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Abstract. Although Modigliani’s life-cycle framework has households build net worth during work years and then spend it down evenly during retirement, survey data seldom seems to bear out the theory’s tidy end-of-life decumulation prediction. Recent work, for example, suggests a stair-step pattern: it is as if a household’s bequeathable wealth falls rather abruptly at about the time of the death of each adult but otherwise stays roughly constant or even rises. The present paper modifies a standard life-cycle model to encompass late-in-life declines in health status, occurring stochastically. We then show how, and why, optimal behavior in the new formulation can include periods of saving after retirement, decumulation of net worth in rapid bursts, and accidental bequests. And, we suggest how these results may explain the patterns that survey data exhibits.

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Introduction. Although Modigliani’s [1986] life-cycle framework has households build net worth during work years and then spend it down evenly during retirement, survey data seldom seems to bear out the theory’s tidy end-of-life decumulation prediction. Recent work by Poterba et al [2011], for example, suggests a stair-step pattern: it is as if a household’s bequeathable wealth falls rather abruptly at about the time of the death of each adult but otherwise stays roughly constant or even rises.¹ The present paper modifies a standard life-cycle model to encompass late-in-life declines in health status, occurring stochastically. We then show how and why optimal behavior in the new formulation can include periods of saving after retirement, decumulation of net worth in rapid bursts, and accidental bequests. And, we suggest a possible explanation for patterns in survey data based upon these results.

This paper focuses on the retirement stage of the life-cycle. Our model has state-dependent utility, limited insurance markets, and Medicaid financed nursing-home care as a fallback for those with low resources but poor health. This paper’s attention centers on old-age declines in health status. We assume that health status is often difficult to quantify, severely limiting the feasibility of private insurance.² We also assume that low health status raises a household’s marginal utility of expenditure — as it leads households to substitute market purchases of personal services for self provision.

Nursing-home care provided by Medicaid constitutes a fallback for elderly households in poor health. While Medicaid-financed care may not be glamorous (O’Brien [2005]), alternatives are extremely costly. Optimal behavior in our model, for instance, may have a middle class household utilizing its own funds in the early stages of poor health status but, in the event of above-average longevity, ultimately turning to Medicaid. Medicaid and annuitization are non-complementary: Medicaid appropriates a recipient’s income, in effect taxing annuity pay outs 100 percent.

Our model can, in principle, shed light on 4 types of data. First, as stated, recent panel data evidence based upon the AHEAD and other sources tends to show level or rising bequeathable net worth for intact one or two-person households, declines for those losing a spouse, and noticeably lower levels for one-person than two-person households (e.g., Poterba et al [2011, 2012], Laibson [2011]). Second, existing work has long noted

¹ Older studies found similar complexities. For instance, Mirer [1979, p.442] concludes, “... the simple form of the life cycle theory of saving, in which wealth is accumulated during working years in order to finance consumption during retirement, is too simple.” Kotlikoff and Summers [1988, p.54] wrote, “Decumulation of wealth after retirement is an essential aspect of the life cycle theory. Yet simple tabulations of wealth holdings by age ... or savings rates by age ... do not support the central prediction that the aged dissave.”

² We might think of the symptoms of non-rectifiable medical conditions as the cause of deterioration in health status. It is then the degree of progress of the symptoms that is difficult to measure objectively. See Section 1 below.
households’ reluctance to annuitize their net worth fully (Yaari [1965], Benartzi et al [2011], Diamond [2004], Davidoff et al [2005], and many others). Third, recent evidence (again derived from the AHEAD) shows that significant fractions of even middle and relatively high permanent income households turn to Medicaid support at advanced ages, with later transitions to Medicaid for those with the highest lifetime incomes (DeNardi et al [2011]). Fourth, survey evidence on middle class bequests and inheritances tends to show measured transfers for 30-40 percent of households (e.g., Laitner and Ohlsson [2001], Laitner and Sonnega [2010]).

Even with Modigliani’s consumption behavior, data on a household’s wealth would not reveal a steady post-retirement decline if the household had fully annuitized — i.e., steady decumulation would then show up in the account balances of the financial intermediaries issuing annuities. See, for example, Blinder Gordon, and Wise [1983]. In practice, however, incomplete annuitization seems the rule. This paper’s analysis examines a household’s optimal decumulation of non-annuitized net worth, emphasizing the role of health status changes.

Friedman and Warshawsky [1990] stress the tradeoff between bequeathing money to children and own consumption in old age, with the pricing of annuities — given adverse selection problems — playing a potentially significant role. For simplicity, the present paper omits intentional bequests. However, our model generates “accidental bequests.” In fact, it can provide a theory of the determinants of such transfers.

Davidoff et al [2005] consider annuitization problems when financial markets are incomplete. Our treatment emphasizes the fallback provided by Medicaid. Sinclair and Smetters [2004] investigate the demand curve for annuities. The present paper shows the consequences of varying degrees of annuitization, though it does not explicitly derive the demand for annuities — we prefer to leave the latter topic for future work that models couples’ behavior as well as singles.

Whereas traditional formulations of health-related risk (Hubbard et al [1995], Sinclair and Smetters [2004]) emphasize budgetary shocks, we assume that health changes in old age affect economic behavior through state-dependent utility. This simplifies our analysis (e.g., Yaari [1965]), and it provides a framework in which agents can choose the magnitude of their responses to health-status changes.

In fact, while standard treatments often rely on numerical solutions, the present paper stresses analytic steps. We believe that our phase diagrams can provide detailed and intuitive descriptions of the optimal behavior of households of different resource levels, and that the simplicity of our approach will facilitate future theoretical generalizations, as well as empirical work. The analysis shows that all households that live long enough become liquidity constrained; it suggests why low-resource households may want to deplete their bequeathable assets at early stages of retirement, whereas those with high resources may want to maintain them or accumulate more; and, it predicts that households will often spend their non-annuitized assets in fairly short bursts. We argue that the last may explain why household decumulation of non-annuitized assets is so difficult to detect in data sets.

1 Special Modeling Elements. This section discusses two specific elements of our approach. The formal modeling begins in Section 2.
State-dependent Utility. We assume health-status dependent utility functions.

Consider a single-person household that is already retired. In this subsection, think of a household with 2 remaining periods of life, \( t = 1, 2 \). In the first, health status is “high”: \( h_1 = H \). In the second, it is “low,” \( h_2 = L \), with probability \( \pi \), and “high,” i.e., \( h_2 = H \), otherwise. The household’s net worth as we start is \( b \), the household has no other resources or resource flow, and the interest rate is 0. Consumption expenditures for \( t = 1 \) or 2 are, respectively, \( x_1 \) and \( x_2 \). Let \( \beta \in (0, 1] \) be the subjective discount factor, and let \( u = u(x) \) be utility flow in good health with

\[
\begin{align*}
    u(x) &= \begin{cases} 
        \frac{[x]^{\gamma}}{\gamma}, & \gamma < 1, \gamma \neq 0, \\
        \ln(x), & \gamma = 0.
    \end{cases}
\end{align*}
\]  
(1)

Following most empirical evidence, we assume

\[
    \gamma < 0.
\]  
(2)

A number of approaches are possible. Hubbard et al [1995], for example, assign exogenous health dependent expenditures \( M \) to \( h_2 = L \) households. A household in our two-period example would solve

\[
    \max_{x_1} \{ u(x_1) + \beta \cdot \pi \cdot u(b - x_1 - M) + \beta \cdot (1 - \pi) \cdot u(b - x_1) \}. \tag{3}
\]

In DeNardi et al [2010], a household in poor health might have a different utility function. In place of (3), we would have

\[
    \max_{x_1} \{ u(x_1) + \beta \cdot \pi \cdot U(b - x_1 - M) + \beta \cdot (1 - \pi) \cdot u(b - x_1) \}. \tag{4}
\]

The present paper uses a simpler version of (4):

\[
    \max_{x_1} \{ u(x_1) + \beta \cdot \pi \cdot U(b - x_1) + \beta \cdot (1 - \pi) \cdot u(b - x_1) \}. \tag{5}
\]

Formulation (5) assumes that poor health causes a change in a household’s utility function from \( u(\cdot) \) to \( U(\cdot) \), and it incorporates the out-of-pocket expenditures associated with poor health into the argument of \( U(\cdot) \).

We focus on the general health status of an individual rather than just his/her medical ailments. If we wanted to emphasize medical emergencies analogous to a broken arm — where treatment could restore good-as-new health status — formulation (3), with \( M \) corresponding to out-of-pocket (OOP) medical expense, would seem appropriate. We study retirement-aged households, however. In that age group, chronic health-status problems seem a more major concern. Consider, for example, difficulties with “activities of daily living” (ADLs), as measured in the HRS — i.e., does the respondent have trouble with (i) bathing, (ii) eating, (iii) dressing, (iv) walking across the room, and (v) getting out of bed. In such cases, individuals will often have long-term OOP medical expenses (as with medication co-pays) but also expenses for hiring assistance or moving to a nursing home. The latter expenses can be quantitatively significant (see below), and they have features of special interest for our modeling.
In particular, formulations (3)-(5) emphasize the role of uninsured risks. Virtually all US households 65 and older have Medicare coverage. Insurance for assistance is much less prevalent, however, presumably because of difficulties in objective verification of patient health status.\footnote{The analysis below does incorporate Medicaid nursing-home care as an option for households. But, we assume that Medicaid attempts to control adverse selection by requiring medical certification of poor health, by requiring that a recipient have essentially no assets and contribute virtually all of his/her income, and by setting care to a very basic level (see, for example, O’Brien [2005]).}

Moreover, an individual may have considerable latitude in choosing how much assistance to purchase, or to live without, which seems quite different from the situation for medical diagnoses. There is no reason to expect an individual to select a level of treatment that precisely restores his/her original utility function \( u(.) \). An individual’s spending on late-in-life health status may depend upon his/her lifetime resources (and, see Section 2, on longevity) as much or more than on his/her specific list of ailments.

This paper adopts a specification for \( U(.) \) in which poor health status lowers utility but raises marginal utility. The argument of \( U(.) \) includes OOP health expenses. Thus, (5) differs from (4) as well as (3). As explained above, we are thinking, in particular, of OOP expenses for alleviating long-term symptoms and obtaining assistance with tasks of daily living, where individuals can exercise choice in allocating their lifetime resources. Put another way, we do not want to distinguish OOP health expenditures from other consumption anymore than we want to distinguish minimal food expense for subsistence from remaining consumption.

Our specification for \( U(.) \) is as follows. There is a household production technology for transforming expenditure, \( x \), into consumption, \( c \):

\[
c ≡ \begin{cases} x, & \text{if } h = H, \\ \omega \cdot x & \omega \in (0, 1), \text{if } h = L. \end{cases}
\]

Then in good health utility from expenditure \( x \) is \( u(x) \), but in poor health utility from \( x \) is

\[
U(x) ≡ u(\omega \cdot x) = [\omega]^\gamma \cdot u(x) ≡ \theta \cdot u(x).
\]

Given \( \gamma < 0 \),

\[
\theta ≡ [\omega]^\gamma > 1.
\]

With (6)-(8) and (2), poor health status indeed lowers utility, i.e., \( U(x) < u(x) < 0 \), but raises marginal utility,

\[
U'(x) = \theta \cdot u'(x) > u'(x) > 0.
\]
consumption, cet par, when \( h = L \), creating an incentive (ii) to allocate more resources to period 2 in order to smooth consumption outcomes. When \( \gamma < 0 \), (8) shows that incentive (ii) predominates.

The outcomes of this paper’s analysis depend heavily on the assumption that poor health raises a household’s marginal utility. The existing literature provides some support. Using formulation (4), DeNardi et al [2010] use a version of \( U(\cdot) \) like (7), and their estimate implies \( \theta > 1 \). French et al [2004] find empirical evidence of a positive relation between a household’s OOP and its lifetime resources. Using a novel direct measure of well-being, Finkelstein et al [2012] find support for state-dependent utility. Their overall results are not inconsistent with (5) — and indeed their formulation leads to (5). See our Appendix I.

Finally, note that formulation (5) offers advantages in terms of analytic tractability. For example, the first-order condition for (3) will not have a closed-form solution. The reverse is true, however, for (5)-(7).

**DISCUSSION.** One way to distinguish health status from medical status alone is to think about OOP costs. HRS measures of OOP health spending are built up from a series of questions covering 9 components: health insurance premiums, drugs, physician costs, nursing home and hospital costs, other, home care, non-medical costs, helper costs, and hospice care. See Marshall et al [2010].

Nursing home and hospital OOP spending can be separated for 2002 and beyond, with about two-thirds attributable to nursing home care. Marshall et al’s mean cost amounts for the last year of life are, respectively, \$2096, 1761, 353, 4731, 384, 687, 790, 1281, and 38 — for a total of \$12,120. We can separate OOP expenses related to medical status, \( M^1 \), from general health status, \( M = M^1 + M^2 = M^1 + (M - M^1) \). The first three cost amounts above plus hospital’s share of the fourth comprise \( M^1 \); the remainder form \( M^2 \).

Our analysis emphasizes the role of \( M^2 \). The preceding figures suggest \( M^2 > M^1 \). The importance of \( M^2 \) becomes even greater when we note that insurance premiums should affect spending in all states and times, and that after the advent of Medicare Part D in 2006 many OOP drug costs can migrate from the second category in our OOP list to the first. Marshall et al write,

Differences in spending by wealth quartile are much larger than income differences with spending in the top wealth quintile equal to \$18,232 compared to \$7,173 in the bottom. These differences appeared to be driven mostly by greater spending for nursing homes, as well as for helpers, home health care, and other sources of spending that likely help maintain the independence of people living at home. [p.4]

And,

Indeed, the ultimate luxury good appears to be the ability to retain independence and remain in one’s home ... through the use of (paid) helpers and home health care assistance. Once admitted to the nursing home, a further benefit of assets is the ability to eschew Medicaid, which many would prefer to avoid ... and to have the

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4 See also Hurd and Rohwedder [2009].

5 Marshall et al [2010, Tab 3a] shows maximum values relative to the mean for all nine OOP components. The ratios are, respectively, 20:1, 60:1, 20:1, 60:1, 20:1, 40:1, 20:1, 50:1, and 1400:1.
ability to purchase more comfortable living arrangements. These types of expenses are generally not amenable to insurance coverage .... [p.26]

Medicaid Financed Nursing Home Care. Nursing home care is so expensive that many middle class households that live long enough may eventually turn to Medicaid. To qualify, an individual must have poor health, virtually no net worth, and must be willing to hand over its income to Medicaid.

A discrete-time analysis does not distinguish stocks from flows. Let $\gamma < 0$ and assume that Medicaid offers a household nursing-home care worth $\bar{x}$. In formulation (5), suppose that a period-2 household finding itself in poor health and accepting Medicaid help would, in effect, turn all of its private resources, i.e., $b - x_1$, over to Medicaid. Then the household faces a non-concave problem at retirement:

$$
\max_{x_1} \left\{ u(x_1) + \beta \cdot \pi \cdot \max\{U(b - x_1), U(\bar{x})\} + \beta \cdot (1 - \pi) \cdot u(b - x_1) \right\}.
$$

We frame our analysis below in continuous time to circumvent the non-concavity. Namely, a household finding itself in poor health at $t_0$ can choose a date $t_1 > t_0$ for turning to Medicaid and use the intervening interval $(t_0, t_1)$ to spend down its bequeathable net worth.

2 Assumptions. We use a stylized model to attempt to gain insight into optimal household behavior during retirement. Key assumptions are as follows.

Consider a single-person household that is already retired. At any age $s$, the household’s health status, $h$, is either “high,” $h = H$, or “low,” $h = L$. The household starts with $h = H$. There is a Poisson process with hazard rate $\Lambda > 0$ such that at the first Poisson event the household drops to health status $L$. Once at $h = L$, the household begins a second Poisson process, with parameter $\lambda > 0$. At the Poisson event for the second process, the household dies.

There is a state verification problem: an agent privately knows when he/she enters state $L$, but the transition is not legally verifiable. Given verification difficulties, our framework assumes that agents cannot purchase or sell annuities or life insurance after the initial moment of analysis.

As suggested in Section 1, we assume that the utility of a household with consumption expenditure $x$ and health status $h$ is

$$
u(x, h) = \begin{cases} 
\frac{|x|^\gamma}{\gamma}, & \text{if } h = H, \\
\theta \cdot \frac{|x|^\gamma}{\gamma}, & \text{if } h = L, 
\end{cases}
$$

with $\gamma < 0$.

In practice, Social Security benefits are by far the most common source of annuity income. SSA benefits cannot legally serve as collateral for borrowing. We assume the same

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6 A less stringent specification would allow households always to add to their annuitized net worth at retirement-age terms. In practice, however, we suspect that additions past retirement age are uncommon. See Sinclair and Smetters [2004].
restriction applies to all annuities. Let $b_t$ be a household’s non-annuitized, or “bequeathable,” private net worth. We assume bankruptcy laws compel $b_t \geq 0$ all $t$.

For simplicity, we assume a fixed (real) interest rate $r_t = r > 0$. It seems realistic to assume that $\lambda$ and $\Lambda$ large enough that $r - (\lambda + \beta) < 0$ and $r - (\Lambda + \beta) < 0$. With a retirement age of 65, for example, an expectation of 15 years of high health status would imply $15 = 1/\Lambda$, or $\Lambda = 0.067$. Sinclair and Smetters [2004] suggest a much higher value for $\lambda$. For convenience, we also assume $r - \beta \geq 0$.

In the model, a household with health status $h = L$ qualifies for Medicaid nursing home care if it turns over to government its annuity income and its entire bequeathable net worth. Let Medicaid nursing home care correspond to consumption flow $\bar{x} > 0$. Acceptance of Medicaid nursing-home care is an absorbing state.

Recapping our assumptions:

a1: Health status is not verifiable; hence, annuities and life insurance can only be purchased at the start of our problem.

a2: Bequeathable net worth must be non-negative: $b_t \geq 0$ all $t \geq 0$.

a3: $\gamma < 0$ and $\theta > 1$.

a4: A household transits from good health status to poor with Poisson hazard $\Lambda$, and from poor health status to death with Poisson hazard $\lambda$. No other transitions have positive probability.

a5: The real interest rate, $r$, is constant, and $r > 0$. We have $\beta > 0$, $r \geq \beta$, and the size of $\lambda$ and $\Lambda$ ensures $r - (\lambda + \beta) < 0$ and $r - (\Lambda + \beta) < 0$.

a6: A household with low health status ($h = L$) can accept Medicaid nursing-home care. To do so, it must transfer its annuity income and all of its bequeathable net worth to Medicaid. Medicaid provides consumption flow $\bar{x}$. We assume there are no transitions from $h = L$ to $h = H$, and that once a $h = L$ household accepts Medicaid it never leaves. For future reference, note that (a5) implies

$$\sigma \equiv \frac{r - (\lambda + \beta)}{1 - \gamma} < 0.$$  \hfill (11)

3 Last Period of Life. We solve our model backward, beginning with the last period of life. A household’s health status is poor in this stage. Rescale the initial last-period-of-life age to 0. As stated, the Poisson hazard rate into death is $\lambda > 0$. Let the age of death be $T$. Beginning-of-period bequeathable net worth is $b_0 = b$, and annuity (real) income is $a$.

Results for this section include:

R1: All second-period households that survive long enough become liquidity constrained — in the sense that they eventually live solely on their annuity income or on Medicaid support. This provides one reason that accidental bequests will not be universal: there will be a finite maximum age beyond which all households have no bequeathable net worth.
R2: Households beginning their second period with positive bequeathable net worth will tend to spend it in a rapid, but controlled, fashion.

We distinguish 2 cases. In the first, households surviving long enough turn to Medicaid. In the second, households never utilize Medicaid.

Case (i): $\bar{x} > a$. At the start of its last period of life, a household chooses a consumption expenditure path $x_t$ all $t \geq 0$ together with an age $T_1$ for accepting Medicaid. (If $T_1 = \infty$, the household never accepts Medicaid.)

The household solves

$$v(a, b) \equiv \max_{x_t, T_1} \{ \int_0^{T_1} \lambda \cdot e^{-\lambda T} \cdot \int_0^T e^{-\beta t} \cdot U(x_t) \, dt \, dT \}$$

subject to: $\dot{b}_t = r \cdot b_t + a - x_t,

$$b_t \geq 0 \quad \text{all} \quad t \geq 0,$$

$$a \quad \text{and} \quad b_0 = b \quad \text{given}.$$

The criterion can be simplified. The last of the three terms yields

$$\int_{T_1}^{\infty} \lambda \cdot e^{-\lambda T} \cdot \int_0^T e^{-\beta t} \cdot U(\bar{x}) \, dt \, dT +$$

$$= \int_{T_1}^{\infty} \int_t^{\infty} \lambda \cdot e^{-\lambda T} \cdot e^{-\beta t} \cdot U(\bar{x}) \, dt \, dT$$

$$= - \int_{T_1}^{\infty} \int_{-\lambda t}^{-\infty} e^z \, dz \cdot e^{-\beta t} \cdot U(\bar{x}) \, dt$$

$$= \int_{T_1}^{\infty} e^{-\lambda t} \cdot e^{-\beta t} \cdot U(\bar{c}) \, dt$$

$$= e^{-(\lambda + \beta)T_1} \cdot \frac{U(\bar{x})}{\lambda + \beta} \equiv \phi^1(T_1).$$

The remaining two criterion terms yield

$$\int_0^{T_1} \lambda \cdot e^{-\lambda T} \cdot \int_0^T e^{-\beta t} \cdot U(x_t) \, dt \, dT + \int_{T_1}^{\infty} \lambda \cdot e^{-\lambda T} \cdot \int_0^{T_1} e^{-\beta t} \cdot U(x_t) \, dt \, dT$$
\[
\begin{align*}
&= \int_0^{T_1} \int_t^\infty \lambda \cdot e^{-\lambda T} \cdot e^{-\beta t} \cdot U(x_t) \, dT \, dt \\
&= -\int_0^{T_1} \left[ \int_{-\lambda t}^{-\infty} e^z \, dz \right] \cdot e^{-\beta t} \cdot U(x_t) \, dt \\
&= \int_0^{T_1} e^{-\lambda \cdot t} \cdot e^{-\beta \cdot t} \cdot U(x_t) \, dt.
\end{align*}
\] (14)

Thus, we can rewrite (12) as

\[
v(a, b) \equiv \max_{x_t,T_1} \left\{ \int_0^{T_1} e^{-(\lambda+\beta)\cdot t} \cdot U(x_t) \, dt + \phi^1(T_1) \right\},
\] (15)

subject to:

\[
\dot{b}_t = r \cdot b_t + a - x_t,
\]

\[
b_t \geq 0 \quad \text{all } \ t \geq 0,
\]

\[
a \quad \text{and} \quad b_0 = b \quad \text{given}.
\]

The first criterion term captures the state dependence of utility on the probability of being alive (Yaari [1965]). The second, \(\phi^1(T_1)\), captures utility under Medicaid.

Formulation (15) is a free endpoint problem (Kamien and Schwartz [1981, sect.7]). We have

**Proposition 1:** In case (i), (15) has a unique solution. We have \(T_1 < \infty\), \(b_{T_1} = 0\), and

\[
x_s = x_0 \cdot e^{\sigma \cdot s} \quad \text{all } \ s \in [0, T_1].
\] (16)

**Proof:** See Appendix.

Several features of the case-(i) solution are

- \(T_1 < \infty\). Given enough longevity, a Medicaid-financed nursing home stay is inevitable.
- At ages past \(T_1\), a household’s bequeathable net worth is 0. Hence, we can say that households that live long enough become “liquidity constrained.”
- The proof of Proposition 1 shows that consumption expenditure drops discontinuously at age \(T_1\), as shown in Figure 1a-b.\(^7\)

The discontinuity in Figure 1a-b occurs despite concave utility. A household in our model could move resources beyond \(T_1\) by postponing its makeup of Medicaid. The flow

\(^7\) Note that Fig 1a-b makes \(x_t\) continuous from the left at \(t = T_1\). Choosing that convention makes \(x_0 = x_0(b)\) continuous at \(b = 0\) — which simplifies the exposition in Section 5.
Fig. 1a: Second-stage-of-retirement optimal consumption and bequeathable net worth for case 1 when $b > 0$

Fig. 1b: Second-stage-of-retirement optimal consumption and bequeathable net worth for case 1 when $b = 0$
rate of its production of utility denominated in time-0 units at $T_1 - \epsilon$ is $\mathcal{H}|_{s=T_1}$. The flow rate of utility loss from postponing Medicaid makeup is

$$-\frac{\partial \phi^1(T_1)}{\partial t} = U(\bar{x}) \cdot e^{-(\lambda+\beta)T_1}.$$  \hfill (17)

Condition (A9) in the proof of Proposition 1 requires that $\mathcal{H}|_{s=T_1}$ and (17) be equal. That does not equalize $U(x_{T_1-\epsilon})$ and $U(x_{T_1+\epsilon})$. The reason is as follows. Obtaining flow $U(x_{T_1-\epsilon})$ requires private resources, whose value, $\mu_{T_1} \cdot [r \cdot b_{T_1} + a - x_{T_1}]$, the Hamiltonian reflects. Since $b_{T_1} = 0$ and $a < x_{T_1}$, that resource cost reduces the advantage of independence from governmental support. Private costs disappear, on the other hand, with acceptance of Medicaid.

The next proposition summarizes several technical features of our solution that simplify our phase-diagram presentations below.

**Proposition 2:** Assume case (i). Consider the solution to Proposition 1. Writing $T_1 = T_1(b)$ we have

$$T_1(b) \text{ strictly increasing in } b, \quad T_1(0) = 0, \quad \lim_{b \to \infty} T_1(b) = \infty. \quad (18)$$

We have $x_{T_1} > \bar{x}$. And, $x_{T_1}$ is a function solely of $a$ and $\bar{x}$.

We have

$$\frac{\partial v(a, b)}{\partial b} > 0 \quad \text{and} \quad \frac{\partial^2 v(a, b)}{\partial b^2} < 0. \quad (19)$$

Let $x_t$ be optimal and $x_0 = x_0(b)$. Let

$$k_0 \equiv \frac{x_{T_1} - a}{x_{T_1}} \cdot \frac{r - \sigma}{-\sigma}.$$  \hfill (20)

Line (11) and the results above show that $k_0 > 0$ and that $k_0$ depends only on $a$ and $\bar{x}$. Then

$$x_0(b) = \begin{cases} \text{convex in } b, & \text{if } k_0 > 1, \\ \text{concave in } b, & \text{if } k_0 \in (0, 1), \end{cases} \quad (21)$$

$$\lim_{b \to \infty} \frac{\partial x_0(b)}{\partial b} = r - \sigma > 0. \quad (22)$$

**Proof:** See Appendix.
**Case (ii):** \( \bar{x} < a \) or \( \bar{x} = a \). In case (ii), annuity income at least equals the consumption expenditure flow \( \bar{x} \) that Medicaid provides. An agent can subsist forever with \( x_t = a \) and \( b_t = 0 \); hence, Medicaid take up no longer occurs.

In case (ii), the household problem is

\[
v(a, b) \equiv \max_{x_t} \int_{0}^{\infty} e^{-(\lambda+\beta) \cdot t} \cdot U(x_t) \, dt \tag{23}
\]

subject to: \( \dot{b}_t = r \cdot b_t + a - x_t \),

\[
b_t \geq 0 \quad \text{all} \quad t \geq 0,
\]

\[
b_0 = b \quad \text{and} \quad a \quad \text{given}.
\]

Ignoring the state-variable constraint \( b_t \geq 0 \) for the moment, the present-value Hamiltonian for (23) is

\[
H \equiv e^{-(\lambda+\beta) \cdot t} \cdot U(x_t) + \mu_t \cdot \left[ r \cdot b_t + a - x_t \right], \tag{24}
\]

with costate \( \mu_t \). We have first-order condition

\[
\frac{\partial H}{\partial x_t} = 0 \iff e^{-(\lambda+\beta) \cdot t} \cdot U'(x_t) = \mu_t \tag{25}
\]

and costate equation

\[
\dot{\mu}_t = -\frac{\partial H}{\partial b_t} \iff \dot{\mu}_t = -r \cdot \mu_t. \tag{26}
\]

Substituting (25) into (26),

\[
-(\lambda + \beta) \cdot e^{-(\lambda+\beta) \cdot t} \cdot U'(x_t) + e^{-(\lambda+\beta) \cdot t} \cdot U''(x_t) \cdot \dot{x}_t = \dot{\mu}_t = -r \cdot \mu_t = -r \cdot e^{-(\lambda+\beta) \cdot t} \cdot U'(x_t)
\]

\[
\iff U''(x_t) \cdot \dot{x}_t = -[r - (\lambda + \beta)] \cdot U'(x_t)
\]

\[
\iff (\gamma - 1) \cdot \frac{\dot{x}_t}{x_t} = -[r - (\lambda + \beta)]. \tag{27}
\]

Assumption (a5) implies \( r - (\lambda + \beta) < 0 \).

Using (27) and the household budget constraint, we draw the phase diagram — see Figure 2. Each dotted curve is a trajectory satisfying the budget constraint and (27). Eq (27) shows that along each, \( x_t > 0 \) all \( t \). Hence, \( \lim_{t \to \infty} b_t = \infty \) as well.

Our conditions above would be sufficient for a solution if transversality condition

\[
\lim_{t \to \infty} \mu_t \cdot b_t = 0 \tag{28}
\]
Fig. 2: Phase diagram trajectories for the unconstrained version of problem (23)
held. It is easy to see that it does not: moving along any trajectory in Figure 2, upon reaching \( x_t = a \) set \( x_s = a \) all subsequent \( s \); then the modified trajectory is feasible and supplies higher consumption at times past \( t \) than the figure’s trajectories.

Case (i), however, suggests a way of reaching a solution. There exists a stationary solution in the phase diagram, as follows.

**Proposition 3:** In case (ii), let any \( T \) be given. Then \((b_t, x_t) = (0, a)\) is a stationary solution for (29):

\[
\max_{x_t} \int_T^\infty e^{-(\lambda+\beta)t} \cdot U(x_t) \, dt \tag{29}
\]

subject to:

\[
\dot{b}_t = r \cdot b_t + a - x_t,
\]

\( b_t \geq 0, \)

\( b_T \) given.

The maximized criterion is

\[
\phi^2(T) \equiv e^{-(\lambda+\beta)T} \cdot \frac{U(a)}{\lambda + \beta}. \tag{30}
\]

**Proof:** See text above.

We can form a solution to (23) from Figure 2 and Proposition 3.

**Proposition 4:** In case (ii), the trajectory in Figure 2 that reaches \((b_t, x_t) = (0, a)\) from above and then remains at \((0, a)\) forever solves problem (23).

**Proof:** See Appendix.

Knowing the form of the solution, we can use the constructive steps in the proof of Proposition 1 to obtain it:
Proposition 5: In case (ii), let

$$v(a, b) \equiv \max_{x_t, T_2} \left\{ \int_0^{T_2} e^{-(\lambda + \beta)\cdot t} \cdot U(x_t) \, dt + \phi^2(T_2) \right\}$$

(31)

subject to: \( \dot{b}_t = r \cdot b_t + a - x_t \),

\( a \) and \( b_0 = b \) given.

A solution to (31) exists with \( b_t \geq 0 \) all \( t \geq 0 \), \( T_2 < \infty \), \( b_{T_2} = 0 \), and

$$x_s = x_0 \cdot e^{\sigma \cdot s} \quad \text{all} \quad s \in [0, T_2].$$

(32)

We have \( x_{T_2} = a \). The solution to (31) is the solution of Proposition 1.

Proof: See Appendix.

Finally, Proposition 2 remains valid, though we can obtain a slightly stronger characterization for the derivative of \( x_0(b) \):

Proposition 6: In case (ii), let \( f(t) \) be as in the proof of Proposition 1. Then \( x_0(b) \) is continuous, strictly increasing, and strictly concave all \( b \geq 0 \) with

$$x_0(0) = a,$$

(33)

$$\frac{\partial x_0(b)}{\partial b} = \frac{1}{\int_0^{T_2(b)} e^{-(r - \sigma) \cdot s} \, ds},$$

(34)

$$\lim_{b \to \infty} \frac{\partial x_0(b)}{\partial b} = r - \sigma.$$

(35)

Proof: See Appendix.

Note the following properties of our solution in case (ii):

- \( T_2 < \infty \). Thus, all households with sufficient longevity eventually live exclusively from their annuity income. Hence, they become “liquidity constrained.”

- In contrast to case (i), optimal consumption is continuous at all ages, including age \( s = T_2 \). See Figure 3a-b. Our discussion of (17) for case (i) shows why a discontinuity does not occur in case (ii).
Fig. 3a: Second-stage-of-retirement optimal consumption and bequeathable net worth for case 2 when $b > 0$

Fig. 3b: Second-stage-of-retirement optimal consumption and bequeathable net worth for case 2 when $b = 0$
DISCUSSION. According to the model then, households exhaust their bequeathable net worth — becoming liquidity constrained — within a finite number of years once they begin the last period of life. If \( \bar{x} > a \), a liquidity constrained household inhabits a Medicaid financed nursing home; if \( \bar{x} \leq a \), the household lives on its private annuity income. If a household dies within the interval strictly between the start of its last period of life and the date it becomes liquidity constrained, it will leave an “accidental bequest;” otherwise its bequest will be zero.

A household beginning its last period with positive bequeathable net worth, \( b_0 > 0 \), tends to spend the latter at a rapid rate — see Figures 1 and 3. Since the hazard rate for death, \( \lambda \), will tend to be very high at this stage of life, the optimal consumption-expenditure profile may be very steep (especially if \( |\gamma| \) is not large), and the consumption-expenditure rate is always bounded below by \( \bar{x} \) (itself a high number relative to middle class resources). Put another way, with no intentional bequests in the model, a brief last period of life leads to rapid asset decumulation.

4 Penultimate Period of Retirement. Households have health status \( h = H \) in the first segment of their retirement.

Results for this section include:

\textbf{R1:} We show how and why penultimate-stage household bequeathable net worth can rise, fall, or remain the same depending on a household’s lifetime resources and annuitization.

\textbf{R2:} A household with low enough annuitization to accept Medicaid may asymptotically deplete its bequeathable net worth in its penultimate period. If such a household remains in good health long enough, it will therefore not have a significant accidental bequest. On the other hand, all households with enough annuity income to avoid Medicaid nursing-home care will leave their penultimate period with positive bequeathable net worth.

A penultimate-stage household solves

\[
V(a, B) \equiv \max_{X_s} \left\{ \int_0^\infty \Lambda \cdot e^{-\Lambda T} \cdot \left[ \int_0^T e^{-\beta t} \cdot u(X_t) \, dt + e^{-\beta T} \cdot v(a, B_T) \right] \, dT \right\} \\
\text{subject to:} \quad \dot{B}_t = r \cdot B_t + a - X_t, \\
B_t \geq 0 \quad \text{all} \quad t \geq 0, \\
\quad a \quad \text{and} \quad B_0 = B \quad \text{given}.
\]

Changing the order of integration, we have

\[
V(a, B) \equiv \max_{X_s} \left\{ \int_0^\infty e^{-\Lambda t} \cdot \left[ u(X_t) + \Lambda \cdot v(a, B_t) \right] \, dt \right\},
\]
subject to:  \[ \dot{B}_t = r \cdot B_t + a - X_t, \]

\[ B_t \geq 0 \quad \text{all} \quad t \geq 0, \]

\[ a \quad \text{and} \quad B_0 = B \quad \text{given}. \]

Section 3 shows that \( v(\cdot, \cdot) \) is concave in \( B \); hence, the criterion integrand of (37) is concave in \((X_t, B_t)\).

Disregarding the state-variable constraint \( B_t \geq 0 \) for the moment, the present-value Hamiltonian for (37) is

\[ \mathcal{H} \equiv e^{-(\Lambda + \beta) \cdot t} \cdot [u(X_t) + \Lambda \cdot v(a, B_t)] + M_t \cdot [r \cdot B_t + a - X_t], \tag{38} \]

with \( M_t \) the costate variable. The first-order condition for \( X_t \) is

\[ \frac{\partial \mathcal{H}}{\partial X_t} = 0 \iff e^{-(\Lambda + \beta) \cdot t} \cdot u'(X_t) = M_t. \tag{39} \]

The costate equation is

\[ \dot{M}_t = -\frac{\partial \mathcal{H}}{\partial B_t} = -e^{-(\Lambda + \beta) \cdot t} \cdot \Lambda \cdot \frac{\partial v(a, B_t)}{\partial B_t} - r \cdot M_t. \tag{40} \]

The budget equation is

\[ \dot{B}_t = r \cdot B_t + a - X_t. \tag{41} \]

We construct a phase diagram for \((B_t, X_t)\). Let \( x_0(b_0) \) be the initial consumption for the household in its second-period-of-retirement, where \( b_0 \) will correspond to the household’s \( B_t \) at the date it exits its penultimate state. The envelope theorem shows

\[ \frac{\partial v(a, B_t)}{\partial B_t} = \mu_0 = U'(x_0(B_t)) = \theta \cdot u'(x_0(B_t)). \tag{42} \]

(Recall that \( \theta > 1 \).) Eqs (39)-(42) imply

\[ u''(X_t) \cdot \dot{X}_t = -(r - (\Lambda + \beta)) \cdot u'(X_t) - \Lambda \cdot \theta \cdot u'(x_0(B_t)). \tag{43} \]

Eqs (41) and (43) determine the phase diagram. The equations for the isoclines are

\[ X_t = \Gamma^1(B_t) \equiv r \cdot B_t + a, \tag{44} \]

\[ X_t = \Gamma^2(B_t) \equiv \Omega \cdot x_0(B_t), \tag{45} \]

where
\[ \Omega \equiv \left[ \frac{\Lambda - r - \beta}{\Lambda \cdot \theta} \right] \in (0, 1). \]

The propositions of Section 3 show that \( \Gamma^2(B) \) is increasing and either concave or convex. Generically, there are two cases.

**Case 1:** \( \Gamma^1(0) > \Gamma^2(0) \). Because \( \Omega \in (0, 1) \), high-annuity households — those of Sect 3, case (ii) — must fall into this category. Low-annuity households may or may not.

In case 1, the phase diagram must look as in Figures 4a-c. The dotted trajectories show candidate solutions. We have

**Proposition 7:** Given case 1, the dotted trajectories of Figures 4a-c are solutions.

**Proof:** See Appendix.

Section 3 characterizes the asymptotic slope for \( \Gamma^2(b) \). When the asymptotic slope for \( \Gamma^2(b) \) exceeds that of \( \Gamma^1(b) \), \( r \), we have Figure 4a. Otherwise, we have Fig 4b-c. In the latter circumstance, to distinguish 4b from 4c, we find the \( \bar{B} \) at which the slope of \( \Gamma^2(\bar{B}) \) is \( r \) and compare the heights of \( \Gamma^1(\bar{B}) \) and \( \Gamma^2(\bar{B}) \).

In case 1, the optimal \( B_t \) either converges to a positive stationary value or it grows throughout the first period of retirement.

**Case 2:** \( \Gamma^1(0) < \Gamma^2(0) \). Case 2 arises, if at all, only with Section 3, case (i).

Figures 5a-c show the possible phase diagrams for case 2. In Fig 5b-c, with a starting value of \( B \) at or to the right of all stationary points, the analysis can follow Case 1. Otherwise, the trajectories curling from northeast to southeast are reminiscent of Section 3, Figure 3. Then we can follow the procedure of Propositions 3-5.

**Proposition 8:** In case 2, \( (B_t, X_t) = (0, a) \) constitutes a stationary solution to problem (37). If we begin this solution at time \( T \), the contribution to lifetime utility is

\[ \Phi(a, T) \equiv \int_T^\infty e^{-(\Lambda+\beta)\cdot t} \cdot [u(a) + \Lambda \cdot v(a, 0)] \, dt = e^{-(\Lambda+\beta)\cdot T} \cdot \frac{u(a) + \Lambda \cdot v(a, 0)}{\Lambda + \beta}. \quad (46) \]

**Proof:** See Appendix.

With Proposition 8, we can show that the solution in Fig 5a, or at the left-hand side of Fig 5b-c, follows the dotted trajectory from the northeast to \( (B_t, X_t) = (0, a) \), and remains at the latter point thereafter. We have
Fig. 4a: Variant of first-period-of-retirement phase diagram in case 1

Fig. 4b: Variant of first-period-of-retirement phase diagram in case 1
Fig. 4c: Variant of first-period-of-retirement phase diagram in case 1
Fig. 5a: Variant of first-period-of-retirement phase diagram in case 2

Fig. 5b: Variant of first-period-of-retirement phase diagram in case 2
Fig. 5c: Variant of first-period-of-retirement phase diagram in case 2
Proposition 9: Suppose case 2 with Fig 5a or the left-hand sides of Fig 5b-c. Pose the problem

\[
\max_{X_t, T_3} \left\{ \int_0^{T_3} e^{-(\Lambda+\beta)\cdot t} \cdot \left[ u(X_t) + \Lambda \cdot v(a, B_t) \right] dt + \Phi(a, T_3) \right\} \tag{47}
\]

subject to: \( \dot{B}_t = r \cdot B_t + a - X_t \),

\( a \) and \( B_0 = B \) given.

A solution to (47) exists with \( B_t \geq 0 \) all \( t \geq 0 \), \( T_3 < \infty \), \( B_{T_3} = 0 \), and \( X_t \) declining \( t \in [0, T_3) \). We also have \( x_{T_3} = a \).

The solution to (47) solves problem (37).

Proof: See Appendix.

DISCUSSION. As in Section 3, we have two cases. Households with low lifetime resources, who will ultimately use Medicaid nursing-home care, are the only ones in case 2. Unless such households begin retirement with high bequeathable net worth (see Figures 5a-c), they will deplete their bequeathable assets within a finite number of years while their health is good. In poor health, they will turn to Medicaid support.

Households with higher lifetime resources will fall into case 1. As Figures 4a-c show, such households will want to increase their bequeathable net worth during good health or at least maintain it at a positive stationary level.

In the end, we think our model suggests an explanation for the slow to nonexistent average decumulation wealth patterns of Mierer, Poterba et al, and others mentioned in the introduction. The explanation has two parts. First, although households in the model have no intentional bequest motive, the analysis implies that some will leave “accidental bequests” as they die while attempting to self-insure against late-in-life low health status. Thus, a cohort’s aggregate of life-cycle savings will never, in fact, be fully depleted from dissaving. Second, our analysis suggests that many households (i.e., at minimum all of those with \( a \geq \bar{x} \)) will want to build and maintain positive bequeathable net worth during their healthy retirement years. On average, good health lasts 5-10 times as long as end-of-life poor health. The data sampling interval of the HRS, for instance, is 2 years. Some households will enter their last period and die between sampling dates. Among households that are successfully sampled during poor health, many will complete much of their wealth depletion and then die before the next survey wave. It seems plausible that other households will be unable to participate in a survey at all while in very poor health, and thus will be lost to attrition. For all of these reasons, it is easy to believe that surveys will often miss the short interval of household net worth decumulation — tending to lead to downward biased estimates of life-cycle dissaving.
5 Conclusion. This paper presents a stylized model of retirement-period life-cycle household behavior. The model emphasizes the role of late-in-life declines in health status. We assume the latter are chronic, difficult to insure, but subject to amelioration through private expenditures on services and assistance. The timing and duration of declines is random.

The model shows that after retirement but while in good health, middle class households may want to maintain, or continue building, their net worth. On the other hand, if and when their health status deteriorates, they will tend to spend down their assets very rapidly. We suggest, in fact, that the comparatively short span, on average, of poor health status — and the potential difficulty of collecting survey data during it — may help explain why evidence of life-cycle dissaving during retirement is hard to find.

Although our model has no intentional bequest behavior, it predicts “accidental bequests” for a fraction of households. It also shows the potential importance of the fallback that Medicaid nursing-home care provides.

Future work will extend the model to include couples, to derive the optimal fraction of life-cycle net worth that a household should annuitize, and to include more health-status levels.
Appendix I

This appendix compares our formulation of state-dependent utility — see Section 1 — with Finkelstein et al [2012].

Finkelstein et al use subjective well-being measures to compare the marginal utility of HRS households with/without medical conditions. Let $x$ be total household spending, $m$ its medical spending (OOP), and, without loss of generality, set the price of medical spending to 1. A Finkelstein et al household in good health has utility (1); a household in poor health solves

$$
\Psi(x) = \phi \cdot \max_m \{ \varphi \cdot u(x - m) + (1 - \varphi) \cdot u(m) \} .
$$

(A1)

Let $\phi > 0$ and $\varphi \in (0, 1)$. Finkelstein et al “estimate the effect of chronic disease on the marginal utility of non-medical consumption, evaluated at a constant level of non-medical consumption” (see Finkelstein et al [2012, p.2 and also eq. 12) and find that the marginal utility falls when health deteriorates. In other words, they estimate that

$$
\frac{\phi \cdot \varphi \cdot u'(x - m)}{u'(x - m)} = \phi \cdot \varphi < 1 .
$$

(A2)

This can be consistent with our assumption that a household’s marginal utility from total expenditure $x$ rises when health declines (see (9)).

To examine the relationship more carefully, note that maximization in (A1) implies

$$
\frac{[1 - \varphi]^{1\gamma}}{[\varphi]^{1\gamma} + [1 - \varphi]^{1\gamma}} \cdot x = m .
$$

So,

$$
\Psi(x) = \phi \cdot \psi \cdot u(x)
$$

(A3)

where

$$
\psi = \varphi \cdot \left[ \frac{[\varphi]^{1\gamma}}{[\varphi]^{1\gamma} + [1 - \varphi]^{1\gamma}} \right]^{\gamma} + (1 - \varphi) \cdot \left[ \frac{[1 - \varphi]^{1\gamma}}{[\varphi]^{1\gamma} + [1 - \varphi]^{1\gamma}} \right]^{\gamma}
$$

$$
= \left[ [\varphi]^{1\gamma} + [1 - \varphi]^{1\gamma} \right]^{1-\gamma} .
$$

The present paper’s assumption about marginal utility is

$$
\frac{\partial \Psi(x)}{\partial x} = \phi \cdot \psi > 1 .
$$

(A4)

Set $\phi = 1$ and $\gamma = -1$ for simplicity. Then for $\varphi = 9/10, 4/5, 1/2$, we have $\psi = 16/10, 9/5, 2$, respectively.

In other words, it is easy to find cases in which (A2) and (A4) hold simultaneously. In fact, a comparison of (A3) and (7) shows that (A1) provides an alternate derivation of our model of (5)-(7).
Appendix II

Proof of Proposition 1. The Hamiltonian for (15) is

\[ H \equiv e^{-(\lambda+\beta) \cdot s} \cdot U(x_s) + \mu_s \cdot [r \cdot b_s + a - x_s], \] (A5)

where \( \mu_s \) is the costate variable. The costate equation is

\[ \dot{\mu}_s = -\frac{\partial H}{\partial b_s} \iff \dot{\mu}_s = -r \cdot \mu_s \iff \mu_s = \mu_0 \cdot e^{-r \cdot s}. \] (A6)

First-order conditions (Kamien and Schwartz [1981, p.143]) include

\[ \frac{\partial H}{\partial x_s} = 0 \iff e^{-(\lambda+\beta) \cdot s} \cdot U'(x_s) = \mu_0, \] (A7)

\[ b_{T_1} \geq 0, \quad \mu_{T_1} \geq \frac{\partial \phi^1(T_1)}{\partial b_{T_1}} \geq 0, \quad b_{T_1} \cdot [\mu_{T_1} - \frac{\partial \phi^1(T_1)}{\partial b_{T_1}}] = 0, \] (A8)

\[ H|_{s=1} + \frac{\partial \phi^1(T_1)}{\partial T_1} = 0. \] (A9)

Step 1. (A7) implies

\[ x_s = x_0 \cdot e^{r \cdot s} \quad \text{all} \quad s \in [0, T_1]. \] (A10)

This establishes (16).

(A6) shows \( \mu_{T_1} > 0 \). In our model, \( \partial \phi^1 / \partial b_{T_1} = 0 \). Hence, (A8) implies

\[ b_{T_1} = 0. \] (A11)

Step 2. We show there is a unique \( x_{T_1} \).

Let \( S = T_1 \). Then

\[ e^{-(\lambda+\beta) \cdot S} \cdot U(x_S) + \mu_S \cdot [b_S + a - x_S] - U(\bar{x}) \cdot e^{-(\lambda+\beta) \cdot S} = 0 \quad \text{from A9} \]

\[ \iff e^{-(\lambda+\beta) \cdot S} \cdot U(x_S) + \mu_S \cdot [a - x_S] - u(\bar{x}) \cdot e^{-(\lambda+\beta) \cdot S} = 0 \quad \text{from A11} \]

\[ \iff e^{-(\lambda+\beta) \cdot S} \cdot U(x_S) + e^{-(\lambda+\beta) \cdot S} \cdot U'(x_S) \cdot [a - x_S] \]

\[ - U(\bar{x}) \cdot e^{-(\lambda+\beta) \cdot S} = 0 \quad \text{from A6} \]

\[ \iff \psi(x_S) \equiv U(x_S) - U(\bar{x}) + U'(x_S) \cdot [a - x_S] = 0. \] (A12)

(A9) then shows \( X_{T_1} \) must be a root of \( \psi(.) \).

By construction, we must have \( x_{T_1} \geq a \). With case (i), \( \bar{x} > a \). We then have

\[ \psi(\bar{x}) = U'(\bar{x}) \cdot [a - \bar{x}] < 0, \] (A13)
\[ \psi'(x) = U''(x) \cdot [a - x] > 0 \quad \text{all} \quad x > a. \quad (A14) \]

This establishes that there will be a unique \( x_{T_1} \geq a \) if and only if \( \psi(\infty) > 0 \).

Let \( \lim \) stand for \( \lim_{x \to \infty} \). We have

\[
\lim \psi(x) > 0 \iff \lim \frac{U(x) - U(\bar{x})}{U'(x) \cdot (x - a)} > 1 \iff \lim \frac{1}{\gamma} \cdot \frac{[x]^\gamma - [\bar{x}]^\gamma}{[x]^\gamma \cdot \frac{x - a}{x}} > 1
\]

\[
\iff \frac{1}{\gamma} \cdot \lim \frac{1 - [\bar{x}^\gamma / x^\gamma]}{1 - \frac{a}{x}} > 1.
\]

The last is true if

\[
\lim \frac{[\bar{x}]^\gamma - 1}{1 - \frac{a}{x}} = \infty.
\]

But,

\[
\lim \frac{[\bar{x}]^\gamma - 1}{1 - \frac{a}{x}} = \lim \frac{[\bar{x}]^\gamma - 1}{1} = \infty. \quad (A15)
\]

This establishes that \( \psi(x) = 0 \) has a unique solution and the solution is larger than \( \bar{x} \). Call the solution \( x_{T_1} \).

**Step 3.** We now show that given \( x_{T_1} \) from Step 2, we have a unique \((x_0, T_1)\) and \( T_1 < \infty \).

From (A10) and Step 2,

\[
x_0 = x_{T_1} \cdot e^{-\sigma \cdot T_1}. \quad (A16)
\]

Let \( f(T) \equiv \int_0^T e^{-(r-\sigma) \cdot s} \, ds \). The budget constraint and (a6) yield

\[
b + \int_0^{T_1} a \cdot e^{-r \cdot s} \, ds = c_0 \cdot f(T_1). \quad (A17)
\]

Given (A16)-(A17),

\[
LHS \equiv b + \int_0^{T_1} a \cdot e^{-r \cdot s} \, ds = c_{T_1} \cdot e^{-\sigma \cdot T_1} \cdot X(T_1) \equiv RHS. \quad (A18)
\]

Think of a graph of the right and left-hand sides of (A18) with respect to \( T_1 \).

For \( T_1 = 0 \), \( LHS = b \). It is easy to see that \( LHS \) is increasing in \( T_1 \) and asymptotically approaches \( b + a / r \). We have

\[
\frac{\partial LHS}{\partial T_1} = a \cdot e^{-r \cdot T_1}.
\]
At $T_1 = 0$, $RHS = 0$. We have
\[
\frac{\partial RHS}{\partial T_1} = -\sigma \cdot RHS + x_{T_1} \cdot c^{-r \cdot T_1}.
\]
Recalling that $\sigma < 0$, we can see that the slope of $LHS$ is positive. $LHS$ diverges to $\infty$ as $T_1 \to \infty$. When $LHS = RHS$, the slope of the latter is larger (recall that in case (i), $x_{T_1} > \bar{x} > a$). Hence, the two curves have a unique intersection. When $b = 0$, we have $T_1 = 0$. Otherwise, $0 < T_1 < \infty$.

The value of $x_0 > 0$ follows from (A16).

**Step 4.** We show that our unique solution to the first-order conditions provides a (unique) solution of (15).

Fixing $T_1 < \infty$ and maximizing over $x_t$, Proposition 1 presents a standard, finite-time-horizon control problem with a concave Hamiltonian. Thus, conditional on $T_1$, we have a (unique) solution. The solution obeys $b_t \geq 0$. We also have a necessary condition for $T_1$, namely, (A9). We have shown that it has a unique solution.

To finish, we show that our solution for $T_1$ is a local maximum. That is to say, it satisfies the second-order necessary condition.

Define $\Psi(T_1) \equiv \psi(x_{T_1})$. We must have $x_{T_1} \geq \bar{x} > a$. Thus, (A14) shows $\psi'(x_{T_1}) > 0$. In Step 3 above, suppose that we impose a value for $T_1$ and solve (A18) for $x_{T_1}$. Then a higher $T_1$ requires a lower $x_{T_1}$. Hence,
\[
\Psi'(T_1) = \psi'(x_{T_1}) \cdot \frac{\partial x_{T_1}}{\partial T_1} = (+)(-) < 0,
\]
establishing the second-order necessary condition.

**Proof of Proposition 2.** In Step 3 of the proof of Proposition 1, an increase in $b$ displaces the graph of $LHS$ upward but does not affect $RHS$. Hence, $\partial T_1 / \partial b > 0$. We also have that $T_1 = 0$ when $b = 0$, and that $T_1 \to \infty$ as $b \to \infty$.

Step 2 in the proof of Proposition 1 shows that $x_{T_1}$ depends upon $a$ and $\bar{c}$ but not $b$. It also shows that $x_{T_1} > a$. In case (i), we also have $a > \bar{x}$.

We have
\[
\frac{\partial v(a, b)}{\partial b} = \mu_0 = u'(c_0) > 0.
\]
We have just shown that $\partial T_1 / \partial b > 0$ and that $x_{T_1}$ is independent of $b$; hence, (A10) shows $\partial x_0 / \partial b > 0$. So,
\[
\frac{\partial^2 v(a, b)}{\partial b^2} = u''(x_0) \cdot \frac{\partial x_0}{\partial b} < 0.
\]

Let $f(T)$ be as in Step 3, proof of Proposition 1. Differentiating (A16) and (A17),
\[
\frac{\partial x_0}{\partial b} = -x_{T_1} \cdot \sigma \cdot e^{-\sigma \cdot T_1} \cdot \frac{\partial T_1}{\partial b},
\]

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\[ 1 + a \cdot e^{-r \cdot T_1} \cdot \frac{\partial T_1}{\partial b} = f(T_1) \cdot \frac{\partial x_0}{\partial b} + x_{T_1} \cdot e^{-r \cdot T_1} \cdot \frac{\partial T_1}{\partial b}. \]

Substituting the first into the second,

\[ 1 + \left( \frac{a - x_{T_1}}{-x_{T_1} \cdot \sigma} \cdot e^{-\sigma \cdot T_1} \right) \cdot \frac{\partial x_0}{\partial b} = f(T_1) \cdot \frac{\partial c_0}{\partial b} \iff \frac{\partial x_0}{\partial b} = \frac{1}{D(T_1)}, \]

where

\[ D(T) \equiv \frac{x_T - a}{x_T} \cdot \frac{1}{-\sigma} \cdot e^{-(r - \sigma) \cdot T} + f(T). \]

Define

\[ k_0 \equiv \frac{x_{T_1} - a}{x_{T_1}} \cdot \frac{r - \sigma}{-\sigma} > 0. \]

Recalling that \( x_{T_1} \) depends only upon \( a \) and \( \bar{x} \),

\[ \frac{dD(T)}{dT} = (1 - k_0) \cdot e^{-(r - \sigma) \cdot T}. \]

We also have

\[ \lim_{T \to \infty} D(T) = \int_0^\infty e^{-(r - \sigma) \cdot s} \, ds = \frac{1}{r - \sigma}, \]

completing the proof. \( \blacksquare \)

**Proof of Proposition 3.** Letting \( \mu_t \) be the costate variable and \( \eta_t \) a Lagrange multiplier for the state-variable constraint \( b_t \geq 0 \), the present-value Hamiltonian is (Kamien and Schwartz [1981, p.215])

\[ H \equiv e^{-(\lambda + \beta) \cdot t} \cdot U(x_t) + \mu_t \cdot [r \cdot b_t + a - x_t] + \eta_t \cdot b_t. \]  \hfill (A21)

The first-order condition for \( x_t \) is

\[ \frac{\partial H}{\partial x_t} = 0 \iff e^{-(\lambda + \beta) \cdot t} \cdot U'(x_t) = \mu_t. \]  \hfill (A22)

The costate equation is

\[ \dot{\mu}_t = -\frac{\partial H}{\partial b_t} \iff \dot{\mu}_t = -r \cdot \mu_t - \eta_t. \]  \hfill (A23)

We also need

\[ \eta_t \geq 0, \]  \hfill (A24)
\[ \eta_t \cdot b_t = 0, \]  
(A25)
as well as the budget and state-variable constraints.

Suppose \((b_t, x_t) = (0, a)\) all \(t \geq T\). (A22) is then satisfied if

\[ \mu_t = e^{-(\lambda + \beta) \cdot t} \cdot U'(a). \]  
(A26)

(A23), in turn, is satisfied when

\[ - (\lambda + \beta) \cdot e^{-(\lambda + \beta) \cdot t} \cdot U'(a) = -r \cdot e^{-(\lambda + \beta) \cdot t} \cdot U'(a) - \eta_t \]

\[ \iff \eta_t = -[r - (\lambda + \beta)] \cdot e^{-(\lambda + \beta) \cdot t} \cdot U'(a). \]  
(A27)

Assumption (a5) implies that \(-[r - (\lambda + \beta)] > 0\). So, (A24) holds. Transversality condition (28) will hold any constant \(b_t\) because \(\mu_t \rightarrow 0\) — see (A26). (A25) will then hold if and only if

\[ b = 0. \]  
(A27)

In the latter case, (30) holds.

Proof of Proposition 4. Write the present value Hamiltonian as in (A21).

Let \(T_2 < \infty\) be the time \((b_t, x_t)\) arrives at \((0, a)\) in Figure 2. For \(t \leq T_2\), set \(\mu_t, x_t,\) and \(b_t\) from (26)-(27) and the budget constraint \(\dot{b}_t = r \cdot b_t + a - x_t\). And, set \(\eta_t = 0\). For \(t > T_2\), set \(\mu_t, x_t, b_t\) and \(\eta_t\) from Proposition 3. Then the first-order condition for \(x_t\), the costate equation, the budget equation, and the state-variable constraint hold all \(t \geq 0\).

We have \(\eta_t \geq 0\). The path of \(\eta_t\) is piecewise continuous. By construction, \(\eta_t \cdot b_t = 0\). Since \(b_t = 0\) all \(t \geq T_2\), transversality condition (28) holds. The costate variable is nonnegative. It is continuous by construction all \(t\). So, the costate is continuous.

Proof of Proposition 5. Proposition 5 follows from Proposition 4.

Proof of Proposition 6. Let \(f(T)\) be as in the proof of Proposition 1. Because \(x_{T_2} = a\) now, the proof of Proposition 1 shows that

\[ \frac{\partial x_0(b)}{\partial b} = \frac{1}{D(T_2)} = \frac{1}{f(T_2)}. \]

We have

\[ \lim_{T \to \infty} f(T) = \frac{1}{r - \sigma}, \]

completing the proof.
Proof of Proposition 7. Let \((B_t, X_t)\) all \(t \geq 0\) follow any of the dotted trajectories in Figure 4a-c. We know \(M_t \cdot B_t \geq 0\). We bound the latter product by a term converging to 0.

In each figure, \(X_t\) rises with \(t\); hence, from (39),

\[
M_t = e^{-(\Lambda + \beta) \cdot t} \cdot u'(X_t) \leq e^{-(\Lambda + \beta) \cdot t} \cdot u'(X_0) .
\]  

(A28)

From budget constraint (41),

\[
[e^{r \cdot t} \cdot B_t] = e^{-r \cdot t} \cdot [a - X_t].
\]

The fundamental theorem of calculus then shows

\[
e^{-r \cdot t} \cdot B_t = B_0 + \int_0^t e^{-r \cdot s} \cdot [a - X_s] \, ds \leq B_0 + \int_0^t e^{-r \cdot s} \cdot a \, ds \leq [B_0 + \frac{a}{r}] .
\]

(A29)

So,

\[
M_t \cdot B_t \leq e^{-(\Lambda + \beta) \cdot t} \cdot u'(X_0) \cdot e^{r \cdot t} \cdot e^{-r \cdot t} \cdot B_t
\]

\[
\leq e^{-(\Lambda + \beta) \cdot t} \cdot u'(X_0) \cdot e^{r \cdot t} \cdot [B_0 + \frac{a}{r}] = e^{r-(\Lambda + \beta) \cdot t} \cdot u'(X_0) \cdot [B_0 + \frac{a}{r}] .
\]

(A30)

By assumption, \(r - (\Lambda + \beta) < 0\). Hence, (A30) converges to 0 as \(t \to \infty\). So, this problem’s transversality condition holds.

The constraint \(B_t \geq 0\) in the original problem is not violated.

Proof of Proposition 8: Letting \(\eta_t\) be a Lagrange multiplier, we use a present value Hamiltonian

\[
\mathcal{H} \equiv e^{-(\Lambda + \beta) \cdot t} \cdot [u(X_t) + \Lambda \cdot v(a, B_t)] + M_t \cdot [r \cdot B_t + a - X_t] + \eta_t \cdot B_t .
\]

(A31)

At \((B_t, X_t) = (0, a)\), the household budget constraint holds. If

\[
M_t = e^{-(\Lambda + \beta) \cdot t} \cdot u'(a),
\]

then (39) holds. The costate equation is

\[
\dot{M}_t = -e^{-(\Lambda + \beta) \cdot t} \cdot \Lambda \cdot \frac{\partial v(a, B_t)}{\partial B_t} - r \cdot M_t - \eta_t .
\]

(A33)

Case 2 requires last-period-of-retirement case (i), which implies, when \(B_0 = 0\),

\[
\frac{\partial v(a, 0)}{\partial B_t} = \theta \cdot u'(x_0(0)) = \theta \cdot u'(x_{T_1}) .
\]

(A34)

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Setting $B_t = 0$ and substituting from (A32)-(A33),

$$- (\Lambda + \beta) \cdot e^{-(\Lambda + \beta) \cdot t} \cdot u'(a) = -e^{-(\Lambda + \beta) \cdot t} \cdot \Lambda \cdot \frac{\partial v(a, 0)}{\partial B_t} - r \cdot e^{-(\Lambda + \beta) \cdot t} \cdot u'(a) - \eta_t$$

$$\iff \eta_t \cdot e^{(\Lambda + \beta) \cdot t} = -\Lambda \cdot \theta \cdot u'(a)T - [r - (\Lambda + \beta)] \cdot u'(a).$$

(A35)

Noting that $r - (\Lambda + \beta) < 0$,

$$\Gamma^2(0) > \Gamma^1(0) \iff \Omega \cdot x_{T_1} > a$$

$$\iff u'(a) > u'(\Omega \cdot x_{T_1}) = \frac{\Lambda \cdot \theta}{\Lambda - (r - \beta)} \cdot u'(x_{T_1})$$

$$\iff -[r - (\Lambda + \beta)] \cdot u'(a) > \Lambda \cdot \theta \cdot u'(x_{T_1}).$$

This shows that $\eta_t$ defined from the last expression in (A35) is positive.

Since $B_t = 0$ all $t$, the transversality condition for this problem holds.

Proof of Proposition 9: The proof exactly follows those of Propositions 4-5.
References


