium bond prices not fully revealing.

The price of a bond with \( n + 1 \) periods to maturity is then given by:

\[ p_t^{n+1} = \log E_t[M_{t+1}^n P_{t+1}^n \Omega_t^j] \]  \hspace{1cm} (A.3)

\[ = -r_t - \frac{1}{2} \Lambda_t^{ij} \Sigma_a \Sigma_a' \Lambda_t^{ij} + A_n + \log E_t[\exp(-\Lambda_t^{ij} a_{t+1}^j + B_n^j X_{t+1} + \sigma_{v,n} v_{t+1}^n) | \Omega_t^j] \]  \hspace{1cm} (A.4)

\[ = -r_t - \frac{1}{2} \Lambda_t^{ij} \Sigma_a \Sigma_a' \Lambda_t^{ij} + A_n + B_n^j \mu^P \]  \hspace{1cm} (A.5)

\[ + \log E_t[\exp(-\Lambda_t^{ij} a_{t+1}^j + B_n^j M X_t + B_n^j N u_{t+1} + \sigma_{v,n} v_{t+1}^n) | \Omega_t^j] \]

To compute the price we first concentrate on the term containing the expectation. Let \( V_n \) be the vector that selects the relevant maturity specific pricing error from \( u_t \). This is such that \( V_n u_t \equiv \sigma_{n,t} v_t^n \).

Then,

\[ \log E_t[\exp(-\Lambda_t^{ij} a_{t+1}^j + B_n^j M X_t + B_n^j N u_{t+1} + \sigma_{v,n} v_{t+1}^n) | \Omega_t^j] \]  \hspace{1cm} (A.6)

\[ = \log E_t[\exp(-\Lambda_t^{ij} a_{t+1}^j + B_n^j M X_t - E(X_t | \Omega_t^j) + B_n^j M E(X_t | \Omega_t^j) + B_n^j N u_{t+1} + \sigma_{v,n} v_{t+1}^n) | \Omega_t^j] \]

\[ = B_n^j M E(X_t | \Omega_t^j) - \Lambda_t^{ij} \Sigma_t^j \]  \hspace{1cm} (A.7)

\[ + \log E_t[\exp((B_n^j - \Lambda_t^{ij}) M X_t - E(X_t | \Omega_t^j) + (B_n^j - \Lambda_t^{ij}) N u_{t+1} + V_n u_{t+1} \]  \hspace{1cm} (A.8)

\[ = B_n^j M E(X_t | \Omega_t^j) + (B_n^j - \Lambda_t^{ij}) N V_n \]  \hspace{1cm} (A.9)

\[ + \frac{1}{2} \left[ (B_n^j - \Lambda_t^{ij}) \Sigma_t + (B_n^j - \Lambda_t^{ij})' + V_n V_n + \Lambda_t^{ij} \Sigma_t \right] \]

\[ = B_n^j M E(X_t | \Omega_t^j) + (B_n^j - \Lambda_t^{ij}) N V_n \]  \hspace{1cm} (A.10)

\[ + \frac{1}{2} \left[ (B_n^j - \Lambda_t^{ij}) \Sigma_t + (B_n^j - \Lambda_t^{ij})' + V_n V_n + \Lambda_t^{ij} \Sigma_t \right] \]
Plugging the evaluated expectation back into the log price expression we obtain that:

\[ p_t^{n+1} = -\delta_0 + A_n + B'_n \mu^P \]

\[ + \frac{1}{2} \left[ (B'_n - \Lambda^P_n \Sigma_{t+1|t} (B'_n - \Lambda^P_n)') + V_n V'_n + \Lambda^P_n \tilde{\Sigma}' \Lambda^j_t - \Lambda^P_n \Sigma_n \Lambda^j_t \right] \]

\[ + (B'_n - \Lambda^P_n) N V'_n - \delta_X X_t + B'_n ME[X_t|\Omega^j_t] \]

\[ = -\delta_0 + A_n + B'_n \mu^P \]

\[ + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n + V_n V'_n \right] - B'_n \Sigma_{t+1|t} \Lambda_0 + B'_n N V'_n - V_n N' \Lambda_j^t \]

\[ - \delta_X X_t + B'_n ME[X_t|\Omega^j_t] \]

\[ = -\delta_0 + A_n + B'_n \mu^P + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n + V_n V'_n \right] - B'_n \Sigma_{t+1|t} \Lambda_0 + B'_n N V'_n \]

\[ - \delta_X X_t + B'_n ME[X_t|\Omega^j_t] - (B'_n \Sigma_{t+1|t} + V_n N') \Lambda_X X_j^t \]

So that if the coefficients \( A_n \) and \( B_n \) follow the difference equations:

\[ A_{n+1} = -\delta_0 + A_n + B'_n \mu^P \]

\[ + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n + V_n V'_n \right] - (B'_n \Sigma_{t+1|t} + V_n N') \Lambda_0 + B'_n N V'_n \]

\[ B_{n+1} = B'_n M H - \delta'_X - (B'_n \Sigma_{t+1|t} + V_n N') \Lambda_X \]

A.2. **Derivations under the risk-neutral measure \( Q \).** Next, we derive the recursion for bond prices under the risk neutral measure for a bond with maturity \( n + 1 \) the bond price must satisfy:

\[ p_t^{n+1} = \log E^Q \left[ e^{-r_t} P_{n+1}^{n+1}|\Omega^j_t] \right] \]

\[ = -r_t + A_n + \log E^Q \left[ \exp(B'_n X_{t+1} + \sigma_{v,n} v_{t+1}^n)|\Omega^j_t] \right] \]

\[ = -r_t + A_n + B'_n \mu^Q + \log E^Q \left[ \exp(B'_n M^Q X_t + B'_n N u_{t+1}^Q + \sigma_{v,n} v_{t+1}^n)|\Omega^j_t] \right] \]

To compute this price we first concentrate on the term containing the expectation.

\[ \log E_t \left[ \exp(B'_n M^Q X_t + B'_n N u_{t+1}^Q + \sigma_{v,n} v_{t+1}^n)|\Omega^j_t] \right] \]

\[ = \log E_t \left[ \exp \left[ B'_n M^Q (X_t - E(X_t|\Omega^j_t) + B'_n M^Q E[X_t|\Omega^j_t] + B'_n N u_{t+1}^Q + \sigma_{v,n} v_{t+1}^n) \right] |\Omega^j_t] \right] \]

\[ = B'_n M^Q E[X_t|\Omega^j_t] + \log E_t[\exp \left[ B'_n M^Q (X_t - E(X_t|\Omega^j_t) + B'_n N u_{t+1}^Q + V_n N'_n) + B'_n N V'_n \right) |\Omega^j_t] \]

\[ = B'_n M^Q E[X_t|\Omega^j_t] + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n + V_n V'_n \right] + B'_n N V'_n \]

The log bond price is then given by:

\[ p_t^{n+1} = -\delta_0 + A_n + B'_n \mu^Q - \delta_X X_t + B'_n M^Q E[X_t|\Omega^j_t] + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n + V_n V'_n \right] + B'_n N V'_n \]
so that the coefficients $A_n$ and $B_n$ can be found by the recursion

$$
A_{n+1} = -\delta_0 + A_n + B_n \mu^Q + \frac{1}{2} \left[ B'_n \Sigma^Q_{t+1|t} B_n + V_n V'_n \right] + B'_n N V'_n \tag{A.25}
$$

$$
B'_{n+1} = B'_n M^Q H - \delta'_X \tag{A.26}
$$

starting from

$$
A_1 = -\delta_0 \tag{A.27}
$$

$$
B_1 = \left[ -\delta'_X \quad 0 \right] \tag{A.28}
$$

since $p_1^t = -r_t$.

### A.3. Identification

The loadings derived under both risk neutral and physical dynamics are by construction equal to each other. We impose restrictions on the risk neutral model to make sure the model is identified under full information and then by matching this recursion we derive the implications for the physical dynamics of these restrictions.

#### A.3.1. Matching recursions for $B$.  

$$
B'_n M H - \delta'_X = (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_X = B'_n M^Q H - \delta'_X \tag{A.29}
$$

$$
B'_n M H - (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_X = B'_n M^Q H \tag{A.30}
$$

and imposing the identifying restriction that that the upper left hand sub matrix of $M^Q$ is diagonal with decreasing elements which ensures that the model is identified under full information.

#### A.3.2. Matching recursions for $A$.  

$$
B'_n \mu^P + \frac{1}{2} \left[ B'_n \Sigma^Q_{t+1|t} B'_n \right] - (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_0 = B'_n \mu^Q + \frac{1}{2} \left[ B'_n \Sigma_{t+1|t} B'_n \right] \tag{A.31}
$$

and imposing the identifying restriction that $\mu^Q = 0$ which ensures that the model is identified under full information we then proceed to solve for $\mu^P$

$$
B'_n \mu^P + \frac{1}{2} \left[ B'_n M^P \Sigma_{t|t} M^P B_n + B'_n N N' B_n \right] - (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_0 = \frac{1}{2} \left[ B'_n M^Q \Sigma_{t|t} M^Q B_n + B'_n N N' B_n \right] \tag{A.32}
$$

which equivalently can be written as

$$
B'_n \mu^P + \frac{1}{2} \left[ B'_n M \Sigma_{t|t} M B_n \right] - (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_0 \tag{A.33}
$$

Rearranging gives

$$
B'_n \mu^P = \frac{1}{2} \left[ B'_n M^Q \Sigma_{t|t} M^Q B_n \right] - \frac{1}{2} \left[ B'_n M \Sigma_{t|t} M B_n \right] + (B'_n \Sigma^t_{t+1|t} + V_n N') \Lambda_0 \tag{A.34}
$$

and solving for $\mu^P$.  

and can be chosen to achieve an arbitrarily close approximation to the limit as

which means that the recursion under \( P \) for \( A \) that imposes the restriction that \( \mu^Q = 0 \) is

Solve for \( B_n^t M^Q \)

where \( H^+ \) is the generalized inverse of \( H \).

\[
\begin{align*}
A_{n+1} &= -\delta_0 + A_n + \frac{1}{2} \left( (B_n^t M - (B_n^t \Sigma_{t+1|t} + V_n N') \Lambda X H^+ ) \Sigma_{t|t} (B_n^t M - (B_n^t \Sigma_{t+1|t} + V_n N') \Lambda X H^+ )' \right) \\
&\quad - \frac{1}{2} \left[ B_n^t M \Sigma_{t|t} MB_n \right] + \frac{1}{2} \left[ B_n^t \Sigma_{t+1|t} B_n^t + V_n V_n' \right] + B_n^t N V_n' \\
\end{align*}
\]

\[
\begin{align*}
A_{n+1} &= -\delta_0 + A_n + \frac{1}{2} \left( (B_n^t \Sigma_{t+1|t} + V_n N') \Lambda X H^+ \right) \Sigma_{t|t} \left( (B_n^t \Sigma_{t+1|t} + V_n N') \Lambda X H^+ \right)' \\
&\quad - B_n^t M \Sigma_{t|t} \left( (B_n^t \Sigma_{t+1|t} + V_n N') \Lambda X H^+ \right)' + \frac{1}{2} \left[ B_n^t \Sigma_{t+1|t} B_n^t + V_n V_n' \right] + B_n^t N V_n' \\
\end{align*}
\]

**Appendix B. Solving the model**

Solving the model implies finding a law of motion for the higher order expectations of \( x_t \) of the form

\[
X_t = MX_{t-1} + Nu_t 
\]

where

\[
X_t = \begin{bmatrix} x_t^{(0)} \\ x_t^{(1)} \\ \vdots \\ x_t^{(k)} \end{bmatrix}, \quad u_t = \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix}
\]

That is, to solve the model, we need to find the matrices \( M \) and \( N \) as functions of the parameters governing the short rate process, the maturity specific disturbances and the idiosyncratic noise shocks. The integer \( k \) is the maximum order of expectation considered and can be chosen to achieve an arbitrarily close approximation to the limit as \( k \rightarrow \infty \). Here, a brief overview of the method is given, but the reader is referred to Nimark (2011) for more details on the solution method.
First, common knowledge of the model can be used to pin down the law of motion for the vector $X_t$ containing the hierarchy of higher order expectations of $x_t$. Rational, i.e. model consistent, expectations of $x_t$ thus implies a law of motion for average expectations $x_t^{(1)}$ which can then be treated as a new stochastic process. Knowledge that other traders are rational, means that second order expectations $x_t^{(2)}$ are determined by the average across traders of the rational expectations of the stochastic process $x_t^{(1)}$. Third order expectations $x_t^{(3)}$ are then the average of the rational expectation of the process $x_t^{(2)}$, and so on. Imposing this structure on all orders of expectations allows us to find the matrices $M$ and $N$. Section XX below describes how this is implemented in practice.

Second, the method exploits that the importance of higher order expectations are decreasing with the order of expectation. This has two components:

(i) The variance of higher order expectations of the factors $x_t$ are bounded by the variance of the true process, or more generally, the variance of $k + 1$ order expectation cannot be larger than the variance of a $k$ order expectation

$$E \left[ x_t^{(k+1)} x_t^{(k+1)}' \right] \leq E \left[ x_t^{(k)} x_t^{(k)}' \right]$$ (B.2)

To see why, note that by the identity

$$x_t^{(k)} \equiv x_t^{(k+1)} + \varepsilon_t^{(k+1)}$$ (B.3)

and the fact that since $x_t^{(k+1)}$ is the average of an optimal estimate of $x_t^{(k)}$ the $k = 1$ order error $\varepsilon_t^{(k+1)}$ must be orthogonal to $x_t^{(k+1)}$ we have that

$$E \left[ x_t^{(k)} x_t^{(k)}' \right] = E \left[ x_t^{(k+1)} x_t^{(k+1)}' \right] + E \left[ \varepsilon_t^{(k+1)} \varepsilon_t^{(k+1)}' \right].$$ (B.4)

Since $E \left[ \varepsilon_t^{(k+1)} \varepsilon_t^{(k+1)}' \right]$ is a covariance

$$E \left[ \varepsilon_t^{(k+1)} \varepsilon_t^{(k+1)}' \right] \geq 0$$ (B.5)

and the inequality (B.2) then follows immediately. (This is an abbreviated version of a more formal proof available in Nimark (2010).)

That the variances of higher order expectations of the factors $x_t$ are bounded is not sufficient for an accurate finite dimensional solution. We also need (ii) that the impact of the expectations of the factors on bond yields are decreasing “fast enough” with the order of expectation. The proof of this result is somewhat involved and readers are referred to the original reference for a proof.

**B.1. Bond yields as a function of the state.** For a given law of motion (??), bond prices are given by

$$p^n_t = A_n + B'_n X_t + v^n_t$$ (B.6)

and the recursions

$$A_{n+1} = -\delta_0 + A_n + B'_n \mu^P + \frac{1}{2} \left[ B'_n \Sigma_{t+1} \mu^P B'_n + V_n V' \right]$$ (B.7)

$$-B'_n \Sigma_{t+1} \Lambda_0 + B'_n NV_n' - V_n N' \Lambda_0$$

$$B_{n+1} = -\delta'_X + B'_n M H - (B'_n \Sigma_{t+1} X + V_n X') \Lambda_X$$ (B.8)
Bond prices thus depend on the law of motion for the law of higher order expectations.

B.2. The law of motion of higher order expectations of the factors. To find the law of motion for the hierarchy of expectations \( X_t \) we use the following strategy. For given \( M, N \) and \( B_n \) in (B.1) and (B.8) we will derive the law of motion for trader \( j \)'s expectations of \( X_t \), denoted \( X^j_{t|t} \equiv E \left[ X_t \mid \Omega^j_t \right] \). First, write the vector of signals \( z^j_t \) as a function of the state, the aggregate shocks and the idiosyncratic shocks

\[
z^j_t = \begin{bmatrix} x^j_t \\ p_t \end{bmatrix}
\]

where the matrix \( D \) is given by

\[
D = \begin{bmatrix}
I_3 & 0 \\
B_1 & \\
& \\
B_N &
\end{bmatrix}
\]

and \( R \) can be partitioned conformably to the aggregate and idiosyncratic shocks

\[
R = \begin{bmatrix} R_u & R_j \end{bmatrix}.
\]

Agent \( j \)'s updating equation of the state \( X^j_{t|t} \) estimate will then follow

\[
X^j_{t|t} = MX^j_{t-1|t-1} + K \left( z^j_t - DMX^j_{t-1|t-1} \right)
\]

Rewriting the observables vector \( z^j_t \) as a function of the lagged state and current period innovations and taking averages across traders using that \( \int \zeta_t(j) dj = 0 \) yields

\[
X_t = MX_{t-1|t-1} + K \left( DNX_{t-1|t-1} + (DN + RA) \epsilon_t + -DMX_{t-1|t-1} \right)
\]

Appending the average updating equation to the exogenous state gives us the conjectured form of the law of motion of \( x_t^{(0:k)} \)

\[
\begin{bmatrix} x_t \\ X_{t|t} \end{bmatrix} = M \begin{bmatrix} x_{t-1} \\ X_{t-1|t-1} \end{bmatrix} + Nu_t
\]

where \( M \) and \( N \) are given by

\[
M = \begin{bmatrix} F^P & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0_{3 \times 3} & 0 \\ 0 & [M - KDM]_- \end{bmatrix} + \begin{bmatrix} 0 \\ [KDM]_- \end{bmatrix}
\]

\[
N = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ [K(DN + RA)]_- \end{bmatrix}
\]

where \([\cdot]_-\) indicates that the last row or column has been canceled to make a the matrix \([\cdot]\) conformable, i.e. implementing that \( x_t^{(k)} = 0 : k > k \). The Kalman gain \( K \) in (B.13) is
given by

\begin{align*}
K &= (PD' + NR') (DPD' + RR')^{-1} \quad \text{(B.19)} \\
P &= M (P - (PD' + NR') (DPD' + RR')^{-1} (PD' + NR')) M' + NN' \quad \text{(B.20)}
\end{align*}

The model is solved by finding a fixed point that satisfies (B.8), (B.17), (B.18), (B.19) and (B.20).

**Appendix C. Computing the cross-sectional variance \( \Sigma_j \)**

The idiosyncratic noise shocks \( \eta^j_t \) are white noise processes that are orthogonal across traders and to the aggregate shocks \( v_t \) and \( \varepsilon_t \). This implies that the cross-sectional variance of expectations is equal to the part of the unconditional variance of trader \( j \)'s expectations that is due to idiosyncratic shocks. This quantity can be computed by finding the variance of the estimates in trader \( j \)'s updating equation (B.13), but with the aggregate shocks \( v_t \) and \( \varepsilon_t \) “switched off”. The covariance \( \Sigma_j \) of trader \( j \)'s state estimate due to idiosyncratic shocks is defined as

\[ \Sigma_j \equiv \mathbb{E} \left( X^j_{t|t} - X_{t|t} \right) \left( X^j_{t|t} - X_{t|t} \right)' \quad \text{(C.1)} \]

and given by the solution to the Lyaponov equation

\[ \Sigma_j = (I - KD) M \Sigma_j M' (I - KD)' + KR_j R_j K' \quad \text{(C.2)} \]

which can be found by simply iterating on (C.2).